Advanced RANS and Near-Wall Turbulence Modelling for High-Speed Flow

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Xinguang Wang

School of Mechanical, Aerospace and Civil Engineering

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Nomenclature

Latin Letters

- a_{ij} Anisotropic stress tensor
- C_f Skin-friction coefficient, $(=\tau_w/\frac{1}{2}\rho U_{\infty}^2)$
- C_p Specific heat coefficient for constant pressure
- C_v Specific heat coefficient for constant volume
- c Speed of Sound
- d Vector between two cell points
- ${\cal E}$ Total energy
- e Internal specific energy
- F_f Volumetric flux
- C_v Specific heat coefficient for constant volume
- H Total enthalpy
- h Specific enthalpy
- k Turbulent kinetic energy, Reynolds or Favre averaged
- \hat{K} Eigenvector in flux-splitting methods
- l_m Mixing length in zero-equation models
- *l* Turbulent length-scale
- P Mean pressure
- P_k Production rate of kinetic energy
- p Pressure
- P_{ij} Production rate in the Reynolds stress equation
- Pr Molecular Prandtl number
- $/textPr_t$ Turbulent Prandtl number
- Q, q_w Wall heat flux
- q Heat flux vector
- R Perfect gas constant

- r Ratio of successive gradients for flux-limited schemes
- Re Reynolds number
- Rt Turbulent Reynolds number
- \mathbf{S}_f Face area vector
- S_{ij} Strain-rate tensor
- S_t Stanton number
- S_{ϕ} Source term
- T Temperature
- t Time
- u Velocity vector
- U_i ith component of mean velocity
- $\overline{u'_i u'_j}$ Reynolds stress tensor in terms of Reynolds averaging
- $\widetilde{u'_{i}u'_{j}}$ Reynolds stress tensor in terms of Favre averaging
- V Volume of control volume
- ω_f Weighting factor
- **x** Location vector
- xI Inviscid impingement point
- x_s Separation point
- x_r Reattachment point
- Y Wall normal distance
- y Coordinate direction or wall normal distance
- y^+ Non-dimensional wall distance, $(=(\tau_w/\rho)^{1/2}y/v)$
- y^* Non-dimensional wall distance, $(=k_v^{1/2}y/v)$

Greek Letters

- $\hat{\alpha}$ Wave strength in flux-splitting methods
- γ Specific heat ratio or blending factor
- δ Boundary layer thickness
- δ^* Displacement thickness
- arepsilon Dissipation rate of kinetic energy
- ε_{ij} Viscous dissipation rate in the Reynolds stress equation
- $\widetilde{\boldsymbol{\epsilon}}$ isotropic eddy dissipation
- θ Momentum thickness
- λ Thermal conductivity

- $\hat{\lambda}$ Eigenvalues in flux-splitting methods
- μ Molecular viscosity
- μ_t Turbulent viscosity
- v Kinematic viscosity
- v_t Turbulent kinematic viscosity
- ρ Density
- $\sigma_{ij}, \underline{\sigma}$ Stress tensor
- τ_w Wall shear stress
- Ψ Flux limiter
- Ω_{ij} Vorticity tensor

Subscript

- f Face value
- P Value at the near wall cell
- w Value at the wall
- 0 Value at the location of the incoming boundary layer
- ∞ Freestream value

Superscript

- imd Intermediate value
- init Initial value
- n New time level
- o Old time level
- ' Fluctuation part in terms of Reynolds averaging
- " Fluctuation part in terms of Favre averaging
- $\overline{()}$ Reynolds averaged value
- $\widetilde{()}$ Favre averaged value

Acronyms

- AWF Analytical Wall Function
- BD Blended Differencing

CD - Central Differencing

CMAWF - Compressibility in the thermal MAWF

EVM - Eddy Viscosity Models

CFD - Computational Fluid Dynamics

CFL - Courant Friedrichs Lewy

CV - Control Volume

DNS - Direct Numerical Simulation

FS - Flux Splitting scheme

GGDH - Generalised Gradient Diffusion Hypothesis

HO - High Order

hyper-CMAWF - CMAWF with hyperbolic assumption for molecular viscosity

JL - Jones and Launder model

KNP - Kurganov Noelle Petrova scheme

KNP - Kurganov Tadmor scheme

para-CMAWF - CMAWF with parabolic assumption for molecular viscosity

LES - Large Eddy Simulation

LS - Launder Sharma model

MAWF - Modified Analytical Wall Function

NLEVM - Non-Linear Eddy Viscosity Model

N-S - Navier-Stokes

RANS - Reynolds Averaged Navier-Stokes

RSM - Reynolds Stress Model

SSG - Speziale Sarkar Gatski model

SST - Shear Stress Transport model

SST - Shear Stress Transport model

SWF - Standard Wall Function

SWTBLI - Shock Wave Turbulent Boundary Layer Interaction

TVD - Total Variation Diminishing

UD - Upwind Differencing

Abstract

This research focuses on the development of wall functions suitable for the prediction of high-speed compressible flows. Wall-functions avoid the need for prohibitively expensive fine near-wall meshes and low-Re models of turbulence which still involve a certain amount of approximation. The conventional log-law-based wall functions, however, have limitations in even incompressible cases, which are further compounded when applied to high-speed compressible flows. The objective of this study is to examine the performance of an advanced analytical wall-function treatment which has been successfully used in a range of incompressible flows and explore how compressibility effects could be accounted for in such approaches.

The starting point was the implementation of the analytical wall function proposed by Craft et al (2002) in OpenFoam and its subsequent use for the prediction of the impinging shock interaction and compression corner cases up to a Mach number of 3. The wall pressure and skin friction results obtained by the original version result in improvements over those of the standard wall function (log-law based) and are close to those obtained by the low-Re number modelling for supersonic flows. However, an unphysical behaviour is encountered when applying it to higher Mach number cases.

A compressible flow version of the analytical wall function is proposed which includes the following modifications: a)inclusion of thermal dissipation terms in the analytical equation for the energy variation over the near-wall cells, b) Variable molecular viscosity (due to temperature variations) over the viscous sub-layer, c) improved variation of the convection terms in the near-wall cell analytical equations.

The resultant model has been applied to the above flows up to Mach numbers of 9 and comparisons drawn with experimental data and with predictions from the log-law based wall functions and from the Low-Re Launder and Sharma model. The present results are consistently closer to the data than those of other wall functions in some instances even better than those of the low-Re number. Improvements are especially noticeable in the prediction of the wall heat flux rates, where the log-law wall function generally predicts too low values in the shock interaction region, while the low-Re model, predicts too high heat transfer rates in the highest Mach number cases, as a result of overpredicting turbulence levels where extremely rapid near-wall temperature variations are found.

Declaration

No portion of the work referred to in the dissertation has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Chapter 1

Introduction

In the design of modern high-speed flight aircraft and re-entry space-exploration vehicles, aerodynamics in the supersonic (M=1.2-5.0) and hypersonic (M>5.0) regime plays a dominating role. Military supersonic aircraft are under development in many countries, such as United States, Russia, and China. In recent years, more attention in the military has been turned to hypersonic flight for the flow characteristics at high-speed and high-altitude which can result in a decrease of the drag because of the low-density of air. Civil supersonic aircraft, such as supersonic transports (SST) and business jets, have attracted the interest of many developed countries because the required fuel consumption is economic.

One of the common occurrences in supersonic and hypersonic flow is shock wave/turbulent boundary layer interactions (SWTBLIs), which lead to significant changes of temperature, density, velocity and Mach number near the wall. In order to reduce the cost of experiments and shorten the research period, the numerical approach, Computational Fluid Dynamics (CFD), has blossomed in recent decades with the rapid development of computer technology. In CFD, turbulence modeling is a key element in the aerodynamic design of advanced aircraft, especially in the complex flow simulations such as those associated with wing-body junctions, or the inlet of airbreathing propulsion systems for high-speed vehicles, where SWTBLIs happen. Specifically, there are hundreds of experiments of SWTBLIs, some of which have been summarized by Settles and Dodson (1994). In the present study, some widely used 2and 3- dimensional experiments, including impinging shock, compression corner, supersonic cavity flow, a cylinder with flare or fin, sharp fin and crossing shock interaction, have been chosen for the present study, in order for the code validation as well as the testing and development of advanced modelling practices for handling the near-wall turbulence effects.

Because of the inevitable presence of low-Reynolds-number turbulence in the near-wall region, turbulence models which are devised for high-Reynolds number turbulence flows are not applicable and have to be modified for the low-Reynolds-number effects. Normally

there are two approaches taken in the near wall turbulence modeling, one of which is Low Reynolds Number models which do explicitly account for viscous and near-wall effects but require a large number of near-wall nodes to resolve the significant changes within the thin viscous sublayer. As a result, most of the computation time may be taken up by the viscous sublayer, even though it may only comprise around 1% of the flow domain. In order to overcome the cost associated with low Reynolds-number models, a wall function approach, which has the advantage of eliminating a considerable number of mesh nodes, is widely used in industrial CFD codes, particularly when simulating flows over complex geometries. Such approaches use algebraic formulae or other low-cost routes to provide the overall resistance of the region to heat and momentum transport.

The most of the conventional wall function approaches are based on assumed log-law profiles for mean velocity and temperature, known as "standard wall functions". However, in practice the near-wall flow profiles may depart quite significantly from these, particularly where there are strong pressure gradients, flow impingement, or separation. In recent years there has been some development of more advanced schemes, designed to account for near-wall convection, pressure gradients, and fluid property variations, although the development and testing of these have been largely restricted to incompressible flows. In this research, the main aim is to examine the performance of these advanced wall-function treatments in high-speed flows and explore how compressibility effects could be accounted for in such approaches. The objective is to utilize these advanced wall-functions to account for the rapid changes and turbulence properties in the near-wall region of complex supersonic or hypersonic flowfields with fast convergence rate, robustness, and reasonable accuracy.

In order to implement the wall functions, the open source CFD package OpenFOAM v2.3.1 and v5.0 are used for the reason that new solvers or utilities can be created by users in a fairly convenient manner. The mathematical basis is developed by the finite volume method which is the most widely used approach in most commercial CFD codes. The convection, diffusion and sources terms of the governing equations are discretized by a variety of finite-difference-type approximations. This converts the integral equations into a system of algebraic equations, which can be solved by a variety of methods.

The study begins by testing the original standard wall function and analytical wall function treatments over a range of 2D supersonic impinging shock and compression corner cases. Then more advanced modifications and refinements to overcome the spikes in the calculation of hypersonic flows and the underestimation of wall heat flux after the separation have been implemented and tested over a range of 2D supersonic and hypersonic cases, drawing comparisons with existing experimental measurements and DNS data, together with

the predictions of the standard wall-function approaches and those from full low-Reynoldsnumber models.

This thesis is organized as follows. In Chapter 2, a range of SWBLIs are introduced, including impinging shock, compression corner, supersonic cavity, a cylinder with flare or fin, sharp fin interaction and crossing shock interaction. In Chapter 3, RANS turbulence models are introduced for incompressible flow, and some of the features of compressibility are discussed towards the end of the chapter. In Chapter 4, the numerical implementation associated with the compressible solvers in OpenFOAM is presented. In Chapter 5, some modifications introduced into the original compressible solver 'rhoCentralFoam' in the present work are the flux splitting methods. Some simple supersonic cases are used to validate this modified solver and the comparison of results from different versions of OpenFoam are also made. Chapter 6 introduces the standard log-law wall functions, together with the analytical wall function. The performance of the analytical wall function is tested when applied to the supersonic impinging shock and compression corner in Chapter 7. The refinements and modifications introduced in the present study to the analytical wall function are described in Chapter 8, together with the initial testing and validation of them. Further tests in supersonic and hypersonic regimes are then reported in Chapter 9 for 2D impinging shock interaction cases and compression corners in Chapter 10. The main conclusions arising from the study, and suggestions for future work areas, are included in Chapter 11.

Chapter 2

Shock Wave Turbulent Boundary Layer Interactions

Supersonic flow and hypersonic flow play a dominant role in the design of modern highspeed flight aircraft and space-exploration re-entry vehicles. One of the main differences between them and subsonic flow is that the upstream flowfield cannot be influenced by flow disturbances downstream since the velocity of the flow stream is greater than the sound speed. This results in the presence of discontinuities in the flowfield associated with abrupt changes in all fluid properties, i.e. density, pressure, temperature, velocity, Mach number etc across them. These discontinuities are referred to as shock waves. In addition to these, other more complex phenomena happen in hypersonic flows, such as viscous effects, high-entropy and temperature effects, and low-density effects at high altitude.

One of the common occurrences in supersonic and hypersonic flow is shock wave/turbulent boundary layer interactions (SWTBLIs), which are associated with the shock-wave deflection near the wall. SWTBLIs are always dependent on the nature of the incoming flow, such as Mach number, Reynolds number, the configuration and the source of the shock. In reviewing these flow features in this chapter, we focus on the mechanisms associated with the various interactions. Rather than an exhaustive study covering all published papers on high speed flows, a selection of previous studies will be referenced that help to bring out the above information, together with identifying suitable test cases for the present study and highlighting a number of model performances that have been well-documented.

Firstly, two-dimensional cases are considered, such as impinging shock interaction, compression corner, and supersonic cavity flow. Then a sharp fin, which is fixed on a cylinder or plate, and generates a three-dimensional interaction will be considered. Crossing shock interactions, which include strong shock-shock interaction and SWTBLI, are finally presented towards the end of this chapter.

2.1 Impinging Shock Interaction

The interaction between an impinging-oblique shock and turbulent boundary layer embodies the problems associated with compressibility, flow separation and turbulence, which are all characteristics of SWTBLIs. In this case, the impinging shock is typically generated by a flat plate or another obstacle with sharp leading edge, and the oblique shock then impinges on (and may be reflected from) a flat wall. For relatively weak interaction cases associated with the incoming Mach number and shock angels (see Fig. 2.1), the boundary layer remains unseparated, and the interaction is embedded well within the boundary layer. Green (1970) studied cases at Mach 2.5, Re_{θ} of 4×10^5 , deflection angles of 2° , 2.5° and 5° , indicating that the reflected wave is a single shock with equal and opposite reflection angle to the incident wave.



Figure 2.1: Shock reflection without boundary-layer separation from Delery and Marvin (1986)

For stronger interactions (see Fig. 2.2), the oblique shock is strong enough to cause separation of the turbulent boundary layer. An increase of pressure takes place at separation and generates the leading reflected shock C2. Then shock C2 intersects the oblique shock C1 at point H, from which emanate the two refracted shocks C3 and C4. The point H can be seen clearly in the Schlieren pictures by Green (1970) at deflection angles of 6.5° , 8° , 9.5° , and 10.5° .

Dupont et al. (2006) have studied characteristic time and length scales of the unsteady reflected shock. Humble et al. (2006) used Particle Image Velocimetry to investigate the



Figure 2.2: Shock reflection with boundary-layer separation from Delery and Marvin (1986)

interaction at Mach 2.1, Re_{θ} 3360 and 5290. The mean velocity profile and deduced skin friction coefficient of the undisturbed boundary layer showed good agreement with theory. Turbulence properties showed the highest turbulence intensity in the region behind the impingement of the incident shock wave.

More recently Pirozzoli and Bernardini (2011) used DNS to study the impinging shock interaction which was examined experimentally by Piponniau et al. (2009) at Mach 2.28. However, the Reynolds number based on momentum thickness was reduced to 2300 from 5100. The global prediction of the flow structure is in good agreement with experiments when comparing the flow properties of velocity and pressure with the experimental data, and the database provided is believed to represent a reliable source for the validation of RANS- and LES-based prediction methods. Jammalamadaka et al. (2014) carried out DNS calculations for three different impinging angles at Mach 2.75. As expected, the separation zone grew in size as the shock angle was increased. They also examined the turbulent kinetic energy budgets, to gain a better understanding of the effect of impinging shock on the turbulence.

Schülein (2004) has studied the impinging shock interaction at Mach number 5.0 and Re_l =37×10⁶/m at the Institute of Fluid Mechanics DLR with the shock-wave generator angles 6° to 14° corresponding to weak-to-strong interactions. The detail of the experimental results of wall pressure, skin friction, station number and flowfields can be found in the report by Schülein (1996). Fedorova and Fedorchenko (2004) performed computations of this case by using the Wilcox (1988) two-parameter turbulence model. Kussoy and Horstmann (1975) have conducted the hypersonic axisymmetric impinging shock interaction at Mach number 7.2 and Re_l =10.9×10⁶/m with the generator at either 7.5° or 15°. Another hypersonic impinging shock interaction at Mach 8.2 has been studied by Kussoy and Horstman (1991). The details of the surface pressure and heat transfer results for wedge angles of 5° to 11° were documented. From the oil visualization results, the 5° case was attached, the 10° case was separated, while there was incipient separation at 8°. The computation of these two hypersonic impinging shock interaction cases, together with other hypersonic cases, was reported by Horstman (1991) using the two-equation $k - \varepsilon$ turbulence model, and also two modifications of the $k - \varepsilon$ turbulence model, which are the two-layer $k - \varepsilon$ model by Rodi (1991) and one arising from the introduction of compressibility effects.

2.2 Compression Corner

Another widely examined SWTBLI model is the 2-dimensional compression corner (see Fig. 2.3). The experiments carried out by Settles (1979), Kuntz (1987), Smits and Muck (1987) are highly cited, mainly because all these papers contain well-defined experimental boundary conditions, adequate documentation data, and a wide range of turning angles.

As an illustration, we consider the experiment at Mach 2.9, Reynolds number 6.3×10^7 /m by Smits and Muck (1987). Three different corner angles, namely 8°, 16° and 20°, represent flow without separation, flow with incipient separation and flow with significant separation respectively. With the increase of corner angle, the interaction of the incoming boundary layer and the shock wave causes and reinforces the oscillation of the shock in the streamwise direction and wrinkles in the spanwise direction. When the shock becomes strong enough, flow separation occurs, and the unsteadiness becomes increasingly important.

For corner angles under 20°, the studies mentioned above restrict the Mach number to around 3, while for corner angles greater than 20°, unsteady SWTBLIs experiments have been performed at a higher Mach number of 5 by Dolling and Erengil (1991) for a 28° compression corner and a Mach number of 9 by Verma (2003) for a 24° compression corner. At higher Mach numbers, the interaction tends to be laminar, and Bazovkin et al. (2009) have studied compression corners of $27^{\circ}-42^{\circ}$ in a laminar hypersonic flow (M_∞=21, Re_∞=6×10⁵/m). Recently, the flow over an 8° ramp has been studied at Mach 8 by Bookey et al. (2005) and Mach 7.2 by Schreyer et al. (2011).

Kussoy and Horstman (1989) have studied the hypersonic flow over an axisymmetric cylinder with a 20°, 30°, 32.5°, or 35° flare at Mach number 7.2 and $\text{Re}_l = 7 \times 10^6/\text{m}$. The experimental data reported includes the surface pressure, heat transfer and limited mean flow filed surveys in the undisturbed and interaction region. Georgiadis et al. (2015) simulated the



Figure 2.3: Sketch of the compression corner flow pattern from Adams (2000)

cases with 20° and 30° flares using the Spalart-Allmaras one-equation model and the Menter family of k- ω two-equation models based on Wind-US and CLF3D.

2.3 2-D Supersonic Cavity Flow

The study of cavity flow dates back to the 1950's, mainly for low-speed applications. In recent decades, the interest in supersonic cavity flow has grown because of the ubiquitous presence of cavity-type flows in applications. Although geometrically simple, cavity flows may occur as a desired or undesired feature, such as weapons bays, landing gear walls and flameholders for scramjets.

Typically, the physical structure of a cavity flow can be classified as closed, open or transitional, depending on the Length-to-Depth ratio (L/D). A closed cavity (L/D greater than 14) contains three distinct flow regions if the cavity is long enough (see Fig. 2.4). The free stream flow separates from the leading edge, then attaches on the floor and separates from the floor, finally re-attaching at the trailing edge. Closed cavities are usually less desirable, because of the high drag which is due to the low pressure at the leading edge and the high pressure at the trailing edge. A short and deep open cavity (L/D between 1 and 10) is shown in Fig. 2.5. The free stream flow separates at the leading edge, then bridges the length of the cavity and re-attaches at the trailing edge. The time-mean pressure distribution in the cavity is relatively constant and slightly above the upstream pressure. Transitional cavities, such as

that shown in Fig. 2.6, occur when the L/D is between 10 and 14 and should be avoided due to the unsteadiness of the flow alternation between the two patterns described above, which often occurs in them.



Figure 2.4: Sketch of the closed cavity flow pattern from ESDU (2004)



Figure 2.5: Sketch of the open cavity flow pattern from ESDU (2004)



Figure 2.6: Sketch of the transitional cavity flow pattern from Wilcox (1990)

There is a large database of published papers about cavity flows. Here, we only concentrate on cavity flow in supersonic and hypersonic flows. For rectangular cavities, there are lots of papers which contain measurements of mean and unsteady pressure. Heller and Bliss (1975) hypothesize that if the open cavity flow is steady in a supersonic regime, then there would be a steady wave structure at the leading and trailing edges (see Fig. 2.7). At the rear bulkhead, the flow stagnates and splits. This will cause the free-stream flow to produce static pressure variations over the cavity mouth which cannot be balanced within the cavity. This would suggest that steady flow in a rectangular cavity is difficult to achieve.



Figure 2.7: Sketch of cavity flow problem from Beranek (1975)

However, Zhang and Edwards (1992) point out that linear stability theory shows that the longitudinal mode of oscillation will not occur beyond Mach 2.7 because the shear layer is then stable. In their experiments, both interferometry and spark Schlieren indicate that the flow is stable at both L/D=1 and 3 cavities at Mach number 3.5. Besides the Mach number, other factors, such as the length-to-depth ratio, upstream boundary layer thickness, and the Reynolds number, also play an important role in determining the oscillations present in the cavity.

2.4 Cylinder with Flare or Fin

A cylinder with flare (Fig. 2.8 left) or fin (Fig. 2.8 right) is used to represent the aerodynamics of vehicles which fly in the supersonic and hypersonic regimes. Marvin and Horstman (1989) have conducted tests on cylinders incorporation a flare with half angles of 20° , 30° , 32.5° , and 35° at Mach 7.05 and Reynolds number of 3800 based on momentum thickness. From the results of surface-oil-film and surface pressure at selected axial positions at 90° intervals, it was concluded that the flow was axisymmetric at the zero angle of attack. Under the same inflow conditions, the flow over a cylinder with fins of half angles of 10° , 15° , and 20° was also investigated.

Dunagan et al. (1986) conducted experiments of the flowfield around a cylinder with flares at the nominal free-stream Mach number of 2.85. Tunnel total pressures of 1.7 and 3.4 atm provided Reynolds number values of 1.8×10^7 and 3.6×10^7 based on model length. Three



Figure 2.8: Sketch of a cylinder with flare (left) and fin (right) from Marvin and Horstman (1989)

cone angles (12.5°, 20° and 30°) were studied giving negligible, incipient, and large scale flow separation respectively. Supporting data were obtained using a 2-D laser velocimeter, as well as mean wall pressure and oil flow measurements. The attached flow case was observed to be steady, while the separated cases exhibited shock unsteadiness. The model with cone angle 12.5° was insensitive to Reynolds number for the extrapolation of the oil-flow data. With the increase of model cone angle and decrease of Reynolds number, the extent of the interaction was seen to increase. CFD simulation results with a $k - \varepsilon$ model achieved good agreement with the experimental data.

Brown et al. (1988) studied the flow of an axisymmetric turbulent boundary layer over a 5.08 cm diameter cylinder that was aligned with the wind tunnel axis, and the sketch map of this model can be found in Fig. 2.9. The boundary layer was compressed by a 30° half-angle conical flare, with the cone axis inclined at an angle α (=0°, 5°, 10°) to the cylinder axis. The inflow conditions were Mach 2.85 and Reynolds number per unit length of 1.6×10⁷/m. The measurements included wall pressures, two-component LDV flowfield data, shadowgraph, and surface-oil flow visualizations.

The test model used by Jeffrey et al. (1994) was a sting-supported cylinder, aligned with the free-stream flow, and a 20° half-angle conical flare offset 1.27 cm from the cylinder centerline. The inflow condition was Mach 2.89 and Reynolds number 1.5×10^7 /m. Surface oil flow, laser light sheet illumination, and Schlieren were used to document the flow topology. The data includes surface-pressure and skin-friction measurements. A numerical simulation of this case can be found in Gaitonde et al. (1997) using a two-equation $k - \varepsilon$ model. The numerical results of surface streamline are shown in Fig. 2.10 (left). A range of computational results was validated by comparing with the experimental data. From the observation of experimental and numerical flowfields, the separation area covered the entire periphery



Figure 2.9: Sketch of a general model with non-zero flare from Brown et al. (1988)

upstream of the juncture. A houseshoe-like vortical structure (see Fig. 2.10 right) was formed at the upper symmetry plane. The legs of this structure wrapped around the juncture and were switched to streamwise direction around the lower symmetry plane. After interaction with the displaced oncoming turbulent boundary layer, a dual scroll-like structure was observed.



Figure 2.10: Surface stream structure (left) and sketch map of the principal vortical structure (right) from Gaitonde et al. (1997)

2.5 Sharp Fin Interaction

The 3-D sharp fin interaction consists of a flat plate with a sharp fin mounted perpendicular on it (see Fig. 2.11). In practice, this type of shock wave/turbulent boundary layer interaction occurs on the supersonic compressible inlet or the wing surface of supersonic aircraft.


Figure 2.11: Sharp fin induced shock wave/boundary-layer interaction from Lee, Settles and Horstman (1992)

The flowfield generated by a single fin interaction is shown by Hsu and Settles (1992) in Fig. 2.12 at M_{∞} =4, α =20°. The results from holographic interferometry were analyzed to yield flowfield density maps in the crossflow plane. A flow structure is obtained which consists of a λ -shock bifurcation astride a large vertical separation region. The separation rolls up the entire incoming boundary layer, forming a low-density vortex core. The rear part of this interaction is dominated by an impinging jet structure which raises the density at the surface to a value higher than at any other point in the flowfield.

Knight et al. (1986) have studied the sharp fin interaction at Mach 3 for a fin angle of 20° , and Reynolds number 9×10^{5} based on boundary layer thickness. The experimental data include surface pressure profiles, surface streamline patterns, and boundary layer profiles of pitot pressure and yaw angle. The algebraic turbulent eddy viscosity model of Baldwin and Lomax and a two-equation model with wall function were used to compute this case. The experimental and numerical results were quite close, and the flow structure was quite the same as described above.

Kim et al. (1990) reported a joint experimental and computational study of skin friction at M=3, α =10° and 16° and M=4, α =16° and 20°. From the results of Laser Interferometer Skin Friction (LISF) meter, the peak skin friction occurred at the rear part of the interaction, where the separated flow attached and rose with the increase of interaction strength. Many other experiments have been conducted at Mach 2, 3, 4, 5, 8 for different angles ranging from 2° to 25° by Lee et al. (1992), Rodi et al. (1992), and Kussoy (1991). All these experiments were designed to generate flows with a variety of shock strength, turbulent boundary layer, and turning angles. The results of the experiments included surface pressure, skin-friction, heat transfer distribution, and surface yaw angles from oil-flow visualization pattern. Most of



Figure 2.12: Shock reflection with boundary-layer separation by Hsu and Settles (1992)

these datasets were, or have subsequently been, used to validate or test computational models and corrections for supersonic and hypersonic flow.

2.6 Crossing Shock Interaction

Complex 3-D crossing shock interactions play an important role in the design of air-breathing propulsion system for high-speed vehicles. Poor design may cause extremely high heat flux in the inlet wall, which will lead to damage of the planes. The geometry contains two symmetric or asymmetric sharp fins which are mounted on a flat plate.

Knight et al. (2003) pointed out some fundamental features of crossing shock interactions. In order to clarify the complicated 3-D flowfield, flow patterns based on experimental planar laser scattering images are plotted at three specific cross-sections in Fig. 2.13, for a case with two sharp fins at 15° angles and Mach number of 4. Cross-section I is located before the intersection of the two single fin interactions, so the flow structures are the same as single fin interactions. Cross-section II is just after the intersection. In the vertical plane, which is assumed to be an inviscid reflection plane, the continuity must be satisfied when the shock wave that intersects this plane reflects from it. The λ -shock structure in cross-section III is reflected from the centre plane and remains intact, though somewhat distorted by SWTBLIS and shock-shock interactions, propagating away from the centre towards the fin surface.

Garrison (1992) studied the flow structure at Mach 3, with fin angles of 7° and 11° and Mach 4, with fin angles of 15° experimentally and numerically (the unit Reynolds number was approximately 80 million per metre). The results of a $k - \varepsilon$ model showed better agreement than those from the B-L turbulence model when compared with the experimental data. However, neither computation fully captured the characteristics of the near-wall phenomenon



Figure 2.13: Crossing shock interaction geometry (Left) and flowfield map (Right) from Knight et al. (2003)

in the complex interaction region. From the observation of experimental and numerical flowfields, there was a complex shock structure overlying a large viscous separated region. This region included a large amount of low-speed and low-stagnation-pressure fluid.

Narayanswami et al. (1993) combined theoretical and experimental studies of $10^{\circ} \times 10^{\circ}$ at Mach 8.3 (Reynolds number based on the boundary layer thickness was 1.7×10^5). The B-L turbulence model gave good results when comparing with the experimental measurements for surface pressure, flow pattern, pitot pressure, and yaw angle profile. However, the computational surface heat transfer in the interaction zone was two times that measured in the experiment. More detail of the flow structures from the summary of the experimental and numerical flowfields was described and is shown in Fig. 2.15. At location 1, λ -shock structures were present, as in the single fin interaction. At location 2, two separation shocks interacted with each other, and a new shock (4) was formed. At locations 3 and 4, the shock structure (4) was strengthened and moved forward. At location 5, this shock moved even further outwards, and close to the rear shock. A new shock structure 5 formed over the interaction central line. At location 6, shocks 3 and 4 crossed each other, and a high-pressure region resulted. The peak plate pressure happened at the interaction central line here. At location 7, two primary shocks crossed, two reflected shocks can be observed clearly, and an expansion region was formed between the reflected shocks and shock structure over the central line (5).

Kussoy et al. (1993) conducted two symmetric experiments (10° and 15°) for crossing shock wave/turbulent boundary layer interactions at Mach 8.3, Reynolds number per unit



Figure 2.14: Sketch map of low structure (left) at different locations (right) from Narayanswami et al. (1993)

(1 – primary shock; 2 – separation shock; 3 – rear shock; 4 and 5 – shocks; 6 – high pressure region; 7 – reflected shock; 8 – expansion region;)

 5.3×10^{6} /m. For both cases, the pressure rose quickly to its maximum value, which was located downstream of the shock crossing point, and then decreased, followed by the expansion fan from the corner of the fins. The normalized heat transfer exhibited the same distribution as pressure. However, a small plateau can be observed near the initial rise for both configurations. Zheltovodov et al. (2001) analyzed symmetric and asymmetric interaction at the Mach number of 4 (Re=88×10⁶ /m, fin angle: $7^{\circ} \times 7^{\circ}$, $7^{\circ} \times 11^{\circ}$, $15^{\circ} \times 15^{\circ}$) and 5 (36.5×10^{6} /m, fin angle: $23^{\circ} \times 23^{\circ}$). The results of $k - \varepsilon$ and $k - \omega$ models agreed well with the main features of their experimental data such as the flowfield topology and pressure distribution but overpredicted the pressure in the mild and strong interaction areas. It was concluded that the stronger the interaction, the worse the numerical prediction of heat flux became.

2.7 Summary

In this chapter, some basic SWTBLI models, including the impinging shock interaction, compression corner, supersonic cavity flow, 3-D sharp fin or flare mounted on a cylinder, sharp fin flow and crossing shock interaction, have been introduced. The basic flow structures of all these cases have been discussed in detail, to provide a good understanding of the physics of these complex phenomena, and a number of previous studies, both experimental and computational, have been highlighted.

Chapter 3

RANS Turbulence Modelling

In engineering applications, turbulence is prevalent in the flows of liquids or gases around airplanes, automobiles, ships, and submarines. One important characteristic of turbulence is its ability to transport and mix fluid more efficiently than the laminar flow. This is demonstrated by a famous experiment reported by Reynolds at the University of Manchester in 1883. Later, Reynolds established that this flow could be characterized by a non-dimensional parameter, now known as the Reynolds number $Re = U_c L_c / v$, where U_c and L_c are characteristic velocity and length scale of the flow, and v is the kinematic viscosity of the fluid.

The turbulent flow can be characterized by random, unsteady, chaotic, and three-dimensional phenomena which are hard to describe and predict. At high Reynolds number, there is a separation of scales, ranging from large-scale motions, which can often be visualized and are strongly influenced by inflow conditions and flow geometry, down to the small-scale motions. These small-scale motions are mostly determined by the rates at which they receive energy from the large scales, and by the viscosity.

Before any further discussion of turbulence modelling, it is useful to introduce the Navier-Stokes equations for continuous media first, which can be obtained from the principles of conservation of mass, momentum, and energy.

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (u\rho) = 0 \tag{3.1}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \bullet [\mathbf{u} (\rho \mathbf{u})] + \nabla p - \nabla \bullet \underline{\sigma} = 0$$
(3.2)

$$\frac{\partial (\rho E)}{\partial t} + \nabla \bullet [\mathbf{u} (\rho E + p)] - \nabla \bullet (\underline{\sigma} \bullet \mathbf{u}) + \nabla \bullet \mathbf{q} = 0$$
(3.3)

where the stress tensor is expressed as $\underline{\sigma} = \mu \left[\nabla \mathbf{u} + \nabla \mathbf{u}^T - \left(\frac{2}{3} \nabla \bullet \mathbf{u}\right) I \right]$, the total specific energy is $E = e + \frac{1}{2} |\mathbf{u}|^2$, and the heat flux vector as $\mathbf{q} = -\lambda \nabla T$. The molecular viscosity is calculated by the Sutherland (1893) equation:

$$\mu = \frac{A_s \sqrt{T}}{1 + T_s/T} \tag{3.4}$$

where, for the air flows considered in this study, the Sutherland coefficient is $A_s = 1.458 \times 10^{-6}$ kg/m-s, and the Sutherland temperature is $T_s = 110.3$ K.

In order to close the system, the equation of state is applied. When the thermally and calorically perfect gas assumption is made, this can be taken as

$$p = \rho RT \tag{3.5}$$

For analyzing turbulent flows, Reynolds (1895) introduced the approach of splitting all the variables into a sum of mean and fluctuating parts. The instantaneous velocity and pressure fields, for example, can be decomposed as the sum of a mean, and a fluctuating part, so that:

$$u_i = U_i + u'_i \quad p = P + p'$$
 (3.6)

Then equations (3.6) are substituted into the continuity and momentum equations, and the resulting equations averaged (the process is now known as Reynolds Averaging), to obtain the RANS equation (i = 1, 2, 3) in incompressible flow:

$$\frac{\partial U_i}{\partial x} = 0 \tag{3.7}$$

$$\frac{\partial U_i}{\partial t} + \frac{\partial \left(U_i U_j \right)}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left(v S_{ij} - \overline{u'_i u'_j} \right)$$
(3.8)

where $S_{ij} = \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}$ is the mean strain-rate tensor. The quantity $\overline{u'_i u'_j}$ is known as the Reynolds-stress tensor, and we denote it by:

$$\tau_{ij} = \rho \overline{u'_i u'_j} \tag{3.9}$$

where τ_{ij} is a symmetric tensor, and thus has six independent unknown components. For general three-dimensional flows, there are now four unknown mean-flow properties (pressure and three velocity components), and six Reynolds-stress components, giving ten unknowns in total. However, we only have one mass equation (3.7) and three momentum conservation equations for a grand total of four. Consequently, more equations are needed to close the system.

Many of the simplest and most widely-used, models for Reynolds stresses fall into the category of the linear eddy-viscosity models, which employ:

$$\overline{u_i'u_j'} = 2/3k\delta_{ij} - v_t S_{ij} \tag{3.10}$$

The quantity k is referred to as specific turbulent kinetic energy, but often just called turbulent kinetic energy, which is defined as:

$$k = \frac{1}{2}\overline{u'_{i}u'_{i}} = \frac{1}{2}\left(\overline{u'^{2}} + \overline{v'^{2}} + \overline{w'^{2}}\right)$$
(3.11)

There are different models for the turbulent viscosity v_t to complete the system, and some of these will be outlined below.

3.1 Zero-Equation Models

Prandtl (1961) put forth the mixing-length hypothesis after his visualization of simple shear flow. In such a flow he proposed modelling the turbulent viscosity as:

$$\mathbf{v}_t = l_m^2 \left| \frac{\partial U}{\partial y} \right| \tag{3.12}$$

where l_m is the mixing-length, taken to vary linearly with distance from the wall. For wallbounded flow, a damping term is needed to account for near-wall viscous effects. Driest (1956) developed a continuous velocity and shear distribution near a smooth wall in order to account for viscous damping effects, taking:

$$l_m = ky \left(1 - \exp\left(-y^+/26\right) \right)$$
(3.13)

Zero-equation models have the advantage of being very simple in concept and can be quite numerically stable. However, these models will work well only for simple two-dimensional shear flow and show weak performance in the more complex flow. The most well-known zero-equation models are Cebeci-Smith (1974) and Baldwin-Lomax (1978) models.

3.2 One-Equation Models

Kolmogorov (1942) introduced the idea that the dissipation rate could be related, via a model coefficient C_D , to the turbulent kinetic energy and a prescribed lengthscale, in order to close the turbulent kinetic energy equation. The turbulence dissipation and viscosity are then:

$$\varepsilon = C_D k^{3/2} / l \tag{3.14}$$

$$v_t = k^{1/2} l = C_D \frac{k^2}{\varepsilon}$$
 (3.15)

where $C_D = 0.08$. In a typical one-equation model, a transport equation for the turbulent kinetic energy is

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = P_k - \varepsilon + \frac{\partial}{\partial x_j} \left[\left(v + v_t / \sigma_k \right) \frac{\partial k}{\partial x_j} \right]$$
(3.16)

where the first term on the right-hand side of equation (3.16) is the production rate of kinetic energy and is given by $P_k = -\overline{u_i u_j} \partial U_i / \partial x_j$. The dissipation rate ε , which is defined as $\varepsilon = v \frac{\partial u'_i \partial u'_j}{\partial x'_j \partial x'_i}$ is modelled by equation (3.14). The last term is related to the diffusive processes.

Baldwin & Barth (1990) and Spalart & Allmaras (1992) have proposed other one-equation models. These two models have shown improved predictive capability for a limited number

of flows with separation. However, when it comes to the complex flows, generally, the one-equation models are not greatly superior to zero-equation models.

Given all these facts, the need for more-nearly universal models, especially for flow with separation, should be reached by seeking a model which takes account of transport effects on the turbulence length scale.

3.3 Two-Equation Models

Two-equation models include not only the computation of k via a transport equation but also effectively of the turbulence length-scale. These models complete and can be used to simulate the flow without prior knowledge of the turbulence structure. The dissipation rate ε or the specific dissipation rate ω is the choice of the second variable to solve a transport equation for in most widely used turbulence models.

3.3.1 Standard $k - \varepsilon$ Model

The standard $k - \varepsilon$ model, usually referenced to Jones and Launder (1972), is a high Reynolds number model in which an equation is solved for ε of the form:

$$\frac{\partial \varepsilon}{\partial t} + U_j \frac{\partial \varepsilon}{\partial x_j} = C_{\varepsilon 1} \frac{\varepsilon}{k} P_k - C_{\varepsilon 2} \frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j} \left[\left(v + v_t / \sigma_{\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_i} \right]$$
(3.17)

The turbulent viscosity is then modelled as

$$v_t = C_\mu k^2 / \varepsilon \tag{3.18}$$

All the closure coefficients are

$$C_{\varepsilon 1} = 1.44$$
 $C_{\varepsilon 2} = 1.92$ $C_{\mu} = 0.09$ $\sigma_{\varepsilon} = 1.3$ $\sigma_{k} = 1$ (3.19)

3.3.2 $k - \varepsilon$ Model of Launder and Sharma

The standard $k - \varepsilon$ model described above is only valid for fully developed turbulent flow. For flows involving solid walls, the local Reynolds number $(=k^{1/2}l/v)$, where $l = k^{3/2}/\varepsilon$ can be small enough that viscous effects cannot be neglected. Launder and Sharma (1974) employed damping functions dependent on R_t , and some additional modelled terms in the transport equations, for the resolution of near-wall turbulent length scales. The transport equations for k and ε are taken as

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j} \left[\rho k U_j - \left(\mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] = P - \rho \varepsilon - \rho D$$
(3.20)

$$\frac{\partial}{\partial t}(\rho\varepsilon) + \frac{\partial}{\partial x_j} \left[\rho\varepsilon U_j - \left(\mu + \frac{\mu_t}{\sigma_{\varepsilon}}\right) \frac{\partial\varepsilon}{\partial x_j} \right] = (C_{\varepsilon 1}P - C_{\varepsilon 2}f_2\rho\varepsilon) \frac{\varepsilon}{k} + \rho E$$
(3.21)

where

$$\mu_t = C_\mu f_\mu \rho \frac{k^2}{\varepsilon} \tag{3.22}$$

where $C_{\varepsilon 1}$, $C_{\varepsilon 2}$, C_{μ} , σ_{ε} , and σ_k are constant values which are the same as in the standard $k - \varepsilon$ model. The proposed damping term functions are,

$$f_{\mu} = \exp \frac{-3.4}{\left(1 + R_t / 50\right)^2} \tag{3.23}$$

$$f_2 = 1.0 - 0.3 \exp\left(-R_t^2\right) \tag{3.24}$$

where $R_t = k^2 / (v\varepsilon)$ is the turbulent Reynolds number, and the extra source terms *D* and *E* are:

$$D = 2v \left(\frac{\partial \sqrt{k}}{\partial y}\right)^2 \tag{3.25}$$

$$E = 2\nu\nu_t \left(\frac{\partial^2 U}{\partial y^2}\right)^2 \tag{3.26}$$

In separated flow, the above ε equation tends to return overly high levels of near-wall turbulence. In order to address this problem, Yap (1987) added an extra source term YC to modify the ε equation:

$$YC = \max\left[0.83\frac{\varepsilon^2}{k}\left(\frac{k^{3/2}/\varepsilon}{2.55Y} - 1\right)\left(\frac{k^{3/2}/\varepsilon}{2.55Y}\right)^2, 0\right]$$
(3.27)

where *Y* is the distance from the wall.

