STABILITY AND STABILISATION OF SWITCHING AND HYBRID DISSIPATIVE SYSTEMS

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Abstract

Stability and Stabilisation of Switching and Hybrid
Dissipative Systems
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A method is proposed to infer stability properties for non-linear switching under continuous state feedback. Continuous-time systems which are dissipative in the multiple storage function sense are considered. A partition of the state space, induced by the cross-supply rates and the feedback function, is used to derive a restriction on switching. Then, conditions are proposed, under which, systems controlled by the feedback function and switching according to the rule are stable. In particular, Lyapunov and asymptotic stability are proved, both in a local and in a global context. Further, it is shown that the approach can be extended when one uses multiple controllers, and, therefore, is able to construct multiple partitions; conditions for this case are also presented. Finally, it is shown that, for the switching families that satisfy the switching rule posited by the results, one is able to find elements (that is, stabilising switching laws for the system) which are non-Zeno. Additional rule-sets that allow this are provided. It is argued that the conditions proposed here are easier to verify and apply, and that they offer additional flexibility when compared to those proposed by other approaches in the literature.

The same infrastructure is used in the study of hybrid systems. For a general class of non-linear hybrid systems, a new property is proposed, that retains some of the properties of dissipativity, but it differs from it, crucially in the fact that it is not purely input-output. For systems having this property, it is shown that the partition used in the switching case can also be used. This, along with a set of conditions allows for the characterisation of the system behaviour in two scenaria. First, when the continuous behaviours and the jumping scheme act co-operatively, leading the system to lower energy levels (from the dissipativity point of view). Second, when the continuous behaviours are allowed to increase the stored energy, but the jumping is able to

compensate this increase. In the first case, it is shown that the equilibrium point under study is stable; in the second, it is shown that the system exhibits a type of attractivity, and, under additional conditions, it is asymptotically stable.

Besides stability, a collection of stabilisation results are given for the case of dissipative switching systems. It is shown that one may design state feedback functions (controllers) with the objective that they satisfy the conditions of the stability theorems in this work. Then, systems under the designed controllers are shown to be stable, provided that the switching adheres to a specific switching rule. This problem is approached using a variety of tools taken from analysis, multi-valued functions and the space of non-switching stabilisation.

In addition to the main results, an extensive overview of the literature in the area of switching and hybrid systems is offered, with emphasis on the topics of stability and dissipativity. Finally, a collection of numerical examples are given, validating the results presented here.

Keywords: Hybrid Systems, Switching Systems, Dissipativity, Stability, Feedback Stabilization, Zeno systems.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Acknowledgements

This section appears early on, but, naturally, it is the last part of this thesis to be written. It marks the end of this project and, with it, the end of a period in my life. Here, I would like to thank some of the people whose help and support were invaluable in this process. I hope that they will appreciate the sentiment, and that they will find it in their hearts to forgive my lackluster prose – this is not a genre in which I have much experience.

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Chapter 1

Introduction

The purpose of this chapter is to give a qualitative outline and motivation for the topics that will be covered in this work. First, a brief presentation of the core concepts of the work, namely, hybrid systems, dissipativity and stability, will be given. Having in mind these concepts, the approach which will be used in this work will be summarised. Finally, a more concrete list will be given, summarising the contributions made in this work and the publications that are associated to it. The chapter is not meant to be a literature overview, and few citations will be given. For a more thourough presentation of the literature, the reader should check Chapters 2 and 4.

1.1   Core Concepts and Motivation

The purpose of this report is to present some results in the area of control engineering, and, in particular, on the topic of hybrid and switching systems. Hybrid and switching systems are types of dynamic systems; that is, they are mathematical objects which are used to model phenomena in which evolution (change with time) occurs. To define a dynamic system, one needs to identify a set of variables, which represent the so-called state of the system. These variables, which might be finite or infinite, must capture all the relevant information about the system. Then, one describes the evolution of the system: the way in which these variables change with time.

Common forms of representation of such systems are ordinary, fractional and stochastic differential equations, in the case of continuous-time systems, for which time is some interval in the real numbers; or difference equations in the case of discrete-time systems, for which the time is an increasing sequence of integers. For the majority of the available literature, the systems under study are assumed to be smooth; that is,

they are described by equations that are as regular (continuous, differentiable etc) as needed, for solutions to the system to exist and be unique. This corresponds to human intuition, as natural phenomena tend to be smooth, in the macroscopic level; and to practical considerations, as many of the available mathematical and computational tools are intended for such systems.

Even though studying smooth systems is easier, it is often the case that use of non-smooth models is necessary – hybrid and switching systems are the results of this line of thought. A hybrid system is a dynamic system which exhibits two broad types of behaviour. First, different modes of continuous (or smooth) evolution of the state; these comprise the continuous behaviour of the system. Second, switches and jumps; these are the types of discrete behaviour of the system. At any given moment, the state evolves in accordance to one and only one of the available modes; this is called the active subsystem. At different points, the active subsystem may change, in which case the system is said to have performed a switch. At different moments, the state of the system may change discontinuously; that is, it may jump somewhere else in the state space. Such behaviour is described as a jump. A system that exhibits both switches and jumps is said to be a hybrid system; for the framework used in this work, this is the most general category of systems. If a system is capable of switches, but it has no jumps, it is said to be a switching system. Conversely, if a system has only a single available continuous behaviour (a sole subsystem), but it is capable of jumps, it is said to be an impulsive or a reset system. While the nomenclature is not definitive, the conceptual outline described here is found throughout the literature of hybrid systems. These ideas, which are given here in an approximate, informal fashion, will be described in detail in the remainder of this work.

The peak in the interest on systems with hybrid behaviours may be attributed to various causes. From the mathematical point of view, the problem was open, and it was posed as a 'new frontier': an area were control theory had produced no results, and for which the mathematical understanding and the available tools were not necessarily available. Indeed, as progress was made, it became apparent that many of the core techniques and tools used in the case of smooth systems cannot be transferred to the hybrid case, except if one makes significant adaptations to them. And, in some cases, the problems become so different that they necessitate the use of entirely new approaches. For instance, many results in the study of hybrid systems relied heavily on the use of tools from multi-valued analysis, whose use in 'mainstream' control theory had been rather marginal.

The study of hybrid systems, however, was not merely a mathematical curiosity. The popularisation of the use of computers in the second part of the twentieth century meant that the prevalence of configurations in which analog systems have to communicate with digital ones has risen significantly. Similarly, the emergence of distributed systems, means that communication times between components of the same system (say, the controller and the plant) cannot always taken to be insignificant. These interactions, as well as others of similar nature, naturally give rise to hybrid behaviours.

The theory of hybrid and switching systems, however, is not solely appropriate to describe the interaction of computers with other systems. Indeed, a variety of other configurations may be better studied and understood in the hybrid context. For instance, simple systems, such as thermostats and gearboxes, as well as more complex ones, such as plane routes [Sastry et al., 1995] and truck convoys [Tomlin et al., 2000] may be posed within the hybrid systems framework. Such systems exhibit an intrinsic multi-modal and/or event-driven character. For instance, a thermostat has a distinct temperature-rise (the ON subsystem) and a temperature-fall phase (the OFF subsystem), which are controlled by the advent of a specific event (the crossing of a temperature limit). Similarly, a group of airplanes is expected to exhibit different behaviours in different settings – when they are close; when they are far away; when they are bound to intersect etc.

Various other examples of applications, describing phenomena within a hybrid framework are available, for instance in the field of automotive systems [Balluchi et al., 2000], human-like robots [Grizzle et al., 2001] and software [De Cicco et al., 2016]. A distinct group of examples are those that come from biology and biochemistry. For biology, it is not uncommon to use hybrid behaviours to describe the highly non-linear, event-controlled behaviour of neurons [Navarro-López et al., 2016] and heart cells [Bartocci et al., 2009]. Hybrid models have also been used to model the evolution of diseases as well as the effect of treatments in a single person. For instance in the case of cancer [Rejniak and Anderson, 2011; Tanaka et al., 2008], and in the case of the sugar cataract [Riley et al., 2009]. Similarly, the spread of infectious disease has been studied using switching models, as shown in [Liu and Stechlinski, 2017]. Other areas, for which such applications have not been developed yet include finance; but, even for those, 'hybridization' seems to be a possible future direction of study [Hommes, 2001].

In this work, systems which exhibit hybrid behaviour will be examined from the point of view of stability properties. Stability is probably the most widely-studied

property in the area of control theory. In the abstract, stability captures the idea that, if the system is perturbed from an equilibrium (a steady-state scenario), it will ultimately return to it. In control engineering, stability is treated as a desirable property; systems that behave in accordance to it are considered to be well-behaved. This is because a variety of useful properties may be encoded as stability properties. The most obvious is asymptotic behaviour; namely, that, eventually, the system state reaches some desirable point. Another is convergence to a subset of the state space, say, a cycle; this is of interest if one wants to show that, eventually, the behaviour of the system becomes oscillatory. Similarly, one may encode tracking behaviour as the stability of the (an) error system; that is, the system under study tracks some given reference, when the error between the two converges to zero.

Given its central nature in control engineering, stability has already been extended to hybrid and switching systems, in a variety of works, which will be discussed in depth in Chapter 2. While the conceptual framework for these extensions is the same, there is no consensus about the details of the implementation. Different definitions assume different models for the systems and their behaviour, and give conditions that are not always comparable. The reason for this disagreement is exactly the complexity of the underlying object: the term 'hybrid behaviour' is very rich, and it includes classes of systems that may be understood better using different sets of assumptions. The purpose of the results in this work is to contribute to this conversation, by proposing a new way to view stability properties in switching and hybrid systems. This new approach brings together some elements that had been used before in the literature: a partition of the state space, use of the energy properties of the system and a bottom-up approach, for which the behaviour of the system is viewed from the point of view of its subsystems.

To say more, stability will be studied from a stabilisation point of view. The core schema of this approach is shown in Figure 1.1: the question of interest is under what conditions a given system becomes stable, when controlled using a given family of feedback functions (v). This question will be approached from two different directions. First, as an analysis question. In this case, the objective will be to derive conditions under which one may conclude that a given v stabilises the system/ Second, as a design question. Here, a collection of methods will be presented, which can be used to design a state feedback input function v in a way that guarantees stability. The two directions are intimately connected in the sense that the design question is answered in terms of the analysis question. That is, one looks for ways to obtain inputs for which the conditions derived from the stability analysis part of the work are known to hold. The

results that will be derived in the process of elaborating these viewpoints will be shown to have advantages, when compared to other results in the literature, which operate under similar sets of assumptions.
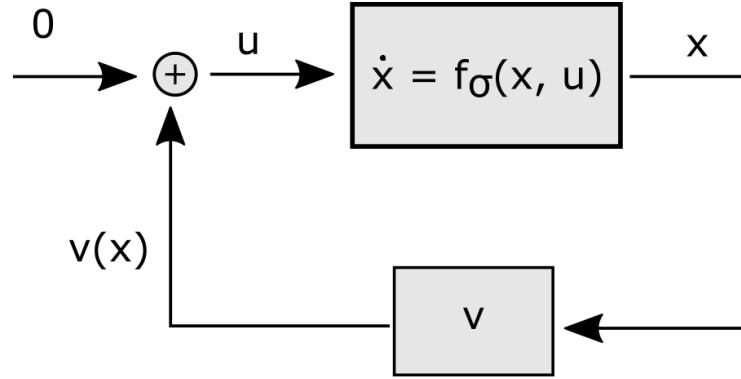
Figure 1.1: Abstract representation of the stabilisation problem.

The present work will offer answers to the stability question mentioned above, making use of the dissipativity framework. This is an energy-based description of system behaviours, which, in the case of control systems, traces back to [Willems, 1972]. The term energy-like means that, within the framework, the trajectories of the system are associated with the evolution of a positive quantity, referred to as the energy of the system. This process is quite common in physics, where one often talks about potential energy of a body induced by gravity, or about voltage. However, in the more general systems framework, the energy itself need not have any concrete physical meaning; rather, energies are defined in a way that encodes desirable properties.

An energy-based description of the system allows one to also describe desirable behaviours. For instance, one may say that, if the system respects some energy limits, it remains within some safe area, or avoids some danger zone. Similarly, they may say that, if the energy does not change very quickly, then a specific level of action is sufficient to keep the system at a certain level. The impulse behind the development of the dissipativity framework is to formalise such intuitions.

A system is said to be dissipative if energy is dissipated along the trajectories of the system. That is, the change in energy is always negative, if it is known that no energy flows from the environment to the system through its inputs and outputs. In the present context, this behaviour is desirable, as it is considered to be 'mild'; if the system is left alone, its natural tendency will be to rest, rather than move away. From the previous description, the connection of stability with dissipativity should already be clear, at least at the conceptual level. Much more will be said on this issue in the

next Chapter.

Again, as in the case of stability, extending this type of property to systems is an involved task. In particular, one has to decide if the energy description under consideration should characterise the whole system, or one/some/all of its subsystems. In the present work, a permissive, bottom-up conception of dissipativity will be used, so that the dissipativity properties of the subsystems determine the dissipativity properties of the system itself. It will be argued that this conception is reasonable, that it offers some flexibility with comparison to more demanding ones, and that it allows for adaptation of the core technique to other contexts, besides stability and stabilisations.

1.2   Contributions and Structure

In one sentence, this work will study dissipative switching and hybrid systems, and it will propose stabilisation techniques for these systems. It differs from pre-existing efforts in the same direction in the nature of the conditions, which, it is argued, offer additional flexibility and are conceptually clearer; and in the specific conception it uses for switching and hybrid systems, which allow it to treat a wider class of systems, including systems for which the subsystems do not possess a common equilibrium. A more detailed outline of the contributions is given here. This outline should also serve as a guide on the structure of the rest of this report.

• **Stabilisation results for switching systems:** The main stabilisation results of this work are presented in Chapter 3. The results are based on a partition of the state space, given by Definition 3.1, and they use the dissipative properties of the system to enforce stable behaviour. It is explained that the Theorems that are based on this approach are both conceptually clearer and easier to apply, when compared to some of the other results that may be found in the literature, and, in particular, when compared with the well-known dissipativity result in [Zhao and Hill, 2008a]. It is also argued that, due to its specific standpoint, from which one designs the switching law taking into account the specific feedback input, the results discussed here offer a novel way to formulate stabilisation conditions. In Chapter 5 it is further shown that these core results may be extended to more complex scenarios, where multiple controllers (and, therefore, partitions) are used.

• **Stabilisation results for hybrid systems:** In Chapter 4, the approach introduced in Chapter 3 is extended to the case of a class of hybrid systems. In particular, hybrid

systems are described in an equation-based framework, capturing both the continuous behaviour and the jumping scheme. It is shown that, for these systems one may use of the partition of Definition 3.1 along with some additional infrastructure, necessary for the jumps, to establish stability under a given state feedback function. To that effect, first a dissipativity-like property is motivated and introduced. For systems which have this property, it is possible to use the partition, in conjunction with some other conditions, to establish a result analogous to Theorem 3.1; this is Theorem 4.1. Further, it is shown that the same infrastructure may be used to show stability and a type of attractivity for hybrid systems that satisfy weaker conditions (in comparison to Theorem 4.1) for their continuous behaviour, compensating with stricter conditions on their jumping; this is Theorem 4.2. It is argued that these results differ substantially from the pre-existing ones, that they share the ease-of-use properties of their switching counterparts, and that the infrastructure developed here allows significant flexibility, which can be used to produce results for scenaria other than stabilisation.

- **Feedback Stabilisation results:** Chapter 6 flips around the point of view of Chapter 3. Here, the conditions of the Theorems of Chapter 3 are taken as design objectives, and the feedback inputs are designed to achieve these objectives. A variety of techniques are proposed to design such controllers, using ideas from analysis as well as the non-switching feedback stability space. It is argued these feedback stability results greatly extend the applicability of the stability ones presented in Chapters 3 and 5, as they allow for a set of alternative options, in case the conditions of those results are not satisfied.

- **Zenoness-related results:** In Chapter 7, the Zeno properties of the switching rule associated to the stabilisation results of Chapter 3 are explored. The purpose of this study is to show that, among the switching laws that are allowed by the rule, one may find non-Zeno ones. To that effect, it is first shown that Zenoness does actually arise in this context. Then, in Lemma 7.2, it is shown that, under some conditions, one may always obtain non-Zeno, stabilising switching rules based on Theorem 3.1, by following a pair of rule-sets, given in Definitions 7.1 and 7.2. These results adapt some of the ideas proposed in [Liu et al., 2010], and they further validate the core choices made in Chapter 3 (and, by extension, in this whole work).

- **Miscellaneous:** Besides the main contributions, a general overview of the state of switching and hybrid systems is offered, and some of the core considerations of this area are discussed. These are found in Chapter 2, focusing mainly on switching systems, and in Chapter 4, which contains more information on hybrid systems.

1.3 Publication list

- **Karalis, P. and Navarro-López, E. M. (2017). Feedback stability for dissipative switched systems.** *IFAC-PapersOnLine*, **50(1):3442 – 3448. 20th IFAC World Congress.**

  This work is a summary of the results of Chapter 3. In particular, it presents the partition of Definition 3.1 and a series of stabilisation results, which are variants of Theorem 3.1 and its asymptotic and global versions, as well as Corollary 3.1.

- **Karalis, P. and Navarro Lopez, E. (2018). Dissipativity and feedback stabilization for switching systems. In review.**

  This work is a summary of the results of Chapters 5 and 6. In it, some of the stabilisation results of Chapter 3 are restated and the partition of Definition 3.1 is described. Then, using this infrastructure, a pair of new results are introduced, using multiple partitions (and controllers). These results are variants of Propositions 5.1 and 5.4. Finally, variants of Theorems 6.1, 6.3 and 6.3 are presented, establishing conditions for feedback dissipativity for a class of switching systems. The work has been submitted for publication.

- **Gomes, C., Karalis, P., Navarro-López, E. M., and Vangheluwe, H. (2017). Approximated stability analysis of bi-modal hybrid co-simulation scenarios. In** *International Conference on Software Engineering and Formal Methods*, **pages 345– 360. Springer.**

  The focus is in the area of co-simulation design. The core idea behind it is that one may use tools from the area of stability analysis to characterise simulation environments for hybrid systems. In particular, it is shown that, for a planar system, one may use Lyapunov functions to guarantee that the if the system under study is stable, the simulated behaviour will retain the same qualitative behaviour (stability), provided that a simulation parameter (delay) is kept under a specific limit. This work is only indirectly related with the results presented in the present report.

Chapter 2

Background and State of the Art

The purpose of this chapter is to set the scene for what is to follow. Here, the systems under consideration will be described in precise terms, the properties under study will be presented, and the stability problem will be formulated. To begin with, some basic concepts of continuous-time, non-switching systems will be introduced, for the sake of completeness and notational coherence. Then, switching systems will be described, along with their basic stability-related properties. Finally, a quick primer on the dissipativity-related results, both for non-switching and for switching systems will be given. In order to keep this chapter reasonably concise, a review of the literature on hybrid systems is deferred until Chapter 4.

2.1   Systems Under Study

In the introduction, it was mentioned that switching and hybrid systems are a generalisation of the simpler types of dynamic systems, usually represented by differential or difference equations. In this work, the focus is put on systems whose core dynamics are given in continuous time. The simplest form of continuous-time dynamic systems, which do not exhibit hybrid behaviour, will be called smooth or conventional systems.

In this work, all systems (smooth and hybrid) will be taken to operate on Euclidean spaces. In particular, the state space is assumed to be some finite dimensional space $X = \mathbb{R}^n$ of finite dimension $n \in \mathbb{N}$. For some function $f : X \mapsto \mathbb{R}^n$ a continuous time dynamic system can be defined as follows:

$$\dot{x}(t) = f(x(t)), \tag{2.1}$$

where $\dot{x} \doteq \frac{dx}{dt}$ denotes the derivative with respect to time, which, in this work, will be represented by the variable t. Systems like these, whose dynamics are do not depend explicitly on time, are called autonomous or time-invariant systems; their 'complement', non-autonomous systems will not be studied in this work.

Equation (2.1), along with an initial condition of the form $x(t_0) = x_0$, with $t_0 \in \mathbb{R}^+$, define an initial value problem. In the abstract, a solution for such a problem is a function that satisfies both these equations; that is, a function whose value at t_0 is fixed, and whose derivative is always given by f. Formally defining solutions for the initial value problem is not a trivial task. It is known ([Hale, 1969]) that it is sufficient to consider as a solution any function $x : \mathbb{R}^+ \mapsto X$, which is absolutely continuous, and which satisfies equation (2.1) almost everywhere. It is also known that, for such a solution to exist and be unique in some interval $[t_0 - \delta, t_0 + \delta]$, for some $\delta > 0$, it is enough for the restriction of f to be Lipschitz continuous in a neighborhood around the initial point $x_0 = x(t_0)$. Additional conditions are available to ensure that such solutions can be extended to infinite intervals, and that they depend continuously on the initial conditions.

Remark 2.1. In the description above, the symbol x is used to represent a solution. That is, x is a function of time, whose image is in the state space, and, for some $t \geq t_0$, $x(t)$ represents the value, at some time instant t, of a trajectory, starting at some $x(t_0) \in X$, and evolving in accordance with equation (2.1). Occasionally, by abuse of notation, x will also be treated as a variable. For instance, it may be said that something happens when $x \in A$, for some $A \subseteq X$, or for all x which satisfy a condition. This 'overloading' of x should cause no confusion, and the right interpretation of x should be clear from the context. In some works in this area, x is kept as a variable, while some alternative notation (such as $\phi^t(x)$) is used to for the solutions. This route is not taken here, as it would make the notation slightly more complex, without providing any significant advantage.

While systems of the form of equation (2.1) are the most basic variety, it is often helpful to consider additional structure, in the form of a communication space. To say more, when modelling real phenomena, it is useful to consider systems with inputs, which drive the evolution externally, and outputs, which might not co-incide with the system state. Such inputs and outputs will also be considered to belong to Euclidean spaces. In particular, consider $U = \mathbb{R}^{m_i}$ and $Y = \mathbb{R}^{m_o}$, for some $m_i, m_o \in \mathbb{N}$, the input and output space, respectively. Then, for a collection of functions $u : \mathbb{R}^+ \mapsto U$, $f : X \times U \mapsto \mathbb{R}^n$ and $h : X \times U \mapsto Y$, one has the following system:

$$\dot{x}(t) = f(x(t), u(t)), \tag{2.2a}$$

$$y(t) = h(x(t), u(t)). \tag{2.2b}$$

For a given input $u : \mathbb{R}^+ \mapsto U$, a function of time, solutions have the same meaning as before; that is, they are absolutely continuous functions, such that equation (2.2) holds almost everywhere. In this case, it is also important to delineate the type of functions u for which such solutions exist. It is known that, if u is a continuous function, and if f is uniformly Lipschitz in its first argument (that is, the Lipschitz constant can be chosen independently of u), and continuous in its second argument, then solutions for system (2.2) exist locally and are unique [Meiss, 2007]. In fact, even weaker conditions can guarantee the same outcome, as seen in [Hale, 1969]. Input functions for which such conditions hold will be referred to as admissible inputs. Since existence and uniqueness issues are not the focus of this work, inputs used here will be taken to be admissible, unless otherwise stated.

Similar definitions can be given, when X, U and Y are only subsets of an Euclidean space. For example, in some setting, it might be interesting to examine the behaviour of a system for only some bounded set of inputs, corresponding to a bounded set U. More generally, the definitions of systems (2.1) and (2.2) can be extended to apply to general differentiable manifolds, as it is done, for instance, in [Isidori, 2013]. Such extensions will not be explicitly addressed here.

Remark 2.2. Similarly to Remark 2.1, the symbol u is defined to be a function of time, but it is occasionally treated as a variable. In contrast, variations of the notation $v : X \mapsto U$ will be used for a feedback controller.

The first type of systems with hybrid behaviour to be addressed here is that of switching systems. For a switching system, the dynamics may change as it evolves through time. In particular, this change has to be a 'discrete event', in the sense that it happens on a single time instant, rather than over time. A switching system, then, is a non-autonomous system, as it depends on time, but not a smooth one, as this dependence is not necessarily continuous.

The simplest forms of such systems studied in the literature are piece-wise linear and piece-wise affine systems, namely systems which switch among dynamics that have the respective property.

In order to represent switching systems, one has to adapt the infrastructure developed for the non-switching case. The first necessary component is a collection of functions $\mathcal{F} = \{f_1, f_2, \ldots, f_N\}$, for some $N \in \mathbb{N}$. The domain of each f_i is $X \times U$ and its range is $\mathbb{R}^n$. The functions are required to have the same properties as those described for f of equation (2.2), so that solutions can be found for each individual f_i. These functions represent the different dynamics which appear in the evolution of the switching system. That is, at every time instant, one and only one of the functions in $\mathcal{F}$ controls the evolution of the system state, in sense of f in equation (2.2). Members of $\mathcal{F}$, will henceforth be referred to as the subsystems of the switching system. In particular, the function that is in control of the state is referred to as the active subsystem, while the rest are the inactive subsystems. While in this case only finite collections of subsystems are considered, an extension in the case of countable or even uncountable subsystems is possible [Liberzon, 2012].

Having introduced the different dynamics, the second component needed to describe switching systems is the method of switching. In its simplest form, a switching law is a function $\sigma : \mathbb{R} \mapsto \mathcal{N} = \{1, 2, \ldots, N\}$, mapping time instants to indices of the functions in $\mathcal{F}$. This is interpreted as follows: if, for some $t \in \mathbb{R}$, $\sigma(t) = j \in \mathcal{N}$, then the subsystem corresponding to f_j is activated. In addition to σ, define the set $\mathcal{T} = \{t_0, t_1, \ldots\}$, which contains the time instants at which the system switches. By convention, the initial time t_0 is also taken to be an element of $\mathcal{T}$. The set $\mathcal{T}$ is defined to be countable, in the general case, but it can also be finite, in which case the system trajectories follow a specific $f \in \mathcal{F}$ after $\bar{t} = \max \mathcal{T}$, with the system reducing into a non-switching one. The case of uncountable sets $\mathcal{T}$ is not well-posed, and it will not be considered.

From all the possible functions σ described above, only those that are constant in between the points of $\mathcal{T}$ are admitted as possible switching laws for a switching system. It is further assumed, by convention, that the admissible σ are right-continuous; that is, as t approaches a value $\hat{t}$ from the right (with $t \geq \hat{t}$), $\sigma(t)$ approaches $\sigma(\hat{t})$. This means that, if some subsystem is active in the interior of some interval (t_k, t_{k+1}), with $t_k, t_{k+1} \in \mathcal{T}$, then it is also active when $t = t_k$. Defined this way, σ may depend directly on time, as in the switching rule 'change mode every two seconds'. Or, it might depend on some function of time, such as the state; as in the rule 'activate this and only this subsystem, when inside some pre-specified area of the state space'. It is also possible to have a mixture of these options.

Beside these components ($\mathcal{F}$, σ, $\mathcal{T}$), one also needs a collection $\mathcal{H} = \{h_1, h_2, \ldots, h_N\}$,

with $h_i : X \times U \mapsto Y$, for all $i \in \mathcal{N}$, corresponding to the output function of each sub-system. The intuition here is that systems with different dynamics might also have different outputs. Then, a switching system can be represented as follows:

$$\dot{x}(t) = f_{\sigma(t)}(x(t), u(t)), \tag{2.3a}$$

$$y(t) = h_{\sigma(t)}(x(t), u(t)). \tag{2.3b}$$

The notation $f_{\sigma(.)}$ is used to denote the member f_i of $\mathcal{F}$, which is active when $\sigma(t) = i$; a similar convention is used for $h_{\sigma(.)}$ As in the case of non-switching systems, the input functions are assumed to be as regular as needed (ie continuous) for the solutions to exist.

Two points have to be made here, concerning switching. First, that, in the switching case, a solution is associated, not only with the input signal u, but also with an (admissible) switching law. Second that, despite the fact that Equation 2.3 makes reference to a specific σ function, a switching system is defined for the whole family of switching laws that have the characteristics that have been discussed above (piece-wise constant and right-continuous functions, taking values in $\mathcal{N}$).

A general treatment of switching systems, as presented here, can be found in [Liberzon, 2012], as well as in [Sun and Ge, 2011]. Some additional results can be found in [Lunze and Lamnabhi-Lagarrigue, 2009], as part of a discussion on the more general hybrid systems. Moreover, various frameworks, differing to the one presented here, can be used to describe the switching behaviour. In the seminal work in [Filippov, 1988], Filippov used differential inclusions (that is, differential equation-like objects whose right-hand side is a set-valued function) to describe a particular type of switching behaviour. To say more, starting with systems with different dynamics on different sides of a surface, he introduced a more permissive concept of solutions, which allowed him to describe a set of behaviours, known as sliding modes. Other work in the same direction can be found in [Goebel et al., 2012], where switching and hybrid systems are represented using differential and difference inclusions. While this framework is conceptually and mathematically clear, and it allows the use of various, powerful mathematical techniques, differential inclusions often tend to be cumbersome to use and counter-intuitive in their behaviour. Another framework which has been used to describe switching systems has been developed in [Haddad et al., 2014]. Here, the concept of left/right-continuous systems is introduced, and these systems are

described as sets of trajectories, possessing certain properties. Using a similar conceptual apparatus, the authors in [Trenn and Willems, 2012] use the behavioural approach developed by Willems to describe switching behaviours. Finally, a class of switching systems can be captured by the formalism of hybrid automata, where the system dynamics and the switching are described as a (modified) finite state automaton. Naturally, the study of this formalism has been more fruitful in cases where this additional automaton structure is of use, such as in applications of mathematical logic to the study of switching systems [Davoren and Nerode, 2000], [Broucke, 1999], [Navarro-López and Carter, 2016].

