ROBUST CONSENSUS FOR NETWORKED SYSTEMS WITH NEGATIVE IMAGINARY PROPERTIES

A thesis submitted to the University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Science and Engineering

2020

Ola Skeik
School of Electrical and Electronic Engineering

Contents

Word count 27,040

List of Tables

List of Figures

The University of Manchester

Ola Skeik
Doctor of Philosophy
Robust consensus for networked systems with negative imaginary properties
January 11, 2020

This thesis investigates multi-agent systems problems for systems with negative imaginary dynamics. Via tools from negative imaginary theory and graph theory, this research addresses the following three multi-agent systems problems: (i) the distributed robust stabilization problem for networked multi-agent systems with strict negative imaginary uncertainties; (ii) the robust output consensus problem for homogeneous multi-agent systems with negative imaginary dynamics; and (iii) a rendezvous problem for multiple wheeled mobile robots, through the development of cooperative control strategies for integrator negative imaginary systems.

The main results of this thesis are summarised as follows. Firstly, a solution to the distributed robust stabilization problem for networked multi-agent systems with strict negative imaginary uncertainties is proposed. The solution includes the derivation of sufficient conditions, in an LMI framework, for the existence of control protocol parameters such that the control protocol robustly stabilizes a networked multi-agent system in presence of strictly negative imaginary uncertainties of certain DC size; and guaranteeing that robust stability is achieved when variations in the network topology occur. Secondly, a solution to the robust output consensus problem for homogeneous multi-agent systems with negative imaginary dynamics is proposed. The solution includes relaxing the assumptions imposed in earlier literature thereby derive robust output consensus conditions under $\mathfrak{L}_2$ external disturbances and model uncertainty which are not restricted and which simplify in the single-input single-output case to provide several insights not easily captured in the multi-input multi-output case. Finally, a solution to a rendezvous problem for nonholonomic wheeled mobile robots via the negative imaginary systems theory is proposed. The solution includes the derivation of necessary and sufficient conditions that guarantee output consensus and output tracking for strongly connected, balanced and directed networks of integrators subject to energy-bounded disturbances using the negative imaginary internal stability theorems and then utilize the results to achieve rendezvous of multiple wheeled mobile robot. Examples are provided in each of the three aforementioned research problems to demonstrate the effectiveness of the associated proposed results. Additionally, experimental results from real-robots are provided for the rendezvous research problem.

This research contributes to the existing literature on cooperative control of multi-agent systems with negative imaginary properties. The research provides a timely and necessary study of the robust stabilization, output consensus and rendezvous problems for networked systems with negative imaginary properties. This current research is important since many practical systems can be modelled as negative imaginary systems and achieving a certain behaviour in multi-agent systems has potential real-world applications.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

i. The author of this thesis (including any appendices and/or schedules to this thesis) owns certain copyright or related rights in it (the "Copyright") and s/he has given The University of Manchester certain rights to use such Copyright, including for administrative purposes.

ii. Copies of this thesis, either in full or in extracts and whether in hard or electronic copy, may be made **only** in accordance with the Copyright, Designs and Patents Act 1988 (as amended) and regulations issued under it or, where appropriate, in accordance with licensing agreements which the University has from time to time. This page must form part of any such copies made.

iii. The ownership of certain Copyright, patents, designs, trade marks and other intellectual property (the "Intellectual Property") and any reproductions of copyright works in the thesis, for example graphs and tables ("Reproductions"), which may be described in this thesis, may not be owned by the author and may be owned by third parties. Such Intellectual Property and Reproductions cannot and must not be made available for use without the prior written permission of the owner(s) of the relevant Intellectual Property and/or Reproductions.

iv. Further information on the conditions under which disclosure, publication and commercialisation of this thesis, the Copyright and any Intellectual Property and/or Reproductions described in it may take place is available in the University IP Policy (see http://documents.manchester.ac.uk/DocuInfo.aspx?DocID=487), in any relevant Thesis restriction declarations deposited in the University Library, The University Library's regulations (see http://www.manchester.ac.uk/library/aboutus/regulations) and in The University's Policy on Presentation of Theses.

Acknowledgements

In the name of ALLAH, the Entirely Merciful, the Especially Merciful.

They said,"Exalted are You; we have no knowledge except what you have taught us.
Indeed, it is You who is the Knowing, the Wise." [2:32] The Noble Quran

First and foremost, all praise and thanks to almighty ALLAH, lords of the worlds, who bestowed upon me guidance, knowledge, strength, courage and patience to carry out and complete this PhD thesis. Without his blessing, this research work could not have been achieved.

I would like to thank my supervisor Alexander Lanzon for giving me this great opportunity to work with him. I would like to thank him for all the support he gave me during the past fours years. Without his full support I would not have been able to reach this stage. It has been a great honour to be working under his supervision.

I must thank The University of Manchester for the President's Doctoral Scholar Award funding which enabled me to undertake this research project. Without this financial support I would not have been able to pursue my PhD.

I also want to express my gratitude to my colleges in the Control Systems Centre, Department of Electrical and Electronic Engineering, especially Gabriela Salcan Reyes, Junyan Hu, Daniel Abara and Suleiman Kurawa for the valuable advices they offered me on different aspects and for making me feel like home.

Special thanks to my family: my parents, my brothers and sister and my parents-in-law for their prayers and support throughout my PhD journey.

Last but not least, I would like to thank my husband, Mohammed, for his endless support, patience, and support through the difficult times of my PhD journey. No words can ever be strong enough to express my gratitude to him for his unconditional love and support.

Publications

1. O.Skeik and A. Lanzon, "Robust output consensus for networks of homogeneous negative imaginary systems," in *Proceedings of the 57th IEEE Conference on Decision and control*, (Miami,FL, USA), pp. 3836-3841, Dec.2018.

2. O.Skeik and A. Lanzon, "Robust stabilization of networked muli-agent systems with strict negative imaginary uncertainties: An LMI approach," in *Proceedings of the 16th European Control Conference*, (Limassol, Cyprus), pp. 167-171, June 2018.

3. O.Skeik, J. Hu, F. Arvin and A. Lanzon,"Cooperative control of integrator negative imaginary systems with application to rendezvous multiple mobile robots," *in Proceedings of the 12th International Workshop on Robot Motion and Control*, (Poznan, Poland), pp. 15-20, July 2019.

4. O. Skeik and A. Lanzon, "Distributed robust stabilization of networked multiagent systems with strict negative imaginary uncertainties," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 14, pp.4845-4858, 2019.

5. O. Skeik and A. Lanzon, "Robust output consensus of homogeneous multi-agent systems with negative imaginary dynamics," *Automatica*, In press.

Notation and Acronyms

Field of numbers

$\mathbb{R}$	real numbers
$\mathbb{R}^n$	real column vectors with n entries
$\mathbb{R}^{n\times m}$	real matrices with n rows and m columns
$\mathbb{C}$	complex numbers
$\mathbb{C}^n$	complex column vectors with n entries
$\mathbb{C}^{n\times m}$	complex matrices with n rows and m columns
j	the imaginary unit, i.e. $j=\sqrt{-1}$

Relational symbols

$\in$	belong to
$\notin$	not an element of
$\subset$	is a proper subset of
$\subseteq$	is a subset of
$\not\subseteq$	is not a subset of
$\cup$	union
$>$	greater than
$\geq$	greater than or equal
$\ngtr$	not greater than
$<$	less than
$\leq$	less than or equal
$\nless$	not less than
$\ll$	much less than

$\neq$	not equal
$\rightarrow$	tends to
$\Rightarrow$	implies
$\Leftarrow$	is implied by
$\Leftrightarrow$	is equivalent to

Matrix operations

0	zero matrix of compatible dimensions
$0_n, 0_{n\times 1}$	is an $n \times 1$ vector with entries 0
$0_{n\times n}$	zero matrix of dimension $n \times n$
1_n	is an $n \times 1$ vector with entries 1
I	identity matrix of compatible dimensions
I_n	identity matrix of dimension $n \times n$
$\mathbb{N}(A)$	null space of matrix A
$\mathbb{R}(A)$	range of matrix A
$\dim(S)$	dimension of S, where S is a subspace of $\mathbb{C}^n$ or $\mathbb{R}^n$
$\mathrm{rank}(A)$	rank of matrix A
A^T	transpose of matrix A
A^*	complex conjugate transpose of matrix A
A^{-1}	inverse of matrix A
A^{-*}	$(A^*)^{-1}$
$A > 0$	positive definite matrix, $x^T Ax > 0$ for all nonzero $x \in \mathbb{R}$
$A \geq 0$	positive semidefinite matrix, $x^T Ax \geq 0$ for all nonzero $x \in \mathbb{R}$
$A > B$	$A - B$ is positive definite matrix
$A \geq B$	$A - B$ is positive semidefinite matrix
$A < 0$	negative definite matrix, $-A$ is positive definite
$A \leq 0$	negative semidefinite matrix, $-A$ is positive semidefinite
$A < B$	$A - B$ is negative definite matrix
$A \leq B$	$A - B$ is negative semidefinite matrix
$\det(A)$	determinant of matrix A
$\mathrm{diag}(A_1, \ldots, A_n)$	block-diagonal matrix with matrices A_i for all $i \in \{1, \ldots, n\}$ on the

	main diagonal
$\lambda_i(A)$	ith eigenvalue of matrix A (when the matrix A has only real eigenvalues)
$\bar{\lambda}(A)$	largest eigenvalue of matrix A (when the matrix A has only real eigenvalues)
$\underline{\lambda}_2(A)$	second smallest eigenvalue of matrix A (when the matrix A has only real eigenvalues)

Miscellaneous

$\forall$	for all
$:$	such that
$\square$	end of proof
$\otimes$	Kronecker product
$\Re[s]$	real part of a complex number $s \in \mathbb{C}$
$[P, K]$	standard positive feedback interconnection between systems P and K
$\lim_{x \to a} f(x)$	limit of $f(x)$ as x tends to a
$\min\limits_{i \in \{1,\dots,n\}} (a_i)$	minimum value of a_i, $i \in \{1, \dots, n\}$
$\max\limits_{i=1,\dots,n} (a_i)$	maximum value of a_i, $i \in \{1, \dots, n\}$
$\sum_{i=1}^{n} a_i$	sum the values of a, starting at a_1 and ending with a_n

Function spaces

$\mathcal{R}^{m \times n}$	space of real, rational, proper transfer function matrices with n inputs and m outputs
$\mathfrak{L}_2[0, \infty)$	subspace of the time domain Lebesgue space $\mathfrak{L}_2(-\infty, \infty)$ of all signals, or vector signals, with bounded energy.
H_2	space of functions of the complex variable s that are analytic for all

	s in the open right half plane (i.e. $s : \Re(s) > 0$), and for which the integrals $\frac{1}{2\pi}\int_{-\infty}^{\infty} x^*(\sigma + j\omega)x(\sigma + j\omega)d\omega$ are uniformly bounded for all $\sigma > 0$
H_∞	subspace of $\mathfrak{L}_\infty$ with functions analytic and bounded in the open right half plane where $\mathfrak{L}_\infty$ is the (Lebesgue) space of all functions essentially bounded on the imaginary axis

Systems operations

$P(j\omega)^*$	complex conjugate transpose of frequency-response function $P(j\omega)$ at each frequency, i.e. $P(j\omega)^* = P(-j\omega)^T$
(A, B, C, D)	state space realisation of $C(sI - A)^{-1}B + D$

Acronyms

CLHP	closed left half plane
LMI	linear matrix inequality
MASs	multi-agent systems
MIMO	multi-input multi-output
NI	negative imaginary
OLHP	open left half plane
SISO	single-input single-output
SNI	strictly negative imaginary
SVD	singular value decomposition
WMR	wheeled mobile robot

Chapter 1

Introduction

1.1 Background

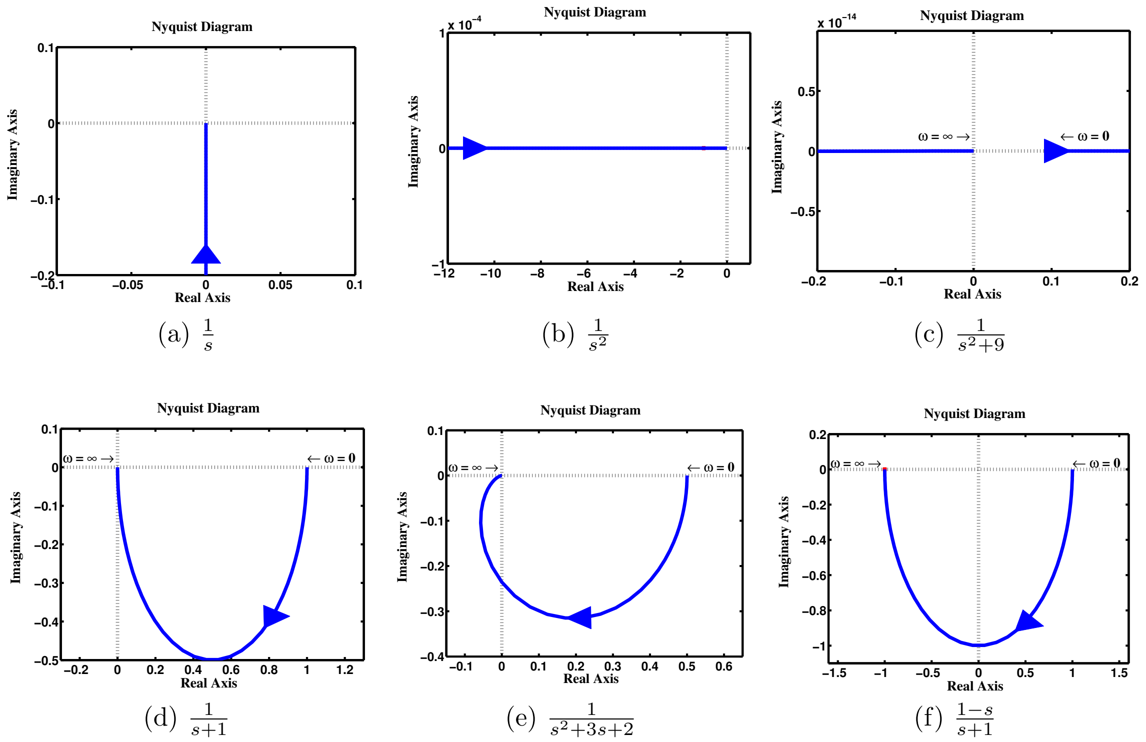
Distributed control of networked multi-agent systems (MASs) and negative imaginary (NI) systems theory are two distinct areas of significant importance to the control systems community.

Distributed control of networked MASs has been an active field of study over the past two decades. Examples of distributed MASs in real-world applications which have a lot of benefits to society include smart grid [1], sensor networks [2], unmanned aerial vehicles [3], and wheeled mobile robots [4], to name a few. The distributed control approach for multi-agent systems is considered more promising compared to the centralized approach due to physical constraints such as limited resources and energy, short wireless communication ranges, narrow bandwidths, and large sizes of agents to control [5]. Furthermore, the motives and advantages of using a distributed control approach for multi-agent systems rather than a centralized approach include flexible scalability, high robustness, and cost reduction in the design, manufacturing, and operation of such systems [6]. Cooperation among agents, through locally shared information, plays a fundamental role in distributed control systems. That is, agents interact locally with each other in order to achieve a desired collective behaviour or a global control objective such as stability of a network, consensus, tracking a reference, rendezvous, synchronization, etc. In other words, distributed control is concerned with the design of distributed control laws (also known as protocols) in order to guarantee a desired collective behaviour or a global control objective for networked MASs. The design of distributed control laws is interesting and challenging due to the need to take into consideration the dynamics of the individual agents and also the interaction among them which is limited to a local neighbour-to-neighbour interaction. Moreover, synthesis of robust control protocols is inevitably required due to uncertainties in the multi-agent dynamics and network environment.

The existing literature on cooperation/distributed control of multi-agent systems is extensive because taking into account different factors lead to different results. For example, factors such as type of agents dynamics whether single integrator, double integrator or general dynamics; communication/network topology among agents whether direct or undirected; available type of shared information whether state or output measurements; the consideration of robust or non-robust design; and the type of desired collective behaviour lead to various analysis and synthesis outcomes. For example, consensus control has been addressed in [7, 8] for single integrators, in [9, 10, 11, 12] for double integrators and in [13, 14, 15, 16] for general dynamics. Consensus/synchronization problems for multi-agent systems using relative output measurements have been addressed in [14, 17, 18, 15, 16]. Robust stability, robust consensus, and robust synchronization of multi-agent dynamical systems with different types of unstructured uncertainties (perturbations due to modelling errors) have been addressed in [19, 20, 18, 17]. Robust consensus under switching topologies has been addressed in [21]. For more details in this area of research, extensive overviews can be found in [22, 23, 5, 24, 25, 26] while books relating to this field of study include [27, 6, 28, 29].

The NI systems theory was first introduced in [30] in the interest to develop, in a systematic framework, robust stability results for flexible structures with co-located force actuators and position sensors. Broadly speaking, negative imaginary systems have the Hermitian imaginary part of their frequency response function matrix negative semi-definite for all positive frequencies excluding frequencies at which a pole exists on the imaginary axis [31]. The transfer functions of NI systems may have a relative degree between zero and two and may have unstable zeros. For single-input single-output (SISO) systems, the NI property of a system can be visualized graphically through Nyquist plots. Specifically, the Nyquist plot of SISO NI systems lie on or below the real axis for all positive frequencies. Strictly negative imaginary

(SNI) systems are an important subset of the NI class. The Nyquist plot for SISO SNI systems strictly lies below the real axis for all positive frequencies and can only touch the real axis at frequencies zero and infinity since the systems are real and rational. Examples of NI systems include single integrator, double integrator and undamped second order systems. The Nyquist plots of such NI systems are shown in Figure 1.1(a) to Figure 1.1(c) respectively. Examples of SNI systems include stable first order systems, lightly damped second order systems and all-pass filters. The Nyquist plots of such SNI systems are shown in Figure 1.1(d) to Figure 1.1(f) respectively.



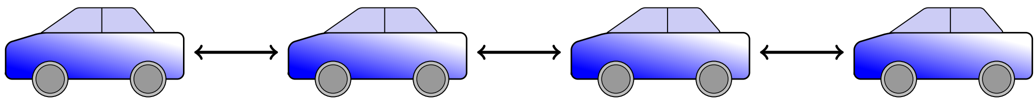
(a) $\frac{1}{s}$ (b) $\frac{1}{s^2}$ (c) $\frac{1}{s^2+9}$

(d) $\frac{1}{s+1}$ (e) $\frac{1}{s^2+3s+2}$ (f) $\frac{1-s}{s+1}$

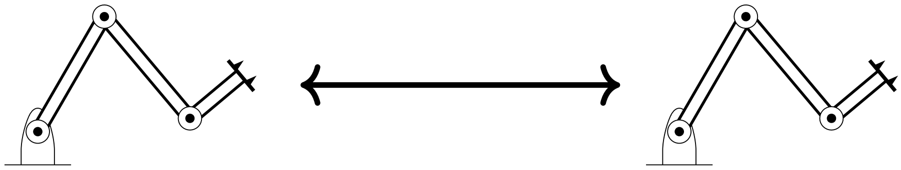
Figure 1.1: Nyquist plots for NI and SNI systems, (a) – (c) are plots for NI systems while (d) – (f) are plots for SNI systems.

In recent years, a great focus has been placed upon the study of NI systems from both a theoretical and application side and rapid developments in this field have been witnessed, such as, for example [32, 33, 34, 35, 36, 37, 38, 39]. Among the areas NI systems literature specifically focuses on include robust stability analysis [34, 40, 41, 30], controller synthesis [42, 43, 44, 45, 46, 47, 48], discrete-time systems [33, 49, 50, 51], nano-positioning systems [36, 52, 53], and applications to multi-agent systems [38, 39, 54].

The study of (robust) cooperative control of multiple NI systems is motived by applications where an individual NI system can not achieve a specific goal on its own, such as controlling and ensuring stability of large vehicle platoons or carrying a large load to a desired destination by means of cooperation among multi-link robotic arms [55, 54, 38, 39] (see Figure 1.2).



(a) large vehicle platoon



(b) multi-link robotic arms

Figure 1.2: Examples of cooperation of NI multi-agent systems.

In this thesis, we focus on investigating multi-agent systems problems for negative imaginary systems. Considering relevant literature, we mention [55, 54, 38, 39]. In [55], the authors analyse the stability of a string of subsystems with NI properties and apply the results to decentralized control of large vehicle platoons. However, the class of systems considered in the study is limited to SISO SNI systems and the interconnection of the subsystems is limited to a string interconnection. In [54], the authors also study the stability of a string of subsystems but for a wider class of systems with NI properties and apply the results to decentralized control of large vehicle platoons. The class of systems considered in [54] are multi-input multi-output (MIMO) NI systems with possible poles on the imaginary axis excluding poles at the origin. Still the interconnection of the subsystems in [54] is limited to a string interconnection. In [38, 39], the authors addressed the robust output feedback consensus problem for networks of homogeneous and heterogeneous NI systems respectively. Specifically, the issue was addressed by reformulating the consensus problem into an internal stability problem, owing to properties of Laplacian matrix of the network graph, and thus providing a solution by means of NI systems robust stability results in [30, 41, 40]. Conditions were derived that guarantee output consensus under $\mathfrak{L}_2$ external disturbances and model uncertainties. However, the works in [38, 39] have several limitations which will be discussed in details in the next section.

1.2 Inadequacy of previous research and identified gaps in literature

Although robust output feedback consensus was investigated in [38, 39] for networks of homogeneous and heterogeneous negative imaginary systems respectively under $\mathfrak{L}_2$ external disturbance and model uncertainties, the work therein has several limitations. Some of the limitations in [38, 39] and identified gaps in the field of robust cooperative control of multiple NI systems are discussed as follows. First, the nominal plants in [38, 39] are assumed NI but the situation, which appears due to physical considerations, where the only knowledge about the system is that its perturbation belongs to a certain class (in this case SNI perturbations) has not been considered. Second, the derived conditions in [38, 39] which guarantee robust output consensus for networked NI systems are only applicable when certain assumptions hold. However, there are cases where these assumptions fail to hold, leading to a need to develop a framework for the robust output consensus problem for networked NI systems which can handle all situations. Third, the works in [38, 39] only deals with undirected graphs to model the interaction among the systems. On the other hand, no previous study has investigated whether output consensus and tracking can be achieved for a certain class of NI systems under directed communication among the systems. Finally, numerical examples provided in the cooperative control of multiple NI systems literature to validate the associated theoretical results are limited to simulation via Matlab. The only work known in this area to have provided experimental validation is given in [56]. Hence, literature in this area generally lacks validating the theoretical results through experiments on real systems.

1.3 Research problems

This thesis aims at addressing the aforementioned identified gaps and limitations. Via tools from NI systems theory and graph theory, this current PhD study will focus on addressing the following multi-agent systems problems:

- the distributed robust stabilization problem for networked multi-agent systems with strict negative imaginary uncertainties and communication among agents in the network modelled by an undirected graph with at least one self-loop;

- the robust output consensus problem for homogeneous multi-agent systems with negative imaginary dynamics that are subject to $\mathfrak{L}_2$ external disturbances and model uncertainties; and communication among systems in the network modelled by an undirected graph in a generalised framework;

- a rendezvous problem for multiple wheeled mobile robots, through the development of cooperative control strategies for integrator NI systems with directional information flow that is balanced and strongly connected.

1.4 Motivation for current research

1.4.1 Distributed robust stabilization problem for networked MASs with SNI uncertainties

The motivating factors for considering to address the distributed robust stabilization problem for networked systems with SNI uncertainties are as follows. Firstly, flexible structures are usually modelled as infinite dimensional distributed parameter systems [40]. It is typical, however, to use a finite dimensional model for control design purpose leading to modelling errors due to unmodelled dynamics. The transfer function of such systems from force inputs and position outputs often poses the NI property and the associated unmodelled dynamics usually belong to the SNI class [30]. It is vital in the control design process to take such unmodelled dynamics in to account, otherwise these unmodelled dynamics may lead to instability and performance degradation of the controlled system. Secondly, recall that in [38, 39] the nominal plants are assumed NI. However, due to physical considerations, in some situations the only thing known about each agent/system is that the perturbation belongs to the SNI class. Consequently, in order to guarantee stability against this class of uncertainties via the robust feedback stability result in [30, 41, 34, 40] a control protocol needs to be designed such that the nominal closed-loop networked system is NI. Although the works in [31, 47, 46, 44] address the case where a system's perturbation is SNI, the works therein only deal with individual systems and are not suitable for networks of systems. Particularly in the literature on NI systems, a systematic robust static state feedback synthesis method for (single) systems with SNI uncertainty was proposed in [47]. On the other hand, no previous study has addressed control synthesis in an LMI framework for robust stability of networked systems with SNI uncertainties. Lastly, dynamic uncertainties due to modelling error have not been much of a focus in the literature on networked multi-agent systems. Few researchers have only considered taking such uncertainties into account in the analysis and synthesis of distributed control protocols. Furthermore, the focus was mainly on uncertainties bounded in H_∞ norm. In this context, recall that robust stability of multi-agent dynamical systems was studied in [19] where three different types of multiplicative perturbations were considered. Robust synchronization of uncertain multi-agent networks was addressed in [18] and [17] with uncertainties in the form of additive perturbations in [18] and in the form of coprime factor perturbations in [17]. Robust consensus control for multi-agent systems involving gap metric uncertainties was investigated in [20]. It is known [30] that, the dynamic uncertainties of NI systems are mainly characterised by phase bounds, where the phase lies between $-\pi$ and 0. Using phase information in control design for lightly damped (i.e. highly resonant) systems is much more effective than using gain information [47, 57]. In other words, for such lightly damped systems gain stabilization, which is dependent on the small-gain theorem, leads to conservative design results [30, 47]. On the other hand, phase stabilization, which ensures stability by restricting the phase of the open-loop system and by which the NI robust stability results were established, yields more powerful, less conservative and robust control systems [31, 57]. Consequently, all the above factors motivate the investigation of the distributed robust stabilization problem for networked systems with SNI uncertainties via tools from NI systems theory.

1.4.2 Robust output consensus problem for homogeneous multi-agent systems with negative imaginary dynamics

Consensus is defined as agents reaching an agreement on a certain quantity of interest [25]. Reaching consensus in MASs is a common and desirable collective behaviour and has been extensively studied in the cooperative control of MASs including the study in [38, 39] for the case where the agents are NI systems. The motivations for this current study comes from the importance of consensus of MAS in real-world applications [23], the many practical systems that can be modelled as NI systems [30, 40], and the establishment of the general internal stability results in [34] by which it is possible to extend the work of [38].

Internal stability of interconnected systems is a fundamental requirement in control systems design [58]. Lightly damped flexible structures with colocated position sensors and force actuators are highly resonant dynamical systems. The modelling of such systems can often belong to the class of negative imaginary systems. Ensuring stability of such systems in face of unmodelled dynamics is quite difficult [30, 31]. Therefore, the literature on NI systems has paid particular attention to robust stability analysis of interconnected NI systems [30, 41, 40, 34]. The first robust stability result for NI systems was established in [30] for stable NI systems and was later found to be valid for NI systems with poles on the imaginary axis excluding poles at the origin in [41]. The result in [30, 41] showed that the internal stability of two systems connected in a positive feedback interconnection where one system is negative imaginary and the other system is strictly negative imaginary is tested via a dc loop gain condition provided two assumptions at infinite frequency hold. Afterwards, conditions for internal stability were proposed in [40] for NI systems with possible poles at the origin despite being proposed under restrictive assumptions. These internal stability results were initially found useful in addressing robust consensus problems for networked NI systems. By using these results, the robust output feedback consensus problem was addressed in [38, 39] for networks of homogeneous and heterogeneous NI systems respectively. Specifically, the issue was addressed by reformulating the consensus problem into an internal stability problem, owing to properties of Laplacian matrix of the network graph, and thus providing a solution by means of the aforementioned NI systems robust stability results in [30, 41, 40]. Conditions were derived that guarantee output consensus under $\mathfrak{L}_2$ external disturbances and model uncertainties. However, the restrictive assumptions under which the aforementioned robust consensus conditions are applicable make the work in [38, 39] not suitable for cases where these assumptions fail to hold. Hence, lacking conditions for all situations, it would not be possible to determine robust output consensus for networked NI systems except for those that satisfy the assumptions in [38, 39]. Consequently, a framework for the robust output consensus problem for networked negative imaginary systems is needed to fill this gap in literature. Recently, new results for internal stability of a positive feedback interconnection of an NI system and an SNI system has been developed in [34] which generalise the existing internal stability results by removing the restrictive assumptions which had been previously imposed. These new results motivate this current research, in which the robust output consensus problem for networks of homogeneous negative imaginary systems is addressed which extends and overcomes limitations in [38].

1.4.3 Rendezvous problem for multiple WMRs

The motivating factors for considering a rendezvous problem for multiple wheeled mobile robots via NI systems theory are as follows. Firstly, rendezvous of multi-agent systems is an important desirable task in cooperative control of multi-agent systems. The rendezvous problem is closely related to the consensus problem with position being the state of interest in rendezvous [24]. In applications where the position is the state of interest rather than velocity, the NI systems theory is deemed effective. Moreover, it is possible to simplify the dynamics of nonholonomic WMRs to a single integrator model via input-output linearisation. Therefore, simple yet sophisticated control laws for integrator NI systems can be designed in a systematic way which can then be directly applied to WMRs. This motives the choice of the NI systems theory to tackle the rendezvous problem for multiple nonholonomic WMRs. Secondly, recall that the robust output consensus problem was addressed in [38] and [39], and the robust output tracking problem was addressed in [39] for networks of NI systems. Nevertheless, the works in [38, 39] are limited to undirected graphs when it comes to modelling the interaction among the systems. This motivates the desire to investigate consensus and tracking problems for certain classes of NI systems with directed information flow among them. Finally, recall that the literature on multi-agent NI systems lacks validation of theoretical results through experiments on real systems. This motivates the interest of validating the developed cooperative control strategies on real-robots.

1.5 Thesis outline

The thesis is divided into six chapters which is organized as follows.

Chapter 1: Introduction

In this chapter, a background on distributed control of networked MASs, NI systems theory, and robust cooperative control of multiple NI systems is first given. Then, inadequacies in previous research and identified gaps in literature are discussed. Afterwards, the research problems that are addressed in this thesis are stated followed by the motivations for carrying out the current research and addressing such problems.

Chapter 2: Preliminaries

In this chapter, the necessary mathematical tools and material that underpin the development of the work in this thesis are presented. The chapter begins with providing some basic concepts in matrix theory followed by some basic concepts on state space systems and linear matrix inequalities respectively. Moreover, foundational results in negative imaginary systems theory are given in this chapter. The chapter is concluded with material related to algebraic graph theory.

Chapter 3 : Distributed robust stabilization of networked multi-agent systems with strict negative imaginary uncertainties

In this chapter, a solution to the distributed robust stabilization problem for networked multi-agent systems with strict negative imaginary uncertainties is proposed. Via transformation techniques and under certain assumptions on the network graph which models the communication among the agents, a result is given that simplifies the problem under consideration. Consequently, sufficient conditions, in an LMI framework, are derived to ensure the existence of control protocol parameters such that the control protocol robustly stabilizes a networked multi-agent system in presence of SNI uncertainties of certain DC size. An algorithm for control protocol design is also provided and the advantages of this design algorithm are discussed. A numerical example is then given to show the usefulness of the proposed results.