3.3.3 $k - \omega$ SST Model

The Shear Stress Transport model of Menter (1994) combines the $k-\omega$ model which is used in the inner boundary layer and the $k-\varepsilon$ in the outer region or in the free shear flow. The original $k-\omega$ equation is multiplied by F_1 and the transformed $k-\varepsilon$ equation is multiplied by $(1-F_1)$, then these two forms are added together to give the SST model:

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j}(\rho u_j k) = -\rho \overline{u'_i u'_j} \frac{\partial u_i}{\partial x_j} - \beta^* \rho \omega k + \frac{\partial}{\partial x_j} \left[(\mu + \sigma_k \mu_t) \frac{\partial k}{\partial x_j} \right]$$
(3.28)

$$\frac{\partial}{\partial t}(\rho\omega) + \frac{\partial}{\partial x_{j}}(\rho u_{j}\omega) = \frac{\gamma\rho}{\mu_{t}}\left(-\rho\overline{u_{i}'u_{j}'}\frac{\partial u_{i}}{\partial x_{j}}\right) - \beta\rho\omega^{2} + \frac{\partial}{\partial x_{j}}\left[(\mu + \sigma_{\omega}\mu_{t})\frac{\partial\omega}{\partial x_{j}}\right] + 2(1 - F_{1})\frac{\rho\sigma_{\omega^{2}}}{\omega}\frac{\partial k}{\partial x_{j}}\frac{\partial w}{\partial x_{j}}$$
(3.29)

where

$$\mu_t = \frac{\rho k}{\max\left(\omega, F_2 \left\|S\right\| / a_1\right)}$$
(3.30)

where $||S|| = \sqrt{S_{ij}S_{ij}/2}$ is the magnitude of the mean-strain rate tensor. The auxiliary function F_2 is defined as,

$$F_2 = \tanh\left[\max\left(2\frac{\sqrt{k}}{0.09\omega y}, \frac{500\mu}{\rho y^2\omega}\right)^2\right]$$
(3.31)

<u>γ</u> ι γι	a_1	eta^*	к	σ_{k1}	σ_{ω_1}
$\left[\beta_1 \left/ eta^* - \sigma_{\omega 1} \kappa^2 \right/ \sqrt{eta^*} ight]$	0.31	0.09	0.41	0.85	0.5
γ2	σ_{k2}	$\sigma_{\omega 2}$	β_2	β_1	
$\beta_2/\beta^* - \sigma_{\omega 2}\kappa^2/\sqrt{\beta^*}$	1.0	0.856	0.0828	0.075	

Table 3.1: Coefficients of SST model

where y is the wall distance. If the original k- ω model coefficients are presented as $\phi_1(\sigma_{k1}, \cdots)$, and the transformed k- ε ones as $\phi_2(\sigma_{k2}, \cdots)$, then the coefficients in the final SST model, $\phi(\sigma_k, \cdots)$, are defined as

$$\phi = F_1 \phi_1 + (1 - F_1) \phi_2 \tag{3.32}$$

The auxiliary function F_1 is effectively a blending function between the model forms:

$$F_1 = \tanh\left(\arg_1^4\right) \tag{3.33}$$

$$\arg_{1} = \min\left[\max\left(\frac{\sqrt{k}}{0.09\omega y}, \frac{500\mu}{\rho y^{2}\omega}\right), \frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^{2}}\right]$$
(3.34)

$$CD_{k\omega} = \max\left(\frac{2\rho\sigma_{\omega 2}}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}, 10^{-20}\right)$$
(3.35)

The model coefficients are given in Table. 3.1.

3.4 Non-linear Eddy Viscosity Models

It should be noticed that the linear eddy viscosity models (EVM) discussed above are very unlikely to perform universal in turbulent flow and can exhibit inaccurate predictions in the non-equilibrium flows. Non-linear eddy viscosity models (NLEVM) have been developed by calculating an algebraic formula which includes linear, quadratic or even higher order combination of strain rate. Pope (1975) introduced non-linear strain terms into the turbulent stress-strain relation as

$$a_{ij} = \overline{u'_i u'_j} / k - \frac{2}{3} \delta_{ij}$$
(3.36)

Craft et al. (1996) introduced the cubic terms into the equation for the quadratic stressstrain relalation, and the equation of anisotropic stress tensor is:

$$a_{ij} = \overline{u'_{i}u'_{j}}/k - \frac{2}{3}\delta_{ij} = -\frac{v_{t}}{k}S_{ij} + c_{1}\frac{v_{t}}{\tilde{\epsilon}}\left(S_{ik}S_{jk} - 1/3S_{kl}S_{kl}\delta_{ij}\right) + c_{2}\frac{v_{t}}{\tilde{\epsilon}}\left(\Omega_{ik}S_{jk} + \Omega_{jk}S_{ik}\right) + c_{3}\frac{v_{t}}{\tilde{\epsilon}}\left(\Omega_{ik}\Omega_{jk} - 1/3\Omega_{kl}\Omega_{kl}\delta_{ij}\right) + c_{4}\frac{v_{t}k}{\tilde{\epsilon}^{2}}\left(S_{ki}\Omega_{lj} + S_{kj}\Omega_{li}\right)S_{kl} + c_{5}\frac{v_{t}k}{\tilde{\epsilon}^{2}}\left(\Omega_{il}\Omega_{lm}S_{mj} + S_{il}\Omega_{lm}\Omega_{mj} - \frac{2}{3}S_{lm}\Omega_{mn}\Omega_{nl}\delta_{ij}\right) + c_{6}\frac{v_{t}k}{\tilde{\epsilon}^{2}}S_{ij}S_{kl}S_{kl} + c_{7}\frac{v_{t}k}{\tilde{\epsilon}^{2}}S_{ij}\Omega_{kl}\Omega_{kl}$$

$$(3.37)$$

where the strain-rate and vorticity tensors in the above equation is as follows:

$$S_{ij} = \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}$$
(3.38)

$$\Omega_{ij} = \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}$$
(3.39)

The isotropic dissipation rate $\tilde{\varepsilon}$ is defined as:

$$\tilde{\varepsilon} = \varepsilon - 2\nu \left(\frac{\partial k^{1/2}}{\partial x_j} \right)^2$$
(3.40)

This model has been used to homogeneous shear flow, straight and curved channel flow, impinging flow and a rotating pipe flow. The comparisons show that this model performs better than a linear eddy-viscosity scheme. In addition to the cubic form of the stress-strain relation, Suga (1995) also developed a formulation for c_{μ} which is depend on the local mean

strain rate. This formulation was found to be numerical unstable for the flow around a sharp corner with separation. An alternative formulation for c_{μ} is proposed by Craft et al. (2000), together with the improvement in the implementation of the non-linear model, to remove this weakness.

3.5 Reynolds Stress Models

Another higher-level turbulence models are the Reynolds stress models (RSM) or second moment closures which solve a separate transport equation for each individual Reynolds stress. The stress transport equation is written as:

$$\frac{\partial \overline{u'_{i}u'_{j}}}{\partial t} + U_{k}\frac{\partial \overline{u'_{i}u'_{j}}}{\partial k} = P_{ij} - \varepsilon_{ij} + \phi_{ij} + d_{ij}^{v} + d_{ij}^{t}$$

$$= -\left(\frac{u'_{i}u'_{k}}{\partial x_{k}} + \overline{u'_{j}u'_{k}}\frac{\partial U_{i}}{\partial x_{k}}\right) + 2v\frac{\partial u'_{i}}{\partial x_{k}}\frac{\partial u'_{j}}{\partial x_{k}} + \frac{\overline{p'}}{p}\left(\frac{\partial u'_{i}}{\partial x_{k}} + \frac{\partial u'_{j}}{\partial x_{k}}\right)$$

$$+ v\frac{\partial^{2}\overline{u'_{i}u'_{j}}}{\partial x_{k}\partial x_{k}} - \frac{\partial}{\partial x_{k}}\left[\overline{u'_{i}u'_{j}u'_{k}} + \overline{p'u'_{i}}/\rho\delta_{jk} + \overline{p'u'_{j}}/\rho\delta_{ik}\right]$$
(3.41)

where P_{ij} denotes the term for stress production, ε_{ij} denotes the viscous dissipation rate, ϕ_{ij} denotes the pressure strain term, d_{ij}^{v} denotes the viscous diffusion term, and d_{ij}^{t} denotes the turbulence diffusion. The production terms P_{ij} are directly expressed in terms of Reynolds stresses and mean velocity derivatives and do not require any modelling. The similar situation is true for the viscous diffusion terms d_{ij}^{v} . An assumption of isotropy of the small dissipative scales in high Reynolds number flows is made, and the dissipation rate is normally modelled as $\varepsilon_{ij} = (2/3) \varepsilon \delta_{ij}$. Therefore, the transport equation for ε (or for ω) is normally similar to the one in the $k-\varepsilon$ (or $k-\omega$) model. Turbulence diffusion terms d_{ij}^{v} and the pressure strain terms ϕ_{ij} need to be modelled. The Generalised Gradient Diffusion Hypothesis (GGDH) of Daly and Harlow (1970) is used to approximated the turbulence diffusion terms d_{ij}^{t} as:

$$\frac{\partial}{\partial x_k} \left[c_s \frac{k}{\varepsilon} \frac{u'_k u'_l}{u'_k u'_l} \frac{\partial \overline{u'_i u'_j}}{\partial x_l} \right]$$
(3.42)

which normally show good performance over simple gradient diffusion models when the the Reynolds stresses are predicted accurately. Normally the most important term requiring modelling is the pressure strain term ϕ_{ij} . Various models for pressure strain term have been developed, for example the high Reynolds number LRR model of Launder, Reece, and Rodi

(1975), the high Reynolds number SSG model of Speziale, Sarkar, and Gatski (1991), the rapid pressure-strain model of Johansson and Hallbäck (1994), the low-Reynolds number Two Component Limit (TCL) model of Craft (1998) and the single-point pressure strain correlation of Mishra and Girimaji (2017).

3.6 Effects of Compressibility

With the rapid development of aircraft and space programs, compressible turbulent flows, in which density varies significantly due to the pressure and temperature variations, have become important, and in modelling these one must take the density fluctuations into account. For compressible flow, the averaging procedure outlined above is defined relative to density-weighted variables. Favre (1965) introduced the Favre, or mass, average to avoid additional terms in the equations. The Favre-average of a variable \tilde{f} is defined as

$$\tilde{f} = \overline{\rho f} / \bar{\rho} \tag{3.43}$$

where overbar denotes Reynolds averaging. Thus the instantaneous value can be decomposed as

$$f = \tilde{f} + f'' \tag{3.44}$$

where the double prime denotes the fluctuating part with respect to Favre averaging. Wilcox (1993) and Gatski and Bonnet (2013) derive and describe the compressible Favre-averaged Navier-Stokes equations. All flow variables are decomposed as follows:

$$u_{i} = \tilde{u}_{i} + u_{i}'' \quad \rho = \bar{\rho} + \rho' \quad p = P + p'$$

$$h = \tilde{h} + h'' \quad e = \tilde{e} + e'' \quad T = \tilde{T} + T'' \quad q_{j} = q_{Lj} + q'_{j}$$
(3.45)

where p, ρ and q_j are decompsed in terms of conventional mean and fluctuating parts. Substituting Equation (3.45) into the original Navier-Stokes equations, and performing the Favre averaging operations, then the Favre-averaged Navier-Stokes equations are obtained: Conservation of the mas

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i} (\bar{\rho} \,\tilde{u}_i) = 0 \tag{3.46}$$

Conservation of the momentum

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{u}_{i}\right) + \frac{\partial}{\partial x_{j}}\left(\bar{\rho}\tilde{u}_{j}\tilde{u}_{i}\right) = -\frac{\partial P}{\partial x_{i}} + \frac{\partial}{\partial x_{j}}\left(\overline{\sigma_{ji}} + \tau_{ji}\right)$$
(3.47)

Conservation of energy

$$\frac{\partial}{\partial t}\left(\bar{\rho}E\right) + \frac{\partial}{\partial x_{j}}\left(\bar{\rho}\tilde{u}_{j}H\right) = \frac{\partial}{\partial x_{j}}\left(-q_{Lj} - q_{Tj} + \overline{\sigma_{ji}'u_{i}'} - \overline{\rho u_{j}''\frac{1}{2}u_{i}''u_{i}''}\right) + \frac{\partial}{\partial x_{j}}\left[\tilde{u}_{i}\left(\overline{\sigma_{ij}} + \tau_{ij}\right)\right]$$
(3.48)

The turbulent kinetic energy equation

$$\frac{\partial}{\partial t}(\bar{\rho}k) + \frac{\partial}{\partial x_{j}}(\bar{\rho}\tilde{u}_{j}k) = \tau_{ij}\frac{\partial\tilde{u}_{i}}{\partial x_{j}} - \bar{\rho}\varepsilon + \frac{\partial}{\partial x_{j}}\left[\overline{\sigma_{ij}u''_{i}} - \overline{\rho u''_{j}\frac{1}{2}u''_{i}u''_{i}} - \overline{p'u''_{j}}\right] - \overline{u''_{i}\frac{\partial P}{\partial x_{i}}} + \overline{p'\frac{\partial u''_{i}}{\partial x_{i}}}$$
(3.49)

where the total energy E and total enthalpy include the kinetic energy and are defined as:

$$E = \tilde{e} + \frac{1}{2}\tilde{u}_i\tilde{u}_i \quad \text{and} \quad H = \tilde{h} + \frac{1}{2}\tilde{u}_i\tilde{u}_i \tag{3.50}$$

The kinetic energy per unit volume of the turbulent fluctuations is defined as

$$\overline{\rho}k = \frac{1}{2}\overline{\rho u_i'' u_i''} \tag{3.51}$$

Normally for one- and two-equation models, the Reynolds-stress tensor is modelled by the Boussinesq approximation for compressible flows, by taking

$$\tau_{ij} = -\overline{\rho u_i'' u_j''} = \mu_t S_{ij} - \frac{2}{3} \overline{\rho} k \delta_{ij}$$
(3.52)

where S_{ij} is the Favre-averaged strain-rate tensor, which is

$$S_{ij} = \left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial \tilde{u}_k}{\partial x_k}$$
(3.53)

Next, the molecular transport of heat is modelled in analogy to the molecular one as:

$$q_{Tj} = \overline{\rho u_j'' h''} = -\frac{\mu_T C_p}{\Pr_T} \frac{\partial \tilde{T}}{\partial x_j} = -\frac{\mu_T}{\Pr_T} \frac{\partial \tilde{h}}{\partial x_j}$$
(3.54)

where Pr_T is the turbulence Prandtl number. The Favre-averaged dissipation rate is given by

$$\bar{\rho}\varepsilon = \overline{\sigma_{ji}\frac{\partial u_i''}{\partial x_i}} \tag{3.55}$$

The dissipation rate is then obtained from a modelled ε equation (in the context of k- ε model, for example). The molecular diffusion $\overline{\sigma_{ij}u''_{i}}$ and turbulent transport $\overline{\rho u''_{j} \frac{1}{2}u''_{i}u''_{i}}$ in the energy equation can be neglected especially in supersonic regime. In the turbulent kinetic energy equation, the most commonly used approximation is:

$$\overline{\sigma'_{ji}u'_{i}} - \overline{\rho u''_{j}\frac{1}{2}u''_{i}u''_{i}} = \left(\mu + \frac{\mu_{T}}{\sigma_{k}}\right)\frac{\partial k}{\partial x_{j}}$$
(3.56)

The pressure diffusion term $\overline{p'u_j'}$ is simply ignored, as it is in the incompressible flows. The pressure-dilatation term, $\overline{p'\frac{\partial u_i''}{\partial x_i}}$, and pressure work term, $\overline{u''\frac{\partial \bar{p}}{\partial x_i}}$, are also normally neglected. This is valid at least in the case of a supersonic turbulent boundary layer Guarini et al., (2000), a supersonic channel flow Huang et al. (1995) and an impinging shock flow Pirozzoli and Bernardini (2011).

3.7 Summary

At the beginning of this section, the Navier-Stokes equations are introduced. Then, the Reynolds averaged equations was introduced, as the most widely-used approach for dealing with industrial turbulent flows. Some of the most widely-used models for representing the Reynolds stresses have been reviewed, concentrating on the particular eddy-viscosity forms. Finally, the effects of compressibility have been considered by introducing the Favre-averaging approach employed in such flows and the effects on the averaged equations and turbulence models.

Chapter 4

Numerical Implementation

In this chapter, the mathematical basis for compressible OpenFOAM solvers is developed, which is based on the finite volume method that is used in most general-purpose CFD codes.

4.1 Finite Volume Method

The purpose of the finite volume method (FVM) is to approximate the differential equations by a set of algebraic equations linking nodal values of the flow variables. The first step of the FVM is to divide the computation domain into a number of control volumes (CV), and then integrate the differential form of the governing equation over each CV. A typical CV, which is shown in Fig. 4.1, does not overlap with its neighbors, and the entire collection of them fills the computational domain. The centroid P of the CV is defined by

$$\int_{V_P} (x - x_P) \, dV = 0 \tag{4.1}$$

The internal face area vector is constructed for each face and points outwards from the cell. The owner and the neighbour cell centres, as shown in Figure 4.1, are denoted with P and N, and the surface vector S_f points to the control volume N.

All transport equations, including the Navier-Stokes equations, can be expressed in the general form as

$$\frac{\partial \phi}{\partial t} + \nabla \bullet [\mathbf{u}\phi] - \nabla \bullet (\Gamma_{\phi} \nabla \phi) = S_{\phi}(\phi)$$
(4.2)



Figure 4.1: Control volume

where $\phi = \phi(\mathbf{x}, t)$ is the transported variable. The first term in equation (4.2) is the unsteady term, the second term is the convection term, whose precise treatment becomes very important when flow fields include discontinuities, the third term is the diffusion term, and the last term is the source term. All these terms will be introduced respectively in the following sections. Before that, temporal and spatial integration of equation (4.2) is applied

$$\int_{t}^{t+\Delta t} \left[\frac{\partial}{\partial t} \int_{V_{P}} \phi dV + \int_{V_{P}} \nabla \bullet \left(\mathbf{u}\phi - \Gamma_{\phi} \nabla \phi \right) dV \right] dt = \int_{t}^{t+\Delta t} \left(\int_{V_{P}} S_{\phi} \left(\phi \right) dV \right) dt \quad (4.3)$$

and using the Gauss theorem $(\int_{V_P} \nabla \bullet \mathbf{A} dV = \int_{S_P} \mathbf{A} \bullet d\mathbf{S})$, then

$$\int_{t}^{t+\Delta t} \left[\frac{\partial}{\partial t} \int_{V_{P}} \phi dV + \int_{S_{P}} \left(\mathbf{u}\phi - \Gamma_{\phi} \nabla \phi \right) \bullet d\mathbf{S} \right] dt = \int_{t}^{t+\Delta t} \left(\int_{V_{P}} S_{\phi}(\phi) dV \right) dt \qquad (4.4)$$

where S_P is the closed surface that bounds the CV.

4.2 Discretization of the Transport Equation

4.2.1 Unsteady Term

The first term in equation (4.4) is the unsteady term which can be written as

$$\int_{t}^{t+\Delta t} \left[\frac{\partial}{\partial t} \int_{V_{P}} \phi\left(\mathbf{x},t\right) dV \right] dt = \int_{V_{P}} \phi\left(\mathbf{x},t+\Delta t\right) dV - \int_{V_{P}} \phi\left(\mathbf{x},t\right) dV$$
(4.5)

The volume integral can be approximated as

$$\int_{V_P} \phi dV = \phi_P V_P \tag{4.6}$$

where $\phi_P = \phi(\mathbf{x}_P)$ which is equivalent to assuming a linear variation of ϕ across the CV, and can be shown (for example by using Taylor series expansion) to be second-order accurate. Assuming that V_P is constant in time, the unsteady term is thus approximated as

$$\int_{t}^{t+\Delta t} \left[\frac{\partial}{\partial t} \int_{V_{P}} \phi\left(\mathbf{x}, t\right) dV \right] dt = \left(\phi_{P}^{n} - \phi_{P}^{o}\right) V_{P}$$
(4.7)

where superscripts n and o stand for the new and old-time levels. The accuracy of time discretization and solution stability depends highly on the time level at which the other terms in the Navier-Stokes equations are evaluated. In a fully explicit scheme, all the remaining terms are evaluated at the old-time level, while in a fully implicit scheme they are all evaluated at the new time level, such as the first order accurate backward Euler implicit method. The Crank-Nicolson method uses an equally weighted combination of the new and old-time level values and can guarantee second-order accuracy in time.

4.2.2 Convection Term

The discretization of the convection term is obtained as

$$\int_{S_P} (\mathbf{u}\phi) \bullet d\mathbf{S} \approx \sum_f \mathbf{S}_f \bullet \mathbf{u}_f \phi_f = \sum_f F_f \phi_f$$
(4.8)

where the subscript f means the value of a variable in the centre of a face, \sum_f means summation over all the faces of a CV, and $F_f = S_f \bullet \mathbf{u}_f$ is the volumetric flux through a face f. This approximation is of second-order accuracy if the assumption of the linear variation of ϕ on a face is made. The face value of the variable is obtained from neighbouring cell centre values using one of several available convection differencing schemes. Some of these alternatives are outlined below.

1. First Order Upwind Differencing (UD)

 ϕ_f is evaluated using a backward- or forward- differencing according to the direction of the flow

$$\phi_f = \begin{cases} \phi_P & F_f \ge 0\\ \phi_N & F_f \le 0 \end{cases}$$
(4.9)

The UD scheme is unconditionally bounded, so does not produce over or undershoots (see Patankar, 1980), though it is only first-order accurate from the Taylor series.

2. Central Differencing (CD)

For unstructured grids, the linear variation of ϕ between the centroids of P and N can be written as

$$\phi_f = \omega_f \phi_P + (1 - \omega_f) \phi_N \tag{4.10}$$

where the weighting factor is defined as

$$\boldsymbol{\omega}_{f} = \frac{|\mathbf{S}_{f} \bullet \mathbf{d}_{fN}|}{|\mathbf{S}_{f} \bullet \mathbf{d}_{PN}|} \tag{4.11}$$

where \mathbf{d}_{PN} denotes the vector from P to N. Ferziger and Perić (2002) show that this approximation is second order accuracy on uniform and non-uniform grids. However, this scheme may produce unphysical oscillations, as is often found with schemes of higher than first order, and is not typically used in RANS-based simulations.

3. Blended Differencing (BD)

The BD scheme combines linearly the UD and CD schemes as

$$\phi_f = (1 - \gamma) \left(\phi_f\right)_{UD} + \gamma \left(\phi_f\right)_{CD} \tag{4.12}$$

or

$$\phi_{f} = \left[(1 - \gamma) \max\left(\operatorname{sgn}\left(F_{f}\right), 0 \right) + \gamma \omega_{f} \right] \phi_{P} + \left[(1 - \gamma) \min\left(\operatorname{sgn}\left(F_{f}\right), 0 \right) + \gamma (1 - \omega_{f}) \right] \phi_{N}$$

$$(4.13)$$

The blending factor γ decides the characteristic of boundedness and accuracy of the solution. When $\gamma=1$, the BD scheme has second-order accuracy and the same property as central differencing. When $\gamma=0$, the BD scheme has first-order accuracy and the same property as unwind differencing.

TVD scheme

Another way to compromise between boundedness and accuracy is by using a flux limiter, which is a procedure that obtains higher than first-order accuracy, but without the oscillation associated with the CD (or some other high order) scheme. Following Sweby (1984), ϕ_f is evaluated in the flux limiting scheme as

$$\phi_f = \left(\phi_f\right)_{UD} + \Psi(r) \left\lfloor \left(\phi_f\right)_{HO} - \left(\phi_f\right)_{UD} \right\rfloor$$
(4.14)

where $(\phi_f)_{HO}$ means the face value ϕ of a high-order scheme, $\Psi(r)$ is the flux limiter function and *r* is the ratio of successive gradients of ϕ . Greenshields et al. (2009) introduce *r* in the case of an unstructured grid as

$$r = 2\frac{\mathbf{d} \bullet (\nabla \phi)_P}{(\nabla_{\mathbf{d}} \phi)_f} - 1 \tag{4.15}$$

where $(\nabla \phi)_P = \left(\frac{1}{V_P} \sum_f \mathbf{S}_f \phi_f\right)$ is the gradient calculated at the centre of the CV, and $(\nabla_{\mathbf{d}} \phi)_f = \phi_N - \phi_P$ is the face gradient in the direction of the area vector \mathbf{S}_f , multiplied by $|\mathbf{d}|$. In most cases, the CD scheme takes the place of HO in equation (4.14), and the flux-limited scheme is written as

$$\phi_f = \left[1 - \Psi\left(1 - \omega_f\right)\right]\phi_P + \Psi\left(1 - \omega_f\right)\phi_N \tag{4.16}$$

Harten (1983) introduces the Total Variation Diminishing (TVD) approach to satisfy the boundedness criteria. The total variation, $TV(\phi^n)$, is defined as

$$TV\left(\phi^{n}\right) = \sum_{f} |\phi_{N}^{n} - \phi_{P}^{n}|$$

$$(4.17)$$

At each time step, the TVD schemes satisfy the following condition Sweby (1984)

$$TV\left(\phi^{n+1}\right) \le TV\left(\phi^{n}\right) \tag{4.18}$$

In order for the above flux limited convection scheme to satisfy the TVD condition in equation (4.18), it can be shown that one needs:

$$0 \le \Psi(r) \le \min(2r, 2) \text{ for } r \ge 0 \tag{4.19}$$

$$\Psi(r) = 0 \text{ for } r < 0 \tag{4.20}$$

The following are some common limiters

1. the limiter of Van Leer (1973)

$$\Psi(r) = \frac{r + |r|}{1 + r}$$
(4.21)

2. the MUSCL limiter of Van Leer

$$\Psi(r) = \max\left[0, \min\left(2r, \frac{r+1}{2}, 2\right)\right]$$
(4.22)

3. The limiter of Van Albada (1997)

$$\Psi(r) = \frac{r^2 + r}{r^2 + 1} \tag{4.23}$$

4. the Minmod limiter of Roe (1986)

$$\Psi(r) = \max\left[0, \min\left(r, 1\right)\right] \tag{4.24}$$

An extended discussion of TVD schemes can be found in Waterson and Deconinck (2007).

• Flux-Splitting (FS) scheme

(1) KNT method

When using the FS scheme, the face interpolation is divided into inward and outward contributions across the face. Kurganov and Tadmor (2000) and Kurganov et al. (2001) FS schemes for unstructured mesh are included in Greenshields et al. (2009). The convection term can be written as

$$(F_{f}\phi_{f})_{FS} = \alpha F_{f+}\phi_{f+} + (1-\alpha)F_{f-}\phi_{f-} + \omega_{f}(\phi_{f-} - \phi_{f+})$$
(4.25)

where the symbols + and - denote the direction of S_f . The third term is used only for the convection term of the momentum equation as numerical diffusion. The diffusive volumetric flux is

$$\omega_f = \max\left(\psi_{f+}, \psi_{f-}\right) \quad \text{KT method} \tag{4.26}$$

$$\omega_f = a (1-a) \left(\psi_{f+} + \psi_{f-} \right) \quad \text{KNT method}$$
(4.27)

The weighting factor is

$$a = \begin{cases} 1/2 & \text{KT Method} \\ \frac{\psi_{f+}}{\psi_{f+} + \psi_{f-}} & \text{KNT Method} \end{cases}$$
(4.28)

The volumetric fluxes which are related to the local speeds of propagation are

$$\psi_{f+} = \max\left(c_{f+} \left| \mathbf{S}_{f} \right| + \phi_{f+}, c_{f-} \left| \mathbf{S}_{f} \right| + \phi_{f-}, 0\right)$$
(4.29)

$$\Psi_{f-} = \max\left(c_{f+} \left| \mathbf{S}_{f} \right| - \phi_{f+}, c_{f-} \left| \mathbf{S}_{f} \right| - \phi_{f-}, 0\right)$$
(4.30)

where $c_{f\pm} = \sqrt{\gamma R T_{f\pm}}$ are the local sound speeds at the face.

(2) Roe-Pike method

In equation (4.4), the inviscid flux vector is defined as:

$$\mathbf{F} = \mathbf{u}\boldsymbol{\phi} \tag{4.31}$$

This flux vector contains the characteristic information propagating through the control volume surface with speed and direction according to the eigenvalues of the system. By splitting \mathbf{F} into parts, each of which contains the information in a particular direction, and applying upwind differencing in consistence with their corresponding eigenvalues, the following expressions are obtained at each face:

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}_L + \sum_{\tilde{\lambda}_i \le 0} \hat{\alpha}_i \tilde{\lambda}_i \hat{\mathbf{K}}^{(i)}$$
(4.32)

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}_R - \sum_{\tilde{\lambda}_i \ge 0} \hat{\alpha}_i \tilde{\lambda}_i \hat{\mathbf{K}}^{(i)}$$
(4.33)

Alternatively, we can also write the numerical flux as:

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2} \left(\mathbf{F}_L + \mathbf{F}_R \right) - \frac{1}{2} \sum_{i=1}^m \hat{\alpha}_i \tilde{\lambda}_i \hat{\mathbf{K}}^{(i)}$$
(4.34)

where the eigenvalues are

$$\hat{\lambda}_1 = \hat{u} - \hat{a}$$
 $\hat{\lambda}_2 = \hat{\lambda}_3 = \hat{\lambda}_4 = \hat{u}$ $\hat{\lambda}_5 = \hat{u} + \hat{a}$ (4.35)

The right eigenvectors are:

$$\hat{\mathbf{K}}^{(1)} = \begin{bmatrix} 1 \\ \hat{u} - \hat{a} \\ \hat{v} \\ \hat{w} \\ \hat{H} - \hat{u}\hat{a} \end{bmatrix} \quad \hat{\mathbf{K}}^{(2)} = \begin{bmatrix} 1 \\ \hat{u} \\ \hat{v} \\ \hat{w} \\ \frac{1}{2}\hat{V}^2 \end{bmatrix} \quad \hat{\mathbf{K}}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \hat{v} \end{bmatrix} \quad (4.36)$$

$$\hat{\mathbf{K}}^{(4)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \hat{w} \end{bmatrix} \qquad \hat{\mathbf{K}}^{(5)} = \begin{bmatrix} 1 \\ \hat{u} + \hat{a} \\ \hat{v} \\ \hat{w} \\ \hat{H} + \hat{u}\hat{a} \end{bmatrix}$$

The wave strengths are:

The Roe average values are:

$$\tilde{\rho} = \sqrt{\rho_L \rho_R}$$

$$\tilde{u} = \frac{\sqrt{\rho_L u_L} + \sqrt{\rho_R} u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

$$\tilde{v} = \frac{\sqrt{\rho_L v_L} + \sqrt{\rho_R} v_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

$$\tilde{w} = \frac{\sqrt{\rho_L w_L} + \sqrt{\rho_R} w_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

$$\tilde{H} = \frac{\sqrt{\rho_L H_L} + \sqrt{\rho_R} H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

$$\tilde{a} = \left((\gamma - 1) \left(\tilde{H} - \frac{1}{2} \tilde{V}^2 \right) \right)^{0.5}$$

$$(4.38)$$

where $\tilde{V}^2 = \tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2$.

For a linearized Riemann problem, only discontinuous jumps are involved, and the Roe-Pike method is a good approximation for contacts and shocks. On the other hand, rarefaction waves carry a continuous change in flow variables, and they tend to spread as time increases. Clearly, the linearized approximation via discontinuous jumps is incorrect, but for the computation setup, only when the rarefaction wave is transonic or sonic the linearized approximations give difficulties with the numerical results exhibiting unphysical and entropy violating discontinuous waves. A general entropy fix is introduced by Harten (1983), and the expression is:

$$\lambda_{i} = \begin{cases} \frac{\lambda_{i}^{2} + \delta^{2}}{2\delta} & \lambda_{i} < \delta\\ \lambda_{i} & \text{elsewhere} \end{cases} \quad i = 1, 2, 5 \tag{4.39}$$

where δ has different formulations, in this report $\delta = \chi \max(\lambda_i)$.

(3) AUSM+ method

An alternative way to compute the numerical flux \mathbf{F} is called the Advection Upstream Splitting Method (AUSM), which is a flux-vector splitting method. This method was first introduced by Liou and Steffen in 1991. This method defines a cell surface Mach number based on characteristic speeds from the neighboring cells. The surface Mach number is used to determine the upwind extrapolation for the convective part of the inviscid fluxes. The pressure term uses another way of splitting. First, the numerical flux is separated to two parts:

$$\mathbf{F}_{j+1/2} = \mathbf{F}_{j+1/2}^c + p_{j+1/2} = \dot{m}_{1/2} \boldsymbol{\psi}^{\pm} + p_{j+1/2}$$
(4.40)

where ψ^{\pm} is calculated in an upwind fashion:

$$\psi^{\pm} = \begin{cases} \psi^{+} & \text{if } \dot{m}_{1/2} > 0\\ \psi^{-} & \text{otherwise} \end{cases}$$
(4.41)

Clearly, the tasks of ASUM-family methods are to define mass and pressure fluxes; below, these are introduced separately

(1) Mass flux

The mass flux at the cell surface can be written as:

$$\dot{m}_{1/2} = u_{1/2} \rho^{\pm} = a_{1/2} M_{1/2} \rho^{\pm} = a_{1/2} M_{1/2} \begin{cases} \rho^{+} & \text{if } M_{1/2} > 0\\ \rho^{-} & \text{otherwise} \end{cases}$$
(4.42)

Using the numerical speed of sound, a pair of 'left' and 'right' Mach numbers can be defined as:

$$M_{L/R} = \frac{u_{L/R}}{a_{1/2}} \tag{4.43}$$

The cell surface Mach number now can be written as:

$$M_{1/2} = M^+_{(m)}(M_L) + M^-_{(m)}(M_L)$$
(4.44)

The split Mach numbers $M_{(m)}^{\pm}$ are a polynomial function of degree m (= 1, 2, 4), as:

$$M_{(1)}^{\pm}(M) = \frac{1}{2}(M \pm |M|)$$
(4.45)

$$M_{(2)}^{\pm}(M) = \pm \frac{1}{4}(M \pm 1)^2 \tag{4.46}$$

and

$$M_{(4)}^{\pm}(M) = \begin{cases} M_{(1)}^{\pm} & \text{if } |M| \ge 1\\ M_{(2)}^{\pm} \left(1 \mp 16\beta M_{(2)}^{\mp}\right) & \text{otherwise} \end{cases}$$
(4.47)

(2) Pressure flux

A general pressure flux is used as:

$$p_{1/2} = P_{(n)}^+(M_L) + P_{(n)}^-(M_L)$$
(4.48)

where n = 1, 3, or 5 corresponds to the degree of polynomials P^{\pm} , as in M^{\pm} . For an accurate solution, fifth-degree polynomials are preferred, and described as:

$$P_{(5)}^{\pm}(M) = \begin{cases} \frac{1}{M}M_{(1)}^{\pm} & \text{if } |M| \ge 1\\ M_{(2)}^{\pm}\left((\pm 2 - M) \mp 16\alpha M_{(2)}^{\mp}\right) & \text{otherwise} \end{cases}$$
(4.49)

Notice that the parameters α , β are set to be 3/16, 1/8 respectively.

4.2.3 Diffusion Term

The diffusion term can be discretized broadly in the same way as the convection term one:

$$\int_{S_P} \left(\Gamma_{\phi} \nabla \phi \right) \bullet d\mathbf{S} = \sum_{f} \Gamma_{\phi f} \mathbf{S}_f \bullet \left(\nabla \phi \right)_f$$
(4.50)

where Γ_{ϕ} denotes the diffusivity and $(\nabla \phi)_f$ is the face gradient of ϕ . If the mesh is orthogonal, it is possible to use the following approximation:

$$\mathbf{S}_{f} \bullet (\nabla \phi)_{f} = \left| \mathbf{S}_{f} \right| \frac{\phi_{N} - \phi_{P}}{|\mathbf{d}|}$$
(4.51)

This approximation uses the two values of ϕ , one either side of the face, and obtains second-order accuracy. If the mesh is non-orthogonal, the accuracy of equation (4.51) would be diminished, for the area vector \mathbf{S}_f , and the vector **d** are then not parallel. In order to maintain second-order accuracy, an alternative is to split the term into two parts

$$\mathbf{S}_{f} \bullet (\nabla \phi)_{f} = \underbrace{\Delta \bullet (\nabla \phi)_{f}}_{\text{orthogonal contribution}} + \underbrace{\mathbf{k} \bullet (\nabla \phi)_{f}}_{\text{non-orthogonal correction}}$$
(4.52)

and the following condition should be satisfied

$$\mathbf{S}_f = \Delta + \mathbf{k} \tag{4.53}$$

where vector Δ means the orthogonal contribution which is parallel with **d**, **k** is the correction term which is chosen to satisfy the equation (4.53). The equation (4.51) is used to calculate the orthogonal part in equation (4.52). One of the many decomposition methods is the minimum correction approach, which is included in Jasak (1996), where the vector parallel to **d** is taken as

$$\Delta = \frac{\mathbf{S}_f \bullet \mathbf{d}}{\mathbf{d} \bullet \mathbf{d}} \mathbf{d} \tag{4.54}$$

and then **k** is calculated using equation (4.53). As the contribution from ϕ_P and ϕ_N decreases, the non-orthogonality increases.

Another approach to calculating the correction term **k** is called the orthogonal-correction approach, which maintains the contribution from ϕ_P and ϕ_N the same as on the orthogonal mesh irrespective of the non-orthogonality, where

$$\Delta = \frac{\mathbf{d}}{|\mathbf{d}|} \left| \mathbf{S}_f \right| \tag{4.55}$$

An alternative approach is an over-relaxed approach, in which the importance of ϕ_P and ϕ_N would increase with the increase of non-orthogonality.

$$\Delta = \frac{\mathbf{d}}{\mathbf{d} \bullet \mathbf{S}_f} \left| \mathbf{S}_f \right|^2 \tag{4.56}$$

Further discussion about their accuracy and stability, when used on non-orthogonal grids, can be found in Jasak (1996).

4.2.4 Source Term

The source term $S_{\phi}(\phi)$ can be a general function of ϕ . To improve the stability of the solution this term can often be linearized, and more detail is given by Patankar (1980). A simple procedure is explained here, in which the source term is split into two parts and written as

$$S_{\phi} = S_u + S_p \phi \tag{4.57}$$

where S_u and S_p can also be functions of ϕ . Using equation (4.2), the volume integral is approximated as

$$\int_{V_P} S_{\phi} dV = S_u V_P + S_p \phi_P V_P \tag{4.58}$$

 S_p should be chosen to be less than zero, allowing this part of the term to be treated implicitly, improving the diagonal dominance of the system matrix, which is strongly related to the convergence rate and stability of the solution.

4.3 Summary

In this chapter, the finite volume method for the discretization of each term in the Navier-Stokes equations has been introduced. Especially for the convection term, a variety of schemes have been described including the flux-limited and flux-splitting methods.

Chapter 5

Introduction to Compressible OpenFoam Solvers

OpenFOAM (Open source Field Operation And Manipulation) is an open source CFD software package including a variety of numerical solvers, and Pre/Psot processing utilities. Here, we only focus on a compressible solver which is a density-based solver, named rhoCentralFoam, which has been employed in the present study.

5.1 RhoCentralFoam

This is a density-based algorithm and the convection term is discretized with the flux-splitting KT and KNP schemes as outlined in section 4.2.2. The procedure is described as follows:

- 1. Initialization of all the variables
- 2. Start of time loop
- 3. The continuity equation is solved for density

$$\frac{\rho_P^n - \rho_P^o}{\Delta t} - \sum_f \left(F_f \rho_f \right)_{FS}^o = 0$$
(5.1)

4. Solve the momentum equation without the diffusion terms, and obtain an intermediate estimate of $(\rho \mathbf{u})^{imd}$

$$\frac{(\boldsymbol{\rho}\mathbf{u})_{P}^{imd} - (\boldsymbol{\rho}\mathbf{u})_{P}^{o}}{\Delta t} + \sum_{f} \left(F_{f} \left(\boldsymbol{\rho}\mathbf{u} \right)_{f} \right)_{FS}^{o} + \sum_{f} \mathbf{S}_{f} \left(p_{f} \right)_{FS}^{o} = 0$$
(5.2)

- 5. Update $\mathbf{u}^{imd} = (\rho \mathbf{u})^{imd} / \rho^n$
- 6. Solve the momentum equation with the diffusion term to obtain \mathbf{u}^n :

$$\frac{\rho_{P}^{n}\mathbf{u}_{P}^{n}-\rho_{P}^{n}\mathbf{u}_{P}^{imd}}{\Delta t}-\frac{\rho_{P}^{n}\mathbf{u}_{P}^{imd}-\rho_{P}^{o}\mathbf{u}_{P}^{o}}{\Delta t}-\sum_{f}\mu_{f}^{o}\mathbf{S}_{f}\bullet\left(\nabla\mathbf{u}\right)_{f}^{n}-\sum_{f}\mu_{f}^{o}\mathbf{S}_{f}\bullet\left[\left(\nabla\mathbf{u}\right)^{T}-\frac{2}{3}\nabla\bullet\mathbf{u}I\right]_{f}^{o}=0$$
(5.3)

- 7. Update $(\rho \mathbf{u})^n = \rho^n \mathbf{u}^n$
- 8. Solve the energy equation for $(\rho E)^{imd}$ without conductive heat flux:

$$\frac{(\rho E)_{P}^{imd} - (\rho E)_{P}^{o}}{\Delta t} + \sum_{f} \left[F_{f} \left(\rho E \right)_{f} \right]_{FS}^{o} - \sum_{f} \mu_{f}^{o} \mathbf{S}_{f} \bullet \left(\nabla \mathbf{u} \right)_{f}^{n} \bullet \left(\mathbf{u}_{f} \right)_{FS}^{o} + \sum_{f} \left[F_{f} p_{f} \right]_{FS}^{o} - \sum_{f} \mu_{f}^{o} \mathbf{S}_{f} \bullet \left[\left(\nabla \mathbf{u} \right)^{T} - \frac{2}{3} \nabla \bullet \mathbf{u} I \right]_{f}^{o} \bullet \left(\mathbf{u}_{f} \right)_{FS}^{o} = 0$$

$$(5.4)$$

- 9. Update $e^{imd} = (\rho E)^{imd} / \rho^n 0.5 (\mathbf{u}^n \bullet \mathbf{u}^n)$, $T^{imd} = e^{imd} / C_v$, $\mu^{imd} (T^{imd})$ (such as Sutherland's Equation), and $\lambda^{imd} = C_p \mu^{imd} / \Pr$
- 10. Solve the energy equation for e^n with the conductive heat flux:

$$\frac{\rho_P^n e_P^n - \rho_P^n e_P^{imd}}{\Delta t} - \frac{\rho_P^n e_P^{imd} - \rho_P^o e_P^o}{\Delta t} - \sum_f \lambda_f^{imd} \mathbf{S}_f \bullet (\nabla T)_f^{imd} = 0$$
(5.5)

- 11. Update $T^n(e^n)$, $\mu^n(T^n)$, $(\rho E)^n = \rho^n[e^n + 0.5(\mathbf{u}^n \bullet \mathbf{u}^n)]$ and p^n from the state equation $p = \rho RT$.
- 12. If the turbulence model is involved in the calculation, for example the k- ε model, the ε and k equations are solved:

$$\frac{\rho_{P}^{n}\varepsilon_{P}^{n} - \rho_{P}^{n}\varepsilon_{P}^{o}}{\Delta t} + \sum_{f} \left[F_{f}(\rho\varepsilon)_{f} \right]_{FS}^{o} - \sum_{f} \left(v + v_{t} / \sigma_{\varepsilon} \right)_{f} S_{f} \bullet \left(\nabla\varepsilon \right)_{f}^{n} = C_{\varepsilon 1} \left[\frac{\varepsilon}{k} P_{k} \right]_{P}^{o} - C_{\varepsilon 2} \left[\frac{\varepsilon}{k} \right]_{P}^{o} \varepsilon_{P}^{n}$$
(5.6)

$$\frac{\rho_P^n k_P^n - \rho_P^n k_P^o}{\Delta t} + \sum_f \left[F_f(\rho k)_f \right]_{FS}^o - \sum_f \left(v + v_t / \sigma_k \right)_f S_f \bullet \left(\nabla k \right)_f^n = \left[P_k \right]_P^o - \left[\rho \frac{\varepsilon}{k} \right]_P^o k_P^n \tag{5.7}$$

The turbulence coefficients in equation (5.6) and (5.7) are listed in equation (3.19).