In the literature, various properties of switching systems have been examined, and numerous applications have been produced. These include stability and dissipativity, which form the central concepts of this work, and which will be discussed presently. They also include the study of reachability and the related properties. Reachability refers, roughly, to the ability of the system trajectories to reach some pre-specified set, starting from some set of initial positions. Works in [Lygeros et al., 1999; Sun et al., 2002; Asarin et al., 2000; Dang and Maler, 1998] have tried to treat this problem formally, while the tools described in [Frehse et al., 2011; Donzé, 2010] offer a more applied approach to the same problem. Viability is a similar idea, and it refers to the property that at least one trajectory starting from the set of initial conditions remains within some target set, for as long as needed. Safety and liveness are more general properties, used in the theory of verification. The first refers to the situation where a property is never true, while the second describes the situation were a property is always known to eventually hold. In the area of dynamic systems, these properties often take the form of avoiding some 'danger' set (safety), or eventually entering some target set (liveness). These properties have been the subject of intense study, in [Prajna and Jadbabaie, 2004] (where the idea of barrier certificates is explored), [Esterhuizen and Lévine, 2016], in [Carter, 2013], [Navarro-López and Carter, 2016] (which discusses the idea of deadness, which is complementary to liveness), [Navarro-López and O'Toole, 2018] and in [Aubin et al., 2001] (which discusses viability). Extensions of the well-known observability and controllability properties, which become highly involved in the switching case, have been proposed in [Bemporad et al., 2000], [Yang and Blanke, 2007], [Collins and Van Schuppen, 2004] and [Ezzine and Haddad, 1989]. Issues of system identification have been examined, for example in [Paoletti et al., 2007], while periodic behaviours have been studied in [Simic et al., 2002]. Issues of optimality have been studied in [Bengea and DeCarlo, 2005], a monograph dedicated on the

topic, as well as in [Xu and Antsaklis, 2004] and [Seatzu et al., 2006]. The optimization parameters in these results may be either the switching instants, or the modes-to-be-activated, or the feedback function. Other results, to be mentioned in passing, include those about systems with delays, such as [Kim et al., 2006], those about fractional order systems, such as [HosseinNia et al., 2013], those on H_∞ control, such as [Zhao and Zhao, 2005]; and application on model predictive control, such as [Oettmeier et al., 2009]. Finally, some results extending switching systems to a stochastic setting can be found in [Cassandras and Lygeros, 2006] [Chatterjee and Liberzon, 2006], [Teel, 2013].

An additional issue that has to be considered when defining switching laws for the systems studied here is the behaviour known as the Zeno phenomenon. This refers to the situation where infinite switches are forced in a finite time interval. Intuitively, this refers to the possibility that the system is forced to perform switches so often that it cannot proceed beyond a specific point. For a simple example, consider a system with two subsystems and the following switching rule: 'switch for the k-th time after $\frac{1}{2^k}$ units of time have passed'. It is clear that, as t approaches 1, the system will be forced to switch progressively faster, and it is not clear what is the correct behaviour when $t = 1$. Zeno behaviour is a serious problem, especially because it can arise for seemingly reasonable rules, and, in the case of state-dependent switching, it is not clear a priori if and when it will arise. Results related to the Zeno behaviour in switching systems can be found for example in [Liu et al., 2010]. Related material, applicable to the more general hybrid systems (see Chapter 4) can be found in [Goebel and Teel, 2008], [Johansson et al., 1999a], [Ames et al., 2006],[Dashkovskiy and Feketa, 2017]. In this work, it is assumed that the systems considered do not exhibit Zeno behaviour. This is a 'soft' assumption: the results presented here (say, stability) are applicable even to systems with Zeno behaviour, but the user should evaluate the conclusions drawn this way, keeping in mind that zenoness might appear. Further discussion of this issue is to follow in Chapter 7.

2.2 Stability for non-switching and switching systems

Stability is an important focus point of this thesis, as well as one of the most fundamental properties in control theory. *Gross modo*, stability refers to the property of the trajectories of a system to remain as close to a pre-specified set of points as needed. The usefulness of such a property becomes clear in a variety of settings. For example,

the problem of keeping some error as close to zero as needed, is a typical problem that can be formulated in terms of stability. The seminal work by Aleksandr Lyapunov in [Lyapunov, 1992] has been the most influential formulation of the core techniques used in the area of stability. Important contributions have also been made by LaSalle [LaSalle, 1960] and Krasovskii [Krasovskii, 1959]. A primer in those, for the case of non-switching systems can be found in any textbook on control theory, such as [Khalil, 2002].

In order to define stability, one needs to have a notion of the points around which the system is expected to behave in the manner described above. These are said to be the equilibrium points of the system. For a system of the form of equation (2.1), one takes the zeros of f to be the equilibrium points of the system. That is, a point $\bar{x} \in X$ is an equilibrium point, if $f(\bar{x}) = 0$. Extending this definition for the case of (switching) system (2.3) is complicated by two factors. First, one has to account for the input. Second, the fact that there are multiple counterparts to f has to be addressed. The following definition is proposed:

Definition 2.1 (Equilibrium points)**.** A **weak equilibrium point** of system (2.3) is an element $\bar{x} \in X$, for which there exists some $\bar{u} \in U$, such that $f_i(\bar{x}, \bar{u}) = 0$, for some $i \in \mathcal{N}$. A **strong equilibrium point** of system (2.3) is an element $\bar{x} \in X$, for which there exists some $\bar{u} \in U$, such that $f_i(\bar{x}, \bar{u}) = 0$, for every value of i. ∎

Given a weak equilibrium $\bar{x}$, one might refer to the equilibrium triplet $(\bar{x}, \bar{u}, i)$. Similarly, for a strong equilibrium point, one has the corresponding equilibrium pair $(\bar{x}, \bar{u})$. While this nomenclature is new, the idea of considering the subsystem equilibria as equilibria of the whole system has been used before, for example, in [Navarro-López and Laila, 2013].

Remark 2.3. While Definition 2.1 is given for switching systems, it is also applicable to system (2.2) (which is a subclass of system (2.3), for which $N = 1$). In this case, there is no distinction of weak and strong equilibrium points, and one refers to equilibrium points in general.

Alternative definitions of equilibria are common in the literature. Indeed, many works, such as [Branicky, 1998a], work without an explicit definition of an equilibrium point, by assuming that the origin is an equilibrium for all the subsystems. An alternative approach would be to restrict attention to equilibria corresponding to zero inputs (that is, zeros of $f_i(\bar{x}, 0) = 0$). In this work, it is argued that such assumptions are both unnatural and restrictive. Indeed, the core motivation for the introduction of

switching systems is that the systems under study might exhibit different modes of behaviour, and assuming a common equilibrium goes against this impulse, by positing that the modes have to be, in some sense, not too different.

Observe that, for every $i \in \mathcal{N}$, multiple equilibrium points might exist. Indeed, it is possible that, under this definition, every point $\bar{x} \in X$ is assignable as a wEP, for some appropriate value $\bar{u}$. Contrarily, in non-degenerate cases, most x are not expected to be sEP, since, in general, $f_i(x, u) \neq f_j(x, u)$. The exact composition and geometry of the equilibrium sets will not be studied further here. Other works that use a similar conception of the equilibrium points include [Simpson-Porco, 2017] and [Hines et al., 2011], and, in the hybrid systems space [Navarro-López and Laila, 2013]. Given this conception of equilibrium points, the following definition of stability is introduced:

Definition 2.2 (Stability). Consider system (2.3), starting at $t_0 \geq 0$, with initial state $x_0 = x(t_0) \in X$, and Σ, a family of piece-wise constant, right continuous switching laws for the system. Take a weak (strong) equilibrium point $\bar{x}$, from a triplet $(\bar{x}, \bar{u}, i)$ (a pair $(\bar{x}, \bar{u})$). Under the constant control $u(t) = \bar{u}$, $\bar{x}$ is said to be:

- **attractive**, if $\lim_{t \to \infty} x(t) = \bar{x}$, for every solution of the system that is defined for every $t \in [0, +\infty)$.

- **Lyapunov stable**, if for each $\varepsilon > 0$, there exists $\delta > 0$, such that, if $\|\bar{x} - x(t_0)\| < \delta$, then $\|\bar{x} - x(t)\| < \varepsilon$, for all $t \geq t_0$.

- **asymptotically stable**, if it is both attractive and stable.

If the conditions hold within some set $E \subseteq \mathbb{R}^n$, invariant in the asymptotic case, with $x_0 \in E$, the respective properties are said to be local. If $E = \mathbb{R}^n$, then they are said to be global. ∎
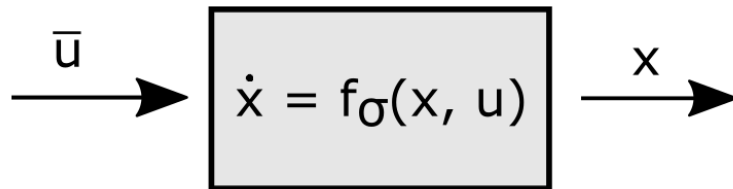
Figure 2.1: Abstract representation of the stability problem of Definition 2.2.

As discussed in the introduction, in the present work, the particular angle from which stability problems will be approached is that of stabilisation. Stabilisation refers

to the problem of rendering a system stable by the use of a specific feedback function. In particular, the focus will be on state feedback functions; that is functions of the state form $v : X \mapsto \mathbb{R}$. This formulation implies that attention will be restricted to possibility of rendering a system stable using static feedback – the related case of designing an auxiliary dynamic system, such that the feedback interconnection with the initial system is stable will not be considered. Evidently, a first requirement on any possible solution of this problem is for v to be such that solutions for the system exist. To that effect, attention will be restricted to feedback functions that are (at least) continuous. This conception of stabilisation is captured by Figure 1.1, given in the previous chapter. It should be compared with Figure 2.1. which pictures the idea described by Definition 2.1. In more precise terms, one may express the stabilisation problem in the following way:

Definition 2.3 (Stabilisation)**.** Consider system (2.3), starting at $t_0 \geq 0$, with initial state $x_0 = x(t_0) \in X$, and Σ, a family of piece-wise constant, right continuous switching laws for the system. Take a weak (strong) equilibrium point $\bar{x}$, from a triplet $(\bar{x}, \bar{u}, i)$ (a pair $(\bar{x}, \bar{u})$). An input function $v : X \mapsto U$ is said to stabilise the system around $\bar{x}$, if the composite system resulting by plugging the input $u'(t) = v(x(t)) + u(t)$ into (2.3) has a stable equilibrium at $\bar{x}$, corresponding to a triplet $(\bar{x}, 0, i)$ (a pair $(\bar{x}, 0)$) according to Definition 2.2. ∎

Remark 2.4. Definition 2.2 is given with respect to an equilibrium point, corresponding to the intuition that one would want the system to converge to some place from which it does not actively try to leave. However, the same definition would work for any set $Q \subseteq \mathbb{R}^n$, given some definition of distance from sets. Usually, one considers $d(x, Q) = \inf\{d(x, q) \mid q \in Q\}$, where $d(x, q)$ is the relevant distance defined for vectors.

Observe that, if v is a constant function, Definition 2.3 reduces to Definition 2.2. More generally, Definition 2.2 is a (conceptual) specialisation of Definition 2.3: the two definitions guarantee the same behaviour, but the former does it for a constant input, the latter does it for any continuous v. Given the conceptual proximity between the properties, there is also some overlap in terminology. In this work, an equilibrium point will be said to be stable, without further qualifications, when referring to Definition 2.2; and it will be said that an equilibrium is stabilised or rendered stable or is stable under an input, when referring to Definition 2.2.

While the core of the definitions proposed here (attractivity and stability) is common in the literature, the terminology occasionally differs. In such works, such as

[Lin and Antsaklis, 2009], stability is decoupled from the swithcing law; that is, only systems which maintain the appropriate behaviour for arbitrary switching are considered to be stable. Then, the term stabilisation is used for the concept of Definition 2.2, where the desired behaviour appears for some family of (admissible) switching laws. While this is a salient distinction to make, the resulting definition of stability is too restrictive, and it will not be used here. In works such as [Margaliot, 2006], an alternative approach is used: stability is not described using a δ-ε-type definition, but rather by positing bounds on some norm of the trajectory. Such ideas are complementary to the ones discussed here, and they will not be pursued further.

The problem of stability in switching systems has been approached from multiple directions. Some core groupings, among the relevant results include the common Lyapunov function approaches, the multiple Lyapunov function approaches and the dwell-time approaches. The first are a simple extension of the corresponding ideas from non-switching system, mentioned above. Briefly, stability is established by use of a function that satisfies Lyapunov-like conditions for all the subsystems of the system; that is, it is positive and it decreases along the trajectories of the subsystems. In [Liberzon, 2012], the authors show that the existence of such a function certifies the stability of the system, as well as a partial converse result, indicating that such functions exist for stable systems. Other common Lyapunov function approaches to the stability problem for switching systems can be found in [Goebel et al., 2012], in a differential inclusion setting, in [Haddad et al., 2014] and in [Liu et al., 2010].

While the approach of common Lyapunov functions is a top-down approach, which considers the behaviour of the system as a whole, the multiple Lyapunov function approach works in the opposite direction, by assigning to each subsystem its own Lyapunov function. In this case, the existence of the Lyapunov functions is not sufficient, so additional conditions are introduced, in order to guarantee that the ensemble of Lyapunov functions is decreasing. The approach was first examined in [Branicky, 1998a],[DeCarlo et al., 2000], where a very general, and rather hard to verify, condition for stability is given. Additional work, along similar lines, has been produced in [Johansson and Rantzer, 1998] and [Pettersson and Lennartson, 2002]. In particular, the first offers a method to construct a piece-wise continuous Lyapunov function, agreeing with the Lyapunov functions of the subsystems in different 'activation' subsets of the state space. The second proposes a partition of the state space, along with a condition, in the spirit of Branicky, for the Lyapunov functions in the boundaries of the areas of the partition. The multiple function approach is, by nature, more flexible, as it allows

different Lyapunov conditions for the subsystems; this makes it more well-suited to the stability concept used in the present work.

Besides the difference in the tools they use, and the type of conditions they posit, single and multiple Lyapunov function approaches differ in another sense. For most of the single Lyapunov function results, the goal is to establish stability under any switching. The existence of a common Lyapunov function means that the behaviour of all the subsystems has some common characteristics; so much so, that switching between them does not disrupt stablity. Approaches with multiple Lyapunov functions, on the contrary, accept that the subsystems might not exhibit common behaviour (in the Lyapunov sense), so they attempt to introduce restrictions in the switching, such that stability is preserved. Such restrictions might be either state-dependent, or time-dependent or both (naturally, since they are restrictions to the admissible types of σ functions).

Dwell-time approaches also intent to restrict the admissible switching laws. In particular, in the relevant works, limits are placed on the admissible length of the time interval between switching instants. The limits might be either upper, in which case, one speaks of 'fast switching', or lower, when 'slow switching' occurs. State dependent rules on the dwell time have also been proposed, notably in [De Persis et al., 2003]. Often the limit is not for the dwell time itself, but for the average length of the different dwell-times in any time interval, as seen for example in [Hespanha and Morse, 1999; Zhai et al., 2000]. More advanced work in this direction allows for different average dwell-times for different subsystems [Zhao et al., 2012]. A slightly different approach is followed in [Kundu et al., 2016], where a more general (global) restriction is placed on the behaviour of the dwell time. It is notable that, on occasion, such approaches are able to incorporate systems for which some or all the subsystems are unstable, as seen, for instance, in [Zhai et al., 2000; Xiang and Xiao, 2014].

In the majority of cases, dwell-time approaches still use Lyapunov functions, but they adapt them toward the specific task of introducing dwell time limits. Other works do not make direct use of Lyapunov functions. For instance, in [Margaliot, 2006], variational techniques are used; in [Agrachev and Liberzon, 2001] and [Agrachev and Liberzon, 2001], the problem is formulated in the language of Lie algebras; and, in [Mancilla-Aguilar, 2000] a criterion using directly the functions in $\mathcal{F}$ is proposed.

Beside Lyapunov stability, various other types of stability for switching systems have also been studied in the literature Input-to-state-stability, described in [Sontag, 1995], captures the idea that the input of a system provides a bound for the norm

of its state, and efforts to use this property in a switching systems setting have been presented in [Vu et al., 2007]. Practical stability is a less demanding stability property. It just requires that, starting from a set Ω_1, the system trajectories converge to some other set, $\Omega_2 \subseteq \Omega_1$. Works on switching systems using this property can be found in [Zhai and Michel, 2003] and [Kuiava et al., 2013]. Another stability property, almost equivalent to practical stability, is region stability, and it is studied in [Podelski and Wagner, 2007], in a hybrid systems setting. The term practical stability is used in some settings, such as [Freeman and Kokotovic, 2008], to describe a property in between what is described with the term here and Lyapunov stability. To say more, this type of practical stability implies that the Lyapunov requirement of Definition 2.2 is satisfied for every ε which is larger than a pre-specified $\hat{\varepsilon}$, rather than for every possible ε. Finally, the works in [Orlov, 2004] study finite time stability, in which convergence to the equilibrium is required to happen within some pre-specified time.

Beside pure stability and stabilisation problems, one may find in the literature various problems of feedback design. That is, problems where a feedback function which renders the system stable is constructed. Important works for the non-switching case include [Clarke et al., 1998], [Clarke et al., 1997] and [Byrnes et al., 1991]. The last one, in particular, was also cited above, since it also deals with the problem of feedback passivity. In the switching case, the problem is rather harder to address. Efforts have been made, for example in [Wu, 2009], [Liu et al., 2011b], for the non-linear case, and in [Sun and Ge, 2003], for linear systems. The non-linear versions of these results are quite restrictive, both in the type of systems they admit and in the ease of use of their conditions. Another approach is offered in [Sanfelice, 2013], for hybrid systems described as differential inclusions. There, the design process is fashioned after the blueprint developed in [Freeman and Kokotovic, 2008], where the problem of inverse optimality is treated. As with other results in the inclusion framework, while a mathematical solution for the problem is obtained, its applicability is difficult to assess.

While the techniques described in this section are appropriate for continuous-time systems and their stability, some of them have direct counterparts in the area of discrete-time systems. For instance, [Branicky, 1998a] phrases some of its results in the discrete-time idiom. Further, some techniques have been developed, which are particular to the discrete time case. Arguably, the most important of those is the framework of the joint spectral radius [Jungers, 2009; Theys et al., 2005]. In the works of this area, such as [Ahmadi et al., 2011], [Ogura and Martin, 2013] and [Jungers and Mason, 2017] linear systems are discussed, and stability results are derived using spectral properties

of the subsystems. The work on path-complete systems has similar goals and considerations, as seen in [Forni et al., 2017] and [Angeli et al., 2017]. Here, the systems are represented with graphs which capture the evolution of the Lyapunov functions of the system. The work in [Kundu and Chatterjee, 2014] also uses graphs to represent the systems, and it establishes stabiltiy, based on a restriction on switching. Finally, other appoaches to show stability for discrete-time systems use results from set theory [Fiacchini and Jungers, 2014] and linear algebra [Fiacchini and Millérioux, 2017].

2.3 Dissipativity for non-switching and switching systems

2.3.1 The non-switching case

In this work, stability properties will be examined within the dissipativity framework. Dissipativity was introduced in the seventies, in the work of Willems [Willems, 1972], with contributions by Hill and Moylan in [Hill and Moylan, 1976, 1980]. It was proposed as a formalisation and generalisation of various pre-existing methods, based on the concept of energy, such as passivity in circuit theory. The driving force behind the development of such energy-based frameworks is that they allow for intuitive descriptions of system behaviours, applicable in a variety of settings and problems. For example, one might use the concept of energy to describe how a particle will move, under some forces, whether it will eventually stop, if it will remain within some bounds, how it will interact with its environment, etc.

For system (2.2), one needs two components to introduce dissipativity. First, the way the system exchanges energy with its environment has to be expressed as a function of its inputs and outputs. This corresponds to the circuit theory conception of power, and the function that performs this mapping is called the supply rate of the system. Given the supply rate, one also needs a storage function, which captures the amount of energy present in the system, as a function of the system state. The following definitions formalise these descriptions:

Definition 2.4 (Dissipativity data). A function $r : U \times Y \mapsto \mathbb{R}$ is a possible supply rate, provided that it is locally integrable along the trajectories of system (2.2). A function $V : X \mapsto \mathbb{R}$ is a possible storage function, provided that it is non-negative everywhere in X. ∎

These definitions are meant to convey the intuition contained in a laymans under-
standing of the concept of energy. With them, some abstract scalar quantity is intro-
duced. This quantity is called the energy of the system, but, in reality, it does not need
to have any physical meaning. This energy might be generated in two ways. It may
either be introduced to the system from its environment, through its inputs and outputs,
or it may be produced by the system itself. In this setting, a dissipative system is a sys-
tem that consumes more energy than it produces. This is a direct generalisation of the
idea of passive systems, found in circuit theory (for instance, [Wing, 2008]). A circuit
is passive when it is composed of passive elements, that is, elements that can store, but
not produce energy (of course, in this case, energy has a physical interpretation). In the
more abstract case of dissipative systems, one also allows systems that produce some
energy, but not all the energy they consume during any of their trajectories. Described
in this way, dissipativity is a form of mild behaviour. If the system does not produce
as much energy as it needs, then, it should be clear that, if no energy is introduced
from the outside, it will eventually reach some low energy state. This is the intuitive
justification for the use of dissipativity-related properties to establish stability. The
description given here corresponds to the following definition.

Definition 2.5 (Dissipativity [Willems, 1972])**.** System (2.2) is said to be dissipative,
with respect to the supply rate r and the storage function V, which satisfy Definition
2.4, provided that the following inequality holds for any trajectory of the system, every
admissible input u and every pair of time instants $t_1, t_2 \in \mathbb{R}$, such that $t_2 \geq t_1$:

$$V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} r(u(\tau), y(\tau)) d\tau. \tag{2.4}$$

■

The inequality of this definition is often referred to as the dissipation inequality. Its
intuitive content should be clear: for any time interval, the change in the energy of the
system is bounded by the integral of the supply rate, or, equivalently, the total energy
supplied to the system during the interval. In [Hill and Moylan, 1976], it is shown that
one can work with a slightly different definition of the dissipativity property, using only
the supply rate; this definition, however, is equivalent to the one given here. Note that,
neither in Definition 2.4, nor in Definition 2.5, it is required that the storage function is
continuous. However, if it happens to be differentiable, then (2.4) takes the following
handy form:

$$\dot{V} = \frac{\partial V}{\partial x} f(x(t), u(t)) \leq s(u(t), y(t)).$$

Here, $\frac{\partial V}{\partial x}$ is the Jacobian of V, calculated at the point $x(t)$.

Depending on the choice of r, different types of dissipativity exist. The most well-known, and most-widely studied of these forms is passivity, which is dissipativity with respect to the supply rate $r(u,y) = y^T u$. Direct extensions of passivity correspond to the supply rate $r(u,y) = y^T u + \varepsilon_1 u^T u + \varepsilon_2 y^T y$, and they are known as strict passivity ($\varepsilon_1 > 0$ and $\varepsilon_2 > 0$), strict-input passivity ($\varepsilon_1 > 0$ and $\varepsilon_2 = 0$) and strict-output passivity ($\varepsilon_1 = 0$ and $\varepsilon_2 > 0$). All these types of passivity are cases of the more general QSR-dissipativity for which the supply rate assumes the form $r(u,y) = y^T Q y + y^T S u + u^T R u$, for matrices Q, r, R with appropriate dimensions. The class of QSR-dissipative systems was first discussed in [Hill and Moylan, 1976], and its advantage is that, for it, an algebraic criterion for dissipativity exists, analogous to the well-known Kalman-Yakubovic-Popov Lemma. Finally, a type of dissipativity, which is particularly interesting, is the property usually called finite-gain stability. Finite-gain stable systems are the subclass of QSR-dissipative systems for which $Q = I$, $S = 0$ and $R = -\varepsilon I$, where I is the identity matrix of appropriate dimension and $\varepsilon > 0$.

Various applications of the dissipativity framework have been developed in the years since its inception. These include various results on the stability of feedback interconnections of dissipative systems, such as those in [Hill and Moylan, 1977] and [Van der Schaft, 1992]. The issue of rendering a system dissipative, with the use of smooth feedback is studied in [Byrnes et al., 1991], for the case of passive systems, and in [Van der Schaft, 1992], for QSR-dissipative systems. A distinct variety of results in the area of dissipativity theory goes by the name of passivity-based control, and it has various applications including the control of dc motors [Sira-Ramirez et al., 1997], chemical processes [Sira-Ramirez and Angulo-Nunez, 1997] and mechanical systems [Ortega et al., 2013]. A related strain of results within the framework of passivity-based control is known as IDA-PBC (interconnection and damping assignment), and it was explored in works including [Ortega et al., 2002] and, more recently, [Ryalat and Laila, 2016]. In these work, state feedback is used to shape the (stored) energy of the system into a new form, which has some desirable properties. Knowledge about these properties is then used to stabilise the system. The work in [Van der Schaft, 2000] offers an overview of various results related with dissipativity, and especially as it applies to the case of the so-called Port Hamiltonian systems, which, it is argued, are good models for a variety of phenomena. The applications of passivity-related properties in the control of robotic systems are also significant; [Ortega and Spong, 1989] and [Arimoto, 1996] are important samples of the work in this area.

An effort to redefine dissipativity, with fewer regularity assumptions on the system, as well as to design continuous feedback functions that render a system QSR-disspative is discussed in [James, 1993; Yuliar et al., 1998]. A slight extensions of the basic definition is discussed in [Chellaboina and Haddad, 2003], where exponential dissipativity is introduced. A more recent, and more radical, work on dissipativity can be found in [Willems, 2007], where the property is expressed within the behavioural framework, developed by Willems during the final years of his life; this reframing corresponds to a proper extension of the class of systems that are said to be dissipative. Other related work includes the use of dissipativity properties within the framework of complex systems [Arcak, 2007], neural networks [Yu and Li, 2001] and fuzzy systems [Uang, 2005].

A relatively more recent trend has been the introduction of properties that are based on dissipativity, but differ substantially from it. First, there exists the property called equilibrium independent dissipativity, discussed in [Simpson-Porco, 2017], which allows the basic theory to incorporate systems whose equilibria depend on the control. Incremental dissipativity, discussed in [Stan et al., 2007] [Liu et al., 2011a], studies systems for which some type of abstract energy, capturing the difference between different trajectories, dissipates with time. The usefulness of such a property stems from the fact that, under some conditions, it allows to conclude that different trajectories will 'forget' their initial conditions and converge to a common trajectory (instead of an equilibrium point). Differential dissipativity, developed in [Forni and Sepulchre, 2013] undertakes a similar task, within the contraction framework, developed by Slotine and Lohmiller [Lohmiller and Slotine, 1998], extending ideas by Demidovich [Pavlov et al., 2004]. In this case, the energy is not defined on the set of trajectories, but, rather, on the tangent bundle of the system. Finally, in [Wu et al., 2013], dissipativity is modified, so that it can be used in a stochastic setting; for that purpose, the dissipation inequality (2.4) is expressed in terms of expectations (both for the left and the right side).

While the focus of this work is on continuous-time systems, some significant dissipativity results from the discrete-time space deserve mention. In particular, in [Lin and Byrnes, 1995] passivity for discrete-time systems is related with stabilization, while in [Byrnes and Lin, 1994; Navarro-López and Fossas-Colet, 2004] the problem of rendering a system passive (feedback equivalence) is treated. In [Monaco and Normand-Cyrot, 1997], sufficient conditions for passivity are proposed, while in [Navarro-López,

2002; Navarro-López et al., 2002] more general dissipative systems are discussed, including systems with general supply rates and QSR-dissipative systems, and KYP-like conditions are proposed for these systems.

The most obvious property that has been associated with dissipativity is stability. The reasoning for that follows from the intuitive description given before, when dissipative behaviour was described as mild. If the system is dissipative, then it does not produce the energy that is necessary for its evolution. Then, if one controls the energy that flows into the system, it should be expected that the system will converge to some state of minimal energy. If it is also the case that this state is an invariant set of the system, this should imply some type of stability. Various characterisations of stability have been proposed, based on this idea, and they are to be found in every one of the introductory works mentioned before. Arguably, the most important and most fundamental of those is the one proposed by Willems himself, in his work introducing the concept [Willems, 1972]. Before stating this result, it is necessary to introduce the following definition.