Chapter 4: Robust output consensus of homogeneous multi-agent systems with negative imaginary dynamics

In this chapter, a solution to the robust output consensus problem for networks of homogeneous negative imaginary systems (with possible poles at the origin) is proposed. The general internal stability results developed in [9] are utilized. By removing certain assumptions which had been imposed earlier in the literature, necessary and sufficient conditions that ensure robust output consensus for networks of homogeneous NI systems under $\mathfrak{L}_2$ external disturbances and model uncertainty are derived. It is also shown that, when the NI systems have no poles at the origin, the derived conditions specialise to those in [38] by either imposing the same two assumptions at infinite frequency or by imposing different assumptions which had not been known previously. Moreover, conditions that ensure robust output consensus for some special cases are provided including specialisation to SISO NI systems. The advantages of these specialised cases are also discussed. A detailed convergence analysis is also provided followed by two examples which demonstrate the effectiveness of the proposed results over earlier results when the assumptions of earlier results do not hold.

Chapter 5: Cooperative control of integrator negative imaginary systems with application to rendezvous multiple mobile robots

In this chapter, a solution to a rendezvous problem for nonholonomic wheeled mobile robots is proposed through the development of cooperative control strategies for integrator NI systems with directional information flow that is balanced and strongly connected. It is shown that a network of homogeneous MIMO integrators with directed information flow that is balanced and strongly connected retains the NI property. Afterwards, necessary and sufficient conditions are derived that guarantee output consensus and output tracking for strongly connected, balanced and directed networks of integrators subject to energy-bounded disturbances using the NI internal stability theorems. Experimental results from both real-robot and simulation are then provided to validate the effectiveness of the proposed theoretical results in solving the rendezvous problem for multiple wheeled mobile robots.

Chapter 6: Conclusions

In this chapter, the contributions of this thesis are summarised and possible directions for future research are discussed.

Chapter 2

Preliminaries

This chapter includes the necessary mathematical tools and material that underpin the development of the work in this thesis. The main topics covered in this chapter are as follows. First, some basic matrix theory is introduced in Section 2.1. Then, some basic concepts on state space systems and linear matrix inequalities are covered in Section 2.2 and Section 2.3 respectively. Foundational results in negative imaginary systems theory is given in Section 2.4. Finally, substantial material on algebraic graph theory is presented in Section 2.5.

2.1 Matrix Theory

In this section, basic material on matrix theory is covered. Material covered in this section can be found in [59, 60, 61, 62, 63].

2.1.1 Miscellaneous

The null space, range and rank of a matrix are defined in the following three definitions respectively.

Definition 2.1 ([59, 60]). *Let $A \in \mathbb{C}^{m\times n}$. Then, the null space of A is defined as the set of all vectors x for which $Ax = 0$. That is*

$$\mathbb{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\}.$$

Definition 2.2 ([59, 60]). *Let $A \in \mathbb{C}^{m\times n}$. Then, the range of A is defined as*

$$\mathbb{R}(A) = \{y \in \mathbb{C}^m : y = Ax, x \in \mathbb{C}^n\}.$$

Definition 2.3 ([60, 61]). *Let $A \in \mathbb{C}^{m\times n}$. Then rank A is defined as*

$$\mathrm{rank}(A) = \dim(\mathbb{R}(A)).$$

Since rank (A)= rank (A^*), then rank (A) equals the maximum number of independent rows or columns.

Diagonal and block diagonal matrices are defined below.

Definition 2.4 ([62]). *A matrix $A \in \mathbb{C}^{n\times n}$ is diagonal if $a_{ij} = 0$ for $i \neq j$.*

Definition 2.5 ([62]). *A block diagonal matrix $A \in \mathbb{C}^{n\times n}$ is of the form*

$$A = \begin{bmatrix} A_{11} & & 0 \\ & \ddots & \\ 0 & & A_{kk} \end{bmatrix} = \mathrm{diag}(A_{11}, \ldots, A_{kk})$$

in which $A_{ii} \in \mathbb{C}^{n_i\times n_i}$, $i = 1, \ldots, k$, $\sum_{i=1}^{k} n_i = n$, and all blocks above and below the block diagonal are zero blocks.

Unitary and Orthogonal matrices are defined as follows.

Definition 2.6 ([62, 61]). *A matrix $A \in \mathbb{C}^{n\times n}$ is unitary if $A^*A = AA^* = I$. A matrix $A \in \mathbb{R}^{n\times n}$ is orthogonal if $A^TA = AA^T = I$.*

The definitions of Hermitian, symmetric and semidefinite matrices are given below.

Definition 2.7 ([62]). *A matrix $A \in \mathbb{C}^{n\times n}$ is said to be Hermitian if $A^* = A$. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be symmetric if $A^T = A$.*

Definition 2.8 ([62, 61]). *A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is*

1. *positive definite if and only if $x^TAx > 0$ for all nonzero $x \in \mathbb{R}$ and it is written as $A > 0$.*

2. *positive semidefinite if and only if $x^TAx \geq 0$ for all nonzero $x \in \mathbb{R}$ and it is written as $A \geq 0$.*

3. *negative definite if $-A$ is positive definite and it is written as $A < 0$.*

4. *negative semidefinite if $-A$ is positive semidefinite and it is written as $A \leq 0$.*

2.1.2 Kronecker product

The Kronecker product is an important notion that has been extensively used throughout this thesis to express the collective networked system. Thus, the definition and properties of Kronecker product are summarized below and can be found in [59, 60, 61].

Definition 2.9. *Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, then the Kronecker product of A and B is defined as the matrix*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}.$$

Lemma 2.1. *Let A, B, C and D be of appropriate dimensions. Then, the following properties hold:*

1. $(I_m \otimes I_n) = I_{mn}$,

2. $(I_n \otimes A) = \mathrm{diag}(A, A, \cdots, A)$,

3. $(A \otimes I_n) = \begin{bmatrix} a_{11}I_n & a_{12}I_n & \cdots & a_{1m}I_n \\ \vdots & \vdots & & \vdots \\ a_{m1}I_n & a_{m2}I_n & \cdots & a_{mm}I_n \end{bmatrix}$,

4. $(\mu A) \otimes B = A \otimes (\mu B) = \mu(A \otimes B)$, *where μ is a scalar,*

5. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$,

6. $(A \otimes B)^T = A^T \otimes B^T$,

7. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$,

8. $A \otimes (B + C) = A \otimes B + A \otimes C$.

2.1.3 Singular value decomposition

The singular value decomposition (SVD) is a powerful tool in matrix analysis and applications. It can be used in analysing gains and directionality in multi-variable systems [63]. One of the advantages of SVD is that it can be applied to every matrix. That is, the SVD is not limited to square matrices and can also be applied to non-square matrices. More details on SVD can be found in [61, 60, 63].

Theorem 2.1. *[61, 60]. Let $A \in \mathbb{C}^{m\times n}$. Then there exists unitary matrices*

$$U = \begin{bmatrix} u_1, u_2, \dots, u_m \end{bmatrix} \in \mathbb{C}^{m\times m}$$
$$V = \begin{bmatrix} v_1, v_2, \dots, v_n \end{bmatrix} \in \mathbb{C}^{n\times n}$$

such that

$$A = U\Sigma V^*, \qquad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & \cdot & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p \end{bmatrix}$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0, \quad p = \min\{m, n\}$$

and σ_i is the ith *singular value.*

The columns of U are called the left singular vectors of A while the columns of V are called the right singular vectors of A.

2.1.4 Eigenvalues and Eigenvectors

The eigenvalues of the Laplacian matrix of a network graph play an important role in consensus control analysis and design. Also, the eigenvalues and eigenvectors of the Laplacian matrix have been used in chapter 4 of this thesis to determine the final convergence of a multiple NI systems in consensus. Hence, a brief explanation of the concept of eigenvalues and eigenvectors is given below. More details can be found in [62, 61].

Definition 2.10 ([62])**.** *Let $A \in \mathbb{C}^{n\times n}$. If a scalar λ and a nonzero vector x satisfying the equation*

$$Ax = \lambda x, \qquad x \in \mathbb{C}^n, x \neq 0, \lambda \in \mathbb{C}$$

then λ is called an eigenvalue of A and x is called an eigenvector of A associated with λ.

Definition 2.11 ([62, 61])**.** *The spectrum of $A \in \mathbb{C}^{n\times n}$ is the set of eigenvalues of A, i.e, the set of all roots of the characteristic polynomial* $\det(A - \lambda I) = 0$.

A nonzero vector $x \in \mathbb{C}^n$ is a right eigenvector of A corresponding to λ an eigenvalue of A if

$$Ax = \lambda x.$$

In a similar manner, a nonzero vector $y \in \mathbb{C}^n$ is a left eigenvector of A corresponding to λ an eigenvalue of A if

$$y^* A = \lambda y^*.$$

The notion of algebraic and geometric multiplicity is given as follows.

Definition 2.12 ([61])**.** *Let λ be an eigenvalue of A. The algebraic multiplicity is the number of times λ appears as a root of the characteristic polynomial* $\det(A - \lambda I) = 0$. *The geometric multiplicity of λ is the number of associated independent eigenvectors which is equal to dimension of the nullspace of $A - \lambda I$;* $\dim \mathbb{N}(A - \lambda I)$.

Generally, the geometric multiplicity is less than or equal to the algebraic multiplicity. In fact if the geometric multiplicity is less than its algebraic multiplicity, then A is said to be defective. If for every eigenvalue of A, the geometric multiplicity is equal to its algebraic multiplicity, then A is said to be diagonalizable.

2.1.5 Jordan block

Every complex matrix can be transformed into a Jordan canonical form. The notion of Jordan canonical form is given in the following theorem.

Theorem 2.2. *[61, 62] Let $A \in \mathbb{C}^{n\times n}$ with $\{\lambda_1, \lambda_2, \ldots, \lambda_{q-1}, \lambda_q\}$ distinct eigenvalues. Then there exists a nonsingular matrix $T \in \mathbb{C}^{n\times n}$ such that*

$$T^{-1}AT = J = \operatorname{diag}(J_1, \ldots, J_q)$$

where each Jordan block matrix J_i $\forall i \in \{1, \ldots, q\}$ has the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & & 0 \\ 0 & \lambda_i & 1 & 0 & & \vdots \\ \vdots & \ddots & \lambda_i & \ddots & \ddots & \\ & & & \ddots & 1 & 0 \\ \vdots & & & \ddots & \lambda_i & 1 \\ 0 & \ldots & & \ldots & 0 & \lambda_i \end{bmatrix}.$$

2.2 State space systems

This section presents some basic concepts in linear control systems theory. The material in this section can be found in [60].

Suppose that $P(s)$ is a real-rational transfer matrix which is proper. Then a state space model (A, B, C, D) such that

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(sI - A)^{-1}B + D$$

is said to be a realisation of $P(s)$. The A, B, C, and D are real constant matrices with appropriate dimensions.

Definition 2.13. *A matrix A is said to be stable or Hurwitz if all its eigenvalues are in the open left half plane, that is, all its eigenvalues have strictly negative real part.*

Theorem 2.3. *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times q}$, then the following are equivalent:*

i) (A, B) is controllable.

ii) The matrix $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has full row rank for all λ in $\mathbb{C}$.

iii) For any eigenvalue λ of A and associated left eigenvector x such that $x^*A = x^*\lambda$, $x^*B \neq 0$.

Theorem 2.4. *Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{q\times n}$, then the following are equivalent:*

i) (C, A) is observable.

ii) The matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank for all λ in $\mathbb{C}$.

iii) For any eigenvalue λ of A and associated right eigenvector y such that $Ay = \lambda y$, $Cy \neq 0$.

Definition 2.14. *A state space realisation (A, B, C, D) of $P(s)$ is said to be minimal if (A, B) is controllable and (C, A) is observable.*

2.3 Linear matrix inequalities

Linear matrix inequalities (LMIs) are a powerful tool that are used to formulate a wide range of control systems problems which can be solved numerically (when difficult to solve analytically) in an efficient way using, for example, interior point techniques [64, 65, 66]. In chapter 3 of this thesis, we use an LMI approach to address the distributed robust stabilization problem for networked multi-agent systems with strict negative imaginary uncertainties. Therefore, some basics about LMIs are gathered below (see [64, 65, 66] for more details).

Definition 2.15 ([64]). *A linear matrix inequality is a constraint of the form*

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i > 0 \tag{2.1}$$

where $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ is the variable and $F_i = F_i^T \in \mathbb{R}^{n\times n}$, $i = 0, \ldots, n$ are given symmetric matrices.

The LMI (2.1) is a convex constraint on x. Thus, its solution set is a convex subset and finding a solution to the LMI (2.1) is called a feasibility problem [65]. For non-strict LMIs, the inequality (2.1) is replaced by a non-strict one, i.e., $F(x) \geq 0$. Also, the inequality (2.1) can be rewritten as $-F(x) < 0$. Furthermore, multiple LMI constraints can be considered as a single LMI constraint since $F_1(x) > 0, F_2(x) > 0, \ldots, F_k(x) > 0$ is equivalent to $F(x) = \text{diag}(F_1(x), \ldots, F_k(x) > 0$. In a wide variety of control problems the decision variable is a matrix variable instead of a scalar. Thus, the inequality(2.1) is considered as $F(X) > 0$ where $X = X^T \in \mathcal{R}^{n\times n}$.
When a control systems problem is formulated in terms of LMIs, the problem can be solved exactly by efficient convex optimization algorithms, known also as "LMI solvers" [65]. YALMIP [67] is a modelling language used to define and solve optimization problems. Moreover, SeDuMi [68] is one of the optimization solvers that YALMIP

depends upon to solve an optimization problem. In chapter 3 of this thesis, we use YALMIP and SeDuMi to solve the LMI constraints to obtain a feasible solution.

2.4 Negative Imaginary Systems

In this section we recall some foundational results in the NI systems theory which have been instrumental in this thesis.

Negative imaginary systems with poles on the imaginary axis excluding poles at the origin are defined as follows.

Definition 2.16 ([41]). *A square, real, rational, proper transfer function matrix $P(s)$ is said to be negative imaginary if*

1. *$P(s)$ has no poles at the origin and in $\Re[s] > 0$;*
2. *$j[P(j\omega) - P(j\omega)^*] \geq 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole of $P(s)$;*
3. *if $j\omega_0$ with $\omega_0 \in (0, \infty)$ is a pole of $P(s)$, then it is a simple pole and the residue matrix $K_0 = \lim_{s \to j\omega_0}(s - j\omega_0)jP(s)$ is Hermitian and positive semidefinite.*

The definition of NI systems has been extended to include poles at the origin in [40]. Negative imaginary systems (with possible poles at the origin) are defined as follows.

Definition 2.17 ([40]). *A square, real, rational, proper transfer function matrix $P(s)$ is said to be negative imaginary if*

1. *$P(s)$ has no poles in $\Re[s] > 0$;*
2. *$j[P(j\omega) - P(j\omega)^*] \geq 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole of $P(s)$;*
3. *if $j\omega_0$ with $\omega_0 \in (0, \infty)$ is a pole of $P(s)$, then it is a simple pole and the residue matrix $K_0 = \lim_{s \to j\omega_0}(s - j\omega_0)jP(s)$ is Hermitian and positive semidefinite;*
4. *if $s = 0$ is a pole of $P(s)$, then $\lim_{s \to 0} s^k P(s) = 0$ $\forall k \geq 3$ and $\lim_{s \to 0} s^2 P(s)$ is Hermitian and positive semidefinite.*

Strictly negative imaginary systems are a subset of the NI class. Strictly negative imaginary systems are defined as follows.

Definition 2.18 ([30]). *A square, real, rational, proper transfer function matrix $K(s)$ is said to be strictly negative imaginary if*

1. $K(s)$ *has no poles in* $\Re[s] \geq 0$;

2. $j[K(j\omega) - K(j\omega)^*] > 0$ *for all* $\omega \in (0, \infty)$.

Let $P(s)$ be an NI system without poles at the origin and $K(s)$ be an SNI system, then the following properties hold (see [30] and also [41, Cor. 3]):

Property 2.1. $P(0) \geq P(\infty)$,

Property 2.2. $K(0) > K(\infty)$.

The following lemma is used to check whether a system belongs to the class of NI or not.

Lemma 2.2 ([47]). *Let (A, B, C, D) be a state space realization of $P(s) \in \mathcal{R}^{m\times m}$ where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$ with $m \leq n$. If* $\det(A) \neq 0$, *$D = D^T$ and there exists a real matrix $Y = Y^T > 0$ such that*

$$AY + YA^T \leq 0 \quad \textit{and} \quad B + AYC^T = 0, \tag{2.2}$$

then $P(s)$ is negative imaginary.

The following theorem is the first main robust stability result for NI systems found in literature. It is valid for NI systems with poles on the imaginary axis excluding poles at the origin. The result states that a necessary and sufficient condition for internal stability of two systems interconnected via positive feedback with one system negative imaginary and the other strict negative imaginary is that their DC loop gain be less than unity provided two conditions at infinite frequency are also satisfied.

Theorem 2.5 ([30, 41]). *Given a negative imaginary transfer function matrix $P(s)$ and a strictly negative imaginary transfer function matrix $\Delta(s)$ that also satisfy $\Delta(\infty) \geq 0$ and $P(\infty)\Delta(\infty) = 0$. Then the positive feedback interconnection $[P(s), \Delta(s)]$ as shown in Figure 2.1 is internally stable if and only if $\bar{\lambda}(P(0)\Delta(0)) < 1$.*

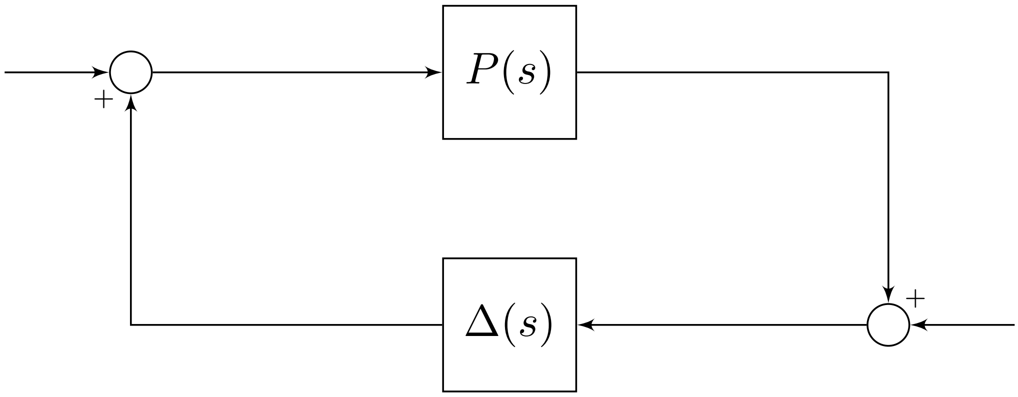


Figure 2.1: Positive feedback interconnection of $P(s)$ and $\Delta(s)$.

The following lemma characterises robust stability for NI systems in the same form as the small gain theorem. This result is a corollary to the principal theorem stated in the literature (also stated above and can be found in [30, Th. 5] or [41, Th. 1]). It was first proposed in [30] for stable NI systems and later shown to be also valid for marginally stable NI systems in [41].

Lemma 2.3 ([30, 41])**.** *Given $\gamma > 0$ and a negative imaginary transfer function matrix $P(s)$. Then the positive feedback interconnection $[\Delta(s), P(s)]$ is internally stable for all strict negative imaginary transfer function matrices $\Delta(s)$ satisfying $\Delta(\infty)P(\infty) = 0$, $\Delta(\infty) \geq 0$ and $\bar{\lambda}\left(\Delta(0)\right) < (1/\gamma)$ (respectively, $\leq (1/\gamma)$) if and only if $\bar{\lambda}\left(P(0)\right) \leq \gamma$ (respectively, $< \gamma$).*

The following lemma provides an internal stability result when the NI system has a single pole at the origin.

Lemma 2.4 ([40])**.** *Let the transfer function matrix $K(s)$ be SNI and the strictly proper transfer function matrix $P(s)$ be NI. Define $P_2 = \lim_{s \to 0} s^2 P(s)$, $P_1 = \lim_{s\to 0} s\left(P(s) - \frac{P_2}{s^2}\right)$, and $P_0 = \lim_{s\to 0}\left(P(s) - \frac{P_2}{s^2} - \frac{P_1}{s}\right)$. Let $P_2 = 0$ and $P_1 \neq 0$. Factorise $P_1 = F_1 V_1^T$ with F_1 and V_1 having full column rank such that $V_1^T V_1 = I$. Suppose that $\mathbb{N}(P_1^T) \subseteq \mathbb{N}(P_0^T)$. Then the closed-loop positive feedback interconnection between $P(s)$ and $K(s)$ is internally stable if and only if*

$$F_1^T K(0) F_1 < 0. \tag{2.3}$$

2.5 Algebraic graph theory

Algebraic graph theory is fundamental in the study of cooperative and consensus control of multi-agent systems. In this section, we present some substantial knowledge on graph theory which has been instrumental in this thesis. Detailed material on graph theory can be found in [6, 26].

Graphs are used to model information exchange among agents in a network. The vertices (also known as nodes) of a graph represent the agents while the edges represent the communication topology (information exchange) among agents. Graphs are categorized as directed or undirected. Furthermore, undirected graphs can either be allowed to have self-loops or not be allowed to have self-loops. Figure 2.2 shows three graphs with different communication topologies.

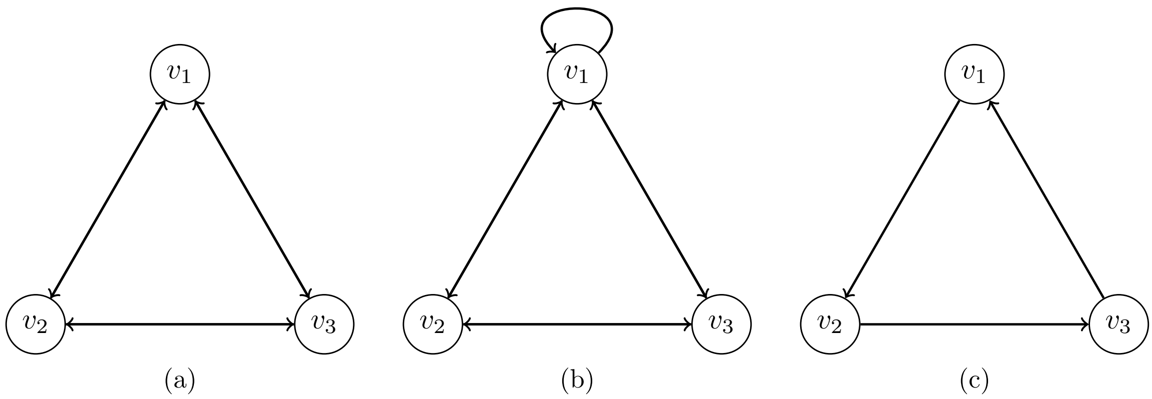


Figure 2.2: Network graphs with different topologies. (a) Undirected graph without self-loops. (b) Undirected graph with a self-loop. (c) Directed graph.

The information flow is bidirectional in an undirected graph whereas directional in a directed graph. Directed communication among agents means that some agents can only receive information while others can only send information to their neighbouring agents. On the other hand, undirected communication means that each agent can both receive and send information to its neighbouring agents. This information can be either the state measurements of agents if available, or the output measurements of agents when the state measurements are not available. If a failure in the communication link among agents occurs (which means additional or removal of some edges from the graph) the network topology is known as switching topology otherwise it is a fixed topology. Furthermore, the dynamics of agents in a network can all be the same, in such case the network is known as a homogeneous network or they may differ from each other, in this case the network is known as a heterogeneous network.

In what follows, a detailed description about undirected and directed graphs is given including the adjacency and Laplacian matrices which play an important role in solving consensus problems.

.

2.5.1 Undirected graphs

An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a non-empty finite vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of unordered pairs of vertices, called edges. An edge in $\mathcal{G}$ is denoted by (v_i, v_j). If $(v_i, v_j) \in \mathcal{E}$, then vertices (i.e., agents) v_i and v_j are adjacent (or neighbours) and can obtain information from each other. The set of neighbours of vertex v_i is defined as $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. An edge (v_i, v_i) is called a self-loop. A loop around vertex v_i means that agent v_i has access to its own absolute measurements. A graph is said to be simple if it contains no self-loops and no repeated edges. For simple graphs, self edges are not allowed, that is, $(v_i, v_i) \notin \mathcal{E}$. A path in a graph from v_i to v_j is a sequence of edges of the form $(v_i, v_{i+1}), (v_{i+1}, v_{i+2}), \ldots, (v_{j-1}, v_j)$. An undirected graph is connected if there is an undirected path between every pair of distinct vertices. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ of $\mathcal{G}$ with self-loops is defined as $a_{ij} = a_{ji} = 1$ if $(v_i, v_j) \in \mathcal{E}$, $a_{ii} = 1$ if v_i has a self-loop, and 0 otherwise. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ of $\mathcal{G}$ with no self-loops is defined as $a_{ij} = a_{ji} = 1$ if $(v_i, v_j) \in \mathcal{E}$, 0 otherwise. The Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ of $\mathcal{G}$ is defined as $l_{ij} = -a_{ij}$, for $i \neq j$ and $l_{ii} = \sum_{j=1}^{N} a_{ij}$ for all $i \in \{1, \ldots, N\}$. Based on the adjacency matrix, this definition can fit for both simple graphs and for graphs with self-loops.

The notation $\hat{\mathcal{L}}$ is hereafter used to indicate the Laplacian matrix associated with a graph with self-loops whilst the notation $\mathcal{L}$ is used to indicate the Laplacian matrix associated with graphs with no self-loops.

It is well known that $\mathcal{L}$ is symmetric and has nonnegative eigenvalues when the graph is undirected, i.e., $\mathcal{L}$ is positive semidefinite. Furthermore, for undirected graphs, zero is a simple eigenvalue of $\mathcal{L}$ and the associated eigenvector is 1_N if and only if the undirected graph is connected [26, 25]. Let μ_i be the ith eigenvalue of an $\mathcal{L}$ associated with an undirected and connected graph. Then the eigenvalues of $\mathcal{L}$ can be arranged as

$$0 = \mu_1 < \mu_2 \le \mu_3 \le \cdots \le \mu_N, \tag{2.4}$$

and we denote $\bar{\lambda}(\mathcal{L})$ as the largest eigenvalue of $\mathcal{L}$ and $\underline{\lambda}_2(\mathcal{L})$ as the second smallest eigenvalue of $\mathcal{L}$, that is

$$\bar{\lambda}(\mathcal{L}) = \mu_N, \qquad \underline{\lambda}_2(\mathcal{L}) = \mu_2. \tag{2.5}$$

The following lemma states that the Laplacian matrix associated with $\mathcal{G}$ with self-loops is positive definite when $\mathcal{G}$ is connected.

Lemma 2.5 ([69]). *For a graph with at least one self-loop, the Laplacian matrix $\hat{\mathcal{L}}$ is positive definite, if the graph is connected.*

2.5.2 Directed graphs

A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a non-empty finite vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of ordered pair of vertices, called edges. An edge in $\mathcal{G}$ is denoted by (v_i, v_j). If $(v_i, v_j) \in \mathcal{E}$, then agent v_j can obtain information from agent v_i ,but not necessarily vice versa. The set of neighbours of vertex v_i is defined as $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. Self edges are not allowed, that is, $(v_i, v_i) \notin \mathcal{E}$. A directed path in a graph from v_i to v_j is a sequence of edges of the form $(v_i, v_{i+1}), (v_{i+1}, v_{i+2}), \ldots, (v_{j-1}, v_j)$. A directed graph is strongly connected if there is a directed path from every vertex to every other vertex. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ of $\mathcal{G}$ is defined as $a_{ij} = 1$ if $(v_j, v_i) \in \mathcal{E}$, while $a_{ij} = 0$ if $(v_j, v_i) \notin \mathcal{E}$. The Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ of $\mathcal{G}$ is defined as $l_{ii} = \sum_{j\neq i} a_{ij}$ and $l_{ij} = -a_{ij}$ for all $i \neq j$. A graph is called balanced if $\sum_{j=1}^{N} a_{ij} = \sum_{j=1}^{N} a_{ji}$ for all i.

Lemma 2.6 ([7, 70]). *If $\mathcal{G}$ is strongly connected, then its Laplacian $\mathcal{L}$ satisfies:*

1. $\mathrm{rank}(\mathcal{L}) = N - 1$;
2. *zero is a simple eigenvalue of $\mathcal{L}$ and 1_N is the corresponding eigenvector, i.e. $\mathcal{L}1_N = 0_N$ and the remaining $N-1$ eigenvalues all have positive real parts.*

Lemma 2.7 ([7]). *A directed graph $\mathcal{G}$ is balanced if and only if $1_N^T\mathcal{L} = 0_N^T$.*

Lemma 2.8 ([71]). *Let $\mathcal{G}$ be a strongly connected and balanced graph. Then, $\mathcal{L}+\mathcal{L}^T \ge 0$ (i.e. positive semidefinite) with zero being its simple eigenvalue.*

Chapter 3

Distributed robust stabilization of networked multi-agent systems with strict negative imaginary uncertainties

3.1 Introduction

In control design, simplified mathematical models are typically used to model real physical systems. This results in discrepancies between the true physical system and the simplified mathematical model which are know as uncertainties (modelling errors). In the control design process, it is vital to take such uncertainties into account so as not to lead to adverse effects such as instability of the controlled system. As a result, stability of a closed-loop system in face of such uncertainties is a fundamental requirement in robust control. In networked multi-agent systems the problem of ensuring stability of the networked system in face of uncertainties in the dynamics of agents is much more complicated due to the additional factors that need to be considered in control protocol design such as the interaction among the agents. Thus, it is worth the effort to consider the study of such networked multi-agent systems problems. We here consider the distributed robust stabilization problem for networked multi-agent systems with SNI uncertainties. Recall that the problem of ensuring a certain collective behaviour in the presence of dynamic uncertainties due to modelling errors has not been much of a focus in the literature on networked multi-agent systems with the few published studies in [19, 20, 18, 17] being limited to uncertainties bounded in H_∞ norm (gain bound). However, in systems where the dynamical uncertainties are characterised by phase bounds, ensuring stability by phase stabilization methods is typical and lead to less conservative design compared with the use of gain stabilization methods [31, 57]. Many practical systems posses the NI properties such as for example lightly damped mechanical systems [30]. The dynamical uncertainties of NI systems typically belong to the SNI class and are mainly characterised by phase bounds. Thus one can effectively use the NI robust stability results, which were established based on phase stabilization, to deal with stability of systems with SNI uncertainties. Hence, we utilize the NI robust stability results in [30, 41] in addressing the considered distributed robust stabilization problem.

In this chapter, a solution to the distributed robust stabilization problem of networked multi-agent systems with SNI uncertainties is proposed. To address the problem, tools from negative imaginary systems theory and graph theory are utilized. Two main results are presented in this chapter. For the first result, it is shown that, under certain assumptions on the network graph, a state, input and output transformation preserves the NI property of the network. This result simplifies the protocol synthesis procedure. For the second result which is based on the first one, sufficient conditions in an LMI framework are derived that ensure the existence of control protocol parameters such that the control protocol robustly stabilizes a networked multi-agent system in presence of SNI uncertainties of certain DC size. Moreover, the synthesised control protocol is shown to ensure robust stability when variations in the network topology occur. An example is provided to show the effectiveness of the proposed results.