13. The turbulence viscosity and thermal conductivity are calculated:

$$\mu_t = \rho C_\mu \frac{k^2}{\varepsilon} \qquad \lambda_t = C_p \frac{\mu_t}{\Pr_t}$$
(5.8)

The effective viscosity and thermal conductivity in the momentum and energy equations are obtained as: $\mu_{eff} = \mu + \mu_t$ and $\lambda_{eff} = \lambda + \lambda_t$.

14. End of the loop.

5.2 Modification Introduced to the RhoCentralFoam Solver

The official version of rhoCentralFoam, which is implemented by Greenshields et al. (2009), is as described above, comprises semi-discrete, non-staggered central schemes and is validated against an analytical solution of the one dimensional shock tube case, two dimensional forward facing step at transient and supersonic flows, a supersonic jet from a circular nozzle and hypersonic flow over a $25^{\circ}-55^{\circ}$ biconic without turbulence. All the cases in this paper is restricted to run at CFL=0.2, and in the OpenFOAM tutorial cases, CFL=0.35. A 4th-order Runge-Kutta method has been introduced to the solver in OpenFoam V2.3.1 for the temporal derivative as part of the present work. Although the original solver employed the KT and KNP flux splitting schemes, in order to explore the performance of these and others in some of the present cases some alternatives have also been introduced into the solver, namely the Roe-Pike and AUSM+ methods, described in Section 4.2.2.

5.3 Solver Validation

Before we explore the potential of wall functions, three simple cases are chosen to validate the developed solver, and in order to select an appropriate numerical setup for the main calculations of this project. Therefore, the solver described above with different flux splitting methods and limiters introduced in Section 4, are compared in three different cases.

5.3.1 Initial Shock Tube Test

The shock tube test, which is available in the tutorial of OpenFoam V2.3.1, is a well-recognised case to test the accuracy of numerical methods in high speed flows. In this case, three different discontinuities are involved simultaneously, namely the expansion fan, the contact discontinuity and shock wave. Here the Sod (1978) shock tube is used, the initial condition is shown in Fig. 5.1, and 100 points are used in the x-direction.



Figure 5.1: Initial conditions for shock tube test

Figure 5.2 (left) shows a comparison of different flux-splitting methods, AUSM+, KNT, and Roe, using the van Leer limiter. The numerical results are compared with the analytical results Anderson (2003) at time 7ms. The three methods all give no oscillatory solution and show similar results. Figure 5.2 (left) displays a comparison of two different limiters,



Figure 5.2: Comparison of different flux-splitting methods (left) using van Leer limiter and different limiters (right) using Roe method

Minmod and van Leer, using the Roe method. The results of the van Leer limiter show less dissipation and are closer to the analytical solution. From this case, the van Leer limiter is recommended for the construction of flux-splitting methods.

5.3.2 Flat Plate Flow

For a M=2.25 supersonic flow over a sharp thin flat plate at zero incidence angle at low Reynolds number, a turbulent boundary layer will develop in the near wall region, and a weak shock will be generated at the leading edge of the flat plate as shown in Fig. 5.3.



Figure 5.3: Inflow conditions and pressure contours for flow over a plate flat

Figure 5.4 displays the comparison of skin friction using three different flux-splitting methods and three different limiters described in section 4.2, predicted by the Launder-Shamma k- ε turbulence model, with Yap correction, as described in section 3.3.2. The methods of AUSM+ and Roe-Pike with three different limiters give very similar results, while the results of the KNT method vary with different limiters, with the Minmod limiter giving much higher skin-friction than the van Albada and van Leer limiters.

5.3.3 Low-Re Impinging Shock Interaction

When a shock wave interacts with a boundary layer, the main effect is a sudden retardation of the flow with subsequent thickness change of the boundary layer and, depending on conditions, separation of the turbulent boundary layer, which is called impinging shock interaction. The iso-lines of mean pressure superimposed on contours of Mach number of low-Re impinging shock interaction is shown in Fig. 5.5. A 200×120 mesh is used by Launder-Shamma k- ε turbulence model with Yap correction, and the non-dimensional near wall cell distance is around 0.6 at the upstream. The detail of case setup will be introduced in Chapter 7.2.


Figure 5.4: Surface skin friction comparison using different flux-splitting methods and limiters

Figure 5.6 shows the skin-friction comparison using different flux-splitting methods and limiters. From the results, all three flux splitting methods give some difference in the interaction zone, while for the AUSM+ and Roe methods, with little influence in the upstream of the interaction domain. The KNT flux-splitting method is quite sensitive to the limiters throughout the most computation domain.



Figure 5.5: Iso-lines of mean pressure superimposed on contours of Mach number for low-Re impinging shock interaction by using Roe-Pike flux-splitting method with van Leer limiter



Figure 5.6: Surface pressure (right) and skin friction (left)comparison of impinging shock interaction using different limiters to construct AUSM+ (top), KNT (middle) and Roe (bottom)

5.4 Code Version Comparison

At the beginning of the research, OpenFOAM v2.3.1 was tested and developed as described above. Since then, this open source CFD toolbox has released many versions of OpenFoam. Version 5.0 was a snapshot of the OpenFOAM development that appeared in 2017, and the differences between this and the earlier version that had been used were explored. Figs. 5.6, 5.7 and 5.8 show comparisons of surface pressure, skin friction and wall heat transfer in the impinging shock interaction case for three different Mach numbers, using the Launder-Sharma turbulence model as described above in each case. The results from the two code versions can be seen to be in close agreement with each other. One shortcoming of version 2.3.1 for complex supersonic and hypersonic flows is that when the computation is restarted at a fixed time (not time zero folder), it fails when revolving the energy equation for some cases, while version 5 does not have such a problem. As it is inevitable to restart the computation for the complex cases using Low-Re models, and the convergence of two versions are same as described, the comparison of wall function approach with the Low-Re models is executable. Since it is often beneficial to be able to stop and restart computations in complex cases, and when testing a variety of modelling approaches, most of the subsequent calculations have been performed using version 5.0 of OpenFOAM, as it gave the same results as the earlier version but was rather more robust when restarting cases.

5.5 Summary

In this chapter, the rhoCentralFoam solver has been introduced, and the modifications that have been introduced to it explained. Three different cases have been used to validate the solver. From the results, the van Leer and van Albada limiters show less dissipation and will be used for the calculations reported in subsequent chapters. The KNT method is quite sensitive to limiters, so it is not recommended in the calculation of complex turbulent flow. Towards the end of the chapter, the Low-Re models result in two different versions of OpenFoam have been compared for different Mach number impinging shock interaction, and the results were found in close agreement with each other.



Figure 5.7: Surface pressure (left), surface skin friction (middle) and wall heat transfer (right) of Ma=3 impinging shock interaction comparison by using different versions of OpenFOAM



Figure 5.8: Surface pressure (left), surface skin friction (middle) and wall heat transfer (right) of Ma=5 impinging shock interaction comparison by using different versions of OpenFOAM



Figure 5.9: Surface pressure (left), surface skin friction (middle) and wall heat transfer (right) of Ma=7.2 impinging shock interaction comparison by using different versions of OpenFOAM

Chapter 6

Wall Functions

The accuracy of numerical results highly depends on the choice of turbulence model, wallfunction, numerical method, and other assumptions. Among all these factors, the treatment of the viscous sublayer, which is very close to the solid wall, plays an important role, since molecular diffusion here results in a significant transport of heat and momentum. There are two broad strategies to the near wall turbulence modelling, one of which is based on the use of Low-Reynolds-Number models which require the use of fine near-wall meshes to resolve the significant changes within the sublayer. As a result, much of the computation time is utilized in the resolution of the near-wall region, which may only be a small region, with the sublayer possibly covering only around 1% computation domain. In order to overcome this expense, and sometimes convergence difficulties associated with very high aspect ratio near-wall cells, the wall-function approach is widely used in advanced CFD codes because of its robustness, fast convergence rate, and reasonable accuracy in a number of situations, which are the main factors when computing complex geometries.

There are different types of wall functions to bridge the gap between the wall and the fully turbulent area. However, most existing wall functions are based on the log-law under local-equilibrium flow conditions. In the section below, the standard wall function is first introduced, because of its wide application in CFD solvers. Some refinements have been made to improve aspects of its near-wall ability, such as two-layer and three-layer wall functions. Besides the log-law type wall functions, some more advanced ones have also been proposed, which do not involve so many assumptions, and the form that has been tested will be described in Section 6.2 and Section 6.3 which have been successively applied for the incompressible flows. The wall functions including the effect of compressibility and heat transfer are introduced in Section 6.4. It is worth to noticed that all the approaches below will be described in the context of the $k-\varepsilon$ model since it is widely used, although most can also be applied with other more advanced models as well. Other wall approaches, such

as non-overlapping domain decomposition (NDD) method by Utyuzhnikov (2005), which decomposes the computational domain into an inner region and the outer region, will not include in this paper.

6.1 Standard Wall Function

This section introduces the log-law velocity and temperature profiles. In the local equilibrium turbulent boundary layer, the law of the wall holds by experimental observations. We can write this symbolically as:

$$U^{+} = \frac{1}{\kappa} \log \left(E y^{+} \right) \quad \text{when} \quad y^{+} > y^{+}_{\nu}$$

$$U^{+} = y^{+} \quad \text{when} \quad y^{+} \le y^{+}_{\nu}$$
(6.1)

with $U^+ = U/(\tau_w/\rho)^{1/2}$ and $y^+ = (\tau_w/\rho)^{1/2}y/v$. In the above equation $y_v^+ \cong 11$, *E* is related to y_v^+ as:

$$y_{\nu}^{+} = \frac{1}{\kappa} \log\left(E y_{\nu}^{+}\right) \tag{6.2}$$

Normally, in the near-wall sublayer, momentum transfer is dominated by molecular viscosity and heat transfer by conductivity. The temperature log law can be expressed similarly to that for U^+ as:

$$T^{+} = \frac{1}{\kappa} \operatorname{Pr}_{t} \log \left(F y^{+} \right) \quad \text{when} \quad y^{+} > y_{\mathrm{T}}^{+}$$

$$T^{+} = \operatorname{Pr} y^{+} \quad \text{when} \quad y^{+} \le y_{\mathrm{T}}^{+}$$
(6.3)

where $T^+ = (T_w - T) \rho C_p (\tau_w / \rho)^{1/2} / q_w$ is the non-dimensional temperature, Pr is the molecular Prandtl number, and *F* must satisfy:

$$\operatorname{Pry}_{\mathrm{T}}^{+} = \frac{1}{\kappa} \operatorname{Pr}_{t} \log \left(F y_{\mathrm{T}}^{+} \right) \tag{6.4}$$

One drawback of the above form is that the use of τ_w in non-dimensionalising U^+ and y^+ leads to problems in regions where the wall shear stress vanishes, or is very small (including,



Figure 6.1: Near-wall grid

for example, stagnation points, and flow separa-

tion/reattachment regions). To avoid this, based on the standard log-law described above, an improved version of log-law based wall function, which is widely used in many commercial CFD software, can be described as:

$$U^{*} = \frac{1}{\kappa^{*}} \log \left(E^{*} y^{*} \right)$$
 (6.5)

with $U^* = Uk_v^{1/2} / (\tau_w / \rho)$, $y^* = yk_v^{1/2} / v$, k_v is the turbulent kinetic energy at the edge of the viscous sublayer, and the von Kármán constant $\kappa^* = 0.4178$, empirical constant $E^* = 9.793$. Normally, k is assumed to be constant across the near-wall fully turbulent region, so k_v is the same as k_p (see Fig. 6.1). For a given wall heat flux, the wall temperature is obtained by:

$$T^{*} = \frac{(T_{w} - T_{P})\rho C_{p}C_{\mu}^{1/4}k_{P}^{1/2}}{q_{w}} = \begin{cases} \Pr C_{\mu}^{1/4}y^{*} & (y^{*} < y_{\nu}^{*})\\ \Pr_{t}\left[\frac{1}{\kappa}\ln\left(E^{*}C_{\mu}^{1/4}y^{*}\right) + P\right] & (y^{*} > y_{\nu}^{*}) \end{cases}$$
(6.6)

where q_w is the heat flux. *P* is a function related to the resistance to heat transfer in the viscous sublayer. The *P*-function by Jayatilleke (1966) and Launder and Spalding (1974) are widely used as shown in equation (6.7) and (6.8) respectively. In this thesis, the equation (6.7) is used in the thermal standard wall function.

$$P = 9.24 \left[\left(\frac{\Pr}{\Pr_t} \right)^{3/4} - 1 \right] \left[1 + 0.28e^{-0.007 \Pr/\Pr_t} \right]$$
(6.7)

$$P = 9.0 \left[\left(\frac{\Pr}{\Pr_t} \right) - 1 \right] \left(\frac{\Pr_t}{\Pr} \right)^{1/4}$$
(6.8)

In order to get the turbulent kinetic energy at point P, the transport equation of k should be solved in the near-wall cell. Since the production and dissipation of k can both vary significantly in this region, suitable approximations should be made for the cell-averaged source and sink terms:

$$\bar{P}_k = \frac{1}{y_n} \int_0^{y_n} P_k dy \quad \bar{\varepsilon} = \frac{1}{y_n} \int_0^{y_n} \varepsilon dy \tag{6.9}$$

 \bar{P}_k and $\bar{\varepsilon}$ are then used in discretizing the kinetic energy equation. Using assumptions based on those above, of a simple boundary layer in local equilibrium, the cell-averaged generation and dissipation rates of *k* over the near-wall cell can be approximated as:

$$\bar{P}_{k} = \frac{\tau_{w}^{2}}{\kappa c_{\mu}^{1/4} \rho k_{p}^{1/2} y_{n}} \log\left(y_{n}/y_{v}\right)$$
(6.10)

$$\bar{\varepsilon} = \frac{1}{y_{\mathrm{n}}} \left[\frac{2k_{\mathrm{p}}}{y_{\mathrm{v}}/v} + \frac{k_{\mathrm{p}}^{3/2}}{c_{l}} \log\left(y_{\mathrm{n}}/y_{\mathrm{v}}\right) \right]$$
(6.11)

Typically, $y_{\nu}^* = y_{\nu} k_{\nu}^{1/2} / \nu$ is taken as 20. The wall shear stress is evaluated at the near-wall node P from the log-law equation (6.5) as:

$$\tau_{\rm w} = \frac{\rho \kappa^* U_P k_P}{\log \left(E^* y_P^* \right)} \tag{6.12}$$

The cell-averaged production and dissipation rates were calculated by assuming constant shear stress and a linear turbulence length scale variation across the near-wall cell.

The main disadvantage of standard wall functions is grid-dependence of the near wall cell, especially when the near wall cell lies within the viscous sublayer. To force the usage of the log law in conjunction with the standard wall functions approach, Grotjans and Menter (1998) recommend to the scalable wall function by introducing a limiter in the calculation of y^* as:

$$y_s^* = max(y^*, y_v^*)$$
(6.13)

The use of scalable wall functions is straightforward by replacing the y^* in the standard wall functions.

6.2 Simple Analytical Wall Function

All the forms of wall functions described above are strongly linked to the log-laws of velocity and temperature in the near-wall region. Smith (1990) developed a novel wall function for the two-equation k - kl model and used this approach to simulate compressible flow involving shock and boundary layer interaction and separation. Boyer and Laurence (2002) proposed a new approach to bridge the gap between low- and high-Reynolds number turbulence modelling. The shape functions, which consisted of the Reichard law for velocity and profiles to match channel-flow DNS, were used to approximate the distribution of velocity, kinetic energy, and dissipation rate of kinetic energy in the near-wall region. But more complex flows are needed to test this approach.

Iacovides et al. (1984) developed The Parabolic Sublayer (PSL) approach which employed a low-Reynolds-number model on a fine grid but assumed that the static pressure stayed constant in a thin layer near the wall. The wall-normal velocity was thus calculated from continuity instead of employing the pressure correction in this region. More advanced wall functions, not based on the assumption of log-laws, have been developed by Gerasimov (2004) and Gant (2003), and these are referred to as the Analytical Wall Function (AWF) and Subgrid-Based Wall Function UMIST-*N* respectively. The AWF is the approach that has been tested in the present work, and is therefore described in detail below. In Chapter 8 specific refinements that have been developed within the present project to extent the approach to compressible flows will be described.

6.2.1 Assumptions of Analytical Wall Function (AWF)

The AWF is based on the analytical solution of the simplified Reynolds equations and takes convection and pressure gradients into account. In the near wall region, the transport equations are simplified by the boundary layer assumptions, which are

- 1. Diffusion of momentum in the direction normal to the wall is significantly greater than that parallel to the wall, so the latter term is neglected from the transport equation.
- 2. The pressure gradient parallel to the wall is constant across the near-wall cells.

Then the simplified momentum and temperature Reynolds equations for a forced convection flow in the near wall x-y plane can be written as:

$$\frac{\partial \left(\rho UU\right)}{\partial x} + \frac{\partial \left(\rho VU\right)}{\partial y} = -\frac{dP}{dx} + \frac{\partial}{\partial y} \left[\left(\mu + \mu_t\right) \frac{\partial U}{\partial y} \right]$$
(6.14)

$$\frac{\partial \left(\rho UT\right)}{\partial x} + \frac{\partial \left(\rho VT\right)}{\partial y} = \frac{\partial}{\partial y} \left[\left(\frac{\mu}{\Pr} + \frac{\mu_t}{\Pr_t}\right) \frac{\partial T}{\partial y} \right]$$
(6.15)

Two additional assumptions are made to solve the above equations analytically:

1. A simple variation of the turbulent viscosity is prescribed.



Figure 6.2: Sketch map of turbulent viscosity

2. The convection terms, which are normal and parallel to the wall, are approximated from nodal variable values (and initially assumed to be constant across the near-wall control volume).

A simple prescription of turbulent viscosity is shown in Fig. 6.2, and can be expressed as:

$$\mu_t = 0 \quad \text{when} \quad y < y_v \tag{6.16}$$

$$\mu_{t} = \mu \alpha \left(y^{*} - y_{v}^{*} \right) = \mu c_{\mu} c_{l} \left(y^{*} - y_{v}^{*} \right) \quad \text{when} \quad y_{n} > y > y_{v}$$
(6.17)

where c_{μ} and c_l are the same as described in the one-equation turbulence model, so $\alpha = c_{\mu}c_l = 0.2295$. The dimensionless wall distance is defined as $y^* = yk_P^{1/2}/v$. The simplification assumptions described above make it possible to solve the momentum and energy equation analytically. Here we only focus on the non-buoyant flow, although the original proposal of Gerasimov (2004) did consider the application to buoyancy-affected flows.

6.2.2 Simple Hydrodynamic Analytical Wall Function

Equation (6.14) can be rewritten using the dimensionless wall distance, y^* as:

$$\frac{\partial}{\partial y^*} \left[(\mu + \mu_t) \frac{\partial U}{\partial y^*} \right] = C \tag{6.18}$$

where

$$C = \frac{\mu^2}{\rho^2 k_P} \left[\frac{\partial \left(\rho UU\right)}{\partial x} + \frac{\partial \left(\rho VU\right)}{\partial y} + \frac{dP}{dx} \right]$$
(6.19)

C is the term which includes the effect of the pressure gradient and convection.

Thus in the viscous sublayer $(y^* < y^*_v)$, the first integration of equation (6.18) using equation (6.16) is:

$$\mu \frac{\partial U_1}{\partial y^*} = C_1 y^* + A_1 \tag{6.20}$$

where

$$C_1 = \frac{\mu^2}{\rho^2 k_P} \left[\frac{\partial \left(\rho UU\right)}{\partial x} + \frac{\partial \left(\rho VU\right)}{\partial y} + \frac{dP}{dx} \right]$$
(6.21)

The second integration of equation (6.18) gives the final expression of the velocity within the inner sublayer:

$$\mu U_1 = \frac{C_1}{2} y^{*2} + A_1 y^* + B_1 \tag{6.22}$$

In the fully turbulent region $(y^* > y_v^*)$, after the first integration the derivative of velocity is given as:

$$\mu \frac{\partial U_2}{\partial y^*} = \frac{C_2 y^* + A_2}{Y} \tag{6.23}$$

where

$$Y = 1 + \alpha \left(y^* - y_v^* \right)$$
(6.24)

$$C_2 = \frac{\mu^2}{\rho^2 k_P} \left[\frac{\partial \left(\rho UU\right)}{\partial x} + \frac{\partial \left(\rho VU\right)}{\partial y} + \frac{dP}{dx} \right]$$
(6.25)

A second integration gives the final expression of the velocity in the fully turbulent region:

$$\mu U_2 = \frac{C_2}{\alpha} \left[y^* - \left(\frac{1}{\alpha} - y_v^*\right) \ln Y \right] + \frac{A_2}{\alpha} \ln Y + B_2$$
(6.26)

In order to obtain the integration constants A_1 , A_2 , B_1 and B_2 , it is reasonable to assume that:

- 1. U_1 is zero at the wall;
- 2. U_2 is equal to U_n at the edge of the near-wall cell;
- 3. At the edge of viscous sublayer, the following continuity conditions should be guaranteed

$$U_1|_{y^*=y^*_v} = U_2|_{y^*=y^*_v}$$
(6.27)

$$\left. \frac{\partial U_1}{\partial y^*} \right|_{y^* = y_v^*} = \left. \frac{\partial U_2}{\partial y^*} \right|_{y^* = y_v^*} \tag{6.28}$$

Applying these conditions, the constants of integration are obtained as

$$A_1 = \frac{\mu_v U_n - N}{\left[\frac{\ln Y_n}{\alpha} + y_v^*\right]} \tag{6.29}$$

$$\begin{bmatrix} -\alpha & +y_{\nu} \end{bmatrix}$$

$$B_1 = 0 \tag{6.30}$$

$$A_2 = (C_1 - C_2) y_v^* + A_1 \tag{6.31}$$

$$B_2 = y_{\nu}^* \left(\frac{C_1}{2} y_{\nu}^* - \frac{C_2}{\alpha} \right) + A_1 y_{\nu}^*$$
(6.32)

where

$$N = \frac{C_2}{\alpha} \left[y_n^* - \left(\frac{1}{\alpha} - y_v^*\right) \ln Y_n \right] + \frac{(C_1 - C_2)}{\alpha} y_v^* \ln Y_n + \left(\frac{C_1}{2} y_v^* - \frac{C_2}{\alpha}\right) y_v^*$$
(6.33)

with

$$Y_n = 1 + \alpha \left(y_n^* - y_v^* \right)$$
(6.34)

Finally, the wall shear stress can be obtained

$$\tau_{wall} = -\mu \frac{\partial U}{\partial y}\Big|_{y=0} = -\frac{\rho \sqrt{k_P}}{\mu} \left[\mu \frac{\partial U}{\partial y^*}\right]_{y^*=0} = -\frac{\rho \sqrt{k_P}}{\mu} A_1$$
(6.35)

An approximation for the cell-averaged production of turbulence kinetic energy can be obtained by

$$\overline{P_k} = -\frac{1}{y_n} \int \overline{uv} \frac{\partial U}{\partial y} dy = -\frac{1}{y_n} \frac{\rho \sqrt{k_P}}{\mu} \int_{y_v^*}^{y_n^*} \mu \alpha \left(y^* - y_v^* \right) \left(\frac{\partial U_2}{\partial y^*} \right)^2 dy^*$$
(6.36)

This term is needed for the kinetic energy transport equation to represent the production of the kinetic energy in a consistent way.

6.2.3 The Expression for the Mean Dissipation Rates

The expression of mean dissipation rate across the cell has been also modified. Jones and Launder (1972) assumed the kinetic energy k is varied with a proportion to y^2 within the viscous sublayer, where it decreases to zero at the wall and increases to k_P at the edge of viscous sublayer. In the fully turbulent region, k is assumed to be constant as the nodal value k_P . Unlike the kinetic energy, the dissipation rate of turbulence energy is not zero at the wall. In the viscous sublayer, Chieng and Launder (1980) had assumed:

$$\varepsilon = 2v \left(\frac{\partial k^{1/2}}{\partial y}\right)^2 \approx \frac{2vk}{y^2} = \frac{2vk_P}{y_v^2}$$
(6.37)

In the fully turbulent region, the dissipation rate is evaluated by:

$$\varepsilon = \frac{k^{3/2}}{c_l y} \tag{6.38}$$

A universal constant c_l is equal to 2.55. Thus the expression for the total average dissipation rate integrates as equation (6.11) for the standard wall function.

In order to avoid the discontinuity of dissipation rate of turbulence kinetic energy in Fig. 6.3 (a), a position is selected as in Fig. 6.3 (b) to make ε continuous at the surface.

$$\frac{k_P^{3/2}}{c_l y_d} = \frac{2\nu k_P}{y_d^2} \quad \text{or} \quad y_d^* = 2c_l = 5.1$$
(6.39)

The mean dissipation rate can then be obtained by integrating this two-part variation to obtain:

$$\bar{\varepsilon} = \frac{1}{y_n} \left[\frac{2k_p^{3/2}}{y_d^*} + \frac{k_p^{3/2}}{2.55} \ln\left(\frac{y_n}{y_d}\right) \right]$$
(6.40)

There is now only one remaining constant, namely the non-dimensional viscosity sublayer thickness y_v^* . From the numerical experiments for fully developed pipe flow, this value is determined as 10.1.



Figure 6.3: Sketch map of ε for (a) the standard and (b) the proposed wall function

6.2.4 Simple Thermal Analytical Wall Function

Equation (6.15) can be rewritten using the dimensionless wall distance:

$$\frac{\partial}{\partial y^*} \left[\left(\frac{\mu}{\Pr} + \frac{\mu_t}{\Pr_t} \right) \frac{\partial T}{\partial y^*} \right] = C_{th}$$
(6.41)

where

$$C_{th} = \frac{\mu^2}{\rho^2 k_P} \left[\frac{\partial \left(\rho UT\right)}{\partial x} + \frac{\partial \left(\rho VT\right)}{\partial y} \right]$$
(6.42)

The same procedure is then carried out as in the case of the momentum equation, and the following equations can be obtained:

In the viscous sublayer $y^* < y_v^*$

$$T_{1} = \frac{\Pr}{\mu} \left[\frac{C_{th1} y^{*2}}{2} + A_{th1} y^{*} \right] + T_{wall}$$
(6.43)

where

$$C_{th1} = \frac{\mu^2}{\rho^2 k_P} \left[\frac{\partial \left(\rho UT \right)}{\partial x} \right]$$
(6.44)

and

$$A_{th1} = -\frac{q_{wall}}{c_p} \frac{\mu^2}{\rho \sqrt{k_P}} \tag{6.45}$$

In the fully turbulent region $y^* > y^*_v$

$$T_{2} = \frac{\Pr}{\mu \alpha_{t}} C_{th2} \left(y^{*} - y_{v}^{*} \right) + \frac{\Pr}{\mu \alpha_{t}} \left[A_{th1} + C_{th1} y_{v}^{*} - \frac{C_{th2}}{\alpha_{t}} \right] \ln Y_{T} + \frac{\Pr y_{v}^{*}}{\mu} \left[\frac{C_{th1}}{2} y_{v}^{*} + A_{th1} \right] + T_{wall}$$
(6.46)

where

$$Y_T = 1 + \alpha_t \left(y^* - y_v^* \right); \quad \alpha_t = \frac{\alpha \operatorname{Pr}}{\operatorname{Pr}_t}$$
(6.47)

$$C_{th2} = \frac{\mu^2}{\rho^2 k_P} \left[\frac{\partial \left(\rho UT\right)}{\partial x} \right]$$
(6.48)

The wall temperature, needed to complete the thermal wall function, can be obtained from the above as:

$$T_{wall} = T_n - \frac{\Pr_{v}}{\mu_{v}} \left[\frac{C_{th2}}{\alpha_t} \left(y_n^* - y_v^* \right) + \frac{\ln Y_{Tn}}{\alpha_t} \left(A_{th1} + C_{th1} y_v^* - \frac{C_{th2}}{\alpha_t} \right) + y_v^* \left(\frac{C_{th1}}{2} y_v^* + A_{th1} \right) \right]$$
(6.49)

and the wall heat flux is related to the wall temperature by following expression:

$$q_{wall} = -\frac{\rho c_p \sqrt{k_P}}{\mu} A_{th1} \tag{6.50}$$

where

$$A_{th1} = \frac{(T_n - T_{wall})\frac{\mu}{\Pr} - \frac{1}{\alpha_t}C_{th2}(y_n^* - y_v^*) - \frac{1}{\alpha_t}\left(C_{th1}y_v^* - \frac{C_{th2}}{\alpha_t}\right)\ln Y_{Tn} - \frac{C_{th2}}{2}y_v^{*2}}{\frac{1}{\alpha_t}Y_{Tn} + y_v^*}$$
(6.51)

and

$$Y_{Tn} = 1 + \alpha_t \left(y_n^* - y_v^* \right)$$
(6.52)

The AWF has been applied to simulate the forced, mixed and natural convection flows by Gerasimov (2004) with the extension of the AWF to buoyant flows. Comparing with the conventional Low-Re models, the computation time is significantly reduced, and its accuracy is generally better than when using standard log-law based wall functions. K. Suga, Craft, and Iacovides (2006) extended the AWF approach to the flows with attachment and separation over smooth and rough walls.



Figure 6.4: Subgrid arrangement within near-wall main-grid control volume

6.3 Subgrid-Based Wall Function (UMIST-N)

This new wall function does not use any assumed profiles of velocity or length scale, in contrast to the log-law based wall functions. Instead, a fine 'subgrid' covering the wall-adjacent control volume (see Figure 6.4) is used to obtain the mean and turbulence parameters by solving simplified boundary-layer-type transport equations. This treatment decouples the numerical solution of the near wall region from that of the main region, and the pressure-correction equation over the subgrid is avoided. So the low convergence problems sometimes associated with the use of very high aspect ratio cells with low Reynolds number models are avoided.

The convection (both parallel and normal to the wall), pressure gradient, diffusion normal to the wall and source terms are taken into account, and the transport equations are solved by the wall function across the subgrid. Gant (2003) tested both a linear and a non-linear k- ε model applied across the subgrid, combined with equations for the wall-parallel velocity components and temperature (where a thermal field is solved). In each maingrid iteration, one subgrid interaction was performed, then the average of production and dissipation of kinetic energy, the wall shear stress and the wall heat flux (or wall temperature) were calculated. The subgrid wall function is called UMIS-N: Unified Modelling through Integrated Sublayer Treatment – a Numerical approach.

In order to simplify the transport equations, assumptions are applied within the subgrid:

1. Only the momentum equations parallel to the wall are solved;

- 2. The diffusion parallel to the wall is negligible compared to that normal to the wall;
- 3. The pressure gradient is assumed to be constant across the near-wall main-grid cell.

For illustration, the simplified 2-dimensional low-Re transport equations in Cartesian tensor form can be expressed below, where y is the wall-normal direction. Wall parallel *U*-momentum

$$\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} = -\frac{dP}{dx} + \frac{\partial}{\partial y} \left[\mu \frac{\partial U}{\partial y} - \rho \overline{uv} \right]$$
(6.53)

Turbulent kinetic energy k

$$\rho U \frac{\partial k}{\partial x} + \rho V \frac{\partial k}{\partial y} = \frac{\partial}{\partial y} \left[\left(\mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial y} \right] + P_k - \rho \varepsilon$$
(6.54)

Isotropic dissipation rate

$$\rho U \frac{\partial \tilde{\varepsilon}}{\partial x} + \rho V \frac{\partial \tilde{\varepsilon}}{\partial y} = \frac{\partial}{\partial y} \left[\left(\mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial \tilde{\varepsilon}}{\partial y} \right] + c_{\varepsilon 1} f_1 P_k \frac{\tilde{\varepsilon}}{k} - c_{\varepsilon 2} f_2 \rho \frac{\tilde{\varepsilon}^2}{k} + \rho Y_c + P_{\varepsilon 3}$$
(6.55)

Temperature T

$$\rho U \frac{\partial T}{\partial x} + \rho V \frac{\partial T}{\partial y} = \frac{\partial}{\partial y} \left[\left(\frac{\mu}{\sigma} + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial y} \right]$$
(6.56)

The production of turbulent kinetic energy, used in equations (6.54) and (6.55), can be given without buoyancy or other fluctuating force fields by:

$$P_{k} = -\rho \overline{u_{i}u_{j}} \frac{\partial U_{i}}{\partial x_{j}} = -\rho \overline{uv} \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) - \rho \overline{u^{2}} \frac{\partial U}{\partial x} - \rho \overline{v^{2}} \frac{\partial V}{\partial y}$$
(6.57)

6.3.1 Lauder-Sharma k-ε model

The Reynolds stress is assumed to be a linear function of the mean strain rate:

$$-\overline{u_i u_j} + \frac{2}{3} k \delta_{ij} = v_t S_{ij} \tag{6.58}$$

where the strain-rate tensor is given by:

$$S_{ij} = \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}$$
(6.59)

and the kinetic eddy-viscosity is

$$\mathbf{v}_t = c_\mu f_\mu \frac{k^2}{\tilde{\varepsilon}} \tag{6.60}$$

Substituting these two expressions into equation (6.53), we can obtain:

$$\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} = -\frac{dP}{dx} + \frac{\partial}{\partial y} \left[(\mu + \mu_t) \frac{\partial U}{\partial y} \right]$$
(6.61)

The gradient production term in equation (6.55) is simplified:

$$P_{\varepsilon 3} = 2\mu v_t \left(\frac{\partial^2 U_i}{\partial x_j \partial x_k}\right)^2 \approx 2\mu v_t \left(\frac{\partial^2 U}{\partial y^2}\right)^2 \tag{6.62}$$

6.3.2 Non-Linear $k - \varepsilon$ model

Gant (2003) tested the UMIST-N procedure in conjunction with the two-equation NLEVM of Craft et al. (2000) have prescribed this turbulence model to obtain the maximum of the numerical stability, which lead to the following expression for the subgrid U-momentum

$$\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} = -\frac{dP'}{dx} + \frac{\partial}{\partial y} \left[\left(\mu + \mu_t' \right) \frac{\partial U}{\partial y} - \rho \overline{uv} \right]$$
(6.63)

where μ_t is the modified eddy-viscosity

More details about the implementation and test case could be found in Gant (2003) and Craft et al. (2004). From the numerical tests of impinging jet and rotation disc flows, this new wall-function showed good agreements with the results of Low-Re turbulence models, while the computation time was an order of magnitude less than the standard Low-Re treatments.

6.4 Wall Functions for Compressible Flows

For the compressible flows, the most widely-used wall function is the standard wall function by considering the variation of density which is adopted in FLUENT software. The uncoupled thermal log-law of wall function is commonly used in the compressible solvers which neglect the coupling effect of velocity and temperature. Therefore, errors happen especially for the complex hypersonic flows where the temperature increases significantly in a thin region near the wall. Nichols and Nelson (2004) developed a wall function with the consideration of the coupled velocity and temperature profiles based on the six fundamental assumptions:

- 1. Analytical expressions of velocity and temperature are available in the lower part of the boundary layer.
- 2. Analytical expressions of the turbulent transport variables are available at the first cell off the wall.
- 3. The pressure gradient normal to the wall is equal to zero.
- 4. The shear stress is constant in the lower part of the boundary layer.
- 5. The wall heat flux is constant in the lower part of the boundary layer.
- 6. There is no chemical reaction in the lower part of the boundary layer.

When applying the standard wall function, it is necessary to avoid the near wall cell falling into the viscous sublayer. Spalding (1961) suggested a unified form for the viscous sublayer and fully turbulent region, which is:

$$y^{+} = u^{+} + e^{-\kappa u^{+}} e^{\kappa u^{+}} - e^{-\kappa B} \left(1 + \kappa u^{+} + \frac{(\kappa u^{+})^{2}}{2} + \frac{(\kappa u^{+})^{3}}{2}\right)$$
(6.64)

where the constants κ and B is taken as 0.4 and 5.5.

Nichols intrduced the compressible velocity law of the wall by White and Christoph (1971) to consider compressibility and heat transfer effects in high-speed flows:

$$u^{+} = (1/2\Gamma) \left\{ \beta + Q \sin\left[\phi + \frac{\sqrt{\Gamma}}{\kappa} \ln\left(\frac{y^{+}}{y_{0}^{+}}\right)\right] \right\}$$
(6.65)

where

$$\Gamma = \frac{ru_{\tau}^2}{2c_p T_w}, \beta = \frac{q_w \mu_w}{\rho_w T_w k_w u_{\tau}}, \phi = \sin^{-1}\left(\frac{-\beta}{Q}\right), Q = (\beta^2 + 4\Gamma)^{1/2}, y_0^+ = e^{-\kappa B}$$

and $r = (Pr)^{1/3}$ is the recovery factor.

Nichols replaced the incompressible factor $e^{-\kappa u^+}e^{\kappa u^+}$ in equation (6.63) by equation (6.64), and the unified wall function including the effects of compressiblity and heat transfer can be obtained as:

$$y^{+} = u^{+} + y^{+}_{white} - e^{-\kappa B} \left(1 + \kappa u^{+} + \frac{(\kappa u^{+})^{2}}{2} + \frac{(\kappa u^{+})^{3}}{2}\right)$$
(6.66)

where

$$y_{white}^{+} = exp\left(\left(\kappa/\sqrt{\Gamma}\right)\left\{\sin^{-1}\left[\left(2\Gamma u^{+} - \beta\right)/Q\right] - \phi\right\}\right) \times e^{-\kappa B}$$
(6.67)

The temperature equation is given by Crocco-Busemann equation as described by White and Christoph [1971], which is:

$$T = T_w [1 + \beta u^+ - \Gamma(u^+)^2]$$
(6.68)

For the adiabatic wall boundary conditions, $q_w = 0$ and $\beta = 0$, then the Crocco-Busemann equation reduces to

$$T = T_w - \frac{ru^2}{2c_p} \tag{6.69}$$

This new wall function includes the effect of compressibility and heat transfer, which makes it possible to obtain wall shear stress and heat transfer in high-speed flows. Afterward, some researchers, such as Nichols and Nelson (2004) and Lee and Stephen (2007), have applied this wall function in the calculation of 2-D high-speed flows.

Abdol-Hamid et al. (1995) took another simple approach of wall function for the compressible flow in the prediction of wall pressure and wall skin friction. This approach modified the turbulent viscosity at the wall only and was applied on a coarse grid with $30 < y^+ < 200$, which is adopted by the commercial software, CFL3D.

6.5 Summary

All the wall functions mentioned above construct a solution in the near wall cells to transfer the wall boundary condition to an interface boundary as summarized by Utyuzhnikov (2005). The standard wall functions, which belong to the log-law type, have been introduced first. The scalable wall function is also introduced in order to reduce the grid dependence. A more advanced wall function, which is based on the simplified Reynolds equations including contributions from the convection and pressure gradient terms, have then been described, namely the Analytical Wall Function (AWF). Another advanced wall function does not involve so many assumptions is also introduced, which is the Subgrid-based wall function (UMIST-N). The dependent variables across the near wall region are determined by solving simplified transport equations in a subgrid construsted across the near wall cell. Although the subgrid wall function is belived to be more accurate than the analytical wall function, it suffers from stability problems especially in the calculation of complex flows. All these two advanced approaches have mainly been developed for incompressible flows, and the main emphasis of the present work has thus been to test the analytical wall function in compressible SWTBLI flows and propose suitable refinements for such flows. Finally, a wall function designed for the compressible flows is introduced, which includes the effect of compressibility and heat transfer.

Chapter 7

Results of 2D Supersonic Flows

In this chapter, the standard wall function and AWF formulations described above have been applied to a number of 2D SWTBLI flows. Results are compared with those from the Low-Reynolds-number Launder-Sharma k- ε model, and available experimental/DNS data, in order to assess their performance prior to considering any further refinements of the AWF for high speed flows. The experimental test case of an impinging shock has been reported by Reda and Murphy (1973), and is used here to test the wall function treatments and their implementation in OpenFOAM v2.3.1. The shock turning angles are 13° , 10° , and 7° , to obtain different separation zones depending on the impinging shock strength. This case is at an incoming Mach number of 2.9 and Re $_{\theta}$ of 47000, and will be referred to here as the High-Reynolds-number (HR) case. Another case will be referred to as the Low-Reynolds-number (LR) case from the DNS of Pirozzoli and Grasso (2006) at an Mach number of 2.25, the turning angle 8° and Re $_{\theta}$ of 3725, and the experiment data at Re $_{\theta}$ of 5350 by Deleuze (1995) and Laurent (1996).

As a second set of cases considered here, the compression corner is a widely used case of shock wave/turbulent boundary layer interaction for validation of RANS models. It consists of a flat plate, followed by a ramp inclined at an angle. The study of Settles and Dodson (1994), for example, considered ramp angles of 16, 20, and 24 at an incoming Mach number of 2.85. For inviscid supersonic flow, the compression corner only generates a single oblique shock at the corner due to flow compression, while for viscous flow, this shock will interact with the upstream turbulence boundary layer, and flow separation will happen when the shock is strong enough.