Definition 2.6 (Positive definiteness). A function $g : X \mapsto \mathbb{R}$ is said to be positive semi-definite when $g(0) = 0$, and, for all $x \in X$, $g(a) \geq 0$. It is said to be positive definite, if the second inequality is strict for $x \neq 0$. If $-g$ is positive (semi-) definite, then g is said to be negative (semi-) definite. ∎

Definition 2.7 ($PD(x)$/$ND(x)$ property). Take some function $g : X \mapsto \mathbb{R}$ and some point $\bar{x} \in X$. g is said to have the $sPD(\bar{x})$ property, if the function $\hat{g} : X \mapsto \mathbb{R}$, defined by $\hat{g}(x) = g(x - \bar{x})$ for every $x \in X$, is positive semi-definite. g is said to have the $PD(\bar{x})$ property if $\hat{g}$ is positive definite. The $sND(\bar{x})$ and $ND(\bar{x})$ properties are defined similarly, corresponding to negative semi-definite and negative definite $\bar{g}$, respectively. ∎

Then, one has the following.

Theorem 2.1 (Lyapunov stability of dissipative systems [Willems, 1972]). *Take system (2.2), dissipative according to Definition 2.5. Assume that $\bar{x} = 0$ is an equilibrium point for the system, corresponding to $\bar{u} = 0$; that is $f(\bar{x}, \bar{u}) = f(0,0) = 0$. Assume, also, that the storage function V is continuous and positive definite. Take an invariant set $E \subseteq \mathbb{R}^n$, such that the initial condition $x(t_0) \in E$. Then, the equilibrium point is Lyapunov stable, locally in E, provided that, under the control $u(t) = \bar{u} = 0$, it holds that*

$$r(x(t), 0) \leq 0, \forall t \geq t_0.$$

∎

The result given here discusses only Lyapunov stability. With appropriate additional assumptions, it is possible to strengthen it, in order to obtain conditions for the asymptotic stability of the equilibrium point. Such a task is undertaken, among others, in [Hill and Moylan, 1980]. In [Hill and Moylan, 1976], the condition on r is translated into an algebraic condition, for the case of QSR-dissipative systems, namely $Q \leq 0$.

Example 2.1. Consider a two-dimensional, single-input system, with state $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, given by the following equations:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -2x_1 - x_2 + u,$$
$$y = x_2.$$

The system could represent, for instance, a spring-damper system with some input, such as gravity. The system is given in the form of equation (2.2), with $f(x,u) = \begin{bmatrix} x_2 \\ -2x_1 - x_2 + u \end{bmatrix}$, and $h(x,u) = x_2$. For this system, $f(0,0) = 0$, so $\bar{x} = 0$ is an equilibrium point, corresponding to $\bar{u} = 0$. Consider the storage function:

$$V(x) = x^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x.$$

Since V is smooth, one may calculate:

$$\frac{\partial V}{\partial x} f(x,u) = 4x_1x_2 - 4x_1x_2 - x_2^2 + ux_2 = uy - y^2 \leq uy.$$

It follows, then, that the system is dissipative, with storage function V and supply rate $r(u,y) = uy$. In other words, it is a passive system.

Regarding stability, consider $u(t) = 0$, for all $t \geq t_0$, with $t_0 \in \mathbb{R}$ the initial time. Then, one has $r(u(t), y(t)) = 0$, for every $t \geq t_0$. Therefore, the condition of Theorem 2.1 is always satisfied, and $\bar{x}$ is a Lyapunov stable equilibrium point of the system. Notice that, $r(0,y) = 0$ is true, even for $y \neq 0$, and, further, y might vanish, even when $x \neq 0$. This indicates the difficulty of extending this result to the asymptotic case, and the need for detectability conditions, which is explored, among others, in [Hill and Moylan, 1980]. □

2.3.2   The switching case

Extending dissipativity definitions in the case of switching systems is a non-trivial task. The difficulties found here are analogous to those faced in the process of extending core stability results to the case of switching systems. As in the stability case, a salient distinction among the various efforts to define dissipativity for switching systems has to do with the number of storage functions used. Some works, including [Haddad and Chellaboina, 2001], [Teel, 2010], [Naldi and Sanfelice, 2011],[Naldi and Sanfelice, 2014] and [Haddad and Sadikhov, 2012] define switching dissipativity using a single storage function, while others, such as [Zhao and Hill, 2008a], [Zhao and Hill, 2008b], [Zefran et al., 2001], [Chen and Saif, 2005], [Pogromsky et al., 1998], [Navarro-López and Laila, 2013], [Karalis and Navarro-López, 2017], [McCourt and Antsaklis, 2010a] introduce multiple storage functions.

To say more, in the single storage function case, the system behaviour is characterised by a single pair of dissipativity data (supply rate and storage function). This means that, in some sense, all the subsystems exhibit the same energy dissipation behaviour. Similar to the case of a common Lyapunov function, single storage function approaches are simple and conceptually clear, but restrictive. They leave out systems which exhibit the kind of mild behaviour that has been discussed previously, but the individual subsystems fail to satisfy the same dissipation inequality. Of the works in the single storage functions setting, [Haddad and Chellaboina, 2001] and [Teel, 2010] work along very similar lines, and, in their definition, follow [Willems, 1972] very closely. The efforts in [Naldi and Sanfelice, 2011, 2014] are conceptually similar, but they are formulated in a differential inclusion setting. A recent and more innovative approach can be found in [Haddad and Sadikhov, 2012]. There, dissipativity is defined using a single storage function, but the system is allowed to have multiple supply rates, only one of which is required to satisfy the inequality at every moment. Concrete definitions in this area tend to be very similar to their non-switching counter-part of Definition 2.5. For instance, the definition given in [Haddad and Chellaboina, 2001] is as follows:

Definition 2.8 (Single storage dissipativity [Haddad and Chellaboina, 2001])**.** System (2.3) is said to be dissipative, with respect to a supply rate $r : U \times Y \mapsto \mathbb{R}$, provided that the following inequality holds for any trajectory of the system, every admissible input

u, every switching law σ and every pair of time instants t_1, $t_2 \in \mathbb{R}$, such that $t_2 \geq t_1$:

$$V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} r(u(\tau), y(\tau)) d\tau. \qquad (2.5)$$

$\blacksquare$

Compare this definition with Definition 2.5. The only additional point is that, in the case of switching case, the inequality is required to hold, not only for every trajectory, but, also, for every possible switching law.

As an alternative to definitions using a single storage function, one may consider definitions which use multiple storage functions. The intuition here is, again, that having each of the subsystems satisfy its own dissipation inequality is already sufficient for a characterisation of the behaviour of the system. In order to define dissipativity in this setting, one first needs the obvious components, already discussed in the non-switching case; namely, a storage function and a supply rate for each subsystem. These have to satisfy their respective dissipation inequalities, which are identical to equation (2.4), and which express the way in which energy is dissipated by the system when a subsystem is active. However, the energy stored in a subsystem, expressed by its storage function, directly depends on the state. Then, even when a subsystem is inactive, its stored energy still changes, following the orbits of the active subsystem. Then, one needs a collection of dissipation inequalities, which capture (and bound) the energy change happening to each inactive subsystem. The supply rate corresponding to these dissipation inequalities is referred to as the cross-supply rate. The work in [Zhao and Hill, 2008a] is one of the more well-known in the area, and it offers the simplest application of the intuition discussed here. The following definition, which will be used in this work, is along similar lines.

Definition 2.9 (Dissipativity data – multiple storage case)**.** A collection of functions $\{r_{ii}\}_{i \in \mathcal{N}}$, such that, for every $i \in \mathcal{N}$, $r_{ii} : U \times Y \mapsto \mathbb{R}$, is a collection of possible supply rates for system (2.3), provided that they are all locally integrable along the system trajectories. Similarly, the collection of functions $\{r_{ij} \mid i, j \in \mathcal{N} \text{ and } i \neq j\}$, such that, for all admissible i, j, $r_{ij} : X \times U \times Y \mapsto \mathbb{R}$ is a collection of possible cross-supply rates, provided that they are all locally integrable along the system trajectories. Finally, a collection of functions $\{V_i \mid i \in \mathcal{N}\}$, such that, for all values of i, $V_i : X \mapsto \mathbb{R}$, is a collection of possible storage functions, provided that they are all positive. $\blacksquare$

Definition 2.10 (Dissipativity)**.** System (2.3) is said to be dissipative, with respect to a collection of supply and cross-supply rates $\{r_{ij} \mid i, j \in \mathcal{N}\}$, as described in Definition

2.9, provided that there exists a collection of storage functions $\{V_i \mid i \in \mathcal{N}\}$, such that the following inequalities are satisfied for all $t_k, t_{k+1} \in \mathcal{T}$, $t_k^1, t_k^2 \in \mathbb{R}$, when $t_k \leq t_k^1 \leq t_k^2 < t_{k+1}$:

1. $V_i(x(t_k^1)) - V_i(x(t_k^2)) \leq \int_{t_k^1}^{t_k^2} r_{ii}(u(t), y(t)) dt$, if $\sigma(t_k) = i$.

2. $V_i(x(t_k^1)) - V_i(x(t_k^2)) \leq \int_{t_k^1}^{t_k^2} r_{ij}(x(t), u(t), y(t)) dt$, if $\sigma(t_k) = j \neq i$.

For the level lines of V_i, the notation $N_i(\varepsilon) = \{x \in X \mid V_i(x) \leq \varepsilon\}$ is used. ∎

In brief, the definition posits that a system is dissipative when all its component subsystems (that is, the members of $\mathcal{F}$) are dissipative (with respect to some arbitrary supply rates), and it introduces the concept of the cross-supply rate, in order to capture the transfer of energy to a component, caused by the activity of another component. This is a very permissive definition; in essence, it just requires that each subsystem is characterised by some energy that evolves in accordance with known bounds.

In Definition 2.10, the supply and cross-supply rates satisfy almost identical inequalities and express energy transfers; they appear, then, to be conceptually similar. There is, however, an important distinction which has to be made between them. A supply rate for some subsystem expresses a property of that subsystem, namely, a deep relation between its inputs, its outputs and its state. A cross-supply rate, on the other hand, is an artifact of the connection: that is, of the fact that two subsystems are components of some switching system. Therefore, while the former is difficult to find, and might not exist, since a subsystem need not have properties of this form, the latter is trivial to find, since the connection is always there (for example, if V_i is differentiable, r_{ij} can always be taken to be $\frac{\partial V_i}{\partial x} f_j$).

The specific formulation of the dissipativity property given in Definition 2.10 is new. Conceptually, however, it is very close to the definition of dissipativity used in [Zhao and Hill, 2008a]. The difference is that, for [Zhao and Hill, 2008a], an additional condition is present, which guarantees that, for some inputs, the system is stable. While this is a desirable property, it has no intuitive basis, as non-switching dissipative systems are allowed to be unstable; and, in essence, it posits conditions/assumptions of the stability results as elements of the definition. Indeed, in this setting, a stability result merely states that, given such a stabilising input, the system becomes stable[1].

---

[1]It is not claimed that proving that is trivial. Just that the specific formulation is the source of some confusion.

It is argued that this approach is conceptually less clear, and that this third condition should be made part of the stability theorem(s).

A slightly more permissive definition of dissipativity is given in [Zefran et al., 2001; Chen and Saif, 2005], for the passive case, and in [Zhao and Hill, 2005], for the finite gain case. In these works, the same dissipation inequality is required for the active subsystem, while the equivalent behaviour for the inactive subsystems are only required to hold on average. That is, the inactive subsystem is required to satisfy Condition 2 of Definition 2.10, but only for whole intervals of the form $[\tau_\lambda, \tau_\kappa]$, with τ_λ, $\tau_\kappa \in \mathcal{T}$, such that the subsystem is inactive in the interval. The work in [Navarro-López and Laila, 2013], extends these results to the general dissipative case, for a more complex class of systems. However, instead of the 'average' version of the second condition of Definition 2.10, the authors posit the following alternative: the energy difference between consecutive activations of a subsystem is bounded by the supply rate of the relevant subsystem. This formulation of the condition is analogous to the form assumed by the dissipativity condition itself, in the case of discrete time systems, as seen, for instance in [Navarro-López, 2002]. These averaging results are less demanding, and, therefore, more well-suited to the study of hybrid automata. That is because for hybrid automata, subsystems are often associated with invariant sets, so that, within an invariant set, one and only one subsystem is active. Then, the evolution of the energy stored in the inactive subsystems is not important, until the trajectories reach the boundaries of an invariant set, in which case a switch happens. Systems of the form (2.3), on the other hand, do not have such spatial restrictions, so a definition along the lines offered here (Definition 2.10) is more appropriate.

As with the non-switching case, dissipativity definitions for switching systems are usually associated with stability results. For the single storage function case, these are similar to Theorem 2.1. To say more, having established that a single supply rate characterises the energy dissipation of the system, the condition that this supply rate is non-positive is sufficient to guarantee stability. All the works mentioned above include such results. For multiple storage function approaches, on the other hand, things are somewhat more involved. To see this, note that, even in the case where each subsystem satisfies a condition equivalent to that of Willems, it might be the case that the active subsystem is introducing so much energy to the inactive ones, that stability is disturbed. The clearest conceptual example where this happens is when two subsystems are stable, but have different, stable equilibria; even if each of them decreases its

own energy, switching makes the system unstable. To prevent such issues, the different stability results found in this area propose conditions related, directly or indirectly to the second condition of Definition 2.10. For the averaging cases, this means that the average behaviour of a subsystem (as described above) is required to be, at most, energy neutral. In this case, the condition is similar to the well-known condition of Branicky, found in [Branicky, 1998a]. For the case of [Zhao and Hill, 2008a], one has a slightly different condition, which is shown in the Theorem below, re-formulated in the notation used here.

Theorem 2.2 (Stabilisation from dissipativity [Zhao and Hill, 2008a])**.** *Take system* (2.3)*, and assume that it is dissipative, in accordance with definition 2.10, and that it has a strong equilibrium point at $\bar{x} \in X$, part of an equilibrium pair $(\bar{x}, \bar{u}) = (0,0)$. Take some continuous input function $v : X \mapsto U$, and a collection of absolutely integrable functions $\{\phi_{ij} : \mathbb{R} \mapsto [0, \infty) \mid i, j \in \mathcal{N}, i \neq j\}$. Assume that all the storage functions are continuous and positive definite. Then, v stabilises $\bar{x}$ for the system* (2.3)*, provided that the following conditions hold:*

- *$r_{ii}(v(x(t)), y(t)) \leq 0$, for all t, and all $i \in \mathcal{N}$ and*

- *$r_{ij}(x(t), v(x(t)), y(t)) \leq \phi_{ij}(t)$, for all t, and all $j \neq i \in \mathcal{N}$.*

$\blacksquare$

The first condition is a typical Willems-style condition for the supply rates of the system, implying that each subsystem dissipates its own energy, when it is active. The second condition is similar in function, but slightly more permissive. For each active-inactive system combination $i - j$, one has a positive ϕ_{ij}. Since ϕ_{ij} is absolutely integrable, if follows that, in total, a finite amount of energy can be transferred to subsystem j, by the activity of subsystem i. This energy, however, cannot be introduced 'too fast', because this would also move the active system outside its previous energy level, a behaviour that is disallowed by the first condition. These two insights are used to produce a δ-ε construction needed for a Lyapunov argument – this is the core of the proof of Theorem 2.2.

Another approach, which falls within the multiple storage function space is the one based on the idea of passivity indices, described in [McCourt and Antsaklis, 2010a] and [McCourt and Antsaklis, 2010b]. Here, the effort is less concentrated on defining the dissipativity framework as such, and more on solving specific problems, including stability, given that all the subsystems are dissipative. The core idea is that one can

express the excess (or the lack) of passivity of a subsystem by a set of numbers (the indices), and then address various problems by introducing algebraic conditions for these numbers. This approach is quite appealing for its applicability to the study of feedback interconnections. Despite its usefulness, it is rather restrictive since it is applicable solely to pairs of passive systems, and because it is difficult to use in setting beyond the interconnection problem.

In [Navarro-López and Laila, 2013], a slight twist is put in the application of dissipativity for stability. The subsystems are taken to form groups, with each group sharing common dissipativity data, and some common equilibria. The concept of total stability is then introduced, corresponding to the possibility that one of the equilibria of one group is stable for every one of the possible trajectories. As stated earlier, this move of allowing multiple non-coinciding equilibria for the subsystems is not very common in the literature. Other related efforts can be found in [Bolzern and Spinelli, 2004], where multiple subsystems are switched in a way that stabilises a (non-equilibrium) point which lays somewhere in between; and [Alpcan and Basar, 2010], where a dwell-time approach is used to guarantee the stability of a set around the union of equilibrium points.

It is appropriate, at this point, to present an application of the result in Theorem 2.2, as an blueprint for the way similar results are applied to switching systems.

Example 2.2. Consider the two dimensional switching system with two modes, discussed in [Zhao and Hill, 2008a]. This system is of form (2.3), with the following data:

$$f_1(x,u) = \begin{bmatrix} x_2 \\ -x_1^3 - x_2 + u \end{bmatrix},$$

$$f_2(x,u) = \begin{bmatrix} x_2 \\ (1 + x_2^2)(-x_1 - x_2 + u) \end{bmatrix},$$

$$h_1(x,u) = h_2(x,u) = x_2.$$

For this system, $(\bar{x}, \bar{u}) = (\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0)$ is an equilibrium pair. The system can be shown

to be dissipative, with the following dissipativity data:

$$V_1(x,u) = \frac{x_1^4}{4} + \frac{x_2^2}{2},$$

$$V_2(x,u) = \frac{x_1^2}{2} + \frac{1}{2}ln(1+x_2^2),$$

$$r_{11}(u,y) = r_{22}(x,y) = uy,$$

$$r_{12}(x,u,y) = x_2(1+x_2^2)(\frac{x_1^3}{1+x_2^2} - x_1) + x_2(1+x_2^2)u,$$

$$r_{21}(x,u,y) = x_2(x_1 - \frac{x_1^3}{1+x_2^2}) + \frac{x_2}{1+x_2^2}u.$$
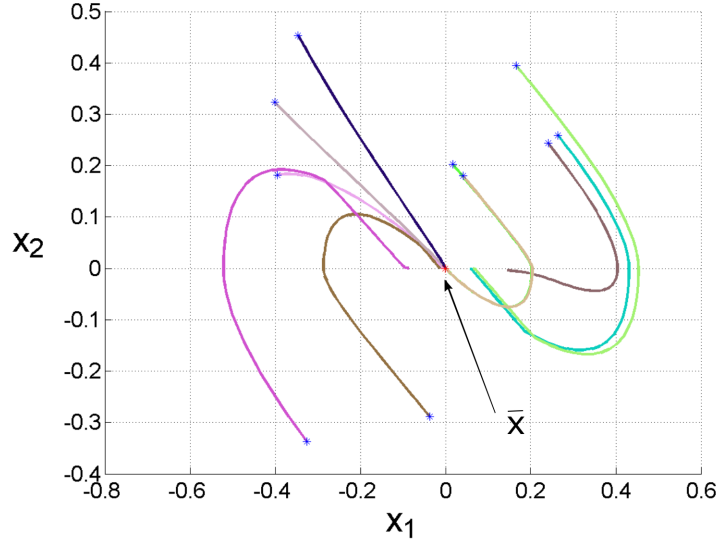
Figure 2.2: Sample trajectories of the system, under the control described in Example 2.2. The equilibrium point is at $\bar{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, and the initial conditions are designated by stars.

To apply Theorem 2.2, take the input function u, which is identically zero. Then, the first condition of Theorem 2.2 always holds. For the second condition, consider the functions $\phi_{21}(t) = e^{-t}$ and $\phi_{12}(t) = e^{-2t}$. The following rules for switching are proposed:

- Keep/Switch to $\sigma(t) = 1$, as long as $0 \leq r_{21}(x(t),0) \leq \phi_{21}(t)$, or $0 \leq r_{12}(x(t),0) > \phi_{12}(t)$.

- Keep/Switch to $\sigma(t) = 2$, as long as $0 \leq r_{12}(x(t),0) > \phi_{21}(t)$, or $0 \leq r_{12}(x(t),0) \leq \phi_{12}(t)$.
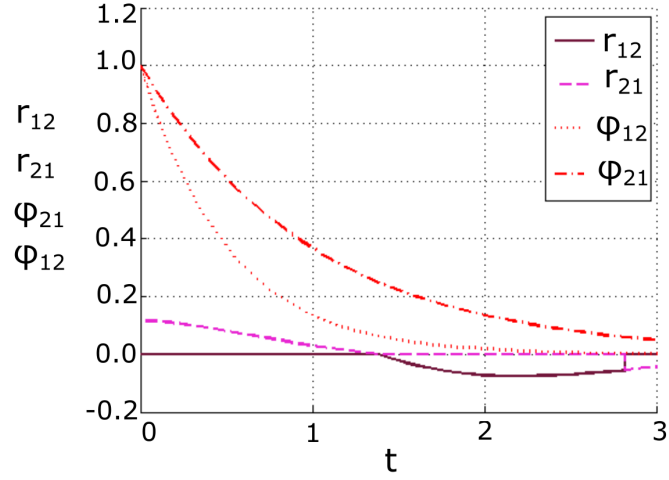
Figure 2.3: Evolution of the cross-supply rates, for one of the trajectories of the system of Example 2.2, shown in Figure 2.2. The ϕ bound for each of them is also given (the red dotted lines).

For these rules, the second condition of Theorem 2.2 always holds. Indeed, subsystem 1 is activated when subsystem 2 would introduce too much energy in subsystem 1 (the second part of the first rule), and it is kept active, as long as it does not transfer too much energy to subsystem 2 (the first part of the rule). Both conditions are, then, satisfied, and one may conclude that the system is Lyapunov stable. Simulations for a variety of initial conditions, chosen uniformly around the equilibrium point, are shown in Figure 2.2, and they validate the claim of Theorem 2.2. In Figure 2.3, it is also shown that, for one of the simulated trajectories, the cross-supply rates may get positive values. □

Chapter 3

Stabilisation for Switching Systems

In the previous chapter, the core components of the framework used in the present work were introduced. Switching systems were presented as the main object of study (along with hybrid systems, to be discussed later), and the two core properties of interest, namely, dissipativity and stability were discussed. In the present chapter, the problem of characterising stability under a given feedback function will be treated, for switching systems which are dissipative. This will lead to a series of stabilisation results, which form the first contribution of the present work.

3.1   Motivation

Before discussing the actual solution proposed here, it is appropriate to give some motivation. To that effect, consider Example 2.2, given as an instance of the application of the results in [Zhao and Hill, 2008a]. In the example, it is stated that the system, under zero input and following a specific set of switching rules, is stable, since it satisfies the conditions of Theorem 2.2. The switching rule used there is a subclass of the following family of rules:

1. Keep subsystem 1 active, when $r_{21}(x(t),0) \leq \phi_{21}(t)$, and $r_{12}(x(t),0) > \phi_{12}(t)$.

2. Keep subsystem 2 active, when $r_{12}(x(t),0) \leq \phi_{12}(t)$, and $r_{21}(x(t),0) > \phi_{21}(t)$.

3. Activate either subsystem, when $r_{12}(x(t),0) \leq \phi_{12}(t)$, and $r_{21}(x(t),0) \leq \phi_{21}(t)$.

The last rule implies that, for any switching law that is acceptable, r_{12} and r_{21} are both under their respective ϕ limits. Given rules like these, one might be tempted to propose a design process for the switching rule σ. First, pick some bounds ϕ_{12} and

ϕ_{21}, that correspond to the desired 'cross-transfer' of energy, and then pick any σ that belongs to the family above, induced by these bounds. This, however, is not possible, for two reasons.

First, while the functions ϕ provide an upper bound on the energy that gets transfered to the inactive subsystem, this bound is not tight. Because the functions ϕ are integrable, their integrals over their domain (here $[t_0, \infty)$) are finite numbers. Roughly, one may say that these numbers correspond to a 'virtual' energy, which is the upper bound of the total energy that may ever be transferred to subsystem i, due to the fact that subsystem j is active (in the case ϕ_{ij}). While this virtual energy represents an upper bound, it is crucial for any admissible, non-zero ϕ function that only a fraction of the available, virtual energy gets transferred to the inactive subsystem. Otherwise, it would be possible that, for some initial conditions, too much energy is transferred, and stability is disturbed. To see this, consider the fact that, while the same ϕ (and, thus, the same total available energy) is used for every initial condition, the initial values of the system energies may get a wide range of values, from infinitesimal, close to the equilibrium, to infinity. Introducing all the available energy cannot be expected to have the same effect in both of these cases. In fact, it is not even certain that, given two options for, say, ϕ_{12}, the one with the lower integral (virtual energy) will result in a family of switching rules for which less energy is cross-transferred.

Further, besides the total virtual energy that is available for cross-transfer, the bounds express some limitations about the temporal profile of the energy transfer. For instance, the exponential functions used in Example 2.2 imply that the energy transfer is front-loaded, in the sense that the energy of inactive subsystems is allowed to increase much more early in time (for t close to 0), than later on (see Figure 3.1). However, it is unclear from where this information might be obtained, given that one does not usually have such detailed information about the time behaviour of the system.

Then, any intuition on picking the bounds seems to vanish, and one is forced to try out different known integrable functions, starting from the zero function, in the hopes that a usable bound will come up. Even worse, from the rules stated above, there is a missing case. The possibility that $r_{21}(x(t), 0) > \phi_{21}(t)$ and $r_{12}(x(t), 0) > \phi_{12}(t)$ hold simultaneously is not considered. For the specific example examined in [Zhao and Hill, 2008a] (as well as the other example used in the same work), this issue does not arise. This is because for the dissipativity data of the system under consideration, it
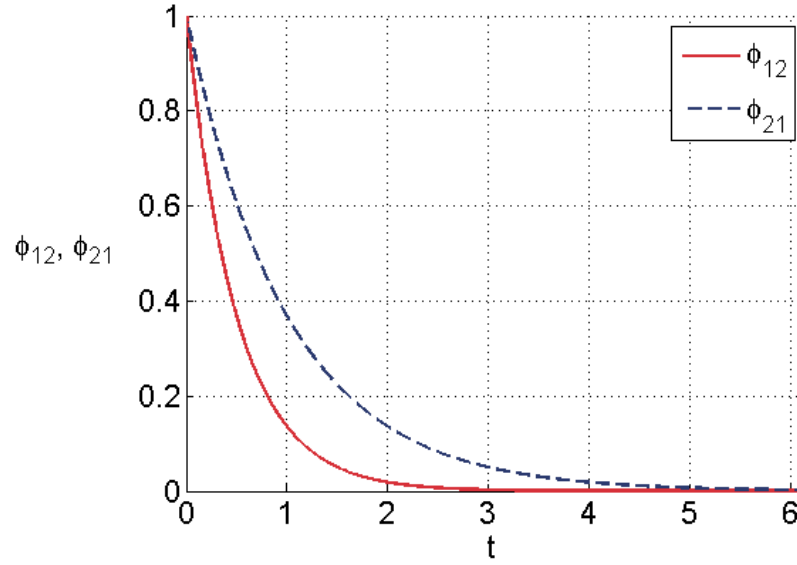
Figure 3.1: Functions ϕ_{12} and ϕ_{21}, the time dependent bounds on the cross-supply rates, needed for Theorem 2.2, when applied in Example 2.2. Energy transfer to the inactive subsystems is allowed only in the early stages of the process.

holds that r_{12} and r_{21} always have opposite signs. To be precise, it holds that

$$r_{12}(x(t),0) = -(1+x_2^2(t))r_{21}(x(t),0).$$

Then, since the bounds are positive, it is impossible to have both r_{12} and r_{21} to cross their respective bounds. In the general case, however, one would have to establish that such a case, where both subsystems are 'disqualified', does not arise.

However, proving that this kind of 'dead-zones' do not exist is not an easy task. The inequalities that need to hold involve both the system state and time, directly. In the absence of extensive information about the trajectories of the system, it is not clear how one would proceed to verify such inequalities. As discussed in Chapter 2, some authors work around this type of issue by requiring that time conditions hold only on average, between switching instants (recall the discussion in Section 2.3.2). In the present case, however, this path is not available. By the nature of the proof, the inequalities need to hold for the whole period of inactivity of the subsystem. It is unclear, then, how one would be able to conclude if some choice of ϕ_{12}, ϕ_{21} is actually usable. Of course, one could try using a time dependent input, designed so as to cause a cancellation of the non-autonomous terms on both sides of the inequalities. However,

introducing a time-dependent term, and then making efforts to cancel it out is a bizarre approach. And, besides, it gives no additional insight about how to chose ϕ. Then, the only *bona fide* option for the bounds is the zero function. This seems to indicate that an alternative viewpoint to the problem might be preferable.

To sum up this part, while Theorem 2.2 furnishes us with a criterion to validate stability under a given input, in actuality, deciding if the criterion is satisfied, and, indeed, picking parameters for which this happens, is a convoluted, potentially impossible task. The intention of the present chapter is to offer a different viewpoint to the characterisation of stabilisation for switching system, which, on the one hand, maintains the core idea of [Zhao and Hill, 2008a] (bounds for the cross-supply rates), but it avoids the issues stressed in this section.

3.2 A State Space Partition

From the preceding discussion, it follows that the time-dependent bounds on the cross-supply rates cause some problems, which one would prefer to avoid. The idea behind the introduction of these bounds, however, still has merit. Indeed, having such bounds allows one to reason about stability in the following way: if a cross-supply rate is bounded by a negative function, it can be concluded that the interaction between the corresponding active and inactive subsystems is energy-decreasing, from the perspective of the active subsystem.