3.2 Problem Formulation

Consider a group of N linear uncertain agents. The dynamics of the ith agent are described by

$$\begin{aligned}\dot{x}_i(t) &= Ax_i(t) + B_1 w_i(t) + B_2 u_i(t),\\ z_i(t) &= C_1 x_i(t),\\ \hat{w}_i(s) &= \Delta_i(s)\hat{z}_i(s),\end{aligned} \tag{3.1}$$

where $x_i(t) \in \mathbb{R}^n$, $w_i(t) \in \mathbb{R}^m$, $u_i(t) \in \mathbb{R}^p$, and $z_i(t) \in \mathbb{R}^m$ are the state, disturbance, control input and controlled output of the ith agent, respectively with $m \leq n$. The matrices $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m}$, $B_2 \in \mathbb{R}^{n\times p}$, $C_1 \in \mathbb{R}^{m\times n}$ are known constant matrices. The transfer function matrix $\Delta_i(s)$ represents the uncertainty in the dynamics of the ith agent where $\hat{w}_i(s)$ and $\hat{z}_i(s)$ are the Laplace transform of $w_i(t)$ and $z_i(t)$ respectively. Suppose that the uncertainty in the dynamics of each agent satisfies the following property and conditions:

Assumption 3.1. *For all $i \in \{1, \ldots, N\}$, the uncertainty $\Delta_i(s)$ is strict negative imaginary and satisfies $\Delta_i(\infty) \geq 0$ and $\bar{\lambda}(\Delta_i(0)) \leq (1/\gamma)$, where $\gamma > 0$ is a pre-specified number.*

The uncertainty thus gives rise to the heterogeneity of the multi-agent system.

Following [69], the control protocol for the ith agent is

$$u_i(t) = cK\left(\sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t)) + a_{ii}x_i(t)\right), \tag{3.2}$$

where $c > 0$ is the coupling strength to be selected, $K \in \mathbb{R}^{p\times n}$ is the control feedback gain matrix to be designed and a_{ij} are the elements of the adjacency matrix with $a_{ii} = 1\ \forall i \in \{1, \ldots, q\}$, and $a_{ii} = 0\ \forall i \in \{q+1, \ldots, N\}$. This protocol structure means that each agent receives the sum of relative state measurements with respect to its neighbours. In addition, a subset of agents receive their own absolute state measurements. Without loss of generality, it is assumed that the first q ($q \ll N$) agents have access to their own absolute state measurements. Consequently, the network graph that models the information exchange among the agents satisfies the following assumption.

Assumption 3.2. *The graph is connected, undirected and at least one vertex has a self-loop.*

Dropping time dependency and Laplace variable dependency where it is clear from the context, it is clear that agent dynamics (3.1) can be rewritten as

$$\begin{aligned} \dot{x} &= (I_N \otimes A)x + (I_N \otimes B_1)w + (I_N \otimes B_2)u, \\ z &= (I_N \otimes C_1)x, \\ \hat{w} &= \Delta(s)\hat{z}, \end{aligned} \tag{3.3}$$

and control law (3.2) can be rewritten as

$$u = (c\hat{\mathcal{L}} \otimes K)x, \tag{3.4}$$

where $x = \begin{bmatrix} x_1^T, \dots, x_N^T \end{bmatrix}^T \in \mathbb{R}^{nN}$, $w = \begin{bmatrix} w_1^T, \dots, w_N^T \end{bmatrix}^T \in \mathbb{R}^{mN}$, $u = \begin{bmatrix} u_1^T, \dots, u_N^T \end{bmatrix}^T \in \mathbb{R}^{pN}$, $z = \begin{bmatrix} z_1^T, \dots, z_N^T \end{bmatrix}^T \in \mathbb{R}^{mN}$, $\Delta(s) = \mathrm{diag}(\Delta_1(s), \dots, \Delta_N(s))$, $\hat{w}$ is the Laplace transform of w, $\hat{z}$ is the Laplace transform of z and $\hat{\mathcal{L}} \in \mathbb{R}^{N\times N}$ is the Laplacian matrix associated with $\mathcal{G}$. By applying protocol (3.4) (or equivalently (3.2) to each agent i in (3.1)) to the uncertain agents (3.3), the resulting uncertain closed-loop networked multi-agent system becomes

$$\begin{aligned} \dot{x} &= \Big((I_N \otimes A) + (c\hat{\mathcal{L}} \otimes B_2K)\Big)\, x + (I_N \otimes B_1)w, \\ z &= (I_N \otimes C_1)x, \end{aligned} \tag{3.5}$$

and

$$\hat{w} = \Delta(s)\hat{z}. \tag{3.6}$$

Note that $\Delta(s)$ is SNI since each $\Delta_i(s), i \in \{1, \dots, N\}$ is SNI and satisfies $\Delta(\infty) \geq 0$ and $\bar{\lambda}(\Delta(0)) \leq 1/\gamma$ by noting that $\bar{\lambda}(\Delta(0)) = \max\limits_{i=1,\dots,N} \bar{\lambda}(\Delta_i(0)) \leq 1/\gamma$. The transfer function matrix of the nominal closed-loop networked multi-agent system from w to z is strictly proper and given by

$$G_{cl}(s) = C_{cl}(sI_{nN} - A_{cl})^{-1}B_{cl}, \tag{3.7}$$

where $A_{cl} = (I_N \otimes A) + (c\hat{\mathcal{L}} \otimes B_2K)$, $B_{cl} = (I_N \otimes B_1)$, $C_{cl} = (I_N \otimes C_1)$ and has an associated DC gain of

$$\bar{\lambda}(G_{cl}(0)) = \bar{\lambda}(C_{cl}(-A_{cl})^{-1}B_{cl}). \tag{3.8}$$

The uncertain networked multi-agent system is depicted in Figure 3.1. According to Lemma 2.3, we can define the distributed robust stabilization problem as follows.

Definition 3.1. *Given* $\gamma > 0$, *control protocol* (3.2) *is said to robustly stabilize the networked system with agent dynamics* (3.1) *against any strict negative imaginary uncertainty satisfying Assumption 3.1 if it is designed such that the transfer function matrix* (3.7) *is negative imaginary and satisfies the DC gain condition* $\bar{\lambda}(G_{cl}(0)) < \gamma$.

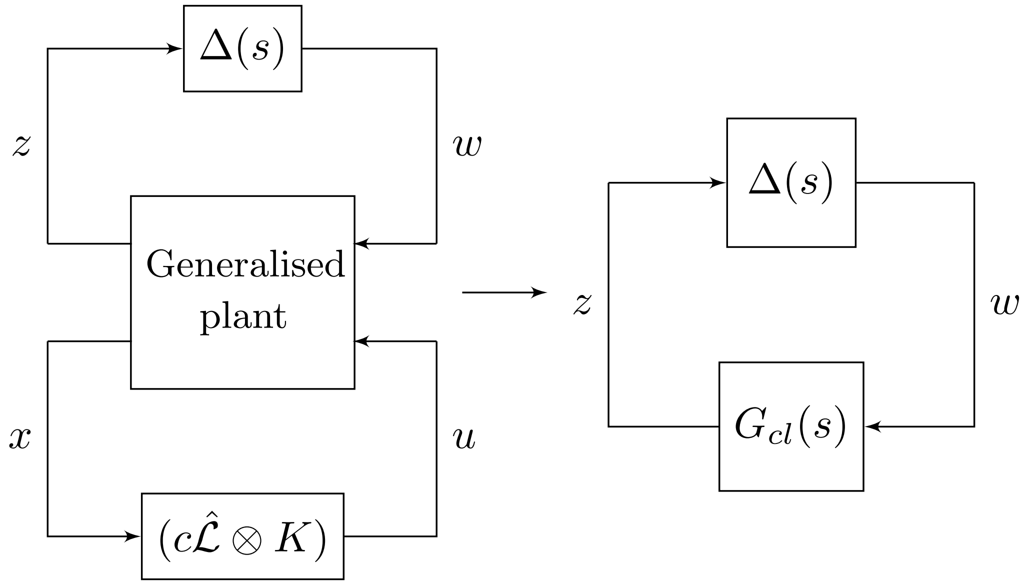


Figure 3.1: Networked multi-agent system with SNI uncertainty.

3.3 Problem reduction and robust protocol synthesis

In order to address the distributed robust stabilization problem, the following technical lemmas are required.

Lemma 3.1. *Let* $U \in \mathbb{R}^{N\times N}$ *be any orthogonal matrix,* $R(s) \in \mathcal{R}^{Nm\times Nm}$ *and let* $\tilde{R}(s) = (U^T \otimes I_m)R(s)(U \otimes I_m)$*. Then the following hold:*

1. $\tilde{R}(s)$ *is NI (resp. SNI) if and only if* $R(s)$ *is NI (resp. SNI).*

2. $\bar{\lambda}(\tilde{R}(0)) = \bar{\lambda}(R(0))$.

Proof. The proof is straight forward from the definition of NI (SNI) systems and properties of orthogonal matrices. □

Lemma 3.2 ([39]). $\mathrm{diag}(R_1(s), \ldots, R_N(s))$ *is NI if and only if* $R_i(s)$ *are all NI for* $i \in \{1, \ldots, N\}$.

The following lemma states that under certain assumptions on the network graph, the NI property is preserved due to transformation.

Lemma 3.3. *Given* $\gamma > 0$ *and assume that the network topology* $\mathcal{G}$ *satisfies Assumption 3.2. Let* $\hat{\mathcal{L}}$ *be the Laplacian matrix of* $\mathcal{G}$ *and let* λ_i *for all* $i \in \{1, \ldots, N\}$ *be* *the eigenvalues of $\hat{\mathcal{L}}$. Then, the transfer function matrix* (3.7) *of the networked system* (3.5) *is negative imaginary and satisfies* $\bar{\lambda}(G_{cl}(0)) < \gamma$ *if and only if for all* $i \in \{1, \ldots, N\}$*, the transfer functions* $\tilde{G}_i(s)$ *of the following* N *isolated subsystems*

$$\begin{aligned} \dot{\tilde{x}}_i &= (A + c\lambda_i B_2 K)\tilde{x}_i + B_1 \tilde{w}_i, \\ \tilde{z}_i &= C_1 \tilde{x}_i, \end{aligned} \tag{3.9}$$

are all negative imaginary and satisfy $\bar{\lambda}(\tilde{G}_i(0)) < \gamma$ *simultaneously, where* $\tilde{G}_i(s) = C_1(sI - A - c\lambda_i B_2 K)^{-1} B_1$.

Proof. The idea of the proof is to transform the networked system (3.5) into a set of block diagonal systems

$$\begin{aligned} \dot{\tilde{x}} &= ((I_N \otimes A) + (c\Lambda \otimes B_2 K))\, \tilde{x} + (I_N \otimes B_1)\tilde{w}, \\ \tilde{z} &= (I_N \otimes C_1)\tilde{x}. \end{aligned} \tag{3.10}$$

in a similar manner to the decomposition approach used in [72, 69, 18, 17] by letting $\tilde{x} = (U^T \otimes I_n)x$, $\tilde{w} = (U^T \otimes I_m)w$, $\tilde{z} = (U^T \otimes I_m)z$ and decomposing $\hat{\mathcal{L}}$ as $U^T \hat{\mathcal{L}} U = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ where $U \in \mathbb{R}^{N \times N}$ is an orthogonal matrix. Consequently, the transfer function matrix of the transformed system (3.10) from $\tilde{w}$ to $\tilde{z}$, which we denote $\tilde{G}_{cl}(s)$, can be expressed as

$$\tilde{G}_{cl}(s) \;=\; \mathrm{diag}\left(\tilde{G}_1(s), \ldots, \tilde{G}_N(s)\right) = (U^T \otimes I_m) G_{cl}(s) (U \otimes I_m). \tag{3.11}$$

The desired conclusion then follows from Lemmas 3.1 and 3.2. $\square$

Remark 3.1. *According to Definition 3.1, the distributed robust stabilization problem is solved by designing a control protocol such that the transfer function matrix* $G_{cl}(s)$ *of the large scale nominal closed-loop networked system is NI and satisfies the DC gain condition. Lemma 3.3 states that a necessary and sufficient condition for* $G_{cl}(s)$ *to be NI and satisfy the DC gain condition is that* N *reduced-order subsystems, where each subsystem has the order of a single agent, satisfy the NI property and DC gain condition simultaneously. Consequently, the previous lemma plays a role in facilitating and simplifying the design procedure where the protocol parameters can be designed based on the reduced-order systems.*

Remark 3.2. *Reduction of the problem as stated in the previous remark is applicable since the uncertainty resulting from transformation remains SNI. If we denote* *the uncertainty of the transformed system as shown in Figure 3.2 by* $\tilde{\Delta}(s)$*, we have* $\tilde{\Delta}(s) = (U^T \otimes I_m)\Delta(s)(U \otimes I_m)$*. Since* $\Delta(s)$ *is SNI and satisfies* $\Delta(\infty) \geq 0$*,* $\bar{\lambda}(\Delta(0)) \leq 1/\gamma$*, then according to Lemma 3.1 so will* $\tilde{\Delta}(s)$ *be SNI and satisfy the corresponding conditions. As a result, under Assumption 3.2 of the network topology, internal stability of the system in Figure 3.1 is equivalent to the internal stability of the system in Figure 3.2.*

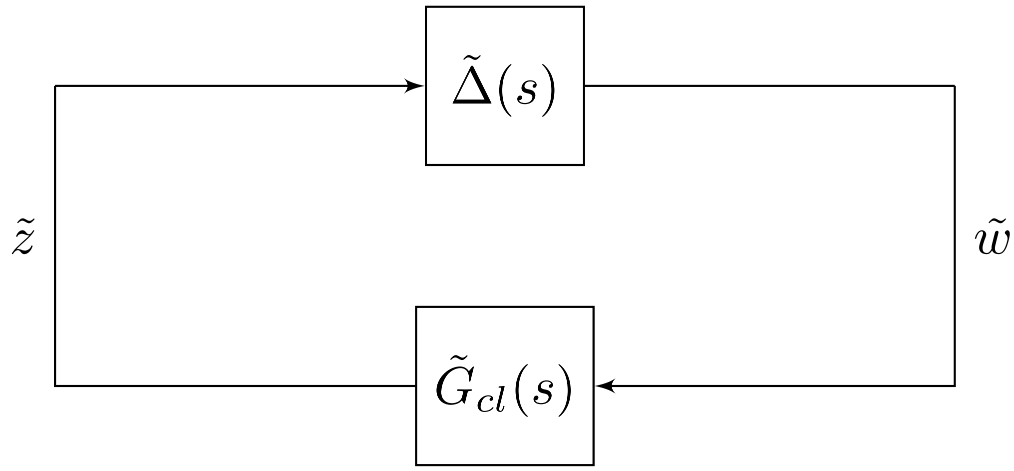


Figure 3.2: Transformed system.

Remark 3.3. *We now give a justification for using graphs with at least one self-loop in this work instead of simple graphs as used in for example [18, 17]. Although the work in [18, 17] assumes simple graphs in their approach to robust synchronization, the results therein are suitable only for the case where the dynamics of the nominal plants have no poles in the open-right half plane. That is, the work therein restricts the matrix* A *from containing eigenvalues with positive real parts. In this work, we impose no such restrictions on the eigenvalues of matrix* A*. Thus, the graph which models the network topology cannot be simple but instead must contain at least one self-loop, as assumed in Assumption 3.2, because a simple connected graph will have one zero eigenvalue (see e.g.,[73]) and thus the subsystem in* (3.9) *corresponding to* $\lambda_1 = 0$ *cannot be controlled to satisfy the NI property.*

Lemma 3.3 reveals that for networked dynamical system (3.5) to satisfy the NI property, it suffices to find a positive scalar c and a gain matrix K such that systems (3.9) satisfy the NI property simultaneously.

Theorem 3.1 below gives sufficient conditions under which a $c > 0$ and a feedback gain matrix K exist such that the networked multi-agent system is robustly stabilized by control protocol (3.2).

Theorem 3.1. *Given $\gamma > 0$, a network topology that satisfies Assumption 3.2 and an uncertain multi-agent system* (3.1) *with $C_1B_2 = 0$, $m \leq n$ and (A, B_2) controllable. If there exists a matrix $Y = Y^T > 0$ and a scalar $\tau > 0$ such that*

$$\begin{aligned}
\begin{bmatrix} AY + YA^T - \tau B_2 B_2^T & B_1 + AYC_1^T \\ B_1^T + C_1 Y A^T & 0 \end{bmatrix} &\leq 0, && (3.12) \\
C_1 Y C_1^T &< \gamma I, && (3.13) \\
\det(AY - \frac{1}{2}\tau B_2 B_2^T) &\neq 0, && (3.14)
\end{aligned}$$

then there exists a feedback gain matrix K and a scalar $c \geq \frac{\tau}{\min\limits_{i\in\{1,\ldots,N\}} \lambda_i}$ such that control protocol (3.2) *robustly stabilizes the networked multi-agent system in the presence of any strict negative imaginary uncertainty satisfying Assumption 3.1. Moreover, a suitable feedback gain matrix K is given by $K = -0.5B_2^T Y^{-1}$.*

Proof. Since the LMI condition (3.12) holds for some matrix $Y > 0$ and some scalar $\tau > 0$ and since $c \geq \frac{\tau}{\lambda_i}$ for all $i \in \{1, \ldots, N\}$, it follows that for all $i \in \{1, \ldots, N\}$

$$\begin{bmatrix} AY + YA^T - c\lambda_i B_2 B_2^T & B_1 + AYC_1^T \\ B_1^T + C_1 Y A^T & 0 \end{bmatrix} \leq$$

$$\begin{bmatrix} AY + YA^T - \tau B_2 B_2^T & B_1 + AYC_1^T \\ B_1^T + C_1 Y A^T & 0 \end{bmatrix} \leq 0 \quad (3.15)$$

as $\lambda_i > 0$ for all $i \in \{1, \ldots, N\}$. This implies that

$$\begin{aligned}
AY + YA^T - c\lambda_i B_2 B_2^T &\leq 0, && (3.16a) \\
B_1 + AYC_1^T &= 0. && (3.16b)
\end{aligned}$$

Furthermore, since $C_1B_2 = 0$ by assumption, then (3.16b) can be written as

$$B_1 + AYC_1^T - 0.5c\lambda_i B_2 B_2^T C_1^T = 0. \quad (3.17)$$

Now let $K = -0.5B_2^T Y^{-1}$. Via simple algebraic manipulation, (3.16a) and (3.17) become

$$\begin{aligned}
(A + c\lambda_i B_2 K)Y + Y(A + c\lambda_i B_2 K)^T &\leq 0, && (3.18a) \\
B_1 + (A + c\lambda_i B_2 K)YC_1^T &= 0, && (3.18b)
\end{aligned}$$

for all $i \in \{1, \ldots, N\}$. Furthermore, (3.14) implies

$$\det(A - 0.5\tau B_2 B_2^T Y^{-1}) \neq 0$$

which is equivalent to

$$\det(A + \tau B_2 K) \neq 0. \tag{3.19}$$

Now since $c\lambda_i \geq \tau$ for all $i \in \{1, \ldots, N\}$, it can be written as $c\lambda_i = \tau + \alpha_i$ where $\alpha_i \geq 0$. Then,

$$\begin{aligned} \det(A + c\lambda_i B_2 K) &= \det(A + \tau B_2 K + \alpha_i B_2 K) \\ &= \det(A + \tau B_2 K) \det(I + (A + \tau B_2 K)^{-1} \alpha_i B_2 K). \end{aligned} \tag{3.20}$$

We need to show that $\det(A + c\lambda_i B_2 K) \neq 0$ for all $i \in \{1, \ldots, N\}$. Towards this end,

$$\det(A + c\lambda_i B_2 K) \neq 0 \;\; \Leftrightarrow \;\; \det(I + (A + \tau B_2 K)^{-1} \alpha_i B_2 K) \neq 0 \tag{3.21}$$

for all $i \in \{1, \ldots, N\}$. It is easily seen that $\det(A + c\lambda_i B_2 K) \neq 0$ for $\alpha_i = 0, i \in \{1, \ldots, N\}$. For $\alpha_i > 0, i \in \{1, \ldots, N\}$ we have

$$\det(I + (A + \tau B_2 K)^{-1} \alpha_i B_2 K) = \alpha_i^n \det\left(\frac{1}{\alpha_i} I + (A + \tau B_2 K)^{-1} B_2 K\right) \tag{3.22}$$

and is nonzero if and only if $\frac{1}{\alpha_i} I + (A + \tau B_2 K)^{-1} B_2 K$ is nonsingular which is satisfied when $\Re\{\lambda_j[(A + \tau B_2 K)^{-1} B_2 K]\} \geq 0 \; \forall j$ since $1/\alpha_i$ for $i \in \{1, \ldots, N\}$ is a positive scalar. Therefore, what is left is to show that $\Re\{\lambda_j[(A + \tau B_2 K)^{-1} B_2 K]\} \geq 0 \; \forall j$.

$$\begin{aligned} AY + YA^T - \tau B_2 B_2^T \leq 0 &\Leftrightarrow (AY - \frac{1}{2}\tau B_2 B_2^T) + (AY - \frac{1}{2}\tau B_2 B_2^T)^T \leq 0 \\ &\Leftrightarrow (AY - \frac{1}{2}\tau B_2 B_2^T)^{-T} + (AY - \frac{1}{2}\tau B_2 B_2^T)^{-1} \leq 0 \\ &\Leftrightarrow (AY - \frac{1}{2}\tau B_2 B_2^T)^{-1} + (AY - \frac{1}{2}\tau B_2 B_2^T)^{-T} \leq 0 \\ &\Rightarrow B_2^T(AY - \frac{1}{2}\tau B_2 B_2^T)^{-1} B_2 + B_2^T(AY - \frac{1}{2}\tau B_2 B_2^T)^{-T} B_2 \leq 0 \\ &\Leftrightarrow [B_2^T(AY - \frac{1}{2}\tau B_2 B_2^T)^{-1} B_2] I + I[B_2^T(AY - \frac{1}{2}\tau B_2 B_2^T)^{-T} B_2] \leq 0 \\ &\Rightarrow \Re\{\lambda_j[B_2^T(AY - \frac{1}{2}\tau B_2 B_2^T)^{-1} B_2]\} \leq 0 \quad \forall j \\ &\Leftrightarrow \Re\{\lambda_j[(AY - \frac{1}{2}\tau B_2 B_2^T)^{-1} B_2 B_2^T]\} \leq 0 \quad \forall j \\ &\Leftrightarrow \Re\{\lambda_j[Y^{-1}(A - \frac{1}{2}\tau B_2 B_2^T Y^{-1})^{-1} B_2 B_2^T]\} \leq 0 \quad \forall j \\ &\Leftrightarrow \Re\{\lambda_j[(A - \frac{1}{2}\tau B_2 B_2^T Y^{-1})^{-1} B_2 B_2^T Y^{-1}]\} \leq 0 \quad \forall j \\ &\Leftrightarrow \Re\{\lambda_j[(A + \tau B_2 K)^{-1} B_2 K]\} \geq 0 \quad \forall j. \end{aligned} \tag{3.23}$$

It follows that

$$\det(A + c\lambda_i B_2 K) \neq 0 \quad \text{for all } i \in \{1, \ldots, N\}. \tag{3.24}$$

Consequently, $\tilde{G}_i(s)$ is negative imaginary for all $i \in \{1, \ldots, N\}$ by Lemma 2.2.

It remains to show that the DC gain of each subsystem is less than γ. Since the LMI condition (3.13) holds, via (3.24) and (3.18b) it follows that

$$\begin{aligned} \gamma I > C_1 Y C_1^T &= C_1(A + c\lambda_i B_2 K)^{-1}(A + c\lambda_i B_2 K) Y C_1^T \\ &= C_1(-A - c\lambda_i B_2 K)^{-1} B_1 = \tilde{G}_i(0) \end{aligned} \tag{3.25}$$

for all $i \in \{1, \ldots, N\}$. Consequently, $\bar{\lambda}(\tilde{G}_i(0)) < \gamma \quad$ for all $i \in \{1, \ldots, N\}$.
From Lemma 3.3 we conclude that $G_{cl}(s)$ is NI and $\bar{\lambda}(G_{cl}(0)) < \gamma$.
Now since $G_{cl}(s)$ is strictly proper, we have $\Delta(\infty)G_{cl}(\infty) = 0$ and since the uncertainty satisfies Assumption 3.1, it follows from Lemma 2.3 that control protocol (3.2) robustly stabilizes the networked multi-agent system. □

Remark 3.4. $C_1 B_2 = 0$ *means that the transfer function from* u_i *to* z_i $\forall i \in \{1, \ldots, N\}$ *has a relative degree strictly greater than unity. This is hence often fulfilled in practice due to strictly proper actuator dynamics and strictly proper plant dynamics.*

Remark 3.5. *By imposing the assumption* $C_1 B_2 = 0$ *in Theorem 3.1, we get simpler solvable conditions* (3.12)–(3.14) *which do not involve the network topology.*

Remark 3.6. *The determinant condition appears because the negative imaginary property excludes poles at the origin. This non-convex condition is not troublesome as a feasible solution for* Y *and* τ *can always be obtained first by solving the LMI conditions and then checking whether the computed values satisfy the determinant condition or not. If they do not, then a small increase in* τ *often resolves the problem.*

Thus, the steps required to design the protocol can be summarized in the following algorithm:

1. Solve the LMI conditions (3.12)–(3.13) for $Y > 0$ and $\tau > 0$. Then, check whether the determinant condition (3.14) is satisfied or not. If not, perturb τ and/or Y to satisfy all of (3.12)–(3.14).

2. Let the feedback gain matrix $K = -0.5 B_2^T Y^{-1}$.

3. Select the coupling strength c to satisfy $c \geq \frac{\tau}{\min\limits_{i \in \{1,\ldots,N\}} \lambda_i}$, where $\lambda_i \; \forall i \in \{1, \ldots, N\}$ are the eigenvalues of $\hat{\mathcal{L}}$ (note that the minimum value of c that can be selected is when the equal sign holds).

Remark 3.7. *The benefit of the aforementioned design procedure is that the feedback gain matrix K is first designed without any knowledge of the network graph. Then, the coupling strength c is adjusted to handle the effect of the network topology. Thus once a feedback gain matrix K is designed, robust stability against SNI uncertainties with certain DC size is achieved via control protocol* (3.2) *for various different network graphs that satisfy the condition $\lambda_i \geq \tau/c$ for all $i \in \{1, \ldots, N\}$. Clearly this inequality is satisfied for a rich class of Laplacian matrices and associated network topologies. Consequently, by selecting a large enough value for the coupling strength c, both robust stability to agents dynamics and robustness to variations in the network topology can be guaranteed.*

Remark 3.8. *Although we build on the work of [69], it is important to observe that the results here are not a specialisation of the results in [69] because we consider a distinct problem from [69]. It is assumed in [69] that the agents are subject to external disturbances in $\mathfrak{L}_2[0, \infty)$ and the problem consider therein is to evaluate the performance of a networked multi-agent system subject to these external disturbances. In this work, we consider the situation where agents are subject to dynamical uncertainties (modelling errors) which belong to the SNI class and the problem here is to maintain stability of the network in the presence of SNI uncertainties with a certain DC size. The authors of [69] study a suboptimal H_∞ control problem, where distributed controllers need to be found such that the H_∞ norm of a transfer function is less than a desired tolerance. Thus it is essential that the gain be small over all frequency ranges. On the contrary, the distributed robust stabilization problem we consider here requires to find distributed controllers such that a transfer function matrix satisfies the NI property and only the DC gain value be restricted on one-side, which is less conservative. Whereas [69] derive conditions for the existence of controllers to have unbounded H_∞ performance region in order to ensure a level of robustness with respect to the communication topology, the results we present in this chapter derive conditions for the existence of controllers that robustly stabilize networked systems in the presence of dynamical uncertainties that* *belong to the SNI class as well as achieving robustness to variations in the network topology.*

Remark 3.9. *It is worth mentioning that the robust stabilization problem we address in this chapter is somehow different and not comparable to the consensus problem addressed for example in [26, 73]. Modelling errors in the agents dynamics are not considered in [26, 73]. Furthermore, the consensus problem addressed therein requires convergence of the states to an unspecified common value depending on the initial state information. This can only be satisfied with simple network graphs where zero is a simple eigenvalue of the Laplacian matrix $\mathcal{L}$ and hence span$\{1_N\}$ is contained in the null space of $\mathcal{L}$, consequently consensus is guaranteed [26]. The robust stabilization problem we address here is mainly concerned with guaranteeing robust stability of the networked system in presence of modelling errors which belong to the SNI class. As stated in Remark 3.3, it is essential the network graph contains at least one self-loop as for simple graphs where zero is a simple eigenvalue of $\mathcal{L}$ the subsystem in* (3.9) *corresponding to this zero eigenvalue of $\mathcal{L}$ cannot be controlled to satisfy the NI property, consequently robust stabilization cannot be guaranteed. Nevertheless, it may be of interest to investigate consensus to a desired reference/trajectory as a next step provided robust stability against SNI uncertainties is satisfied first for the networked system. However, this may be challenging and not straightforward and is beyond the scope of this thesis.*

3.4 Numerical Example

The example in [47] is modified in order to design distributed controllers for systems with heterogeneous SNI uncertainties. Consider a group of $N = 6$ uncertain systems connected over a network topology. The block diagram of the ith uncertain systems in depicted in Figure 3.3 and the network topology that models the communication among the systems and the associated Laplacian matrix are shown in Figure 3.4.

Each of the six systems contains an uncertain flexible structure with co-located force actuation and position sensing and thus the transfer function of the ith flexible structure $M_i(s)$ $\forall i \in \{1, \ldots, 6\}$ is strictly negative imaginary. For control design purpose, $M_i(s)$ has been replaced by unity gain and the resulting modelling error $\Delta_i(s) = M_i(s) - 1$ is an SNI uncertainty as shown in Figure 3.3. It is assumed that

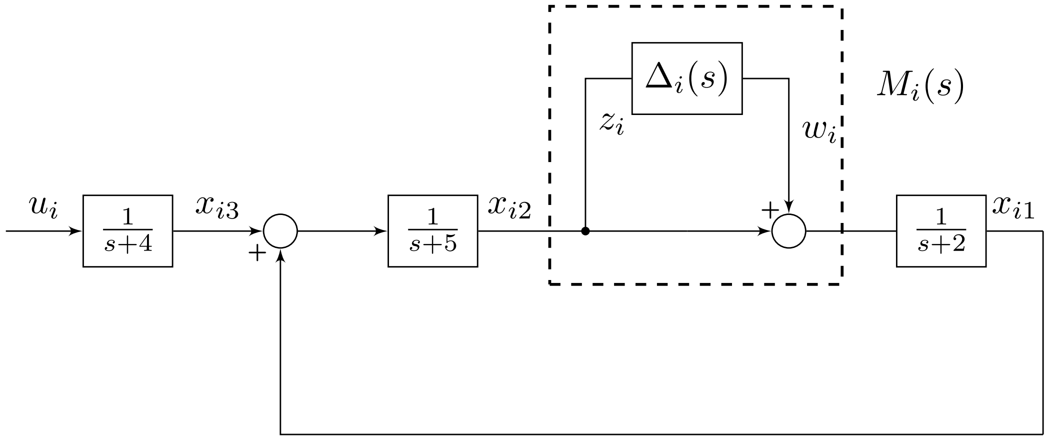
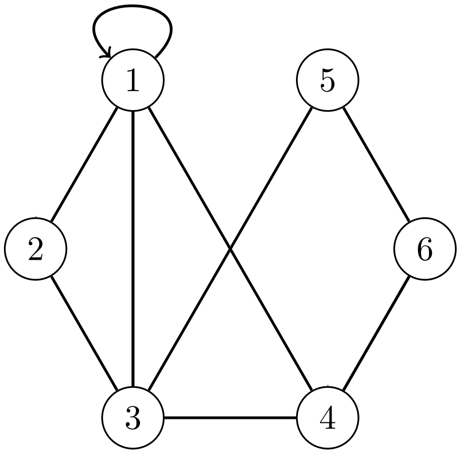


Figure 3.3: Block diagram of the ith uncertain system to be controlled.



$$\hat{\mathcal{L}} = \begin{bmatrix} 4 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 & 0 \\ -1 & 0 & -1 & 3 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Figure 3.4: Network topology and associated Laplacian matrix.