7.1 Results of High-Reynolds-Number Impinging Shock

The working medium is assumed to be a perfect gas, and air properties, such as the gas constant $R=287.06m^2S^{-2}K^{-1}$, specific heat coefficients for constant pressure $C_p=1004.06J/(kgK)$ and molecular Prandtl number Pr=0.7 are used in the calculation. Sutherland's law, $\mu = 1.458 \times 10^{-6} \text{ T}^{3/2}$ /(T+110.3), is used to evaluate the dynamic viscosity. The computational domains and a sample grid are shown in Fig. 7.1, where LR denotes the low-Re case and HR denotes the high-Re case.



Figure 7.1: Computational domains for 2-D supersonic impinging shock interaction from Asproulias (2014)

The lower part of inlet boundary profiles is a developed turbulent boundary layer from the calculation of flat plate flow at momentum thickness θ_0 =0.082cm with 1.5% freestream turbulence intensity and a ratio of μ_t/μ =10. In the upper part of the inlet boundary which is listed in Table 7.1, the inviscid oblique shock relation equations in Anderson (2010) are used to calculate the flow variables downstream of the shock for the given turning angle 13°, 10°, and 7°, and the results are listed in Table 7.2. At the wall, an isothermal no-slip wall boundary condition is used, and the wall temperature is fixed at 271K. At top and outlet boundaries, Neumann conditions are used for all components.

To test the grid independence, Fig. 7.2 shows results obtained with both wall function approaches on grids ranging from 100×50 to 200×100 , with the near-wall size 1.0×10^{-4} m. The dimensionless wall distance y+ is about 15 at the upstream to guarantee the near-wall cell out of the viscous sublayer. The results indicate that the used meshes give grid-independent

2.9 47000 0.082 1.69 689010 291 271	M_{∞}	$Re_{\theta 0}$	θ_0 , cm	δ_0 , cm	$p_{t0}, N/m^2$	T_{t0}, \mathbf{K}	T_w, \mathbf{K}
	2.9	47000	0.082	1.69	689010	291	271

Table 7.1: Inflow conditions for the high-Re case

Table 7.2: After oblique shock elements for the high-Re case

$\beta(^{\circ})$	<i>M</i> ₂	<i>u</i> _{2<i>x</i>} , m/s	<i>u</i> _{2y} , m/s	p ₂ , <i>N/m</i> ²	<i>T</i> ₂ , K
13	2.28	531.79	122.77	53069	142.73
10	2.42	553.34	97.57	43929	133.85
7	2.56	571.99	70.28	36023	125.69

solutions. For the Low-Re Launder-Sharma model, a 200×90 mesh is used with the first cell size about 4.0×10^{-6} m and the y+ around 0.6 at the upstream.



Figure 7.2: Grid independency study for the high Reynolds number 13° impinging shock interaction using $k - \varepsilon$ model with Standard Wall Function (SWF) and Analytical Wall Function (AWF)

Figure 7.5 shows iso-lines of mean pressure superimposed on contours of Mach number at impinging angles of 13° , 10° and 7° by the *k*- ε model with standard wall function on the finest grid. As the shock angle increases, the shock strength and size of the separation bubble increase, the thickness of the boundary layer due to the SWBLI is enhanced, and the structure of the reflected shock is shifted further upstream of the inviscid impingement point, which is related to the angle of shock generator.

For a quantitative assessment of the wall-functions, Figure 7.6 compares the numerical results of wall pressure and skin friction with experiment. For the 7° case, the wall pressure

predicted by all models agrees well with the experimental measurements. There is no separation from the numerical or experimental results indicated by the skin friction C_f . The results of the k- ε model with standard wall function have the weakest interaction strength, which is indicated by the minimum point of skin friction, while the results of the k- ε model with analytical wall function have a stronger interaction strength, which is closer to the experimental data.

For the 10° case, all the models return similar wall pressure, which fits the experimental data well. From the skin friction results, the numerical and experimental data show that there is a small separation zone near the inviscid impingement point. The results of the Lauder-Sharma (LS) model are in good agreement with the experiment, the results of the k- ε model with analytical wall function are close to the results of the low-Re number model (LS), and the standard wall function approach tends to underestimate the skin friction in the interaction zone.

For the 13° case, generally, all the models give a good prediction of the wall pressure, but slight differences are present near the beginning of the interaction. The LS model tends to predict the start point of interaction closer to the experiment than the wall function approaches. The standard wall function approach gives slightly closer results to the LS model than the analytical wall function approach, but the two different wall function approaches do not show much difference. From the skin friction, the results of all the tested near-wall treatments fit the experiment well, and bigger separation bubble is predicted by the LS model, while the wall function approaches give similar size to each other. From all the comparisons, the skin friction of the analytical wall function approach tends to give closer results to the LS model, and is closer to the experiment, while the standard wall function approach tends to underestimate the strength of interaction.

Figure 7.7 displays comparisons of the mean velocity profiles at eight different streamwise locations for the 13° case. In general, all the models give good predictions at most locations. Locations 1 and 8 are located outside of the separation zone, and similar velocity results are predicted. At location 2, the LS model predicts the flow has separated, while the results of the wall function approaches and experiment still show unseparated flow. At locations 3, 4 and 5, the numerical results tend to underestimate the streamwise velocity, while at locations 6 and 7, they tend to overestimate it. Figure 7.8 shows the corresponding comparison of the streamwise Reynolds stress at the same eight different streamwise locations as in Figure 7.7. From the results, the numerical treatments underestimate the Reynolds streamwise stress at all eight locations, especially in the middle of the separation zone. In the EVMs, the streamwise Reynolds stress is obtained by $\widetilde{u''u''} = (2/3)k$ for most of these flow locations, where the flow

is shear-dominated. The underestimates of the streamwise stress in a boundary-layer type flow is usually because that the normal stresses will not be isotropic.

7.2 Results of Low-Reynolds-Number Impinging Shock

The DNS inflow conditions for this case are listed in Table 7.3. The oblique shock relation equations were again used to calculate the elements after the impinging shock, and the results are listed in Table 7.4. As in the previous case, these were used in the upper part of the inlet boundary, and the lower part of the inlet boundary profiles were a developed turbulent boundary layer from the calculation of flat plate flow at momentum thickness $\theta_0=0.147$ mm with 1.5% freestream turbulence intensity and a ratio of $\mu_t/\mu=10$.

Table 7.3: Inflow conditions for the low-Re case

M_{∞}	$Re_{\theta 0}$	θ_0 , cm	δ_0 , cm	$p_{\infty}, N/m^2$	<i>T</i> ∞, K
2.25	3725	0.147	2	23813	169.4

The computational domain was shown in Figure 7.1. Structured grids were again used, similar to those of the previous case, but with the near-wall grid adjusted to account for the much lower Reynolds number. Near-wall grid cell spacings were around y+=15 for the high-Re k- ε model with wall function approaches, and y+=0.6 for the LS model. The boundary conditions are similar to those applied for the high Reynolds impinging shock interaction, except an adiabatic condition is used for the wall temperature. The grid-independence studies in Figure 7.9 show that the k- ε model using analytical wall function and standard wall function converge when the grid number increases.

Figure 7.10 displays the Mach number and turbulent kinetic energy contours from the high-Re k- ε model using the analytical wall function (AWF) and standard wall function (SWF) and the LS k- ε model with the Yap correction (LS). The main structure of the flowfields, which contain an impinging shock, a reflected shock and a small separation bubble in between, has no obvious difference between all three models. The LS model tends to give larger interaction region.

Figures 7.11 and 7.12 display the wall pressure and skin friction comparison with the DNS results and experiment. The LS predicts the wall pressure at the start of interaction well

Table 7.4: After oblique shock elements for the low-Re case

M_2	<i>u</i> _{2<i>x</i>} , m/s	<i>u</i> _{2<i>y</i>} , m/s	p ₂ , <i>N/m</i> ²	<i>T</i> ₂ , K	
1.94	536.97	76.28	37984	193.8	

with the DNS and experiment and returns the similar results as the wall function approaches at the downstream. The SWF approach fails to predict a separation zone, while the LS model returns a larger separation zone than the DNS results, and the AWF approach returns a slightly smaller zone than the DNS, although the beginning of the interaction and separated flow region is located a little downstream of the DNS results, as can be seen in both the wall pressure and skin friction plots. However, the reattachment point of the AWF fits well with the LS and both appear to be fairly close to the experimental results. Downstream from the interaction zone, all the three models capture the recovery of skin-friction, and the AWF and LS are in good agreement with the DNS results, though they are a bit further from the experimental results.

Figure 7.13 shows the contours of mean streamwise velocity predicted by the k- ε model with AWF and SWF and the LS k- ε model with the Yap correction (LS), and the locations of six lines at which profiles will be compared in the following figures. xI is the inviscid impinging point, x_s is the start of separation, and x_R is the reattachment point. Figure 7.14 displays comparisons of mean streamwise velocity profiles for the low Reynolds number impinging shock at these six locations. At location 1, which is located upstream of the interaction zone, the undisturbed turbulent boundary layers by using the RANS models are in good agreement with the DNS one. At locations 4 and 6, downstream of the separation point, the LS model predicts a thicker boundary layer, while the AWF and SWF give a comparable thickness to the DNS one. At location 9, the velocity distribution is quite similar to location 6. At locations 10 and 13, downstream of the reattachment point, the DNS data.

Figure 7.15 displays a comparison of kinetic energy profiles at the above six locations. At locations 1 and 13, all three models return the similar profiles to each other. Within or near the separation (locations 4, 6, 9, 10), the AWF is closer to the results of the LS model. Figure 7.16 displays the dissipation rate ε at six locations. From the comparison, all three models fit each other well, and the AWF is closer to the LS model, compared with the SWF.

7.3 Results of 2D Compression Corner

A perfect gas is assumed as the working medium. The inflow conditions by Smits and Muck (1987) are summarized in Table 7.5. Structured grids are used for this simple geometry, refined to the corner in the streamwise direction and refined normal to the wall, and the value of y+ at the near-wall cell centres is around 15 for the high-Re turbulence model with wall functions, and 0.6 for the low-Re turbulence model. The computational domain and a sample grid are shown in Fig. 7.3.

β	M_{∞}	$Re_{\theta 0}$	θ_0 , cm	δ_0 , cm	$p_{t\infty}, N/m^2$	$U_{\infty},m/s$	<i>T</i> ∞, K	T_w, \mathbf{K}
16°	2.85	81900	0.13	2.6	2.29×10^{4}	576	102.1	282
20°	2.85	81900	0.13	2.5	2.32×10^{4}	562	98.3	274
24°	2.84	75600	0.12	2.3	2.36×10^{4}	569	100.3	276

Table 7.5: Inflow conditions for compression corner



Figure 7.3: Computational domains for 2-D Compression Corner

The inlet boundary profiles are those of developed boundary, obtained from a separate flat plate calculation, at momentum thickness θ_0 =0.082cm with 1.5% freestream turbulence intensity and a ratio of μ_t/μ =10. At the wall, an isothermal no-slip wall boundary condition is used, and the wall temperature is fixed as shown in Table 7.5. At top and outlet boundaries, Neumann conditions are used for all components.

Figure 7.4 shows grid-independence studies for the high-Re k- ε with analytical wall function and standard wall function on grids ranging from 60×40 to 140×80 , with the near-wall size 1.5×10^{-4} m. The dimensionless wall distance y+ is about 15 at the upstream to guarantee the near-wall cell out of the viscous sublayer. The results indicate that the used meshes give grid-independent solutions. For the Low-Re Launder-Sharma model, a 280×250 mesh is used with the first cell size about 3.5×10^{-6} m and the y+ around 0.6 at the upstream.

Figure 7.17 displays iso-lines of mean pressure superimposed on contours of Mach number at ramp angles 16° , 20° and 24° by the LS model. As the ramp angle increases, the shock strength and size of separation bubble increase, the thickness of boundary layer due to the SWBLI is enhanced, and the separation shock is shifted further upstream.



Figure 7.4: Grid independency study for 24° compression corner from Asproulias (2014)

Figure 7.18 displays the comparison of wall pressure and skin friction predicted by the RANS models with the experimental measurements. For the 24° case, in general, the results of wall pressure by the wall function approaches and LS model are in good agreement with experiment, except for a slight difference near the beginning of the interaction. The start position of interaction predicted by the AWF is a bit downstream than the results of the SWF approach and the LS model, which are closer to the experiment. From the skin friction, the results by all models fit the experimental data reasonably well. The LS model gives the slightly larger separation bubble, while both wall function approaches give similar size to each other.

For the 20° case, all the models predict wall pressure reasonably well when compared with the experiment, but the point at which the pressure starts to increase, which can be defined as the start of the interaction, falls a bit downstream than that measured in the experiments. The LS model tends to predict the start of interaction slightly closer to the experiment as the 24° case. From the skin friction results, the numerical and experimental data show that the size of separation zone is much smaller than the 24° case. It is quite obvious that the drop in skin friction, associated with flow separation, occurs a little more upstream with the LS model and the AWF approach than with the standard wall function, in both the 24° cases.

For the 16° case, the wall pressure predicted by all three models fits the experiment well. The skin friction profile shows that both the low-Re model and the AWF approach predict a small separation zone, although it is not so clear from the experimental data whether such a feature should be present Figure 7.19 displays mean velocity comparisons at eight different locations for the ramp angle of 24° . Location 1 is located outside of the separation zone, and similar velocity results are predicted by all three approaches. Location 2 is the start of the separation zone, and the AWF results return a thinner boundary layer than the experiment since at this location the flow is still undisturbed by the AWF approach. Locations 3 and 4 are both located in the separation bubble and have negative values of streamwise velocity. The reattachment point of the experiment is located upstream of the numerical predictions as seen in Figure 7.18, and this is confirmed in the velocity profile at location 5, where the experimental data shows only positive velocity values, while the numerical results still have negative values. Locations 6, 7 and 8 are all located downstream of the separation bubble, and the numerical results tend to underestimate the mean velocity, as a result of having reattached rather too late.

Figure 7.20 shows mean velocity comparisons at eight different locations for the ramp angle of 20°. Locations 1 and 8 are located outside of the separation zone, and similar velocity results with experiment are predicted. At locations 2, 3 and 4, the numerical methods tend to overestimate the mean velocity. At location 5, negative velocity is predicted in the near wall region by both the experimental data and all three near-wall modelling treatment. At locations 6, 7 and 8, the numerical results fit the experimental data well.

7.4 Summary

In this Chapter, the LS model, and the high-Re k- ε model with standard wall function and analytical wall function have been tested in supersonic impinging shock interaction flows at two different Reynolds numbers and in compression corners. The main conclusions are:

For the high-Re impinging shock interaction case:

- 1. When the impinging shock is strong enough, the interaction zone has a separation bubble, whose size increases as the impinging angle grows. For the 7° case, there is no separation from the numerical or experimental results.
- 2. The wall pressure, predicted by the LS model, the high-Re k- ε model with standard wall function (SWF) and analytical wall function (AWF), is in good agreement with the experiment.
- 3. For the skin friction results, the skin friction of analytical wall function approach tends to give closer results to the LS model, and is close to the experiment, while the standard wall function approach tends to underestimate the strength of interaction.

- 4. From the mean velocity comparison, all the models give good predictions to the experiment at all eight locations generally, especially at locations 1 and 8, which are located outside of the separation zone.
- 5. From the streamwise Reynolds stress comparison, the numerical methods underestimate the Reynolds streamwise stress at eight all locations, especially in the middle of the separation zone. In the EVMs, the streamwise Reynolds stress is obtained by $\widetilde{u''u''} = (2/3)k$ for most of these flow locations, where the flow is shear-dominated. The underestimates of the streamwise stress in a boundary-layer type flow is usually because that the normal stresses are not isotropic.

For the low-Re impinging shock interaction case:

- From the skin friction comparison, the SWF underestimates the SWTBLI and fails to reproduce the separation bubble in the near wall area, while the LS model returns a larger separation zone. The AWF approach gives a smaller zone than the DNS results, since the beginning of the interaction zone predicted by the AWF lies a bit downstream to that of the DNS data. However, the reattachment point of the AWF fits well with the LS and experimental results.
- 2. From the mean streamwise velocity comparison, the results from using the RANS models are in good agreement with the DNS results, especially upstream and downstream of the interaction zone.
- 3. From the turbulent kinetic energy and dissipation rate comparison, all three models give the similar profiles to each other at the upstream and downstream of the interaction zone. Within or near the separation, the AWF is closer to the results of the LS model.

For the compression corner case:

- 1. The LS model predicts that there is a separation bubble near the corner in all three cases, with the size of the bubble increasing with ramp angle. The SWF approach tends to underestimate the SWTBLI, and for the 16° case it fails to reproduce the separation bubble.
- 2. The wall pressure, predicted by all three models, is in good agreement with the experimental data, except at the start of the interaction zone, which the AWF tends to predict as being a little further downstream than the other near-wall modelling treatments.

- 3. For the skin friction results, the standard wall function approach tends to underestimate the strength of interaction, especially when the separation is weak.
- 4. From the mean velocity comparison for the ramp angles 24° and 20°, all the models give good prediction at all locations, especially where the locations are outside of the separation zone.



Figure 7.5: Iso-lines of mean pressure superimposed on contours of Mach number for high-Re impinging shock interaction



Figure 7.6: Mean wall pressure (right) and skin-friction (left) distribution at different impinging angles 13° (top), 10° (middle) and 7° (bottom)



Figure 7.7: Comparison of mean velocity profiles for the 13° High-Re impinging shock interaction at eight streamwise locations. AWF (blue solid line), SWF (red dashed line), LS model (block dashed-dot-dot line) and experiment (black circle)



Figure 7.8: Comparison of Reynolds streamwise stress for the High-Re impinging shock interaction at eight streamwise locations. AWF (blue solid line), SWF (red dashed line), LS model (block dashed-dot-dot line) and experiment (black circle)


Figure 7.9: Grid independency study for the low Reynolds number 13° impinging shock interaction



Figure 7.10: Mach (left) and kinetic energy (right) contours of low-Re impinging shock interaction using $k - \varepsilon$ model with analytical wall function (top) and standard wall function (middle) and LS $k - \varepsilon$ model (bottom)



Figure 7.11: Mean wall pressure comparison with DNS results and experiment for low-Re impinging shock interaction



Figure 7.12: Skin-friction distribution comparison with DNS results and experiment for low-Re impinging shock interaction



Figure 7.13: Streamwise velocity contour and a sketch of probe locations for low-Re impinging shock interaction



Figure 7.14: Comparison of mean streamwise velocity profiles for the low Reynolds number impinging shock at six streamwise locations, AWF (blue dashed) SWF (red dashed-dot-dot) LS (black solid) and DNS results (block circle)



Figure 7.15: Comparison of turbulent kinetic energy for the low Reynolds number impinging shock at six streamwise locations, AWF (blue dashed) SWF (red dashed-dot-dot) and LS (black solid)



Figure 7.16: Comparison of turbulent dissipation rate for the low Reynolds number impinging shock at six streamwise locations, AWF (blue dashed) SWF (red dashed-dot-dot) and LS (black solid)



Figure 7.17: Iso-lines of mean pressure superimposed on contours of Mach number for compression corner at different ramp angles



Figure 7.18: Mean wall pressure (right) and skin-friction (left) distribution at different ramp angles 16° (top), 20° (middle) and 24° (bottom)



Figure 7.19: Comparison of mean velocity profiles for the 24° compression corner at eight streamwise locations. AWF (blue solid line), SWF (red dashed line), LS model (black dashed-dot-dot line) and experiment (black circle)



Figure 7.20: Comparison of mean velocity profiles for the 20° compression corner at eight streamwise locations. AWF (blue solid line), SWF (red dashed line), LS model (black dashed-dot-dot line) and experiment (black circle)

Chapter 8

Modifications to AWF

The analytical wall function described above uses the simplified momentum and energy equations and turbulence viscosity assumption to obtain analytical profiles of the mean velocity and temperature, as a function of y*, across the near-wall control volume, and hence to calculate the wall shear stress, the cell-averaged production and dissipation rates of turbulent kinetic energy, and the wall temperature or wall heat (depending on thermal boundary condition). One of the main simplifications introduced to the momentum and energy equations is that the convection terms (and the pressure gradient term in the momentum equation) are treated as constants in the near-wall cell, and approximated using nodal values as described in Section 6.2. As one might expect, the approximation of these terms (and particularly the convection ones) can have a significant effect on the analytical solutions in complex flows. Some methods of approximating these terms have been explored by Gerasimov (2004) and others, mainly in the context of buoyancy-influenced flows. In the current work, some further refinements to these have been developed, particularly considering compressible flows. Some additional features have also been introduced to the model, such as the inclusion of a viscous dissipation term in the simplified energy equation, and these developments are reported in this chapter.

8.1 Modifications to Convection Term in AWF (MAWF)

8.1.1 Modifications to the Momentum Convection Term

Although the Analytical Wall function as described in Chapter 6 works well for the Mach=3 cases, there is some unphysical behaviour encountered when applying it to higher Mach number cases. As an example, Fig. 8.1 shows results from an impinging shock interaction



Figure 8.1: Wall properties of Ma=5 impinging shock interaction using AWF

case at Mach 5, with shock deflection angle of 14°. Large spikes can be seen in both skin friction and wall heat flux when the form of AWF described in Chapter 6 is employed. These spikes were found to mainly occur as a result of the strong velocity gradients at the start of separation feeding into the convection terms, and this is confirmed by the red line in Fig. 8.1, which shows no such spikes when the convection terms were simply not included in the AWF formulation.



Figure 8.2: Convection terms and pressure gradient in the wall-parallel momentum equation near the start of separation

(-X represents $\rho U \partial U / \partial x$; -Y represents $\rho V \partial U / \partial y$; -XY represents $\rho U \partial U / \partial x + \rho V \partial U / \partial y$; -p represents $\partial p / \partial x$;)

In the initial implementation of AWF, the convection terms are constants across each separation region of the near wall cells. To explore how appropriate such an approximation is, Fig. 8.2 shows profiles of the wall-parallel momentum convection terms, (and the pressure gradient) calculated from the LS models, across the near-wall flow at a location close to the start of separation. At the wall, as might be expected, both the wall-normal and wall-parallel convection contributions are zero, and they then increase in magnitude as one moves away from the wall. This might suggest that taking the above approximation of convection terms being constant in the near-wall region would lead to an overestimation of their influence very close to the wall. In order to try improving the near-wall modelling of the convection terms we can note that immediately adjacent to the wall one should expect them to increase quadratically with wall distance (since $U \sim y$ and $V \sim y^2$ immediately adjacent to the wall). To account for this, a parabolic expression has therefore been taken to represent the convection terms in the near wall momentum equation, so the equation (6.13) is rewritten as:

$$\frac{\partial}{\partial y^*} \left[(\mu + \mu_t) \frac{\partial U}{\partial y^*} \right] = D y^{*2} + C \tag{8.1}$$

where:

$$D = \frac{\mu^2}{\rho^2 k_P} \frac{1}{y_n^{*2}} \left[\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} \right] \quad C = \frac{\mu^2}{\rho^2 k_P} \frac{dP}{dx}$$
(8.2)

and the convection terms are approximated across the two regions $0 < y^* < y^*_{\nu}$ and $y^* > y^*_{\nu}$. In the viscous sublayer ($0 < y^* < y^*_{\nu}$), the first integration of equation (8.1) now yields:

$$\mu \frac{\partial U_1}{\partial y^*} = \frac{D_1 y^{*3}}{3} + C_1 y^* + A_1 \tag{8.3}$$

where

$$D_1 = \frac{\mu^2}{\rho^2 k_P} \frac{1}{y_n^{*2}} \left[\rho_P U_1 \frac{\partial U}{\partial x} + \rho_P V_1 \frac{\partial U}{\partial y} \right] \quad C_1 = \frac{\mu^2}{\rho^2 k_P} \frac{dP}{dx}$$
(8.4)

The second integration gives:

$$U_1 = \frac{1}{\mu_w} \left[\frac{D_1 y^{*4}}{12} + \frac{C_1 y^{*2}}{2} + A_1 y^* \right]$$
(8.5)

In the fully turbulent layer $(y^* > y_v^*)$, the first integration of equation is:

$$\frac{\partial U_2}{\partial y^*} = \frac{1}{\mu_v} \left[\frac{D_2}{3} \frac{y^{*3}}{Y} + C_2 \frac{y^*}{Y} + \frac{A_2}{Y} \right]$$
(8.6)

where:

$$D_2 = \frac{\mu^2}{\rho^2 k_P} \frac{1}{y_n^{*2}} \left[\rho_P U_2 \frac{\partial U}{\partial x} + \rho_P V_2 \frac{\partial U}{\partial y} \right] \quad C_2 = \frac{\mu^2}{\rho^2 k_P} \frac{dP}{dx}$$
(8.7)

The second integration gives:

$$U_{2} = \frac{1}{\mu_{\nu}} \left[\frac{D_{2}}{3} \frac{1}{\alpha^{3}} \left[\frac{\alpha^{2}}{3} y^{*3} - \frac{\alpha \left(1 - \alpha y_{\nu}^{*}\right)}{2} y^{*2} + \left(1 - \alpha y_{\nu}^{*}\right)^{2} y^{*} - \left(1 - \alpha y_{\nu}^{*}\right)^{3} \frac{1}{\alpha} \ln Y \right] + C_{2} \frac{1}{\alpha} \left[y^{*} - \left(1 - \alpha y_{\nu}^{*}\right) \frac{1}{\alpha} \ln Y \right] + A_{2} \frac{1}{\alpha} \ln Y + B_{2} \right]$$
(8.8)

From the near wall cell boundary condition, the coefficients in the above equations are:

$$A_{1} = \frac{\mu_{\nu}U_{n} - N}{\frac{\ln Y_{n}}{\alpha} + y_{\nu}^{*}}$$

$$(8.9)$$

$$-D_{2})\frac{y_{\nu}^{*3}}{\alpha} + (C_{1} - C_{2})y_{\nu}^{*} + A_{1}$$

$$(8.10)$$

$$A_2 = (D_1 - D_2)\frac{y_{\nu}^{*3}}{3} + (C_1 - C_2)y_{\nu}^* + A_1$$
(8.10)

$$B_{2} = \left[\frac{D_{1}y_{\nu}^{*4}}{12} + \frac{C_{1}y_{\nu}^{*2}}{2} + A_{1}y_{\nu}^{*}\right] - \frac{D_{2}}{3\alpha^{3}}\left[\frac{\alpha^{2}}{3}y_{\nu}^{*3} - \frac{\alpha\left(1 - \alpha y_{\nu}^{*}\right)}{2}y_{\nu}^{*2} + \left(1 - \alpha y_{\nu}^{*}\right)^{2}y_{\nu}^{*}\right] - C_{2}\frac{1}{\alpha}y_{\nu}^{*}$$

$$(8.11)$$

The coefficient N in (8.9) is defined by:

$$N = \frac{D_2}{3} \frac{1}{\alpha^3} \left[\frac{\alpha^2}{3} y_n^{*3} - \frac{\alpha (1 - \alpha y_v^*)}{2} y_n^{*2} + (1 - \alpha y_v^*)^2 y_n^* - (1 - \alpha y_v^*)^3 \frac{1}{\alpha} \ln Y_n \right] + C_2 \frac{1}{\alpha} \left[y_n^* - (1 - \alpha y_v^*) \frac{1}{\alpha} \ln Y_n \right] + \frac{1}{\alpha} \ln Y_n \left[\frac{1}{3} y_v^{*3} (D_1 - D_2) + (C_1 - C_2) y_v^* \right] + \frac{D_1 y_v^{*4}}{12} + \frac{C_1 y_v^{*2}}{2} - \frac{D_2}{3\alpha^3} \left[\frac{\alpha^2}{3} y_v^{*3} - \frac{\alpha (1 - \alpha y_v^*)}{2} y_v^{*2} + (1 - \alpha y_v^*)^2 y_v^* \right] - C_2 \frac{1}{\alpha} y_v^*$$

The wall shear stress is:

$$\tau_{wall} = -\mu \frac{\partial U}{\partial y}\Big|_{y=0} = -\frac{\rho_v \sqrt{k_P}}{\mu_v} \left[\mu \frac{\partial U}{\partial y^*}\right]_{y^*=0} = -\frac{\rho_v \sqrt{k_P}}{\mu_v} A_1$$
(8.12)

and the cell-averaged production of turbulence kinetic energy is:

$$\bar{P}_{k} = \frac{1}{y_{n}} \frac{\rho \sqrt{k_{P}}}{\mu} \int_{y_{v}^{*}}^{y_{n}^{*}} \mu \alpha \left(y^{*} - y_{v}^{*}\right) \left(\frac{\partial U_{2}}{\partial y^{*}}\right)^{2} dy^{*}$$

$$(8.13)$$

where $\partial U_2 / \partial y^*$ is now given by equation (8.6).

8.1.2 Modifications to the Energy Convection Term

a. Linear Assumption

For the convection terms in the energy equation, Figure 8.3 shows the near-wall variation of these from the LS model at the same locations as those shown in Fig. 8.2. Again the terms go to zero at the wall. To account for this, a similar treatment to that adopted above for the momentum equation has been followed, except that in this case the convection terms are taken to vary linearly near the wall (since U \sim y, and T is assumed to be O(1)):

$$\frac{\partial}{\partial y^*} \left[\left(\frac{\mu}{\Pr} + \frac{\mu_t}{\Pr_t} \right) \frac{\partial T}{\partial y^*} \right] = \mathbf{D}_{th} y^* \tag{8.14}$$

where:

$$D_{th} = \frac{\mu^2}{\rho^2 k_P} \frac{1}{y_n^*} \left(\frac{\partial \left(\rho UT\right)}{\partial x} + \frac{\partial \left(\rho VT\right)}{\partial y} \right)$$
(8.15)



Figure 8.3: Convection terms in the energy equation near the start of separation (-X represents $\rho U \partial T / \partial x$; -Y represents $\rho V \partial T / \partial y$; -XY represents $\rho U \partial T / \partial x + \rho V \partial T / \partial y$;)

In the viscous sublayer $(y^* < y^*_v)$, the first integration of equation (8.14) gives:

$$\frac{\mu_w}{\Pr} \frac{\partial T_1}{\partial y^*} = \frac{D_{th1}}{2} y^{*2} + A_{th1}$$
(8.16)

where

$$D_{th1} = \frac{\mu_w^2}{\rho_w^2 k_P} \frac{1}{y_v^*} \left(\rho U \frac{\partial T}{\partial x} + \rho V \frac{\partial T}{\partial y} \right)$$
(8.17)

The coefficient in (8.16) can be obtained by the definition of wall heat flux:

$$A_{th1} = -\frac{q_{wall}}{c_p} \frac{\mu_w}{\rho_w \sqrt{k_P}}$$

The second integration gives:

$$T_{1} = \frac{\Pr}{\mu_{w}} \left(\frac{D_{th1}}{6} y^{*3} + A_{th1} y^{*} \right) + T_{w}$$
(8.18)

In the fully turbulent region $(y^* > y_v^*)$, the first integration of (8.14) gives:

$$\left(\frac{\mu_{\nu}}{\Pr} + \frac{\mu_{t}}{\Pr_{t}}\right) \frac{\partial T_{2}}{\partial y^{*}} = \frac{D_{th2}}{2} y^{*2} + A_{th2}$$
(8.19)

where:

$$D_{th2} = \frac{\mu_v^2}{\rho_v^2 k_P} \frac{1}{y_n^*} \left(\rho U \frac{\partial T}{\partial x} + \rho V \frac{\partial T}{\partial y} \right)$$
(8.20)

The second integration gives:

$$T_{2} = \frac{\Pr}{\mu_{w}} \left\{ \frac{D_{th2}}{2} \frac{1}{\alpha_{t}^{2}} \left[\frac{\alpha_{t}}{2} y^{*2} - (1 - \alpha_{t} y_{v}^{*}) y^{*} + (1 - \alpha_{t} y_{v}^{*})^{2} \frac{1}{\alpha_{t}} \ln Y_{T} \right] + \frac{A_{th2}}{\alpha_{t}} \ln Y_{T} + B_{th2} \right\}$$
(8.21)

The coefficients are:

$$B_{th2} = A_{th1} y_v^* + BT + \frac{\mu_v}{\Pr} T_w$$
(8.22)

$$A_{th2} = \left(\frac{D_{th1}}{2} - \frac{D_{th2}}{2}\right) y_{\nu}^{*2} + A_{th1}$$
(8.23)

where BT in (8.22) represents:

$$BT = \frac{D_{th1}}{6} y_{\nu}^{*3} - \frac{D_{th2}}{2} \frac{1}{\alpha_t^2} \left[\frac{\alpha_t}{2} y_{\nu}^{*2} - (1 - \alpha_t y_{\nu}^*) y_{\nu}^* \right]$$

Then substituting (8.22) and (8.23) into (8.21):

$$T_{2} = \frac{\Pr}{\mu_{w}} \left\{ \frac{D_{th2}}{2} \frac{1}{\alpha_{t}^{2}} \left[\frac{\alpha_{t}}{2} y^{*2} - (1 - \alpha_{t} y_{v}^{*}) y^{*} + (1 - \alpha_{t} y_{v}^{*})^{2} \frac{1}{\alpha_{t}} \ln Y_{T} \right] + A_{th1} y_{v}^{*} + BT + \frac{\ln Y_{T}}{\alpha_{t}} \left(\left(\frac{D_{th1}}{2} - \frac{D_{th2}}{2} \right) y_{v}^{*2} + A_{th1} \right) \right\} + T_{w}$$
(8.24)

When $y^* = y_n^*$, and $T_2 = T_n$:

$$T_{2} = \frac{\Pr}{\mu_{w}} \left\{ \frac{D_{th2}}{2} \frac{1}{\alpha_{t}^{2}} \left[\frac{\alpha_{t}}{2} y^{*2} - (1 - \alpha_{t} y_{v}^{*}) y^{*} + (1 - \alpha_{t} y_{v}^{*})^{2} \frac{1}{\alpha_{t}} \ln Y_{T} \right] + A_{th1} y_{v}^{*} + BT + \frac{\ln Y_{T}}{\alpha_{t}} \left(\left(\frac{D_{th1}}{2} - \frac{D_{th2}}{2} \right) y_{v}^{*2} + A_{th1} \right) \right\} + T_{w}$$
(8.25)

For the isothermal wall boundary conditions, the wall heat flux is then given by:

$$q_{wall} = -\frac{\rho_v C_p \sqrt{k_P}}{\mu_v} A_{th1}$$
(8.26)

where A_{th1} can be obtained from (8.25)

$$A_{th1} = \frac{1}{\frac{\ln Y_{Tn}}{\alpha_t} + y_{\nu}^*} \left\{ \frac{\mu_{\nu}}{\Pr} (T_n - T_{\nu}) - \frac{D_{th2}}{2} \frac{1}{\alpha_t^2} \left[\frac{\alpha_t}{2} y_n^{*2} - (1 - \alpha_t y_{\nu}^*) y_n^* + (1 - \alpha_t y_{\nu}^*)^2 \frac{1}{\alpha_t} \ln Y_{Tn} \right] -BT - \frac{\ln Y_{Tn}}{\alpha_t} \left(\frac{D_{th1}}{2} - \frac{D_{th2}}{2} \right) y_{\nu}^{*2} \right\}$$
(8.27)

b. Parabolic Assumption

The attempt of the parabolic assumption also applies to the convection term in the simplified energy equation (since U~y, and $\partial T/\partial x$ is assumed to be O(y)):

$$\frac{\partial}{\partial y^*} \left[\left(\frac{\mu}{\Pr} + \frac{\mu_t}{\Pr_t} \right) \frac{\partial T}{\partial y^*} \right] = D_{th} y^{*2}$$
(8.28)

In the viscous sublayer $(y^* < y^*_{\nu})$, the first integration gives:

$$\frac{\partial}{\partial y^*} \left[\left(\frac{\mu}{\Pr} + \frac{\mu_t}{\Pr_t} \right) \frac{\partial T}{\partial y^*} \right] = D_{th} y^{*2}$$
(8.29)

where

$$D_{th1} = \frac{\mu_w^2}{\rho_w^2 k_P} \frac{1}{y_v^{*2}} \left(\rho U \frac{\partial T}{\partial x} + \rho V \frac{\partial T}{\partial y} \right)$$
(8.30)

The second integration gives:

$$T_{1} = \frac{\Pr}{\mu_{w}} \left(\frac{D_{th1}}{12} y^{*4} + A_{th1} y^{*} \right) + T_{w}$$
(8.31)

In the fully turbulent region $(y^* > y_v^*)$, the first integration of the simplified energy equation gives:

$$\left(\frac{\mu}{\Pr} + \frac{\mu_t}{\Pr_t}\right) \frac{\partial T_2}{\partial y^*} = \frac{D_{th2}}{3} y^{*3} + A_{th2}$$
(8.32)

where:

$$D_{th2} = \frac{\mu_v^2}{\rho_v^2 k_P} \frac{1}{y_n^{*2}} \left(\rho U \frac{\partial T}{\partial x} + \rho V \frac{\partial T}{\partial y} \right)$$
(8.33)

The second integration gives:

$$T_{2} = \frac{\Pr}{\mu_{w}} \left\{ \frac{D_{th2}}{3} \frac{1}{\alpha_{t}^{3}} \left[\frac{\alpha_{t}^{2}}{3} y^{*3} - \frac{\alpha_{t} \left(1 - \alpha_{t} y_{v}^{*}\right)}{2} y^{*2} + \left(1 - \alpha_{t} y_{v}^{*}\right)^{2} y^{*} - \left(1 - \alpha_{t} y_{v}^{*}\right)^{3} \frac{1}{\alpha_{t}} \ln Y_{T} \right] + \frac{A_{th2}}{\alpha_{t}} \ln Y_{T} + B_{th2} \right\}$$

$$(8.34)$$

The coefficients are:

$$B_{th2} = A_{th1} y_{\nu}^* + BT + \frac{\mu_{\nu}}{\Pr} T_{\nu}$$
(8.35)

$$A_{th2} = (\mathbf{D}_{th1} - D_{th2})\frac{y_{\nu}^{*3}}{3} + A_{th1}$$
(8.36)

where

$$BT = \frac{D_{th1}}{12}y^{*4} - \frac{D_{th2}}{3}\frac{1}{\alpha_t^3} \left[\frac{\alpha_t^2}{3}y_v^{*3} - \frac{\alpha_t\left(1 - \alpha_t y_v^*\right)}{2}y_v^{*2} + \left(1 - \alpha_t y_v^*\right)^2 y_v^*\right]$$

Then substituting (8.35) and (8.36) into (8.34):

$$T_{2} = \frac{\Pr}{\mu_{w}} \left\{ \frac{D_{th2}}{3} \frac{1}{\alpha_{t}^{3}} \left[\frac{\alpha_{t}^{2}}{3} y^{*3} - \frac{\alpha_{t} \left(1 - \alpha_{t} y_{v}^{*}\right)}{2} y^{*2} + \left(1 - \alpha_{t} y_{v}^{*}\right)^{2} y^{*} - \left(1 - \alpha_{t} y_{v}^{*}\right)^{3} \frac{1}{\alpha_{t}} \ln Y_{T} \right] + A_{th1} y_{v}^{*} + BT + \frac{\ln Y_{T}}{\alpha_{t}} \left(\left(D_{th1} - D_{th2}\right) \frac{y_{v}^{*3}}{3} + A_{th1} \right) \right\} + T_{w}$$

$$(8.37)$$

When $y^* = y_n^*$, and $T_2 = T_n$:

$$T_{w} = T_{n} - \frac{\Pr}{\mu_{v}} \left\{ \frac{D_{th2}}{3} \frac{1}{\alpha_{t}^{3}} \left[\frac{\alpha_{t}^{2}}{3} y_{n}^{*3} - \frac{\alpha_{t} \left(1 - \alpha_{t} y_{v}^{*}\right)}{2} y_{n}^{*2} + \left(1 - \alpha_{t} y_{v}^{*}\right)^{2} y_{n}^{*} - \left(1 - \alpha_{t} y_{v}^{*}\right)^{3} \frac{1}{\alpha_{t}} \ln Y_{Tn} \right] + A_{th1} y_{v}^{*} + BT + \frac{\ln Y_{Tn}}{\alpha_{t}} \left(\left(\frac{D_{th1}}{3} - \frac{D_{th2}}{3} \right) y_{v}^{*3} + A_{th1} \right) \right\}$$

$$(8.38)$$

For the isothermal wall boundary conditions, the wall heat flux is obtained from equation (8.26), where A_{th1} can be obtained from (8.38) as

$$A_{th1} = \frac{1}{\frac{\ln Y_{Tn}}{\alpha_t} + y_v^*} \left\{ \frac{\mu_v}{\Pr} (T_n - T_w) - BT - \frac{\ln Y_{Tn}}{\alpha_t} \left(\left(\frac{D_{th1}}{3} - \frac{D_{th2}}{3} \right) y_v^{*3} + (C_{th1} - C_{th2}) y_v^* \right) - \frac{D_{th2}}{3} \frac{1}{\alpha_t^3} \left[\frac{\alpha_t^2}{3} y_n^{*3} - \frac{\alpha_t \left(1 - \alpha_t y_v^* \right)}{2} y_n^{*2} + \left(1 - \alpha_t y_v^* \right)^2 y_n^* - \left(1 - \alpha_t y_v^* \right)^3 \frac{1}{\alpha_t} \ln Y_{Tn} \right] \right\}$$
(8.39)

To illustrate the impact of the above changes to the convection term modelling, Fig. 8.4 shows the predictions of skin friction and wall heat flux for the Ma=5 impinging shock case. The modified AWF (labelled as MAWF) does clearly reduce the spikes seen from the original AWF formulation. The MAWF with parabolic assumptions in the convection terms in the energy equation returns more smooth wall heat flux results than the MAWF with linear assumption.