A natural step in the direction of keeping this basic argument structure, while avoiding the pitfalls of time-dependent bounds would be to adopt state-dependent ones, having the form: $\psi : X \mapsto \mathbb{R}$. This would yield inequalities (that is, switching rules) of the following form (again, using Example 2.2):

$$\text{If } r_{21}(x(t),0) \leq \psi_{21}(x(t)), \text{ then, DO SOMETHING.}$$

The exact nature of the main clause (DO SOMETHING) of this conditional statement will be discussed later in this section. For now, two remarks are necessary. First, the lack of intuition mentioned in the previous section, regarding the choice of ϕ, vanishes in this case. Indeed, now one does not have to decide if the cross-transfer of energy happens early or late in the trajectory, but only if the cross-transfer is more or less powerful in some subset of the state space. For autonomous systems, such as the ones discussed in this work, this viewpoint is a more natural one. Second, this phrasing of

the condition implies a type of partition of the state space. The idea of splitting the state space into subsets, and deriving switching rules based on those subsets has been used in the relevant literature, for instance in [Liu et al., 2010], [Peleties and DeCarlo, 1992], [Pogromsky et al., 1998] and, more recently, in [Liu and Stechlinski, 2018]. This works, however, either assume the existence of a single storage function, or work with a simpler class of systems. It is the purpose of this work to expand on this idea and further study its applicability.

Beside the motivation set out in the previous section, another assumption of the work in [Zhao and Hill, 2008a] deserves some attention; namely, the fact that the authors only consider systems with a single, common equilibrium point. This assumption is crucial for the argument made in that work to hold. That is because, *grosso modo*, the authors show stability by proving that the energies corresponding to all the subsystems eventually reach zero, and any energy increase caused by inactivity is not 'too large'. For a system whose subsystems have distinct equilibria, however, it would make sense, intuitively, to assume that the infimum of the energies of individual subsystems are located around their respective equilibria. Then, little justification is left for the assumption that all the energies of the subsystems eventually decrease to zero. A workaround to this issue is proposed in [Alpcan and Basar, 2010], where stability is established with respect to a set which contains all the individual equilibrium points. While this is a reasonable and smart approach, the results in this work will resemble more closely the one proposed in [Lu and Brown, 2010]. There, instead of treating all the subsystems as equal, one of them is selected and it takes a privileged position in establishing stability. The energy of this 'primary' subsystem is required to decrease and eventually reach zero, while, for the rest of the subsystems, much more permissive restrictions are in place, in order to guarantee that they behave in a mild way. As will be shown later in this chapter, this choice allows for the results to be applicable to a considerably wider class of systems.

As a starting point for the exposition of the results in this chapter, consider the following definition:

Definition 3.1 (State space partition). Take a weak equilibrium point $\bar{x}$, from a triplet $(\bar{x}, \bar{u}, i)$ of System (2.3) and a continuous function $v : X \mapsto U$. Use $\mathcal{J}$ to denote $\mathcal{P}(\mathcal{N})$, the powerset of $\mathcal{N}$. The partition $\mathcal{K}[\bar{x}, v, \psi]$ is a collection of sets $\{K_J\}_{J \in \mathcal{J}}$, defined as

follows:

$$K_J = \{x \in X \mid r_{ij}(x, v(x), h_j(x, v(x))) \leq \psi(x) \forall j \in J\}$$

$$\bigcap$$

$$\{x \in X \mid r_{ij}(x, v(x), h_j(x, v(x))) > \psi(x) \forall j \in \mathcal{N}/J\}.$$

■

Remark 3.1. While the partition is given for a weak equilibrium point, it can also be constructed in the strong equilibrium point case. In the strong case, one has a pair $(\bar{x}, \bar{u})$, instead of a triplet $(\bar{x}, \bar{u}, i)$, and they may use any subsystem ($i \in \mathcal{N}$) in the formation of the partition.

Some remarks on the notation are due. First, the function ψ is the state-dependent bound mentioned above, and the elements of the partition are sets K_J (for instance, $K_{\{1,2\}}$). The subscript signifies the subsystems for which the cross-supply rates (with respect to subsystem i) are below the bound ψ. For instance, if $\mathcal{N} = \{1, 2, 3\}$, the set $K_{\{1,2\}}$ consists of all the points for which r_{i1} and r_{i2} are dominated by ψ, but r_{i3} dominates it; here, i is the subsystem corresponding to the equilibrium point $\bar{x}$, used to build the partition. By construction, the sets in the partition are disjoint; that is, for any $J_1, J_2 \in \mathcal{P}(\mathcal{N})$, it holds that $K_{J_1} \cap K_{J_2} = \emptyset$. Further, since all the possible combinations (for the cross-supply rates to be over/under the bound) are included, the partition covers the whole state space; that is $\bigcup_J K_J = \mathbb{R}^n$. Figure 3.2 provides an example of such a partition, for the planar case.

Following the discussion above, the partition is defined with respect to a pre-specified subsystem i, to which the equilibrium point corresponds. The partition is formed by the supply rate and the cross-supply rates that correspond to this subsystem. Evidently, one can have as many partitions as subsystems. A similar effect could have been achieved by introducing a collection of bounds, one for every pairing of i with the rest of the subsystems. This would be conceptually identical to the approach taken here, albeit slightly more general. Here, the simpler approach is chosen, so that the notation does not become over-burdened. Finally, while the partition can be constructed for any function ψ, continuous functions are used, anticipating some requirements for regularity that arise in the subsequent results.

Besides the equilibrium and the bound, the input function is also a factor in the formation of the partition. Since system (2.3) is autonomous, restricting attention to state-dependent inputs is not unreasonable. Another option would have been the use of
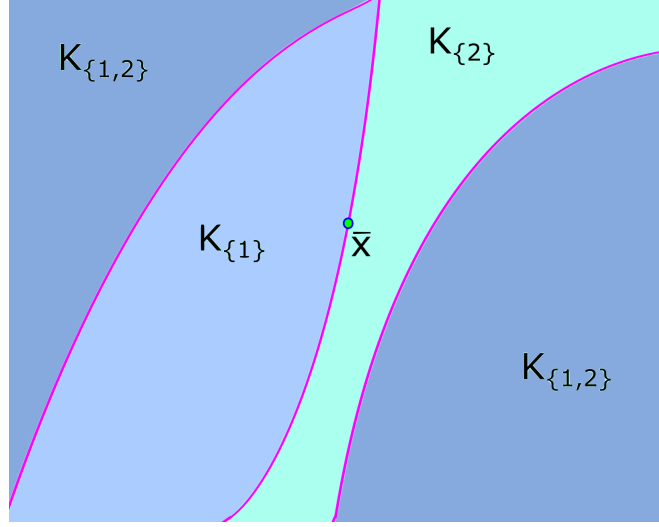
Figure 3.2: Generic partition for a system with two subsystems, labeled 1 and 2. The labelled areas of the partition correspond to different elements of $\mathcal{K}$, described in Definition 3.1.

output feedback. Such results have been studied, for example in [El-Farra et al., 2005] and [Sun and Ge, 2003]. In this work, it is assumed that the full state of the system is always available, even though, in reality, this might not always be the case. For some studies on the notion of observability that is appropriate for switching systems can be found in the relevant literature outlined in the previous section; such notions, however, will not be further explored here. A more involved approach could also be introduced, using both time-dependent inputs and time-dependent bounds ψ. Many of the arguments made in this work could be transferred to this setting with few changes.

3.3 Main Stabilisation Results

In principle, the partition may be formed for positive, negative or mixed-sign ψ functions. However, if ψ is taken to be negative, then it captures the idea that, in some area in the state space, some cross-supply rates do not become positive. This information can be used to make a straightforward Lyapunov-style argument for stability. The gist of the argument is as follows: if one takes ψ to be non-positive, then it is possible, under some circumstances, to extract from the partition a restriction for the switching law, such that, for systems that adhere to the restriction, stability ensues. This discussion is captured by the following pair of results.

Theorem 3.1 (Stabilisation from dissipativity)**.** *Consider system* (2.3), *and a weak*

equilibrium point $\bar{x}$, from a triplet $(\bar{x}, \bar{u}, i)$. Take a compact set $E \subseteq X$, such that $\bar{x} \in E$. Assume that system (2.3) is dissipative according to Definition 2.10, and that V_i is continuous and it has the $PD(\bar{x})$ property. Take some initial condition $x(t_0) = x_0 \in E$ and the continuous feedback controller $v : X \mapsto U$. Take some non-positive function $\psi : X \mapsto \mathbb{R}$. Form the partition $\mathcal{K}[\bar{x}, v, \psi]$, as given in Definition 3.1. Consider the following:

- ***Condition:*** *$(E \cap K_\emptyset)/\{\bar{x}\} = \emptyset$;*

- ***Switching Rule:*** *if, for some t, $x(t) \in K_J \cap E$, then $\sigma(t) \in J$.*

If the system that satisfies the condition, and the switching law σ is restricted by the switching rule, then the controller v stabilises the equilibrium point $\bar{x}$, locally in E. ∎

Proof. First, for any trajectory satisfying the condition and evolving under the switching law proposed here, the stored energy V_i is decreasing. To see this, take $\tau_2 \geq \tau_1$. Then, it holds that

$$
\begin{aligned}
V_i(x(\tau_2)) &\leq V_i(x(\tau_1)) + \int_{\tau_1}^{\tau_2} r_{i\sigma(s)}(x(s), v(x(s)), h_{\sigma(s)}(x(s), v(x(s)))) ds \\
&\leq V_i(x(\tau_1)) + \int_{\tau_1}^{\tau_2} \psi(x(s)) ds \\
&\leq V_i(x(\tau_1)),
\end{aligned}
\tag{3.1}
$$

where the second inequality follows from the condition, the switching law and the structure of the partition.

A Lyapunov-style argument is, then, used to establish stability for the composite system (that is, the system under v). To see this, note that, for every sphere of radius ε, $B(\varepsilon, \bar{x}) \subseteq E$, one can find $\mu = \min_{x \in S(\varepsilon, \bar{x})} V_i(x)$. Such a μ always exists, as the sphere is compact and V_i is continuous. Then some $\mu' < \mu$ can always be found, such that the level line $N_i(\mu')$, with $\mu' < \mu$, is wholly contained within $B(\varepsilon, \bar{x})$. For any δ such that $B(\delta, \bar{x})$ is contained within $N_i(\mu')$, one is certain (since V_i is decreasing) that, any trajectory with initial state $x(t_0) \in B(\delta, \bar{x})$ will always remain within $B(\varepsilon, \bar{x})$, hence $\bar{x}$ is Lyapunov stable in E. □

Theorem 3.2 (Asymptotic stabilisation from dissipativity)**.** *Consider the same setup as in Theorem 3.1, but with a function ψ that has the $ND(\bar{x})$ property. Assume, additionally, that $\bar{x}$ belongs to the interior of E, and that E is forward-invariant for the system; that is, for every trajectory with initial condition $x(t_0) \in E$, it is true that*

x(t) ∈ E, for all t ≥ t₀. Then, x̄ is rendered asymptotically stable in E, for system (2.3), under the controller v, provided that it satisfies the condition of Theorem 3.1, and that it switches according to the associated switching rule. ∎

Proof. Lyapunov stability for the composite system follows by the argument given for Theorem 3.1. For asymptotic stability, convergence to $\bar{x}$ is also needed.

By assumption, V_i is continuous and lower-bounded by 0, and, as shown above, it is also decreasing along the trajectories of the system. By the monotone convergence theorem [Jost, 2006], one has that a strictly decreasing, lower-bounded sequence of real numbers always converges. It is deduced, then, that $V_i(x(.))$ converges to some $\kappa_{x_0} \geq 0$, which depends on the initial condition $x(t_0) = x_0$, as $t \to \infty$. The point to which the sequence converges depends on the initial condition. It is conceivable, then, that, for some initial conditions, the trajectories might converge, but not to zero, and, therefore, not to $\bar{x}$. One needs to show that this is not possible.

For that purpose, assume that, for some initial condition x_0, $\kappa_{x_0} = \kappa > 0$. Take some $\delta > 0$, such that $B(\delta, \bar{x}) \subset N_i(\kappa) \cap E$. The convergence of V_i implies that the trajectory will never enter $B(\delta, \bar{x})$. Take $V_i(x_0) = \kappa_0 \geq \kappa$, and

$$W = N_i(\kappa_0) \cap E - B(\delta, \bar{x}).$$

The set W is compact, since $N_i(\kappa_0)$ is closed, by definition; E is compact, by assumption; and $B(\delta, \bar{x})$ is open, by definition. Take $\mu_i(x) = \max_{j \in \mathcal{N}}\{r_{ij}(x, v(x), h_j(x, v(x))\}$, which is continuous, since the cross-supply rates are continuous. It follows that $\gamma = \max_{x \in W} \mu_i(x)$, the supremum of a continuous function over a compact set, always exists. Further, it follows that $\gamma < 0$, since it is the maximum over negative values.

Then, by the second part of Inequality ((3.1)), it holds that

$$V_i(x(\tau_2)) \leq V_i(x(\tau_1)) + \gamma(\tau_2 - \tau_1). \tag{3.2}$$

For τ_2 large enough, this implies that $V_i(x(\tau_2))$ becomes negative. This is a contradiction, caused by assuming that $\kappa > 0$. It holds, then, that, for every initial condition, $\kappa = 0$ and, since $V_i \to 0$, $\lim_{t \to \infty} x(t) = \bar{x}$. The proof is, then, complete. □

These results are local, since E is required to be compact. Evidently, it would be useful to have global versions. The only obstacle to that is the fact that, in the general case, the level lines of the storage function V_i might be non-compact sets. As a result, the first part of W, in the in the proof of Theorem 3.2 may be non-compact too, as

neither of its components is known to be compact. In order to overcome this issue, it is common in the literature to require that V_i has the well-known radial unboundedness property, described in the following definition.

Definition 3.2 (Radially unbounded functions)**.** A function $V : X \mapsto \mathbb{R}$ is said to be radially unbounded, when $\|x\| \to \infty$ implies that $V(x) \to \infty$. ∎

It is known (say, from [Khalil, 2002]), that functions that are radially unbounded have compact level lines. Then, one may obtain global versions of Theorems 3.1 and 3.2, just by requiring that V_i has this property.

Theorem 3.3 (Global stabilisation from dissipativity)**.** *Consider the same set-up as in Theorem 3.1 (Theorem 3.2). Take $E = \mathbb{R}^n$, and assume that V_i is radially unbounded. Then, $\bar{x}$ is Lyapunov stable (respectively, asymptotically stable) in E, for system* (2.3), *under the control v, provided that the system satisfies the condition of Theorem 3.1, and that it switches according to the associated switching law.* ∎

3.4   Discussion and Additional Results

Having stated the main theorems of this Chapter, it is appropriate to add some comments on their function. The core idea behind them is as follows. For some feedback law $v(x)$, and a parameter-like function $\psi(x)$, the partition can be used to pick out subsystems that exhibit a mild behaviour; that is, those whose activation is energy neutral, from the perspective of V_i. The switching rule guarantees that the active subsystem always belongs to the subset of (energy-neutral) subsystems. While the term is used in the singular referred to as a switching rule, it would be more accurate to say that it is a family of switching rules; for every K_J, an element of the partition, any switching among the elements of J is allowed.

The role of the condition is to prevent a situation similar to the one described in Section 3.1, regarding the approach of [Zhao and Hill, 2008a]. That is, it guarantees that there is no point for which all the available cross-supply rates are over the bound ψ. Recall that, if a cross-supply rate is over the bound, then the corresponding subsystems may not be activated 'safely', without increasing the stored energy. In other words, the condition ensures that, everywhere in the state space, at least an energy-neutral subsystems exists and can be picked, without violating the switching rule. This subsystem acts, then, as a 'fall-back' option, from the point of view of Theorems of this section, in the sense that it allows for a choice of switching that does not cause

an energy increase. The most obvious way to achieve this goal is by requiring that the supply rate corresponding to V_i (that is, r_{ii}) is always under the bound ψ. Such an approach would correspond to the usual stability condition for dissipative systems, as seen, for example, in [Willems, 1972]. It would also mirror the choice made in [Zhao and Hill, 2008a], as seen in Theorem 2.2, as well as the intuitive expectation that the energy increases (for a stable subsystem) would be caused rather than compensated by the interconnection with other subsystems. For Theorem 3.1, however, this particular assumption is not made; any cross-supply rate can act as the fall-back, and, indeed, different cross-supply rates may fulfill that role in different subsets of the state space. Following this part of the discussion, one gets the following result, which specialises Theorem 3.1 to the case where some pre-specified cross-supply rate r_{ij} is always below the bound.

Corollary 3.1 (Stabilisation: special case)**.** *Take the same setup as in Theorem 3.1. Consider the partition $\mathcal{K}[\bar{x}, v, \psi]$, as given in Definition 3.1. For some $k \in \mathcal{N}$, take the following pair of restrictions:*

- ***Condition:*** $r_{ik}(x, v(x), h_j(x, v(x))) \leq \psi(x)$ *for all $x \in X/\{\bar{x}\}$;*

- ***Switching Rule:*** *if, for some t, $x(t) \in K_J$, then $\sigma(t) \in J \cup \{k\}$.*

For a system that satisfies the condition and switches in accordance with the switching rule, $\bar{x}$ is Lyapunov stable in E. ∎

The preceding discussion makes clear that this result is strictly less general than Theorem 3.1. The same reformulation may be produced for the case of Theorems 3.2 and 3.3, to the same effect. Such a task will be avoided here, as it would not add to the presentation.

In the context of Theorems 3.1 and 3.2, the function ψ plays a role that is intuitively clear and well-defined. Indeed, choosing between different ψ functions can be done in a sensible way as follows. If, for some choices ψ and ψ', one has one always dominates the other (say, for all x, one has $\psi(x) \geq \psi'(x)$), then, establishing the result for ψ' implies that the cross-supply rates are bounded further from 0, and, therefore, energy gets dissipated quicker. It follows, then, that one should expect quicker convergence to $\bar{x}$, when compared to the case of ψ. At the same time, however, one should expect that it may be trickier to show that the condition of Theorem 3.1 holds.

The results of this Chapter, so far, concerned either neighborhoods of the equilibrium point (local versions), or the whole state space (global versions). However, the

energy decrease argument used in proving Theorem 3.2 can be applied on arbitrary subsets of the state space.

Lemma 3.1 (Energy decrease from dissipativity)**.** *Take a system* (2.3) *with an equilibrium point $\bar{x}$ from a triplet $(\bar{x}, \bar{u}, i)$ and some input feedback function v. Suppose that the system is dissipative, and that the storage function V_i is continuous and non-negative. Build the partition $\mathcal{K}[\bar{x}, v, \psi]$, for some continuous function $\psi : X \mapsto \mathbb{R}$. Take some arbitrary set $E \subseteq X$, and assume that, in the time interval $[t_0, T]$, the system state remains in E; that is $x(t) \in E$ for all $t \in [t_0, T]$. Then, for any $t \leq t' \in [t_0, T]$, it holds that:*

$$V_i(x(t_0)) \geq V(x(t)) \geq V(x(t')).$$

∎

Proof. This is implied by the proof of Theorem 3.1. □

In this case, energy decrease is not enough to conclude stability, since stability is only meaningful on neighborhoods of equilibria (in the wide sense: X is still a neighborhood). It is possible, however, to draw conclusions about some aspects of the behaviour of the system, while it remains in the set. In particular, it can be shown that, as it evolves, the system will tend to leave the set E. Such a result is useful, in case one wants to establish that the system will not spend all of the remaining time in some set. This description evokes the liveness and deadness properties, which are discussed in [Carter, 2013] and [Navarro-López and Carter, 2016]. Further, this information is useful, because it allows a jigsaw approach to stability: one where the state space is covered by sets of arbitrary shape (that is, not necessarily neighbourhoods of the equilibrium point), each of which is characterised by different switching rules. Conceptually, this is also similar to a technique used in the area of model predictive control (for non-switching systems), known as explicit mpc, and presented, for instance, in [Bemporad et al., 2002]. There, the (optimisation) problem at hand is posed in such a way that a partition of the state-space into polygons emerges. This partition is then used to construct piece-wise affine, explicit solutions for the problem solution (the solution is the analogue of $u(t)$ for some instant t), which may be calculated 'offline' (in conventional mpc, the calculation happens 'online', at each time instance). The following result is an application of Lemma 3.1, and it captures some of the preceding discussion.

Corollary 3.2 (Behaviour in arbitrary set E)**.** *Take a system* (2.3) *with some equilibrium point $\bar{x}$ from a triplet $(\bar{x}, \bar{u}, i)$ and some input feedback function v. Suppose that the system is dissipative, and that the storage function V_i is continuous, it has the $ND(\bar{x})$ property, and it is radially unbounded. Build the partition $\mathcal{K}[\bar{x}, v, \psi]$, for some continuous function $\psi : X \mapsto \mathbb{R}$. Take some set $E \subseteq X$, and an initial condition $x(t_0) \in E$. Then, for system* (2.3)*, satisfying the condition of Theorem 3.1, and switching according to the associated switching law, one of the following occurs:*

- *There exists $T > t_0$, such that $x(T) \notin E$.*

- $\lim_{t \to \infty} x(t) = \bar{x}$.

∎

Proof. By the same argument used in Theorem 3.2, V_i is known to be strictly decreasing on E. This allows for two distinct behaviours for every trajectory of $\mathcal{H}$. Either the trajectory leaves E, since remaining within it indefinitely would eventually make V_i negative (same as the proof of Theorem 3.2); or it reaches $\bar{x}$, remaining within E the whole time. No other point $\hat{x}$ in the boundary of E can be approached the same way, because of the fact that ψ has the $ND(\bar{x})$ property, so, in some neighbourhood of $\hat{x}$ it would have to always be larger than some constant $\lambda > 0$. But remaining indefinitely in such a neighbourhood would be impossible, since it would, again, make V_i negative. $\square$

Remark 3.2. Corollary 3.2 is given only in the style of Theorem 3.2, for a ψ has the $ND(\bar{x})$ property. This is necessary. If one allows for ϕ to be $sND(\bar{x})$, a third type of behaviour might come to be, if E contains any non-wandering sets (say, cycles). Additionally, V_i is required to be radially unbounded, in order to allow for non-compact sets E. Evidently, one can relax this conditions, by putting restrictions on the characteristics of the E sets.

3.5   Numerical Examples

A starting point to discuss the different ways of applying the results of the previous section is Example 2.2, which was discussed extensively in Section 3.1.

Example 3.1. As it was pointed in Example 2.2, $\bar{x} = 0$ is a strong equilibrium point for the system of Example 2.2, corresponding to the equilibrium pair $(\bar{x}, \bar{u}) = (\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0)$.

Further, the following dissipativity data (Definition 2.9) hold for it (repeated here for convenience):

$$V_1(x,u) = \frac{x_1^4}{4} + \frac{x_2^2}{2},$$

$$V_2(x,u) = \frac{x_1^2}{2} + \frac{1}{2}ln(1+x_2^2),$$

$$r_{11}(u,y) = r_{22}(x,y) = uy,$$

$$r_{12}(x,u,y) = x_2(1+x_2^2)(\frac{x_1^3}{1+x_2^2} - x_1) + x_2(1+x_2^2)u,$$

$$r_{21}(x,u,y) = x_2(x_1 - \frac{x_1^3}{1+x_2^2}) + \frac{x_2}{1+x_2^2}u.$$

In order to construct the partition of Definition 3.1, one needs a feedback law, a bound and a pre-selected subsystem. Since the equilibrium point is strong, any of the subsystems can be used; here, take subsystem 1. In this case, Lyapunov stability will be established; then, consider $\psi(x) = 0$ for all $x \in X$, which is non-positive, as needed. For the condition of Theorem 3.1, one needs to show that $K_\emptyset \subseteq \{\bar{x}\}$. This condition is automatically satisfied for the trivial feedback controller $v(x) = 0$, for all $x \in X$. That is because

$$r_{11}(x,u) = v(x)x_2 = 0.$$

In fact, in this case $K_\emptyset = \emptyset$. It follows that stability is guaranteed, if the switching law is in accordance to the switching rule of Theorem 3.1. The choice of data made here results in the partition $\mathcal{K}[\bar{x}, v, \psi]$, which, in the two mode case, depends solely on the state x and the sign of the quantity $r_{11}(x, v(x)) - \psi(x)$. Then, one can visualise the partition by plotting the quantity; this is done in Figure 3.3.

The resulting rule is then the following: any subsystem or sequence of subsystems can be active in the grey area, while only subsystem 1 is allowed to be active in the red area (the yellow ball is added for visibility and it represents a neighborhood of the equilibrium point $\bar{x}$). Figure 3.4 shows the trajectories of the system, resulting for a random (uniform) selection of initial conditions from a square area with a side of 20 units around the equilibrium point. For these simulations, the switching law is as follows:

- in the red area, subsystem 1 is always active;

- in the grey area, either one of the subsystems 1 and 2 gets activated;
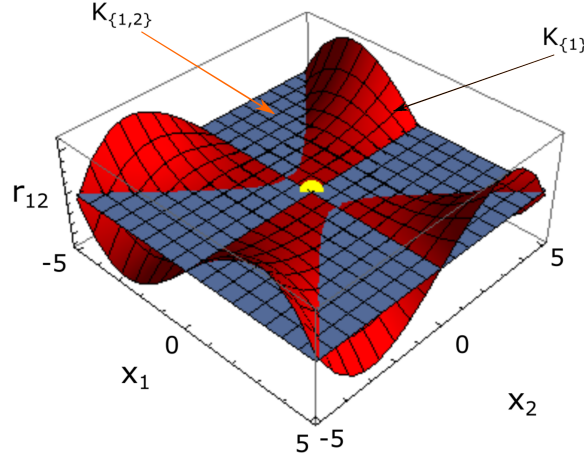
Figure 3.3: Visualisation of the partition for Example 3.1. A plot of r_{12} (the red surface) and the bound ψ (gray plane). The yellow ball is a neighborhood of the equilibrium point $\bar{x}$. The arrows point to the areas that correspond to elements of the partition $\mathcal{K}$.

- if activated in the red area, the subsystems remain active for a random time interval that lasts at most 0.1 time units, or until the state enters the grey area;

- when the state exists the red area, a random choice is made and one of the two systems is activated.

All the random choices made in this process are uniform. In this particular case, the trajectories seem to actually converge to $\bar{x}$; this asymptotic behaviour is not predicted by the theorem. In Figure 3.9, it is shown that the stored energy V_1 is indeed non-increasing, as expected for a Lyapunov stable equilibrium. Even though the simulations shown here are made for a collection of initial conditions in a neighborhood of $\bar{x}$, the conclusion holds globally, in accordance to Theorem 3.3, since V_1 is radially unbounded. In Figure 3.6, one can also see the switching rule that is used to produce one of the trajectories of Figure 3.4.

One may also apply Theorem 3.1, this time with respect to subsystem 2. The resulting partition, $\mathcal{K}'[\bar{x}, u, \psi]$ is different to the one shown in Figure 3.3, as it is constructed using r_{21}, rather than r_{12}. This partition is shown in Figure 3.7. Even though the partition is different, the exact same thought process (and switching rules) described above can also be applied here in order to conclude stability.   □

The application of the results in the case of asymptotic stabilisation is similar. The next example substantiates this point.

Figure 3.4: Sample trajectories for the system of Example 3.1. Randomly selected initial conditions (shown as stars). Equilibrium point at $\bar{x} = \begin{bmatrix} 0,0 \end{bmatrix}^T$.

Figure 3.5: Evolution of stored energy for the system of Example 3.1. Energies corresponding to the trajectories of Figure 3.4.
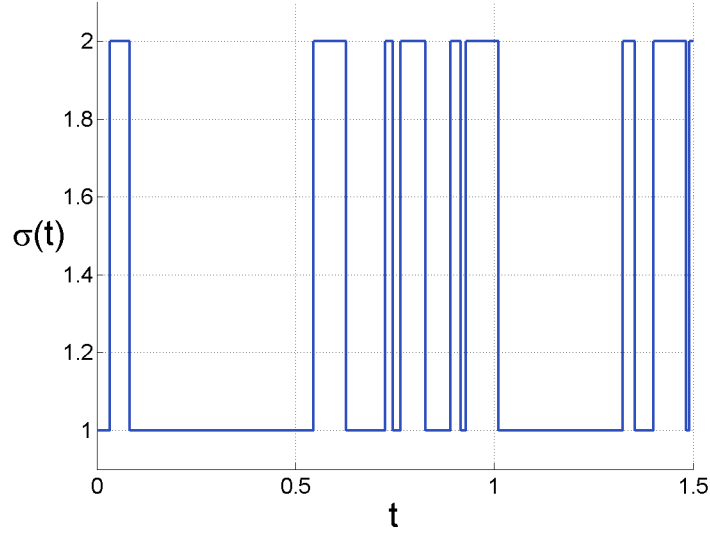
Figure 3.6: Sample switching signal for the system of Example 3.1. It corresponds to the trajectory of the same colour, shown in Figure 3.4.
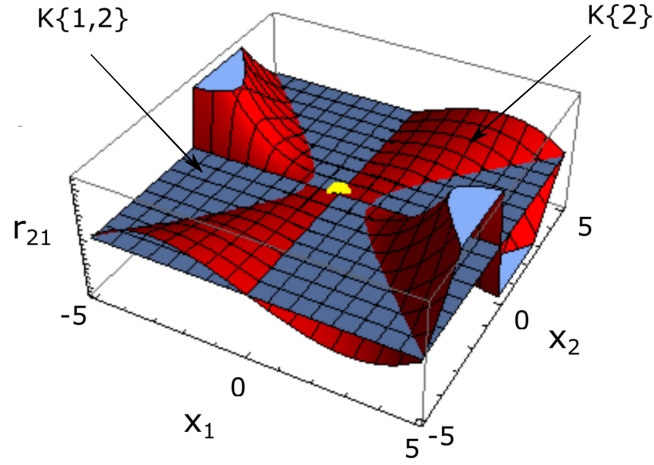
Figure 3.7: Visualisation of a different partition for Example 3.1, constructed wrt subsystem 2. A plot of r_{21} (the red surface) and the bound ψ (gray plane). The yellow ball is a neighborhood of the equilibrium point $\bar{x}$. Compare with Figure 3.3.