$\Delta_i(s)$ satisfies Assumption 3.1 with $\gamma = 1$. Via results of this chapter, parameters K and c of distributed control protocol (3.2) can be designed to ensure robust stability of the closed-loop networked system against SNI uncertainties and also ensure robustness to variations in the network topology.

To this end, the dynamics of the ith system can be obtained from Figure 3.3 in the form of (3.1) with $x_i = [x_{i1}^T, x_{i2}^T, x_{i3}^T]^T$ and matrices

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -5 & 1 \\ 0 & 0 & -4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

It is easy to see that $C_1 B_2 = 0$, $m \leq n$ and (A, B_2) is controllable. It can also be seen from Figure 3.3 that the transfer function from u_i to z_i has a relative degree strictly greater than unity which emphasises the statement in Remark 3.4.

Using the YALMIP [67] and SeDuMi [68] toolboxes to solve the LMI conditions as according to step 1 in the algorithm, we obtain the feasible solutions

$$Y = \begin{bmatrix} 3.7674 & 0.4547 & -0.4949 \\ 0.4545 & 0.0909 & 0 \\ -0.4946 & 0 & 0.4946 \end{bmatrix} > 0,$$

and $\tau = 2.7377$. We check that $\det(AY - \frac{1}{2}\tau B_2 B_2^T) = -3.3474 \neq 0$. No perturbations to Y and τ are necessary. From step 2 in the algorithm, the feedback gain matrix is given by

$$K = -\frac{1}{2}B_2^T Y^{-1} = \begin{bmatrix} -0.5 & 2.5 & -1.5 \end{bmatrix}.$$

The minimum eigenvalue of $\hat{\mathcal{L}}$ in Figure 3.4 is 0.1266. We hence select the coupling strength c according to step 3 in the algorithm to be $c = 43$ (twice the minimum value). Thus, Theorem 3.1 states that the control protocol with the values of K and c as computed above robustly stabilizes the networked system against any SNI uncertainty having a DC gain less than or equal to unity.

To illustrate this and to avoid construction of the 18th order ($N = 6$ and $n = 3$) overall plant dynamics in (3.5), we simply demonstrate that each of the N subsystems $\tilde{G}_i(s)$ given by (3.9) within the transformed overall plant dynamics (3.10) are all individually NI and satisfy the DC gain conditions. Figure 3.5 gives the Nyquist plots of the 6 subsystems; that is $\tilde{G}_i(s)$, $\forall i \in \{1, \dots, 6\}$. It is clear from the plots that the systems have a negative imaginary frequency response. Furthermore, Table 3.1 gives the explicit transfer functions for $\tilde{G}_i(s)$ from which it is easy to verify that each $\tilde{G}_i(s)$ is stable. Table 3.1 also shows that the nonsingular determinant condition (3.24) is satisfied for each of the 6 subsystems, and that the DC gains of $\tilde{G}_i(s)$, $i \in \{1, \dots, 6\}$ are all equal to 0.1 since they have been all set to be equal to $C_1 Y C_1^T$ which are less than unity.

Moreover, control protocol (3.2) with the same values of K and c as designed above also guarantees a level of robustness to variations in the network topology. To see this, consider the four different network topologies in Figure 3.6 constructed by adding or/and removing links from the original network topology in Figure 3.4. Control protocol (3.2) with the same values of K and c as designed above is guaranteed to

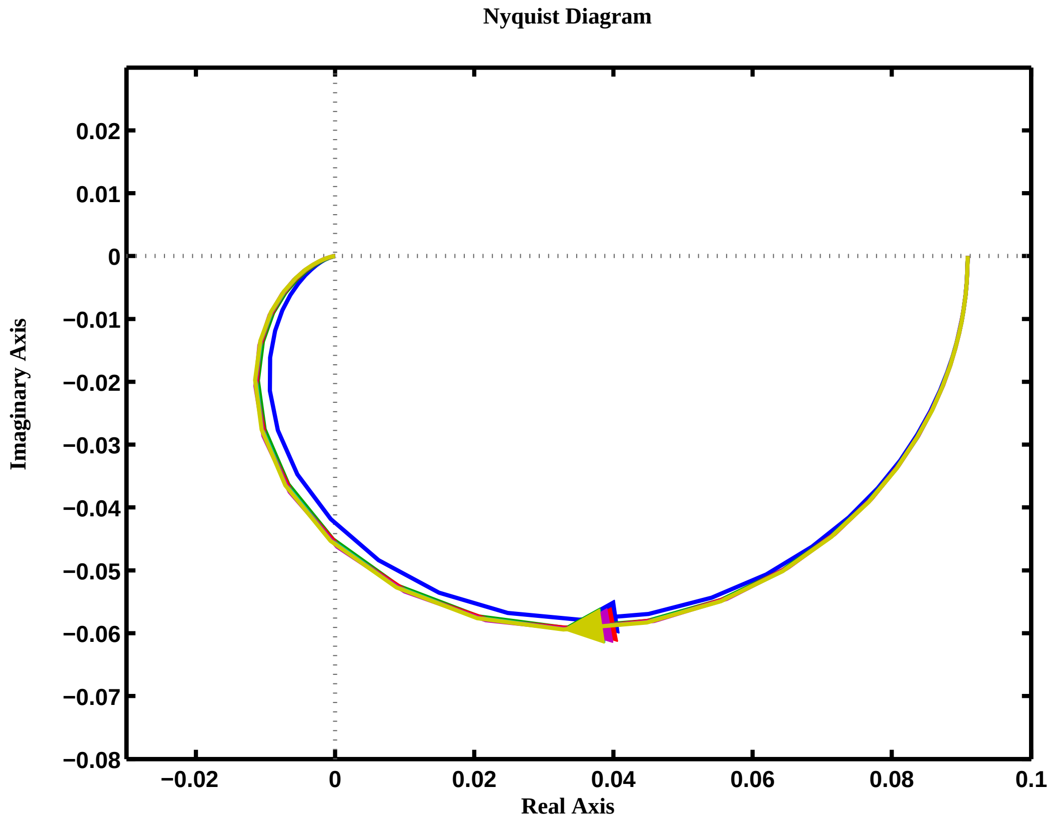


Figure 3.5: Nyquist plots of $\tilde{G}_i(s), \forall i \in \{1, \ldots, 6\}$, $c = 43$.

Table 3.1: Verifying the NI property for $\tilde{G}_i(s)$, $i \in \{1, \ldots, 6\}$

i	λ_i	$\tilde{G}_i(s)$	$\tilde{G}_i(0) = C_1 Y C_1^T$	$\det(A + c\lambda_i B_2 K)$
1	0.1266	$\frac{s+9.504}{s^3+19.23\,s^2+82.98\,s+104.5}$	0.1	-104.5
2	1.2205	$\frac{s+57.05}{s^3+90.29\,s^2+462.9\,s+627.6}$	0.1	-627.6
3	2.3293	$\frac{s+105.2}{s^3+162.3\,s^2+847.9\,s+1158}$	0.1	-1157.7
4	3.0647	$\frac{s+137.2}{s^3+210.1\,s^2+1103\,s+1509}$	0.1	-1509.4
5	4.9643	$\frac{s+219.8}{s^3+333.5\,s^2+1763\,s+2418}$	0.1	-2417.6
6	5.2945	$\frac{s+234.1}{s^3+355\,s^2+1875\,s+2575}$	0.1	-2575.5

achieve robust stability for all four various different networked systems (i.e. agents may be connected over any of the four network topologies) in the presence of SNI uncertainties with DC gains less than or equal to unity since $c = 43$ is greater than the minimum value of c corresponding to each of these network graphs. The minimum values of c which correspond to the network graphs of Figure 3.6a to Figure 3.6d are 18.764, 25.1627, 40.8283, 34.3070 respectively.

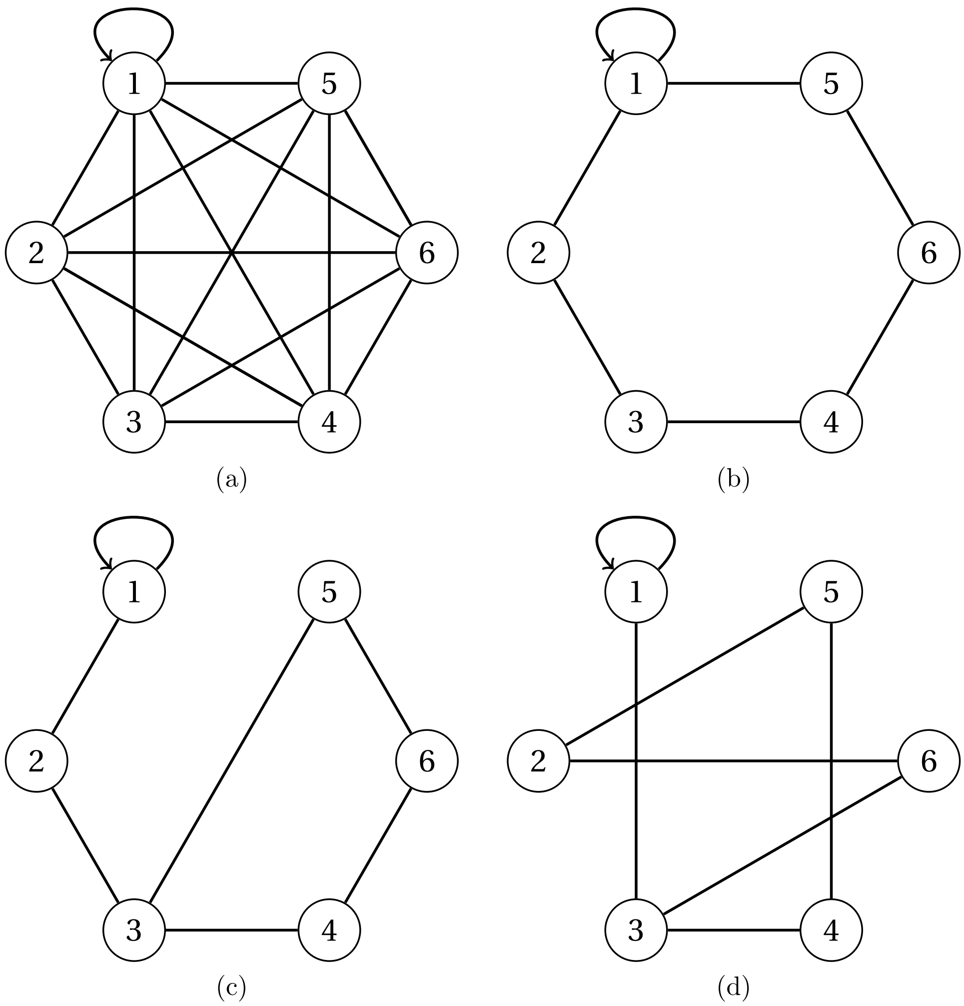


Figure 3.6: Four different network topologies.

We can also easily demonstrate that for some specific uncertainties the conclusion holds. For instance, choose $\Delta_1(s) = 0.5/(s+1)$, $\Delta_2(s) = (1-s)/(1+s)$, $\Delta_3(s) = 1/(s+3)$, $\Delta_4(s) = 1/(s^2+3s+2)$, $\Delta_5(s) = 1/(s+1)^2$, $\Delta_6(s) = (0.5s+1)/(s^2+s+1)$ which are SNI. $\Delta(s)$ in Figure 3.1 has $\bar{\lambda}(\Delta(0)) = 1 \leq 1/\gamma$. A pole-zero map of $G_{cl}(I - \Delta G_{cl})^{-1}$ is shown in Figure 3.7 for the original network topology in Figure 3.4. Since all closed-loop poles are in the left half plane, we conclude that the heterogeneous perturbed closed-loop system of Figure 3.1 is internally stable.

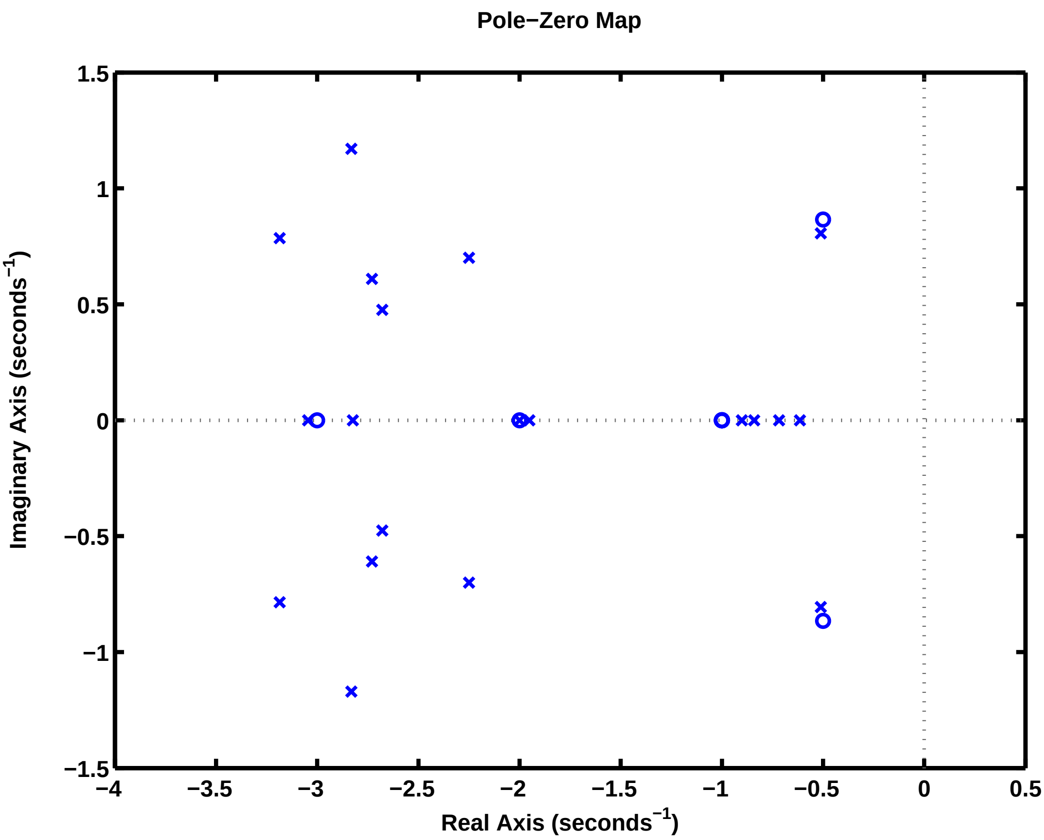


Figure 3.7: Poles and zeros of $G_{cl}(I - \Delta G_{cl})^{-1}$ corresponding to original network topology; poles are marked by x, and zeros are marked by o.

3.5 Summary

This chapter studied the distributed robust stabilization problem for networked multi-agent systems with strict negative imaginary uncertainties. It was shown that a state, input and output transformation preserves the NI property of the network when the network topology is modelled by an undirected graph with self-loops. This result was shown to be useful in control protocol design as the problem simplified to finding parameters which ensured that each of the multiple reduced-order systems satisfy the NI property. The synthesis procedure involved the design of two separate parameters; one which is a scalar that handles the effect of the network topology and the other one was a state feedback gain matrix. The advantage of this design procedure lies in the ability of the control protocol to maintain robust stability in the face of SNI uncertainties for different network topologies by simply appropriately adjusting this coupling scalar while leaving the state feedback gain matrix unchanged. A numerical example was given to show the usefulness of the results.

Chapter 4

Robust output consensus of homogeneous multi-agent systems with negative imaginary dynamics

4.1 Introduction

This chapter contributes to the existing literature on cooperative control of multiple NI systems. (Robust) cooperative control of multiple NI systems is motivated by applications where an individual NI system cannot achieve a desired collective behaviour on its own. Consensus, where agents cooperate to reach an agreement, is one of the most important and desirable collective behaviours due to the potential real-world applications it may have [23]. Consensus of MAS has been studied widely by many researchers. In terms of agents dynamics both homogeneous and heterogeneous dynamics have been considered and in terms of the shared information both state and output information have been considered such as, for example, [22, 14, 38, 39, 7]. In this chapter, we address the robust output consensus problem for multiple homogeneous NI systems. First, we focus here on homogeneous NI dynamics since the null space property of the Laplacian matrix, by which a collective behaviour is governed by, only exists for homogeneous dynamics. Furthermore, although [39] overcomes this issue and considers heterogeneous NI dynamics it comes at the expense of providing some robust output consensus conditions that are sufficient but not necessary. Therefore, by considering homogeneous NI dynamics, we are able here to obtain necessary and sufficient conditions for robust output consensus. Second, we use here relative output measurements as it is practically more significant since full state information is not always available.

In [14, 15, 16] consensus problems for homogeneous MAS using relative output measurements were addressed. However, unlike [14, 15, 16] which use state space techniques and observer-based consensus protocols where protocol design involves the solution of Riccati equations or/and linear matrix inequalities, the robust output consensus conditions we propose here are much simpler since they depend on the dc and infinite frequency gains of the systems as well as the network graph but not on the precise dynamics of the systems. Meanwhile, most closely related to our work is [38] and the motivations for this current study comes from the importance of consensus of MAS in real-world applications [23], the many practical systems that can be modelled as NI systems [30, 40], and the establishment of the general internal stability results in [34] by which it is possible to extend the work of [38]. While the work of [38] has successfully considered the robust output consensus problem for networked homogeneous NI systems, it has certain limitations in terms of the imposed assumptions. A principal limitation of [38] is that for NI systems with no poles at the origin, two assumptions at infinite frequency need to hold before the robust output consensus condition can be considered, while for systems with poles at the origin, the NI systems are limited to being strictly proper, matrix factorisation is required, and null space conditions need to be satisfied before the robust output consensus conditions can be considered. As a result, it not possible to determine robust output consensus for networked NI systems with the results of [38] when such assumptions do not hold.

In this chapter, we build on and extend the work of [38]. Similar to [38] we address the robust output consensus problem as an internal stability problem for networked NI systems subject to external disturbances and model uncertainties but different from [38] we use the generalised internal stability results in [34] to do so rather than those in [30, 41, 40] thereby extending the results of [38]. Therefore, the advantages of this current work over [38] and the main contributions of this chapter to existing knowledge are summarised as follows: (i) we relax the assumptions imposed in [38] thereby derive robust output consensus conditions which are not restricted; (ii) one distinct advantage that unfolds in our work is that not only do the derived conditions specialise to those in [38] by imposing the same two assumptions at infinite frequency but also specialise to those in [38] by imposing different assumptions which were unknown in [38]; (iii) the derived conditions simplify in the SISO case providing several insights which are not easily captured in the MIMO case (for SISO NI systems with no poles at origin) and are less sensitive to the network graph that models the interconnection of the systems (for SISO NI systems with poles at origin); and (iv) we show that consensus for some networked NI systems including a network of robotic arms cannot be determined by the results in [38] but can easily be concluded via the results of this chapter.

4.2 Problem description

Consider a network of N homogeneous NI systems with external disturbances acting on each system. The dynamics of the ith NI system are described as

$$y_i = d_{o_i} + P(s)\left(d_{in_i} + u_i\right) \quad \forall i \in \{1, \ldots, N\} \tag{4.1}$$

where $P(s)$ is an $n \times n$ transfer function matrix of the ith NI system, u_i, y_i, d_{in_i} and d_{o_i} are all vector signals with "n" elements and d_{in_i} and d_{o_i} are also energy-bounded in an H_2 (or in the time domain $\mathfrak{L}_2[0,\infty)$) sense. The signals u_i, y_i, d_{in_i} and d_{o_i} denote control input, output of the ith NI system, input and output disturbances respectively. It is assumed that relative output measurements with respect to neighbouring agents are available to each system. The network graph which models the information exchange among the systems is assumed fixed and satisfies the following assumption:

Assumption 4.1. *The network graph $\mathcal{G}$ is undirected and connected.*

Following [38], the distributed control protocol for the ith NI system is given by

$$\begin{aligned} u_i &= K(s)z_i, \\ z_i &= \sum\nolimits_{j=1}^{N} a_{ij}(y_i - y_j), \end{aligned} \quad \forall i \in \{1, \ldots, N\} \tag{4.2}$$

where $K(s)$ is the transfer function matrix of an SNI feedback controller, z_i represents the signal of relative measurements of neighbouring agents with respect to system i and a_{ij} denotes the elements of the adjacency matrix associated with the network graph $\mathcal{G}$. The collective network dynamics can thus be written as

$$y = d_o + \left(I_N \otimes P(s)\right)\left(d_{in} + u\right), \tag{4.3}$$

and

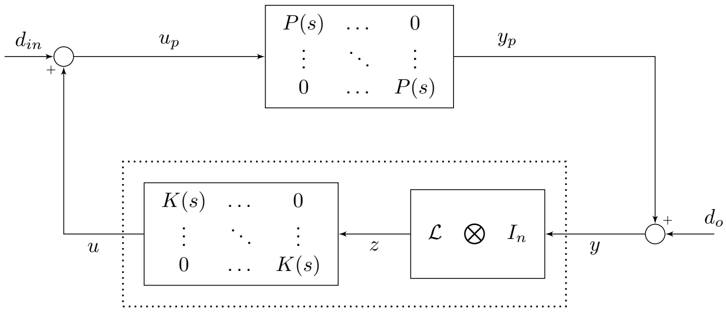


Figure 4.1: Real physical system with real disturbances.

$$
\begin{aligned}
u &= (I_N \otimes K(s))z, \\
z &= (\mathcal{L} \otimes I_n)y,
\end{aligned}
\tag{4.4}
$$

where $z = \left[z_1^T, \ldots, z_N^T\right]^T$, $y = \left[y_1^T, \ldots, y_N^T\right]^T$, $u = \left[u_1^T, \ldots, u_N^T\right]^T$, $d_{in} = \left[d_{in_1}^T, \ldots, d_{in_N}^T\right]^T$ and $d_o = \left[d_{o_1}^T, \ldots, d_{o_N}^T\right]^T$ are all vector signals with "nN" elements and d_{in} and d_o are also energy-bounded in an H_2 (or in the time domain $\mathfrak{L}_2[0, \infty)$) sense. $\mathcal{L} \in \mathbb{R}^{N \times N}$ is the Laplacian matrix associated with the network graph $\mathcal{G}$. A block diagram of the closed loop networked MAS is depicted in Figure 4.1. This figure represents the block diagram of the real physical system with real disturbances. In this chapter we address the robust output consensus problem for networks of NI systems as an internal stability problem. To this end, via block diagram algebra, it is possible to move the block $(\mathcal{L} \otimes I_n)$ in Figure 4.1 right past the summing junction to obtain a mathematical equivalent system as shown in Figure 4.2. It can be seen that the disturbances on

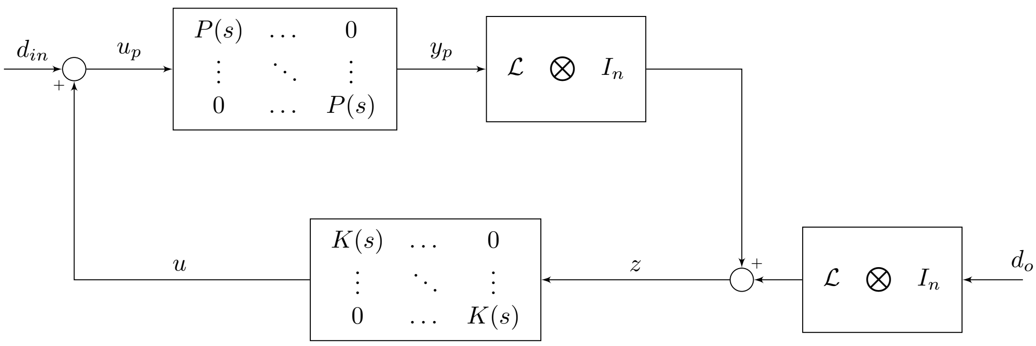


Figure 4.2: Mathematical equivalent system.

signal z in Figure 4.2 are a subset of the disturbances acting on the signal y in Figure 4.1 due to $\mathcal{L}$ being rank deficient. Let w_o be the disturbances acting on signal z, let $\bar{P}(s) = (\mathcal{L} \otimes I_n)(I_N \otimes P(s))$ denote the transfer function matrix from u_p to z and let $\bar{K}(s) = I_N \otimes K(s)$ denote the transfer function matrix from z to u. According to [38, Lemma 3], $\bar{P}(s)$ is NI if and only if $P(s)$ is NI with $\mathcal{G}$ satisfying Assumption 4.1. Similarly, $\bar{K}(s)$ is SNI since $K(s)$ is SNI. Then, the internal stability framework we consider, for addressing the output consensus problem, is given in Figure 4.3. That

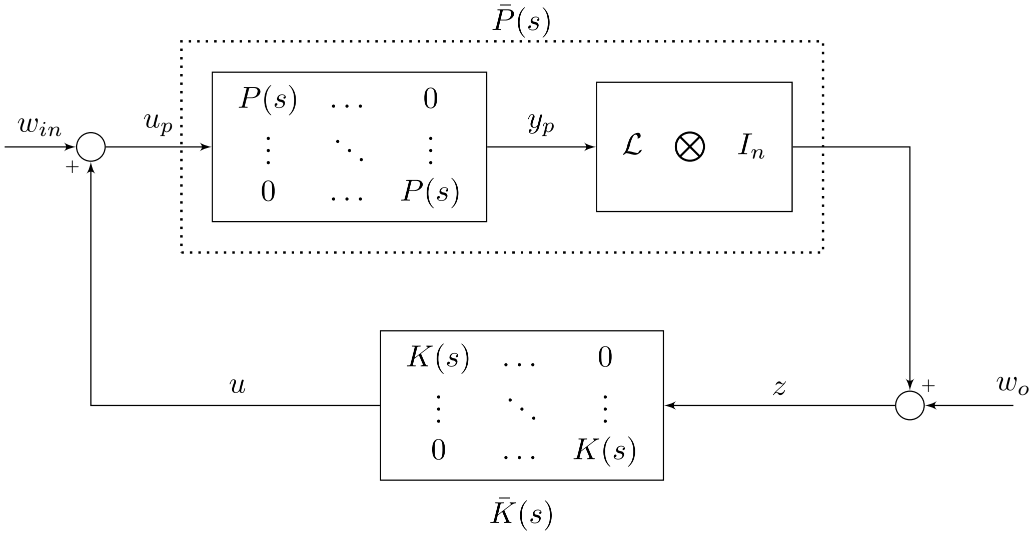


Figure 4.3: Internal stability framework.

is, we address the output consensus problem as an internal stability problem for the interconnection $[\bar{P}(s), \bar{K}(s)]$ where the plant $\bar{P}(s)$ is black-boxed thus we are being silent about the signal y_p.

Remark 4.1. *Internal stability of the interconnection $[\bar{P}, \bar{K}]$ guarantees that for all bounded inputs (w_{in}, w_o), the outputs (u_p, z) are bounded. (see e.g. [60]). Then, internal stability on the signals u_p and z is equivalent to consensus on signal y in Figure 4.1 via properties of $\mathcal{L}$ and via rank deficiency in the matrix $\mathcal{L} \otimes I_n$. On the other hand, internal stability of the interconnection $[\bar{P}, \bar{K}]$ does not imply asymptotic stability of the state space description; since $\bar{P}(s)$ is unobservable. Thus, the nature of the output signal y (or y_p) is depended on the block $(I_N \otimes P(s))$ and hence the final convergence trajectory of the output will depend on the dynamics of $P(s)$. For example, if $P(s)$ is an integrator and $u_p \to 0$, then $y_p \to$ a constant value, if $P(s)$ has stable dynamics and $u_p \to 0$, then $y_p \to 0$, etc. This will be discussed further in Section 4.4.*

The following remark discusses how model uncertainties are captured in this framework.

Remark 4.2. *Model uncertainties are captured in this framework by noting that any additive NI perturbations to a nominal NI system results in an NI perturbed system. Other forms of feedback uncertainties are also possible that preserve NI properties (see e.g. [35], [54]). Hence, $P(s)$ is regarded interchangeably as a nominal or perturbed plant as long as it fulfils the robust output consensus conditions As an example, consider a family of NI plant dynamics with the same dc and infinite frequency gains. Different systems such as $P_1(s) = (s+6)/(s+1.8)$ and $P_2(s) = ((s+1)(s+2)(s+3))\,/\,((s^2+2s+3)(s+0.6))$ both belong to this set. As will be shown in the next section, the consensus conditions depend on the dc and infinite frequency gains of the systems as well as the network graph but not on the precise dynamics of the systems.*

The robust output consensus problem we consider is defined as follows.

Definition 4.1 ([38, 39])**.** *For a family of NI plant dynamics and for all $\mathfrak{L}_2[0,\infty)$ disturbances acting on the plant input and/or output, robust output consensus is said to be achieved with distributed control protocol* (4.2) *for a network of NI systems if there exists $\epsilon_i(t) \in \mathfrak{L}_2[0,\infty)$ $\forall i \in \{1,\dots,N\}$ such that $y_i(t) \to y_{ss}(t)+\epsilon_i(t)$ $\forall i \in \{1,\dots,N\}$, where $y_{ss}(t)$ is the final convergence trajectory. Note that $\epsilon_i(t) = 0$ $\forall t$ and $\forall i \in \{1,\dots,N\}$ when there are no external disturbances.*

Our objective is to derive conditions for robust output consensus of a network of homogeneous NI systems under $\mathfrak{L}_2$ external disturbances and model uncertainty by using the general internal stability results in [34].

4.3 Robust output consensus

4.3.1 Networked homogeneous NI systems with no poles at the origin

The following theorem gives conditions under which robust output consensus is achieved for networked NI systems with no poles at the origin.

Theorem 4.1. *Consider a network of homogeneous NI systems $P(s)$ without poles at the origin, a network graph $\mathcal{G}$ that satisfies Assumption 4.1 and an SNI feedback controller $K(s)$ for each NI agent. Let μ_i for all $i \in \{1, \dots, N\}$ be the eigenvalues of the Laplacian matrix $\mathcal{L}$ associated with $\mathcal{G}$ ordered as stated in* (2.4). *Then, the following three statements are equivalent:*

(a) robust output consensus is achieved via control protocol (4.4) *for networked system* (4.3) *as shown in Figure 4.1 (or in a distributed manner* (4.2) *for each system* (4.1)*) under any external disturbances $d_{in}, d_o \in \mathfrak{L}_2^{nN}[0, \infty)$ and model uncertainty that retains the NI property of the perturbed system $P(s)$;*

(b) the set of conditions

$I_n - \mu_i P(\infty)K(\infty)$ *is nonsingular ,*

$\bar{\lambda}[[I_n - \mu_i P(\infty)K(\infty)]^{-1}[\mu_i P(\infty)K(0) - I_n]] < 0,$ *and*

$\bar{\lambda}[[I_n - \mu_i K(0)P(\infty)]^{-1}[\mu_i K(0)P(0) - I_n]] < 0,$

are satisfied for all $i \in \{2, \dots, N\}$;

(c) the set of conditions

$I_n - \mu_i P(\infty)K(\infty)$ *is nonsingular,*

$\bar{\lambda}[[\mu_i P(0)K(\infty) - I_n][I_n - \mu_i P(\infty)K(\infty)]^{-1}] < 0,$ *and*

$\bar{\lambda}[[\mu_i K(0)P(0) - I_n][I_n - \mu_i K(\infty)P(0)]^{-1}] < 0,$

are satisfied for all $i \in \{2, \dots, N\}$.