Figure 8.4: Wall skin friction and heat flux for Ma=5 impinging shock interaction using MAWF with linear and parabolic assumptions for the convection terms in the energy equation

8.2 The Compressibility in the Thermal MAWF (CMAWF)

Although the MAWF with the parabolic assumption for convection terms both in momentum equation and energy equation in Chapter 8.1 performs well for the calculation of surface pressure and skin friction, while in Fig. 8.4, it does not reproduce the wall heat flux as the LS results, especially the downstream, where there is a much larger difference. The AWF above is based on the simplified Reynolds equations (Equ. 6.13 and 6.14) which take into account the important effects, such as pressure gradients and convective transport. Such simplification works well for low-speed flows as described in Gerasimov (2004) and Gant (2003). However, the simplified form employed for the energy equation may neglect some important terms for the high-speed near-wall flows. Figure 8.4 shows the temperature distribution in the undisturbed near-wall region at different Mach numbers. The hypersonic Ma=8 case shows that it increases first and then drops to the inflow temperature, which returns the positive wall heat flux, while for the supersonic flows, the temperature drops from the wall temperature to inflow temperature gradually, which will return the negative wall heat flux. The differences shown in the profiles imply about that the simplified energy equation might be a weakness in the AWF and some other terms might play an important role in the near wall region for the high-speed flows, especially for the hypersonic flows.



Figure 8.5: The undisturbed temperature distribution in the near wall region at Ma=3 by Smits and Muck (1987), Ma=5 by Schülein et al. (2015) and Ma=8 by Kussoy and Horstman (1991)

8.2.1 Simplification of the Energy Equation

The original AWF employed a simplified form of the internal energy equation. As described in Chapter 6.2, the energy equation in compressible solvers is based on the total energy equation:

$$\frac{\partial (\rho E)}{\partial t} + \nabla \bullet [\mathbf{u} (\rho E + p)] - \nabla \bullet (\underline{\sigma} \bullet \mathbf{u}) + \nabla \bullet \mathbf{q} = 0$$
(8.40)

In order to be consistent with the main code, it is better to work with a simplified form of this total energy equation when dealing with highly compressible cases. In the original thermal AWF, only the thermal convection terms in (8.40) are involved. In order to examine the relative size of various terms in the energy equation, Figure 8.6 compares thermal convection terms, the pressure gradient and viscous dissipation terms using k- ε model with AWF wall function at Ma=3, 5 and 8.2 impinging shock interaction cases. From the figure, the dissipation term plays a dominant role in the near wall region for high-speed flow.



Figure 8.6: Comparison of terms in the energy equation at Ma=3 (left), 5 (middle) and 8.2 (right) impinging shock interaction cases

When the viscous dissipation term $\nabla \bullet (\underline{\sigma} \bullet \mathbf{u})$ is expanded, it can be seen to contain a large number of velocity related terms:

$$\nabla \bullet (\underline{\sigma} \bullet \mathbf{u}) = \mu \begin{bmatrix} \frac{2}{3} \left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \right)^2 \right] \\ + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 \\ + \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{3} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] u \\ + \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{3} \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] v \\ + \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + \frac{1}{3} \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] w \end{bmatrix}$$
(8.41)

In order to introduce a simplified form of this into the near-wall energy equation for the AWF the comparison has been made when velocity is equal to (u, 0, 0) and (u, v, w) along the near wall cells at Mach=3, 5 and 8.2 impinging shock interaction cases in Fig. 8.7. From the comparison, two lines are close to each other, which means that the wall-parallel velocity plays a dominant role in the calculation of the dissipation term for 2D flows. As the AWF is developed for the supersonic and hypersonic 2D flows, let $\mathbf{u} = (u, 0)$, and take it to the shear stress:

$$\underline{\boldsymbol{\sigma}} = \boldsymbol{\mu} \left[\nabla \mathbf{u} + \nabla \mathbf{u}^T - \left(\frac{2}{3} \nabla \bullet \mathbf{u} \right) I \right] = \boldsymbol{\mu} \left| \begin{array}{c} \frac{\partial u}{\partial x} & 0\\ \frac{\partial u}{\partial y} & 0 \end{array} \right| + \boldsymbol{\mu} \left| \begin{array}{c} \frac{1}{3} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ 0 & -\frac{2}{3} \frac{\partial u}{\partial x} \end{array} \right| = \underline{\boldsymbol{\sigma}}_1 + \underline{\boldsymbol{\sigma}}_2 \quad (8.42)$$

where the shear stress is separated into two parts. The dissipation term in energy equation is calculated separately as $\nabla \bullet (\underline{\sigma} \bullet \mathbf{u}) = \nabla \bullet (\underline{\sigma}_1 \bullet \mathbf{u}) + \nabla \bullet (\underline{\sigma}_2 \bullet \mathbf{u})$, which are convenient forms to evaluate by the OpenFoam solver.



Figure 8.7: Dissipation comparison at different Mach number at Ma=3 (left), 5 (middle) and 8.2 (right) impinging shock interaction cases

$$\nabla \bullet (\underline{\sigma}_1 \bullet \mathbf{u}) = \mu \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] u \right] + \left(\frac{\partial u}{\partial x} \frac{\partial \mu}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial \mu}{\partial y} \right) u \quad (8.43)$$

$$\nabla \bullet (\underline{\sigma}_2 \bullet \mathbf{u}) = \mu \left[\frac{1}{3} \left(\frac{\partial u}{\partial x} \right)^2 + \left[\frac{1}{3} \frac{\partial^2 u}{\partial x^2} \right] u \right] + \left(\frac{1}{3} \frac{\partial u}{\partial x} \frac{\partial \mu}{\partial x} \right) u$$
(8.44)

As we can expect that the velocity gradient in the wall normal direction is much more significant than it in the wall parallel direction, $\nabla \bullet (\underline{\sigma}_1 \bullet \mathbf{u}) - 3\nabla \bullet (\underline{\sigma}_2 \bullet \mathbf{u})$ represents the viscous dissipation term in the wall normal direction.

$$\nabla \bullet (\underline{\sigma}_1 \bullet \mathbf{u}) - 3\nabla \bullet (\underline{\sigma}_2 \bullet \mathbf{u}) = \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial^2 u}{\partial y^2} u + u \frac{\partial u}{\partial y} \frac{\partial \mu}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} u\right)$$
(8.45)

In Fig. 8.8, T1 represents equation (8.43), T2 represents equation (8.44), T1+T2 represents the whole dissipation term by the shear stress (8.42), and T1-3×T2 is equal to equation (8.45). From the comparison, equation (8.44) is negligible compared to equation (8.43), which means that equation (8.43) plays a dominant role to represent the dissipation in the near wall region at high-speed flows. For high-speed flows, the speed drops to zero quickly in the near wall region, which implies that the viscous dissipation terms in the wall normal direction may be more important even than the convection terms in the near-wall layers. The simplified energy equation is better to include the viscous dissipation terms in the wall normal direction as equation (8.45).



Figure 8.8: Different parts of the dissipation at Ma=3 (left), 5 (middle) and 8.2 (right) impinging shock interaction cases

8.2.2 The Thermal AWF with a Dissipation Term (CMAWF)

In order to take the viscous dissipation into account within the AWF framework, one additional term as equation (8.45) has been added in the thermal AWF, and the simplified energy equation in the near wall region can then be written as:

$$\frac{\partial \left(\rho U E + U p\right)}{\partial x} + \frac{\partial \left(\rho V E + V p\right)}{\partial y} - \frac{\partial}{\partial y} \left(\mu U \frac{\partial U}{\partial y}\right) = \frac{\partial}{\partial y} \left[\lambda \frac{\partial T}{\partial y}\right]$$
(8.46)

where the thermal conductivity is:

$$\lambda = C_p \left(\frac{\mu}{\Pr} + \frac{\mu_t}{\Pr_t}\right) \tag{8.47}$$

Equation (8.46) can be rewritten as:

$$\frac{1}{C_p} \left[\frac{\partial \left(\rho U E + U p \right)}{\partial x} + \frac{\partial \left(\rho V E + V p \right)}{\partial y} \right] - \frac{1}{C_p} \frac{\partial}{\partial y} \left(\mu U \frac{\partial U}{\partial y} \right) = \frac{\partial}{\partial y} \left[\left(\frac{\mu}{\Pr} + \frac{\mu_t}{\Pr_t} \right) \frac{\partial T}{\partial y} \right]$$
(8.48)

In the previous section, the representation of the thermal convection terms that assumed a parabolic variation for them near the wall gave slightly better results than those using the linear variation, so that has been retained in the present treatment of them (tests with a linear form employed within the approach outlined below again gave similar conclusions to those drawn above). To derive the CMAWF, the same broad integration procedure has been used as previously. In the viscous sublayer ($y^* < y^*_v$), where $\mu_t = 0$, equation (8.48) becomes:

$$\frac{\partial}{\partial y^*} \left(\frac{\mu}{\Pr} \frac{\partial T_1}{\partial y^*} \right) = \mathcal{D}_{th1} y^{*2} - \frac{1}{C_p} \frac{\partial}{\partial y^*} \left(\mu_\nu U_1 \frac{\partial U_1}{\partial y^*} \right)$$
(8.49)

where

$$D_{th1} = \frac{\mu_{\nu}^2}{C_p \rho_{\nu}^2 k_P} \frac{1}{y_{\nu}^{*2}} \left(\frac{\partial \left(\rho U E + U p\right)}{\partial x} + \frac{\partial \left(\rho V E + V p\right)}{\partial y} \right)$$
(8.50)

The first integration of (8.49) gives:

$$\frac{\mu_{w}}{\Pr}\frac{\partial T_{1}}{\partial y^{*}} = \frac{D_{th1}}{3}y^{*3} + A_{th1} - \frac{1}{C_{p}}\mu_{v}U_{1}\frac{\partial U_{1}}{\partial y^{*}}$$
(8.51)

In order to approximate the final term in equation (8.51), the expressions obtained for the analytical velocity profile and its gradient are used (equations (8.3) and (8.5)). Substituting these into equation (8.51) and rearranging results in:

$$\frac{\partial T_1}{\partial y^*} = \frac{\Pr}{\mu_w} \left\{ \frac{D_{th1}}{3} y^{*3} + A_{th1} - \left(N_7 y^{*7} + N_5 y^{*5} + N_4 y^{*4} + N_3 y^{*3} + N_2 y^{*2} + N_1 y^* \right) \right\}$$
(8.52)

where

$$N_{7} = \frac{1}{C_{p}\mu_{w}} \frac{D_{1}^{2}}{36} \quad N_{5} = \frac{1}{C_{p}\mu_{w}} \frac{C_{1}D_{1}}{4} \quad N_{4} = \frac{1}{C_{p}\mu_{w}} \frac{5}{12}A_{1}D_{1}$$
$$N_{3} = \frac{1}{C_{p}\mu_{w}} \frac{C_{1}^{2}}{2} \qquad N_{2} = \frac{1}{C_{p}\mu_{w}} \frac{3A_{1}C_{1}}{2} \quad N_{1} = \frac{1}{C_{p}\mu_{w}}A_{1}^{2}$$

The second integration then gives:

$$T_{1} = \frac{\Pr}{\mu_{w}} \left(\frac{D_{th1}}{12} y^{*4} + A_{th1} y^{*} - \left(\frac{N_{7}}{8} y^{*8} + \frac{N_{5}}{6} y^{*6} + \frac{N_{4}}{5} y^{*5} + \frac{N_{3}}{4} y^{*4} + \frac{N_{2}}{3} y^{*3} + \frac{N_{1}}{2} y^{*2} \right) \right) + T_{w}$$
(8.53)

In the fully turbulent region $(y^* > y_v^*)$, equation (8.48) can be rewritten as:

$$\frac{\partial}{\partial y^*} \left(\frac{\mu}{\Pr} \frac{\partial T_2}{\partial y^*} \right) = \mathcal{D}_{th2} y^{*2} - \frac{1}{C_p} \frac{\partial}{\partial y^*} \left(\mu_v U_2 \frac{\partial U_2}{\partial y^*} \right)$$
(8.54)

where:

$$D_{th2} = \frac{\mu_v^2}{C_p \rho_v^2 k_P} \frac{1}{y_n^{*2}} \left(\frac{\partial \left(\rho U E + U p\right)}{\partial x} + \frac{\partial \left(\rho V E + V p\right)}{\partial y} \right)$$
(8.55)

The first integration of (8.53) gives:

$$\left(\frac{\mu_{\nu}}{\Pr} + \frac{\mu_{t}}{\Pr_{t}}\right) \frac{\partial T_{2}}{\partial y^{*}} = \frac{D_{th2}}{3} y^{*3} + A_{th2} - \frac{1}{C_{p}} \mu_{\nu} Y U_{2} \frac{\partial U_{2}}{\partial y^{*}}$$

This equation can be rewritten as:

$$\frac{\partial T_2}{\partial y^*} = \frac{\Pr}{\mu_v} \left(\frac{D_{th2}}{3} \frac{y^{*3}}{Y_T} + C_{th2} \frac{y^*}{Y_T} + A_{th2} - \frac{1}{C_p} \frac{1}{Y_T} \mu_v Y U_2 \frac{\partial U_2}{\partial y^*} \right)$$
(8.56)

The second integration gives:

$$T_{2} = \frac{\Pr}{\mu_{\nu}} \left\{ \frac{D_{th2}}{3} \frac{1}{\alpha_{t}^{3}} \left[\frac{\alpha_{t}^{2}}{3} y^{*3} - \frac{\alpha_{t} \left(1 - \alpha_{t} y_{\nu}^{*}\right)}{2} y^{*2} + \left(1 - \alpha_{t} y_{\nu}^{*}\right)^{2} y^{*} - \left(1 - \alpha_{t} y_{\nu}^{*}\right)^{3} \frac{1}{\alpha_{t}} \ln Y_{T} \right] \right. \\ \left. + \frac{A_{th2}}{\alpha_{t}} \ln Y_{T} + B_{th2} - \frac{1}{\mu_{\nu} C_{p}} \int_{y_{\nu}^{*}}^{y^{*}} \frac{1}{Y_{T}} \mu_{\nu} U_{2} \left(\mu_{\nu} Y \frac{\partial U_{2}}{\partial y^{*}}\right) dy^{*} \right\}$$

$$\left. \left. + \frac{A_{th2}}{\alpha_{t}} \ln Y_{T} + B_{th2} - \frac{1}{\mu_{\nu} C_{p}} \int_{y_{\nu}^{*}}^{y^{*}} \frac{1}{Y_{T}} \mu_{\nu} U_{2} \left(\mu_{\nu} Y \frac{\partial U_{2}}{\partial y^{*}}\right) dy^{*} \right\}$$

$$\left. \left. + \frac{A_{th2}}{\alpha_{t}} \ln Y_{T} + B_{th2} - \frac{1}{\mu_{\nu} C_{p}} \int_{y_{\nu}^{*}}^{y^{*}} \frac{1}{Y_{T}} \mu_{\nu} U_{2} \left(\mu_{\nu} Y \frac{\partial U_{2}}{\partial y^{*}}\right) dy^{*} \right\}$$

$$\left. \left. + \frac{A_{th2}}{\alpha_{t}} \ln Y_{T} + B_{th2} - \frac{1}{\mu_{\nu} C_{p}} \int_{y_{\nu}^{*}}^{y^{*}} \frac{1}{Y_{T}} \mu_{\nu} U_{2} \left(\mu_{\nu} Y \frac{\partial U_{2}}{\partial y^{*}}\right) dy^{*} \right\}$$

$$\left. \left. + \frac{A_{th2}}{\alpha_{t}} \ln Y_{T} + B_{th2} - \frac{1}{\mu_{\nu} C_{p}} \int_{y_{\nu}^{*}}^{y^{*}} \frac{1}{Y_{T}} \mu_{\nu} U_{2} \left(\mu_{\nu} Y \frac{\partial U_{2}}{\partial y^{*}}\right) dy^{*} \right\}$$

$$\left. + \frac{A_{th2}}{\alpha_{t}} \ln Y_{T} + B_{th2} - \frac{1}{\mu_{\nu} C_{p}} \int_{y_{\nu}^{*}}^{y^{*}} \frac{1}{Y_{T}} \mu_{\nu} U_{2} \left(\mu_{\nu} Y \frac{\partial U_{2}}{\partial y^{*}}\right) dy^{*} \right\}$$

$$\left. + \frac{A_{th2}}{\alpha_{t}} \ln Y_{T} + B_{th2} - \frac{1}{\mu_{\nu} C_{p}} \int_{y_{\nu}^{*}}^{y^{*}} \frac{1}{Y_{T}} \mu_{\nu} U_{2} \left(\mu_{\nu} Y \frac{\partial U_{2}}{\partial y^{*}}\right) dy^{*} \right\}$$

$$\left. + \frac{A_{th2}}{\alpha_{t}} \ln Y_{T} + B_{th2} - \frac{1}{\mu_{\nu} C_{p}} \int_{y_{\nu}^{*}}^{y^{*}} \frac{1}{Y_{T}} \mu_{\nu} U_{2} \left(\mu_{\nu} Y \frac{\partial U_{2}}{\partial y^{*}}\right) dy^{*} \right\}$$

Substituting the analytical velocity and the analytical velocity wall-normal gradient equations (8.6) and (8.8) into the viscous dissipation term in equation (8.57), then:

$$\begin{split} &\int_{y_{\nu}^{*}}^{y^{*}} \frac{1}{Y_{T}} \mu_{\nu} U_{2} \left(\mu_{\nu} Y \frac{\partial U_{2}}{\partial y^{*}} \right) dy^{*} \\ &= \frac{1}{Y_{T}} \left\{ \frac{D_{2}}{3} \frac{1}{\alpha^{3}} \left[\frac{\alpha^{2}}{3} y^{*3} - \frac{\alpha \left(1 - \alpha y_{\nu}^{*}\right)}{2} y^{*2} + \left(1 - \alpha y_{\nu}^{*}\right)^{2} y^{*} - \left(1 - \alpha y_{\nu}^{*}\right)^{3} \frac{1}{\alpha} \ln Y \right] \right. \\ &+ C_{2} \frac{1}{\alpha} \left[y^{*} - \left(1 - \alpha y_{\nu}^{*}\right) \frac{1}{\alpha} \ln Y \right] + A_{2} \frac{1}{\alpha} \ln Y + B_{2} \left\{ \left(\frac{D_{2}}{3} y^{*3} + C_{2} y^{*} + A_{2} \right) dy^{*} \right\} \end{split}$$

As noted that the above equation is hard to integrate explicitly, in the code, this term is integrated numerically. The coefficients B_{th2} and A_{th2} in equation (8.57) are:

$$B_{th2} = A_{th1}y^* + BT + \frac{\mu_v}{\Pr}T_w$$
 (8.58)

$$A_{th2} = \left(\frac{D_{th1}}{3} - \frac{D_{th2}}{3}\right) y_{\nu}^{*3} + A_{th1}$$
(8.59)

where

$$BT = \frac{D_{th1}}{12} y_{\nu}^{*4} + \frac{C_{th1}}{2} y_{\nu}^{*2} - \left(\frac{N_7}{8} y_{\nu}^{*8} + \frac{N_5}{6} y_{\nu}^{*6} + \frac{N_4}{5} y_{\nu}^{*5} + \frac{N_3}{4} y_{\nu}^{*4} + \frac{N_2}{3} y_{\nu}^{*3} + \frac{N_1}{2} y_{\nu}^{*2}\right) \\ - \frac{D_{th2}}{3} \frac{1}{\alpha_t^3} \left[\frac{\alpha_t^2}{3} y_{\nu}^{*3} - \frac{\alpha_t \left(1 - \alpha_t y_{\nu}^*\right)}{2} y_{\nu}^{*2} + \left(1 - \alpha_t y_{\nu}^*\right)^2 y_{\nu}^*\right] - \frac{1}{\alpha_t} C_{th2} y_{\nu}^*$$

Substituting (8.58) and (8.59) into (8.57), then:

$$T_{2} = \frac{\Pr}{\mu_{\nu}} \left\{ \frac{D_{th2}}{3} \frac{1}{\alpha_{t}^{3}} \left[\frac{\alpha_{t}^{2}}{3} y^{*3} - \frac{\alpha_{t} \left(1 - \alpha_{t} y_{\nu}^{*}\right)}{2} y^{*2} + \left(1 - \alpha_{t} y_{\nu}^{*}\right)^{2} y^{*} - \left(1 - \alpha_{t} y_{\nu}^{*}\right)^{3} \frac{1}{\alpha_{t}} \ln Y_{T} \right] \right. \\ \left. + \frac{\ln Y_{T}}{\alpha_{t}} \left[\left(\frac{D_{th1}}{3} - \frac{D_{th2}}{3} \right) y_{\nu}^{*3} + A_{th1} \right] + A_{th1} y_{\nu}^{*} \right. \\ \left. + BT - \frac{1}{\mu_{\nu} C_{p}} \int_{y_{\nu}^{*}}^{y^{*}} \frac{1}{Y_{T}} \mu_{\nu} U_{2} \left(\mu_{\nu} Y \frac{\partial U_{2}}{\partial y^{*}} \right) dy^{*} \right\} + T_{w}$$

$$(8.60)$$

When $y^* = y_n^*$, and $T_2 = T_n$:

$$T_{w} = T_{n} - \frac{\Pr}{\mu_{v}} \left\{ \frac{D_{th2}}{3} \frac{1}{\alpha_{t}^{3}} \left[\frac{\alpha_{t}^{2}}{3} y_{n}^{*3} - \frac{\alpha_{t} \left(1 - \alpha_{t} y_{v}^{*}\right)}{2} y_{n}^{*2} + \left(1 - \alpha_{t} y_{v}^{*}\right)^{2} y_{n}^{*} - \left(1 - \alpha_{t} y_{v}^{*}\right)^{3} \frac{1}{\alpha_{t}} \ln Y_{Tn} \right] \right. \\ \left. + \frac{\ln Y_{Tn}}{\alpha_{t}} \left[\left(\frac{D_{th1}}{3} - \frac{D_{th2}}{3} \right) y_{v}^{*3} + A_{th1} \right] + A_{th1} y_{v}^{*} + BT - \frac{1}{\mu_{v} C_{p}} \int_{y_{v}^{*}}^{y_{n}^{*}} \frac{1}{Y_{T}} \mu_{v} U_{2} \left(\mu_{v} Y \frac{\partial U_{2}}{\partial y^{*}} \right) dy_{v}^{*} \right\}$$

$$(8.61)$$

For the isothermal wall boundary conditions:

$$q_{wall} = -\frac{\rho_v C_p \sqrt{k_P}}{\mu_v} A_{th1}$$
(8.62)

where A_{th1} can be obtained from (8.61) as:

$$A_{th1} = \frac{1}{\frac{\ln Y_{Tn}}{\alpha_t} + y_v^*} \left\{ \frac{\mu_v}{\Pr} \left(T_n - T_w \right) - \frac{D_{th2}}{3} \frac{1}{\alpha_t^3} \left[\frac{\alpha_t^2}{3} y_n^{*3} - \frac{\alpha_t \left(1 - \alpha_t y_v^* \right)}{2} y_n^{*2} + \left(1 - \alpha_t y_v^* \right)^2 y_n^* - \left(1 - \alpha_t y_v^* \right)^3 \frac{1}{\alpha_t} \ln Y_{Tn} \right] - \frac{\ln Y_{Tn}}{\alpha_t} \left(\frac{D_{th1}}{3} - \frac{D_{th2}}{3} \right) y_v^{*3} - BT + \frac{1}{\mu_v C_p} \int_{y_v^*}^{y_n^*} \frac{1}{Y_T} \mu_v U_2 \left(\mu_v Y \frac{\partial U_2}{\partial y^*} \right) dy^* \right\}$$
(8.63)

8.2.3 The Modification to the Dissipation Terms in the Main Code

From section 8.2.1 it became apparent that the viscous dissipation term in the energy equation can play an important role in the near-wall region, and in Section 8.2.2 an approximation for the term has been introduced into the analytical temperature profile within the AWF approach. The other place where the term does, potentially need accounting for is when considering the discretized energy equation over the near-wall cell. In the standard discretization the term will be evaluated at the cell centre, and then multiplied by the cell volume. However,

because of the rapid variation of velocity gradients across the region covered by the near-wall cell, such an approach is likely to lead to an inaccurate representation of it. The approach taken here has therefore been to develop a cell-averaged value for the term, based on the analytical profiles for the velocity and its gradient across the near-wall cell. This can then be used as a source term within the discretized energy equation for the near-wall cell. The viscous dissipation terms in the near wall region are constructed as:

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}U\right) = \frac{\rho^2 k_P}{\mu_w^2} \left[\mu\left(\frac{\partial U}{\partial y^*}\right)^2 + U\frac{\partial}{\partial y^*}\left(\mu\frac{\partial U}{\partial y^*}\right)\right]$$

To obtain a cell averaged value for the term, it is integrated over the two regions ($0 < y^* < y^*_v$ and $y^* > y^*_v$), to yield:

$$\overline{\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}U\right)} = \frac{\rho^{2}k_{P}}{\mu_{w}^{2}}\frac{1}{y_{n}^{*}}\left[\int_{0}^{y_{v}^{*}}\mu_{w}\left(\frac{\partial U_{1}}{\partial y^{*}}\right)^{2}dy^{*} + \int_{0}^{y_{v}^{*}}\mu_{w}U_{1}\frac{\partial^{2}U_{1}}{\partial y^{*2}}dy^{*} + \int_{y_{v}^{*}}^{y_{n}^{*}}\mu_{w}Y\left(\frac{\partial U_{2}}{\partial y^{*}}\right)^{2}dy^{*} + \int_{y_{v}^{*}}^{y_{n}^{*}}U_{2}\frac{\partial}{\partial y^{*}}\left(\mu\frac{\partial U_{2}}{\partial y^{*}}\right)dy^{*}\right]$$

$$(8.64)$$

The equations of the analytical velocity and the analytical velocity gradient (8.3) and (8.5) in the viscous sublayer, and (8.6) and (8.8) in the fully turbulent layer are substituted into equation (8.64) to obtain the cell-averaged viscous dissipation for AWF. In principle, a similar treatment could be applied to form a cell-averaged expression for use with the standard wall-function in the near-wall cells. In this case, of course, the velocity profile approximations would come from an assumed linear and log-law profile shape ($0 < y^* < y^*_{\nu}$ and $y^* > y^*_{\nu}$ respectively). This term could then be evaluated as:

$$\overline{\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}U\right)} = \frac{\tau_w^2}{\mu_w}\frac{1}{y_n^*}\left[y_v^* + \frac{1}{\kappa^2 E^2}\left(\frac{1}{y_v^*} - \frac{1}{y_n^*}\right) - \frac{1}{\kappa^2 E}\int_{y_v^*}^{y_n^*}\ln\left(Ey^*\right)\frac{1}{y^{*2}}dy^*\right]$$
(8.65)

To illustrate the effect that the above modifications have on the model predictions, Fig. 8.9 shows wall pressure, skin friction and heat flux results for the Mach 5 impinging shock interaction case, with shock deflection angle of 14°. The results labelled CMAWF are those including the viscous dissipation term in the analytical temperature equation, whilst those labelled CMAWF-D also include the cell-averaged viscous dissipation term in the near-wall-cell discretized equation, as described above. Similarly, the SWF-D results include



Figure 8.9: Wall properties comparison for Ma=5 impinging shock interaction case using CMAWF (left) and SWF (right) approaches

the equivalent term in the standard wall function. Then you can start describing the changes seen in the predictions. The main things are to note are that the addition of the viscous dissipation in only the analytical temperature expression leads to later separation (and only a small recirculation zone). Including the contributions in both the analytical profile and the main discretized equations give wall pressure and skin friction results fairly close to the MAWF ones but leads to the wall heat flux levels downstream of reattachment that is significantly closer to the LS ones than the original MAWF predicted.

For the comparison of SWF in Figure 8.9 (right), the addition of an approximated cellaveraged viscous dissipation term in the near-wall discretized equation does not appear to improve the predictions. This is likely due to a combination of the fact that the term is based on the assumed log-law velocity profile (which is unlikely to be accurate around the shock interaction region), and the simple nature of the other assumptions built into the SWF. In the later simulations, the standard wall function is therefore applied in its original form, as described in Chapter 6, without the above modification to account for viscous dissipation.

8.3 Accounting for the Variation in Fluid Properties

In the formulations of the AWF considered above, the transport properties such as the density, temperature, and pressure have been treated as constant across the near wall cells. In practice, for the hypersonic flows, the fluid properties vary significantly, especially where the strong SWBLIs happen, and strong temperature gradients always occur in the near wall region.

To illustrate the effect of the variation in molecular viscosity near the wall, Figure 8.10 shows predicted results using the LS model with the Yap correction for an impinging shock at Ma=5 and at Ma=8.2. Results are shown both with a constant value used for the viscosity and with Sutherland's law employed to give a temperature-dependent viscosity. At the lower Mach number, there are only rather minor differences in the predicted skin friction and wall heat flux, whilst these become rather larger in the higher Mach number case, as a result of the stronger near-wall temperature variation in hypersonic flows.

The variable properties accounted in the AWF involves two parts. One is deciding where to evaluate the viscosity and density values used in the definition of y^* and also in the model of μ_t , while the other is accounting for the variation of μ when integrating the simplified transport equations across the near-wall layer. Figure 8.11 shows the velocity and temperature distribution in the near wall region at the upstream, separation zone and downstream for M=8.2 impinging shock interaction. From the comparison, the velocity variation in the near wall region increases gradually, while the un-dimensional temperature decreases first and then increase to the inflow temperature, which means that the temperature variation has strong

gradient in the near wall region for the hypersonic SWBLIs, especially in the interaction region where the gradient of velocity and temperature gradients in the wall parallel direction and wall-normal direction become significant. Also, the variation of molecular viscosity at the separation zone and downstream have a difference with the constant values, which is consistent with the skin-friction and wall heat flux in Fig. 8.12.



Figure 8.10: Wall properties comparison using LS model with the Yap correction with constant molecular viscosity or Sutherland's Law for the Ma=5 (left) and Ma=8.2 (right) impinging shock interactions

In the previous sections, the non-dimensional distance y^* for the near wall cells is defined using the wall properties on the wall and the kinetic energy at cell P, as the kinetic energy is assumed as constant in the near wall cells.

$$y^* = \frac{y\sqrt{k_P}}{\mu_w}\rho_w \tag{8.66}$$

For some early studies of analytical wall functions, the fluid properties at node P are used for the definition of wall non-dimensional distance.



Figure 8.11: Velocity (top) and temperature (bottom) distribution at the upstream(left), separation zone (middle) and downstream (right) of the flowfield

$$y^* = \frac{y\sqrt{k_P}}{\mu_P}\rho_P \tag{8.67}$$

Also for the expression of turbulence viscosity is using the nodal value of viscosity:

$$\mu_t = \mu_P \alpha \left(y^* - y_v^* \right) \tag{8.68}$$

In the calculations, the above evaluations of y^* and μ_t , the numerical results are griddependence because the fluid properties might change rapidly with the distance from the wall. Gerasimov (2004) decides to evaluate these values at the edge of viscosity sublayers. The temperature at the edge of viscosity sublayer T_v is calculated from the thermal analytical wall function, and ρ_v and μ_v are obtained from T_v . Given that the best evaluation of kinetic energy is the nodal value, the non-dimensional distance is evaluated by:

$$y^* = \frac{y\sqrt{k_P}}{\mu_v}\rho_v \tag{8.69}$$

Accordingly, the turbulence viscosity is obtained by:

$$\mu_t = \mu_v \alpha \left(y^* - y_v^* \right) \tag{8.70}$$

In order to represent the strong gradients of temperature, the variation of the fluid molecular viscosity is taken into account for the analytical wall functions. Gerasimov (2004) did explore the use of linear, hyperbolic and parabolic variation assumptions for the molecular viscosity across the near-wall viscous layer, as show in Figure 8.12 and detailed in equations (8.71), (8.72) and (8.73) respectively.

$$\mu = \mu_{\nu} + b_{\mu} \left(y^* - y_{\nu}^* \right) \text{ where } b_{\mu} = \frac{\mu_{\nu} - \mu_{w}}{y_{\nu}^*}$$
(8.71)

$$\mu = \frac{\mu_{\nu}}{1 + b_{\mu} (y^* - y_{\nu}^*)} \quad \text{where} \quad b_{\mu} = \frac{\mu_{w} - \mu_{\nu}}{\mu_{w} y_{\nu}^*} \tag{8.72}$$

$$\mu = \frac{\mu_w}{1 + b_\mu y^* \left(y^* - 2y_\nu^*\right)} \quad \text{where} \quad b_\mu = \frac{1}{y_\nu^{*2}} \left(1 - \frac{\mu_w}{\mu_\nu}\right) \tag{8.73}$$



Figure 8.12: Variation of molecular viscosity in the near wall cell using linear (left), hyperbolic (middle) and parabolic (right) assumptions

These same formulations have been considered in the present work, for application in the high-speed flows considered here. For the linear assumption, when taken to the simplified momentum and energy equations, it is impossible to obtain a stable numerical solution after second integration. In order to make the derivatives easier, it is better to place the variation of dimensionless distance on the denominator. In this section, only the parabolic and hyperbolic assumptions are considered.

8.3.1 Parabolic Assumption

a. Hydrodynamic Analytical Wall Function

The simplified momentum equation is

$$\frac{\partial \left(\rho UU\right)}{\partial x} + \frac{\partial \left(\rho VU\right)}{\partial y} = -\frac{dP}{dx} + \frac{\partial}{\partial y} \left[\left(\mu + \mu_t\right) \frac{\partial U}{\partial y} \right]$$
(8.74)

which can be rewritten as:

$$\frac{\partial}{\partial y^*} \left[(\mu + \mu_t) \frac{\partial U}{\partial y^*} \right] = D y^{*2} + C$$
(8.75)

where

$$D = \frac{\mu^2}{\rho^2 k_P} \frac{1}{y_n^{*2}} \left[\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} \right] \quad C = \frac{\mu^2}{\rho^2 k_P} \frac{dP}{dx}$$
(8.76)

In the viscous sublayer ($y^* < y^*_{\nu}$), the first integration of equation (8.75), where $\mu_t = 0$, leads to:

$$\mu \frac{\partial U_1}{\partial y^*} = \frac{D_1 y^{*3}}{3} + C_1 y^* + A_1 \tag{8.77}$$

where

$$D_1 = \frac{\mu_v^2}{\rho_v^2 k_P} \frac{1}{y_v^{*2}} \left[\rho_P U_1 \frac{\partial U}{\partial x} + \rho_P V_1 \frac{\partial U}{\partial y} \right] \quad C_1 = \frac{\mu_v^2}{\rho_v^2 k_P} \frac{dP}{dx}$$
(8.78)

Using the parabolic formulation for the molecular viscosity across the sublayer gives:

$$\mu = \frac{\mu_{w}}{1 + b_{\mu} y^{*} (y^{*} - 2y_{\nu}^{*})} \quad \text{where} \quad b_{\mu} = \frac{1}{y_{\nu}^{*2}} \left(1 - \frac{\mu_{w}}{\mu_{\nu}} \right)$$

Equation (8.77) can then be rewritten as:

$$\frac{\partial U_1}{\partial y^*} = \frac{1}{\mu_w} \left[1 + b_\mu y^* \left(y^* - 2y_\nu^* \right) \right] \left[\frac{D_1 y^{*3}}{3} + C_1 y^* + A_1 \right]$$
(8.79)

The second integration gives:

$$U_{1} = \frac{1}{\mu_{w}} \left[\frac{D_{1}y^{*4}}{12} + \frac{C_{1}y^{*2}}{2} + A_{1}y^{*} + b_{\mu} \left(\frac{D_{1}}{18}y^{*6} - \frac{2D_{1}}{15}y^{*}_{\nu}y^{*5} + \frac{C_{1}}{4}y^{*4} + \frac{(A_{1} - 2C_{1}y^{*}_{\nu})}{3}y^{*3} - A_{1}y^{*}_{\nu}y^{*2} \right) \right]$$
(8.80)

In the fully turbulent layer $(y^* > y_v^*)$, the first integration of the simplified momentum equation is:

$$\frac{\partial U_2}{\partial y^*} = \frac{1}{\mu_v} \left[\frac{D_2}{3} \frac{y^{*3}}{Y} + C_2 \frac{y^*}{Y} + \frac{A_2}{Y} \right]$$
(8.81)

where:

$$D_2 = \frac{\mu_v^2}{\rho_v^2 k_P} \frac{1}{y_n^{*2}} \left[\rho_P U_2 \frac{\partial U}{\partial x} + \rho_P V_2 \frac{\partial U}{\partial y} \right] \quad C_2 = \frac{\mu_v^2}{\rho_v^2 k_P} \frac{dP}{dx}$$
$$Y = 1 + \alpha \left(y^* - y_v^* \right)$$

The second integration gives:

$$U_{2} = \frac{1}{\mu_{\nu}} \left\{ \frac{D_{2}}{3} \frac{1}{\alpha^{3}} \left[\frac{\alpha^{2}}{3} y^{*3} - \frac{\alpha \left(1 - \alpha y_{\nu}^{*}\right)}{2} y^{*2} + \left(1 - \alpha y_{\nu}^{*}\right)^{2} y^{*} - \left(1 - \alpha y_{\nu}^{*}\right)^{3} \frac{1}{\alpha} \ln Y \right] + C_{2} \frac{1}{\alpha} \left[y^{*} - \left(1 - \alpha y_{\nu}^{*}\right) \frac{1}{\alpha} \ln Y \right] + A_{2} \frac{1}{\alpha} \ln Y + B_{2} \right\}$$
(8.82)

From the near wall cell boundary conditions, the coefficients in the above equations are:

$$A_{1} = \frac{\mu_{\nu}U_{n} - N}{\frac{1}{\alpha}\ln Y_{n} + \frac{\mu_{\nu}}{\mu_{w}}y_{\nu}^{*} - \frac{2\mu_{\nu}}{3\mu_{w}}b_{\mu}y_{\nu}^{*3}}$$
(8.83)

$$A_2 = (D_1 - D_2)\frac{y_v^{*3}}{3} + (C_1 - C_2)y_v^{*} + A_1$$
(8.84)

$$B_{2} = \frac{\mu_{\nu}}{\mu_{w}} \left[\frac{D_{1}y_{\nu}^{*4}}{12} + \frac{C_{1}y_{\nu}^{*2}}{2} + A_{1}y_{\nu}^{*} - b_{\mu}y_{\nu}^{*3} \left(\frac{7}{90}D_{1}y_{\nu}^{*3} + \frac{5}{12}C_{1}y_{\nu}^{*} + \frac{2}{3}A_{1} \right) \right] - \left[\frac{D_{2}}{3} \frac{1}{\alpha^{3}} \left[\frac{\alpha^{2}}{3}y_{\nu}^{*3} - \frac{\alpha(1-\alpha y_{\nu}^{*})}{2}y_{\nu}^{*2} + (1-\alpha y_{\nu}^{*})^{2}y_{\nu}^{*} \right] + C_{2}\frac{1}{\alpha}y_{\nu}^{*} \right]$$
(8.85)

The coefficient N in (8.83) represents:

$$N = \frac{D_2}{3} \frac{1}{\alpha^3} \left[\frac{\alpha^2}{3} y_n^{*3} - \frac{\alpha \left(1 - \alpha y_v^*\right)}{2} y_n^{*2} + \left(1 - \alpha y_v^*\right)^2 y_n^* - \left(1 - \alpha y_v^*\right)^3 \frac{1}{\alpha} \ln Y_n \right] \right] \\ + C_2 \frac{1}{\alpha} \left[y_n^* - \left(1 - \alpha y_v^*\right) \frac{1}{\alpha} \ln Y_n \right] + \frac{1}{\alpha} \ln Y_n \left(\frac{1}{3} y_v^{*3} \left(D_1 - D_2\right) + \left(C_1 - C_2\right) y_v^* \right) \right] \\ + \frac{\mu_v}{\mu_w} \left[\frac{D_1 y_v^{*4}}{12} + \frac{C_1 y_v^{*2}}{2} - b_\mu y_v^{*4} \left(\frac{7}{90} D_1 y_v^{*2} + \frac{5}{12} C_1 \right) \right] \\ - \left[\frac{D_2}{3} \frac{1}{\alpha^3} \left[\frac{\alpha^2}{3} y_v^{*3} - \frac{\alpha \left(1 - \alpha y_v^*\right)}{2} y_v^{*2} + \left(1 - \alpha y_v^*\right)^2 y_v^* \right] + C_2 \frac{1}{\alpha} y_v^* \right]$$
(8.86)

b. Thermal Analytical Wall Function

The energy equation with the thermal dissipation terms in the near wall cell is:

$$\frac{\partial \left(\rho U E\right)}{\partial x} + \frac{\partial \left(\rho V E\right)}{\partial y} + \frac{\partial \left(U p\right)}{\partial x} + \frac{\partial \left(V p\right)}{\partial y} - \frac{\partial}{\partial y} \left(\mu U \frac{\partial U}{\partial y}\right) = \frac{\partial}{\partial y} \left[\lambda \frac{\partial T}{\partial y}\right]$$
(8.87)

which can be written as:

$$\frac{\partial}{\partial y^*} \left[\left(\frac{\mu}{\Pr} + \frac{\mu_t}{\Pr_t} \right) \frac{\partial T}{\partial y^*} \right] = D_{th} y^{*2} - \frac{1}{C_p} \frac{\partial}{\partial y^*} \left(\mu U \frac{\partial U}{\partial y^*} \right)$$
(8.88)

Thus in the viscous sublayer ($y^* < y_v^*$), where $\mu_t = 0$, equation (8.87) can be rewritten as:

$$\frac{\partial}{\partial y^*} \left(\frac{\mu}{\Pr} \frac{\partial T_1}{\partial y^*} \right) = \mathcal{D}_{th1} y^{*2} - \frac{1}{C_p} \frac{\partial}{\partial y^*} \left(\mu U_1 \frac{\partial U_1}{\partial y^*} \right)$$
(8.89)

where

$$D_{th1} = \frac{\mu_{\nu}^2}{C_p \rho_{\nu}^2 k_P} \frac{1}{y_{\nu}^{*2}} \left(\rho_P U_1 \frac{\partial E}{\partial x} + \rho_P V_1 \frac{\partial E}{\partial y} + \frac{\partial (Up)}{\partial x} + \frac{\partial (Vp)}{\partial y} \right)$$
(8.90)

The first integration of (8.89) gives:

$$\frac{\mu}{\Pr}\frac{\partial T_1}{\partial y^*} = \frac{D_{th1}}{3}y^{*3} + A_{th1} - \frac{1}{C_p}\mu U_1\frac{\partial U_1}{\partial y^*}$$
(8.91)

which, with the assumed parabolic formulation for μ , can be rewritten as:
$$\frac{\mu_{w}}{\Pr}\frac{\partial T_{1}}{\partial y^{*}} = \left[1 + b_{\mu}y^{*}\left(y^{*} - 2y_{\nu}^{*}\right)\right] \left[\frac{D_{th1}}{3}y^{*3} + A_{th1}\right] - \frac{1}{C_{p}}\mu\frac{\partial U_{1}}{\partial y^{*}}U_{1}\left[1 + b_{\mu}y^{*}\left(y^{*} - 2y_{\nu}^{*}\right)\right]$$
(8.92)