Example 3.2. Consider a system with the general form (2.3), and the following dynamics:

$$f_1(x,u) = \begin{bmatrix} -4x_1 + 3x_1 sin^2 x_1 + x_1 x_2 + x_1 u \\ -x_1^2 - 3x_2 sin^2 x_1 - 4x_2 + x_2 u \end{bmatrix}, \tag{3.3}$$

$$f_2(x,u) = \begin{bmatrix} x_1^2 x_2 + x_1 u \\ -x_1^3 + 2x_2 + x_2 u \end{bmatrix}, \tag{3.4}$$
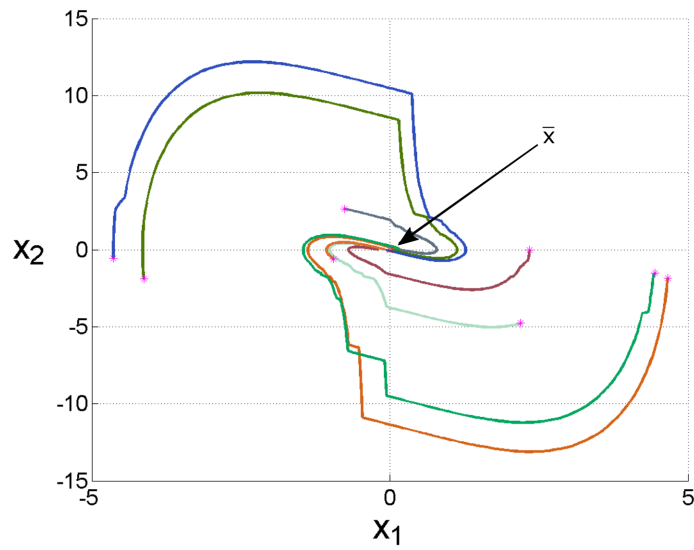
and $h_1(x,u) = h_2(x,u) = x_1^2 + x_2^2$.
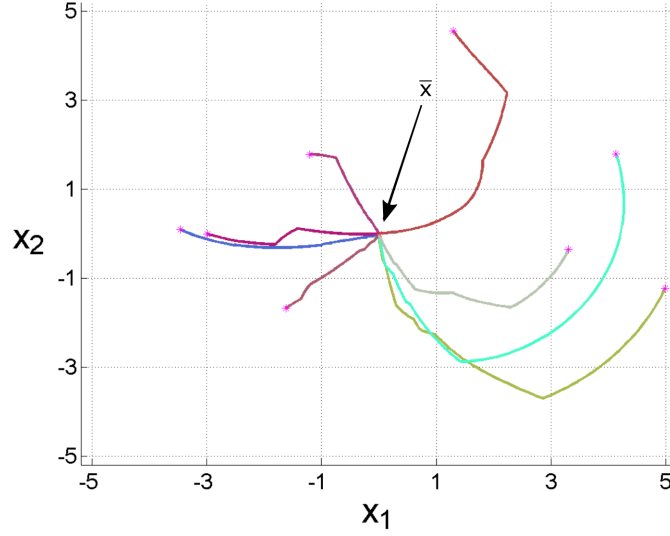
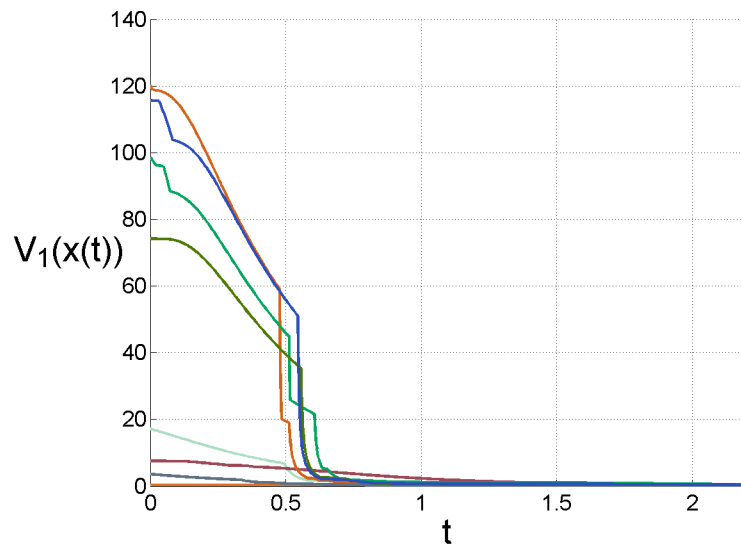Figure 3.8: Sample trajectories for the system of Example 3.2. Randomly selected initial conditions (shown as stars). Equilibrium point at $\bar{x} = \begin{bmatrix} 0, 0 \end{bmatrix}^T$.

The relevant dissipativity data are as follows:

$$V_1(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2,$$

$$r_{11}(x,u) = -4y + uy,$$

$$r_{12}(x,u) = 2x_2^2 + x_2^2 u + x_1^2 u.$$

Take a constant $v(x) = -3$ and $\psi(x) = -x_1^2 - x_2^2$. Then, for the equilibrium point $\bar{x}$ from the pair $(\bar{x}, \bar{u}) = (\begin{bmatrix} 0 & 0 \end{bmatrix}^T, -3)$ (which, in this case, is strong), form the partition $\mathcal{K}[\bar{x}, v, \psi]$. In this case, it holds that

$$r_{11}(x, v(x)) = -7y^2 \leq \psi(x),$$

for all $x \in X$, and, therefore, $K_{\emptyset}$ is known to be empty. Further, since

$$r_{12}(x, v(x)) = -x_1^2 - 3x_2^2 \leq \psi(x),$$

it also follows that $K_{\{1\}}$ and $K_{\{2\}}$ are empty too, and, then, that $K_{\{1,2\}} = X$. In this case, visualising the partition is redundant. By Theorem 3.2, $\bar{x}$ is asymptotically stable under any switching law. Again, sample trajectories of this system, for randomly-generated

initial conditions and switching law are shown in Figure 3.8.
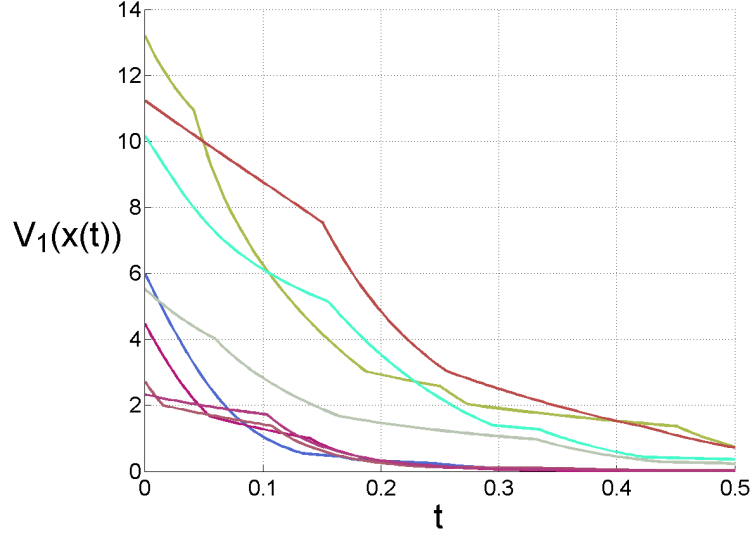
Figure 3.9: Evolution of stored energy for the system of Example 3.2. Energies corresponding to the trajectories of Figure 3.8.

□

In the previous examples, the Theorems of this chapter are used to stabilise certain equilibrium points. Both cases, however, are concerned with the stability of strong equilibrium points. In fact, these are the sole equilibrium points corresponding to the chosen feedback functions. These restrictions are not necessary for the results to be applicable, as will be shown in the following example.

Example 3.3. One can obtain a system with multiple weak equilibrium points by considering a system whose subsystems exhibit the same dynamics, but they are centered around different equilibrium points. Such a system would assume the form:

$$\dot{x}(t) = A(x(t) - \bar{x}_{\sigma(t)}) + Bu(t) \tag{3.5}$$

where A, B and $\bar{x}_k$ (the set of equilibrium points: one for each $k \in \mathcal{N}$) have appropriate dimensions, and the output is still to be defined. In the present case, take the following pair of matrices:

$$A = \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

Observe that A is a Hurwitz matrix (its eigenvalues are negative), and the pair A,

B is controllable. Take the weak equilibria x_i, from triplets $(\bar{x}_i, 0, i)$, with $i = 1, 2, 3$,

$$\bar{x}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \bar{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \bar{x}_3 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

From the theory of non-switching dissipative systems [Brogliato et al., 2007] it is known that, for some appropriately chosen matrix P, both the subsystems are dissipative with supply rate $r_{ii}(u, y) = u^T y$, and with storage functions of the form

$$V_i = (x - \bar{x}_i)^T P (x - \bar{x}_i),$$

provided that the output functions assume the form

$$h_i(x, u) = C^T (x - \bar{x}_i).$$

In particular, the matrices P and C have to satisfy the following conditions:

$$PA + A^T P = -Q,$$
$$C = PB,$$

where Q is some positive definite matrix. It is known that a unique solution P that satisfies these conditions can always be found, provided that A is a stable matrix. In this case, choosing the identity matrix as Q yields the following:

$$P = \begin{bmatrix} 0.5 & -\frac{1}{3} \\ -\frac{1}{3} & \frac{7}{12} \end{bmatrix}, \quad C = PB = \begin{bmatrix} \frac{1}{12} \\ \frac{1}{8} \end{bmatrix}.$$

These storage functions and dynamics are associated with the following cross-supply rates:

$$r_{12}(x, u) = 2(x - \bar{x}_1)^T PA(x - \bar{x}_2) + 2(x - \bar{x}_1)^T PBu,$$

$$r_{13}(x, u) = 2(x - \bar{x}_1)^T PA(x - \bar{x}_3) + 2(x - \bar{x}_1)^T PBu,$$

$$r_{21}(x, u) = 2(x - \bar{x}_2)^T PA(x - \bar{x}_1) + 2(x - \bar{x}_2)^T PBu,$$

$$r_{23}(x, u) = 2(x - \bar{x}_2)^T PA(x - \bar{x}_3) + 2(x - \bar{x}_2)^T PBu,$$

$$r_{31}(x, u) = 2(x - \bar{x}_3)^T PA(x - \bar{x}_1) + 2(x - \bar{x}_3)^T PBu,$$

$$r_{32}(x, u) = 2(x - \bar{x}_3)^T PA(x - \bar{x}_2) + 2(x - \bar{x}_3)^T PBu.$$
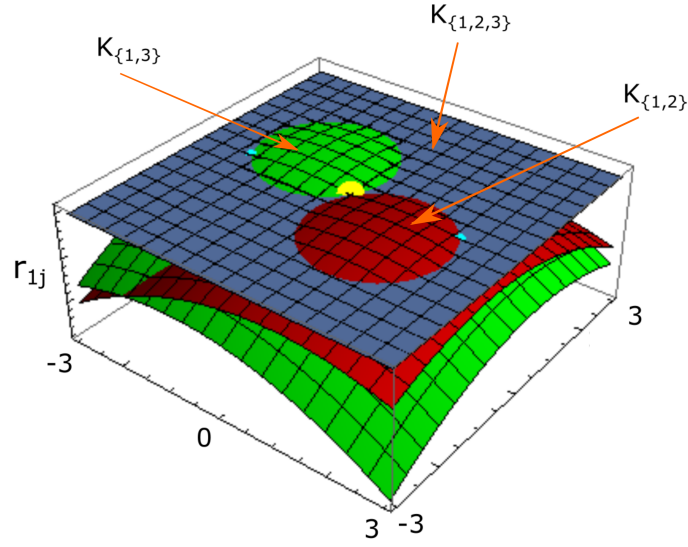
Figure 3.10: Visualization of (a part of) the partition for the system of Example 3.3. Red for r_{12} and green for r_{13}. The large yellow ball represents (a neighborhood of) $\bar{x}_1$; the smaller cyan ones, x_2, x_3. The arrows point to the areas that correspond to elements of the partition $\mathcal{K}$.

Assume that the goal is to stabilise one of the equilibrium points, say x_2. For $\psi(x) = 0$, for all x, and using as a feedback function the output of subsystem 2, that is $v(x) = -C^T(x - \bar{x}_2)$, one obtains the partition $\mathcal{K}[x_2, v, \psi]$. For this partition, it holds that $r_{22}(x, v(x)) = -h_2^2(x) \leq \psi(x)$, for all $x \in X$. Then, by Corollary 3.1, x_2 is stabilised, provided that the switching law is followed. Again, the form of the partition depends on the signs of the cross-supply rates, in the following way: if any of them is less than zero, the corresponding subsystem or subsystem 2 may be active; if they are both positive, then only subsystem 2 may be active. In Figure 3.10 the cross-supply rates are plotted in an area (a square with a side of 10 units) around the equilibrium points. In Figure 3.11, simulated trajectories of the system for random initial conditions are shown, exhibiting the stable behaviour of the system around x_2. In Figure 3.11 one of the randomly-generated switching sequences is shown; they are also randomly-generated, using the approach described in Example 3.1. □

Remark 3.3. While, in the process of analysing Example 3.3, the partition is visualised, that this is not a necessary step. The knowledge that r_{22} is always dominated by ψ is sufficient to be able to apply Corollary 3.1.
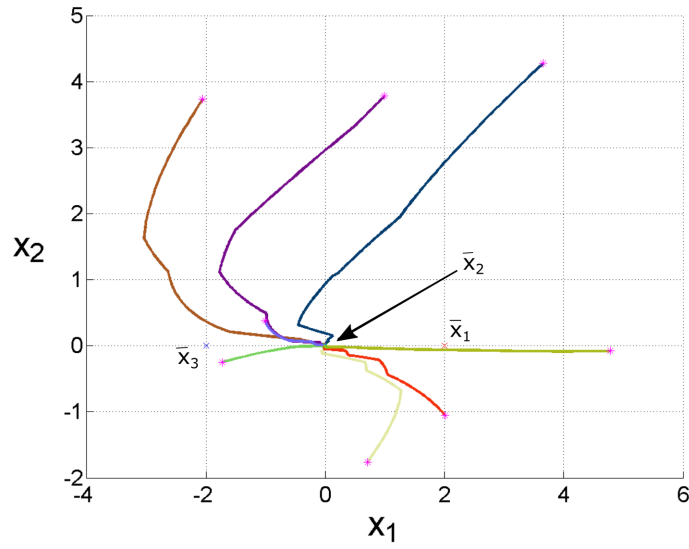
Figure 3.11: Sample trajectories of the system of Example 3.3. Randomly selected initial conditions (magenta stars). Stability around $\bar{x}_2 = \begin{bmatrix} 0, 0 \end{bmatrix}^T$. The other equilibrium points are also marked.
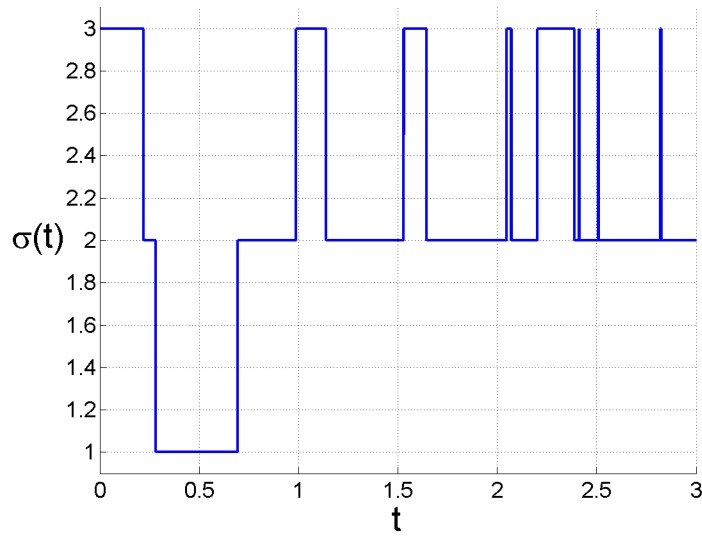
Figure 3.12: Switching law for one of the sample trajectories system of Example 3.3.

Overview

The main contribution of this chapter was a collection of stabilisation results for dissipative, switching, non-linear systems. It has been shown that one may use the dissipativity properties of system (2.3) to build a partition of the state space (Definition 3.1).

This partition may be used to design switching restrictions, which, if applied, force the trajectories of the system to be energy-decreasing. Based on this idea, local, global and asymptotic stabilisation results were proved (Theorems 3.1, 3.2 and 3.3, respectively). It was shown that these results address some of the issues found in the previous literature. In particular, it was argued that the conditions on the cross-supply rates placed by the results of this chapter are more intuitive and easier to use than others that have been proposed in the literature, and, especially Theorem 2.2. It was also shown that, in contrast to much of the existent literature, the approach proposed here is able to accommodate systems whose subsystems have distinct, non-coinciding equilibrium points.

Chapter 4

Stabilisation for Hybrid Systems

In Chapter 3, the core result of this work was given for the case of switching systems. Here, this results will be extended to the wider category of hybrid systems. Before that, as promised in Chapter 2, some introductory material on hybrid and impulsive systems will be given, along with a review of the relevant literature, which was not included in that Chapter. While this separation may seem unorthodox, it was argued that splitting the background overview in this manner allows for a more accessible distribution of the material.

4.1   Overview of Impulsive Systems

In the Introduction, it was explained that the term hybrid systems refers to a class of systems which exhibit two types of discontinuous behaviour. First, similar to switching systems, discussed in Chapter 2, hybrid systems have continuous dynamics which may change abruptly (instantaneously) in time. That is, a collection of functions, representing different continuous dynamics is available, and, out of those, one and only one is in control of the state in every time instant. This aspect of hybrid systems, then, is conceptually identical to switching systems, and no new notation is necessary to represent it.

   Second, hybrid systems exhibit state discontinuities, which are usually referred to as jumps, or as resets. Systems which are characterized only by this type of behaviour, without any switching, are usually called impulsive systems or reset systems. The second term is usually understood to be narrower, and it refers to systems in which (part of) the state returns to some pre-specified value. Examples of this category are systems that force a reset (a jump in the state), when the error between the state and

some desired value crosses some threshold. Impulsive systems, on the other hand, are an application of the study of impulsive differential equations, presented, among others in [Lakshmikantham et al., 1989], [Samoilenko and Perestyuk, 1995] and [Bainov and Simeonov, 1993]. As such, they tend to be more flexible about the types of jumps that are allowed. These may include jumps in constant or variable time intervals, as well as state-dependent jumps.

The first works in the area of impulsive/reset systems are significantly older, compared to the main bulk of scientific production in the area. They include the works in [Witsenhausen, 1966], [Pavlidis, 1967] and [Clegg, 1958], which use some of the ideas discussed in this Chapter, although not the more modern nomenclature. Important works in the area of impulsive systems are [Haddad et al., 2014] and [Li et al., 2005], which are textbooks, and they intend to provide a summary of the efforts in this space. Other works include [Sun et al., 2003], which deals with asymptotic stability, [Hespanha et al., 2008], which discusses input-to-state stability properties and [Yang and Chua, 1997], where the author explores the interaction of impulsive behaviour with chaotic dynamics. Important works in the area of reset control systems include [Beker et al., 2004], [Nešić et al., 2008], [Guo et al., 2009] and [Baños et al., 2011], where the problem of feedback design is discussed, and [Carrasco et al., 2010], where passivity is used to analyse these systems. The textbook [Baños and Barreiro, 2011] offers a broader outlook in the area.

Building on the notational infrastructure of Equation (2.2), one only needs a way to represent the jumps, in order to have a mathematical model of an impulsive system. To that effect, consider a set $\mathcal{T} = \{t_0, t_1, \dots\}$, containing all the reset times of the system. For every one of these time instants, the change in the state is given by an appropriate, continuous function $g : X \times U \mapsto X$. In the non-impulsive case (discussed in Chapter 2), it was mentioned that, under appropriate assumptions, solutions for the initial value problem exist and they are absolutely continuous. For the impulsive case, this property is expected to hold between jumps, but it clearly cannot hold on the time instants included in $\mathcal{T}$. Instead, the solutions are taken to be right-continuous. That is for the time instants of $\mathcal{T}$, the 'proper' value of the solution is taken to be the post-jump value – $x(\hat{t}) = \lim_{t \searrow \hat{t}} x(t)$.

Given these elements, one obtains the following set of equations:

$$\dot{x}(t) = f(x(t), u(t)), \text{ for } t \in \mathbb{R}/\mathcal{T} \tag{4.1a}$$

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = g(x(t_k^-), u(t_k^-)), \text{ for } t_k \in \mathcal{T}, \tag{4.1b}$$

$$y(t) = h(x(t), u(t)). \tag{4.1c}$$

Here, the $\pm$ super-scripts refer to the one-side limit of the underlying quantity, when time tends to the argument of this quantity. For instance, $x(t_k^-) = \lim_{t \nearrow t_k} x(t)$ – the same interpretation can be applied to the other symbols. Note that, since x is taken to be right-continuous, $x(t_k^+) = x(t_k)$. The superscirpt $+$ is used to empahsise the jump, and it is not strictly necessary. The definition is given for a system with inputs and outputs, but, evidently, one may write the equations without them, to obtain an uncontrolled system, analogous to system (2.1).

In the formulation of Equation (4.1), Zenoness is an issue of interest. It should be intuitively clear that, if the elements of $\mathcal{T}$ are not required to be strictly increasing, it is possible to have an infinite sequence of jumps. For instance, if $\mathcal{T}$ corresponds to the times in which the state $x(t)$ enters some set $Q \subseteq X$, then, if, for some k, it holds that:

$$x(t_k^-) + g(x(t_k^-), u(t_k^-)) \in Q,$$

then multiple (here, two) jumps will happen at the same time instant. Even in the case where the sequence of elements in $\mathcal{T}$ is strictly increasing, it is still possible that Zenoness ensues, if the sequence has a convergence point. Similarly, one may envision a situation where infinite jumps happen, resulting in the system never progressing beyond t_k. The simplest form of this case would be having two points x_a and $x_b \in X$, such that, for some $u \in U$, it holds that:

$$x_a + g(x_b, u) = x_b,$$
$$x_b + g(x_a, u) = x_a.$$

If the jumping scheme is such that both these points force jumps (say, because jumping is state-dependent), then an infinite sequence of jumps ($x_a \to x_b \to x_a$) arises. In most cases, one would like to avoid such a scenario. To that effect, it is possible to introduce geometric criteria, as well as requirements on the properties of g. This is done, among others, in [Nersesov and Haddad, 2008] and [Simić et al., 2000]. In the present work

the matter is treated implicitly. It is assumed that the jumping scheme is always available, and any state-dependent restrictions are rules to be followed, rather than intrinsic characteristics of the system. Then, it can always be assumed that the interesting sets $\mathcal{T}$ of jumping times are such that multiple jumps do not happen at the same instant. That is, it is assumed that the elements of $\mathcal{T}$ are strictly increasing, and that, if $\mathcal{T}$ is a (countably) infinite set, then $\lim_{k\to\infty} t_k = \infty$, where $t_k \in \mathcal{T}$.

For system (4.1), the stability and stabilisation definitions given in Chapter 2 (Definition 2.2) can still be used. In the abstract, stability for such a system implies that the interaction of the jumps and the continuous dynamics 'pushes' the system toward some equilibrium point. The relationship between the two aspects of the system may be co-operative (when both the continuous and the discrete part have a stabilising effect), or antagonistic (when the continuous dynamics tend to stabilise the system, but the jumps lead it away from the equilibrium, or the other way around). Evidently, for an equilibrium point $\bar{x}$, from a pair $(\bar{x}, \bar{u})$ (strong, since there is only one 'subsystem') of system (4.1) to be stable, one would want to eliminate the possibility that, after the system reaches the equilibrium but then jumps away. This would imply, at least, that $g(\bar{x}, \bar{u}) = 0$. In this work, this is not a strict requirement for the equilibrium point. Instead, it is made part of the results themselves. This is in keeping with the viewpoint taken in the description of switching systems, and, for which, the different parts of the system (subsystems, jumps) are available always (in space as well as in time), in principle, and whatever restrictions are necessary are enforced at Theorem-level.

The vast majority of the results cited to this point are relevant to the topic of stabilisation, either by designing the input, or by controlling the frequency of the jumps. A class of results also applies dissipativity related results to impulsive and reset systems. These include [Haddad et al., 2014], where dissipativity is discussed in a formulation which is heavily inspired by [Willems, 1972], [Carrasco et al., 2010] which uses passivity and it is more geared towards linear controller design, and [Liu et al., 2007], which discusses the matter in more general terms. Here, for the sake of conciseness, a separate definition of dissipativity will not be given, anticipating the introduction of a definition for hybrid systems (Definition 4.2), which is more general, and can be specialised to the case of system (4.1).

4.2   Overview of Hybrid Systems

In essence, one builds the infrastructure for hybrid systems by combining the ideas be-
hind (2.3) (the generic switching system), with the approach presented in the previous
section, whose end-result in Equation (4.1). Three issues need to be addressed. First,
in the case of hybrid systems, the set of times for which events happen is different
to the set on which state discontinuities appear. In particular, one has two sets, $\mathcal{T}^c$
and $\mathcal{T}^d$, both taking values in $\mathbb{R}^+$, and, of those sets, the first represents the switch-
ing times, while the second contains the jump times. $\mathcal{T}$ is used to represent the union
of these sets. Further, one has to decide if each of the subsystems corresponds to a
separate type of jumps, or if, in the hierarchy of the system, the reset 'mechanism' is
at the same hierarchy level as the subsystems, meaning that the same reset behaviour
persists, regardless of the active subsystem. Here, the second path is taken, for the
sake of simplicity. However, this is not necessarily the most common approach in the
literature.

Finally, one has to decide what degree of autonomy to assign to the jump mecha-
nism of the system, with respect to the inputs and outputs. To be more precise, it has to
be decided if the continuous and the discrete parts of the system share the same inputs
and outputs, or if a different pair corresponds to each. Conceptually, this corresponds
to the idea that the jumps may be dependent on the appearance of events, external to the
system (for instance, the user pushes a button), which are not the same as the conven-
tional inputs of the system. In such an approach, the jumping component of the system
is, in some sense, an additional subsystem, which is active only on $\mathcal{T}^d$. The privileged
status of this subsystem is expressed in the fact that it is allowed its own, separate input
and outputs. This approach of splitting the system inputs and outputs into a discrete
and a continuous one is quite popular in the literature, and it has been used in works
as divergent as [Liu et al., 2007], [Teel, 2010] and [Branicky et al., 1998]. However, it
will not be taken here, in the interest of simplicity, and, instead, it will be assumed that
a unified input and output exist for the system.

Given those observations, it follows that, to represent a hybrid system, one needs
to bring together the infrastructure developed for (2.3) ($\mathcal{F}$, $\mathcal{O}$), with that shown in the
definition of impulsive systems (4.1). The resulting description for hybrid systems

takes the following form:

$$\dot{x}(t) = f_\sigma(t)(x(t), u(t)), \tag{4.2a}$$

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = g(x(t_k^-), u(t_k^-)), \text{ for } t_k \in \mathcal{T}_d, \tag{4.2b}$$

$$y(t) = h_\sigma(t)(x(t), u(t)), \text{ for } t \notin \mathcal{T}_d. \tag{4.2c}$$

Again, the $\pm$ superscripts denote the value of the quantity (here, x and u), as the variable t approaches t_k from the right or the left, respectively. Additionally, whenever the meaning is clear, the following convention is used: $x^+ = x(t_k^+) = \lim_{t \searrow t_k} x(t)$; similarly for $u(t_k^+)$ and similarly for the minus superscript. Again, by right-continuity, $x^+ = x(t^+) = x(t)$; the $+$ superscript is used solely for convenience. Using this notation, the third part of Equation (4.2a) assumes the following form:

$$\Delta x(t_k) = x^+ - x^- = g(x^-, u^-), \text{ for } t_k \in \mathcal{T}_d.$$

The literature of hybrid systems is largely overlapping with that of switching systems. This is to be expected – if the set $\mathcal{T}^d$ is empty, then (4.2a) reduces to (2.3). For this reason, many of the relevant works have already been discussed in Chapter 2. The approach taken here may be described as an equation-based one, in the sense that that differential and difference equations describe the different components of the system. This is in line with the works in [Branicky et al., 1998] and [Liu and Stechlinski, 2018], as well as the more general framework in [Ye et al., 1998]. The work in [Haddad et al., 2014], which uses the so-called left/right-continuous systems is similar in conception and properties, although it is slightly more rigorous. In more theoretical matters, the work in [Simić et al., 2000] takes a similar view of hybrid systems, and uses it to develop a more formal geometric description for them; in [Ames, 2006], the author uses category theory to describe hybrid behaviours; and, in [Hyun and Verriest, 2015], hybrid systems are described using non-standard analysis.