Proof. We begin by proving the equivalence of conditions (a) and (b). Let $\bar{P}(s) = \mathcal{L} \otimes P(s)$ and $\bar{K}(s) = I_N \otimes K(s)$. Now $\bar{P}(s)$ is NI by [38, Lemma 3] and has no poles at the origin since $P(s)$ has no poles at origin. Also, $\bar{K}(s)$ is SNI since $K(s)$ is SNI. Via Remark 4.1 and as in the proof of [38, Th. 1], the internal stability of $[\bar{P}(s), \bar{K}(s)]$ in Figure 4.3 implies output consensus (Figure 4.1) when $d_{in} = d_o = 0$, by noting that $z \to 0 \Leftrightarrow y \to 1_N \otimes y_{ss}$ since Assumption 4.1 holds. According to [34, Th. 9], $[\bar{P}(s), \bar{K}(s)]$ is internally stable if and only if

$$\begin{aligned}
&I_{Nn} - \bar{P}(\infty)\bar{K}(\infty) \text{ is nonsingular,}\\
&\bar{\lambda}[[I_{Nn} - \bar{P}(\infty)\bar{K}(\infty)]^{-1}[\bar{P}(\infty)\bar{K}(0) - I_{Nn}]] < 0, \text{ and}\\
&\bar{\lambda}[[I_{Nn} - \bar{K}(0)\bar{P}(\infty)]^{-1}[\bar{K}(0)\bar{P}(0) - I_{Nn}]] < 0.
\end{aligned}$$

Now $\mathcal{L}$ is a real symmetric matrix due to Assumption 4.1. Thus, $\mathcal{L}$ can be written as $\mathcal{L} = U\Lambda U^T$ where U is an orthogonal matrix and Λ is a diagonal matrix with eigenvalues of $\mathcal{L}$ on the diagonal. Then,

$$\begin{aligned}
& I_{Nn} - \bar{P}(\infty)\bar{K}(\infty) \\
& = I_{Nn} - (\mathcal{L} \otimes P(\infty))(I_N \otimes K(\infty)) \\
& = I_{Nn} - (\mathcal{L} \otimes P(\infty)K(\infty)) \\
& = I_{Nn} - (U\Lambda U^T \otimes P(\infty)K(\infty)) \\
& = (U \otimes I_n)[I_{Nn} - (\Lambda \otimes P(\infty)K(\infty))](U^T \otimes I_n) \\
& = (U \otimes I_n)\mathrm{diag}(I_n - \mu_i P(\infty)K(\infty))(U^T \otimes I_n) \\
& \qquad\qquad\qquad\qquad\qquad \forall i \in \{1, 2, \ldots, N\}.
\end{aligned}$$

So,

$$\begin{aligned}
& I_{Nn} - \bar{P}(\infty)\bar{K}(\infty) \text{ is nonsingular} \\
& \Leftrightarrow I_n - \mu_i P(\infty)K(\infty) \ \forall i \in \{2, \ldots, N\} \text{ is nonsingular} \\
& \quad \text{(due to the fact that } U \text{ and } U^T \text{ are nonsingular} \\
& \quad \text{matrices and for } \mu_1 = 0,\ I_n \text{ is nonsingular).}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& \bar{\lambda}[[I_{Nn} - \bar{P}(\infty)\bar{K}(\infty)]^{-1}[\bar{P}(\infty)\bar{K}(0) - I_{Nn}]] < 0 \\
& \Leftrightarrow \bar{\lambda}[[I_{Nn} - (\mathcal{L} \otimes P(\infty)K(\infty))]^{-1}[(\mathcal{L} \otimes P(\infty)K(0)) - I_{Nn}]] < 0 \\
& \Leftrightarrow \bar{\lambda}[[I_{Nn} - (U\Lambda U^T \otimes P(\infty)K(\infty))]^{-1}[(U\Lambda U^T \otimes P(\infty)K(0)) - I_{Nn}]] < 0 \\
& \Leftrightarrow \bar{\lambda}[(U \otimes I_n)[I_{Nn} - (\Lambda \otimes P(\infty)K(\infty))]^{-1}(U^T \otimes I_n)(U \otimes I_n) \\
& \qquad\qquad\qquad \times\ [(\Lambda \otimes P(\infty)K(0)) - I_{Nn}](U^T \otimes I_n)] < 0 \\
& \Leftrightarrow \bar{\lambda}[[I_{Nn} - (\Lambda \otimes P(\infty)K(\infty))]^{-1}[(\Lambda \otimes P(\infty)K(0)) - I_{Nn}]] < 0 \\
& \Leftrightarrow \max_{i=1,\ldots,N} \bar{\lambda}[[I_n - \mu_i P(\infty)K(\infty)]^{-1}[\mu_i P(\infty)K(0) - I_n]] < 0 \\
& \quad \text{(since the matrix in the previous step is block diagonal)} \\
& \Leftrightarrow \bar{\lambda}[[I_n - \mu_i P(\infty)K(\infty)]^{-1}[\mu_i P(\infty)K(0) - I_n]] < 0 \qquad \forall i \in \{2, \ldots, N\} \\
& \quad \text{(since for } \mu_1 = 0, \text{the condition is trivally fulfilled)}
\end{aligned}$$

and

$$
\begin{aligned}
&\bar{\lambda}[[I_{Nn} - \bar{K}(0)\bar{P}(\infty)]^{-1}[\bar{K}(0)\bar{P}(0) - I_{Nn}]] < 0 \\
&\Leftrightarrow \bar{\lambda}[[I_{Nn} - \mathcal{L} \otimes K(0)P(\infty)]^{-1}[\mathcal{L} \otimes K(0)P(0) - I_{Nn}]] < 0 \\
&\Leftrightarrow \bar{\lambda}[[I_{Nn} - (U\Lambda U^T \otimes K(0)P(\infty))]^{-1}[(U\Lambda U^T \otimes K(0)P(0)) - I_{Nn}]] < 0 \\
&\Leftrightarrow \bar{\lambda}[(U \otimes I_n)[I_{Nn} - (\Lambda \otimes K(0)P(\infty))]^{-1}(U^T \otimes I_n)(U \otimes I_n) \\
&\qquad\qquad\qquad\qquad \times\ [(\Lambda \otimes K(0)P(0)) - I_{Nn}](U^T \otimes I_n)] < 0 \\
&\Leftrightarrow \bar{\lambda}[[I_{Nn} - (\Lambda \otimes K(0)P(\infty))]^{-1}[(\Lambda \otimes K(0)P(0)) - I_{Nn}]] < 0 \\
&\Leftrightarrow \max_{i=1,\ldots,N} \bar{\lambda}[[I_n - \mu_i K(0)P(\infty)]^{-1}[\mu_i K(0)P(0) - I_n]] < 0 \\
&\quad \text{(since the matrix in the previous step is block diagonal)} \\
&\Leftrightarrow \bar{\lambda}[[I_n - \mu_i K(0)P(\infty)]^{-1}[\mu_i K(0)P(0) - I_n]] < 0 \qquad \forall i \in \{2, \ldots, N\} \\
&\quad \text{(since for } \mu_1 = 0, \text{the condition is trivally fulfilled)}\,.
\end{aligned}
$$

The proof for robust output consensus under external disturbances and model uncertainties then follows similarly to that in the proof of [38, Th.1] (see also Remark 4.2 for model uncertainties). The equivalence of conditions (a) and (c) can be proved in a similar manner by applying [34, Th. 14] instead of [34, Th. 9]. □

Remark 4.3. *The first and second conditions within conditions (b) of Theorem 4.1 guarantee that the matrix $I_n - \mu_i K(0)P(\infty)$ in the third condition is nonsingular $\forall i \in \{2, \ldots, N\}$.*

Remark 4.4. *Unlike [38], both set of conditions (b) and (c) of Theorem 4.1 include the nonzero eigenvalues of the Laplacian matrix $\mathcal{L}$. Thus, it can be concluded that the nonzero eigenvalues of $\mathcal{L}$ play a central role in achieving output consensus for networks of NI systems when the assumptions $P(\infty)K(\infty) = 0$ and $K(\infty) \geq 0$ of [38] are relaxed.*

The following corollary shows that the conditions of Theorem 4.1 not only specialise to that in [38] by imposing the same two assumptions at infinite frequency but also specialise to that in [38] by imposing different assumptions which were not known previously in [38].

Corollary 4.1. *Let the hypotheses of Theorem 4.1 hold and furthermore let either (i) $P(\infty)K(\infty) = 0$ and $K(\infty) \geq 0$, or (ii) $P(\infty)K(\infty) = 0$ and $P(0) > 0$, or* *(iii)* $P(\infty) = 0$ *hold. Then, robust output consensus is achieved via control protocol* (4.4) *for networked system* (4.3) *as shown in Figure 4.1 (or in a distributed manner* (4.2) *for each system* (4.1)*) under any external disturbances* $d_{in}, d_o \in \mathfrak{L}_2^{nN}[0,\infty)$ *and model uncertainty that retains the NI property of the perturbed system* $P(s)$ *if and only if*

$$\bar{\lambda}[P(0)K(0)] < \frac{1}{\bar{\lambda}(\mathcal{L})}.$$

Proof. For the case that (i) holds, the result is a direct consequence of the equivalence of conditions (a) and (b) of Theorem 4.1 and the lines of the proof here follow those of [34, Cor. 12] with the difference of having the eigenvalues of $\mathcal{L}$ included here. Furthermore, $\bar{\lambda}[P(0)K(0)] < 1/\mu_i \ \forall i \in \{2,\dots,N\} \Leftrightarrow \bar{\lambda}[P(0)K(0)] < 1/\bar{\lambda}(\mathcal{L})$ since (2.4) holds by Assumption 4.1. Also, $\bar{\lambda}[P(0)K(0)] < 1/\bar{\lambda}(\mathcal{L}) \Leftrightarrow \mu_i P(0) < K(0)^{-1} \quad \forall i \Rightarrow \mu_i P(\infty) < K(0)^{-1}$ (since $P(\infty) \leq P(0)$ via [41, Cor. 3]) and $\mu_i > 0 \quad \forall i \in \{2,\dots,N\}$ and $\mu_1 = 0$ via Assumption 4.1. For the case that (ii) holds, the result is a direct consequence of the equivalence of conditions (a) and (c) of Theorem 4.1 and the lines of the proof here follow those of [34, Cor. 15] with the difference of having the eigenvalues of $\mathcal{L}$ included here. Furthermore, $\bar{\lambda}[P(0)K(0)] < 1/\mu_i$ $\forall i \in \{2,\dots,N\} \Leftrightarrow \bar{\lambda}[P(0)K(0)] < 1/\bar{\lambda}(\mathcal{L})$ since (2.4) holds due to Assumption 4.1. Also, $\bar{\lambda}[P(0)K(0)] < 1/\bar{\lambda}(\mathcal{L}) \Leftrightarrow \mu_i K(0) < P(0)^{-1} \quad \forall i \Rightarrow \mu_i K(\infty) < P(0)^{-1} \quad \forall i$ (since $K(\infty) < K(0)$ via [41, Cor. 3]) and $\mu_i > 0 \quad \forall i \in \{2,\dots,N\}$ and $\mu_1 = 0$ via Assumption 4.1. For the case that (iii) holds, the result is a direct consequence of the equivalence of conditions (a) and (b) of Theorem 4.1 with $P(\infty) = 0$ and by (2.4). □

4.3.2 SISO specialisation: no poles at the origin

The following theorem shows that when the NI systems are SISO, the robust output consensus conditions of Theorem 4.1 can be simplified as follows.

Theorem 4.2. *Consider a network of homogeneous SISO NI systems* $P(s)$ *without poles at the origin, a network graph* $\mathcal{G}$ *that satisfies Assumption 4.1 and an SNI feedback controller* $K(s)$ *for each NI agent. Let* μ_i *for all* $i \in \{1,\dots,N\}$ *be the eigenvalues of the Laplacian matrix* $\mathcal{L}$ *associated with* $\mathcal{G}$ *ordered as stated in* (2.4)*. Then, robust output consensus is achieved via control protocol* (4.4) *for networked system* (4.3) *as shown in*

Figure 4.1 (or in a distributed manner (4.2) *for each system* (4.1)*) under any external disturbances* $d_{in}, d_o \in \mathfrak{L}_2^{nN}[0,\infty)$ *and model uncertainty that retains the NI property of the perturbed system* $P(s)$ *if and only if any of the following five conditions holds:*

1) $P(0)K(0) < 1/\bar{\lambda}(\mathcal{L})$ *and* $P(\infty)K(\infty) < 1/\bar{\lambda}(\mathcal{L})$ *;*

2) $K(\infty) > 0$ *and* $P(\infty)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$*;*

3) $K(0) < 0$ *and* $P(0)K(0) > 1/\underline{\lambda}_2(\mathcal{L})$*;*

4) $K(\infty) > 0$ *and there exists* $i \in \{i \in \{2,\dots,N-1\}\colon\ \mu_i \neq \mu_{i+1}\}$ *such that* $P(0)K(0) < 1/\mu_i$ *and* $P(\infty)K(\infty) > 1/\mu_{i+1}$ *;*

5) $K(0) < 0$ *and there exists* $i \in \{i \in \{2,\dots,N-1\}\colon\ \mu_i \neq \mu_{i+1}\}$ *such that* $P(0)K(0) > 1/\mu_{i+1}$ *and* $P(\infty)K(\infty) < 1/\mu_i$*.*

Proof. Since $n = 1$, conditions (b) of Theorem 4.1 become $P(\infty)K(\infty) \neq 1/\mu_i$, $\frac{\mu_i P(\infty)K(0)-1}{1-\mu_i P(\infty)K(\infty)} < 0$ and $\frac{\mu_i K(0)P(0)-1}{1-\mu_i K(0)P(\infty)} < 0$ $\forall i \in \{2,\dots,N\}$. Furthermore, since (2.4) holds, these three conditions reduce to either conditions i), ii), or iii) below.

i) $P(0)K(0) < 1/\bar{\lambda}(\mathcal{L})$, $P(\infty)K(\infty) < 1/\bar{\lambda}(\mathcal{L})$ and $P(\infty)K(0) < 1/\bar{\lambda}(\mathcal{L})$;

ii) $P(0)K(0) > 1/\underline{\lambda}_2(\mathcal{L})$, $P(\infty)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$ and $P(\infty)K(0) > 1/\underline{\lambda}_2(\mathcal{L})$;

iii) There exists $i \in \{i \in \{2,\dots,N-1\}\colon\ \mu_i \neq \mu_{i+1}\}$ such that $1/\mu_{i+1} < P(0)K(0) < 1/\mu_i$, $1/\mu_{i+1} < P(\infty)K(\infty) < 1/\mu_i$ and $1/\mu_{i+1} < P(\infty)K(0) < 1/\mu_i$.

Likewise, conditions (c) of Theorem 4.1 with (2.4) lead to either conditions I), II), or III) below.

I) $P(0)K(0) < 1/\bar{\lambda}(\mathcal{L})$, $P(\infty)K(\infty) < 1/\bar{\lambda}(\mathcal{L})$ and $P(0)K(\infty) < 1/\bar{\lambda}(\mathcal{L})$;

II) $P(0)K(0) > 1/\underline{\lambda}_2(\mathcal{L})$, $P(\infty)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$ and $P(0)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$;

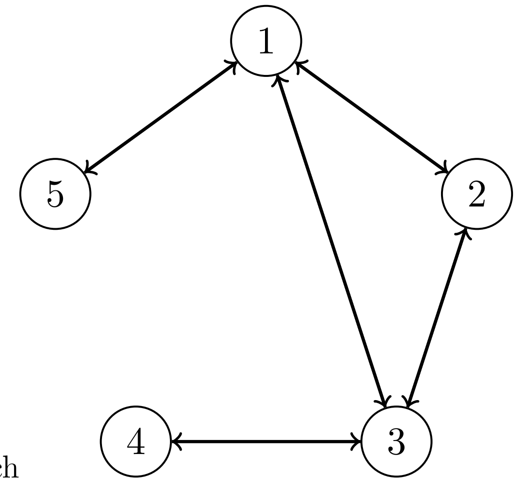
III) There exists $i \in \{i \in \{2,\dots,N-1\}\colon\ \mu_i \neq \mu_{i+1}\}$ such that $1/\mu_{i+1} < P(0)K(0) < 1/\mu_i$, $1/\mu_{i+1} < P(\infty)K(\infty) < 1/\mu_i$ and $1/\mu_{i+1} < P(0)K(\infty) < 1/\mu_i$.

($\Leftarrow$) Condition 1) implies both conditions i) and I) via [34, Lemma 5]. Condition 2) together with using properties of [41, Cor. 3] imply $P(0) \geq P(\infty) > 0$ and $K(0) > K(\infty) > 0$ respectively. Then we get $P(0)K(0) \geq P(\infty)K(0)$, $P(0)K(\infty) \geq$

$P(\infty)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$, $P(0)K(0) > P(0)K(\infty)$, and $P(\infty)K(0) > P(\infty)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$ which together imply conditions ii) and II). Likewise, condition 3) together with using properties of [41, Cor. 3] imply $P(\infty) \leq P(0) < 0$ and $K(\infty) < K(0) < 0$ respectively. Then we get $1/\underline{\lambda}_2(\mathcal{L}) < P(0)K(0) < P(0)K(\infty)$, $P(\infty)K(0) < P(\infty)K(\infty)$, $1/\underline{\lambda}_2(\mathcal{L}) < P(0)K(0) \leq P(\infty)K(0)$ and $P(0)K(\infty) \leq P(\infty)K(\infty)$ which together imply conditions ii) and II). Condition 4) together with using properties of [41, Cor. 3] imply $P(0) \geq P(\infty) > 0$ and $K(0) > K(\infty) > 0$ respectively. Then we get $1/\mu_i > P(0)K(0) \geq P(\infty)K(0)$, $P(0)K(\infty) \geq P(\infty)K(\infty) > 1/\mu_{i+1}$, $1/\mu_i > P(0)K(0) > P(0)K(\infty)$ and $P(\infty)K(0) > P(\infty)K(\infty) > 1/\mu_{i+1}$ which together give conditions iii) and III). Likewise, condition 5) together with using properties of [41, Cor. 3] imply $P(\infty) \leq P(0) < 0$ and $K(\infty) < K(0) < 0$ respectively. Then we get $P(0)K(\infty) > P(0)K(0) > 1/\mu_{i+1}$, $1/\mu_i > P(\infty)K(\infty) > P(\infty)K(0)$, $P(\infty)K(0) > P(0)K(0) > 1/\mu_{i+1}$ and $1/\mu_i > P(\infty)K(\infty) > P(0)K(\infty)$ which together imply conditions iii) and III).
($\Rightarrow$) Both conditions i) and I) reduce to condition 1). Now consider the five cases as in the proof of [34, Th. 17] which are $0 < K(\infty) < K(0)$, $0 = K(\infty) < K(0)$, $K(\infty) < 0 < K(0)$, $K(\infty) < K(0) = 0$ and $K(\infty) < K(0) < 0$. Only the first and last cases are allowed by conditions ii), iii), II) and III) as the three middle cases violate them. Consequently, it is easy to see that condition ii) [resp. II)] imply either condition 2) or 3) while condition iii) [resp. III)] imply either condition 4) or 5). $\square$

The following example shows the effectiveness of Theorem 4.2.

Example 4.1. *Given five homogeneous SISO NI systems each with transfer function* $P(s) = \frac{1}{s^2+5} + 2$. *We consider the connection of these NI systems over the network topology shown in Figure 4.4. The associated Laplacian matrix* $\mathcal{L}$ *is also shown in Figure 4.4. The nonzero eigenvalues of* $\mathcal{L}$ *arranged as in* (2.4) *are* $\{0.6972, 1.3820, 3.6180, 4.3028\}$. *Consider distributed control protocol* (4.2) *with the following SNI feedback controller* $K(s) = \frac{1}{s+5} + d$ *where* d *is a tuning parameter and* $d \neq 0$. *We use Theorem 4.2 to study the effect of tuning parameter* d *on achieving robust output consensus. It is important to note that since* $P(\infty)K(\infty) = 2d \neq 0$, *the results in [38, Th. 1] cannot be used to determine whether robust output consensus of the networked NI systems can be achieved or not. The values of* d *are chosen as* 0.2, 0.4, 0.6, 4, *and* -4. *It*



$$\mathcal{L} = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

chch

Figure 4.4: Network graph and associated Laplacian matrix.

is easy to check that for $d = 0.2$ *and* $d = 0.6$, *conditions 1)–5) of Theorem 4.2 fail to hold. Thus, we conclude that robust output consensus is not achieved with these values. For* $d = 0.4$, *condition 4) is satisfied; for* $d = 4$, *condition 2) is satisfied; and for* $d = -4$, *condition 1) is satisfied. Thus, we conclude that robust output consensus is achieved with these values. However, it is important to note that Condition 4) of Theorem 4.2 involves the knowledge of all nonzero eigenvalues (hence more sensitive to the network graph) whereas conditions 1) and 2) of Theorem 4.2 depend only on the knowledge of the largest and second smallest eigenvalue of* $\mathcal{L}$ *respectively (hence less sensitive to network graph).*

Remark 4.5. *Although SNI controller synthesis for performance is not explicitly covered in this chapter, Example 4.1 gives an indication how an SNI controller to each NI system in the SISO case can be selected to reduce the effect of the network graph which is not apparent in the MIMO case. It can be deduced from Theorem 4.2 and Example 4.1 that it is preferable to select the SNI controller in protocol* (4.2) *to satisfy either one of the first three conditions in Theorem 4.2 and avoid satisfying the last two conditions of Theorem 4.2 in order to minimize the effect of the network graph on robust output consensus since an estimate for the second smallest and largest eigenvalues of* $\mathcal{L}$ *would only be needed. Furthermore, unlike the MIMO case, a Nyquist plot interpretation can be drawn for the SISO case in a similar manner as in [34] but with the difference that the crucial point here is no longer* $+1$. *Theorem 4.2 indicates that robust output consensus is achieved via condition 1) when the Nyquist plot of* $P(s)K(s)$

starts and ends to the left of $1/\bar{\lambda}(L)$, *and via either condition 2) or 3) when the Nyquist plot of* $P(s)K(s)$ *starts and ends to the right of* $1/\underline{\lambda}_2(\mathcal{L})$ *and additionally* $P(0)$, $P(\infty)$, $K(0)$ *and* $K(\infty)$ *all have the same sign; either positive or negative. Moreover, robust output consensus is achieved via condition 4) when the Nyquist plot of* $P(s)K(s)$ *starts to the left of* $1/\mu_i$ *and ends to the right of* $1/\mu_{i+1}$ *for an* $i \in \{i \in \{2,\dots,N-1\}$*:* $\mu_i \neq \mu_{i+1}\}$ *and additionally* $P(0)$, $P(\infty)$, $K(0)$ *and* $K(\infty)$ *all have positive signs and via condition 5) when the Nyquist plot of* $P(s)K(s)$ *starts to the right of* $1/\mu_{i+1}$ *and ends to the left of* $1/\mu_i$ *for an* $i \in \{i \in \{2,\dots,N-1\}$*:* $\mu_i \neq \mu_{i+1}\}$ *and additionally* $P(0)$, $P(\infty)$, $K(0)$ *and* $K(\infty)$ *all have negative signs.*

4.3.3 Networked homogeneous NI systems with poles at the origin

The following theorem gives conditions under which robust output consensus is achieved for networked NI systems with possible poles at the origin.

Theorem 4.3. *Consider a network of homogeneous NI systems* $P(s)$*, a network graph* $\mathcal{G}$ *that satisfies Assumption 4.1, and an SNI feedback controller* $K(s)$ *for each NI agent. Let* μ_i *for all* $i \in \{1,\dots,N\}$ *be the eigenvalues of the Laplacian matrix* $\mathcal{L}$ *associated with* $\mathcal{G}$ *ordered as stated in* (2.4)*. Let* $\Psi < 0$ *be such that* $\bar{\lambda}[P(\infty)\Psi] < 1/\bar{\lambda}(\mathcal{L})$*. Then, the following three conditions are equivalent:*

(a) robust output consensus is achieved via control protocol (4.4) *for networked system* (4.3) *as shown in Figure 4.1 (or in a distributed manner* (4.2) *for each system* (4.1)*) under any external disturbances* $d_{in}, d_o \in \mathfrak{L}_2[0,\infty)$ *and model uncertainty that retains the NI property of the perturbed system* $P(s)$*;*

(b) the set of conditions

$I_n - \mu_i P(\infty)K(\infty)$ *is nonsingular,*

$\bar{\lambda}[[I_n - \mu_i P(\infty)K(\infty)]^{-1}[\mu_i P(\infty)K(0) - I_n]] < 0,$ *and*

$\bar{\lambda}[\lim_{s\to 0}[[I_n - \mu_i \Psi P(\infty)][I_n - \mu_i K(s)P(\infty)]^{-1}\,[\mu_i K(s)P(s) - I_n][I_n - \mu_i \Psi P(s)]^{-1}]] < 0,$

are satisfied for all $i \in \{2,\dots,N\}$;

(c) the set of conditions

$I_n - \mu_i P(\infty)K(\infty)$ *is nonsingular,*

$\bar{\lambda}[\lim_{s\to 0}[[I_n - \mu_i P(s)\Psi]^{-1}[\mu_i P(s)K(\infty) - I_n][I_n - \mu_i P(\infty)K(\infty)]^{-1}[I_n - \mu_i P(\infty)\Psi]]] < 0,$

and $\bar{\lambda}[\lim_{s\to 0}[[\mu_i K(s)P(s) - I_n][I_n - \mu_i K(\infty)P(s)]^{-1}] < 0,$
are satisfied for all $i \in \{2, \ldots, N\}$.

Proof. We begin by proving the equivalence of conditions (a) and (b). Recall that $\bar{P}(s) = \mathcal{L} \otimes P(s)$ in Figure 4.3 is NI and now has poles at the origin since $P(s)$ has poles at the origin and $\bar{K}(s) = I_N \otimes K(s)$ in Figure 4.3 is SNI since $K(s)$ is SNI. Also, recall that the internal stability of $[\bar{P}(s), \bar{K}(s)]$ implies output consensus when $d_{in} = d_o = 0$. Thus, we shall prove the internal stability of $[\bar{P}(s), \bar{K}(s)]$ and then the proof for robust output consensus runs as before. Let $\bar{\Psi} = (I_N \otimes \Psi)$. We have $\bar{\Psi} < 0$ if and only if $\Psi < 0$. Also, $\bar{\lambda}[\bar{P}(\infty)\bar{\Psi}] = \bar{\lambda}[\mathcal{L} \otimes P(\infty)\Psi] < 1$ if and only if $\bar{\lambda}[P(\infty)\Psi] < 1/\bar{\lambda}(\mathcal{L})$. Hence by applying [34, Th. 24], $[\bar{P}(s), \bar{K}(s)]$ is internally stable if and only if

$$\begin{aligned}
&I_{Nn} - \bar{P}(\infty)\bar{K}(\infty) \text{ is nonsingular,}\\
&\bar{\lambda}[[I_{Nn} - \bar{P}(\infty)\bar{K}(\infty)]^{-1}[\bar{P}(\infty)\bar{K}(0) - I_{Nn}]] < 0, \text{ and}\\
&\bar{\lambda}[\lim_{s\to 0}[[I_{Nn} - \bar{\Psi}\bar{P}(\infty)][I_{Nn} - \bar{K}(s)\bar{P}(\infty)]^{-1}\\
&\qquad\qquad \times [\bar{K}(s)\bar{P}(s) - I_{Nn}][I_{Nn} - \bar{\Psi}\bar{P}(s)]^{-1}]] < 0.
\end{aligned}$$

Recall that $\mathcal{L}$ is a real symmetric matrix due to Assumption 4.1. Thus, by applying the same transformation as in Theorem 4.1 we arrive at conditions (b) of this theorem. It is not difficult to verify that the equivalence of conditions (a) and (c) can be proved in a similar manner by applying [34, Th. 26] rather than [34, Th. 24]. □

It is important to show that the limits in Theorem 4.3 are finite $\forall i \in \{2, \ldots, N\}$. To this end, we begin by stating a modified version of [34, Lemma 28].

Lemma 4.1. *Let the hypotheses of Theorem 4.3 hold and furthermore consider* $I_n - \mu_i P(\infty)K(\infty)$ *nonsingular and* $\bar{\lambda}[\lim_{s\to 0}[[I_n - \mu_i P(s)\Psi]^{-1}[\mu_i P(s)K(\infty) - I_n]$ $[I_n - \mu_i P(\infty)K(\infty)]^{-1}[I_n - \mu_i P(\infty)\Psi]]] < 0$ $\forall i \in \{2, \ldots, N\}$. *Then,* $\lim_{s\to 0}[[I_n - \mu_i \Psi P(s)][I_n - \mu_i K(\infty)P(s)]^{-1}]$ $\forall i \in \{2, \ldots, N\}$ *is finite and nonsingular.*

Proof. The proof is omitted since it follows similar arguments as in the proof of [34, Lemma 28]. □

Remark 4.6. *For the limit in conditions (b) of Theorem 4.3: Since* $\mu_i P(s)$ *is NI* $\forall i \in \{2, \ldots, N\}$, $\Psi < 0$ *and* $\bar{\lambda}[\mu_i P(\infty)\Psi] < 1$ $\forall i \in \{2, \ldots, N\}$, *then Lemma 20 of*

[34] can be employed to show that both $[I_n - \mu_i \Psi P(s)]^{-1}$ *and* $\mu_i P(s)[I_n - \mu_i \Psi P(s)]^{-1}$ *have no poles at the origin* $\forall i \in \{2, \dots, N\}$*. Hence,* $\lim_{s\to 0}[[\mu_i K(s)P(s) - I_n][I_n - \mu_i \Psi P(s)]^{-1}]]$ *is finite* $\forall i \in \{2, \dots, N\}$*. For the limits in conditions (c) of Theorem 4.3: The limit in the second condition is finite* $\forall i \in \{2, \dots, N\}$ *since* $\lim_{s\to 0}[[I_n - \mu_i P(s)\Psi]^{-1}[\mu_i P(s)K(\infty) - I_n]]$ *is finite* $\forall i \in \{2, \dots, N\}$ *by [34, Lemma 20] while the limit in the third condition is finite* $\forall i \in \{2, \dots, N\}$ *by [34, Lemma 20] and Lemma 4.1 above since* $\lim_{s\to 0}[[\mu_i K(s)P(s) - I_n][I_n - \mu_i K(\infty)P(s)]^{-1}] = \lim_{s\to 0}[[\mu_i K(s)P(s) - I_n][I_n - \mu_i \Psi P(s)]^{-1}[I_n - \mu_i \Psi P(s)][I_n - \mu_i K(\infty)P(s)]^{-1}]$ $\forall i \in \{2, \dots, N\}$*.*

When the networked NI systems have a single or double pole at the origin in all directions, the robust output consensus conditions will neither depend on the nonzero eigenvalues of $\mathcal{L}$ nor on the matrix Ψ as follows.