The coefficient in (8.92) can be obtained from the definition of wall heat flux:

$$A_{th1} = -\frac{q_{wall}}{c_p} \frac{\mu_v}{\rho_v \sqrt{k_P}}$$

The second integration gives:

$$T_{1} = \frac{\Pr}{\mu_{w}} \left\{ \begin{array}{c} \frac{D_{th1}y^{*4}}{12} + A_{th1}y^{*} + b_{\mu} \left(\frac{\frac{D_{th1}}{18}y^{*6} - \frac{2D_{th1}}{15}y^{*}_{\nu}y^{*5} + \frac{C_{th1}}{4}y^{*4}}{+\frac{(A_{th1} - 2C_{th1}y^{*}_{\nu})}{3}y^{*3} - A_{th1}y^{*}_{\nu}y^{*2}} \right) \\ -\int_{0}^{y^{*}} \frac{1}{C_{p}} \mu \frac{\partial U_{1}}{\partial y^{*}} U_{1} \left[1 + b_{\mu}y^{*} \left(y^{*} - 2y^{*}_{\nu} \right) \right] dy^{*} \end{array} \right\} + T_{w}$$
(8.93)

The integration of the thermal dissipation term is calculated numerically, using the analytical expressions for U_1 and its derivative from equations (8.79) and (8.80). In the fully turbulent region $(y^* > y_v^*)$, equation (8.87) can be rewritten as:

$$\frac{\partial}{\partial y^*} \left[\left(\frac{\mu_v}{\Pr} + \frac{\mu_t}{\Pr_t} \right) \frac{\partial T_2}{\partial y^*} \right] = D_{th2} y^{*2} - \frac{1}{C_p} \frac{\partial}{\partial y^*} \left(\mu U_2 \frac{\partial U_2}{\partial y^*} \right)$$
(8.94)

where:

$$D_{th2} = \frac{\mu_{\nu}^2}{C_p \rho_{\nu}^2 k_P} \frac{1}{y_n^{*2}} \left(\rho_P U_2 \frac{\partial E}{\partial x} + \rho_P V_2 \frac{\partial E}{\partial y} + \frac{\partial (Up)}{\partial x} + \frac{\partial (Vp)}{\partial y} \right)$$
(8.95)

The first integration of (8.94) gives:

$$\frac{\partial T_2}{\partial y^*} = \frac{\Pr}{\mu_v} \left(\frac{D_{th2}}{3} \frac{y^{*3}}{Y_T} + \frac{A_{th2}}{Y_T} - \frac{1}{C_p} \frac{1}{Y_T} \mu_v Y U_2 \frac{\partial U_2}{\partial y^*} \right)$$
(8.96)

The second integration gives:

$$T_{2} = \frac{\Pr}{\mu_{\nu}} \left\{ \begin{array}{l} \frac{D_{th2}}{3} \frac{1}{\alpha_{t}^{3}} \left[\frac{\alpha_{t}^{2}}{3} y^{*3} - \frac{\alpha_{t}(1 - \alpha_{t} y_{\nu}^{*})}{2} y^{*2} + (1 - \alpha_{t} y_{\nu}^{*})^{2} y^{*} - (1 - \alpha_{t} y_{\nu}^{*})^{3} \frac{1}{\alpha_{t}} \ln Y_{T} \right] \\ + \frac{A_{th2}}{\alpha_{t}} \ln Y_{T} + B_{th2} - \frac{1}{\mu_{\nu} C_{p}} \int_{y_{\nu}^{*}}^{y^{*}} \frac{1}{Y_{T}} \mu_{\nu} U_{2} \left(\mu_{\nu} Y \frac{\partial U_{2}}{\partial y^{*}} \right) dy^{*} \end{array} \right\}$$

$$(8.97)$$

where the thermal dissipation term again integrated numerically, this time using the analytical expressions for U_2 and its derivative from equations (8.81) and (8.82). The coefficients in equation (8.97) are:

$$B_{th2} = \frac{\mu_{\nu}}{\mu_{w}} \left(y_{\nu}^{*} - \frac{2}{3} b_{\mu} y_{\nu}^{*3} \right) A_{th1} + BT + \frac{\mu_{\nu}}{\Pr} T_{w}$$
(8.98)

$$A_{th2} = \left(\frac{D_{th1}}{3} - \frac{D_{th2}}{3}\right) y_{\nu}^{*3} + A_{th1}$$
(8.99)

where:

$$BT = \frac{\mu_{\nu}}{\mu_{w}} \begin{bmatrix} \frac{D_{th1}y^{*4}}{12} - b_{\mu} \left(\frac{7D_{th1}}{90} y_{\nu}^{*6} + \frac{5C_{th1}}{12} y_{\nu}^{*4} \right) \\ -\int_{0}^{y_{\nu}^{*}} \frac{1}{C_{p}} \mu \frac{\partial U_{1}}{\partial y^{*}} U_{1} \left[1 + b_{\mu} y^{*} \left(y^{*} - 2y_{\nu}^{*} \right) \right] dy^{*} \end{bmatrix} \\ -\frac{D_{th2}}{3} \frac{1}{\alpha_{t}^{3}} \begin{bmatrix} \frac{\alpha_{t}^{2}}{3} y_{\nu}^{*3} - \frac{\alpha_{t} (1 - \alpha_{t} y_{\nu}^{*})}{2} y_{\nu}^{*2} + (1 - \alpha_{t} y_{\nu}^{*})^{2} y_{\nu}^{*} \end{bmatrix}$$

Then take (8.98) and (8.99) into (8.97):

$$T_{2} = \frac{\Pr}{\mu_{\nu}} \left\{ \begin{array}{l} \frac{D_{th2}}{3} \frac{1}{\alpha_{t}^{3}} \left[\frac{\alpha_{t}^{2}}{3} y^{*3} - \frac{\alpha_{t}(1 - \alpha_{t} y_{\nu}^{*})}{2} y^{*2} + (1 - \alpha_{t} y_{\nu}^{*})^{2} y^{*} - (1 - \alpha_{t} y_{\nu}^{*})^{3} \frac{1}{\alpha_{t}} \ln Y_{T} \right] \\ + \frac{\ln Y_{T}}{\alpha_{t}} \left(\left(\frac{D_{th1}}{3} - \frac{D_{th2}}{3} \right) y_{\nu}^{*3} + A_{th1} \right) \\ + \frac{\mu_{\nu}}{\mu_{w}} \left(y_{\nu}^{*} - \frac{2}{3} b_{\mu} y_{\nu}^{*3} \right) A_{th1} + BT - \frac{1}{\mu_{\nu} C_{p}} \int_{y_{\nu}^{*}}^{y^{*}} \frac{1}{Y_{T}} \mu_{\nu} U_{2} \left(\mu_{\nu} Y \frac{\partial U_{2}}{\partial y^{*}} \right) dy^{*} \end{array} \right\} + T_{w}$$

$$(8.100)$$

When $y^* = y_n^*$, and $T_2 = T_n$:

$$T_{w} = T_{n} - \frac{\Pr}{\mu_{v}} \left\{ \begin{array}{l} \frac{D_{th2}}{3} \frac{1}{\alpha_{t}^{3}} \left[\frac{\alpha_{t}^{2}}{3} y_{n}^{*23} - \frac{\alpha_{t}(1 - \alpha_{t} y_{v}^{*})}{2} y_{n}^{*2} + (1 - \alpha_{t} y_{v}^{*})^{2} y_{n}^{*} - (1 - \alpha_{t} y_{v}^{*})^{3} \frac{1}{\alpha_{t}} \ln Y_{Tn} \right] \\ + \frac{\ln Y_{Tn}}{\alpha_{t}} \left(\left(\frac{D_{th1}}{3} - \frac{D_{th2}}{3} \right) y_{v}^{*3} + A_{th1} \right) \\ + \frac{\mu_{v}}{\mu_{w}} \left(y_{v}^{*} - \frac{2}{3} b_{\mu} y_{v}^{*3} \right) A_{th1} + BT - \frac{1}{\mu_{v} C_{p}} \int_{y_{v}^{*}}^{y_{n}^{*}} \frac{1}{Y_{T}} \mu_{v} U_{2} \left(\mu_{v} Y \frac{\partial U_{2}}{\partial y^{*}} \right) dy^{*} \right) \right\}$$

$$(8.101)$$

For the isothermal wall boundary conditions:

$$q_{wall} = -\frac{\rho_{\nu}C_p\sqrt{k_P}}{\mu_{\nu}}A_{th1}$$
(8.102)

where A_{th1} can be obtained from (8.101) as:

$$\begin{aligned} A_{th1} &= \frac{1}{\frac{\ln Y_{Tn}}{\alpha_t} + \frac{\mu_v}{\mu_w} \left(y_v^* - \frac{2}{3} b_\mu y_v^{*3} \right)} \left\{ \frac{\mu_v}{\Pr} \left(T_n - T_w \right) \right. \\ &- \frac{D_{th2}}{3} \frac{1}{\alpha_t^3} \left[\frac{\alpha_t^2}{3} y_n^{*23} - \frac{\alpha_t \left(1 - \alpha_t y_v^* \right)}{2} y_n^{*2} + \left(1 - \alpha_t y_v^* \right)^2 y_n^* - \left(1 - \alpha_t y_v^* \right)^3 \frac{1}{\alpha_t} \ln Y_{Tn} \right] \right. \\ &- C_{th2} \frac{1}{\alpha_t} \left[y_n^* - \left(1 - \alpha_t y_v^* \right) \frac{1}{\alpha_t} \ln Y_{Tn} \right] - \frac{\ln Y_{Tn}}{\alpha_t} \left(\left(\frac{D_{th1}}{3} - \frac{D_{th2}}{3} \right) y_v^{*3} + \left(C_{th1} - C_{th2} \right) y_v^* \right) \right. \\ &- BT + \frac{1}{\mu_v C_p} \int_{y_v^*}^{y_n^*} \frac{1}{Y_T} \mu_v U_2 \left(\mu_v Y \frac{\partial U_2}{\partial y^*} \right) dy^* \right\} \end{aligned}$$

$$(8.103)$$

8.3.2 Hyperbolic Assumption

a. Hydrodynamic Analytical Wall Function

In the viscous sublayer ($y^* < y_v^*$), the first integration of equation (8.74), where $\mu_t = 0$, is:

$$\mu \frac{\partial U_1}{\partial y^*} = \frac{D_1 y^{*3}}{3} + C_1 y^* + A_1 \tag{8.104}$$

where

$$D_1 = \frac{\mu_v^2}{\rho_v^2 k_P} \frac{1}{y_v^{*2}} \left[\rho_P U_1 \frac{\partial U}{\partial x} + \rho_P V_1 \frac{\partial U}{\partial y} \right] \quad C_1 = \frac{\mu_v^2}{\rho_v^2 k_P} \frac{dP}{dx}$$
(8.105)

Substituting the hyperbolic assumption of equation (8.72) into (8.104):

$$\frac{\partial U_1}{\partial y^*} = \frac{1}{\mu_{\nu}} \left[\left(\frac{D_1 y^{*3}}{3} + C_1 y^* + A_1 \right) + b_{\mu} \left(y^* - y_{\nu}^* \right) \left(\frac{D_1 y^{*3}}{3} + C_1 y^* + A_1 \right) \right]$$
(8.106)

The second integration gives:

$$U_{1} = \frac{1}{\mu_{\nu}} \left(\begin{array}{c} \frac{D_{1}y^{*4}}{12} + \frac{C_{1}}{2}y^{*2} + A_{1}y^{*} \\ +b_{\mu} \left(\frac{D_{1}y^{*5}}{15} - \frac{D_{1}y^{*}_{\nu}}{12}y^{*4} + \frac{C_{1}}{3}y^{*3} + \frac{(A_{1} - C_{1}y^{*}_{\nu})}{2}y^{*2} - A_{1}y^{*}_{\nu}y^{*} \right) \right)$$
(8.107)

In the fully turbulent layer $(y^* > y_v^*)$, the first integration of the simplified momentum equation gives:

$$\frac{\partial U_2}{\partial y^*} = \frac{1}{\mu_v} \left[\frac{D_2}{3} \frac{y^{*3}}{Y} + C_2 \frac{y^*}{Y} + \frac{A_2}{Y} \right]$$
(8.108)

The second integration gives:

$$U_{2} = \frac{1}{\mu_{\nu}} \begin{bmatrix} \frac{D_{2}}{3} \frac{1}{\alpha^{3}} \left[\frac{\alpha^{2}}{3} y^{*3} - \frac{\alpha(1-\alpha y_{\nu}^{*})}{2} y^{*2} + (1-\alpha y_{\nu}^{*})^{2} y^{*} - (1-\alpha y_{\nu}^{*})^{3} \frac{1}{\alpha} \ln Y \right] \\ + C_{2} \frac{1}{\alpha} \left[y^{*} - (1-\alpha y_{\nu}^{*}) \frac{1}{\alpha} \ln Y \right] + A_{2} \frac{1}{\alpha} \ln Y + B_{2} \end{bmatrix}$$
(8.109)

From the near wall cell boundary conditions, the coefficients in the above equations are:

$$A_{1} = \frac{\mu_{\nu}U_{n} - N}{\frac{\ln Y_{n}}{\alpha} + y_{\nu}^{*} - 0.5b_{\mu}y_{\nu}^{*2}}$$
(8.110)

$$A_2 = (D_1 - D_2)\frac{y_{\nu}^{*3}}{3} + (C_1 - C_2)y_{\nu}^* + A_1$$
(8.111)

$$B_{2} = \frac{D_{1}y_{\nu}^{*4}}{12} + \frac{C_{1}}{2}y_{\nu}^{*2} + A_{1}y_{\nu}^{*} - b_{\mu} \left(\frac{D_{1}}{60}y_{\nu}^{*5} + \frac{C_{1}}{6}y_{\nu}^{*3} + \frac{1}{2}A_{1}y_{\nu}^{*2}\right) - \left[\frac{D_{2}}{3}\frac{1}{\alpha^{3}}\left[\frac{\alpha^{2}}{3}y_{\nu}^{*3} - \frac{\alpha(1-\alpha y_{\nu}^{*})}{2}y_{\nu}^{*2} + (1-\alpha y_{\nu}^{*})^{2}y_{\nu}^{*}\right] + C_{2}\frac{1}{\alpha}y_{\nu}^{*}\right]$$
(8.112)

The coefficient N in (8.110) represents:

$$N = \frac{D_2}{3} \frac{1}{\alpha^3} \left[\frac{\alpha^2}{3} y_n^{*3} - \frac{\alpha(1 - \alpha y_v^*)}{2} y_n^{*2} + (1 - \alpha y_v^*)^2 y_n^* - (1 - \alpha y_v^*)^3 \frac{1}{\alpha} \ln Y_n \right] + C_2 \frac{1}{\alpha} \left[y_n^* - \left(\frac{1}{\alpha} - y_v^* \right) \ln Y_n \right] + \frac{1}{\alpha} \ln Y_n \left((D_1 - D_2) \frac{y_v^{*3}}{3} + (C_1 - C_2) y_v^* \right) + \left(\frac{D_1 y_v^{*4}}{12} + \frac{C_1}{2} y_v^{*2} - b_\mu \left(\frac{D_1}{60} y_v^{*5} + \frac{C_1}{6} y_v^{*3} \right) \right) - \left[\frac{D_2}{3} \frac{1}{\alpha^3} \left[\frac{\alpha^2}{3} y_v^{*3} - \frac{\alpha(1 - \alpha y_v^*)}{2} y_v^{*2} + (1 - \alpha y_v^*)^2 y_v^* \right] + C_2 \frac{1}{\alpha} y_v^* \right]$$
(8.113)

b. Thermal Analytical Wall Function

The simplified energy equation can be rewritten as:

$$\frac{\partial}{\partial y^*} \left[\left(\frac{\mu}{\Pr} + \frac{\mu_t}{\Pr_t} \right) \frac{\partial T}{\partial y^*} \right] = D_{th} y^{*2} - \frac{1}{C_p} \frac{\partial}{\partial y^*} \left(\mu U \frac{\partial U}{\partial y^*} \right)$$
(8.114)

where

$$D_{th} = \frac{\mu^2}{C_p \rho^2 k_P} \frac{1}{y^{*2}} \left(\frac{\partial \left(\rho U E\right)}{\partial x} + \frac{\partial \left(\rho V E\right)}{\partial y} + \frac{\partial \left(U p\right)}{\partial x} + \frac{\partial \left(V p\right)}{\partial y} \right)$$
(8.115)

Thus in the viscous sublayer ($y^* < y^*_v$), where $\mu_t = 0$, equation (8.114) can be rewritten as:

$$\frac{\partial}{\partial y^*} \left(\frac{\mu}{\Pr} \frac{\partial T_1}{\partial y^*} \right) = \mathcal{D}_{th1} y^{*2} - \frac{1}{C_p} \frac{\partial}{\partial y^*} \left(\mu U_1 \frac{\partial U_1}{\partial y^*} \right)$$
(8.116)

where

$$D_{th1} = \frac{\mu_{\nu}^2}{C_p \rho_{\nu}^2 k_P} \frac{1}{y_{\nu}^{*2}} \left(\rho_P U_1 \frac{\partial E}{\partial x} + \rho_P V_1 \frac{\partial E}{\partial y} + \frac{\partial (Up)}{\partial x} + \frac{\partial (Vp)}{\partial y} \right)$$
(8.117)

The first integration of (8.116) gives:

$$\frac{\mu}{\Pr}\frac{\partial T_1}{\partial y^*} = \frac{D_{th1}}{3}y^{*3} + A_{th1} - \frac{1}{C_p}\mu U_1\frac{\partial U_1}{\partial y^*}$$
(8.118)

Again, substituting the viscosity variation from equation (8.72) into (8.118) now gives:

$$\frac{\partial T_1}{\partial y^*} = \frac{\Pr}{\mu_{\nu}} \left[\left(1 + b_{\mu} \left(y^* - y_{\nu}^* \right) \right) \left(\frac{D_{th1}}{3} y^{*3} + A_{th1} \right) - \left(1 + b_{\mu} \left(y^* - y_{\nu}^* \right) \right) \frac{1}{C_p} \mu U_1 \frac{\partial U_1}{\partial y^*} \right]$$
(8.119)

The second integration gives:

$$T_{1} = \frac{\Pr}{\mu_{\nu}} \left[\begin{array}{c} \frac{D_{th1}y^{*4}}{12} + A_{th1}y^{*} + b_{\mu} \left(\frac{D_{th1}y^{*5}}{15} - \frac{D_{th1}y^{*}}{12}y^{*4} + \frac{A_{th1}}{2}y^{*2} - A_{th1}y^{*}_{\nu}y^{*} \right) \\ -\int_{0}^{y^{*}} \frac{1}{C_{p}} \left(1 + b_{\mu} \left(y^{*} - y^{*}_{\nu} \right) \right) \mu U_{1} \frac{\partial U_{1}}{\partial y^{*}} dy^{*} \end{array} \right] + T_{w} \quad (8.120)$$

In the fully turbulent region $(y^* > y_v^*)$, the simplified energy equation can be rewritten as:

$$\frac{\partial}{\partial y^*} \left[\left(\frac{\mu_v}{\Pr} + \frac{\mu_t}{\Pr_t} \right) \frac{\partial T_2}{\partial y^*} \right] = D_{th2} y^{*2} - \frac{1}{C_p} \frac{\partial}{\partial y^*} \left(\mu U_2 \frac{\partial U_2}{\partial y^*} \right)$$
(8.121)

where:

$$D_{th2} = \frac{\mu_{\nu}^2}{C_p \rho_{\nu}^2 k_P} \frac{1}{y_n^{*2}} \left(\rho_P U_2 \frac{\partial E}{\partial x} + \rho_P V_2 \frac{\partial E}{\partial y} + \frac{\partial (Up)}{\partial x} + \frac{\partial (Vp)}{\partial y} \right)$$
(8.122)

The first integration of equation (8.121) gives:

$$\left(\frac{\mu_{\nu}}{\Pr} + \frac{\mu_{t}}{\Pr_{t}}\right)\frac{\partial T_{2}}{\partial y^{*}} = \frac{D_{th2}}{3}y^{*3} + A_{th2} - \frac{1}{C_{p}}\mu U_{2}\frac{\partial U_{2}}{\partial y^{*}}$$
(8.123)

The second integration gives:

$$T_{2} = \frac{\Pr}{\mu_{\nu}} \left\{ \begin{array}{l} \frac{D_{th2}}{3} \frac{1}{\alpha_{t}^{3}} \left[\frac{\alpha_{t}^{2}}{3} y^{*3} - \frac{\alpha_{t}(1 - \alpha_{t}y_{\nu}^{*})}{2} y^{*2} + (1 - \alpha_{t}y_{\nu}^{*})^{2} y^{*} - (1 - \alpha_{t}y_{\nu}^{*})^{3} \frac{1}{\alpha_{t}} \ln Y_{T} \right] \\ + \frac{A_{th2}}{\alpha_{t}} \ln Y_{T} + B_{th2} - \frac{1}{\mu_{\nu}C_{p}} \int_{y_{\nu}^{*}}^{y^{*}} \frac{1}{Y_{T}} \mu_{\nu}U_{2} \left(\mu_{\nu}Y \frac{\partial U_{2}}{\partial y^{*}} \right) dy^{*} \end{array} \right\}$$

$$(8.124)$$

The coefficients are:

$$B_{th2} = A_{th1} y_{\nu}^{*} - \frac{1}{2} b_{\mu} A_{th1} y_{\nu}^{*2} + BT + \frac{\mu_{\nu}}{\Pr} T_{w}$$
(8.125)

$$A_{th2} = \left(\frac{D_{th1}}{3} - \frac{D_{th2}}{3}\right) y_{\nu}^{*3} + A_{th1}$$
(8.126)

where:

$$BT = \frac{D_{th1}y_{\nu}^{*4}}{12} - \frac{D_{th1}}{60}b_{\mu}y_{\nu}^{*5} - \int_{0}^{y_{\nu}^{*}}\frac{1}{C_{p}}\left(1 + b_{\mu}\left(y^{*} - y_{\nu}^{*}\right)\right)\mu U_{1}\frac{\partial U_{1}}{\partial y^{*}}dy^{*}$$
$$- \frac{D_{th2}}{3}\frac{1}{\alpha_{t}^{3}}\left[\frac{\alpha_{t}^{2}}{3}y_{\nu}^{*3} - \frac{\alpha_{t}\left(1 - \alpha_{t}y_{\nu}^{*}\right)}{2}y_{\nu}^{*2} + \left(1 - \alpha_{t}y_{\nu}^{*}\right)^{2}y_{\nu}^{*}\right]$$

Then take (8.125) and (8.126) into (8.124):

$$T_{2} = \frac{\Pr}{\mu_{\nu}} \left\{ \begin{array}{l} \frac{D_{th2}}{3} \frac{1}{\alpha_{t}^{3}} \left[\frac{\alpha_{t}^{2}}{3} y^{*3} - \frac{\alpha_{t}(1 - \alpha_{t} y_{\nu}^{*})}{2} y^{*2} + (1 - \alpha_{t} y_{\nu}^{*})^{2} y^{*} - (1 - \alpha_{t} y_{\nu}^{*})^{3} \frac{1}{\alpha_{t}} \ln Y_{T} \right] \\ + \frac{\ln Y_{T}}{\alpha_{t}} \left(\left(\frac{D_{th1}}{3} - \frac{D_{th2}}{3} \right) y_{\nu}^{*3} + A_{th1} \right) + A_{th1} y_{\nu}^{*} - \frac{1}{2} b_{\mu} A_{th1} y_{\nu}^{*2} + BT \\ - \frac{1}{\mu_{\nu} C_{p}} \int_{y_{\nu}^{*}}^{y_{\nu}^{*}} \frac{1}{Y_{T}} \mu_{\nu} U_{2} \left(\mu_{\nu} Y \frac{\partial U_{2}}{\partial y^{*}} \right) dy^{*} \right] \right\}$$

$$(8.127)$$

In order to complete the thermal wall function, the final expression for the wall temperature needs to be substituted into the code for prescribed heat-flux boundary conditions. From equation (8.127), when $y^* = y_n^*$, and $T_2 = T_n$:

$$T_{w} = T_{n} - \frac{\Pr}{\mu_{v}} \left\{ \begin{array}{l} \frac{D_{th2}}{3} \frac{1}{\alpha_{t}^{3}} \left[\frac{\alpha_{t}^{2}}{3} y_{n}^{*3} - \frac{\alpha_{t}(1 - \alpha_{t} y_{v}^{*})}{2} y_{n}^{*2} + (1 - \alpha_{t} y_{v}^{*})^{2} y_{n}^{*} - (1 - \alpha_{t} y_{v}^{*})^{3} \frac{1}{\alpha_{t}} \ln Y_{Tn} \right] \\ + \frac{\ln Y_{Tn}}{\alpha_{t}} \left(\left(\frac{D_{th1}}{3} - \frac{D_{th2}}{3} \right) y_{v}^{*3} + A_{th1} \right) + A_{th1} y_{v}^{*} - \frac{1}{2} b_{\mu} A_{th1} y_{v}^{*2} + BT \\ - \frac{1}{\mu_{v} C_{p}} \int_{y_{v}^{*}}^{y_{n}^{*}} \frac{1}{Y_{T}} \mu_{v} U_{2} \left(\mu_{v} Y \frac{\partial U_{2}}{\partial y^{*}} \right) dy^{*} \right] \right\}$$

$$(8.128)$$

For the isothermal wall boundary conditions:

$$q_{wall} = -\frac{\rho_v c_p \sqrt{k_P}}{\mu_v} A_{th1}$$
(8.129)

where A_{th1} can be obtained from (8.128) as:

$$A_{th1} = \frac{1}{\left(\frac{\ln Y_{Tn}}{\alpha_t} + y_v^* - \frac{b_{\mu}}{2}y_v^{*2}\right)} \left\{ \frac{\mu_v}{\Pr} \left(T_n - T_w\right) - \frac{D_{th2}}{3} \frac{1}{\alpha_t^3} \left[\frac{\alpha_t^2}{3} y_n^{*3} - \frac{\alpha_t \left(1 - \alpha_t y_v^*\right)}{2} y_n^{*2} + \left(1 - \alpha_t y_v^*\right)^2 y_n^* - \left(1 - \alpha_t y_v^*\right)^3 \frac{1}{\alpha_t} \ln Y_{Tn} \right] - \frac{\ln Y_{Tn}}{\alpha_t} \left(\frac{D_{th1}}{3} - \frac{D_{th2}}{3} \right) y_v^{*3} - BT + \frac{1}{\mu_v C_p} \int_{y_v^*}^{y_n^*} \frac{1}{Y_T} \mu_v U_2 \left(\mu_v Y \frac{\partial U_2}{\partial y^*} \right) dy^* \right\}$$
(8.130)

In order to test the effect of the above formulations, Figure 8.13 shows predicted results using them, and the CMAWF with constant viscosity, for Mach 3, 5 and 8.2 impinging shock cases. For the Mach number 3 case, the molecular viscosity variation has virtually no influence on the near wall properties, mainly because the analytical temperature, and only very slight differences in predictions between the approaches are seen around the interactions region. Slight difference only happens in the interaction region where the temperature gradient is much stronger than the upstream and downstream region. For the Mach number 5 case, the differences in the model predictions are more obvious than for the Mach 3 case, but still fairly small. However, at a Mach number of 8.2 the differences between the predictions using a constant viscosity and those using a variable form are quite significant, because of the stronger near-wall temperature gradients present in this case. Overall, the modelling of the variable viscosity decreases the predictions of wall skin friction and wall heat flux, bringing the latter much closer to the experimental data. As the inflow Mach increases, the assumptions become more important. For both assumptions, the numerical results show little difference at lower Mach numbers. Between the two forms adopted for the viscosity variation (parabolic and hyperbolic) there is little difference in the results. The hyperbolic one returns very slightly lower skin friction and wall heat flux levels, arguably slightly closer to the measured data, but the difference is quite marginal. In the calculations, both forms are used and compared. From the analytical molecular viscosity variation, the parabolic assumption gives closer prediction to the LS results, and it is recommended in the applications.

8.4 Summary of Modified Wall Functions

The development of the wall functions is implemented based on the structured finite-volume flow solver based on OpenFoam2.3 and 5.0. The wall shear stress and the wall heat flux for isothermal boundary condition (or wall temperature for adiabatic wall boundary condition) are replaced using the wall functions equation described above. For the turbulence parameters, the *k*-equation is solved in the near wall cell with the modified \bar{P}_k and $\bar{\varepsilon}$. Many modifications



Figure 8.13: Wall properties comparison using hyperbolic and parabolic assumptions for Ma=3 (top) Ma=5 (middle) and Ma=8.2 (bottom) impinging shock interactions

have been made compared to the original AWF as described in this Chapter. In order to give a general idea of the implementation of all the wall functions in this report, a short introduction to each method is given below.

8.4.1 Standard Wall Function (SWF)

The standard wall function is mainly described in chapter 6.1 based on the log-law velocity and temperature profiles. The wall shear stress is obtained from equation (6.12). The cell-averaged generation and dissipation rates of k are obtained from equation (6.10) and (6.11). The wall heat flux or wall temperature is obtained from equation (6.6) for the isothermal or adiabatic wall boundary condition.

8.4.2 Analytical Wall Function (AWF)

The original AWF was introduced in Chapter 6 in detail. The wall shear stress is obtained from Equation (6.34) with A_1 by equation (6.28), or by equation (A.2) when $y_n^* < y_v^*$. The cell-averaged production of k is obtained by equation (6.35). The mean dissipation rate of k is described in chapter 6.2.3 and is obtained by equation (6.39). The thermal analytical wall function is applied, depending on the thermal wall boundary conditions, as:

Adiabatic wall boundary condition:

When $y_n^* > y_v^*$, T_{wall} is obtained from equation (6.48). When $y_n^* < y_v^*$, T_{wall} is obtained from equation (A.4).

Isothermal wall boundary condition:

When $y_n^* > y_v^*$, q_{wall} is obtained from Equation (6.49) with A_{th1} by equation (6.50). When $y_n^* < y_v^*$, q_{wall} is obtained from Equation (A.5) with A_{th1} by equation (A.6).

8.4.3 Modified Analytical Wall Function (MAWF)

The modified analytical wall function was described in Section 8.1, with the parabolic assumption for the convection terms in the simplified momentum equation, and parabolic assumption for the convection terms in the simplified energy equation. The mean dissipation rate of *k* is calculated with the same equations as in the AWF. The wall shear stress is obtained from Equation (8.12) with A_1 by equation (8.9), or by equation (B.2) when $y_n^* < y_v^*$. The thermal analytical wall function depends on the wall boundary conditions:

Adiabatic wall boundary condition:

When $y_n^* > y_v^*$, T_{wall} is obtained from equation (8.37). When $y_n^* < y_v^*$, T_{wall} is obtained from equation (B.7).

Adiabatic wall boundary condition:

When $y_n^* > y_v^*$, q_{wall} is obtained from equation (8.63) with equation (8.39). When $y_n^* < y_v^*$, q_{wall} is obtained from equation (8.63) with equation (B.8).

From the comparison of MAWF-linear and MAWF-para at the end of Chapter 8.1, MAWF-para, which means parabolic assumptions are used for both convection terms in momentum and energy equations, is recommended in the future computation.

8.4.4 Compressibility in the thermal MAWF (CMAWF)

The CMAWF was introduced in Section 8.2 by adding thermal dissipation terms in the simplified energy equation to overcome the underestimation of wall heat flux by the MAWF. The wall shear stress and the mean dissipation rate of k are calculated with the same equations as in the MAWF. The thermal analytical wall function with the energy dissipation term is used is applied, depending on the wall boundary conditions, as:

Adiabatic wall boundary condition:

When $y_n^* > y_v^*$, T_{wall} is obtained from equation (8.61). When $y_n^* < y_v^*$, T_{wall} is obtained from equation (C. 2).

Adiabatic wall boundary condition:

When $y_n^* > y_v^*$, q_{wall} is obtained from equation (8.62) with A_{th1} by equation (8.63). When $y_n^* < y_v^*$, q_{wall} is obtained from equation (8.62) with A_{th1} by equation (C.3).

The viscous dissipation term in the discretized energy equation for the near-wall cell is also replaced by a cell-averaged value calculated from the analytical solution by equation (8.64).

8.4.5 CMAWF with Assumptions for Molecular Viscosity

The CMAWF with hyperbolic or parabolic assumptions for molecular viscosity was introduced in Section 8.3.

a. Parabolic Assumption (para-CMAWF)

The wall shear stress is obtained from Equation (8.12) with A_1 by equation (8.83), or by equation (D.3) when $y_n^* < y_v^*$, and the production of *k* is calculated by the analytical velocity gradient in the fully turbulent layer as (8.81). The mean dissipation rate of *k* is calculated with the same equations as MAWF. The thermal analytical wall function with the energy dissipation term is applied, depending on the wall boundary conditions, as:

Adiabatic wall boundary condition:

When $y_n^* > y_v^*$, T_{wall} is obtained from equation (8.101). When $y_n^* < y_v^*$, T_{wall} is obtained from equation (D. 5).

Adiabatic wall boundary condition:

When $y_n^* > y_v^*$, q_{wall} is obtained from equation (8.102) with A_{th1} by equation (8.103). When $y_n^* < y_v^*$, q_{wall} is obtained from equation (8.102) with A_{th1} by equation (D.6).

b. Hyperbolic Assumption (hyper-CMAWF)

The wall shear stress is obtained from Equation (8.11) with A_1 by equation (8.110), or by equation (E.1) when $y_n^* < y_v^*$, and the production of k is calculated by the analytical velocity gradient in the fully turbulence layer as (8.108). The mean dissipation rate of k is calculated with the same equations as MAWF. The thermal analytical wall function with the energy dissipation term is applied, depending on the wall boundary conditions, as:

Adiabatic wall boundary condition:

When $y_n^* > y_v^*$, T_{wall} is obtained from equation (8.128). When $y_n^* < y_v^*$, T_{wall} is obtained from equation (E. 2).

Adiabatic wall boundary condition:

When $y_n^* > y_v^*$, q_{wall} is obtained from equation (8.129) with A_{th1} by equation (8.130). When $y_n^* < y_v^*$, q_{wall} is obtained from equation (8.129) with A_{th1} by equation (E.3).

All these two assumptions are taken forward in the calculations of hypersonic cases, since for the supersonic flows the effect on the wall skin-friction and wall heat flux is negligible from the comparison in Section 8.3.

8.5 Summary

In this Chapter, a number of refinements have been explored and tested in the wall functions.

The first refinement is an approximation of the convection terms, in order to reduce the spikes that were seen around the shock interaction zone predicted by the original AWF. By assuming a near-wall parabolic variation in these terms the spikes were significantly reduced (or eliminated in some cases), which is named as MAWF.

As noticed that the MAWF does not reproduce the wall heat flux as the LS results, especially in the downstream region. The near wall terms in the energy equation are compared, and the thermal dissipation terms are significantly important for compressible flows. A simplified energy equation from the total energy equation with the thermal dissipation terms is used to obtain the analytical temperature in the near wall cell, together with the replacement of the thermal dissipation terms by the analytical solution in the main code, which is named as CMAWF.

For the hypersonic flows, where the temperature gradient is stronger than the supersonic flows, the molecular viscosity with hypersonic and parabolic assumptions is made in order to account for the effect of the strong gradients, referred to as hyper-CMAWF and para-CMAWF.

All the wall functions described above have been tested and implemented in the Open-FOAM v5.0. The CMAWF is recommended for supersonic flows, and para-CMAWF or hyper-CMAWF is recommended for the hypersonic flows.

Chapter 9

2D and Axisymmetric Impinging Shock Interaction

In this chapter, a number of the 2D and axisymmetric impinging shock interaction cases from supersonic to hypersonic are selected to evaluate the wall functions described above under OpenFoam v5.0. From the conclusion of Chapter 8 the most advanced CMAWF described in section 8.4.5 is used, but for simplicity, this will just be referred to as the AWF in the present chapter. The initial study by Settles and Dodson (1991) examined more than one hundred experimental studies of shock wave interaction with boundary layers from Mach number 3 to 13. High-quality data of hypersonic and supersonic SWBLIs for turbulence modelling are extremely scarce, and only 5 hypersonic experiments and 7 supersonic experiments are considered as acceptable. An updated version has been examined and summarized by Settles and Dodson (1994). Another useful summary for hypersonic flows is a result of deliberations with AGAED WG 18. The summary of all these acceptable results from all the sources mentioned above can be found in the experimental report of Ma=5 impinging shock interaction by Schülein et al. (2005). Four acceptable supersonic and hypersonic experiments have been selected from the references to be considered in the present study, which are listed in Table 9.1.

9.1 Case Setup

The freestream static temperature for all these four cases is quite low because the high enthalpy environments in the wind tunnels are hard to reproduce. In addition, all the test gas is air, so the working medium is assumed as a perfect gas, and air properties, such as the gas constant $R=287.06m^2S^{-2}K^{-1}$, specific heat coefficients for constant pressure

Experimental Configuration	Reference	Ma	$\operatorname{Re}_{\theta}$
ET PX	Reda and Murphy (1973) Schülein (2004) Kussoy and Horstman (1991)	2.9 5.0 8.18	47000 5920 4600
330 cm 20.3 cm diam 0° 	Kussoy and Horstmann, (1975)	7.2	13600

 $C_p=1004.06J/(kgK)$, molecular Prandtl number Pr=0.7 are used in the calculations, and Sutherland's law, $\mu=1.458\times10^{-6}T^{3/2}/(T+110.3)$, is used to evaluate the dynamic viscosity.

Structured grids are used for all four cases, because of the simple geometries. All grids are uniform in the streamwise direction and refined normal to the wall, and the value of y+ is about 25 for the high-Reynolds-number turbulence model with wall function approaches, and 0.6 for the low-Reynolds-number turbulence models.

The high Reynolds number k- ε model with wall functions and the low Reynolds number Lauder-Shamma (LS) model with the Yap correction are used for all four cases. The specification of a freestream turbulence intensity (*I*) is used to calculate the upstream turbulent kinetic energy from

$$k = \frac{3}{2} \left(|U|I \right)^2 \tag{9.1}$$

where I=1.5% for all 2-D impinging shock interaction cases. The dissipation of k is often determined by an assumed value for the ratio of turbulent to laminar viscosity μ_t/μ :

$$\varepsilon = C_{\mu} \frac{\rho k^2}{\mu} \left(\frac{\mu_t}{\mu}\right)^{-1} = C_{\mu} \frac{Re_{\infty}k^2}{|U|} \left(\frac{\mu_t}{\mu}\right)^{-1}$$
(9.2)

where $\mu_t/\mu=10$. For the axisymmetric case, μ_t/μ is taken equal to 100, in order to obtain an early transition from laminar to turbulent flow. Craft, Launder, and Suga (1997) suggest that the freestream intensity has little effect on the surface properties in the fully developed turbulent region, at least in the case of low-speed flows. For the supersonic and hypersonic flows, it is expected that there should be little effect of freestream turbulence levels on the near wall properties in the fully turbulent region where the shock wave/turbulent boundary layer interactions happen.

9.1.1 Ma=3 Impinging Shock Interaction

For the Ma=3 impinging shock interaction, the case is as described in Section 7.2. The mesh and boundary condition are as shown in figure 7.1, the freestream flow condition is listed in Table 7.1, and the flow properties across the oblique shock are listed in table 7.2. For the LS model with the Yap correction, the non-dimensional wall distance of the near wall node the upstream of separation is $y+\approx0.6$, while for the wall function approach, $y+\approx25$, which is much coarser than the grids in Chapter 7. The grid-independence study results for the Mach=3.0 impinging shock interaction are shown in Fig. 9.1 and 9.2 using the LS model or $k-\varepsilon$ model with wall function approaches. Overall, grid-independent results are obtained for all methods. The 200 × 90 grid is used for the LS model, while the 150 × 60 grid is used for wall function approaches.



Figure 9.1: Grid-independence study for the Mach=3.0 13^o impinging shock interaction predicted by the LS model with the Yap correction

9.1.2 Ma=5 Impinging Shock Interaction

For the Ma=5 impinging shock interaction, Schülein (2004) has obtained the surface pressure, skin-friction and wall heat flux as well as flow visualizations. The freestream flow condition



Figure 9.2: Grid-independence study for the Mach=3.0 13° impinging shock interaction predicted by the $k - \varepsilon$ model with AWF (left) and SWF (right)

is listed in Table 9.2, and the flow properties across the oblique shock are listed in Table 9.3 using the oblique shock relationships.

M_{∞}	$Re_{\theta 0}$	θ_0, mm	δ_0 , mm	$p_{t\infty}$, (MPa)	$T_{t\infty}, \mathbf{K}$	T_w, \mathbf{K}
5.0	5920	0.16	3.8	2.12	410	300

Table 9.2: Inflow conditions for Ma=5 impinging shock interaction

The computation mesh is similar to the Ma=3 impinging shock interaction case, except that the oblique shock elements are applied on the top boundary (Instead of the inlet boundary), in order to get the impinging interaction region at the required location. The inlet profiles, such as the velocity, pressure, and temperature are obtained from the flat plate computation under the same conditions as in Table 9.2 at the momentum thickness 16mm.

Grid-independence test results are shown in Fig. 9.3 and 9.4 for all methods. For the further computations and comparisons, the 240×80 grid is used for the LS model with the Yap correction, while the 120×45 grid is used for the wall function approaches.

$\beta(^{\circ})$	M_2	<i>u</i> _{2<i>x</i>} , m/s	<i>u</i> _{2y} , m/s	$p_2, N/m^2$	<i>T</i> ₂ , K
14	3.60	748.02	186.50	17605	113.97
10	4.00	779.93	137.52	12200	97.60
6	4.39	804.26	84.53	8080	84.28

Table 9.3: After oblique shock elements for Ma=5 impinging shock interaction



Figure 9.3: Grid-independence study for the Mach= $5.0 \ 14^{\circ}$ impinging shock interaction predicted by the LS model with the Yap correction



Figure 9.4: Grid-independence study for Mach=5.0 14° impinging shock interaction predicted by the *k*- ε model with AWF (left) or SWF (right)

9.1.3 Ma=7.2 Axisymmetric Impinging Shock Interaction

For the Ma=7.2 axisymmetric impinging shock interaction, the experiment geometry is shown in Fig. 9.4. The experimental instruments included pressure taps, transient thin-skin technique (surface heat transfer), surface shear, survey mechanism, pitot pressure probes, static pressure probes, and total temperature probes. This experiment was conducted in the Ames 3.5-Foot Hypersonic Wind Tunnel. The nominal free-stream test conditions were: total



Figure 9.5: Experimental configuration by Kussoy and Horstmann (1975)

temperature = 695K, total pressure = 34 atm, free-stream unit Reynolds number = 10.9×10^6 m⁻¹, free-stream Mach number = 7.2. The local free-stream conditions ahead of the incident shock are listed in Table 9.4 for shock generator angles of 7.5° and 15°. The undisturbed boundary layer information is listed in Table 9.5.