The work in the area of differential inclusions, already mentioned in Chapter 2 is central in the area of hybrid systems. Works such as [Goebel et al., 2012] develop a versatile framework for the description of hybrid systems, representing the continuous behaviours as a differential inclusion, and the jumps as a difference inclusion. In this case, the hybrid behaviours (both in continuous modes and schemes of jumping) are subsumed within the differential/difference inclusion. Working on this framework, various researchers have produced a wide array of results on hybrid systems, including results on the concept of solutions for hybrid systems, such as [Goebel and Teel, 2006]

and [Sanfelice et al., 2006]; results on stability and feedback stability such as [Sanfelice et al., 2007], [Goebel, 2014] and [Sanfelice, 2013]; results on Zenoness, such as [Dashkovskiy and Feketa, 2017]; and some more exotic works, such as [Biemond et al., 2016] and [Sanfelice and Teel, 2008].

While the inclusion framework is very powerful in deriving results, putting it into application is a rather involved task. Indeed, the use of multi-valued functions eliminates a large part of the intuition one may have about the function of the system, and, because these are much more complex mathematical objects than 'conventional' functions, it is often difficult to reason accurately about them. Therefore, with the exception of some simple examples, such as the bouncing ball, which is used in the majority of the works in this area, and some work on power converters ([Albea et al., 2015]), the application of such results has been limited.

Finally, the hybrid automata framework is also widely used to represent hybrid behaviours. The idea here is that the hybrid behaviours of the system are captured in the form of a finite state automaton. The automaton is described by a graph, and each of the nodes of the graph corresponds to different continuous dynamics (the term 'discrete locations' is often used, corresponding to the term 'subsystems', used in the present work). The graph has edges between two nodes, if and only if switching from one to the other is possible. Additionally, the edges are given labels, which include information on jumping. The graph for an automaton of this type is given in Figure 4.1. In the space of hybrid automata, one usually speaks of executions, which are objects combining the information contained in the solutions and switching laws used in the present work.

Works such as [Lygeros et al., 2003] and [Van Der Schaft and Schumacher, 2000], give a summary of the formalism used for the automata models. Some applications of this framework, in the areas of verification and reachability, have already been discussed in Chapter 2. More works in the same direction include [Doyen et al., 2018], [Asarin et al., 2002] and the tool in[Henzinger et al., 1997]. Other efforts include works from the area of chaotic dynamics [Navarro-López and Barajas-Ramírez, 2010], the area of robotics [Egerstedt, 2000] and the study of Zenoness [Johansson et al., 1999b]. Finally, a variety of results have been produced in the area of hybrid system simulation, including the works in [Bourke and Pouzet, 2013], [Benveniste et al., 2017] and [Gomes et al., 2017].

The works in [Haddad et al., 2014], [Naldi and Sanfelice, 2011], [Naldi and Sanfelice, 2014], [Navarro-López and Laila, 2013] apply dissipativity concepts directly to
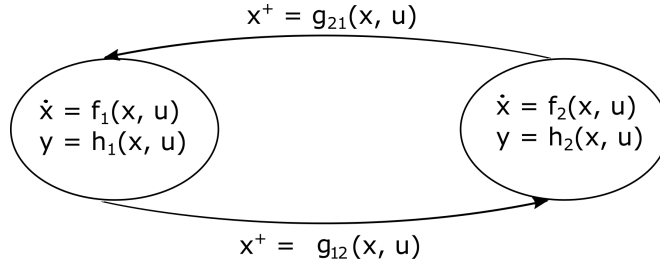
Figure 4.1: Representation of a generic hybrid automaton as a graph. The nodes represent subsystems; the edges represent the switches and the jumps.

hybrid systems. Of those, the first two use a single storage function for the whole system, and they require that it satisfies a pair of energy-balance inequalities. The first is identical to the dissipation inequality used in Theorem 2.1. The second is an inequality that has to hold on $\mathcal{T}^d$, that is, in the case of jumps, and it constraints the change in the stored energy that is caused by each jump. For the multiple storage function approach, showcased in the other two citations, the same consideration discussed about switching systems also apply. Here, a definition that combines Definition 2.10 with the one used in [Liu et al., 2007], for the case of impulsive systems is introduced.

Definition 4.1 (Dissipativity data - Hybrid case). A collection of functions $\{r_{ii}\}_{i\in\mathcal{N}}$, such that, for every $i \in \mathcal{N}$, $r_{ii} : U \times Y \mapsto \mathbb{R}$, is a collection of possible **supply rates** for system (2.3), provided that they are all locally integrable along the system trajectories. Similarly, the collection of functions $\{r_{ij} \mid i, j \in \mathcal{N} \text{ and } i \neq j\}$, such that, for all admissible i, j, $r_{ij} : X \times U \times Y \mapsto \mathbb{R}$ is a collection of possible **cross-supply rates**, provided that they are all locally integrable along the system trajectories. A collection $\{q_i \mid i \in \mathcal{N}\}$, such that $q_i : X \times U \times Y \mapsto \mathbb{R}$ is a collection of possible, **discrete supply rates**, if every one of them is locally absolutely summable. Finally, a collection of functions $\{V_i \mid i \in \mathcal{N}\}$, such that, for all values of i, $V_i : X \mapsto \mathbb{R}$, is a collection of possible storage functions, provided that they are all positive. ∎

Definition 4.2 (Flow dissipativity). System (4.2a) is said to be flow dissipative, with respect to a collection of supply and cross-supply rates $\{r_{ij} \mid i, j \in \mathcal{N}\}$ and discrete supply rates $\{q_i \mid i \in \mathcal{N}\}$, having the properties described in Definition 4.1, provided that there exists a collection of storage functions $\{V_i \mid i \in \mathcal{N}\}$, such that the following inequalities are satisfied for all $t_k, t_{k+1} \in \mathcal{T}$, $t_k^1, t_k^2 \in \mathbb{R}$, when $t_k \leq t_k^1 \leq t_k^2 < t_{k+1}$:

1. $V_i(x(t_k^1)) - V_i(x(t_k^2)) \leq \int_{t_k^1}^{t_k^2} r_{ii}(u(t), y(t))dt$, if $\sigma(t_k) = i$.

2. $V_i(x(t_k^1)) - V_i(x(t_k^2)) \leq \int_{t_k^1}^{t_k^2} r_{ij}(x(t), u(t), y(t))dt$, if $\sigma(t_k) = j \neq i$.

Additionally, for every $t_k \in \mathcal{T}^d$, the following is required to hold for every $i \in \mathcal{N}$:

3. $V_i(x^+) - V_i(x^-) \leq q_i(x^-, u^-, y^-)$, for $t_k \in \mathcal{T}_d$.

■

It is important to point out that the property described here is a dissipativity-like property, rather than a dissipativity property *per se*. A common thread running through all the non-switching definitions of dissipativity is that the property has to be an input-output property, independent of the state representation of the system. That is why it is often given in an alternative formulation (alternative to Definition 2.5, that is), in which the storage function, which depends on the state, does not appear (see [Hill and Moylan, 1976]). For switching systems, in a multiple storage function setting, this conception of dissipativity (as an input-output property) is maintained in the requirement that the supply rates which bound the evolution of the energy of a subsystem, when the subsystem itself is active (in the present work, the functions r_{ii}) , do not depend directly on the state. In an impulsive setting, it is still possible to maintain this property for a reasonably large class of systems. In the language of Definition 4.1, a dissipative impulsive system would have a single discrete supply rate, which is independent of the state; that is, $q : U \times Y \mapsto \mathbb{R}$. Then, the (single) storage function V would be required to satisfy inqualities of the form of 1 and 3, from Definition 4.2.

For the true hybrid case, however, it is difficult to envision such a configuration. The expectation that the same jumping scheme, described by g, will satisfy dissipation inequalities for the stored energies of multiple subsystems, is not realistic. As stated previously, the input-output dependence is meant to convey the idea that dissipativity is a deep property of the system and its trajectories. Conversely, there is no logical necessity for which the jumping mechanism and the continuous evolution of a hybrid system should be connected. To see this, consider the simplest example of a hybrid system used in the literature, the bouncing ball. This is a ball, falling under the effect of gravity, until it meets the floor. Then, when it reaches the floor, it bounces: its motion changes direction, and it starts ascending. For such a system, the continuous evolution and the jumping depend on wholly different elements. The former depends on the mass of the ball and the force of gravity, while the latter depends on the elasticity of the ball and the floor. Then, expecting that they give rise to the same mode of energy dissipation is not intuitive.

One might be tempted to circumvent this problem, by introducing multiple modes of jumping in the definition of hybrid systems. Then, instead of g, used in Equation

(4.2a), one would have a collection of g_{ij}, each mapping $X \times U$ to X, and each describing the jumping scheme that is active, when one switches from subsystem i to subsystem j. This is done, for instance in [Navarro-López and Laila, 2013] and [Liu and Stechlinski, 2018]. Then, one might require that only specific jumping schemes (say, those entering or leaving i) should satisfy a true dissipation inequality (that is, without direct dependence of the cross-supply rate on the state). The problem with this approach is that, if taken seriously, it causes a great inflation of functions. Besides the N^2 different jumping functions, one would need N^3 discrete supply rates (one for N^2 for each storage function). This situation might be remedied in various ways (for instance, [Navarro-López and Laila, 2013] takes some of the cross-supply rates to be identical to some of the supply rates), and it is usually less dire in the hybrid automata case, where only a subset of the possible switches is usually allowed. However, in the general case, this approach seems to pose more problems than it resolves, and, therefore, it is not taken in this work.

Instead, a quasi-dissipativity property is used, in the form of Definition 4.2. The name flow-dissipativity is used to convey that the new property is related to dissipativity, in as much as the continuous behaviour of the system (the flows) does satisfy true dissipation inequalities (Condition 1, in the Definition), but it is not a traditional dissipativity property. In [Naldi and Sanfelice, 2011], the term flow-passivity is used, to cater to similar considerations in a single storage function setting. However, as far as the author is aware, the property presented here has not been used before in the literature.

The intuition behind Definition 4.2 is similar to the one discussed in the switching case. A hybrid system in the form (4.2a) is dissipative when it is characterised by a compendium of energies, which behave in a known way along the trajectories of the system. In particular, in the periods between jumps, this behaviour is characterised by the supply and cross-supply rates, in the same manner as with switching systems. And, for the jumps, the discrete supply rate and the corresponding dissipation inequality are introduced. Because the jump is instantaneous, this last dissipation inequality, has the form used for discrete-time, dissipative systems (see [Navarro-López, 2002]). The difference is that, in the present case the inequality represents a balance, satisfied by the values of the stored energy before and after the jumping instant (t_k), rather than that the value of the energy in consecutive jumps (this is the discrete-time system case).

4.3 Stabilisation results for Hybrid Systems

For hybrid systems, within the framework presented in the previous section, the same ideas as in Chapter 3 may be used to derive stabilisation conditions. In the abstract, the system is stable around some equilibrium, when the dynamics force a motion toward this point. One way to guarantee this forcing is to phrase it in terms of energy. For a hybrid system, this would mean one of the following scenaria. Either both the jumps and the continuous dynamics of the system act in a co-operative way to guarantee that the trajectories evolve in an energy-decreasing way. Or, the continuous dynamics cause an energy decrease that pulls towards the infimum of the energy (on the equilibrium point), while at the same time, counter-acting the energy-increasing influence of the jumps. Finally, it is possible that the opposite happens – energy decreasing jumps counter-act the influence of the continuous dynamics, and move the trajectories toward the equilibrium point.

In the framework developed in this work, the first of this cases can be expressed in a straightforward manner. The energy decrease properties of the continuous dynamics can be expressed using the partition of Definition 3.1, and a similar partition may be developed for the jumping, resulting to an additional behavioural rule for the system. The following theorem captures this intuition.

Theorem 4.1 (Stabilisation from flow dissipativity I)**.** *Consider system* (4.2a)*, and a weak equilibrium point $\bar{x}$, from a triplet $(\bar{x}, \bar{u}, i)$. Take a compact, forward-invariant set $E \subseteq X$, such that $\bar{x} \in E$. Assume that system* (2.3) *is flow dissipative according to Definition 4.2, and that V_i is continuous and that it has the $sPD(\bar{x})$ property . Take some initial condition $x(t_0) = x_0 \in E$ and the continuous feedback controller $v : X \mapsto U$. Take some non-positive, continuous function $\psi : X \mapsto \mathbb{R}$. Consider the partition $\mathcal{K}[\bar{x}, v, \psi]$, as given in Definition 3.1. Assume that the following condition holds.*

- ***Condition:*** $(E \cap K_{\emptyset})/\{\bar{x}\} = \emptyset$. ∎

Consider the following set:

$$\Lambda = \{x \in X \mid q_i(v(x), h(x, v(x)) \leq \psi(x)\}.$$

For it and $\mathcal{K}$, take the following rules on switching and jumping:

- ***Switching Rule:*** *if, for some t, $x(t) \in K_J \cap E$, then $\sigma(t) \in J$.*

- ***Jumping Rule:*** *no jumps happen in X/Λ.*

If the system operates under both the switching rule and the jumping rule, then v
stabilises the equilibrium point $\bar{x}$ in E. If, in addition, ψ has the $ND(\bar{x})$ property, then
the (induced) stability is asymptotic.

Proof. For any time instants t_1 $t_2 \in \mathbb{R}$, with $t_1 \leq t_2$, use $N(t_1, t_2)$ to refer to the subset
of $\mathcal{T}^d$, whose members are in the interval $[t_1, t_2)$. Then, one may write, for V_i, the
following inequality:

$$V_i(x(t_2)) - V_i(x(t_1)) \leq \int_{t_1}^{t_2} r_{i\sigma(t)}(x(s), v(x(s)), y(x(s), v(x(s)))) ds +$$
$$\sum_{\tau \in N(t_1, t_2)} q_i(x^-, v(x^-), y(x^-, v(x^-))).$$

If the jumping rule is maintained, the sum on the right hand side is negative, and it
can be omitted – the inequality will still hold without it. In this case, the inequality has
exactly the form of that in the proof of Theorems 3.1 and 3.2, and the same arguments
used there to establish Lyapunov and asymptotic stability of $\bar{x}$ for the composite system
can also be used here. □

In Theorem 4.1, both the partition and the set Λ are built with respect to the same
bound ψ. However, this is not instrumental to the proof, and one may use different
bounds (say, ψ^c and ψ^d) for each of them. In this case, asymptotic stabilisation can
be concluded even when only one of the bounds has the $ND(\bar{x})$ property (instead of
$sND(\bar{x})$ property). In particular, this is always the case if ψ^c is $ND(\bar{x})$. It is also the
case, when ψ^c is $sND(\bar{x})$, ψ^d is $ND(\bar{x})$, provided that it can be guaranteed that infinite
jumps actually take place.

In the discussion of impulsive systems, it was stated that, while it is not required in
principle that no jumps from an equilibrium point are possible, this restriction is part
of the Theorems. Indeed, the jumping rule does not allow any jump away from the
equilibrium point; any such jump would increase the stored energy V_i, and, therefore,
it would imply that q_i is over its bound (ψ) for this specific jump, breaking the jumping
rule.

Example 4.1. Consider a system, for which the continuous part is identical to that used
in Example 2.2. That is, a system in form (4.2a), with the two continuous subsystems

given as follows:

$$f_1(x,u) = \begin{bmatrix} x_2 \\ -x_1^3 - x_2 + u \end{bmatrix},$$

$$f_2(x,u) = \begin{bmatrix} x_2 \\ (1+x_2^2)(-x_1 - x_2 + u) \end{bmatrix},$$

$$h_1(x,u) = h_2(x,u) = x_2.$$

Assume that this system is connected with a jumping subsystem, which is linear, and it is described by an equation that takes the following form:

$$\Delta x = (A_d - I)x^- + B_d u. \tag{4.3}$$

Take

$$A_d = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Here, the jumping is just a multiplication of each of the states by a scalar. In this case,

$$q_1(x,u) = -\frac{15x_1^4}{16} + -0.91\frac{x_2^2}{2} + \frac{u^2}{2} + \frac{0.3x_2 u}{2},$$

and, therefore, for $q_1(x,0) \leq 0$ for all $x \in X$. Then, in this case, $\Lambda = X$, and jumps may happen everywhere.

The partition $\mathcal{K}$ for this system, corresponding to zero bound ϕ and input v, has already been shown in Example 3.1 (Figure 3.3), and the Condition of Theorem 4.1 is known to hold. Then, if the jumping rule of Theorem 4.1 is followed, the equilibrium is found to be Lyapunov stable.

Figure 4.2 shows some sample trajectories of the system, for which stability ensues, as expected. The points of jumps are also marked on the trajectories. The initial conditions are chosen in a uniformly random manner, using the rand function of Matlab. In intervals of varying length (again, chosen randomly), the integration is stopped, the possibility for either a jump or a switch is examined, a choice is made, and the integration continues. This is the same approach used in Example 3.1, with the addition that jumps are also possible. In Figure 4.3, the active subsystem is shown along with the time-stamps for (some of) the jumps for one of the simulated trajectories.

If one considers, instead, the following matrices:

$$A_d = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.2 \end{bmatrix} \quad B_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

(which correspond to a different kind of scaling – here, x_2 is taken away from the equilibrium), then $\Lambda' \neq X$, and more attention is required. In Figure 4.4, the set Λ' is shown in some area around the equilibrium point $\bar{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$. From this figure, one may deduce the jumping rule of the Theorem, which, if followed, ensures stability. Indeed, when the jumps follow the rule (that is, they only happen in the coloured area of Figure 4.4, marked as Λ'), and the switches happen as described in the first part of this example, the system is still stable. $\qquad\square$
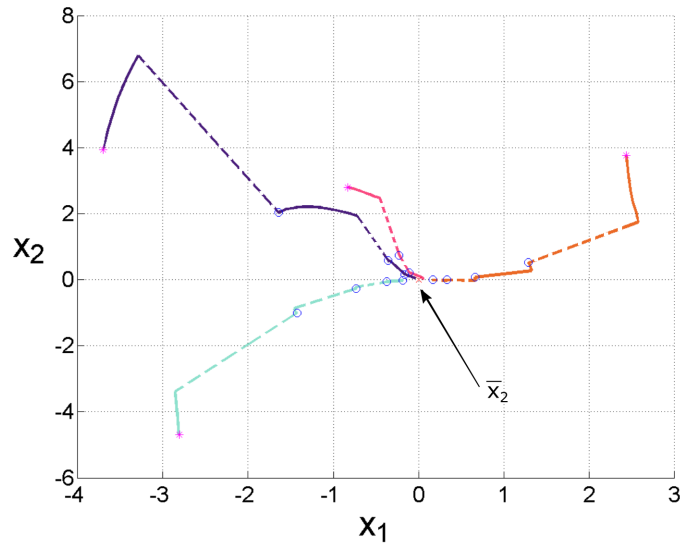
Figure 4.2: Sample trajectories for Example 4.1. Circles point out the end-points of jumps. The equilibrium point is shown in the middle.

The result of Theorem 4.1 is not surprising. From its predecessor, Theorem 3.1, it is known that, for a well-behaved system (that is, one having the properties described in the theorem), the input and the switching law can be picked together, in a way that guarantees stability. The additional conditions of Theorem 4.1 ensure that, for the given jumping scheme, this energy decrease is not disrupted. Naturally, this configuration seems redundant – it is easy to imagine a case in which the jumps add some energy, but not enough to cause instability; or, conversely, that stabilizing jumps come so often that the energy increase caused by the continuous subsystems. One may
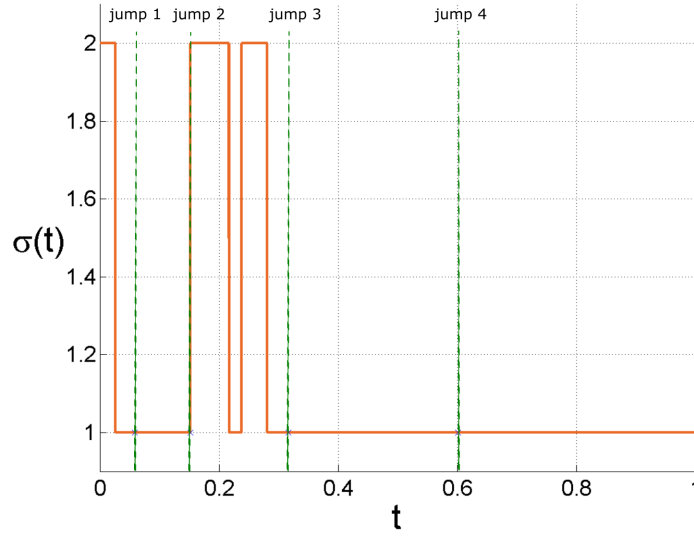
Figure 4.3: Switching law for Example 4.1. The time stamps of some of the events are also marked.

obtain such results by adapting similar ideas from two different spaces. First, impulsive systems, where the idea that an unstable underlying system can be stabilized by forcing frequent jumps has been discussed in [Liu et al., 2007], with the idea tracing back to [Liu, 1994]. Second, from the space of dwell-time approaches, discussed in Chapter 2. Some of those approaches may be applied to systems with unstable (thus, energy increasing) subsystems; for instance, [Zhai et al., 2001]. In the remainder of this sections, a result inspired by works on impulsive systems will be presented.

Consider the case for which the cross-supply rates are not bounded by some function ψ which is $ND(\bar{x})$, but rather by some function that is continuous and positive. Use this to build the partition of Definition 3.1, for the equilibrium point $\bar{x}$ and some input v. Then, it is clear that some modes of switching, which are allowed by the partition, will still cause an increase of the stored energy of the system. For a system without the possibility of jumps, this would mean that stability may not be guaranteed, in general. However, for a hyrbid system, it is possible to ask the question if, for an appropriately chosen jumping scheme, it is possible to make the system behave in a stable manner. Naturally, for this to happen, the net effect of each of the jumps to the overall energy should be negative; this, however, is not enough. For this configuration to work, it is also necessary that the combined effect of the jumps has to be such that it counter-acts the energy increase caused by the continuous subsystems. The magnitude of this energy increase, however, is dependent on the time between jumps. Then, one
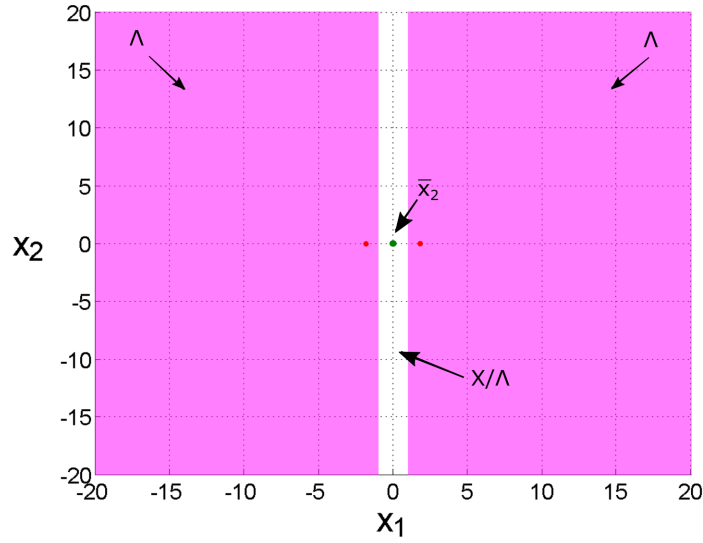
Figure 4.4: Set Λ' for q_1, for the system of Example 4.1, under the second jumping scheme. No jumps are allowed in the uncoloured (white) area, in accordance with Theorem 4.1.

may conclude that, in order to use this type of argument to establish stabilisation conditions, some restriction to the frequency of switching is also needed. The following theorem introduces a result in this direction.

Theorem 4.2 (Stabilisation from flow dissipativity II). *Consider system (4.2a), and a weak equilibrium point $\bar{x}$, from a triplet $(\bar{x}, \bar{u}, i)$. Assume that system (2.3) is flow dissipative according to Definition 4.2, with V_i being continuous and it has the $PD(\bar{x})$ property. Take a continuous feedback controller $v : X \mapsto U$. Consider the following bound functions:*

$$\psi(x) = cV_i(x), \tag{4.4}$$

$$\psi^d(x) = -dV_i(x), \tag{4.5}$$

with c, d positive constants, and with $d < 1$. For the first bound (ψ), form the partition $\mathcal{K}[\bar{x}, v, \psi]$, as given in Definition 3.1, and, with the second bound (ψ^d), construct the following set:

$$\Lambda = \{x \in X \mid q_i(x, v(x), h(x, v(x))) \leq \psi^d(x)\}.$$

Take some $E \subseteq \Lambda \subseteq X$, with the initial state $x(t_0) \in E$, and for which the following holds:

- ***Condition*** $(E \cap K_\emptyset)/\{\bar{x}\} = \emptyset$;

Consider the following pair of rules, constraining jumping and switching:

- ***Switching Rule:*** *if, for some t, $x(t) \in K_J \cap E$, then $\sigma(t) \in J$.*

- ***Jumping Rule:*** *The time τ_m between any two consecutive jumps is strictly less than $-\frac{\ln(1-d)}{c}$.*

Take the time instants given by any sequence given by the rules:

$$\tau_0 \geq t_0,$$
$$\tau_{k+1} = \tau_k + \tau_m, \text{ for } k \in \mathcal{N}.$$

For any two elements of this sequence, say τ_{k+1} and τ_k, for which it holds that $x(t) \in E$ for every $t \in [\tau_k, \tau_{k+1}]$, one has that

$$V_i(\tau_{k+1}) < V_i(\tau_k).$$

If $x(t) \in E$ for every $t \in [\tau_k, \infty]$, for some $k \in \mathbb{N}$, then the state converges to $\bar{x}$. If, additionally, E is a neighbourhood of $\bar{x}$, then $\bar{x}$ is asymptotically stable.

∎

Proof. First, since $E \subseteq \Lambda$, jumps can happen as often as needed. As in the proof of Theorem 4.1, one may write the following inequality:

$$V_i(x(\tau_2)) - V_i(x(\tau_1)) \leq \int_{\tau_1}^{\tau_2} r_{i\sigma(t)}(x(s), v(x(s)), y(x(s), v(x(s)))) ds +$$
$$\sum_{\tau \in N(\tau_1, \tau_2)} q_i(x^-, v(x^-), y(x^-, v(x^-))).$$

Take two consecutive instances of the sequence described in the theorem, say τ_1 and τ_2. Then, it is known that there is at least one jump, happening on this interval; that is, $N(\tau_1, \tau_2) \neq \emptyset$. Assume that there is a single jump in the interval, happening at some point $t_j \in [\tau_1, \tau_2]$. Then, for the interval $[\tau_1, t_j)$, the sum on the right hand side of the inequality vanishes, and one has:

$$V_i(x(t_j)) - V_i(x(\tau_1)) \leq \int_{\tau_1}^{t_j} r_{i\sigma(t)}(x(s), v(x(s)), y(x(s), v(x(s)))) ds$$
$$\leq \int_{\tau_1}^{t_j} \psi(x) ds$$
$$\leq \int_{\tau_1}^{t_j} c V_i(x(s)) ds.$$

The second inequality follows by the switching rule, while the third from 4.4. By the Gronwall Inequality [Khalil, 2002], this yields the following:

$$V_i(x(t_j^-)) - V_i(x(\tau_1)) \leq \int_{\tau_1}^{t_j} ce^{c(s-\tau_1)}V_i(x(\tau_1))ds$$
$$= cV_i(x(\tau_1))\int_{\tau_1}^{t_j} e^{c(s-\tau_1)}ds$$
$$= cV_i(x(\tau_1))[\frac{e^{c(s-\tau_1)}}{c}]_{\tau_1}^{t_j}$$
$$= V_i(x(\tau_1))[e^{c(t_j-\tau_1)} - 1]$$
$$\Rightarrow V_i(x(t_j^-)) \leq V_i(x(\tau_1))e^{c(t_j-\tau_1)}.$$

Using the same argument, but for the interval $(t_j, \tau_2]$, where, similarly, no jumps happen, one has:

$$V_i(x(\tau_2)) \leq V_i(t_j^+)e^{c(\tau_2-t_j^+)}.$$

For the jump, using the fact that the state is within Λ, one has:

$$V_i(t_j^+) - V(t_j^-) \leq -dV(t_j^-) \Rightarrow$$
$$V_i(t_j^+) \leq (1-d)V(t_j^-).$$

Putting everything together, one has:

$$V_i(x(\tau_2)) \leq (1-d)V_i(x(\tau_1))e^{c(t_j-\tau_1)}e^{c(\tau_2-t_j^+)}$$
$$= (1-d)V_i(x(\tau_1))e^{c\tau_m}$$
$$< (1-d)V_i(\tau_1)e^{-ln(1-d)}$$
$$= V_i(x(\tau_1)),$$

where in the last equalities, the fact that $\tau_m < -\frac{ln(1-d)}{c}$ has been used. This is the required result. The same construction can be applied in the case where multiple jumps appear. There, one would have multiple break points like t_j, but the process would remain the same. While the construction was made for two specific points in the sequence, the same arguments can be used for every pair of consecutive points.