Corollary 4.2. *Consider a network of homogeneous strictly proper NI systems* $P(s)$*, a network graph* $\mathcal{G}$ *that satisfies Assumption 4.1, and an SNI feedback controller* $K(s)$ *for each NI agent. Assume one of the following conditions hold:*

1) $\lim_{s\to 0} s^2 P(s)$ *is nonsingular;*

2) $\lim_{s\to 0} s^2 P(s) = 0$ *and* $\lim_{s\to 0} sP(s)$ *is nonsingular.*

Then, robust output consensus is achieved via control protocol (4.4) *for networked system* (4.3) *as shown in Figure 4.1 (or in a distributed manner* (4.2) *for each system* (4.1)*) under any external disturbances* $d_{in}, d_o \in \mathfrak{L}_2[0, \infty)$ *and model uncertainty that retains the NI property of the perturbed system* $P(s)$ *if and only if* $K(0) < 0$*.*

Proof. Since $P(\infty) = 0$, conditions (b) in Theorem 4.3 reduce to

$$\bar{\lambda}[\lim_{s\to 0}[[\mu_i K(s)P(s) - I_n][I_n - \mu_i \Psi P(s)]^{-1}]] < 0 \tag{4.5}$$

while conditions (c) in Theorem 4.3 reduce to

$$\begin{aligned} &\bar{\lambda}[\lim_{s\to 0}[[I_n - \mu_i P(s)\Psi]^{-1}[\mu_i P(s)K(\infty) - I_n]] < 0, \text{and} \\ &\bar{\lambda}[\lim_{s\to 0}[[\mu_i K(s)P(s) - I_n][I_n - \mu_i K(\infty)P(s)]^{-1}] < 0 \end{aligned} \tag{4.6}$$

$\forall i \in \{2, \dots, N\}$. The proof is then similar to the proof of [34, Cor. 32] but when applied either to condition (4.5) or (4.6). $\square$

Remark 4.7. *A procedure was provided in [34] for selecting Ψ by first decomposing $P(\infty)$ as $P(\infty) = Q\Lambda Q^T$, where Q is an orthogonal matrix and Λ is diagonal, then selecting Λ_1 as a diagonal matrix with negative elements such that $\Lambda - \Lambda_1 > 0$ so that $\Psi = Q\Lambda_1^{-1}Q^T$ fulfills the required condition $\bar{\lambda}[P(\infty)\Psi] < 1$ as this is equivalent to $P(\infty) - \Psi^{-1} > 0$. Since the nonzero eigenvalues of $\mathcal{L}$ play an important role in the robust output consensus conditions, we hence need to select Λ_1 such that $\Lambda - \frac{1}{\bar{\lambda}(\mathcal{L})}\Lambda_1 > 0$ since the condition $\bar{\lambda}[P(\infty)\Psi] < 1$ is replaced here by $\bar{\lambda}[P(\infty)\Psi] < 1/\bar{\lambda}(\mathcal{L})$.*

4.3.4 SISO specialisation: must have pole(s) at the origin

The following corollary shows that when the NI systems are SISO and must either have a single or double pole at the origin, the robust output consensus conditions of Theorem 4.3 can be simplified as follows.

Corollary 4.3. *Consider a network of homogeneous SISO NI systems $P(s)$ with $s = 0$ being a single or double pole of $P(s)$, a network graph $\mathcal{G}$ that satisfies Assumption 4.1 and an SNI feedback controller $K(s)$ for each NI agent. Then, robust output consensus is achieved via control protocol* (4.4) *for networked system* (4.3) *as shown in Figure 4.1 (or in a distributed manner* (4.2) *for each system* (4.1)*) under any external disturbances $d_{in}, d_o \in \mathfrak{L}_2^{nN}[0,\infty)$ and model uncertainty that retains the NI property of the perturbed system $P(s)$ if and only if either one of the following conditions hold:*

1) $P(\infty)K(\infty) < 1/\bar{\lambda}(\mathcal{L})$ and $K(0) < 0$;

2) $P(\infty)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$ and $K(\infty) > 0$.

Proof. The two conditions in this corollary can be obtained either via conditions (b) or conditions (c) of Theorem 4.3. First, we give the proof via conditions (c) of Theorem 4.3. Since $n = 1$, conditions (c) of Theorem 4.3 become $P(\infty)K(\infty) \neq 1/\mu_i$ $\forall i \in \{2, \dots, N\}$, $\frac{K(\infty)}{1-\mu_i K(\infty)P(\infty)} < 0$ $\forall i \in \{2, \dots, N\}$, and $\frac{K(0)}{K(\infty)} > 0$ after writing $P(s)$ in its Laurent series form and taking the limit at $s \to 0$. Consider the three cases for the first condition: $P(\infty)K(\infty) > 1/\mu_i$ $\forall i \in \{2, \dots, N\}$, $P(\infty)K(\infty) < 1/\mu_i$ $\forall i \in \{2, \dots, N\}$ and there exists $i \in \{i \in \{2, \dots, N-1\}\colon \mu_i \neq \mu_{i+1}\}$ such that $1/\mu_{i+1} < P(\infty)K(\infty) < 1/\mu_i$. The third case is violated by the second condition. Hence we consider the first two cases. These two cases with (2.4) and the NI property

$K(0) > K(\infty)$ lead to the three conditions being reduced to either condition 1) or 2) in this corollary. Second, we give the proof via conditions (b) of Theorem 4.3. Since $n = 1$, conditions (b) of Theorem 4.3 become $P(\infty)K(\infty) \neq 1/\mu_i$, $\frac{\mu_i P(\infty)K(0)-1}{1-\mu_i P(\infty)K(\infty)} < 0$ and $\frac{K(0)}{1-\mu_i P(\infty)K(0)} < 0 \; \forall i \in \{2,\dots,N\}$ after writing $P(s)$ in its Laurent series form and taking the limit at $s \to 0$. The three aforementioned cases are considered where the third case is violated by the third condition. Hence, these three conditions with (2.4) lead to either one of the following conditions

i) $P(\infty)K(\infty) < 1/\bar{\lambda}(\mathcal{L})$, $P(\infty)K(0) < 1/\bar{\lambda}(\mathcal{L})$, and $K(0) < 0$;

ii) $P(\infty)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$, $P(\infty)K(0) > 1/\underline{\lambda}_2(\mathcal{L})$, and $K(0) > 0$.

The equivalence between i) and 1) and the equivalence between ii) and 2) can be concluded via similar arguments as in the proof of [34, Cor. 30] □

4.4 Convergence analysis

In this section, we study convergence of the networked systems under distributed control protocol (4.2). Recall form Remark 4.1 that internal stability of the interconnection $[\bar{P}, \bar{K}]$ does not imply asymptotic stability of the state space description since $\bar{P}(s)$ is unobservable. Thus, we are interested in analysing the final steady state trajectory of the networked system. We show that the same conclusion as in [38] for the final convergence can be drawn here which states that the steady state behaviour of the closed loop networked system is determined by the eigenvalues of the closed loop networked system on the imaginary axis. In doing so, the external disturbances and model uncertainty will not be considered in this section.

Let a minimal realisation for the ith NI system $P(s)$ be

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i \\ y_i &= Cx_i + Du_i \end{aligned}, \qquad i \in \{1,\dots,N\} \tag{4.7}$$

and a minimal realisation for the ith SNI controller $K(s)$ be

$$\begin{aligned} \dot{\bar{x}}_i &= \bar{A}\bar{x}_i + \bar{B}\bar{u}_i \\ \bar{y}_i &= \bar{C}\bar{x}_i + \bar{D}\bar{u}_i \end{aligned}, \qquad i \in \{1,\dots,N\} \tag{4.8}$$

where $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times n}$, $C \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{n\times n}$, $\bar{A} \in \mathbb{R}^{q\times q}$, $\bar{B} \in \mathbb{R}^{q\times n}$, $\bar{C} \in \mathbb{R}^{n\times q}$ and $\bar{D} \in \mathbb{R}^{n\times n}$. Define $R = (I_{Nn} - \mathcal{L} \otimes D\bar{D})$. Unlike [38], the closed loop system, with the assumption $D\bar{D} = 0$ removed, is now given by

$$\begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \end{bmatrix} = \Psi_{cl} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \tag{4.9}$$

where Ψ_{cl} is defined as

$$\Psi_{cl} = \begin{bmatrix} (I_N \otimes A) + (\mathcal{L} \otimes B\bar{D})R^{-1}(I_N \otimes C) & (I_N \otimes B\bar{C}) + (\mathcal{L} \otimes B\bar{D})R^{-1}(I_N \otimes D\bar{C}) \\ (\mathcal{L} \otimes \bar{B})R^{-1}(I_N \otimes C) & (I_N \otimes \bar{A}) + (\mathcal{L} \otimes \bar{B})R^{-1}(I_N \otimes D\bar{C}) \end{bmatrix}.$$

Define $\tilde{R}_i = (I_n - \mu_i D\bar{D})$ and $\tilde{S}_i = (I_n - \mu_i \bar{D}D)$. The following Lemma yields information about the spectrum of Ψ_{cl}, which can be considered a generalisation of [38, Lemma 5].

Lemma 4.2. *Let μ_i be the ith eigenvalue of $\mathcal{L}$ associated with eigenvector $v^i_{\mathcal{L}}$. The spectrum of Ψ_{cl} is given by the union of the spectra of the following matrices:*

$$\psi_i = \begin{bmatrix} A + \mu_i B\bar{D}\tilde{R}_i^{-1}C & B\bar{C} + \mu_i B\bar{D}\tilde{R}_i^{-1}D\bar{C} \\ \mu_i \bar{B}\tilde{R}_i^{-1}C & \bar{A} + \mu_i \bar{B}\tilde{R}_i^{-1}D\bar{C} \end{bmatrix} \qquad \forall i \in \{1, \dots, N\}.$$

Furthermore, let $\begin{bmatrix} v_1^{i\,*} & v_2^{i\,*} \end{bmatrix}^*$ be an eigenvector of ψ_i. Then, the corresponding eigenvector of Ψ_{cl} is $\begin{bmatrix} v^i_{\mathcal{L}} \otimes v^i_1 \\ v^i_{\mathcal{L}} \otimes v^i_2 \end{bmatrix}$.

Proof. By Assumption 4.1, the Laplacian matrix can be decomposed as $\mathcal{L} = U\Lambda U^T$ where $U = [u_1, \dots, u_N]$.Therefore,

$$\begin{aligned}
&(\mathcal{L} \otimes \bar{D})R^{-1} \\
&= (I_{Nn} - \mathcal{L} \otimes \bar{D}D)^{-1}(\mathcal{L} \otimes \bar{D}) \\
&= (U \otimes I_n)(I_{Nn} - \Lambda \otimes \bar{D}D)^{-1}(U^T \otimes I_n)(\mathcal{L} \otimes \bar{D}) \\
&= \begin{bmatrix} (u_1 \otimes I_n), & \dots, & (u_N \otimes I_n) \end{bmatrix} \operatorname{diag}(\tilde{S}_i^{-1}) \begin{bmatrix} (u_1^T \otimes I_n) \\ \vdots \\ (u_N^T \otimes I_n) \end{bmatrix} (\mathcal{L} \otimes \bar{D}) \qquad \forall i \in \{1, \dots, N\} \\
&= \left[\sum\nolimits_{i=1}^{N} (u_i \otimes I_n)\tilde{S}_i^{-1}(u_i^T \otimes I_n)\right] (\mathcal{L} \otimes \bar{D}).
\end{aligned}$$

Similarly, $(\mathcal{L}\otimes I_n)R^{-1} = [\sum_{i=1}^{N}(u_i\otimes I_n)\tilde{R}_i^{-1}(u_i^T\otimes I_n)](\mathcal{L}\otimes I_n)$, and the entries of Ψ_{cl} can be expressed as

$$
\begin{aligned}
\Psi_{cl_{11}} &= (I_N\otimes A) + (I_N\otimes B)\left[\sum_{i=1}^{N}(u_i\otimes I_n)\tilde{S}_i^{-1}(u_i^T\otimes I_n)\right](\mathcal{L}\otimes \bar{D}C)\\
\Psi_{cl_{12}} &= (I_N\otimes B\bar{C}) + (I_N\otimes B)\left[\sum_{i=1}^{N}(u_i\otimes I_n)\tilde{S}_i^{-1}(u_i^T\otimes I_n)\right](\mathcal{L}\otimes \bar{D}D\bar{C})\\
\Psi_{cl_{21}} &= (I_N\otimes \bar{B})\left[\sum_{i=1}^{N}(u_i\otimes I_n)\tilde{R}_i^{-1}(u_i^T\otimes I_n)\right](\mathcal{L}\otimes C)\\
\Psi_{cl_{22}} &= (I_N\otimes \bar{A}) + (I_N\otimes \bar{B})\left[\sum_{i=1}^{N}(u_i\otimes I_n)\tilde{R}_i^{-1}(u_i^T\otimes I_n)\right](\mathcal{L}\otimes D\bar{C}).
\end{aligned}
$$

Without loss of generality, let $v_{\mathcal{L}}^i$ be a normalized eigenvector of $\mathcal{L}$ associated with the eigenvalue μ_i. Also, since the columns of U are a set of N orthonormal vectors, each of which is an eigenvector of $\mathcal{L}$, it follows that $u_j^T v_{\mathcal{L}}^i = 0 \quad \forall i\neq j, u_i^T v_{\mathcal{L}}^i = 1$ and $u_i = v_{\mathcal{L}}^i$. Let λ_{ψ_i} be the eigenvalue of ψ_i. We consequently obtain

$$
\begin{aligned}
&\Psi_{cl}\begin{bmatrix} v_{\mathcal{L}}^i\otimes v_1^i \\ v_{\mathcal{L}}^i\otimes v_2^i\end{bmatrix}\\
&= \begin{bmatrix} v_{\mathcal{L}}^i\otimes (Av_1^i + \mu_i B\bar{D}\tilde{R}_i^{-1}Cv_1^i + B\bar{C}v_2^i + \mu_i B\bar{D}\tilde{R}_i^{-1}D\bar{C}v_2^i) \\ v_{\mathcal{L}}^i\otimes (\mu_i\bar{B}\tilde{R}_i^{-1}Cv_1^i + \bar{A}v_2^i + \mu_i\bar{B}\tilde{R}_i^{-1}D\bar{C}v_2^i)\end{bmatrix}\\
&= \begin{bmatrix} v_{\mathcal{L}}^i\otimes \lambda_{\psi_i}v_1^i \\ v_{\mathcal{L}}^i\otimes \lambda_{\psi_i}v_2^i\end{bmatrix} = \lambda_{\psi_i}\begin{bmatrix} v_{\mathcal{L}}^i\otimes v_1^i \\ v_{\mathcal{L}}^i\otimes v_2^i\end{bmatrix}
\end{aligned}
$$

which shows that λ_{ψ_i} is also an eigenvalue of Ψ_{cl} with $\begin{bmatrix} v_{\mathcal{L}}^i\otimes v_1^i \\ v_{\mathcal{L}}^i\otimes v_2^i\end{bmatrix}$ being the associated eigenvector. □

The importance of Lemma 4.2 is that it characterises the spectrum of Ψ_{cl} which plays an essential role in determining the final convergence of system (4.9). In what follows we show that the steady-state behaviour of the closed loop system (4.9) is in particular determined by the eigenvalues of A on the imaginary axis. For $\mu_1 = 0$, $\psi_1 = \begin{bmatrix} A & B\bar{C} \\ 0 & \bar{A}\end{bmatrix}$. For $\mu_i > 0\ \forall i\in\{2,\ldots,N\}$, $\psi_i = \begin{bmatrix} A & B\bar{C} \\ 0 & \bar{A}\end{bmatrix} + \mu_i\begin{bmatrix} B\bar{D} \\ \bar{B}\end{bmatrix}\tilde{R}_i^{-1}\begin{bmatrix} C & D\bar{C}\end{bmatrix}$. The eigenvalues of ψ_1 are the union of the eigenvalues of A and $\bar{A}$ which are in the CLHP and OLHP, respectively. For $\mu_i > 0\ \forall i\in\{2,\ldots,N\}$ and $\det(A)\neq 0$, using [41, Lemma 7] and [41, Lemma 8], ψ_i can be written as

$$\psi_i = \begin{bmatrix} A & B\bar{C} \\ 0 & \bar{A} \end{bmatrix} + \mu_i \begin{bmatrix} B\bar{D} \\ \bar{B} \end{bmatrix} \tilde{R}_i^{-1} \begin{bmatrix} C & D\bar{C} \end{bmatrix} = \Phi T_i$$

where $T_i = \begin{bmatrix} Y^{-1} - \mu_i C^* \bar{D}\tilde{R}_i^{-1} C & -C^* \tilde{S}_i^{-1} \bar{C} \\ -\mu_i \bar{C}^* \tilde{R}_i^{-1} C & \bar{Y}^{-1} - \mu_i \bar{C}^* \tilde{R}_i^{-1} D\bar{C} \end{bmatrix}$, $\Phi = \begin{bmatrix} AY & 0 \\ 0 & \bar{A}\bar{Y} \end{bmatrix}$, and in a similar manner as [34, Th. 9] ψ_i is Hurwitz if and only if conditions (b) of Theorem 4.1 are satisfied. For $\mu_i > 0\ \forall i \in \{2, \ldots, N\}$ and $\det(A) = 0$, it can be verified that

$$\begin{bmatrix} A_1 & B_1\bar{C}_1 \\ 0 & \bar{A}_1 \end{bmatrix} + \begin{bmatrix} B_1\bar{D}_1 \\ \bar{B}_1 \end{bmatrix} R_1^{-1} \begin{bmatrix} C_1 & D_1\bar{C}_1 \end{bmatrix} = \psi_i$$

where $A_1 = A + \mu_i B\Psi(I - \mu_i D\Psi)^{-1}C$, $B_1 = B(I - \mu_i \Psi D)^{-1}$, $C_1 = \mu_i(I - \mu_i D\Psi)^{-1}C$, $D_1 = \mu_i D(I - \mu_i \Psi D)^{-1}$, $\bar{A}_1 = \bar{A}$, $\bar{B}_1 = \bar{B}$, $\bar{C}_1 = \bar{C}$, $\bar{D}_1 = \bar{D} - \Psi$ and $R_1^{-1} = (I - D_1\bar{D}_1)$. Also, since (A, B, C, D) is a minimal realisation of $P(s)$ it follows that (A_1, B_1, C_1, D_1) is a minimal realisation of $P_1(s) = \mu_i(I - \mu_i P(s)\Psi)^{-1}P(s)$ which is an NI system without poles at the origin via [34, Lemma 20] $\forall i \in \{2, \ldots, N\}$. Moreover, it is obvious that $(\bar{A}_1, \bar{B}_1, \bar{C}_1, \bar{D}_1)$ is a minimal realisation of $K_1(s) = K(s) - \Psi$ which is SNI. Consequently, $\det(A_1) \neq 0$ and [41, Lemma 7], [41, Lemma 8] can be employed to write ψ_i as

$$\Phi_i T_i = \begin{bmatrix} A_1 & B_1\bar{C}_1 \\ 0 & \bar{A}_1 \end{bmatrix} + \begin{bmatrix} B_1\bar{D}_1 \\ \bar{B}_1 \end{bmatrix} R_1^{-1} \begin{bmatrix} C_1 & D_1\bar{C}_1 \end{bmatrix} = \psi_i$$

where $T_i = \begin{bmatrix} {Y_{1i}}^{-1} - C_1^* \bar{D}_1 R_1^{-1} C_1 & -C_1^* S_1^{-1} \bar{C}_1 \\ -\bar{C}_1^* R_1^{-1} C_1 & \bar{Y}_1^{-1} - \bar{C}_1^* R_1^{-1} D_1 \bar{C}_1 \end{bmatrix}$, $\Phi_i = \begin{bmatrix} A_1 Y_{1i} & 0 \\ 0 & \bar{A}_1\bar{Y}_1 \end{bmatrix}$, $R_1 = (I - D_1\bar{D}_1)$ and $S_1 = (I - \bar{D}_1 D_1)$. Hence, ψ_i is Hurwitz if and only if $\forall i \in \{2, \ldots, N\}$, $I - P_1(\infty)K_1(\infty)$ is nonsingular, $\bar{\lambda}[[I - P_1(\infty)K_1(\infty)]^{-1}[P(\infty)_1 K_1(0) - I]] < 0$, and $\bar{\lambda}[[I - K_1(0)P_1(\infty)]^{-1}[K_1(0)P_1(0) - I]] < 0$ if and only if conditions (b) of Theorem 4.3 are satisfied in a similar manner as [34, Th. 9] and via similar arguments as in the proof of [34, Th. 24], respectively. Thus, the eigenvalues of Ψ_{cl} on the imaginary axis are the eigenvalues of A on the imaginary axis and all the remaining eigenvalues of Ψ_{cl} are in the OLHP. Let n_0 be the number of eigenvalues of Ψ_{cl} on the imaginary axis denoted by λ_A and let v_A^r and v_A^l be the right and left eigenvectors of A associated with λ_A (note that they can also represent respectively the generalised right and left eigenvectors of A associated with λ_A in case the algebraic multiplicity is greater then the geometric multiplicity). The steady state expression of system (4.9) is given below.

Theorem 4.4. *The steady state trajectory of system* (4.9) *is given by*

$$\begin{bmatrix} x(t) \\ \bar{x}(t) \end{bmatrix} \xrightarrow{t\to\infty} \begin{bmatrix} w_1 & \dots & w_{n_0} \end{bmatrix} e^{J't} \begin{bmatrix} v_1^* \\ \vdots \\ v_{n_0}^* \end{bmatrix} \begin{bmatrix} x(0) \\ \bar{x}(0) \end{bmatrix} \tag{4.10}$$

where J' is the Jordan block associated with λ_A, and $\forall j \in \{1,\dots,n_0\}$

$$w_j = \begin{bmatrix} 1_N \otimes v_A^r \\ 0_{Nq\times 1} \end{bmatrix}, v_j = \begin{bmatrix} 1_N \otimes \frac{1}{N} v_A^l \\ 1_N \otimes \frac{1}{N}(\lambda_A I_q - \bar{A})^{-*}\bar{C}^* B^* v_A^l \end{bmatrix}$$

are the right and left eigenvectors of Ψ_{cl} associated with λ_A.

Proof. Similar to proof of [38, Th. 2]. $\square$

Since we are concerned with output consensus, internal stability guarantees that $y \to 1_N \otimes y_{ss}$. Thus, the final output convergence is given by

$$\begin{aligned} y(t) &= R^{-1}(I_N \otimes C)x(t) + R^{-1}(I_N \otimes D\bar{C})\bar{x}(t) \\ Ry(t) &= (I_N \otimes C)x(t) + (I_N \otimes D\bar{C})\bar{x}(t) \\ 1_N \otimes y_{ss} &= \begin{bmatrix} (I_N \otimes C) & (I_N \otimes D\bar{C}) \end{bmatrix} \begin{bmatrix} x(t) \\ \bar{x}(t) \end{bmatrix}. \end{aligned}$$

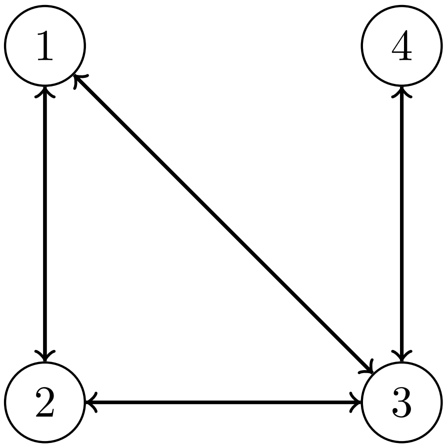
4.5 Illustrative examples

In this section, we give two examples to demonstrate the effectiveness of the robust output consensus results proposed in this chapter. In each example, four NI systems are considered. The network topology that models the interaction among the NI systems and its associated Laplacian matrix are given in Figure 4.5.

4.5.1 Without poles at the origin

The transfer function matrix of the four NI systems and the SNI feedback controller for each NI agent are

$$P(s) = \begin{bmatrix} \frac{2s^2+2.2}{s^2+0.6} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}, \quad K(s) = \begin{bmatrix} \frac{1}{s^2+15s+20} & 0 \\ 0 & \frac{-2s-9}{s+5} \end{bmatrix},$$



$$\mathcal{L} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Figure 4.5: Network Graph and associated Laplacian matrix.

respectively. Although $P(\infty)K(\infty) = 0$, $K(\infty) \ngeq 0$. Thus, the results in [38, Th. 1] cannot be used to determine robust output consensus of the networked NI systems. On the other hand, since $P(0) > 0$ we conclude via Corollary 4.1 (case (ii)) that robust output consensus is achieved for the NI systems since $\bar{\lambda}[P(0)K(0)] = 0.18 < (1/\bar{\lambda}(\mathcal{L})) = 0.24$ as shown in Figure 4.6 with external disturbances acting on the systems.

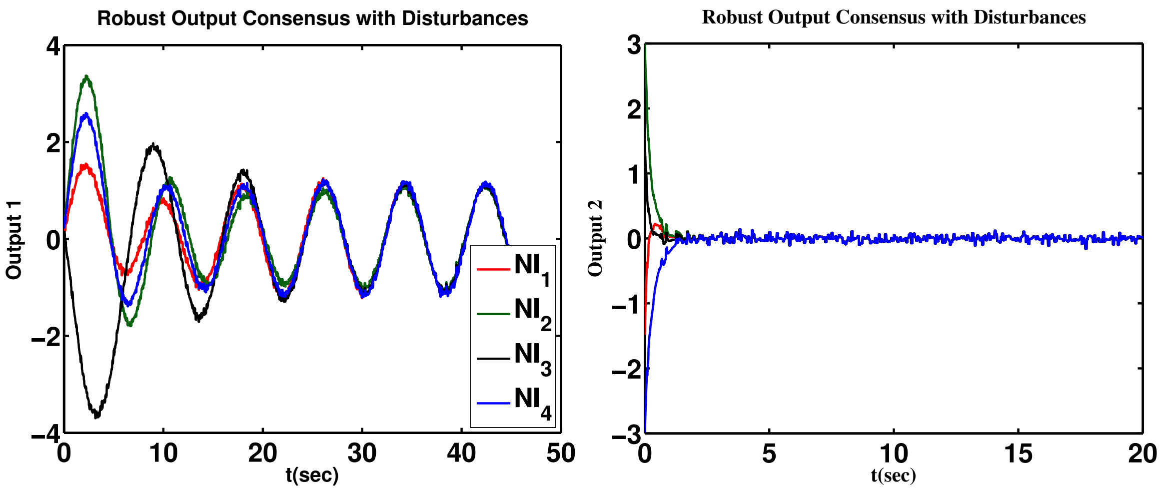


Figure 4.6: Robust output consensus for networked NI systems.

4.5.2 With poles at the origin

Here we consider four homogeneous flexible robotic arms. Figure 4.7 shows the model of the ith robotic arm. The arm is modelled by slewing beam with co-located piezoelectric actuator and sensor and is driven by a motor pinned to one of its ends

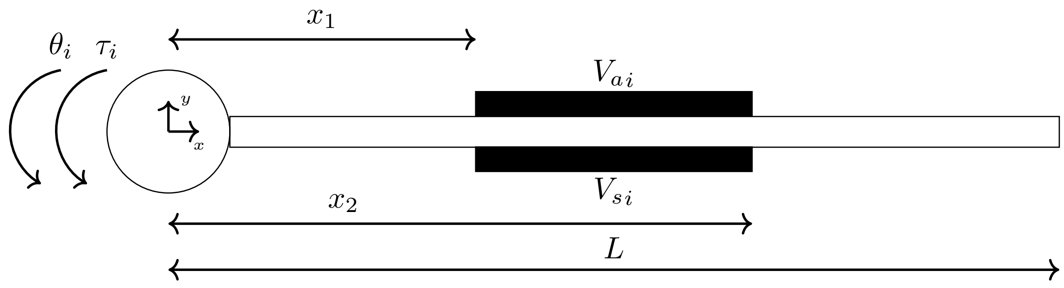


Figure 4.7: Schematic diagram of a slewing beam equivalent to ith robotic arm [40].

[74]. Thus, the ith robotic arm has two inputs (V_{ai}, τ_i) which represent voltage and torque applied to the piezoelectric actuator and motor respectively, and two outputs (V_{si}, θ_i) which represent voltage sensed by the piezoelectric sensor and the motor hub angle respectively $\forall i \in \{1, \dots, 4\}$. Whereas the robotic arm has an infinite dimensional model, for purpose of control design a finite dimensional model can be approximated. A finite dimensional model $P_i(s) = P(s)$ $\forall i \in \{1, \dots, 4\}$ for the flexible robotic arms, taking the first resonant mode into account (see [40] for more details), is obtained as

$$P(s) = \begin{bmatrix} P_{\tau,\theta}(s) & P_{V_a,\theta}(s) \\ P_{\tau,V_s}(s) & P_{V_a,V_s}(s) \end{bmatrix} = \begin{bmatrix} \frac{3.231s^2+1.618}{s^2(s^2+3.4^2)} & \frac{3.5573\times10^{-4}}{s^2+3.4^2} \\ \frac{3.5573\times10^{-4}}{s^2+3.4^2} & \frac{2.35}{s^2+3.4^2} \end{bmatrix}. \quad (4.11)$$

It can be verified by Definition 2.17 that (4.11) is NI with two poles at the origin. Consider the SNI controller in [40]

$$K(s) = \begin{bmatrix} \frac{-4.29s^2-231.5s-5.11.9}{s^2+62.13s+232.4} & \frac{15s-247.5}{s^2+62.13s+232.4} \\ \frac{15s-247.5}{s^2+62.13s+232.4} & \frac{-2.22s^2-117.9s-162}{s^2+62.13s+232.4} \end{bmatrix}.$$

Although $P(s)$ has poles at the origin and is strictly proper, the results in [38] cannot be used to determine robust output consensus for the robotic arms since $\mathbb{N}(P_2) \nsubseteq \mathbb{N}(P_0)$ where $\mathbb{N}$ denotes the null space and P_0, P_2 are the coefficients in the Laurent series expansion of $P(s)$ around the zero. On the other hand, robust output consensus for the robotic arms can be easily concluded via Theorem 4.3 of this chapter. In fact, we need only check condition (4.5) since $P(\infty) = 0$. Now since it is possible to select $\Psi = K(0) < 0$ (see [34]), we check that (4.5) is satisfied $\forall i \in \{2, 3, 4\}$, i.e. $\bar{\lambda}[\lim_{s\to0}[[\mu_i K(s)P(s) - I_n][I_n - \mu_i \Psi P(s)]^{-1}]] = -1 < 0$ $\forall i \in \{2, 3, 4\}$. Observe that with this choice of Ψ, we were able to easily determine robust output consensus without knowledge of the eigenvalues of $\mathcal{L}$. Note that [39, Th. 15], which captures robust output consensus for the heterogeneous case, is much more complicated to use

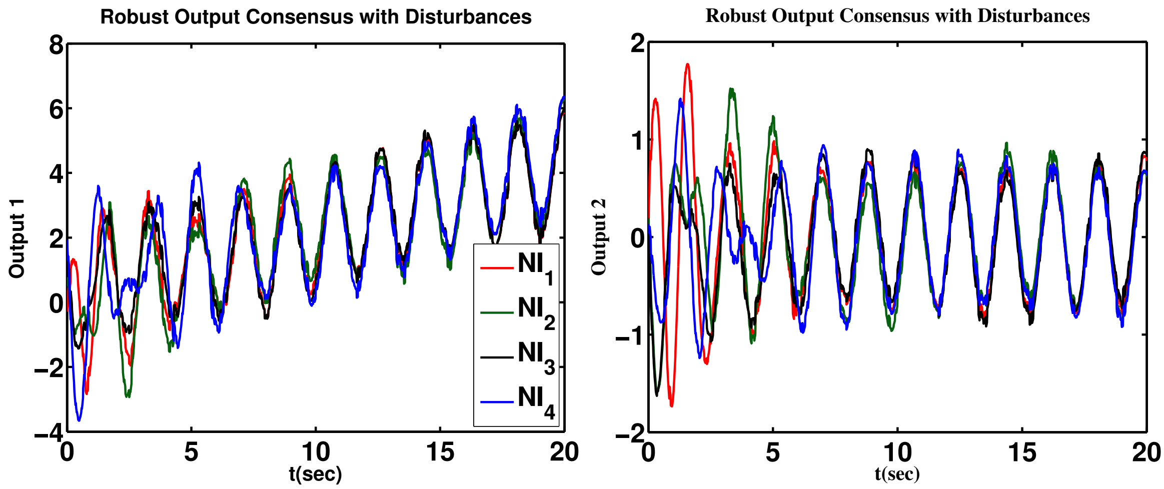


Figure 4.8: Robust output consensus for networked robotic arms.

since the conditions are checked for the augmented networked plant and controller which increase in dimension by increasing the number of connected agents. Moreover, matrix factorisation is needed and additional conditions need to be satisfied, such as non-singularity and sign semidefiniteness, for specific matrices before being able to determine whether output consensus is achieved or not. Simulation results, using the finite dimensional model (4.11), are shown in Figure 4.8. The initial conditions have been arbitrarily chosen as $[1, 0.1, 2, 0.2, 3, 0.3]^T$, $[-1, 0.1, -2, 0.2, -3, 0.3]^T$, $[2.5, 0.5, -2.5, 0.6, -3.2, 0.1]^T$, $[2, 0.3, -5.5, 0.4, -1, 0.2]^T$ for the four NI systems respectively, and $[0, 0]^T$ for all four SNI controllers. It can be seen from Figure 4.8 that robust output consensus is achieved with external disturbances.