As noticed in Fig. 9.5, the experimental configuration is an axisymmetric cylinder with an axisymmetric shock generator. The wedge type is used for the front and back boundaries as suggested in the OpenFOAM tutorial, and the wedge angle is restricted to below 5° . In this case, the shock generator is included in the computation, and the mesh used for the

Table 9.4: Local freestream conditions for Ma=7.2 impinging shock interaction

$\beta(^{\circ})$	M∞	P∞	T∞	T _w	ρ_{∞}	U∞	T _o
7.5°	6.71	607	70.6	300	0.03	1129	695
15°	6.86	607	67.8	300	0.0312	1132	695

Table 9.5:	The undisturbed	boundary lay	er information	n for Ma=7.2	impinging sh	lock interac-
tion						



Figure 9.6: Computation domains for the Ma=7.2 β =15° impinging shock interaction



Figure 9.7: Grid-independence study for the Mach=7.2 15° impinging shock interaction predicted by the LS model with the Yap correction

wall function approach is shown in Figure 9.6. From the experimental report by Kussoy and Horstmann (1975), the flowfields near the shock generator are still laminar. In order to generate the impinging shock and expansion wave near the fin angle, a coarse mesh is used near the shock generator with y+=40 at the tip, while a fine mesh is used upstream of separation with y+=0.5. The mesh is refined towards the cylinder and generator from the mid-line of the computation domain as shown in Fig. 9.6. For the grid-independence study for the LS model with the Yap correction, as shown in Fig. 9.7, the mesh in the y-direction from the mid-line to the upper boundaries is fixed for all three meshes, and the number in the y-direction refers only to the mesh from the mid-line to the flat wall. From the comparison of the three meshes, the numerical results become converged as the grid number increases. For the later comparisons, the medium mesh, 420×105 , is used for the shock generator



Figure 9.8: Grid-independence study for the Mach=7.2 15° impinging shock interaction predicted by the $k - \varepsilon$ model with AWF (left) and SWF (right)

angles 7.5° and 15°. The grid-independence study results for the wall-function are shown in Fig. 9.8. The numerical results of the AWF are more grid-independent than the results of SWF, particularly since the SWF method fails to reproduce the separation region when using a coarse mesh. Overall, the 280×40 grid is used for the computation with wall function approaches, and this does reproduce a separated flow region using both the AWF and SWF methods.

9.1.4 Ma=8.2 Impinging Shock Interaction

Both the impinging shock interaction and double fin interaction at Mach number 8 were tested in the NASA Ames Research Center 3.5-ft hypersonic wind tunnel. The test bed is a sharp flat plate, 76cm wide, 220cm long and 10cm thick. A 2-D sketch of the experimental configuration is shown in Fig. 9.9 (top and middle), and the red region is the numerical simulation domain with a developed boundary layer. The entire test bed was water-cooled, and a constant surface temperature of 300K was maintained during a run. At 187cm from the leading edge, pitot pressure, static pressure and total temperature surveys were taken for the undisturbed boundary layer. The natural transition occurred between 50cm to 100cm from the leading edge. The impinging shock interacts with the fully developed turbulent boundary layer on the flat plate. From the experimental observation, separation happens when the wedge-shock generator angle is 10°, but not for the 5° case. The experimental data include surface pressure and heat transfer. The nominal free-stream test conditions were: total temperature 1166K, total pressure 60atm, and Mach number 8.2, and the detail of the freestream conditions are listed in Table 9.6. The mesh employed for the LS model is similar

M∞	$P_{\infty}(Pa)$	$T_{\infty}(K)$	$T_w(K)$	$U_{\infty}(m/s)$	$\delta(cm)$	δ *(cm)	$\theta(cm)$
8.18	430	81	300	1476	3.7	1.59	0.094

Table 9.6: Freestream conditions for Ma=8.18 impinging shock interaction

to that used in the Mach=7.2 impinging shock case, with refinements around both the shock generator wall and flat plate wall as shown in Fig. 9.9 (bottom).



Figure 9.9: Experimental configuration of 5° (top) and 10° (middle) by Kussoy and Horstman (1991) and the grid (bottom) for the Ma=8.2 10° impinging shock interaction

Figure 9.10 shows the grid-independence study for the LS model with the Yap correction, where the mesh in the y-direction from the mid-line to the upper boundaries is fixed for all three meshes, and the number in the y-direction refers only to the mesh from the mid-line to the flat wall. From the comparison, the numerical results become converged as the grid number increases. To illustrate the effect of y+ in the interaction region, two finer near-wall meshes were generated with y+=0.2 and y+=0.1 at the upstream. Figure 9.11 shows the y+ distribution along the wall and wall heat flux results for the Ma=8.2 impinging shock interaction using a 480×105 mesh. The numerical results show good grid independent. For the further computations, the medium mesh, 320×70 , is used for the shock generator angles of 5° and 10° using the LS model with the Yap correction. Figure 9.12 shows the grid independence for the wall function approaches, and the 160×60 mesh is used for the further computations.



Figure 9.10: Grid-independence study for Mach=8.2 impinging shock interaction at shock generator angles of 5° (left) and 10° (right) predicted by the LS model with the Yap correction



Figure 9.11: The y+ distribution (left) and wall heat flux (right) for the Ma=8 impinging shock interaction using a 480×105 mesh



Figure 9.12: Grid-independence study for Mach=8.2 impinging shock interaction at shock generator angle of 10° with AWF (left) and SWF (right)

9.2 **Results**

9.2.1 Ma=3 Impinging Shock Interaction

Figure 9.13 displays the Mach number contours for shock generator angles of 7° , 10° and 13° by the LS model with the Yap correction on 200×90 grid. As the shock generator angles increase, the impinging shock becomes stronger, the size of the separation zone increases, and the separation point is shifted further upstream. From the Mach number contours predicted by AWF and SWF in Fig. 9.14, both wall function approaches can capture the SWTBLIs in the near wall region, and the flow structures are similar to the sketch of shock impinging shock interaction in Fig. 2.2. However, the SWF predicts a bigger separation zone than the AWF results and LS results, which will become obvious from the later comparison.

Figure 9.14 compares the numerical results of wall pressure, skin friction and wall heat flux with experiments. For the 7° case, there is no flow separation seen in either the numerical or experimental results. The pressure distribution by both wall function approaches is close to the LS and experimental results. For the skin friction, the SWF returns the lowest values in the upstream and downstream regions. For the wall heat flux, the prediction by SWF is much lower than the results of LS and AWF, even upstream of the interaction. For the AWF results, they match the LS results well in the upstream region, while they are lower than the previous version of AWF as shown in the comparison of Chapter 8.

For the 10° case, all approaches return the same pressure and fit the experimental results well. For the skin friction, the wall function approaches return less separation than the LS results, which is obviously different from the numerical results in Chapter 7, in which the vanLeer limiter and a 200×100 mesh, are used in the computation. It is mainly because of the limiter difference since the vanAlbada limiter is used in this chapter and produce more dissipation than the vanLeer limiter. For the wall heat flux, the SWF returns totally different results to those of the LS results, while the AWF results are much closer to the low-Re model results.

For the 13° case the pressure contours how that the start of separation predicted by the SWF is earlier than the that of the LS model and AWF, while the LS model predicts that reattachment occurs later than the SWF and AWF. Broadly the same conclusion can be seen from the wall heat flux, but in this case, the SWF predicts very different results from the other modelling approaches. Overall, all three methods give a good prediction of the pressure with slight differences around the start of separation.

Figure 9.16 displays predicted and measured near-wall velocity profiles at eight different streamwise locations. Overall, all the approaches return good predictions at all eight locations.

At location 2, the SWF and LS predict the flow has separated, while the AWF is still unseparated. In the separation zone, locations 3-5, the numerical results tend to underestimate the mean velocity, while in the downstream region, locations 6-8, they tend to overestimate the mean velocity. The equivalent non-dimensional predicted temperature profiles are compared in Fig. 9.17. At location 1, the AWF returns a similar temperature distribution as the LS model, while the SWF returns lower unidimensional temperature than the others. At location 2, the SWF has already separated, and returns negative near wall unidimensional temperature, meaning that the wall heat flux becomes positive. At locations 3-8, all methods predict a negative unidimensional temperature, with the AWF being in general closer agreement with the LS results, particularly in the downstream region.

The analytical velocity and temperature profiles are obtained by the AWF in the near-wall cells, compared to the LS profiles, in the upstream and downstream regions are shown in Fig. 9.18. These two locations are x=0.116m and 0.4m which, from the Mach number contour plots of Fig. 9.13 can be seen to be far from the flow separation region. From the comparison, the analytical temperature and velocity fit the LS results perfectly in the upstream location. In the downstream location, the temperature and velocity at the edge of the first cell have slight differences, but in the near wall region the analytical solution has a similar gradient to the LS approach. Especially for the temperature distributions in the near wall region, the gradient is close to zero for this case, and the analytical solution can reproduce such gradient perfectly. On the other hand, the thermal SWF, which follows the log-law distribution, would not be expected to fit the distributions of temperature in the supersonic flow, at least for this Mach=3.0 case.

9.2.2 Ma=5 Impinging Shock Interaction

Figure 9.19 displays the Mach number contour and iso-lines of mean pressure at Mach 5, for shock generator angles $\beta = 6^{\circ}$, 10° and 14° predicted by LS model with the Yap correction on 420×105 grid.

From the Mach number contours, there is no separation for the 6° and 10° cases, while for the 14° case, flow separation does occur, as it did in the Mach 3.0 β =13° case. When we compare Fig. 9.19 with Fig. 9.13 at β =10° with different Mach numbers, it becomes apparent that the impinging shock angle increases as the inflow Mach number increases. In Fig. 9.20, the near wall temperature and velocity are compared between the LS model and wall function approaches at the same momentum thickness. From the near wall velocity comparison, the streamwise velocity at the near-wall cell predicted by AWF is closer to the LS model and experimental results. For the wall temperature distribution, the non-dimensional temperature is negative at the wall and then increases gradually with wall distance, which means that the wall heat flux is positive at the upstream location. The AWF approach predicts the similar near-wall cell temperature as the LS model and experimental results, while the SWF approach tends to underpredict the wall temperature at the upstream location.

In Fig. 9.21, the Mach number contours from the wall function approaches and low-Re model approaches show that the SWF predicts a larger flow separation region than the AWF and LS model, as was also seen in the Mach 3 case. From the surface pressure, skin-friction and wall heat flux comparison in Fig 9.22, the numerical results underestimate the separation when compared with the experimental data. To compare these results with predictions reported by others, Ali Pasha and Sinha (2008) reported results of the same case using a standard k- ω model, and with the addition of an empirical term, nominally developed to account for shock unsteadiness. Their results are also included in Fig. 9.22(c). In general, the standard k- ω model returns a small separation bubble and high skin friction in the downstream region, while the modified k- ω model improved the size of separation bubble, but returned a downstream skin friction values much lower than those from the LS model or experiments. In this section, the Stanton number is also compared, which is calculated by:

$$St = \frac{h}{\rho_{\infty}U_{\infty}C_p} = \frac{q_w}{\rho_{\infty}U_{\infty}C_p(T_{\infty} - T_w)}$$

which means that the Stanton number represents the wall heat flux. From the comparison at different shock generator angles, the AWF and SWF return different signs of wall heat flux, which is the same as seen in the Mach 3 case. Overall, the AWF approach returns the similar wall heat flux as that predicted by the LS model, while the SWF approach fails to predict the wall heat flux in upstream and downstream regions.

In Fig. 9.23a, the near velocity comparison for $\beta=6^{\circ}$ shows that the velocity in the downstream region by all approaches fit the experiment well, although the SWF returns slightly lower values than the AWF, leading to the slightly lower skin friction in Fig 9.22. The non-dimensional temperature variation by LS model decrease to negative values first and increase to the downstream temperature after the reflected shock. From Fig 9.22, the AWF and LS have the same sign of wall heat flux, which means that the analytical near-wall temperature should show a similar tendency to the LS profile. In Fig 9.23b, the near temperature for $\beta=6^{\circ}$ by AWF is closer to that of the LS model and experiments than the SWF, which returns a positive non-dimensional temperature which means that the SWF predicts a negative wall heat flux at all three downstream locations. For the comparison at $\beta=10^{\circ}$ in Fig. 9.24, the four locations are all downstream of separation region, and the same conclusions can be drawn as in the $\beta=6^{\circ}$ case. For the strong separation at $\beta=14^{\circ}$ in

Fig. 9.25, the first-cell temperature by the SWF and AWF at x-xI=26cm is close to the wall temperature, but have opposite signs to each other.

The analytical near-wall solution in the upstream and downstream regions is compared in Fig 9.26 for the β =14° case. For the velocity profile, the analytical solution of AWF is close to the LS results, leading to the skin friction predictions also being similar. For the temperature, similar profiles from the AWF and LS are also noticed. For this Mach 5 case, the non-dimensional temperature decreases to negative values first and then increases as the inlet or post-reflected shock temperatures. Overall, the AWF can reproduce the drop of temperature in the viscous sublayer and predict broadly the similar wall heat flux as LS model, while the log-law based thermal SWF fails to predict the wall heat flux accurately.

9.2.3 Ma=7.2 Axisymmetric Impinging Shock Interaction

Figure 9.27 displays the Mach number contours and iso-lines of mean pressure at Mach 7.2, for shock generator angles of 7.5° and 15° by LS model with the Yap correction on 240×80 grid. The shock generator is included in the computation domain in this case because now both the shock generated and the expansion fan produced from the corner of the shock generator interact with the incoming boundary layer. The expansion wave can be expected to reduce the pressure, skin-friction and wall heat flux when compared with the previous two cases.

Figure 9.28 shows the pressure contours for the 7.5° case predicted by the AWF, SWF, and LS approaches, respectively. The start of separation is between 48cm and 54cm from the leading point of the shock generator, and the AWF predicts the earliest separation, followed by the SWF and then the LS. Six locations are selected to compare the near wall velocity and temperature, as shown in Figs. 9.29 and 9.30 respectively. For the velocity comparison, the first cell velocity predicted by the SWF is slightly lower than that from the AWF and LS approaches in the upstream locations. The para-AWF and hyper-AWF in the figures denote the AWF approach with the parabolic or hyperbolic assumptions to the molecular viscosity as described in Section 8.3. Overall, the para-AWF and hyper-AWF return the same near wall velocity distribution, which also fit the LS results well. The SWF predicts a lower velocity in all six locations. The near wall gradient suggested by the experimental measurements is larger than all the numerical results at locations x=42, 48, 54 and 60cm. For the temperature comparison in Fig. 9.30, the measured non-dimensional temperature at all locations first decreases as one moves away from the wall, and then increases at a slower rate at all locations from the experimental data. The temperature predicted by the LS fits well with the experiments in the upstream and downstream locations. In the interaction region, the numerical results from all three modelling approaches underpredict the near wall gradient.

The analytical near-wall solution is compared in Figs. 9.31 and 9.32 at an upstream location and a downstream location for the β =7.5° case. From the comparison, the analytical solution in the near wall cell fits well with the experimental data and LS results. Especially for the temperature variation, the analytical solution reproduces the sharp gradient in the near-wall region and returns a similar near-wall temperature variation to the LS results.

Figure 9.33 displays the pressure contours for the 15° case by AWF, SWF and LS approaches, respectively and the interaction region starts around x=35.5cm. Six locations are selected, spinning from the upstream to the downstream regions, at which to compare the near wall properties. For the velocity comparison in Fig. 9.34, the numerical results have a slight difference from the experimental data, except the location x=35.5, where the flow is separated. The LS results are closer to the experimental data in most locations, and the para-AWF and hyper-AWF tend to improve it compared to the AWF results and tend to return the similar results as the LS results. At x=35.5cm, the LS returns negative velocity in the near wall region, which means the flow separation has begun by this location, while the AWF results return positive velocity which is close to the experimental results. For the temperature comparison in Fig. 9.35, a strong gradient of temperature variation exists in the near wall region in the experimental data, and the LS results can obtain such a strong gradient in all locations. The first cell temperature by wall function approaches varies from model to model. At most locations, the AWF results tend to predict the similar temperature as the experimental data, while the SWF seems to have the similar tendency as the LS results.

In order to validate whether the AWF can reproduce the near wall gradients of velocity and temperature, three locations, x=20cm, 35.5cm and 65cm, are chosen at which to compare the analytical results with the LS and experimental results. As noticed above, the parabolic and hyperbolic assumption return quite similar results to each other, so for clarity only the para-AWF results are included in Fig. 9.36 (at x=20cm), 9.37 (at x=35.5cm) and 9.38 (at x=65cm). At the upstream location x=20cm, the AWF produces similar near-wall gradients of velocity and temperature to the LS results, which means that the wall shear stress and wall heat flux predicted by the two models are also similar. At x=35.5cm, the para-AWF fits the experiment better than the LS. At x=65cm, the velocity by the numerical calculation is similar to each other but smaller than the experimental values, and the LS shows that the non-dimensional temperature drops first and increase in the near wall region. The AWF can reproduce this tendency and fits the experiment better than the LS.

The surface pressure, skin-friction and wall heat flux are compared by different approaches in Fig 9.39. For the 7.5° case, the numerical approaches all predict the similar peak pressure that matches the experimental value well. For the skin-friction, the para- and hyper-AWF have some fluctuations in the interaction region, and they predict smaller peak

skin-friction than the AWF and LS results. The SWF and para- (hyper-) AWF predict the similar peak skin-friction to each other, which is closer to the experiment. For the wall heat flux, the parabolic (or hyperbolic) assumptions of molecular viscosity decrease the peak wall heat flux and make the peak value closer to the experiment. The SWF underpredicts the wall heat flux in both the upstream and the downstream regions.

For the 15° case, Roy and Blottner (2006) have presented results from a number of other workers who used the k- ε Launder-Sharma, k- ε Jones-Launder, k- ε Rodi, and k- ω Wilcox. These numerical results are also included in Fig 9.39(b) for reference. The AWF tends to give higher peak pressure than the LS and SWF results. The LS results by Huang and Coakley (1993) tend to overpredict the wall pressure in the upstream region, and overpredict the peak value by 30%, while the JL model by Horstman (1991) obtains the same upstream pressure as the LS results by Huang and Coakley (1993) and overpredict the peak value by 20%. For the skin-friction, the AWF tends to give similar peak value as the LS results, while the SWF tends to underpredict the peak value compared to the LS results. All numerical approaches overpredict the peak experimental values by a factor of two. For the wall heat flux, the para-(or hyper-) AWF tend to return lower peak values of AWF. The LS method in this report is used with the Yap correction and tends to give more accurate results than the LS model without the lengthscale correction. Overall, the numerical results return higher values of peak pressure, skin-friction, and wall heat flux than the experimental results. The AWFs have some fluctuations in the interaction region, and the para- (hyper-) AWF tends to decrease the peak values. The SWF tends to predict the lowest wall heat flux values throughout the computation domain as was also seen in the Ma=3 and Ma=5 cases.

9.2.4 Ma=8.2 Impinging Shock Interaction

In this case, the shock generator is again included in the computation geometry, as it was in the Mach 7.2 case, in order to produce both the impinging shock from its leading edge and the expansion fan from its corner. Figure 9.40 displays the Mach number contours and iso-lines of mean pressure at Mach 8.2, shock generator angles of 5° and 10° predicted by the LS model with the Yap correction on a 320×70 grid. On the flat plate, there is no separation for these two cases. Figure 9.41 displays Mach number contours of the 10° case predicted by the different wall function approaches. All methods can capture the interaction at the bottom wall and show little difference between their flowfields.

Figure 9.42 displays the near-wall velocity and temperature comparison for the Ma=8.2 flat plate upstream boundary layer at momentum thickness θ =0.094cm. The first cell velocity by SWF is lower than LS results, while the para- (or hyper-) AWF returns higher velocity than the LS results. However, all approaches obtain a fairly similar near-wall velocity profile.

For the temperature distribution, the non-dimensional temperature decreases to negative values first and then increases to the far-field temperature. The first cell temperature by SWF is much lower than the LS and experimental results. The AWFs tend to return values similar to those of the LS and experimental measurements.

Figure 9.43a displays the comparison of surface results for the Ma=8.2 impinging shock interaction at the shock generator angle of 5° . For the surface pressure, all approaches return similar results to the experimental values, although the SWF tends to overpredict it in the downstream region. For the skin-friction, the AWF with hyperbolic and parabolic assumptions for the molecular viscosity tend to return smaller values than the AWF. The AWF and SWF obtain the similar skin-friction to each other and cannot capture the drop of skin-friction predicted by the LS. For wall heat flux, the hyper- (or para-) AWF obtains lower values than the LS and AWF results, and the values are much closer to the experimental measurements, especially in the downstream region. The SWF, although it overpredicts the near-wall temperature, returns a similar wall heat flux as the experimental measurements. From the thermal SWF Equ. (6.6), the wall heat flux is mainly decided by the first cell temperature to the experimental values, but significantly underpredicted the wall heat flux. In this case, however, the non-dimensional temperature is significantly underpredicted while the corresponding wall heat flux is close to the experimental measurements.

Figure 9.43b displays the wall surface results for the shock generator angle of 10° , this time also including numerical results reported by Horstman (1991) using k- ε Rodi (1991) and Smith (1996) using the k-l model. For the wall pressure, the k- ε Rodi tends to underestimate the peak pressure compared to the other models and wall function approaches. The SWF tends to get the highest peak value among all the methods. For the skin-friction, the SWF and AWF return similar results to the LS results, while the AWF cannot reproduce the separation predicted by the LS, and the SWF separation begins further upstream than the LS results. The AWF with parabolic or hyperbolic assumption returns lower skin-friction in the interaction region while returning similar values to the LS model in the upstream and downstream regions. They also return a similar separation region to the LS. For the wall heat flux, the various models return different peak values. The AWF returns the highest values which are close to the LS results. Among all methods, the AWF with parabolic or hyperbolic assumption fits the experimental measurements well throughout the domain length. The k- ε Rodi predicts a similar wall heat flux as the experimental measurements in the upstream and interaction regions but underestimates it in the downstream region by up to 20% as summarised by Roy and Blottner (2006).

Figure 9.44 shows near-wall velocity and temperature comparisons at different locations for the 10° shock generator case. Overall, the first cell velocity by wall function approaches is similar to the velocity distribution predicted by the LS model and experimental measurements. The near wall velocity by the SWF is lower than the LS, while the AWF is higher at all four locations. For the near-wall temperature, the SWF returns much lower values at all locations than the LS results. The AWF returns similar value to the LS in the upstream regions, lower values than the LS at around x=35mm, and higher values than the LS further downstream.

Figure 9.45 shows the analytical solution compared with the LS results for the 10° case. In the upstream region, the analytical velocity and temperature gradients by the AWF are larger than the LS results, while the hyper- (or para-) AWF fit the LS well. As a result, the AWF underestimates the wall heat flux and skin-friction in the upstream region. In the downstream region, the same tendency happens as the upstream. At location x=35cm, the hyper- (or para-) AWF have lower gradients of velocity and non-dimensional temperature than the LS results and returns the lowest wall heat flux and skin-friction compared to AWF and LS results.

Figure 9.46 displays the analytical molecular viscosity and turbulent viscosity in the near wall cells at different locations. For the AWF, the molecular viscosity is assumed to be constant in the near wall cell, while the para-AWF and hyper-AWF mean the viscosity has a parabolic or hyperbolic variation in the viscous sublayer as described in Section 8.3. From the comparison, the para-AWF is much closer to the LS results at all sampled locations. For the turbulent viscosity, the para-AWF and hyper-AWF predict similar results to the LS results. In the interaction region, the temperature rises quickly, and as a consequence so does the near-wall molecular viscosity. The para-AWF and hyper-AWF can capture this feature via their analytical expressions for the molecular viscosity. The molecular viscosity, which is calculated by the analytical temperature at the edge of viscosity sublayer, is used to evaluate the turbulence viscosity by equation (8.68). The rapid change of temperature in the near-wall region will affect the turbulence viscosity as described in Section 8.3 via the assumption of molecular viscosity and turbulence viscosity. The LS model has to model the rapid change of temperature to the damping terms via the molecular viscosity and then goes to the k and ε equations. The resolution of Lauder-Sharma $k \cdot \varepsilon$ is then used to construct the μ_t by equation (3.22). It seems that the LS response slowly to the sudden change of temperature in the near wall region, which will result in the overpredict of wall heat flux as shown in Figure 9.43.

9.2.5 Further Discussion of Numerical Results

It is obvious from the observation of the impinging shock interaction cases that the temperature variation in the near-wall region varies at different Mach numbers. In order to understand the difference between supersonic and hypersonic flows, some of the near wall properties are compared at different Mach numbers with the largest shock generator angles of each case, as displayed in Fig. 9.47. In the upstream and downstream regions, the non-dimensional temperature tends to drop to lower negative values as the Mach number increases from the LS results. For the SWF, the wall heat flux is obtained from the assumption of a log-law in the near wall region, and the accuracy of the first cell temperature will decide the accuracy of the wall heat flux as shown from the wall heat flux equation (6.6). For the AWF, the analytical temperature at different Mach number have shown more accurate prediction as the LS results, especially the at the upstream and downstream locations as displayed in the previous four sections.

9.3 Summary

In this section, the modified analytical wall function as described in Chapter 8 has been evaluated in supersonic and hypersonic impinging shock interaction flows. The main conclusions are:

- 1. The increase of shock angle will produce a stronger impinging shock. When the impinging shock is strong enough, a separation bubble develops, such as in the Ma=3 β =13°, Ma=5 β =13° and Ma=7.2 β =15° cases.
- 2. The wall pressure, predicted by the LS model, the *k*- ε model with AWF (para- and hyper- AWF for hypersonic cases) and SWF, fit the experimental data well, except in the case of the axisymmetric impinging shock interaction at Ma=7.2 and β =15°.
- 3. The skin friction for the Ma=3 case and Ma=5 case by the LS model fit the experimental data well across the range from strong to weak interactions. In the upstream region, the AWF is closer to the LS results. The SWF separates earlier than the LS and re-attaches also earlier than LS, while the AWF separates later and re-attaches earlier than LS. However, the SWF fails to reproduce the separation of the weak interactions. For the axisymmetric Ma=7.2 case, all the approaches over-predict the peak skin-friction, but in the upstream and downstream regions, all give good predictions. The AWF with hyperbolic and parabolic assumptions to the molecular viscosity tends to decrease the peak values of skin-friction for both Ma=7.2 and Ma=8 cases, which makes it closer to the experimental data compared to the AWF with constant molecular viscosity.
- 4. The wall heat flux by SWF is mainly decided by the near wall cell temperature from the thermal SWF equation (6.6). For the Ma=3, 5 and 7.2 cases, the SWF fails to

predict the wall heat flux over the entire plate length. For the Ma=8.2 case, the SWF does not predict non-dimensional temperature accurately when compared with the experimental data in the upstream region but fits the experimental wall heat flux well. Overall, the modified AWF in Section 8.2 returns good agreement with the experiment wall heat flux for Ma=3 and Ma=5 case. For the hypersonic cases, the AWF with the parabolic and hyperbolic assumption for the molecular viscosity as described in Section 8.3 decrease the peak wall heat flux and fit the experiment well.

- 5. The numerical results of mean velocity in the near-wall region from all the modelling approaches generally match the experimental well, especially in the upstream and downstream locations. As the start locations of the interaction are predicted differently by different approaches, the predicted mean velocity in this region does show greater differences between the different models.
- 6. For the near-wall non-dimensional temperature, the AWF give closer results with the LS than the SWF for Ma=3 and 5 cases in general. Especially in the upstream and downstream regions, the AWF fits the LS and experiment perfectly, since there is no separation and interaction. For the hypersonic cases, the AWFs fit the experimental data and LS well in the upstream and downstream region. In the interaction region, the AWFs fit the experimental data better than the SWF at most locations.
- 7. For the near wall analytical velocity at Ma=3.0 and 5.0, the AWF returns similar near-wall velocity gradient to the LS in the upstream and downstream regions, which means that the wall shear stress also closely matches that of the LS model. For hypersonic cases, the AWFs fit the LS and experimental data well in the upstream and downstream regions, while in the interaction region, the para- (or hyper-) AWFs is closer to the experimental data than the LS. For the Ma=8.2 case, the para- (hyper-) AWF capture the sharp near wall velocity gradient, which is close to the LS in the upstream and downstream regions. In the interaction region, the para- (hyper-) AWF return smaller gradients than the LS, which means they also return lower wall shear stress values by para- (hyper-) AWF, as shown by the skin-friction comparison.
- 8. For the near wall analytical temperature, the non-dimensional temperature gradient is close to zero in the near wall region at Ma=3.0, which agrees well with the LS model results. For hypersonic flows, the gradients at the wall are negative and then turn to positive at all locations. Overall, the AWFs return a negative gradient at the wall. For Ma=7.2 and 8.2 cases, para- (hyper-) AWF show differences from the AWF

with constant molecular viscosity, since there are significant temperature variations across the near-wall region.

Overall, all methods return good wall pressure predictions and show good agreement with the experiment except in the Ma=7.2 β =15° case. the AWFs predict the wall heat flux accurately compared to the experiments for the Ma=3.0, 5.0 and 8.2 cases. For the axisymmetric Ma=7.2 case, all numerical approaches, including those reported in previous studies, overestimate the peak wall heat flux. The thermal SWF is based on the log-law assumption and fails to predict the wall heat flux at all locations for Ma=3.0, 5.0 and 7.2 cases. For the Ma=8.2 case, it returns much smaller near wall cell temperature than the experimental measurements in the upstream region, but returns similar wall heat flux as the experiments.



Figure 9.13: Iso-lines of mean pressure superimposed on Mach number contours for Ma=3 impinging shock interaction with shock generator angles of $\beta=7^{\circ}$ (top), $\beta=10^{\circ}$ (middle), $\beta=13^{\circ}$ (bottom) predicted by the LS model with the Yap correction



Figure 9.14: Mach number contours for Ma=3 impinging shock interaction with shock generator angle of 13° predicted by the $k - \varepsilon$ model with AWF (top) and SWF (bottom)



Figure 9.15: Surface distributions of of wall pressure (top), skin-friction (middle) and wall heat flux (bottom) comparison at different locations for Ma=3 impinging shock interaction with shock generator angles of 7° (left), 10° (middle) and 13° (right)


Figure 9.16: Near-wall velocity comparison at different locations for Ma=3 impinging shock interaction with shock generator angle β =13°



Figure 9.17: Near-wall temperature comparison at different locations for Ma=3 impinging shock interaction with shock generator angle β =13°



Figure 9.18: The analytical velocity and temperature comparison at the upstream (top), separation region (middle) and downstream (bottom) of Ma=3 impinging shock interaction with shock generator angle β =13°



Figure 9.19: Iso-lines of mean pressure superimposed on the contours of Mach number Ma=5 impinging shock interaction with shock generator $\beta=6^{\circ}$ (top), $\beta=10^{\circ}$,(middle), and $\beta=14^{\circ}$ (bottom) using LS model with the Yap correction



Figure 9.20: Near-wall velocity (right) and temperature (left) comparison in the upstream boundary layer of the Ma=5 flat plate



Figure 9.21: The Mach number contour for Ma=5 impinging shock interaction with shock generator $\beta = 14^{\circ}$ by $k - \varepsilon$ model with AWF (top) and SWF (bottom)



Figure 9.22: Near-wall pressure (top), skin-friction (middle) and wall heat flux (bottom) comparison for Ma=5 impinging shock interaction with shock generator $\beta = 6^{\circ}$ (left) $\beta = 10^{\circ}$ (middle) and $\beta = 14^{\circ}$ (right)



Figure 9.23: Near-wall velocity (left) and temperature (right) comparison at different locations for Ma=5 impinging shock interaction with shock generator angle β =6°



Figure 9.24: Near-wall velocity (left) and temperature (left) comparison at different locations for Ma=5 impinging shock interaction with shock generator angle β =10°



Figure 9.25: Near-wall velocity (left) and temperature (right) comparison at different locations for Ma=5 impinging shock interaction using LS model with the Yap correction with shock generator angle β =14°



Figure 9.26: Analytical velocity (left) and temperature (right) comparison at selected upstream and downstream locations for the Ma=5 β =14° impinging shock interaction



Figure 9.27: Iso-lines of mean pressure superimposed on the contours of Mach number for Ma=7.2 impinging shock interaction using LS model with the Yap correction with shock generator angle β =7.5° (top) and β =15° (bottom)



Figure 9.28: Pressure contours of Ma=7.2 impinging shock interaction with shock generator angle β =15° predicted by the LS model with the Yap correction (bottom) and $k - \varepsilon$ model with SWF (middle) and AWF (top)



Figure 9.29: Near-wall velocity comparison at different locations for the Ma=7.2 β =7.5° impinging shock interaction



Figure 9.30: Near-wall temperature comparison at different locations for the Ma=7.2 β =7.5° impinging shock interaction



Figure 9.31: The analytical velocity (left) and temperature (right) comparison at an upstream location for the Ma=7.2 β =7.5° impinging shock interaction



Figure 9.32: The analytical velocity (left) and temperature (right) comparison at a downstream location for the Ma=7.2 β =7.5° impinging shock interaction



Figure 9.33: Pressure contours of Ma=7.2 impinging shock interaction with shock generator angle β =15° predicted by the LS model with the Yap correction (bottom) and $k - \varepsilon$ model with SWF (middle) and AWF (top)



Figure 9.34: Near-wall velocity comparison at different locations for the Ma=7.2 β =15° impinging shock interaction



Figure 9.35: Near-wall temperature comparison at different locations for the Ma=7.2 β =15° impinging shock interaction



Figure 9.36: The analytical velocity (left) and temperature (right) comparison at x=20cm for the Ma=7.2 β =15° impinging shock interaction



Figure 9.37: The analytical velocity (left) and temperature (right) comparison at x=35.5cm for the Ma=7.2 β =15° impinging shock interaction



Figure 9.38: The analytical velocity (left) and temperature (right) comparison at x=65cm for the Ma=7.2 β =15° impinging shock interaction



Figure 9.39: Surface distributions of of wall pressure (top), skin-friction (middle) and wall heat flux (bottom) comparison for the Ma=7.2 impinging shock interaction with shock generator angles β =7.5° (left) and β =15° (right)



Figure 9.40: Iso-lines of mean pressure superimposed on Mach number contours for Ma=8.2 impinging shock interaction with shock generator angles $\beta=5^{\circ}$ (top) and $\beta=10^{\circ}$ (bottom) predicted by the LS model with the Yap correction



Figure 9.41: Iso-lines of mean pressure superimposed on Mach number contours for Ma=8.2 impinging shock interaction with shock generator angles β =10° (bottom) predicted by the AWF (top), para-AWF (top middle), hyper-AWF (bottom middle) and SWF (bottom)



Figure 9.42: Near-wall velocity (left) and temperature (right) comparison at boundary layer momentum thickness of 0.094cm for the Ma=8.2 flat plate computation



Figure 9.43: Surface distributions of of wall pressure (top), skin-friction (middle) and wall heat flux (bottom) comparison for the Ma=8.2 impinging shock interaction with shock generator angles β =5° (right) and β =10° (left)



Figure 9.44: Near-wall velocity (left) and temperature (right) comparison at different locations for the Ma=8.2 β =10° impinging shock interaction



Figure 9.45: The analytical velocity (left) and temperature (right) comparison at different locations for the Ma=8.2 β =10° impinging shock interaction



Figure 9.46: The molecular viscosity (right) and turbulent viscosity (right) in the near wall cell at different locations for the Ma=8.2 β =10° impinging shock interaction



Figure 9.47: The temperature comparison in the upstream (left) and downstream (right) regions at different Mach number predicted by the $k - \varepsilon$ model with SWF (top), AWF (middle) and LS model (bottom)

Chapter 10

2D and Axisymmetric Compression Corner

In this chapter, the 2D and axisymmetric compression corner cases, ranging from supersonic to hypersonic flow conditions, are selected to evaluate the wall functions described above, using OpenFoam v5.0. The most advanced AWF in Chapter 8, CMAWF, which include the modifications to the convection terms in both simplified momentum and energy equations, and the inclusion of the thermal dissipation terms both in the main code and thermal wall function code, is evaluated by comparison with the Low-Re Launder-Sharma model and experimental data. The cases in this section are mainly from the reports by Settles and Dodson (1994) and by Schülein et al. (2015). Three suitable compression corner experiments have been selected from the references and listed in Table 10.1.

10.1 Case Setup

The case setup of the Ma=3 and Ma=7.2 compression corner is similar to that described in Section 9.1. However, in the Ma=9.2 case the test gas is nitrogen, which is taken as a perfect gas with properties of gas constant $R=296.8m^2S^{-2}K^{-1}$, specific heat coefficients for constant pressure $C_p=1039J/(kgK)$. The molecular Prandtl number Pr is 0.7, which is the same as air since the molecular weight is the same as air. The version of Sutherland's law in OpenFoam is taken from the books by Crane (1988) and CRC (2014), as $\mu=1.406\times10^{-6}T^{3/2}/(T+111)$. Structured grids are used for all three cases, and the grids are refined normal to the wall and the ramp corner. The non-dimensional near-wall cell sizes are similar to Chapter 9, namely y+ values of around 25 for the high-Reynolds-number turbulence model with wall

Experimental Configuration	Reference	Ma	$\operatorname{Re}_{\theta}$
$M = 9.2$ $\theta = 15^{\circ}, 30^{\circ}, 38^{\circ}$ $x_{c} = 56 \text{ cm}$	Smits and Muck (1987) Coleman and Stollery (1972)	3.0 9.2	75600 3800
			Re_L
FLOW 8 = 2 cm 20 cm DIA 139 cm	Kussoy and Horstman (1989)	7.2	4.7×10 ⁵ /cm

Table 10.1: Inflow conditions for compression corner cases

function approaches and 0.6 for the low-Reynolds-number turbulence models. The freestream turbulence quantities are evaluated in the same way as described in Section 9.1.

10.1.1 Ma=3 Compression Corner

For the supersonic compression corner, the freestream conditions are listed in Table 7.5, and the computation domain and boundary conditions are the same as shown in Fig. 7.2. For the LS model with the Yap correction, the non-dimensional wall distance of the near wall node at upstream of separation is $y+\approx0.6$, while for the wall function approach, $y+\approx25$, which is much coarser than the grids in Chapter 7. The results of a grid independence study for the Mach=3.0 compression corner are displayed in Fig. 10.1 and 10.2 using LS model or $k-\varepsilon$ model with wall function approaches. Overall, essentially grid independence results are obtained for both models. In subsequent comparisons the 280 × 250 grid is used for the LS model, while the 100 × 60 grid is used for the wall function approaches.

10.1.2 Ma=7 Axisymmetric Compression Corner

The Ma=7 axisymmetric compression corner measurements were conducted by Kussoy and Horstman (1989) in a 3.5-Foot Hypersonic Wind Tunnel with the experiment geometry as shown in Fig. 10.3. The experimental data include the surface pressure, wall heat-transfer



Figure 10.1: Grid-independence study for 24° compression corner at Mach=3.0 predicted by the LS model with the Yap correction



Figure 10.2: Grid-independence study for 24° compression corner at Mach=3.0 predicted by the $k - \varepsilon$ model with AWF (left) and SWF (right)

distributions and some flowfield properties in the undisturbed and the interaction region. The nominal free-stream test conditions were: total temperature = 900K, total pressure = 34 atm, free-stream unit Reynolds number = 7×10^6 m⁻¹, and free-stream Mach number = 7.2. Four flares were tested with angles of 20°, 30° 32.5° and 35°. From the oil-flow visualization technique, the flowfiled of flare angle 20° is un-separated, while separated for other three flare angles. In this report, the cased with the flare angles of 20° and 30° are chosen for the numerical tests.

Table 10.2: Local freestream conditions for Ma=7 axisymmetric compression corner

Ma _∞	P _o (KPa)	$T_o(K)$	P _∞ (Pa)	$T_{\infty}(K)$	$U_{\infty}(m/s)$	$T_w(K)$
7.11	2515	888.9	550.13	80.0	1274.7	311



Figure 10.3: Experimental configuration by Kussoy and Horstman (1989)

Georgiadis and Rumsey (2015) has compared the full-geometry case and no-cone geometry with a straight inflow section to match the boundary layer thickness in the experiments. From the numerical results, these two approaches revealed some differences, but the simplified no-cone geometry is deemed to be sufficient. The computation starts at x=-75cm upstream of the ramp corner. The inflow condition is adjusted slightly as in Table 9.2 with the boundary conditions as shown in Fig 10.4. A fine mesh is used for the low-Re models with refinement to the ramp wall so that the y+ on the ramp is lower than 1.0 as shown in Fig.4. For the wall function approaches, a simple mesh with uniform grid ratio is used and the y+ in the upstream region is around 25. To handle the axisymmetric geometry in OpenFoam, the grid shown in Fig.10.4 is extruded by rotating it through a 2-deg angle to form a small wedge shape, with OpenFoam's "wedge" boundary conditions applied on the periodic faces.



Figure 10.4: Computation domains for Ma=7 axisymmetric compression corner



Figure 10.5: Grid-independence study for 20° (right) and 30° (left) axisymmetric compression corner at Mach=7 with LS model with the Yap correction



Figure 10.6: Grid-independence study for 20° axisymmetric compression corner at Mach=7 using the $k - \varepsilon$ model with AWF (left) and SWF(right)

From the results of the grid-independence study for the LS model with the Yap correction in Fig. 10.5, the numerical results show great gird independence. For the wall function approaches, the numerical results of skin-friction show that it is largely grid-independent, except for some sharp spikes that appear around the ramp corner in the case of the AWF. These arise mainly because the convection terms near the corner show much bigger fluctuation when refined the mesh to the corner. For the subsequent comparisons the 300×120 grid is used for the Low-Re model computation, while the 100×50 grid is used for the wall function approaches.

Table 10.3 compares the displacement thickness and momentum thickness from the experiments by Kussoy and Horstman (1989) and numerical predictions by Georgiadis and Rumsey (2015) to those of these computations. The momentum thickness predicted by

Georgiadis and Rumsey (2015) is about 21% lower than the experiment, while the LS returns closer results with only 5% underprediction.