For the convergence to $\bar{x}$, given the energy decrease on the elements of the sequence $\{\tau_k\}$, one can use the same argument used in the proof of Theorem 3.2. In brief: since the sequence of $V_i(x(\tau_k))$ is strictly decreasing, and by the continuity of V_i, it can only converge to $\bar{x}$, for which $V_i(\bar{x}) = 0$.

For Lyapunov stability under the control v, it follows from the above discussion that, for a system starting at some energy level $\hat{V}$, the maximal attainable energy is $e^{c\tau_m}\hat{V}$. It follows that, for any energy level, one may find some initial energy in such a way that trajectories starting within the initial level never cross the given level. This observation can be levelled in the usual way (as is done in the proof of Theorem 3.1) to establish Lyapunov stability. This, along with attractivity (convergence to $\bar{x}$) give asymptotic stability and the proof is complete. $\qquad\square$

The result is given here for the case of a specific selection of bounds (ψ and ψ^d), which guarantee that the stored energy evolves within some exponential bounds. To be precise, the choice of ψ made in Theorem 4.2 implies that, while the stored energy V_i is allowed to increase, this increase is required to be at most exponential. This is a weak condition, when compared with that of Theorem 4.1, even though they assume the same form. Instead of requiring that the continuous subsystems are 'stabilising' (that is, they cause dissipation of energy) as is done in Theorem 4.1, the condition in Theorem only requires that energy is not stored too rapidly. While the chosen bounds preform these functions, it is possible to extend Theorem 4.2, by using a technique that has been used in the impulsive systems case, and, in particular, in [Liu et al., 2007].

The choice of ψ^d is made, so that this potential increase is counteracted. It ensures that, after every jump, the energy is only a fraction of what it was before. In this context, only values of d in the interval $(0,1)$ are meaningful. Picking $d = 1$ would imply that every jump transports the system to the equilibrium point, while $d = 0$ would mean that the jumps are energy neutral, in which case they could not act in the required way (that is, counter-balancing the energy increase caused by the continuous subsystems). Finally, values of d in $(1,\infty)$ are meaningless, as they would imply that, after each jump, the energy has to become negative.

The two constants are connected. A large c would mean that a fast increase of energy may be caused by the continuous part of the system. For a given maximum time between jumps (τ_m), this situation would necessitate a large value of d (that is, one close to 1), so that the strongly dissipative jumps balance out that increase. Alternatively, for a given d, a large value of c would imply (by the jumping rule), that high frequency jumping is necessary to cause the needed energy decrease. The energy decrease argument does not hold for any two time instants, but rather on time instants on the given family of sequences. This is not surprising. If the continuous subsystems can cause an increase to the energy of the system, then, one may always find time instants that are close enough (so that no jump happens in between), and for which the stored

energy does not decrease. Due to the effect of the jumps, however, if one 'samples' in appropriate intervals, they are able to observe the energy decrease described by the Theorem 4.2.

For the result of Theorem 4.2 to hold, motion has to happen within the set E, for which information is available. This may not be the case for two reasons: either the continuous evolution takes the state out of E, or the jumping does. If it is known that this does not happen (that is, E is forward invariant and contains $\bar{x}$), then they may be dropped – energy decrease holds for every τ_i, and $\bar{x}$ is an asymptotically stable equilibrium point of the system under the control v.

Example 4.2. Consider a hybrid system, whose continuous part is identical to the one described in Example 3.3. That is, a system with a triplet of functions f_i, assuming the following form:

$$f_i(x,u) = \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}(x - \bar{x}_i) + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}u.$$

The equilibrium points are $\bar{x}_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, $\bar{x}_1 = \begin{bmatrix} -2 & 0 \end{bmatrix}^T$ and $\bar{x}_1 = \begin{bmatrix} 2 & 0 \end{bmatrix}^T$.

Consider a jumping switching scheme, defined as follows, for matrices A_d and B_d of appropriate dimensions:

$$x^+ = A_d(x^- - \bar{x}_1) + B_d u. \tag{4.6}$$

In the notation of (4.2a), this is:

$$g(x,u) = (A_d - I)(x^- - \bar{x}_1) + B_d u.$$

The idea behind this choice is that, for appropriately chosen matrices A_d and B_d, this scheme may be stabilising with respect to the equilibrium point $\bar{x}_1$. It is a reasonable question, then, if such matrices exist, and how one may find them.

Recall that, for subsystem 1, the stored energy is $V_1(x) = (x - \bar{x}_1)^T P(x - \bar{x}_1)$. For this storage function, one has:

$$V_1(x^+) - V_1(x^-) = (x^- - \bar{x})^T A_d^T P A(x^- - \bar{x}) + 2(x^- - \bar{x})^T A_d^T P B u - (x^- - \bar{x})^T P(x^- - \bar{x}).$$

For these conditions, the system is flow dissipative, in accordance with Definition 4.2, with

$$q_1(x,v(x)) = (x - \bar{x}_1)^T (A_d^T P A_d - P)(x - \bar{x}_1) + 2(x - \bar{x}_1)^T A_d^T P B_d v + v^T B_d^T P B_d v.$$

Consider the zero input $v \equiv 0$. Then, it holds that $\Lambda = X$, as given in Theorem 4.2, provided that the following holds:

$$A_d^T P A_d - P < -dP.$$

This Lyapunov-style inequality is derived and discussed in [Navarro-López, 2002]. However, in the present case, the problem is posedin the opposite direction. That is, P is given, and A_d is sought.

In this low-dimensional case, one may find A_d by a brute-forcing-like search, which yields the following:

$$A_d = \begin{bmatrix} 0.28 & 0.61 \\ 0.17 & 0.47 \end{bmatrix}.$$

Of course, one may also write the corresponding equations, and solve for (the elements of) A_d, either numerically or analytically.

For this choice of A_d, and any B_d, the uncontrolled system satisfies the conditions of Theorem 4.2 with $d = 0.2$ and $c = 0$. Therefore, one may conclude that the equilibrium point x_1 is asymptotically stable, when infinite jumps are known to happen; that is because $\tau_m \to \infty$, and the frequency constraint is satisfied for any collection of jumps. Note that, in Example 3.3, the zero input only gives Lyapunov stability – that is because one is forced to chose $\psi(x) = 0$ for all $x \in X$. Here, the presence of jumps means that attractivity is guaranteed by other means (that is, by using the jumps). □

Overview

In this chapter, a pair of stabilisation results were proved, for a class of non-linear hybrid systems. The hybrid systems under study are described in an equations-based formalism, and they have the form (4.2a). For these systems, a new property is introduced, called flow-dissipativity (Definition 4.2). The property retains some of the characteristics of dissipativity, but diverging from it in the fact that it is not purely input-output. Based on this property and the partition described in the previous Chapter (Definition 3.1, one is able to describe the continuous behaviour of the system, as well as the jumping scheme in terms of (stored) energy. In Theorem 4.1 it is shown that, under certain conditions, if both the jumping scheme and the continuous flows are restricted to be energy-decreasing, the equilibrium point under study is stable. In Theorem 4.2 this approach is altered. It is shown that, if the continuous flows are allowed

to be energy-increasing, one may still conclude stability (as well as a form of attractivity) under the given controller, provided that the jumping scheme is able to compensate for that energy increase. Besides the main results, an overview of the literature of the space of impulsive and hybrid systems was presented in this chapter.

Chapter 5

Stabilisation Results with Multiple Controllers

In Chapter 3, the core result of this work was presented. It was shown that, by constructing the partition of Definition 3.1, one may posit conditions for stabilisation for a variety of possible scenarios. In the present chapter, additional aspects of these stability results are explored. In particular, it is shown that one may use multiple partitions, corresponding to different feedback functions to establish stability in a more complex setting that in the previous chapter. These partitions may be usable everywhere in the state space; this is shown in Section 5.1. Or, they may be available only in specific domains/subsets of the state space; this is examined in Section 5.2.

5.1   Multiple Controllers Available Everywhere

The stabilisation results of Chapter 3 are built around the idea of a state space partition. In brief, the partition is constructed, some of its properties are examined, and, if they are found to hold, a switching law is chosen, which, in combination with a feedback function lead to stability. As shown in Example 3.1, it is possible to construct different partitions, corresponding to different equilibrium points, and, in this manner, draw conclusions about the stability of these equilibrium points. Similarly, one may construct different partitions, corresponding to different feedback functions; then, stability of a single equilibrium under each of those controllers can be assessed. Theorem 3.1 and the related results, however, cannot be applied in the case where two or more controllers are active within the subset E of the state space, on which stability is studied. This is a significant shortcoming. The assumption that the same controller

is applicable to every subsystem is often unnatural, as the subsystems might exhibit highly dissimilar dynamics. Further, in the literature of switching systems, it is a common assumption that different subsets of the state space correspond to different modes of control. It is useful, then, to consider the possibility of extending the use of the partition to systems under multiple controllers.

The first way to approach this issue to assume that multiple controllers for the system are available. That is, a collection of feedback control functions (like v of Theorem 3.1) are given, and they are available for activation everywhere in the area of interest. For each of these controllers, a partition in the style of Definition 3.1 may be constructed, for a chosen equilibrium point. Then, the arguments of the previous Chapter may be applied to each of the partitions separately. In particular, it should be intuitively clear, then, that, under some conditions, the energy decrease property shown in Lemma 3.1 must hold, at least for as long as the controller that was used to form the partition is active. But, if the energy decrease argument holds for each one of the available controllers, it is expected that it will also hold when one alternates among them. If this is shown to be true, then one may argue that the equilibrium point is stable, in the same way that was used in Theorem 3.1.

There are two differences between this process and the one in Theorem 3.1. First, the switching rule is now more complex, as it should capture the idea that different restrictions correspond to different controllers (and partitions). Second, and more importantly, the conditions of Theorem 3.1 has to be modified. Even if some subset of E belongs to $K_\emptyset$ (the set for which every cross-supply is over the bound, and no subsystem may be activated without disturbing stability) for one of the partitions, it might still be possible to stabilise the system, provided that one of the other partitions offers alternative options. The process is akin to navigating, alternatively consulting multiple different maps. Each map dictates different controls, and, if some care is taken, they may lead to the destination (the equilibrium point). But, the unlucky navigator might end up in a danger-zone; an area where none of the available maps gives any useful information. The condition of Theorem 3.1 has to be modified to ensure that such danger-zones do not exist in the area of interest.

To be more precise, consider, first, the relatively simple case of two controllers, say $u_a : X \mapsto U$ and $u_b : X \mapsto U$. In this case, it is useful to consider an extended switching law, $\sigma' : \mathbb{R} \mapsto \mathcal{N} \times \{a, b\}$, which assigns to each time instant an active controller as well as an active subsystem. As in the case of the switching law σ, the two elements of σ' should exhibit at most countable points of discontinuity, and be constant in between.

For a choice of equilibrium point e, and some ψ function, each one of the controllers induces a different partition. These partitions are denoted with $\mathcal{K}_a$ and $\mathcal{K}_b$, respectively; a superscript is used to distinguish elements of each partition (say, K_j^a). The core idea here is that when a controller is active, the restriction introduced by the corresponding partition causes the energy of the chosen subsystem to be non-increasing; and that, if one can always choose a combination of a controller and a subsystem in accordance with these restrictions (that is, no danger-zones exist), then stability is guaranteed for the composite system. The following result captures this idea.

Proposition 5.1 (Stabilisation with two controllers)**.** *Consider System* (2.3)*, dissipative in accordance with Definition 2.10. Suppose that it has some weak equilibrium point $\bar{x}$, part of a triplet $(\bar{x}, \bar{u}, i)$, and that V_i is continuous and it has $PD(\bar{x})$. Take a pair of continuous, non-negative $(PD(\bar{x}))$ functions $\psi_a : X \mapsto \mathbb{R}$ and $\psi_b : X \mapsto \mathbb{R}$. Consider the two partitions $\mathcal{K}_a = \mathcal{K}[\bar{x}, v_a, \psi_a]$, $\mathcal{K}_b = \mathcal{K}[\bar{x}, v_b, \psi_b]$, corresponding to two continuous, feedback controllers v_a and v_b. The equilibrium point $\bar{x}$ is Lyapunov (asymptotically) stable for the system under these controllers, if the following hold:*

- ***Condition:*** $K_{\emptyset}^a \cap K_{\emptyset}^b \subseteq \{\bar{x}\}$.

- ***Switching Rule:*** *if, for some t, $x(t) \in K_{J_a}^a \cap K_{J_b}^b$, then $\sigma'(t) \in (J_a \times \{a\}) \cup (J_b \times \{b\})$.*

■

Proof. The switching rule introduced here implies that, whenever one of the controllers gets activated, its corresponding switching rule is followed. Then, from the proof of Theorems 3.1, it follows that the following energy inequalities hold

$$V_i(x(\tau_2)) \leq V_i(x(\tau_1)) + \int_{\tau_1}^{\tau_2} \psi_a(x(s))ds,$$

$$V_i(x(\tau_3)) \leq V_i(x(\tau_4)) + \int_{\tau_3}^{\tau_4} \psi_b(x(s))ds,$$

for every interval $[\tau_1, \tau_2]$ ($[\tau_3, \tau_4]$), where controller a (b) is active. One may introduce a function $\psi' : X \times \mathbb{R} \mapsto \mathbb{R}$, such that:

$$\psi'(x,t) = \begin{cases} \psi_a(x), & \text{for } \sigma'(x) \in \mathcal{N} \times \{a\}, \\ \psi_b(x), & \text{for } \sigma'(x) \in \mathcal{N} \times \{b\}, \end{cases}$$

in order to combine the above inequalities into one:

$$V_i(x(\tau_6)) \leq V_i(x(\tau_5)) + \int_{\tau_5}^{\tau_6} \psi'(x(s), s) ds,$$

for any interval $[\tau_5, \tau_6]$. This inequality has the same properties as inequality (3.1). Therefore, the same argument may be used *mutatis mutandis*, to conclude Lyapunov and asymptotic stability. In particular, the condition plays the same role as its counterpart in Theorem 3.1, guaranteeing that ψ' has the $sND(\bar{x})$ or the $ND(\bar{x})$ property as needed. $\qquad\square$

The argument here has been given for the case of two controllers, but they also apply to any scenario where a finite number of controllers are used. Then, the result takes the following form.

Proposition 5.2 (Stability with M controllers). *Consider system (2.3), and make the same assumptions about it as in Proposition 5.1. In this case, take a collection of M of continuous feedback controllers $v_q : X \mapsto U$ and continuous, $sND(\bar{x})$ $(ND(\bar{x}))$ functions $\psi_q : X \mapsto \mathbb{R}$, for $q \in \{1, 2, \ldots, M\}$. Consider the M associated partitions $\mathcal{K}^q = \mathcal{K}[\bar{x}, v_q, \psi_q]$. If the following conditions hold, the equilibrium point $\bar{x}$ is Lyapunov (asymptotically) stable for the system under the given controllers.*

- ***Condition:*** *$\bigcap_q K_\emptyset^q \subseteq \{\bar{x}\}$.*

- ***Switching Rule:*** *if, for some t, $x(t) \in \bigcap_q K_{J_q}^a$, then $\sigma'(t) \in \bigcup_q (J_q \times \{q\})$.*

$\qquad\blacksquare$

Propositions 5.1 and 5.2 provide a proper extension of the results of Chapter 3, and, in particular, Theorem 3.1 in the following sense. The set of switching laws that are allowed by the restriction of Proposition 5.1 are a superset of those that could be derived when using Theorem 3.1 separately, for each of the controllers. In particular, the intersection of the sets $K_\emptyset$ mean that the area in which the condition is possibly not satisfied is now smaller, and, therefore, the condition becomes easier to satisfy. Using the same analogy as before, while an area may be a danger-zone, if one uses a single map, using another map may give useful information about navigating it.

The results here can be derived from Theorem 3.1 in another way. One may consider M distinct copies of each subsystem, each controlled by a controller $v' : X \mapsto U$ of the form (using the notation of Proposition 5.2) $v'(x) = u_q(x) + v(x)$, for some common control $v : X \mapsto U$. Then, this augmented system (which has NM subsystems) can

be used in the setting of Theorem 3.1, for some partition that depends on v. The two formulations are equivalent, and the present one is chosen because it is clearer.

Example 5.1. Consider again the system of Example 2.2, which was discussed at length in the previous chapters. In Example 3.1, it was shown that the system under the control $u_a(x) = 0$ satisfies the condition of Proposition 5.1 about $K_\emptyset^a$ (in the notation of Proposition 5.1). Similarly, under the controller $u_b(x) = -x_2$, the same condition is satisfied for $K_\emptyset^b$ ($= \emptyset$), due to the fact that

$$r_{11}(x, u_b(x)) = -x_2^2 \leq \psi(x) = 0.$$

It follows that the condition of Proposition 5.1 holds, and switching according to the switching rule stabilises the system.
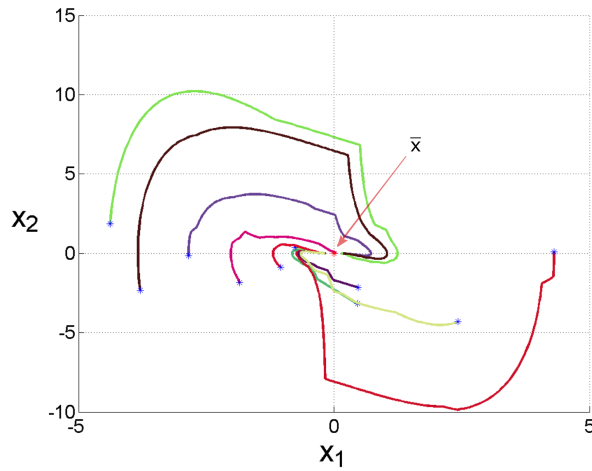
Figure 5.1: Sample trajectories for the system described in Example 5.1. Randomly-selected initial conditions, shown as stars. Equilibrium point at $\bar{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

Figure 5.1 shows some sample trajectories for a randomly-generated switching signal that obeys the rules. In particular, the active subsystem is integrated under the action of one of the two controllers (say, a), for a small interval of randomly chosen length up to 0.2 time units, or until the switching rule forces a change. Every time a change happens, a random choice (in essence, a coin flip) is made, in order to decide which of the available controllers will be used. As seen in Figure 5.2, the stored energy ($V_1(x) = 0.25x_1^4 + 0.5x_2^2$) for these random trajectories is non-increasing, as expected.

□
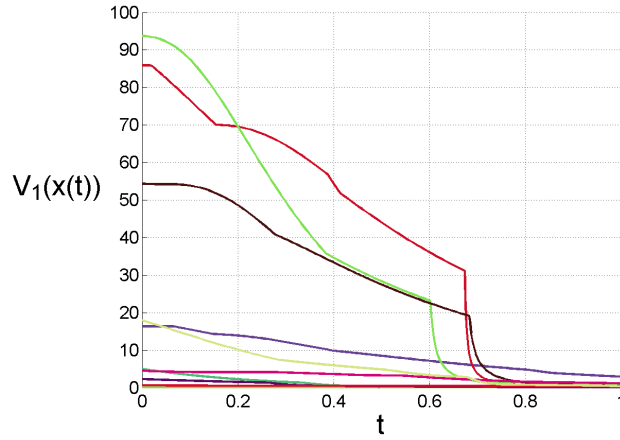
Figure 5.2: Evolution of the stored energy V_1, for the system described in Example 5.1. Colour-coded with the corresponding trajectories, shown in Figure 5.1

5.2 Multiple Controllers Available Locally

For Proposition 5.1, the operating assumption is that the collection of controllers under study is available everywhere in the state space. This is the more natural adaptation of Theorem 3.1 in the multiple controller setting, and the one that is more attuned to the view of switching systems taken in this work. It is not, however, the only option. One may assume that each controller is 'tied' to some subset of the state space; that is, it is only available within its own 'domain'. The simplest form of this description would be to say that, in the different subspaces defined by a plane, different controllers are available. For each of these controllers, and an equilibrium point, one may build the partition of Definition 3.1; in this case, each partition will only be relevant in the area where the associated controller is active.

Now, assume that, in every one of this subspaces, the behaviour of the system is constrained (using the partition) in the way specified by Lemma 3.1. Then, one of the stored energies will decrease, until the system reaches the equilibrium point, or until it leaves the area of activation of the current controller. The same process may be used in the new area, entered by the system, and (at least, under some conditions) one should expect that the net effect of the process will be an aggregate decrease in the stored energy, and, as a result, a motion toward the equilibrium point.

The idea described here is, in essence, to use Lemma 3.1 multiple times, in different areas of the state space. In the application of this idea, two steps are needed. First, one has to use a 'local' energy decrease result to conclude that, overall, the energy

decreases. This is relatively straightforward, at least in the case of strict decrease.

A more elusive problem is created by the fact that the introduction of a surface where switching is (potentially) forced may cause the appearance of sliding modes (see [Filippov, 1988]). That is, the trajectory may move, for some time, along the switching surface, as it is being 'pushed' toward it by the dynamics on the sides of the surface. Such a behaviour would cause the emergence of a new kind of dynamics for the system (that is, the dynamics of moving along the switching surface). The properties of this dynamics need to be understood, since it is possible that they disrupt the tendency of the system to dissipate energy, and, therefore, make it unstable.

It is worth highlighting that this is not the usual sliding mode setting. Most work on sliding modes assumes that, away from the sliding surface, the dynamics of the system are not switching. That is, the surface is a manifold on which two smooth systems 'compete'. In the case described here, however, the systems on the sides of the surface are themselves switching. Therefore, while ideas from sliding mode control will be employed, this ideas do not map directly to the scenario under study.

Here, sliding behaviour will be examined only in a simplified case, where two controllers are available on the sides of a switching surface. However, the same considerations apply in more complex cases. First, take a smooth function $p : X \mapsto \mathbb{R}$, and define three sets as follows:

$$Q_+ = \{x \in X \mid p(x) > 0\}, \tag{5.1a}$$

$$Q_- = \{x \in X \mid p(x) < 0\}, \tag{5.1b}$$

$$Q_0 = \{x \in X \mid p(x) = 0\}. \tag{5.1c}$$

Roughly, sliding along the surface Q_0 may happen, if, around it, the velocity vectors point to opposite directions. In particular, for the controllers $v_+ : X \mapsto U$ and $v_- : X \mapsto U$, active in Q_+ and Q_- respectively, one gets the vectors $f_{i+}(x) \doteq f_i(x, v_+(x))$ and $f_{i-}(x) \doteq f_i(x, v_-(x))$, corresponding to each $i \in \mathcal{N}$. Then, sliding may happen in some area $\hat{Q} \subseteq Q_0$, provided that in $\hat{Q}$ the following holds:

$$\frac{\partial p}{\partial x} f_{j-} > 0, \text{ and } \frac{\partial p}{\partial x} f_{k+} < 0, \text{ for some } j, k \in \mathcal{N}. \tag{5.2}$$

That is to say, sliding is possible when, from the 'side' of Q_-, one of the subsystems (here, j), pushes the system state toward the switching surface, while, from the 'side' of Q_+, one of the subsystems (here, k) also pushes toward Q_0. Note that the claim is

not that sliding necessarily happens. Rather, it is that a specific pair of subsystems may cause it to appear; and, these subsystems may be forced to be active by the switching restrictions placed by the theorems.

The existence of sliding modes means that the arguments of Chapter 3 cannot be applied, in the sense that the emergent behaviour of the system is not necessarily dissipative. Indeed, while in the sliding mode, the inequalities of Definition 2.10 need no longer hold, and, therefore the composite system is no longer dissipative. As a workaround, one might consider introducing an assumption that ensures the inequalities of (5.2) do not hold. As will be explained, however, such a step is not necessary. In particular, the following result can be proved.

Proposition 5.3 (Stabilisation for two control domains)**.** *Consider System (2.3), dissipative in accordance with Definition 2.10. Suppose that it has some weak equilibrium point $\bar{x}$, from a triad $(\bar{x}, \bar{u}, i)$, and that V_i is differentiable and it has the $PD(\bar{x})$ property. Consider some smooth function p, and define Q_- and Q_+ as in (5.1). Take two continuous feedback functions v_- and v_+, and the associated partitions $\mathcal{K}_+ = \mathcal{K}[\bar{x}, v_+, \psi_+]$ and $\mathcal{K}_- = \mathcal{K}[\bar{x}, v_-, \psi_-]$; again, elements of the partitions are distinguished by the respective superscripts. Assume that the following conditions hold:*

- $(K_{\emptyset}^+ \cap Q_+)/\{\bar{x}\} = \emptyset,$

- $(K_{\emptyset}^- \cap Q_-)/\{\bar{x}\} = \emptyset,$

with ψ_+ and ψ_- which have the $sND(\bar{x})$ property. Consider the following switching scheme:

- *Follow the switching rule induced by $\mathcal{K}^-$ in Q_-.*

- *Follow the switching rule induced by $\mathcal{K}^+$ in Q_+.*

Then, $\bar{x}$ is stablised. If, additionally, ψ_+ and ψ_- are $ND(\bar{x})$, $\bar{x}$ is stabilised asymptotically. ∎

Proof. As stated above, two issues need to be addressed. First, that the energy decrease argument of Theorem 3.1 still holds. To see that, consider the modified inequality (3.1) with $\rho : \mathbb{R} \times X \mapsto \mathbb{R}$:

$$\rho(t,x) = r_{i\sigma(t)}(x, v_-(x), y(t))I_{Q_-}(x) + r_{i\sigma(t)}(x, v_+(x), y(t))I_{Q_+}(x),$$

where $I_{Q_-} : X \mapsto \{0,1\}$ and $I_{Q_+} : X \mapsto \{0,1\}$ are the indicator functions of the respective sets, and $r_{i\sigma(t)}$ is the (cross-)supply rate corresponding to the system that is active at time t (and whose index is $\sigma(t)$). This yields:

$$
\begin{aligned}
V_i(x(\tau_2)) &\leq V_i(x(\tau_1)) + \int_{\tau_1}^{\tau_2} \rho(s, x(s)) ds \\
&\leq V_i(x(\tau_1)) + \int_{\tau_1}^{\tau_2} I_{Q_-}(x(s)) \psi_-(x(s)) + I_{Q_+}(x(s)) \psi_+(x(s)) ds \\
&\leq V_i(x(\tau_1)).
\end{aligned}
$$

In this case, as in Theorem 3.1, the switching rule guarantees that the last inequality holds, since the active subsystem is always chosen in such a way that the corresponding cross-supply rate is dominated by the appropriate ψ function; and the pair of conditions ensure that this can happen everywhere in the sets of interest. Further, when ψ_- and ψ_+ are $ND(\bar{x})$, the inequalities above become strict, and a similar argument to the one used in Theorem 3.2 can be used to conclude that the stability is asymptotic.

Concerning the possibility of sliding, consider a set $\hat{Q} \subseteq Q_0$, on which sliding is allowed by the switching rule. This would imply a configuration shown in Figure 5.3. That is, it would require that some indices $j, k \in \mathcal{N}$ exist, such that the subsystems j and k are allowed to be active in the neighborhood of every point in $\hat{Q}$. It follows, then, that the cross-supply rates corresponding to these subsystems are dominated by their respective ψ bounds in this area. However, by [Filippov, 1988], $\hat{f} : X \mapsto \mathbb{R}^n$, the sliding dynamics that are in operation in this case have to be an element of the convex combination of the dynamics on the sides of Q_0 (see Figure 5.4). It follows that, (in the case of a differentiable storage function V_i),

$$
\frac{\partial V_i}{\partial x} \hat{f}(x) = \frac{\partial V_i}{\partial x} [a f_k^+ + (1-a) f_j^-], \text{ for } a \in (0,1)
$$

(where the arguments for f_k^+ and f_j^- have been dropped). As a result, the evolution along the sliding surface happens in accordance with a virtual cross-supply rate, which is always bounded by a convex combination of ψ_- and ψ_+. Then, even if sliding modes are allowed by the rules, and they actually occur, they have to belong to the class that does not invalidate the energy decrease argument made above. In other words, any sliding mode that would cause an increase in the system energy is disallowed by the switching rule. □
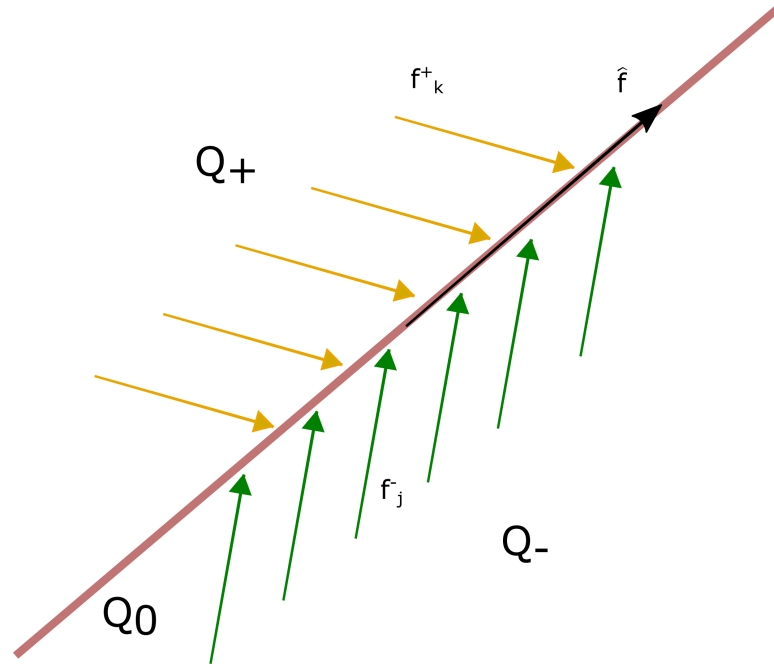
Figure 5.3: Sample arrangement of the system dynamics around the surface Q_0, when sliding occurs. On the Q_- side, the (green) arrows represent $f_j^- = f_j(x, v_-(x))$, while, on the Q_+ side, the (yellow) arrows stand for $f_k^+ = f_k(x, v_+(x))$, where j, k are any (all) subsystem indices.