4.6 Summary

Necessary and sufficient conditions for robust output consensus were proposed for multiple homogeneous NI systems which are subject to $\mathfrak{L}_2$ external disturbances and model uncertainties by utilising the recently published robust NI stability results. Advantages of the proposed results over existing results in literature were discussed. It was shown that the derived conditions specialise to those in earlier literature by either imposing the same assumptions at infinite frequency or by imposing different ones which had not been known previously. It was also shown that the derived conditions simplify in the SISO case. Furthermore, it was shown that the steady state behaviour of the closed loop networked system is determined by the eigenvalues of the closed loop networked system on the imaginary axis which is in agreement with conclusions of earlier studies. The results were enhanced by several examples such that the effectiveness of the proposed results over earlier results were apparent when the assumptions of earlier results do not hold.

Chapter 5

Cooperative control of integrator negative imaginary systems with application to rendezvous multiple mobile robots

5.1 Introduction

The study of multiple mobile robot cooperation falls under the more general field of study: cooperative control of multi-agent systems. It has been recognized as an important field of study due to the various applications mobile robots have which directly benefit society such as cooperative rescue missions in hazardous environments. Therefore, cooperative control of multi-robot systems has been studied widely by researchers over the past decade such as in [75, 4, 76] to name a few. Rendezvous of multiple mobile robots is a common desirable task that needs to be achieved via the design of appropriate distributed control laws. Specifically, to achieve rendezvous, multiple robots cooperate with each other in order to reach the same position. The rendezvous problem has been studied extensively over the past decade such as in [77, 78, 79, 80] to name a few.

In this chapter, a rendezvous problem for multiple nonholonomic wheeled mobile robots is tackled via the negative imaginary systems theory. Some of the advantages of using the NI systems theory to tackle such a problem are as follows. First, recall that the NI systems theory is deemed effective in applications where position is the state of interest. Second, but not least, recall that the dynamics of nonholonomic WMRs can be simplified to a single integrator model via input-output linearisation making it possible to develop simple yet sophisticated control laws for integrator NI systems in a systematic way which can be directly applied to WMRs. Cooperative control strategies are proposed in this chapter for integrator NI systems which are then implemented on real-robots to achieve rendezvous. The theoretical work in this chapter builds on [38] and [39]. Recall that robust output consensus was addressed in [38] and [39] for networks of homogeneous and heterogeneous NI systems respectively by reformulating the consensus problem into an internal stability problem. Also, recall that robust output tracking for multiple NI systems was addressed in [39]. However, one drawback associated with the work in [38, 39] is that it only deals with undirected graphs to model the interaction among the systems and hence is not applicable to directed graphs. Nevertheless, we prove in this chapter that for MIMO integrator NI systems and for directed graphs that are balanced and strongly connected the consensus and tracking problems can be guaranteed via the NI internal stability theorems. Consequently, the results are utilized to achieve rendezvous for multiple nonholonomic WMRs.

The remainder of this chapter is organized as follows. First, we prove that multiple integrator NI systems with directional information flow that is balanced and strongly connected retain the NI property. Subsequently, we derive conditions, using NI systems theory, such that output consensus and cooperative tracking are guaranteed for a network of integrators with strongly connected, balanced and directed information flow subject to energy-bounded disturbances. Finally, experimental results from both real-robot and simulation are provided to validate the effectiveness of the proposed theoretical results in solving a rendezvous problem for multiple WMRs.

5.2 Problem Formulation

Consider a group of N homogeneous MIMO integrators with external disturbances acting on each system. The dynamics of the ith system are described as

$$y_i = w_{o_i} + P(s)(u_i + w_{in_i}) \qquad \forall i \in \{1, \dots, N\}, \tag{5.1}$$

where $P(s) = \frac{k}{s} I_m$ is the transfer matrix with $k > 0$, u_i and y_i are the control input and output of the ith system respectively. Also, w_{in_i} and w_{o_i} are input and output disturbances which are energy-bounded in an H_2 (or in the time domain $\mathfrak{L}_2[0, \infty)$) sense. The information flow among the integrators is modelled by a directed graph $\mathcal{G}$ which is assumed balanced and strongly connected.

Following [38], the distributed control law is given by

$$\begin{aligned} u_i &= K(s) z_i, \\ z_i &= \sum_{j=1}^{N} a_{ij}(y_i - y_j), \end{aligned} \tag{5.2}$$

where $K(s)$ is a SNI feedback controller to each system, z_i denotes the signal of relative output measurements and a_{ij} are the elements of the adjacency matrix associated with the network graph $\mathcal{G}$. The collective network dynamics can be written as

$$y = w_o + (I_N \otimes P(s))(u + w_{in}), \tag{5.3}$$

and

$$\begin{aligned} u &= (I_N \otimes K(s)) z, \\ z &= (\mathcal{L} \otimes I_m) y, \end{aligned} \tag{5.4}$$

where $\mathcal{L}$ is the Laplacian matrix associated with the network graph $\mathcal{G}$. The closed loop networked system is shown in Figure 5.1. By moving the block $(\mathcal{L} \otimes I_m)$ past

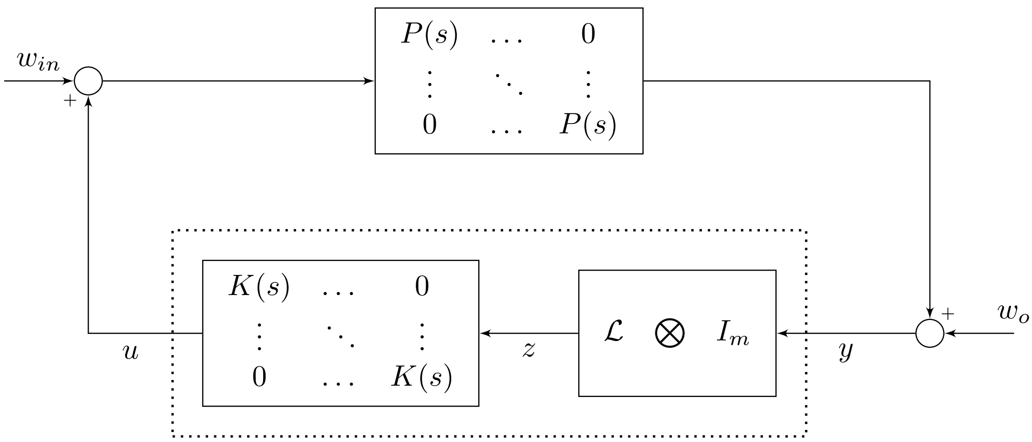


Figure 5.1: Real physical networked system; general framework adopted from [38] but $\mathcal{L}$ here is for directed, strongly connected and balanced graphs.

the summing junction and letting $\bar{w}_o = (\mathcal{L} \otimes I_m) w_o$, we obtain the block diagram of Figure 5.2. Note that $\bar{w}_o$ is a subset of w_o since $\mathcal{L}$ is rank deficient. Consider

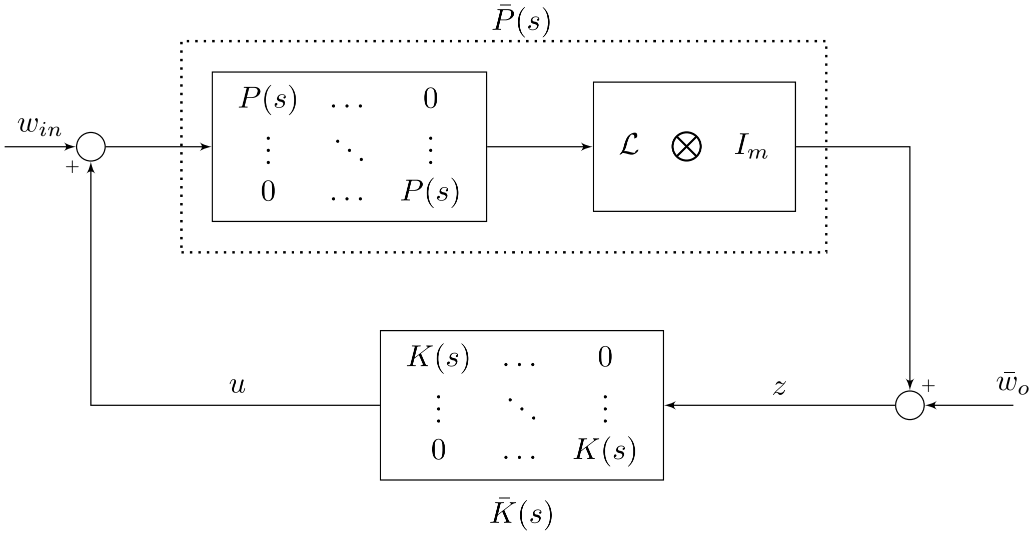


Figure 5.2: Internal stability framework; general framework adopted from [38] but $\mathcal{L}$ here is for directed, strongly connected and balanced graphs.

the augmented plant $\bar{P}(s)$ to be the integration of the agents dynamics with the network topology; i.e. the transfer function from u to z. Then $\bar{P}(s)$ can be written as $\bar{P}(s) = (\mathcal{L} \otimes I_m)(I_N \otimes P(s)) = (\mathcal{L} \otimes P(s))$. Also, let $\bar{K}(s) = (I_N \otimes K(s))$ which is SNI since $K(s)$ is SNI.

Our objectives is first to show that $\bar{P}(s)$ is NI when $P(s)$ is a MIMO integrator and $\mathcal{L}$ corresponds to a directed, strongly connected and balanced graph. Then, we derive conditions using NI systems theory for which output consensus and cooperative tracking are achieved. This can be done since internal stability of the interconnection $[\bar{P}(s), \bar{K}(s)]$ in Figure 5.2 is equivalent to consensus on the output y in Figure 5.1 by the properties and the rank deficiency of the Laplacian matrix $\mathcal{L}$.

5.3 Main results

5.3.1 Output consensus

The following lemma states that the augmented plant $\bar{P}(s)$ is NI for a network of homogeneous MIMO integrators and *directed* information flow that is balanced and strongly connected.

Lemma 5.1. *Consider a network of MIMO integrators with directed information flow $\mathcal{G}$ that is balanced and strongly connected. Then, $\bar{P}(s)$ is negative imaginary.*

Proof. Since the network consists of MIMO integrators with $P(s) = \frac{k}{s} I_m$, it follows that $P(j\omega)^* = -P(j\omega)$. We need to show that $\bar{P}(s)$ is NI according to Definition 2.17. In fact we only need to show that $\bar{P}(s)$ satisfy conditions 1, 2 and 3 of Definition 2.17 since the agents are integrators. First, since $P(s)$ has a pole at the origin, then $\bar{P}(s)$ will have its poles at the origin as well. Consequently, condition 1 is satisfied. To show condition 2 is satisfied we have

$$\begin{aligned} j\left(\bar{P}(j\omega) - \bar{P}(j\omega)^*\right) &= j\left((\mathcal{L} \otimes P(j\omega)) - (\mathcal{L}^* \otimes P(j\omega)^*)\right) \\ &= j\left((\mathcal{L} \otimes P(j\omega)) - (\mathcal{L}^* \otimes -P(j\omega))\right) \\ &= j\left((\mathcal{L} \otimes P(j\omega)) + (\mathcal{L}^* \otimes +P(j\omega))\right) \\ &= j\left((\mathcal{L} + \mathcal{L}^*) \otimes P(j\omega)\right) \\ &= j\left((\mathcal{L} + \mathcal{L}^*) \otimes \frac{k}{j\omega} I_m\right) \\ &= \frac{k}{\omega}\left((\mathcal{L} + \mathcal{L}^*) \otimes I_m\right) \\ &= \frac{k}{\omega}\left((\mathcal{L} + \mathcal{L}^T) \otimes I_m\right) \geq 0 \quad \forall \omega > 0, \end{aligned}$$

is positive semidefinite by Lemma 2.8, by properties of Kronecker product and by noting that $\mathcal{L}$ is real.
Finally we have that $\lim_{s\to 0} s^r \bar{P}(s) = \lim_{s\to 0} s^r (\mathcal{L} \otimes \frac{k}{s} I_m) = 0_{Nm\times Nm}$ for all $r \geq 3$ and $\lim_{s\to 0} s^2 \bar{P}(s) = \lim_{s\to 0} s^2 (\mathcal{L} \otimes \frac{k}{s} I_m) = 0_{Nm\times Nm}$.
Accordingly, $\bar{P}(s)$ is NI. □

The following theorem gives a condition under which output consensus is achieved for a network of integrators with information flow that is balanced and strongly connected.

Theorem 5.1. *Consider a network of MIMO integrators with directed information flow $\mathcal{G}$ that is balanced and strongly connected. Let $K(s)$ be a strictly negative imaginary feedback controller for each integrator. Then, output feedback consensus is achieved via control protocol* (5.4) *for networked system* (5.3) *as shown in Figure 5.1 (or in a distributed manner* (5.2) *for each system* (5.1)*) under any external disturbances $w_{in}, w_o \in \mathfrak{L}_2[0,\infty)$ if and only if $K(0) < 0$.*

Proof. Fist we need to prove that $[\bar{P}(s), \bar{K}(s)]$ as depicted in Figure 5.2 is internally stable using Lemma 2.4. Then, internal stability of the interconnection $[\bar{P}(s), \bar{K}(s)]$

in Figure 5.2 is equivalent to consensus on output y in Figure 5.1 by properties of Laplacian matrix $\mathcal{L}$ and due to $(\mathcal{L} \otimes I_m)$ being rank deficient. To this end, we have already shown in Lemma 5.1 that $\bar{P}(s)$ is NI. Also, $\bar{K}(s) = (I_N \otimes K(s))$ is SNI since $K(s)$ is SNI. For $P(s) = \frac{k}{s} I_m$ we have $P_2 = 0_{m\times m}, P_1 = kI_m$, and $P_0 = 0_{m\times m}$. This gives $\bar{P}_2 = (\mathcal{L} \otimes P_2) = 0_{Nm\times Nm}$, and $\bar{P}_1 = (\mathcal{L} \otimes P_1) = (\mathcal{L} \otimes kI_m) \neq 0_{Nm\times Nm}$. Also, $\mathbb{N}(\bar{P}_1^T) = \mathbb{N}(\mathcal{L}^T \otimes P_1^T) = \{1_N \otimes b : b \in \mathbb{R}^{m\times 1}\} \cup \{0_{Nm\times 1}\}$ and $\mathbb{N}(\bar{P}_0^T) = \mathbb{N}(\mathcal{L}^T \otimes P_0^T) = \{1_N \otimes b : b \in \mathbb{R}^{m\times 1}\} \cup \{c \otimes d : c \in \mathbb{R}^{N\times 1}, d \in \mathbb{R}^{m\times 1}\}$ due to Lemma 2.7 and Lemma 2 in [38]. Consequently, $\mathbb{N}(\bar{P}_1^T) \subseteq \mathbb{N}(\bar{P}_0^T)$. As $\mathcal{G}$ is strongly connected we have $\mathrm{rank}(\mathcal{L}) = N - 1$. Thus, the singular value decomposition of $\mathcal{L}$ will have the form $\mathcal{L} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma V_1^T$, with $U_1 \in \mathbb{R}^{N\times N-1}$, $0 < \Sigma \in \mathbb{R}^{N-1\times N-1}$, and $V_1^T \in \mathbb{R}^{N-1\times N}$. Subsequently, $\bar{P}_1$ can be written as $\bar{P}_1 = (\mathcal{L} \otimes kI_m) = (U_1\Sigma \otimes k)(V_1 \otimes I_m)^T = \bar{F}_1 \bar{V}_1^T$.

Hence, $\bar{F}_1^T \bar{K}(0) \bar{F}_1 = (U_1\Sigma \otimes k)^T (I_N \otimes K(0))(U_1\Sigma \otimes k) = (\Sigma^T U_1^T U_1 \Sigma \otimes kK(0)k) = (\Sigma^T\Sigma \otimes k^2 K(0)) < 0$ if and only if $K(0) < 0$ since $\Sigma^T\Sigma > 0$ with full rank of $N - 1$ and $k > 0$. We conclude that $[\bar{P}(s), \bar{K}(s)]$ is internally stable if and only if $K(0) < 0$. The proof that internal stability of $[\bar{P}(s), \bar{K}(s)]$ implies output consensus with and without external disturbances is similar to that of Theorem 1 in [38] with the only difference being that here $\mathcal{L}1_N = 0_N$ holds because $\mathcal{G}$ is strongly connected. $\square$

5.3.2 Cooperative tracking to a pre-defined fixed reference

In this section we show that the NI stability theory can be used to solve a cooperative tracking problem for multiple integrator NI systems. The distributed control law now becomes

$$\begin{aligned} u_i &= K(s) z_i, \\ z_i &= \sum_{j=1}^{N} a_{ij}(y_i - y_j) + d_i(y_i - r), \end{aligned} \tag{5.5}$$

for all $i \in \{1, \ldots, N\}$ where r is the fixed reference and $d_i = 1$ if agent i is connected to the reference and $d_i = 0$ otherwise. The collective dynamics of the control law can be written as

$$\begin{aligned} u &= (I_N \otimes K(s)) z, \\ z &= ((\mathcal{L} + D) \otimes I_m)(y - 1_N r). \end{aligned} \tag{5.6}$$

The plant from u to z is $\bar{P}(s) = ((\mathcal{L} + D) \otimes I_m)(I_N \otimes P(s)) = ((\mathcal{L} + \mathcal{D}) \otimes P(s))$ which is NI following similar steps as in Lemma 5.1 and noting that $(\mathcal{L} + \mathcal{D}) + (\mathcal{L} + \mathcal{D})^T$ is positive definite since $\mathcal{D}$ is nonzero diagonal matrix, and by Lemmas 2.6 and 2.8.

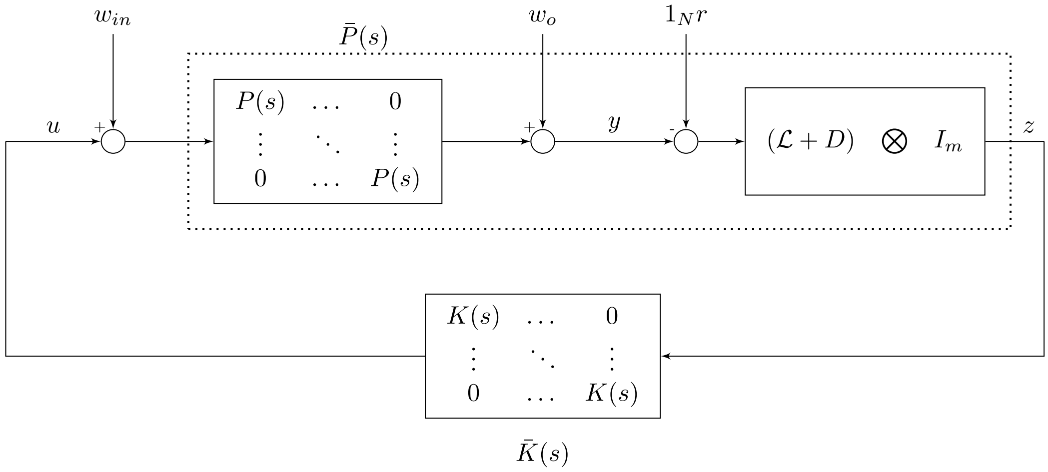


Figure 5.3: Cooperative output tracking block diagram.

The following theorem gives a condition under which cooperative output tracking to a fixed reference is achieved for a network of MIMO integrators with directed information flow that is balanced and strongly connected.

Theorem 5.2. *Consider a network of MIMO integrators with directed information flow $\mathcal{G}$ that is balanced and strongly connected. Let $K(s)$ be a strictly negative imaginary feedback controller for each integrator. Then, cooperative output tracking to a fixed reference r is achieved via control protocol* (5.6) *for networked system* (5.3) *as shown in Figure 5.3 (or in a distributed manner* (5.5) *for each system* (5.1)*) under any external disturbances $w_{in}, w_o \in \mathfrak{L}_2[0, \infty)$ if and only if $K(0) < 0$.*

Proof. Following similar steps as in the proof of Theorem 5.1, $[\bar{P}(s), \bar{K}(s)]$ is internally stable if and only if $K(0) < 0$. Subsequently, internal stability of $[\bar{P}(s), \bar{K}(s)]$ yields $z \to 0$ in (5.6). That is, $z_i \to 0\ \forall i \in \{1, \ldots, N\}$ in (5.5). Hence, $y_i \to y_j\ \forall j \in \mathcal{N}_i, j \neq i$ with $d_i = 0$ and $y_i \to r$ when $d_i = 1$. Consequently, cooperative output tracking to a fixed reference r is achieved, i.e., $y_i = y_j = r\ \forall i \in \{1, \ldots, N\}$ and $\forall j \in \mathcal{N}_i, j \neq i$. □

Remark 5.1. *A simple DC gain condition, that can easily be satisfied, guarantees output consensus and tracking via the distributed control law proposed in this chapter. This can be considered an advantage over other existing protocols in literature, for* *example [77] where state space techniques and finding suitable gain matrices to satisfy certain conditions are required in the design.*

5.4 Application

In this section we apply the theoretical results of Section 5.3.2 to achieve rendezvous of nonholonomic WMR.

5.4.1 Robot model

Consider a group of $N = 3$ homogeneous WMR. The kinematic model of the ith robot is given by

$$\begin{aligned} \dot{x}_i &= v_i \cos\phi_i, \\ \dot{y}_i &= v_i \sin\phi_i, \qquad \forall i \in \{1,2,3\} \\ \dot{\phi}_i &= w_i. \end{aligned} \tag{5.7}$$

where (x_i, y_i) is the position of the ith robot and ϕ_i is its orientation. Also, v_i and ω_i are the linear and angular velocities of the ith robot respectively. In this chapter, we are interested in the rendezvous problem defined as follows:

Problem statement 5.1. *Find a distributed control law v_i and w_i for each WMR such that all WMR reach a pre-defined fixed rendezvous position (x_r, y_r) which is available only to some of the WMR.*

In order to apply NI cooperative tracking results of Section 5.3.2 to the WMR, we apply input output linearisation such that the system between the new input and output is linear (see e.g. [81, 82] for more information). Define two new outputs to be controlled as

$$\begin{aligned} \tilde{x}_i &= x_i + l\cos\phi_i \\ \tilde{y}_i &= y_i + l\sin\phi_i \end{aligned}, \qquad \forall i \in \{1,2,3\} \tag{5.8}$$

where $l \neq 0$ is the distance from $(\tilde{x}_i, \tilde{y}_i)$ to (x_i, y_i). Using the kinematic model (5.7), the new dynamics of the WMR is given by

$$\begin{bmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{y}}_i \end{bmatrix} = F(\phi_i) \begin{bmatrix} v_i \\ \omega_i \end{bmatrix} \tag{5.9}$$

where $F(\phi_i) = \begin{bmatrix} \cos\phi_i & -l\sin\phi_i \\ \sin\phi_i & l\cos\phi_i \end{bmatrix}$. Define new inputs to be u_{1i}, u_{2i}. Using these two new control variables we get

$$\begin{bmatrix} v_i \\ \omega_i \end{bmatrix} = F(\phi_i)^{-1} \begin{bmatrix} u_{1i} \\ u_{2i} \end{bmatrix} \tag{5.10}$$

and the robot's linearised model becomes

$$\begin{aligned} \dot{\tilde{x}}_i &= u_{1i} \\ \dot{\tilde{y}}_i &= u_{2i}. \end{aligned}, \qquad \forall i \in \{1, 2, 3\} \tag{5.11}$$

The dynamic of system (5.11) is linear and decoupled and corresponds to an integrator NI system; i.e. $P(s) = \frac{1}{s} I_2$ which implies that the algorithms of the previous section can be directly applied.

5.4.2 Simulation results

The initial positions (in metres) and orientation (in degrees) are chosen as $(0.7, 0.4, 0)$, $(0.3, 0.4, 0)$ and $(0.5, 0.1, 0)$ for the three robots. Also, we assume in this simulation that $l = 0.03$ m. The final rendezvous position (in metres) is $(0.5, 0.3)$. Note that with this approach the orientation is left uncontrolled. The network graph that models the communication links among the three robots is shown in Figure 5.4. Also, the

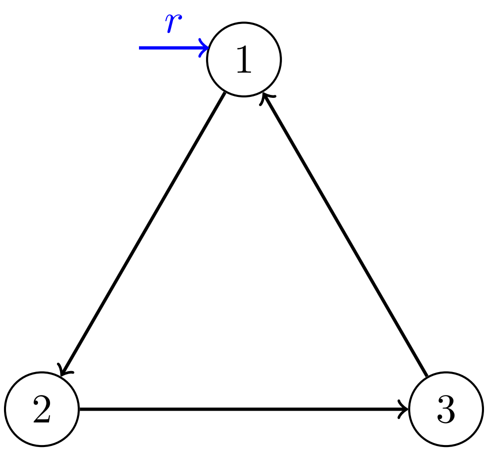


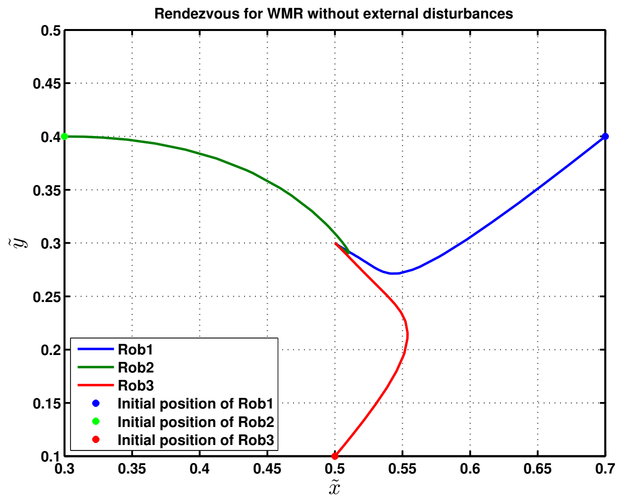
Figure 5.4: Network graph.

reference is available only to system 1. Hence, the Laplacian matrix associated with

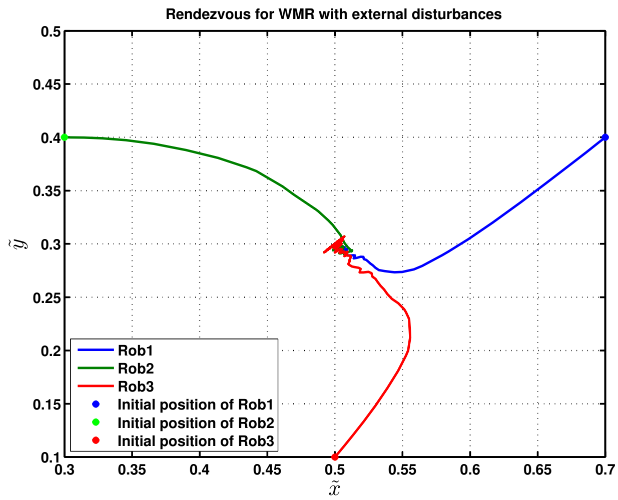
$\mathcal{G}$ and the diagonal matrix $\mathcal{D}$ are

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \qquad \mathcal{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

An SNI controller for each robot is chosen as $K(s) = \frac{-3s-2}{s+1} I_2$ with zero initial condition which satisfies $K(0) = -2I_2 < 0$. Consequently, rendezvous is achieved via the cooperative output position tracking results of Theorem 5.2 as shown in Figure 5.5 without and with external input disturbances. Also, Figure 5.6 shows a time plot of Figure 5.5(b) which confirms convergence to the final rendezvous position.

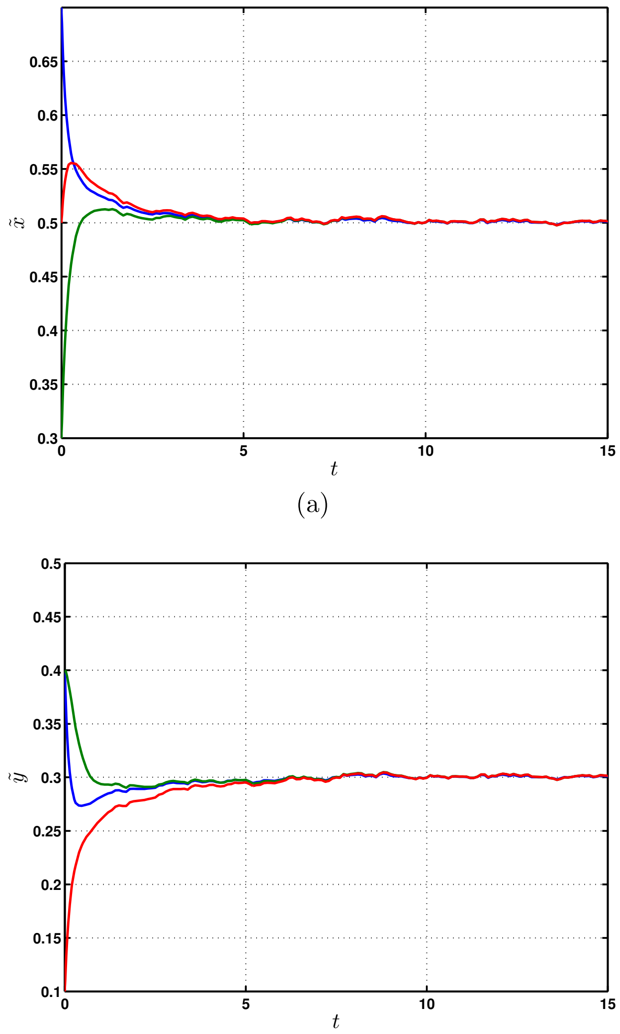


(a)



(b)

Figure 5.5: Rendezvous for WMR, (a) without disturbances and (b) with disturbances.