Case	δ^{*} (cm)	θ_0 (cm)
Experimental Data	0.735	0.0642
SA by Georgiadis and Rumsey (2015)	0.793	0.0526
SST-V by Georgiadis and Rumsey (2015)	0.762	0.0502
LS with the Yap correction	0.771	0.0612

Table 10.3: Comparison of inflow boundary layer thickness at s=-6cm

10.1.3 Ma=9.2 Compression Corner

The experiments by Coleman and Stollery (1972) were conducted at a Mach number of 9.22 in the Imperial College no. 2 Gun Tunnel using nitrogen as the test gas and the experimental configuration is shown in Fig. 10.7. The free stream and surface temperatures were 64.5K and 295K, respectively. The free stream Reynolds number was 0.47×10^6 /cm. The inflow conditions are listed in Table 10.4.



Figure 10.7: Experimental configuration by Coleman and Stollery (1972)

The experimental gas is nitrogen, and the gas properties are similar to air, and have been listed towards the beginning of this section. The numerical setup of nitrogen is described at the beginning of this section. The computation meshes for the corner angles of 15° and 34° are shown in Fig. 10.8. The boundary conditions are similar to those of the previous case, although since this is a 2D planar geometry the 'wedge' arrangement and boundary conditions are not employed.

Table 10.4: Local freestream conditions for Ma=9.22 compression corner

M∞	$P_{\infty}(Pa)$	$T_{\infty}(K)$	$T_w(K)$	$U_{\infty}(m/s)$	$\rho_{\infty}(\text{kg/m}^3)$	$\beta(^{\circ})$
9.22	2473.8	64.5	295	1509.4	0.129	15, 26, 32, 34, 38

The grid-independence study results for the LS model with the Yap correction in Fig. 10.9 shows that there is little difference in results when refined the mesh in the x- and

y-directions. For the wall function approaches, when the number of mesh cells is doubled to 80 in the y-direction, the numerical results show little difference between meshes, as shown in Fig. 10.10. For the later comparison the 240×150 grid is used for the Low-Re model computation, while the 180×80 grid is used for the wall function approaches.



Figure 10.8: The computation domain and mesh for the Ma=9.2 compression corners at β =15° (top) and β =34° (bottom) for the Low-Re models



Figure 10.9: Grid-independence study for the Mach=9.2 compression corners at β =15° (right) and β =34° (left) predicted by the LS model with the Yap correction



Figure 10.10: Grid-independence study for the Mach=9.2 β =38° compression corner predicted by th *k* – ε model with AWF (right) and SWF (left)

10.2 Results

10.2.1 Ma=3 Compression Corner

Figure 10.11 displays the iso-lines of mean pressure superimposed on Mach number contour at ramp angles of 16°, 20° and 24° by the LS model with the Yap correction on the 280×250 grid. As the ramp angles increase, the separation bubbles move further upstream. Figure 10.12 shows the Mach number contours for $\beta=24^\circ$ case predicted by using the AWF and SWF. Both wall function approaches do capture the ramp shock and separation bubble clearly. Obviously, the SWF returns larger separation bubbles compared with the AWF and LS.

Figure 10.13 compares the numerical results of wall pressure, skin friction and wall heat flux with experiments. For the wall pressure, all the numerical approaches fit the experimental data well. The SWF returns a larger interaction region than the AWF and LS for the β =20° and 24° cases, but while it agrees well with the experimental measurements for the β =20° case, the AWF and LS fit the experimental data better in the β =24° case.

For the skin friction, all numerical approaches predict a similar distribution of the skinfriction in general. The AWF agrees with the LS for all ramp angles. For the β =16° and 20° cases, the separation region is much smaller than the β =24° case. The SWF returns smaller skin-friction than that predicted by the LS upstream and downstream of the interaction region. In the downstream region the SWF is closer to the experiment results, while upstream of the interaction zone the AWF and LS are closer to the experiment. For the β =24° case, all the numerical approaches overestimate the skin-friction in the downstream region beyond s=4cm. The wall heat flux predicted by the AWF is closer to that predicted by the LS, while the SWF results are very different to the two other approaches, for all ramp angles, as was also seen in the cases in the previous chapter.

Figure 10.14 displays the near wall velocity comparison with experiments at eight different streamwise locations for the β =24° case. In the upstream region, all numerical approaches return the similar results as the experiment. In the upstream region, around s=-3.3cm, the SWF predicts larger near wall velocity than the AWF and LS, while the AWF fits the LS well throughout the flow domain, and both are closer to the experimental data than the SWF results are. From the comparison at location s=1.016cm, it can be concluded that the experimental results show the flow reattaches earlier than the numerical models predict, since the measurements show no reversed flow here, whilst the models still do. Figure 10.15 shows the near wall temperature comparison with experiments at eight different streamwise locations for the β =24° case. The AWF fits the LS results well at all locations, while the SWF underestimates the non-dimensional temperature at all locations, even in the upstream region.

Figure 10.16 displays the near wall velocity comparison for the β =20° case. The SWF predicts the interaction region to start further upstream than the other two models do, and the skin friction comparison of Fig. 10.13 showed this to broadly agree with the experimental location. At s=-1.11cm and s=-0.635cm, the SWF predicts that the flow has already separated, and the near-wall velocity profile it predicts is closer to the experimental measurements than the LS and AWF results are. At the downstream locations beyond s=1.27cm, the SWF overestimate the near wall velocity as the β =20° case. Figure 10.17 displays the near wall temperature comparison for the β =20° case. Overall, the similar conclusion can be made as the β =24° case. The SWF returns smaller non-dimensional wall temperature than that predicted by the AWF and LS, which agree with each other well at all locations.

Figures 10.18 and 10.19 show the analytical velocity and temperature comparison at three locations (one upstream of the interaction, one around the separation region, and one further downstream) for the β =24° case and the β =20° case respectively. The analytical velocity and temperature predicted by the AWF are similar near wall gradients to these predicted by the LS in the first near wall cell at all three locations.

10.2.2 Ma=7 Axisymmetric Compression Corner

Figure 10.20 shows Mach number contours for ramp angles of 20° and 30° predicted by the LS model with the Yap correction on the 300×120 grid. As the ramp angles increase, there is a small separation near the ramp corner. Figure 10.21 shows Mach number contours at

 β =30° predicted by the AWF and SWF. Both wall function approaches do capture the ramp shock but fail to reproduce the small separation bubble seen in the LS results.

Figure 10.21 compares the numerical results of wall pressure, skin friction and wall heat flux with experiments at β =20° and 30°. Results are also shown from simulation reported by Georgiadis and Rumsey (2015) using the one-equation Spalart-Allmaras (SA) and *k*- ω SST model. The wall function approaches using a high-Re turbulence model with coarser near-wall grid cells return lower wall pressures than the low-Re approaches in the interaction region. For the β =20° case, all the numerical results underestimate the wall pressure. For the β =30° case, the SST results show a peak pressure which is almost 20% higher than the experiment, while all the other methods return lower wall pressure than the experiment.

For the skin friction, all the numerical approaches return similar skin friction levels in the upstream region of the flow domain. There is a small separation region around the compression corner visible in the LS results for both ramp angles. The AWFs also predict a flow recirculation around the corner, but return a rather more negative skin friction in the revered flow region than the LS model. The SWF does not predict any separation in either ramp angle case. Downstream of the interaction zone, the AWF and SWF return the same values as each other, while the para-AWF and hyper-AWF return lower skin friction than the AWF, as was also seen in the hypersonic impinging shock interactions at Chapter 9. Overall, the skin friction predicted by the AWF with the hyperbolic and parabolic variation of molecular viscosity is closer to the LS results for both the 30° case in the downstream region.

For the wall heat flux, the SWF underestimates it, while other approaches fit the experimental data well in the upstream region for both ramp angles. For the β =20° case, the SA, SST, and AWF return similar wall heat flux to each other which is larger than the experimental data. The para- and hyper- AWF also return a similar wall heat flux to each other, which is closer to the LS results and the experimental data in the downstream region. The SWF returns much lower wall heat flux than other approaches. For the β =30° case, the SST returns a peak wall heat flux which is almost double the value measured experimentally. The wall heat flux predicted by the AWF is larger than the experimental data, as it was in the β =20° case, while the SA obtains a smaller value than the experiment. The wall heat flux by para- and hyper- AWF is a bit lower than the experimental measurements, again as was found in the β =20° case. Overall, the LS predicts precious wall heat flux downstream of both ramp angles. The SWF fails to predict the wall heat flux accurately throughout most of the flow domain. The wall heat fluxes predicted by the para- and hyper- AWF are similar to the experimental data for both cases.

Figure 10.23 displays the near wall velocity and temperature comparison with experiments at four different streamwise locations for the $\beta = 20^{\circ}$ case. Upstream location s=-6cm, the para- and hyper- AWF return larger first-cell velocity, while the SWF returns a lower value than the LS. Generally, all numerical approaches return similar results to the experimental measurements. The first-cell temperate predicted by the SWF is much larger than the LS and experiment. The AWFs return similar values to the experimental data. On the ramp, the numerical approaches return high gradients of velocity and temperature. For the velocity, the LS results fit the experimental data well, while the para- and hyper-AWF tend to close the LS data at the first cell. For the near-wall temperature, all the numerical approaches tend to overestimate the near-wall temperature. The wall function approaches return similar temperature distribution to each other at three different locations on the ramp. For these three locations, the numerical approaches return quite similar profiles at each of the three locations, while the wall temperature in the experiment varies from location to location. The wall heat flux from the experiment at these three locations is more or less the same, while the temperature distribution in the experiment suggests that the near-wall temperature gradient should be different at each location. This would not appear to be entirely consistent with the reported wall heat flux being nearly constant across the three locations.

Figure 10.24 shows the analytical velocity and temperature comparison with the LS data at four locations for the β =20° case. The analytical velocity and temperature profiles are close to the LS predicted behaviour in the first near-wall cell at all four locations. In particular, the analytical temperature on the ramp does capture the sharp gradient of temperature seen in the low-Re model results.

10.2.3 Ma=9.2 Compression Corner

Figure 10.25 shows Mach number contours at ramp angles of 15°, 26°, 32° and 38° by the LS model with the Yap correction on the 240×150 grid. Flow separation is only seen in the ramp corner β =38°.The increased grid non-orthogonality is what causes the slight irregularities in the contours seen around the shock over the ramp in the highest angle case. Figure 10.26 shows Mach number contours at β =15° and β =38° obtained by using the AWF and SWF. Both wall function approaches do capture the ramp shock and the AWF returns a smaller interaction region than that seen in the SWF and LS results. For the strong interaction case, the ramp shock by the wall function approaches is also wavy as the LS results, and the post-shock region seems irregular.

Figure 10.21 compares the numerical results of wall pressure, skin friction and wall heat flux with experiments at ramp angles of $\beta = 15^{\circ}$, 26° , 32° 34° and 38° . Also included in the figure are results for the 15° case from Huang and Coakley (1993), using the *k*- ε

Launder-Sharma without Yap correction and k- ω Wilcox models. All three low-Re models return a good prediction of surface pressure with slight underestimation at the ramp by $k \cdot \varepsilon$ LS and $k \cdot \omega$ Wilcox. The high-Re $k \cdot \varepsilon$ model with wall function approaches tends to underestimate the wall pressure in the interaction region at the ramp. For the skin-friction, the AWF tends to predict higher values than the LS models in the upstream region. The paraand hyper- AWF tend to return lower skin-friction than the AWF, giving values closer to the LS results in the upstream and downstream regions. For the wall heat flux, the LS with the Yap correction fits the experimental data well, while the LS results by Huang and Coakley (1993) yields wall heat transfer almost 25% higher than the experiment. The para- and hyper-AWF return lower wall heat flux values than the AWF with constant viscosity, agreeing well with the LS results in the regions upstream and downstream of the corner. However, they tend to underestimate the wall heat flux by around 8% in the interaction region. The SWF returns lower wall heat flux along the whole wall length, underestimating it by around 50% in the downstream region. For the $\beta = 26^{\circ}$ case, the interaction is not strong enough to separate the flow around the corner. The same conclusions can be made as in the $\beta = 15^{\circ}$ case, except that the wall heat flux predicted by the SWF is close to the hyper-AWF results.

For the β =32° case, the start of interaction region moves forward to the flat plate. The LS model returns an accurate wall pressure when compared to the experimental measurements, while the wall function approaches underestimate the peak pressure in the interaction region, as they did in the β =15° case. The LS model returns the lowest values, as it did in the previous two cases. Introducing the parabolic or hyperbolic viscosity variation with temperature into the AWF again results in a reduction in wall skin friction, particularly in the downstream region. For the Wall heat flux, the LS model predicts higher value than the experimental data, as it overpredicted the value in the β =15° case. The para- and hyper- AWF returns closer heat flux to the experimental data, which are more accurate than the LS results in the downstream region.

For the β =34° case, the numerical results by Horstman (1991) and Huang and Coakley (1993) are also included in the comparison. Rodi *k*- ε model gives the accurate wall pressure compared to the experimental measurements, while the other methods fail to capture the peak pressure. The wall skin-friction is similar to the previous case. The peak wall heat flux by the *k*- ε LS model (without the Yap correction) is almost two times higher than the experiment, and the *k*- ε Rodi model overpredicts the peak value by around 25%, while the LS model overpredicts the peak value within 5%. In the downstream region, para- and hyper-AWF predict the heat flux in good agreement with the experimental measurements.

For the $\beta = 38^{\circ}$ case, there is obvious separation around the ramp corner. The LS model underpredicts the extent of separation but gives accurate peak pressure compared to the

experimental measurements. The wall function approaches fail to predict the peak pressure accurately, as was also found in the $\beta = 34^{\circ}$ case. From the skin-friction, it can be seen that the SWF returns a larger separation region than the AWFs, and reruns higher peak skin friction with more than 50% overprediction than that predicted by other wall functions. In the downstream region, the para- and hyper AWF give lower skin friction as the LS model, which is the same as the previous cases, while the AWF and SWF return the similar skin friction as the LS results. For the wall heat flux, the SWF shows the opposite tendency to that seen in the lower ramp angle cases and returns wall heat flux values 40% higher than in the experiment. The LS returns much higher peak wall heat flux as the experiment at a location a bit further upstream, which is also seen from the LS model results of 34° case by Huang and Coakley (1993). The sudden increase of peak wall heat flux by the LS model with the Yap correction might be because of the large separation at this ramp angle, or because the LS model tends to give a higher wall heat flux near the flow re-attachment region as described by Huang and Coakley (1993). The para- and hyper-AWF fit the experiment better than other approaches in the downstream region, and tend to underestimate the wall heat flux by around 15%.

Figure 10.28 displays the near wall velocity and temperature comparison at three different streamwise locations for the β =15° case. At the upstream, s=-5cm location, the para- and hyper- AWF fit the LS profiles better than the AWF or SWF do in the near-wall region. The SWF gives much larger near-wall temperature than the LS model, while the AWFs gives lower values. At the ramp corner, s=0cm, the velocity distribution is similar to that seen further upstream. The non-dimensional temperature by the SWF is larger than that predicted by the LS in the upstream region, while the AWFs tend to give results similar to those of the LS. The same conclusion can also be drawn from the results at the further downstream locations.

Figure 10.29 displays the near wall velocity and temperature comparison at three different streamwise locations for the β =38° case. At the upstream, s=-5cm location, the flow stays undisturbed and similar near-wall velocity and temperature are obtained as in the previous β =15° case. At s=0cm, the flow is separated and the size of separation region predicted by the SWF is close to that shown in the LS results, so it is not surprising that the near-wall velocity shows that the SWF results fit the LS ones better than the AWFs results do. The same tendency is also seen in the temperature comparison. In the downstream region, the distribution of velocity and temperature show some fluctuations, as noted when discussing the Mach number contours of Fig. 10.25. Similar results are seen from all wall function approaches.
Figures 10.30 and 10.31 show the analytical velocity and temperature comparison with the LS data at three locations for the β =15° and 38° cases. The analytical velocity and temperature return similar near-wall gradients and fit the LS results well in the first near wall cell, except for the analytical velocity by para- and hyper- AWF at s=0cm in the β =15° case, where they return negative skin friction as shown in Fig. 10.27. Generally speaking, all analytical solutions do capture the sharp gradients of velocity and temperature near the wall, and the AWF with hyperbolic or parabolic variations for the molecular viscosity fit the LS better than the AWF with a constant molecular viscosity in most positions.

10.3 Summary

Predictions from the modified analytical wall function in Chapter 8, in addition to those from the LS model with the Yap correction, have been compared with several supersonic and hypersonic compression corner interaction flows. The available data for comparison includes surface pressure, skin-friction, heat transfer and some mean near wall properties such as velocity and temperature. The main conclusions are:

- 1. The increase of ramp angle will produce stronger ramp shock, and flow separation happens near the ramp corner when the ramp shock is strong enough. The separation happens for Ma=3 compression corner angles of β =16°, 20°, and 24°, Ma=7 axisymmetric compression corner angles of β =20°, and 30°, and Ma=9.2 compression corner angles of β =32°, 34°, and 38°.
- 2. For the Ma=3 compression Corner, the LS model returns accurate wall pressure compared to the experimental data at all ramp angles. The SWF predicts a larger interaction region and agrees with the experimental data well for the β =20° case, while the AWF and LS fit the experimental measurements better for the β =24° case. For the Ma=7 and Ma=9.2 cases, the wall function approaches fit the LS well in the upstream and downstream regions. For the Ma=7 cases, all the numerical models underpredict the wall pressure compared with the experiments.
- 3. The skin friction for the Ma=3 case predicted the LS model fits the experimental data well from strong to weak interactions. The AWF approaches are close to the LS results while the SWF predicts a larger interaction region. For the axisymmetric Ma=7 and Ma=9.2 cases, all the approaches return similar skin friction upstream of the interaction region, while the para- and hyper- AWF tend to be similar to the LS results in the downstream region, and the AWF and SWF overestimate the skin-friction.

- 4. As mentioned in the previous chapter, the wall heat flux by the thermal SWF is evaluated by Equ. (6.6) based on the log-law. For all the cases, the SWF fails to predict the wall heat flux accurately throughout the domain. Overall, the modified AWF in Chapter 8.2 returns a good agreement with the experimental data for the Ma=3 case. For two hypersonic cases, the AWF with the parabolic and hyperbolic assumption or the molecular viscosity as described in Chapter 8.3 decreases the peak wall heat flux and fits the experiment well.
- 5. Generally speaking, the numerical results of mean velocity in the near-wall region, from all the models, approach the experimental measurements, especially in the upstream and downstream locations. The accuracy of the extent of interaction will result in some differences of mean velocity distributions in the interaction region. The AWFs give closer non-dimensional temperature to that predicted by the LS than the SWF for all supersonic and hypersonic cases, especially in the upstream and downstream regions.
- 6. The AWFs capture the sharp gradients of velocity and temperature in the near wall region for all cases and predict the closer wall skin-friction and wall heat flux to the experimental data for almost all cases. For the hypersonic cases, the analytical solution of para- (hyper-) AWF improves the near wall velocity and temperature distributions by compared to those returned by the AWF, and approach the LS results at most locations.

For the compression corner interactions, all numerical methods show accurate wall pressure, except approximately 20% underestimation for the Ma=7.0 axisymmetric compression corner. For the hypersonic flows, the modifications to the AWF, such as the parabolic assumptions to the convection terms in the simplified momentum and energy equations, the dissipations term in the thermal AWF and parabolic or hyperbolic assumptions to the molecular viscosity, give better predictions of skin-friction and wall heat flux. The analytical solution in the near wall region does capture the rapid gradients of velocity and temperature as these by the LS model.



Figure 10.11: Iso-lines of mean pressure superimposed on Mach number contours for Ma=3 compression corner with ramp angle $\beta = 16^{\circ}$ (top), 20° (middle) and 24° (bottom) predicted by the LS model with the Yap correction



Figure 10.12: Iso-lines of mean pressure superimposed on Mach number contours for Ma=3 β =24° compression corner predicted by the *k* – ε model with AWF (top) and SWF (bottom)



Figure 10.13: Surface distribution of wall pressure (top), skin-friction (middle) and wall heat flux (bottom) comparison for Ma=3 compression corner with ramp angles β =16° (left), 20° (middle) and 24° (right)



Figure 10.14: Near-wall velocity comparison at different locations for the Ma=3 compression corner with a ramp angle β =24°



Figure 10.15: Near-wall temperature comparison at different locations for the Ma=3 compression corner with a ramp angle β =24°



Figure 10.16: Near-wall velocity comparison at different locations for the Ma=3 compression corner with a ramp angle β =20°



Figure 10.17: Near-wall temperature comparison at different locations for the Ma=3 compression corner with a ramp angle β =20°



Figure 10.18: Analytical velocity (left) and temperature (right) comparison at different locations for the Ma=3 compression corner with a ramp angle β =24°



Figure 10.19: Analytical velocity (left) and temperature (right) comparison at different locations for the Ma=3 compression corner with a ramp angle β =20°



Figure 10.20: Mach number contours for Ma=7 axisymmetric compression corner with ramp angles β =20° (left) and β =30° (right) predicted by the LS model with the Yap correction



Figure 10.21: Mach number contours for Ma=7 axisymmetric compression corner with ramp angles β =20° (left) and β =30° (right) predicted by the $k - \varepsilon$ model with AWF (left) and SWF (right)



Figure 10.22: Surface distribution of wall pressure (top), skin-friction (middle) and wall heat flux (bottom) comparison for the Ma=7 axisymmetric compression corner with ramp angles β =20° (right) and β =30° (left)



Figure 10.23: The near wall velocity (left) and temperature (right) comparison for the Ma=7 axisymmetric compression corner with a ramp angle β =20°



Figure 10.24: The analytical velocity (left) and temperature (right) comparison for the Ma=7 axisymmetric compression corner with a ramp angle β =20°

10.3 Summary



Figure 10.25: Mach number contours for the Ma=9.22 compression corner with ramp angles β =15°(top left), 26°(top right), 32°(bottom left) and 38°(bottom right) predicted by the LS model with the Yap correction



Figure 10.26: Mach number contours for the Ma=9.22 compression corner with ramp angles β =15°(top) and 38°(bottom) predicted by the *k* – ε model with SWF (left) and AWF (right)



Figure 10.27: Surface distribution of wall pressure (left), skin-friction (middle) and wall heat flux (right) comparison for the Ma=9.22 compression corner with ramp angles β =15° (row 1), 26° (row 2), 32° (row 3), 34° (row 4) and 38° (row 5)



Figure 10.28: The near wall velocity (left) and temperature (right) comparison for the Ma=9.22 compression corner with a ramp angle β =15°



Figure 10.29: The near wall velocity (left) and temperature (right) comparison for the Ma=9.22 compression corner with a ramp angle β =34°



Figure 10.30: The analytical velocity (left) and temperature (right) comparison for the Ma=9.22 compression corner with a ramp angle β =15°



Figure 10.31: The analytical velocity (left) and temperature (right) comparison for the Ma=9.22 compression corner with a ramp angle β =34°

Chapter 11

Conclusion & Future Work

11.1 Preliminary Remark

There are two approaches taken in the near wall turbulence modelling, which are Low Reynolds Number models and High-Reynolds k- ε model with wall function approaches, to account for the influence of low-Reynolds-number effects on the flow near the wall. The main aim of this project is to examine the performance of wall-function treatments which have been successfully used in a range of incompressible flows and explore how compressibility effects could be accounted for in such approaches. An advanced analytical wall-function is modified to account for the rapid changes of velocity and temperature in the near-wall region of complex supersonic or hypersonic flow fields with fast convergence rate, robustness, and reasonable accuracy.

For high-speed flow applications, the Reynolds number may be much higher than that in many incompressible flows, which results in very thin near-wall viscous layers. Especially for the shock wave turbulent boundary layer interaction, all fluid properties, such as density, pressure, temperature, velocity, Mach number etc, change a lot in the interaction zone, so modifications and corrections are introduced to the original analytical wall function, such as the continuity of convection terms and thermal dissipation terms in energy equations as describe in Chapter 8.

The open source CFD package OpenFoam v2.3.1 and v5.0 are used in the present work for the reason that new solvers or utilities can be created by users in a fairly convenient manner. The mathematical basis is developed by the finite volume method which is the most widely used approach in most commercial CFD codes. The convection, diffusion and sources terms of the governing equations are discretized by a variety of finite-difference-type approximations. This converts the integral equations into a system of algebraic equations, which can be solved by a variety of methods. In order to assess the performance of the wall functions, the effort has been made in the scope of 2-D supersonic and hypersonic cases.

11.2 Conclusions

This report has described the evaluation of the performance of the standard wall function approach and analytical wall function approach, both implemented within OpenFoam V2.3.1 and subsequently V5, by predicting the 2-D impinging shock interaction and compression corner. The Launder-Sharma k- ε model with the Yap correction is used to compare with the results of the high-Re k- ε model with the wall function approaches. Prior to the modelling explorations the effectiveness of the convective discretisation schemes within OpenFoam was also tested for the models and cases relevant to this investigation. The tests with different grids indicated that the used meshes gave grid-independent solutions, and the main conclusions are:

- Some modifications have been made to rhoCentralFoam solver, such as the Roe-Pike and AUSM+ flux-splitting methods in the construction of convection term. Three different cases have been used to validate the solver. From the results, the Roe-Pike method with van Albada limiter is recommended in the calculation of complex turbulence flow. The Low-Re model results in the two different versions of OpenFoam show good agreement with each other.
- 2. For the high-Re impinging shock interaction case, the current simulations predict that when the impinging shock is strong enough, the interaction zone has a separation bubble, whose size increases as the impinging angle grows, in agreement with what has been reported experimentally.
- 3. For the low-Re impinging shock interaction case, the SWF approach underestimates the SWTBLI and fails to reproduce the separation bubble in the near wall region.
- 4. From the results of compression corner by using the LS model, all three cases, namely the 16°, 20°, and 24°, have a separation bubble near the corner, and the size increases with the ramp angle. The SWF approach tends to underestimate the SWTBLI, especially for the 16° case, it fails to reproduce the separation bubble.
- 5. In all cases the wall pressure, predicted by all three models, namely the LS model and the high-Re k- ε model with SWF and AWF, is in good agreement with the experiment.

6. From the mean velocity comparisons in all cases, all three models generally result in good predictions at different locations, especially where the locations are outside of the separation zone.

The overall picture that emerges from the above comparisons is as follows. First all three models display a good prediction of wall pressure. However, the SWF approach tends to underestimate the SWTBLI, and for the 16° compression corner and low-Re impinging shock interaction case, where small separation happens, it fails to reproduce the separation bubble. The analytical wall function of Craft et al (2002) on the other hand, is able to reproduce the shock-induced flow separation and returns predictions similar to those of the low-Re model.

The original analytical wall function described in Chapter 6 uses the simplified momentum and energy equations and turbulence viscosity assumption to calculate the wall shear stress, the production of the turbulent kinetic energy, the wall temperature under isothermal conditions, or the wall heat under adiabatic conditions. When applied to the hypersonic cases, the original AWF resulted in severe oscillations near the interaction and larger wall heat flux than expected. This was traced to the assumption of spatially constant convection in the simplified analytical momentum and energy equations. It was subsequently corrected by imposing a parabolic variation of the convection terms with wall distance.

Another predictive flaw of the original AWF was the under-prediction of wall heat flux for cases with Mach number higher than 5. This was found to be caused by the analytical energy equation omitted the effects of viscous dissipation. This was addresses by the inclusion of a simplified form of the viscous dissipation term (which only includes velocity gradients in the wall-normal direction).

Finally, a further predictive weakness of the AWF at the highest Mach numbers was the over prediction of the wall heat flux. The cause of this was very strong near-wall temperature variations, resulting in strong near-wall molecular viscosity variations, not taken into account in the original AWF. This was addressed by the inclusion of a variable molecular viscosity across the viscous-sub-layer, based on the analytical temperature variation over this region.

The predictions resulting from the inclusion of all the above modifications to the AWF are found to be in close agreement with both the experimental data and the Lauder-Sharma model predictions and in fact for the very high Mach numbers (7-9) the present predictions are even closer to the experimental data than those of the Low-Re model.

11.3 Future Work

The SWTBLI cases in Chapter 2 were reported as a group of flows with different and important features to be modelled. There are experimental and DNS cases reported of such

flows with abrupt changes, such as the unsteadiness of the shock system in the compression corner by Dolling and Murphy (1983) and the 3-D sharp fin interaction by Hsu and Settles (1992). For all these cases, the turbulent boundary layer interacts with the shock and causes separation, reattachment, expansion, and reflected shocks, which result in a rather complicated interaction and lag between the mean and turbulent flowfields, so the standard wall functions are not expected to perform all that well. As noted above, in some cases, the standard wall function is believed to be able to accurately predict the surface pressure but fails to predict the wall heat flux and reproduce the separation for some cases. It will be informative to apply the compressible form of the AWF proposed here to these cases as well.

Although the analytical wall function approach performs better than the standard wall function so far, it will also be beneficial to test other forms of advanced wall function approach, called subgrid-based wall function by Gant (2002).

Besides the two equation models used in this thesis, other low-Re number models enable accurate CFD computations for a range of difficult flows, such as non-linear eddy-viscosity models and second-moment closures. In order to overcome the inevitable high computational cost, it is recommended to use this new AWF with these advanced RANS models for the compressible flows.

It is believed that the work proposal above involves a lot of extensive challenging work (which thus characterizes a Ph.D. course) and has the ability to generate results and findings that will be of relevance and use in predicting high-speed turbulent flows in an industrial context.

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Appendix A

AWF Approach When $y_n^* < y_v^*$

a. Hydrodynamic Analytical Wall Function

After the second integration of the simplified momentum equation, the analytical velocity in the near wall region is

$$\mu U_1 = \frac{C_1}{2} y^{*2} + A_1 y^* \tag{A.1}$$

The continuity condition $U_1|_{y^*=y_n^*} = U_n|_{y^*=y_n^*}$ is applied, and the coefficient A_1 is calculated by:

$$A_1 = \frac{1}{y_n^*} \left(\mu_w U_n - \frac{C_1}{2} y_n^{*2} \right)$$
(A.2)

The wall shear stress is obtained by the definition as equation (6.32), and the production of k is equal to zero since the turbulence viscosity is assumed to zero in the viscous sublayer.

b. Thermal Analytical Wall Function

After the second integration of the simplified energy equation, the analytical temperature in the near wall region is

$$T_{1} = \frac{\Pr}{\mu_{w}} \left(\frac{C_{th1} y^{*2}}{2} + A_{th1} y^{*} \right) + T_{wall}$$
(A.3)

When $y^* = y_n^*$, $T_1 = T_n$. The wall temperature can be obtained from the above as:

$$T_{w} = T_{n} - \frac{\Pr}{\mu_{w}} \left(\frac{C_{th1} y_{n}^{*2}}{2} + A_{th1} y_{n}^{*} \right)$$
(A.4)

and the wall heat flux is related to the wall temperature by following expression:

$$q_{wall} = -\frac{\rho c_p \sqrt{k_P}}{\mu} A_{th1} \tag{A.5}$$

where

$$A_{th1} = \frac{1}{y_n^*} \left(\frac{\mu_w}{\Pr} \left(T_n - T_{wall} \right) - \frac{C_{th1} y_n^{*2}}{2} \right)$$
(A.6)

Appendix B

MAWF Approach When $y_n^* < y_v^*$

a. Hydrodynamic Analytical Wall Function

When $y_n^* < y_v^*$, the analytical velocity in the near wall region is

$$U_1 = \frac{1}{\mu_w} \left(\frac{D_1 y^{*4}}{12} + \frac{C_1 y^{*2}}{2} + A_1 y^* \right)$$
(B.1)

From the near wall cell boundary condition, the unknown coefficient in the above equations is:

$$A_1 = \frac{\mu_w}{y_n^*} U_n - \frac{C_1}{2} y_n^* - \frac{D_1 y_n^{*3}}{12}$$
(B.2)

b. Thermal Analytical Wall Function with linear assumption

When $y_n^* < y_v^*$, the analytical temperature in the near wall region is

$$T_1 = \frac{\Pr}{\mu_w} \left(\frac{D_{th1}}{6} y^{*3} + \frac{C_{th1}}{2} y^{*2} + A_{th1} y^* \right) + T_w$$
(B.3)

When $y^* = y_n^*$

$$T_w = T_n - \frac{\Pr}{\mu_w} \left(\frac{D_{th1}}{6} y_n^{*3} + \frac{C_{th1}}{2} y^{*2} + A_{th1} y_n^* \right)$$
(B.4)

Or for the isothermal wall condition:
$$A_{th1} = \frac{\mu_w}{\Pr y_n^*} \left(T_n - T_w \right) - \frac{D_{th1}}{6} y_n^{*2} - \frac{C_{th1}}{2} y^*$$
(B.5)

c. Thermal Analytical Wall Function with parabolic assumption When $y_n^* < y_v^*$, the analytical temperature in the near wall region is

$$T_1 = \frac{\Pr}{\mu_w} \left(\frac{D_{th1}}{12} y^{*4} + \frac{C_{th1}}{2} y^{*2} + A_{th1} y^* \right) + T_w$$
(B.6)

When $y^* = y_n^*$

$$T_w = T_n - \frac{\Pr}{\mu_w} \left(\frac{D_{th1}}{12} y_n^{*4} + \frac{C_{th1}}{2} y^{*2} + A_{th1} y_n^* \right)$$
(B.7)

Or for the isothermal wall condition:

$$A_{th1} = \frac{\mu_w}{\Pr y_n^*} \left(T_n - T_w \right) - \frac{D_{th1}}{12} y_n^{*3} - \frac{C_{th1}}{2} y^*$$
(B.8)

Appendix C

CMAWF Approach When $y_n^* < y_v^*$

In the case of $y_n^* < y_v^*$, the analytical temperature can be obtained from the simplified energy equation with thermal dissipation terms:

$$T_{1} = \frac{\Pr}{\mu_{w}} \left(\begin{array}{c} \frac{D_{th1}}{12} y^{*4} + \frac{C_{th1}}{2} y^{*2} + A_{th1} y^{*} \\ -\left(\frac{N_{7}}{8} y^{*8} + \frac{N_{5}}{6} y^{*6} + \frac{N_{4}}{5} y^{*5} + \frac{N_{3}}{4} y^{*4} + \frac{N_{2}}{3} y^{*3} + \frac{N_{1}}{2} y^{*2} \right) \right) + T_{w}$$
(C.1)

When $y^* = y_n^*$

$$T_{w} = T_{n} - \frac{\Pr}{\mu_{w}} \left(\frac{D_{th1}}{12} y_{n}^{*4} + \frac{C_{th1}}{2} y^{*2} + A_{th1} y_{n}^{*} - \left(\frac{N_{7}}{8} y_{n}^{*8} + \frac{N_{5}}{6} y_{n}^{*6} + \frac{N_{4}}{5} y_{n}^{*5} + \frac{N_{3}}{4} y_{n}^{*4} + \frac{N_{2}}{3} y_{n}^{*3} + \frac{N_{1}}{2} y_{n}^{*2} \right) \right)$$
(C.2)

Or for the isothermal wall condition:

$$A_{th1} = \frac{\mu_v}{\Pr y_n^*} \left(T_n - T_w \right) - \frac{D_{th1}}{12} y_n^{*3} - \frac{C_{th1}}{2} y_n^* + \left(\frac{N_7}{8} y_n^{*7} + \frac{N_5}{6} y_n^{*5} + \frac{N_4}{5} y_n^{*4} + \frac{N_3}{4} y_n^{*3} + \frac{N_2}{3} y_n^{*2} + \frac{N_1}{2} y_n^{*1} \right)$$
(C.3)

Appendix D

Para-CMAWF Approach When $y_n^* < y_v^*$

When $y_n^* < y_v^*$, the molecular viscosity with parabolic assumption is:

$$\mu = \frac{\mu_w}{1 + b_\mu y^* (y^* - 2y_n^*)} \quad \text{where} \quad b_\mu = \frac{1}{y_n^{*2}} \left(1 - \frac{\mu_w}{\mu_n} \right) \tag{D.1}$$

a. Hydrodynamic Analytical Wall Function

After the second integration, the analytical velocity is obtained as:

$$U_{1} = \frac{1}{\mu_{w}} \begin{bmatrix} \frac{D_{1}y^{*4}}{12} + \frac{C_{1}y^{*2}}{2} + A_{1}y^{*} \\ +b_{\mu} \left(\frac{D_{1}}{18}y^{*6} - \frac{2D_{1}}{15}y_{n}^{*}y^{*5} + \frac{C_{1}}{4}y^{*4} + \frac{(A_{1} - 2C_{1}y_{n}^{*})}{3}y^{*3} - A_{1}y_{n}^{*}y^{*2} \end{bmatrix}$$
(D.2)

When $y^* = y_n^*$, $U_1 = U_n$

$$A_{1} = \frac{1}{y_{n}^{*} - b_{\mu} \frac{2}{3} y_{n}^{*3}} \left[\mu_{w} U_{n} - \frac{D_{1} y_{n}^{*4}}{12} - \frac{C_{1} y_{n}^{*2}}{2} + b_{\mu} \left(\frac{7D_{1}}{90} y_{n}^{*6} + \frac{5C_{1}}{12} y_{n}^{*4} \right) \right]$$
(D.3)

b. Thermal Analytical Wall Function

When $y_n^* < y_v^*$, the analytical temperature is obtained after the second integration:

$$T_{1} = \frac{\Pr}{\mu_{w}} \begin{bmatrix} \frac{D_{th1}y^{*4}}{12} + \frac{C_{th1}y^{*2}}{2} + A_{th1}y^{*} \\ +b_{\mu} \left(\frac{D_{th1}}{18}y^{*6} - \frac{2D_{th1}}{15}y^{*}_{n}y^{*5} + \frac{C_{th1}}{4}y^{*4} + \frac{(A_{th1} - 2C_{th1}y^{*}_{n})}{3}y^{*3} - A_{th1}y^{*}_{n}y^{*2} \right) \\ -\int_{0}^{y^{*}} \frac{1}{C_{p}} \mu \frac{\partial U_{1}}{\partial y^{*}} U_{1} \left[1 + b_{\mu}y^{*} \left(y^{*} - 2y^{*}_{n} \right) \right] dy^{*} \end{bmatrix} + T_{w}$$
(D.4)

When $y^* = y_n^*$ and $T_1 = T_n$, for adiabatic wall condition, the wall temperature is:

$$T_{w} = T_{n} - \frac{\Pr}{\mu_{w}} \begin{bmatrix} \frac{D_{th1}y_{n}^{*4}}{12} + \frac{C_{th1}y_{n}^{*2}}{2} + A_{th1}y_{n}^{*} \\ +b_{\mu} \left(\frac{D_{th1}}{18}y_{n}^{*6} - \frac{2D_{th1}}{15}y_{n}^{*}y_{n}^{*5} + \frac{C_{th1}}{4}y_{n}^{*4} + \frac{(A_{th1} - 2C_{th1}y_{n}^{*})}{3}y_{n}^{*3} - A_{th1}y_{n}^{*}y_{n}^{*2} \right) \\ -\int_{0}^{y_{n}^{*}} \frac{1}{C_{p}} \mu \frac{\partial U_{1}}{\partial y^{*}} U_{1} \left[1 + b_{\mu}y^{*} \left(y^{*} - 2y_{n}^{*} \right) \right] dy^{*}$$
(D.5)

Or for the isothermal wall condition:

$$A_{th1} = \frac{1}{y_n^* - \frac{2}{3}b_\mu y_n^{*3}} \left\{ \frac{\mu_w}{\Pr} \left(T_n - T_w \right) - \left[\begin{array}{c} \frac{D_{th1}y_n^{*4}}{12} + \frac{C_{th1}y_n^{*2}}{2} - b_\mu y_n^{*4} \left(\frac{7D_{th1}}{90} y_n^{*2} + \frac{5C_{th1}}{12} \right) \\ -\int_0^{y_n^*} \frac{1}{C_p} \mu \frac{\partial U_1}{\partial y^*} U_1 \left[1 + b_\mu y^* \left(y^* - 2y_n^* \right) \right] dy^* \end{array} \right] \right\}$$
(D.6)

Appendix E

Hyper-CMAWF Approach When $y_n^* < y_v^*$

When $y_n^* < y_v^*$, the molecular viscosity with hyperbolic assumption can be written as:

$$\mu = \frac{\mu_n}{1 + b_\mu (y^* - y_n^*)} \quad \text{where} \quad b_\mu = \frac{\mu_w - \mu_n}{\mu_w y_n^*} \tag{E.1}$$

a. Hydrodynamic Analytical Wall Function

$$A_{1} = \frac{1}{y_{n}^{*} - \frac{1}{2}b_{\mu}y_{n}^{*2}} \left[\mu_{n}U_{1} - \left(\frac{D_{1}y_{n}^{*4}}{12} + \frac{C_{1}}{2}y_{n}^{*2} - b_{\mu}y_{n}^{*3}\left(\frac{D_{1}y_{n}^{*2}}{60} + \frac{C_{1}}{6}\right) \right) \right]$$
(E.2)

b. Thermal Analytical Wall Function

When $y^* = y_n^*$ and $T_1 = T_n$, for adiabatic wall condition, the wall temperature is:

$$T_{w} = T_{n} - \frac{\Pr}{\mu_{n}} \begin{bmatrix} \frac{D_{th1}y_{n}^{*4}}{12} + \frac{C_{th1}}{2}y_{n}^{*2} + A_{th1}y_{n}^{*} - b_{\mu}y_{n}^{*2} \left(\frac{D_{th1}y_{n}^{*3}}{60} + \frac{C_{th1}}{6}y_{n}^{*} + \frac{1}{2}A_{th1}\right) \\ -\int_{0}^{y_{n}^{*}} \frac{1}{C_{p}} \left(1 + b_{\mu} \left(y^{*} - y_{n}^{*}\right)\right) \mu U_{1} \frac{\partial U_{1}}{\partial y^{*}} dy^{*} \end{bmatrix}$$
(E.3)

For the isothermal wall boundary conditions, A_{th1} is:

$$A_{th1} = \frac{1}{y_n^* - \frac{b_\mu}{2} y_n^{*2}} \left\{ \begin{array}{l} \frac{\mu_n}{\Pr} \left(T_n - T_w \right) - \left(\frac{D_{th1} y_n^{*4}}{12} + \frac{C_{th1}}{2} y_n^{*2} \right) + b_\mu y_n^{*2} \left(\frac{D_{th1} y_n^{*3}}{60} + \frac{C_{th1}}{6} y_n^{*} \right) \\ + \int_0^{y_n^*} \frac{1}{C_p} \left(1 + b_\mu \left(y^* - y_v^* \right) \right) \mu U_1 \frac{\partial U_1}{\partial y^*} dy^* \end{array} \right\}$$
(E.4)