(a)

(b)

Figure 5.6: Positions of the three robots with disturbances

5.4.3 Experimental results

To validate the proposed control mechanism using real-robot experiments, miniature mobile robots, *Mona* [83] which is a low-cost and open-source mobile robot platform, were deployed. Mona has been developed for use in the research of swarm robotics [84]. Experiments were conducted in a rectangular arena with dimensions of $1.4\,\mathrm{m} \times 0.9\,\mathrm{m}$ which has been developed for study on long-term swarm robotics scenarios [85]. An open-source multi-robot tracking system [86] which tracks both the position and orientation of the robot using an overhead camera was used. Via a unique circular tag attached on the surface of each robot, the position and orientation can be tracked. The position information is sent to the controller via a ROS communication framework.

Three Mona robots where used in the experiment. The initial positions of the three robots were $(0.6942, 0.376)$ m , $(0.3296, 0.4415)$ m, and $(0.4409, 0.1194)$ m. The final rendezvous position was $(0.5, 0.3)$ m. Also, $l = 0.03$ m for the robots. The same network graph of Figure 5.4 was used. The controller to each Mona robot was the same as the one designed in Section 5.4.2. Snapshots of the positions of the three Mona robots during the experiment at different time durations is shown in Figure 5.7. As can be seen from the figure, the Mona robots rendezvous at the desired reference position. The position trajectory of the three Mona robots is shown in Figure 5.8. Note that in order to avoid collision during the experiment, each robot stops when it is near the rendezvous point (0.05 m range). Finally, the control inputs are shown in Figure 5.9. It is apparent that the control inputs switch between their extreme values over the entire control time interval. That is, the control inputs saturate because the control magnitude is limited due to physical consideration. Such control inputs are known as bang-bang control inputs. An advantage of bang-bang control inputs is that maximum effort is applied so that the final trajectory is reached in minimum time. More details on bang-bang control can be found in [87].

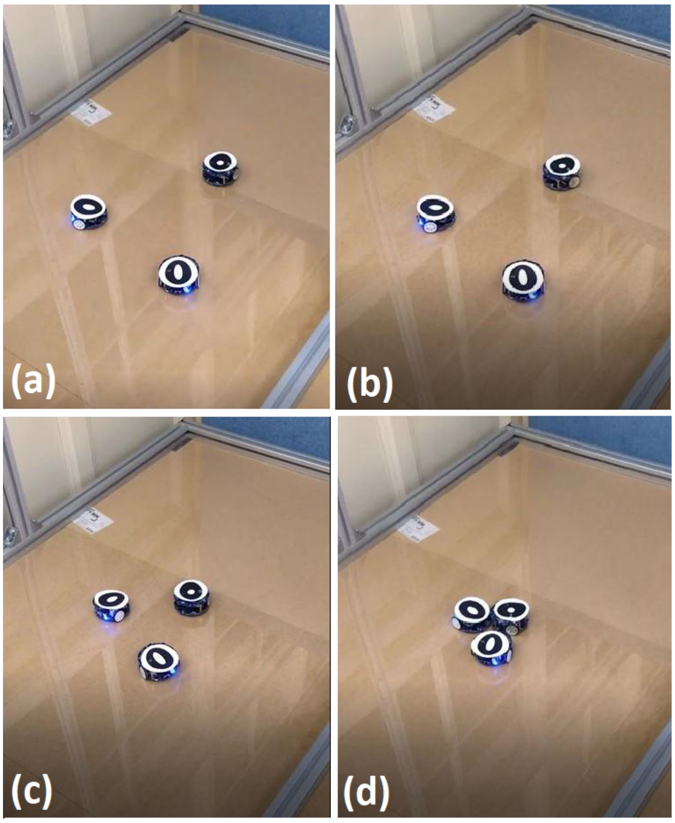


Figure 5.7: Position snapshots of the 3 Mona robots at different time durations starting from initial position and ending in rendezvous. (a) $t = 0$ s. (b) $t = 2$ s. (c) $t = 5$ s. (d) $t = 11$ s.

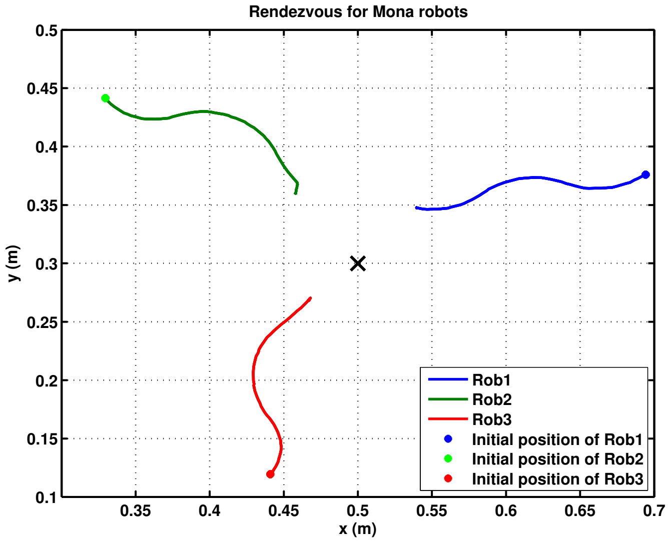
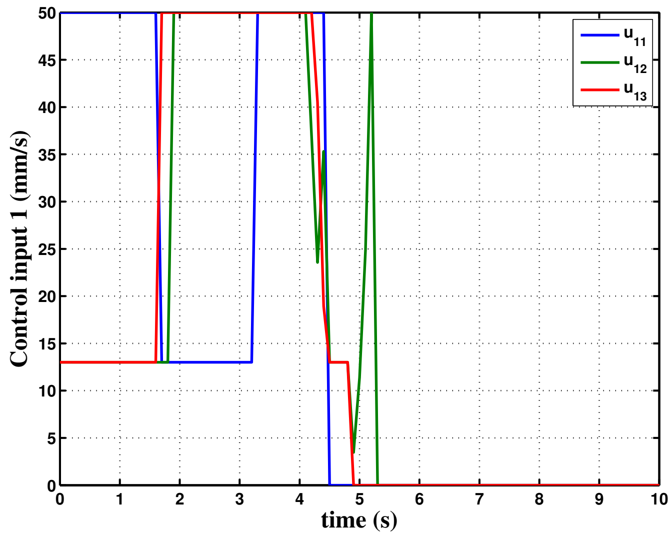
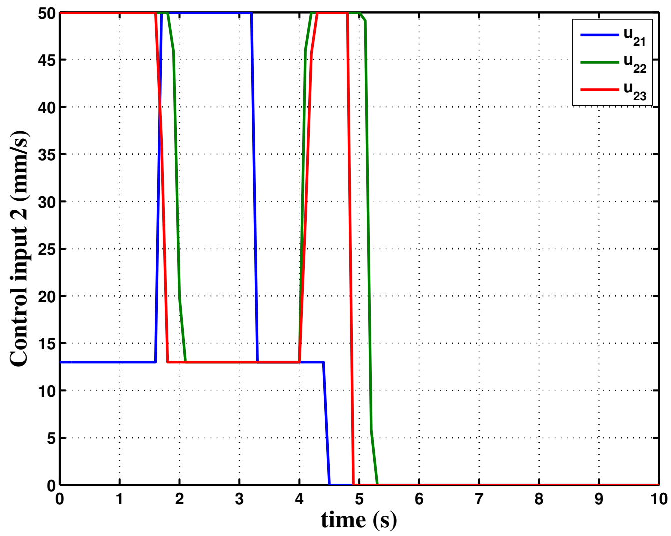


Figure 5.8: Position trajectory and rendezvous of three Mona robots to final position $(0.5, 0.3)$ m marked in cross.



(a)



(b)

Figure 5.9: The control input which represents the desired (reference) velocity to each motor. (a) Control input 1. (b) Control input 2.

5.5 Summary

In this chapter, the cooperative control problem for a network of integrators with directed information flow that is balanced and strongly connected was addressed via negative imaginary systems theory. Moreover, a rendezvous problem for multiple wheeled mobile robots was tackled via the proposed cooperative tracking results. First, a directed, strongly connected and balanced network of integrators was shown to retains the NI property. Consequently, a condition was derived that guarantees output consensus for the networked system under external disturbances by use of internal stability results from NI systems theory. It was also shown that cooperative tracking to a pre-defined fixed reference can also be achieved for the networked system in a similar manner. Finally, experimental results, using Mona robots, as well as simulation results showed that rendezvous is achieved via the distributed control protocol proposed with communication among the robots being modelled by directed graphs that are balanced and strongly connected.

Chapter 6

Conclusions

This thesis makes several noteworthy contributions to the study of networked multi-agent systems with negative imaginary properties. These contributions are summarised in this chapter alongside with suggestions on possible directions for future research. Prior to exploring and developing the work of this thesis, existing studies on this topic were very limited which were reported in [38, 39]. The main aim of this thesis is to fill in the gap that existed in the literature regarding negative imaginary systems in a multi-agent framework. The thesis addresses three problems, namely (i) the distributed robust stabilization problem of networked multi-agent systems with strict negative imaginary uncertainties, (ii) the robust output consensus problem for homogeneous multi-agent systems with negative imaginary dynamics, and (iii) a rendezvous problem for nonholonomic wheeled mobile robots, and proposes a solution to each of them.

6.1 Contributions

The main contributions of this thesis are summarised below.

1. Propose a solution to the distributed robust stabilization problem of networked multi-agent systems with strict negative imaginary uncertainties (results of Chapter 3). Particularly,

    - show that under the assumption that the network graph is connected and undirected with at least one self-loop, a state, input and output transformation preserves the negative imaginary property. Thus, show that a necessary and sufficient condition for the transfer function matrix of the nominal closed-loop networked system to be NI and satisfy a DC gain condition is that multiple reduced-order equivalent systems be NI and satisfy a DC gain condition simultaneously;

- derive sufficient conditions, in an LMI framework, for the existence of control protocol parameters such that the control protocol robustly stabilizes a networked multi-agent system in presence of SNI uncertainties of certain DC size;
- provide an algorithm to design the control protocol parameters; which are a positive scalar that handles the effect of the network topology and a state feedback gain matrix;
- ensure robust stability when variations in the network topology occur by simply appropriately adjusting the positive scalar while leaving the state feedback gain matrix unchanged;
- provide a numerical example to show the usefulness of the proposed results.

2. provide a solution to the robust output consensus problem for multiple homogeneous negative imaginary systems by means of recently published robust NI stability results (results of Chapter 4). Particularly,

    - we relax the assumptions imposed in [38] thereby derive robust output consensus conditions under $\mathfrak{L}_2$ external disturbances and model uncertainty which are not restricted;
    - one distinct advantage that unfolds in our work is that not only do the derived conditions specialise to those in [38] by imposing the same two assumptions at infinite frequency but also specialise to those in [38] by imposing different assumptions which were unknown in [38];
    - the derived conditions simplify in the SISO case providing several insights which are not easily captured in the MIMO case (for SISO NI systems with no poles at origin) and are less sensitive to the network graph that

models the interconnection of the systems (for SISO NI systems with poles at origin);

- provide an analysis for the final steady state trajectory of the networked systems and proof that it is in agreement with conclusions of earlier studies; and
- provide examples that demonstrate the capability of the proposed robust output consensus results over earlier results when earlier assumptions fail to hold.

3. Propose a solution to a rendezvous problem for nonholonomic wheeled mobile robots via the negative imaginary systems theory (results of Chapter 5). Particularly,

   - show that the NI property is preserved for multiple MIMO integrator systems with directional information flow that is balanced and strongly connected;
   - derive necessary and sufficient conditions that guarantee output consensus and output tracking for strongly connected, balanced and directed networks of integrators subject to energy-bounded disturbances using the NI internal stability theorems;
   - utilize the aforementioned results to achieve rendezvous of multiple WMR. Specifically, provide experimental results from both real-robot and simulation to validate the effectiveness of the proposed theoretical results in solving a rendezvous problem for multiple WMR.

6.2 Directions for future research

Possible directions for future research are briefly discussed below.

- One of the main aspects of this thesis is addressing the robust output consensus problems in networks of continuous-time negative imaginary systems. However, the thesis did not consider addressing the robust output consensus problem in networks of discrete-time negative imaginary systems. Recently, the theory on discrete-time negative imaginary systems has been developed [51, 88, 49]. As indicated in [51], the theory is promising because modern applications depend significantly on digital control. Thus, it would be of interest to explore the robust output consensus problem in networks of discrete-time negative imaginary systems.

- It is unavoidable that components of a control system wear out over time. Thus, it is of importance to detect any faulty component and also isolate it before an entire failure of a system occurs. For this reason, fault detection and isolation (FDI) has been an important field of study in the control systems community. A survey on this topic can be found in [89]. Particularly, there has been a growing trend towards the study of FDI of multi-agent systems. A distributed simultaneous fault detection and consensus control problem was addressed in [90]. With the methodology proposed in [90], each agent can detect both its faults and its neighbours faults and isolate them. Therefore, a possible area of future research would be to study the problem of distributed FDI in networks of homogeneous NI systems and investigate whether a methodology can be developed such that each NI system can detect and isolate its own and its neighbouring agents faults.

- Controller synthesis for a class of NI systems was addressed in [45] via a data-driven approach. The proposed controller methodology in [45] guarantees that the controller satisfies the NI property at every frequency point thus the stability of the closed loop system is ensured via the NI internal stability results in [30, 31]. This study provides an insight for future research directions as it would be interesting to investigate whether distributed control protocols can be synthesised for networked NI systems using a data-driven approach.

Bibliography

[1] G. Merabet, M. Essaaidi, H. Talei, M. Abid, N. Khalil, M. Madkour, and D. Benhaddou, “Applications of multi-agent systems in smart grids: A survey,” in *International Conference on Multimedia Computing and Systems -Proceedings*, pp. 1088–1094, 2014.

[2] F. Derakhshan and S. Yousefi, “A review on the applications of multiagent systems in wireless sensor networks,” *International Journal of Distributed Sensor Networks*, vol. 15, no. 5, 2019.

[3] J. Hu and A. Lanzon, “An innovative tri-rotor drone and associated distributed aerial drone swarm control,” *Robotics and Autonomous Systems*, vol. 103, pp. 162–174, May 2018.

[4] W. Dong, “Tracking control of multiple-wheeled mobile robots with limited information of a desired trajectory,” *IEEE Transactions on Robotics*, vol. 28, no. 1, pp. 262–268, 2012.

[5] Y. Cao, W. Yu, W. Ren, and G. Chen, “An overview of recent progress in the study of distributed multi-agent coordination,” *IEEE Transactions on Industrial Informatics*, vol. 9, pp. 427–438, Feb. 2013.

[6] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[7] R. Olfati-Saber and R. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Transactions on Automatic Control*, vol. 49, pp. 1520–1533, Sept. 2004.

[8] G. F. Young, L. Scardovi, and N. E. Leonard, "Robustness of noisy consensus dynamics with directed communication," in *Proceedings of the 2010 American Control Conference*, pp. 6312–6317, June 2010.

[9] T. Meng, Y. Xie, and Z. Lin, "Consensus of second-order multi-agent systems under unknown but bounded measurement noises," *Systems and Control Letters*, vol. 133, 2019.

[10] W. Ren, "On consensus algorithms for double-integrator dynamics," *IEEE Transactions on Automatic Control*, vol. 53, pp. 1503–1509, July 2008.

[11] W. Ren and E. Atkins, "Distributed multi-vehicle coordinated control via local information exchange," *International Journal of Robust and Nonlinear Control*, vol. 17, pp. 1002–1033, 2007.

[12] W. Yu, G. Chen, and M. Cao, "Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems," *Automatica*, vol. 46, pp. 1089–1095, June 2010.

[13] Y. Huo, Y. Lv, X. Wu, and Z. Duan, "Fully distributed consensus for general linear multi-agent systems with unknown external disturbances," *IET Control Theory and Applications*, vol. 13, no. 16, pp. 2595–2609, 2019.

[14] X. Li, Y. C. Soh, and L. Xie, "Output-feedback protocols without controller interaction for consensus of homogeneous multi-agent systems: A unified robust control view," *Automatica*, vol. 81, pp. 37–45, July 2017.

[15] Z. Li, X. Liu, P. Lin, and W. Ren, "Consensus of linear multi-agent systems with reduced-order observer-based protocols," *Systems and Control Letters*, vol. 60, no. 7, pp. 510–516, 2011.

[16] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 57, no. 1, pp. 213–224, 2010.

[17] H. Jongsma, H. L. Trentelman, and M. Camlibel, "Robust synchronization of coprime factor perturbed networks," *Systems and Control Letters*, vol. 95, pp. 62–69, Sept. 2016.

[18] H. L. Trentelman, K. Takaba, and N. Monshizadeh, "Robust synchronization of uncertain linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 58, pp. 1511–1523, June 2013.

[19] S. Hara and H. Tanaka, "D-stability and robust stability conditions for LTI systems with generalized frequency variables," in *49th IEEE Conference on Decision and Control (CDC)*, pp. 5738–5743, Dec. 2010.

[20] L. Alvergue, G. Gu, and L. Qiu, "Output consensus control for multi-agent systems in the presence of gap metric uncertainties," in *54th IEEE Conference on Decision and Control (CDC)*, (Osaka, Japan), pp. 5581–5586, Dec. 2015.

[21] R. Aldana-López, D. Gómez-Gutiérrez, M. Defoort, J. Sánchez-Torres, and A. Muñoz-Vázquez, "A class of robust consensus algorithms with predefined-time convergence under switching topologies," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 17, pp. 6179–6198, 2019.

[22] S. Knorn, Z. Chen, and R. H. Middleton, "Overview: Collective Control of Multiagent Systems," *IEEE Transactions on Control of Network Systems*, vol. 3, pp. 334–347, Dec. 2016.

[23] Y. Chen, J. Lu, X. Yu, and D. J. Hill, "Multi-agent systems with dynamical topologies: Consensus and applications," *IEEE Circuits and Systems Magazine*, vol. 13, pp. 21–34, Jan. 2013.

[24] G. Antonelli, "Interconnected dynamic systems: An overview on distributed control," *IEEE Control Systems Magazine*, vol. 33, pp. 76–88, Feb. 2013.

[25] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[26] W. Ren, R. Beard, and E. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Systems Magazine*, vol. 27, no. 2, 2007.

[27] Z. Li and Z. Duan, *Cooperative Control of Multi-agent System: A Consensus Region Approach*. CRC Press, 2015.

[28] W. Ren and R. W. Beard, *Distributed Consensus in Multi-vehicle Cooperative Control*. Springer-Verlag, 2008.

[29] J. S. Shamma, *Cooperative control of distributed multi-agent systems*. John Wiley and Sons, Ltd, 2007.

[30] A. Lanzon and I. R. Petersen, "Stability robustness of a feedback interconnection of systems with negative imaginary frequency response," *IEEE Transactions on Automatic Control*, vol. 53, pp. 1042–1046, May 2008.

[31] I. R. Petersen and A. Lanzon, "Feedback control of negative-imaginary systems," *IEEE Control Systems Magazine*, vol. 30, pp. 54–72, Oct. 2010.

[32] P. Bhowmick and S. Patra, "On decentralized integral controllability of stable negative-imaginary systems and some related extensions," *Automatica*, vol. 94, pp. 443–451, 2018.

[33] H. J. Chen and A. Lanzon, "Closed-loop stability analysis of discrete-time negative imaginary systems," *Systems & Control Letters*, vol. 114, pp. 52–58, Apr. 2018.

[34] A. Lanzon and H. J. Chen, "Feedback stability of negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 62, pp. 5620–5633, Nov. 2017.

[35] A. Ferrante, A. Lanzon, and L. Ntogramatzidis, "Foundations of not necessarily rational negative imaginary systems theory: Relations between classes of negative imaginary and positive real systems," *IEEE Transactions on Automatic Control*, vol. 61, pp. 3052–3057, Oct. 2016.

[36] I. R. Petersen, "Negative imaginary systems theory and applications," *Annual Reviews in Control*, vol. 42, pp. 309–318, 2016.

[37] J. Xiong, A. Lanzon, and I. R. Petersen, "Negative imaginary lemmas for descriptor systems," *IEEE Transactions on Automatic Control*, vol. 61, pp. 491–496, Feb. 2016.

[38] J. Wang, A. Lanzon, and I. R. Petersen, "Robust output feedback consensus for networked negative-imaginary systems," *IEEE Transactions on Automatic Control*, vol. 60, pp. 2547–2552, Sept. 2015.

[39] J. Wang, A. Lanzon, and I. R. Petersen, "Robust cooperative control of multiple heterogeneous negative-imaginary systems," *Automatica*, vol. 61, pp. 64–72, Nov. 2015.

[40] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "Generalizing negative imaginary systems theory to include free body dynamics: Control of highly resonant structures with free body motion," *IEEE Transactions on Automatic Control*, vol. 59, pp. 2692–2707, Oct. 2014.

[41] J. Xiong, I. R. Petersen, and A. Lanzon, "A negative imaginary lemma and the stability of interconnections of linear negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 55, pp. 2342–2347, Oct. 2010.

[42] G. Salcan-Reyes, A. Lanzon, and I. R. Petersen, "Controller synthesis to render a closed loop transfer function strongly strictly negative imaginary," in *Proceedings of the 57th IEEE Conference on Decision and Control*, (Miami, FL, USA), pp. 2988–2993, Dec. 2018.

[43] G. Salcan-Reyes and A. Lanzon, "On negative imaginary synthesis via solutions to riccati equations," in *Proceedings of the 16th European Control Conference*, (Limassol, Cyprus), pp. 870–875, June 2018.

[44] J. Xiong, J. Lam, and I. R. Petersen, "Output feedback negative imaginary synthesis under structural constraints," *Automatica*, vol. 71, pp. 222–228, Sept. 2016.

[45] M. A. Mabrok and I. R. Petersen, "Controller synthesis for negative imaginary systems: A data driven approach," *IET Control Theory and Applications*, vol. 10, no. 12, pp. 1480–1486, 2016.

[46] M. A. Mabrok, A. Kallapur, I. R. Petersen, and A. Lanzon, "A generalized negative imaginary lemma and Riccati-based static state-feedback negative imaginary synthesis," *Systems and Control Letters*, vol. 77, pp. 63–68, Mar. 2015.

[47] Z. Song, A. Lanzon, S. Patra, and I. R. Petersen, "A negative-imaginary lemma without minimality assumptions and robust state-feedback synthesis for uncertain negative-imaginary systems," *Systems and Control Letters*, vol. 61, pp. 1269–1276, Dec. 2012.

[48] Z. Song, A. Lanzon, S. Patra, and I. R. Petersen, "Towards controller synthesis for systems with negative imaginary frequency response," *IEEE Transactions on Automatic Control*, vol. 55, pp. 1506–1511, Jun. 2010.

[49] M. Liu and J. Xiong, "Bilinear transformation for discrete-time positive real and negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 63, no. 12, pp. 4264–4269, 2018.

[50] M. Liu and J. Xiong, "Properties and stability analysis of discrete-time negative imaginary," *Automatica*, vol. 83, pp. 58–64, 2017.

[51] A. Ferrante, A. Lanzon, and L. Ntogramatzidis, "Discrete-time negative imaginary systems," *Automatica*, vol. 79, pp. 1 – 10, 2017.

[52] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "Spectral Conditions for Negative Imaginary Systems With Applications to Nanopositioning," *IEEE/ASME Transactions on Mechatronics*, vol. 19, pp. 895–903, June 2014.

[53] B. Bhikkaji and S. O. R. Moheimani, "Fast scanning using piezoelectric tube nanopositioners: A negative imaginary approach," in *2009 IEEE/ASME International Conference on Advanced Intelligent Mechatronics*, (Singapore), pp. 274–279, July 2009.

[54] H. J. Chen, A. Lanzon, and I. R. Petersen, "Conditions for preserving negative imaginary properties in feedback interconnections and an application to multi-agent systems," in *Proceedings of the 25th Mediterranean Conference on Control and Automation*, (Valletta, Malta), pp. 1143–1147, July 2017.

[55] C. Cai and G. Hagen, "Stability analysis for a string of coupled stable subsystems with negative imaginary frequency response," *IEEE Transactions on Automatic Control*, vol. 55, pp. 1958–1963, Aug. 2010.

[56] V. Tran, M. Garratt, and I. R. Petersen, "Formation control of multi-UAVs using negative-imaginary systems theory," in *2017 Asian Control Conference, ASCC 2017*, pp. 2031–2036, 2018.

[57] K. Liu, M. Ono, X. Li, and M. Wu, "Robust performance synthesis for systems with positive-real uncertainty and an extension to the negative-imaginary case," *Automatica*, vol. 82, pp. 194–201, 2017.

[58] Q. Wang, T. H. Lee., and J.B.He, "Internal stability of interconnected systems," *IEEE Transactions on Automatic Control*, vol. 44, no. 3, pp. 593–596, 1999.

[59] P. Lancaster and M. Tismenetsky, *The theory of matrices: with applications*. Computer science and applied mathematics, Orlando: Academic Press, 2nd ed., 1985.

[60] K. Zhou, J. C. Doyle, and K. Glover, *Robust and optimal control*. Prentice-Hall, Inc., 1996.

[61] A. J. Laub, *Matrix Analysis for Scientists and Engineers*. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics (SIAM), 2005.

[62] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge University Press, 2nd ed., 2013.

[63] S. Skogestad and I. Postlethwaite, *Multivariable feedback control: Analysis and design*. John Wiley & Sons, 2nd ed., 2001.

[64] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, USA: Society for Industrial and Applied Mathematics (SIAM), 1994.

[65] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox For Use with MATLAB*. The Mathworks, 1995.

[66] C. Scherer and S. Weiland, "Linear matrix inequalities in control." Lecture Notes, Dutch Institute for Systems and Control,The Netherlands, 2015.

[67] J. Lofberg, "YALMIP : a toolbox for modeling and optimization in MATLAB," in *2004 IEEE International Symposium on Computer Aided Control Systems Design*, (Taipei, Taiwan), pp. 284–289, IEEE, 2004.

[68] J. F. Sturm, "Using SeDuMi 1.02 , A Matlab toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11, no. 1, pp. 625–653, 1999.

[69] Z. Li, Z. Duan, and G. Chen, "On H_∞ and H_2 performance regions of multi-agent systems," *Automatica*, vol. 47, pp. 797–803, Apr. 2011.

[70] P. Lin and Y. Jia, "Average consensus in networks of multi-agents with both switching topology and coupling time-delay," *Physica A: Statistical Mechanics and its Applications*, vol. 387, no. 1, pp. 303–313, 2008.

[71] W. Liu, S. Zhou, Y. Qi, and X. Wu, "Leaderless consensus of multi-agent systems with Lipschitz nonlinear dynamics and switching topologies," *Neurocomputing*, vol. 173, pp. 1322–1329, 2016.

[72] J. Fax and R. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1465–1476, Sept. 2004.

[73] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, pp. 655–661, May 2005.

[74] H. R. Pota and T. E. Alberts, "Multivariable transfer functions for a slewing piezoelectric laminate beam," *Journal of Dynamic Systems, Measurement and Control, Transactions of the ASME*, vol. 117, no. 3, pp. 352–359, 1995.

[75] G. Antonelli, F. Arrichiello, F. Caccavale, and A. Marino, "Decentralized time-varying formation control for multi-robot systems," *International Journal of Robotics Research*, vol. 33, no. 7, pp. 1029–1043, 2014.

[76] W. Ren, H. Chao, W. Bourgeous, N. Sorensen, and Y. Chen, "Experimental validation of consensus algorithms for multivehicle cooperative control," *IEEE Transactions on Control Systems Technology*, vol. 16, no. 4, pp. 745–752, 2008.

[77] X. Ding, "On dynamic distributed control and its application to multi-robot systems," in *Chinese Control Conference, CCC*, pp. 8863–8868, July 2017.

[78] R. Zheng and D. Sun, "Rendezvous of unicycles: A bearings-only and perimeter shortening approach," *Systems and Control Letters*, vol. 62, no. 5, pp. 401–407, 2013.

[79] D. V. Dimarogonas and K. J. Kyriakopoulos, "On the Rendezvous Problem for Multiple Nonholonomic Agents," *IEEE Transactions on Automatic Control*, vol. 52, pp. 916–922, May 2007.

[80] J. Lin, A. Morse, and B. Anderson, "The Multi-Agent Rendezvous Problem," in *Proceedings of the 42nd IEEE Conference on Decision and Control*, (Maui, Hawaii USA), pp. 1508–1513, Dec. 2003.

[81] S. G. Tzafestas, *Introduction to Mobile Robot Control.* Elsevier, 2014.

[82] A. De Luca, G. Oriolo, and M. Vendittelli, *Control of Wheeled Mobile Robots: An Experimental Overview*, pp. 181–226. Berlin, Heidelberg: Springer Berlin Heidelberg, 2001.

[83] F. Arvin, J. Espinosa, B. Bird, A. West, S. Watson, and B. Lennox, "Mona : an Affordable Open-Source Mobile Robot for Education and Research," *Journal of Intelligent & Robotic Systems*, May 2018.

[84] F. Arvin, S. Watson, A. Emre Turgut, J. Espinosa, T. Krajnik, and B. Lennox, "Perpetual robot swarm: Long-term autonomy of mobile robots using on-the-fly inductive charging," *Journal of Intelligent & Robotic Systems*, Oct. 2017.

[85] F. Arvin, T. Krajník, and A. E. Turgut, "PΦSS: An Open-source Experimental Setup for Real-world Implementation of Swarm Robotic Systems in Long-term Scenarios," in *5th International Conference Modelling and Simulation for Autonomous*, Oct. 2018.

[86] T. Krajník, M. Nitsche, J. Faigl, P. Vaněk, M. Saska, L. Přeučil, T. Duckett, and M. Mejail, "A practical multirobot localization system," *Journal of Intelligent & Robotic Systems*, vol. 76, pp. 539–562, Dec. 2014.

[87] F. L. Lewis, D. Vrabie, and V. L. Syrmos, *Optimal Control.* John Wiley & Sons, Inc., 3 ed., 2012.

[88] M. Liu and J. Xiong, "Properties and stability analysis of discrete-time negative imaginary systems," *Automatica*, vol. 83, pp. 58 – 64, 2017.

[89] I. Hwang, S. Kim, Y. Kim, and C. Seah, "A survey of fault detection, isolation, and reconfiguration methods," *IEEE Transactions on Control Systems Technology*, vol. 18, no. 3, pp. 636–653, 2010.

[90] M. Davoodi, N. Meskin, and K. Khorasani, "Simultaneous fault detection and consensus control design for a network of multi-agent systems," *Automatica*, vol. 66, pp. 185–194, 2016.

....