While Proposition 5.3 establishes that the existence of sliding modes does not impact stability, it may be of interest to find conditions under which such behaviours do not appear. To that effect, one may consider the following slightly modified sets:

$$\tilde{Q}_+ = \{x \in X \mid p(x) \geq -\varepsilon\}, \tag{5.3a}$$

$$\tilde{Q}_- = \{x \in X \mid p(x) \leq \varepsilon\}, \tag{5.3b}$$

where ε is some positive constant. In this case, $\tilde{Q}_+ \cap \tilde{Q}_- \neq \emptyset$; the two sets have an area of overlap, in $\{x \in X \mid |p(x)| \leq \varepsilon\}$. The rough idea here is that this area of overlap acts as a boarder between the two regions, separating the surface of switching on the one direction $(+ \to -)$ from that of switching to the other $(- \to +)$. The following result captures this idea.

Proposition 5.4 (Stabilisation for two overlapping control domains)**.** *Consider system* (2.3)*, and make the same assumptions as those in Proposition 5.3 about $\bar{x}$ and V_i. Consider some continuous function $p : X \mapsto \mathbb{R}$, and define $\tilde{Q}_-$ and $\tilde{Q}_+$ as in* (5.3)*. Take two continuous feedback functions $v_- : X \mapsto U$ and $v_+ : X \mapsto U$ and two bound*
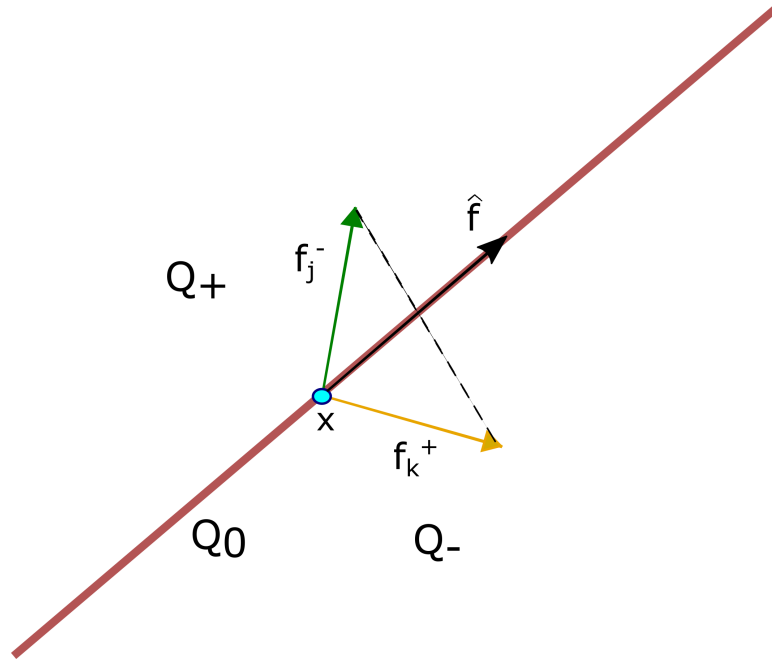
Figure 5.4: The range of possible sliding dynamics, in the formulation of [Filippov, 1988], on a point x of the surface Q_0, for which the dynamics on the sides of Q_0 are represented by the green (for Q_-) and yellow (for Q_+) arrows.

functions $\psi_+ : X \mapsto \mathbb{R}$ and $\psi_- : X mapsto \mathbb{R}$ which are sND($\bar{x}$). Construct the associated partitions $\mathcal{K}^+ = \mathcal{K}[\bar{x}, v_+, \psi_+]$ and $\mathcal{K}^- = \mathcal{K}[\bar{x}, v_-, \psi_-]$. Assume that the following conditions hold:

- $K_0^+ \cap \tilde{Q}_+ \subseteq \{\bar{x}\}$,

- $K_0^- \cap \tilde{Q}_- \subseteq \{\bar{x}\}$.

Consider the following switching scheme:

- *Follow the switching rule induced by $\mathcal{K}^-$ in $\tilde{Q}_-/\tilde{Q}_+$.*

- *Follow the switching rule induced by $\mathcal{K}^+$ in $\tilde{Q}_+/\tilde{Q}_-$.*

- *Follow the switching rule induced by $\mathcal{K}^-$ when entering $\tilde{Q}_+ \cap \tilde{Q}_-$ from $\tilde{Q}_-$.*

- *Follow the switching rule induced by $\mathcal{K}^+$ when entering $\tilde{Q}_+ \cap \tilde{Q}_-$ from $\tilde{Q}_+$.*

Then, $\bar{x}$ is stabilised. If, additionally, ψ_+ and ψ_- are ND($\bar{x}$), then $\bar{x}$ is stabilised asymptotically. ∎

Proof. The arguments used in the case of Proposition 5.3 also apply here. □
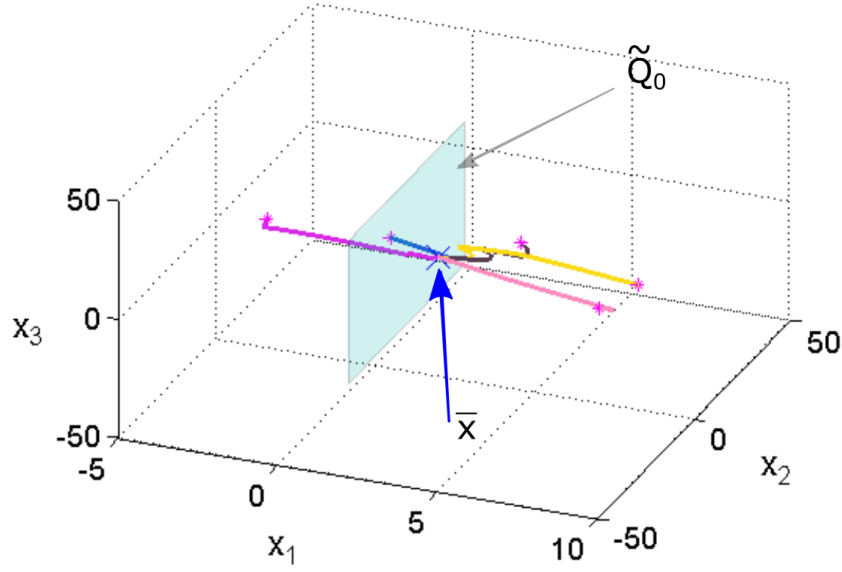
Figure 5.5: Sample trajectories of the system described in Example 5.2. The equilibrium point is $\bar{x} = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}^T$, while the red stars indicate the randomly-chosen initial conditions. The blue plane represents $\tilde{Q}_0$ (see (5.3)), for $c = 1$.

Example 5.2. In order to illustrate the application of Proposition 5.4, consider a system with two subsystems, each of which has dynamics of the following form:

$$\dot{x} = f_i(x, u), \text{ for } i \in \{1, 2\}.$$

Take the following definitions:

$$f_1(x, u) = A(x - \bar{x}) + Bu,$$

$$f_2(x, u) = \begin{bmatrix} -x_2 - x_3 \\ x_1 + 0.2x_2 \\ 0.2 + x_2 x_3 - 5.7 x_3 + u. \end{bmatrix}$$

.

For the first subsystem, A and B take the following values:

$$A = \begin{bmatrix} -4.6740 & 1.6918 & -0.3927 \\ 0.6120 & -3.0957 & 4.8164 \\ 3.8187 & -1.3108 & -3.4360 \end{bmatrix},$$

$$B^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

For this system, any triad of the form $(\bar{x}, 0, 1)$ is a weak equilibrium point, which is also stable (for the dynamics of the subsystem), since the matrix A is Hurwitz. The second subsystem also has a pair of weak equilibria, corresponding to $u = 0$, which will not be calculated here, but they are known to be unstable (see [Cvitanović et al., 2016]). As stated before, subsystem 1, which is linear, stable and controllable, can be shown to be passive with respect to a storage of the form

$$V_1(x) = (x - \bar{x})^T P(x - \bar{x}),$$

provided that one considers an output function of the form

$$y = h_1(x) = h_2(x) = Cx = (PB)^T(x - \bar{x}).$$

In this case, one has

$$P = \begin{bmatrix} 0.2679 & 0.0943 & 0.1819 \\ 0.0943 & 0.1670 & 0.1089 \\ 0.1819 & 0.1089 & 0.2774 \end{bmatrix}.$$

For the second subsystem, one may use the trivial storage $V_2 \equiv 0$. In this case, the interesting dissipativity data of Definition 2.9 are as follows:

$$r_{11}(x, u) = yu,$$

$$r_{12}(x, u) = 2(x - \bar{x})^T P f_2(x, u).$$

Consider the two controllers $v_-(x) = 0$, for all $x \in X$ and $v_+(x) = -y$, active in the regions defined by the hyper-surface $Q_0 = \{x \in X \mid p(x) = x_1 = c\}$, for some constant c. Since both v_- and v_+ ensure that r_{11} is always negative, the first two conditions of Proposition 5.4 hold everywhere. Then, any implementation of the switching scheme described in Proposition 5.4 results in stability around the equilibrium, when used along with the given feedback functions v_-, v_+.

In this case, consider the weak equilibrium $(\begin{bmatrix} 2, 0, 0 \end{bmatrix}^T, 0, 1)$, and define the sets $\tilde{Q}_-$ and $\tilde{Q}_+$ with $\varepsilon = 0.001$. The sole requirement for implementing the switching scheme is for one to keep track of r_{12} and the distance from Q_0. Figure 5.5 shows the evolution of some sample trajectories for the system under this scheme, for $c = 1$, with the additional stipulation that, beside the forced switches related to Q_0 and r_{12},

the possibility of switching is also checked every 0.5 time units. In Figure 5.6, the scenario where $\bar{x}$ belongs to Q_0 is also presented.
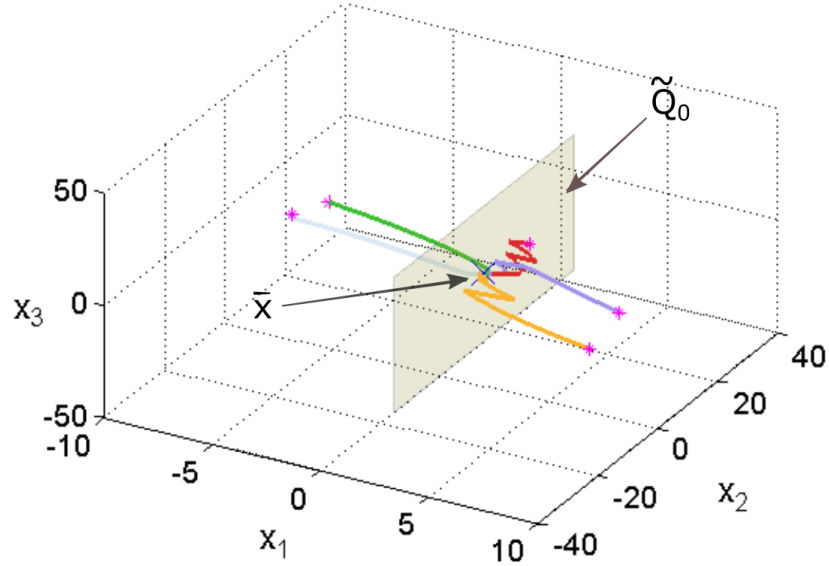
Figure 5.6: Sample trajectories of the system described in Example 5.2. Equilibrium at $\bar{x} = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}^T$ The plane represents $\tilde{Q}_0$, for $c = 2$. Compare with Figure 5.5, where the equilibrium point is not on the switching surface.

Note that, in this case, while the second subsystem is unstable, the switching rules do allow for its activation, in some areas of the state space. This is not just a theoretical possibility, but something that appears in simulations, as seen in Figure 5.7. $\qquad\square$

Overview

In this section, an extension of the results of Chapter 3 was proposed. It was pointed out that considering a single feedback function v, as is done in Theorem 3.1 is restrictive, and methods to work around this issue were explored. Two results were proposed along these lines. First, with Proposition 5.1, the scenario were a pair of controllers are available everywhere in the state space was examined. It was shown that a modest extension of the technique used in Theorem 3.1 allows one to derive conditions for stabilisation for this scenario. Subsequently, the scenario where two controllers are available, but each of them is active in a specific domain, was examined. In Proposition 5.3, conditions for stabilisation for this scenario were derived. In particular, it was

Figure 5.7: Sample switching law, corresponding to one of the trajectories of Figure 5.6.

shown that the emergence of sliding dynamics which destabilise the system is not possible. While the main results of this section are given for two different controllers, it is argued that they can be extended to the case of any finite collection of controllers.

Chapter 6

Feedback Stabilisation Design

In Chapter 3, a method to validate that a given feedback input function stabilises a switching system was introduced. It was shown that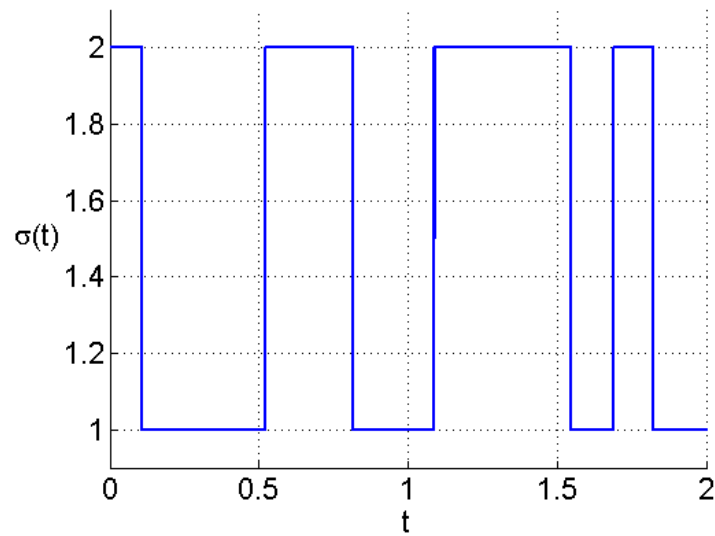, for a dissipative system, one may use a given input feedback function to construct a partition of the state space, and then use this partition to find which of the possible switching signals result in stable behaviour. Having explored some of the more advanced features of the method in Chapters 5 and 7, it is now appropriate to discuss the problem of feedback stabilization design. In the present setting, this problem takes the following form: for a given dissipative system, it will be asked if the methods developed in the previous chapters can be used to pick a feedback function that allows one to conclude stability.

6.1 Problem statement and results for a general cross-supply rate

A common two-step process underpins the results of Chapter 3. First, one establishes that a specific family of switching restrictions are possible, in the sense that they are applicable everywhere in the set of interest E; then, the actual restrictions are either checked or designed against this family. It should be noted that the first step depends wholly on the chosen feedback function v, as this determines the partition $\mathcal{K}$. This process suggests the following method to design a stabilising feedback for the switching system. If v is designed so that the first step of the process is possible, or, equivalently, the condition of Theorem 3.1 (the condition on $K_{\emptyset}$) is satisfied, then one may find a switching signal that satisfies the restriction of the rule of the same Theorem, and, therefore stability may be concluded. To say more, the idea is that, if a feedback

function can be found, for which at least one of the cross-supply rates is under the bound ψ (that is, a 'fall-back' option always exists), then a switching signal can also be picked, such that the equilibrium point $\bar{x}$ is stabilised. This idea allows one to adapt various techniques from the area of feedback stabilisation for non-switching systems to design a feedback that stabilises of system (2.3).

For the convenience of the reader, the general form of the systems under study, introduced in Chapter 2, is repeated here.

$$\dot{x}(t) = f_{\sigma(t)}(x(t), u(t)), \tag{6.1a}$$

$$y(t) = h_{\sigma(t)}(x(t), u(t)). \tag{6.1b}$$

For this system, the running assumption is that it is dissipative. Then, the collection of supply/cross-supply rates will be used to construct stabilising feedback functions, in the way outlined above. For the purposes of this section, no assumption is made for the form assumed by any of the cross-supply rates; for this reason, the results of the sections are referred as the 'general case' results. This should be contrasted with the next section, for which (at least one of) the cross-supply rates are assumed to be affine and linear in the control.

The problem discussed here concerns the stability of an equilibrium point $\bar{x}$ from a triplet $(\bar{x}, i, \bar{u})$, of a dissipative, switching system in the form (6.1). From the point of view of Theorem 3.1 and its sister results, the stabilisation design problem is essentially the problem of picking a feedback function v, such that, for the corresponding partition $\mathcal{K}[\bar{x}, v, \psi]$, formed using some given bound function ψ, the condition of the Theorem holds; that is

$$K_{\emptyset} \cap E \subseteq \{\bar{x}\}.$$

Finding such a feedback function is equivalent to the problem of finding a continuous function which takes its values within the values of the following multi-valued function $W_i : X \rightsquigarrow U$:

$$W_i(x) = \{u \in U \mid \exists j \in \mathcal{N}, \text{ s.t } r_{ij}(x, u) \leq \psi(x)\}. \tag{6.2}$$

The dependence of W_i on ψ will not be specifically pointed out; this should cause no confusion. Posing the problem in this manner follows the lead of Theorem 3.1, in the sense that, for every x, it is enough that any one of the cross-supply rates are upper-bounded by ψ. One may, instead, follow the logic of Corollary 3.1, for which the requirement is that a specific cross-supply rate (say, the one corresponding to subsystem j) is always below the bound. This yields a simplified version of W_i, denoted by

$W_{ij} : X \rightsquigarrow U$, which can be used to make the problem more tractable. W is defined as follows:

$$W_{ij}(x) = \{u \in U \mid r_{ij}(x,u) \leq \psi(x), j \in \mathcal{N}\}. \tag{6.3}$$

Again, the dependence on ψ is omitted. As discussed with respect to Corollary 3.1, this scenario is more conservative; finding a continuous function taking values in W_i may be possible for systems for which the same is impossible in W. However, the simplified version allows for the use of some mathematical techniques which would be unavailable, if W_i were used.

In the field of multi-valued analysis, problems of this type are known as selection problems, and in the general case, they are rather hard to solve. An obvious necessary condition for the existence of a solution to the problem studied here is for W_i (or, W_{ij}) to have non-empty values for every x in some part of the state space. This condition is, in general, non-trivial. However, when discussing design, it is a reasonable assumption to make; in its absence, the problem cannot be solved for the given $\psi(x)$, and, therefore, no design process is relevant.

A well-known result in this area is Michael's Theorem, first proved in [Michael, 1956]. A formulation of this result is the following:

Theorem 6.1 (Michael's theorem [Michael, 1956])**.** *Suppose that A is a metric space and B is a Banach space, and take a multi-valued function $F : A \mapsto B$. Assume that F is lower semi-continuous (lsc) in A; that is, for every $a \in A$, it holds that for every $U \subseteq B$, such that $u \in F^{-1}(U)$, there exists some $V \subseteq A$, a neighborhood of a, such that $V \subseteq F^{-1}(U)$. Further, assume that F has non-empty convex, closed values. Then, there exists a continuous selection of F; that is, a continuous function $f : A \mapsto B$, such that, for all $a \in A$, $f(a) \in F(a)$.* ∎

While the result holds for more general spaces, as stated above, only Euclidean spaces will be discussed in this work. Lower semi-continuity, and its sister property, upper semi-continuity are the most common regularity properties one encounters in the study of multi-valued functions. An equivalent description of lower semi-continuity would be to say that, for every point in the pre-image of an open set U, one may find an open neighborhood V, for which all the points have images under F that intersect U; that is, for $a \in V$, it is true that $F(a) \cap U \neq \emptyset$.

Even in the best case, the result of Theorem 6.1 gives only a partial solution to the problem at hand. When applied on any multi-valued function F, it may establish that a continuous selection of F does exist. However, it does not give a method to construct

such a function. For a constructive result, one needs a more demanding result, stated here:

Theorem 6.2 (Minimal selection continuity [Freeman and Kokotovic, 2008])**.** *Suppose that A is a metric space and B is a Hilbert space of finite dimension, and take a multi-valued function $F : A \mapsto B$. Assume that F is lower semi-continuous (lsc) in A. Further, assume that F has non-empty convex, closed values (these are the same as in Theorem 6.1). Additionally, assume that the graph of F is closed. That is, the set $grp(F) = \{(a,b) \mid a \in A, b \in F(a)\} \subseteq A \times B$ is closed. Then, the minimal selection $m : A \mapsto B$, defined as follows, is continuous:*

$$m(a) = arg\{\min \|b\| \mid b \in F(a)\}.$$

∎

Theorem 6.2 has stricter requirements. It states that, for some of the multi-valued functions, one may go beyond showing the existence of selections (with Theorem 6.1), and construct a specific selection; namely, the function that assigns to every $a \in A$ the element of $F(a)$ which has the smallest norm.

By applying Theorem 6.2 to W_{ij} from (6.3), the following result is obtained.

Proposition 6.1 (Feedback design – general case I)**.** *Take system (6.1), dissipative according to Definition 2.10, with a single input ($m_i = 1$), and a weak equilibrium point $\bar{x}$ from a triplet $(\bar{x}, i, \bar{u})$. Consider some set $Q \subseteq X$, and assume that there exists at least one $k \in \mathcal{N}$, such that r_{ik} is convex with respect to its second argument (u), for every $x \in Q$. Take the continuous state feedback input $v : X \mapsto U$. Assume, also, that the following holds:*

$$\forall x \in Q, \exists u \text{ s.t. } r_{ik}(x, u) < \psi(x).$$

Then, a feedback function $u(x)$ exists, such that, for $\mathcal{K}[\bar{x}, v, a]$, it holds that $(K_{\emptyset} \cap Q)/\{\bar{x}\} = \emptyset$. Additionally, v is continuous in Q, with the possible exception of $\bar{x}$. ∎

Proof. To begin with, the assumptions regarding r_{ik} imply that the following set-valued map is non-empty in Q:

$$W(x) = \{u \in U \mid r_{ik}(x, u) < \psi(x)\}. \tag{6.4}$$

Due to the continuity and convexity of r_{ik} and the continuity of a, one can conclude

by [Freeman and Kokotovic, 1996, Proposition 4.4] that the set-valued function

$$\bar{W}(x) = \{u \in U \mid r_{ik}(x,u) \leq a(x)\} \tag{6.5}$$

is lower semi-continuous, and that it has non-empty, closed, convex values in Q. This is enough to conclude the existence of a continuous selection of $\bar{W}(x)$ on $X/\{\bar{x}\}$, by Theorem 6.1, but not enough to construct such a selection.

Consider the following feedback function:

$$v(x) = \begin{cases} \bar{u}, \text{ if } x = \bar{x}, \\ arg\{\min_{u \in \bar{W}(x)} \|u\|\}, \text{ if } x \neq \bar{x}, \end{cases} \tag{6.6}$$

the minimal selection of $\bar{W}(x)$.

Observe that the graph of $\bar{W}$, that is $Grp(\bar{W}) = \{(x,u) \mid u \in \bar{W}(x)\}$, is closed. This follows from by Lemma 6.1, which is given after the present proof.

Since $\bar{W}$ is lower semi-continuous and its graph is closed, it follows from [Aubin and Frankowska, 2009, Proposition 9.3.2] that the graph of $v(x)$ is closed relative to $X/\bar{x}$. Further, by [Aubin and Frankowska, 2009, Lemma 9.3.1], if follows that $\|v(x)\|$ is upper semi-continuous in $X/\bar{x}$, which implies that $v(x)$ is locally bounded in the same area. If follows, then, by [Fuller, 1968, Theorem 3.4], that $u(x)$ is continuous everywhere in Q, with the possible exception of $\bar{x}$.

For this choice of continuous $v(x)$, it is clear that, for the partition $\mathcal{K}[\bar{x}, v, a]$, $K_{\emptyset} = \emptyset$, as required. $\qquad\square$

Lemma 6.1. *The graph of the set-valued function $\bar{W}$, given in 6.5, is closed.*

Proof. This can be shown by contradiction. To see that, assume that $Grp(\bar{W})$ is not closed. By the definition of closedness, this is equivalent to the assumption that there exists a sequence $\{(x_j, u_j)\}$, with $(x_j, u_j) \in Grp(\bar{W})$, converging to some $(\check{x}, \check{u}) \notin Grp(\bar{W})$. This, in turn, implies that

$$\Psi(\check{x}, \check{u}) = r_{ik}(\check{x}, \check{u}) - \psi(\check{x}) = \varepsilon > 0.$$

By the continuity of Ψ, if follows that there exists a neighborhood $V \subseteq X \times U$ of $(\check{x}, \check{u})$, such that $\Psi(x,u) > 0$ everywhere in V. This, however, means that the sequence of points $\{(x_j, u_j)\}$, which belong to $Grp(\bar{W})$, can never enter V. For any of the points of the sequence to enter V, it would mean that Ψ is simultaneously greater and

smaller than zero, which is impossible. But, if the sequence does not enter V, it cannot converge to $(\check{x}, \check{u})$, as initially assumed. Then, such a sequence, converging to $(\check{x}, \check{u})$ does not exist, and the set $Grp(\bar{W})$ is closed. $\qquad \square$

The idea here is clear, albeit not particularly intuitive. Under some conditions, the minimal selection of $\bar{W}$ ensures that the condition of Theorem 3.1 is satisfied, and, therefore, that stability, or. at least, the energy decrease property described in Lemma 3.1 may be concluded. Then, for every $x \in X$, the choice of $v(x)$ is the solution of a convex optimisation problem. In some cases, it might be possible to formulate this problem as a parametric optimization, with x in the role of the parameter. This process, however, is not always applicable, and, in any case, it is outside the scope of this work. The interested reader should consult the work presented in [Gal, 1995].

Proposition 6.1 is given for the case of a system with a single input, because it makes the notation and presentation slightly simpler. The same argument applies to the case of multiple inputs, provided that the convexity requirement is satisfied for the argument u of r_{ik}, which is now a vector.

One may apply Proposition 6.1 in the following manner.

Example 6.1. Assume that, for system (6.1), one of the cross-supply rates takes the following form:

$$r_{ik}(x, u) = p(x) + q_1(x)u + q_2(x)u^2,$$

for some continuous functions p, q_1, q_2 of appropriate dimensions. The goal is to find some feedback function v, such that, in some $Q \subseteq X$, Lemma 3.1 can be applied, for some appropriate function ψ. This, in turn, would show that, under v, the evolution of the system within Q decreases the stored energy V_i, provided that the switching law is picked from the family specified in Theorem 3.1.

In the form given above, r_{ik} is convex with respect to its second argument, in every point of X where $q_2(x) \geq 0$. Consider the discriminant of $\Psi(x, u) = r_{ik}(x, u) - \psi(x)$, viewed as a second degree polynomial of u:

$$\Delta(x) \doteq q_1^2(x) - 4q_2(x)(p(x) - \psi(x)).$$

For any point $\hat{x} \in X$, for which $\Delta(\hat{x})$ is positive, there is always some $\hat{u} \in U$ for which $\Psi(\hat{x}, \hat{u}) < 0$, and, therefore, $W(\hat{x})$ is always non-empty (both W and Ψ are defined in the proof of Proposition 6.1). Then, by Theorem 6.1, it can be concluded that the minimal selection of $\bar{W}$ is continuous, and that it is such that, for $\mathcal{K}[\bar{x}, v, \psi]$, it holds

that $(K_\emptyset \cap Q)/\{\bar{x}\}$ is empty. Then, the condition of Lemma 3.1 holds, and it can be shown that, for any time interval $[\tau_1, \tau_2]$, for which the state remains within Q, the stored energy V_i decreases, provided that the switching law adheres to the restrictions (the switching rule) described in Theorem 3.1. If additional information is available on Q (say, if it is a neighborhood of $\bar{x}$), stability can also be concluded, using Theorem 3.1. $\qquad\square$

Proposition 6.1 guarantees the existence of a feedback function with the desired properties described in Example 6.1, but it does not guarantee that this feedback function is continuous at $\bar{x}$. This is a token of the fact that, in general, one cannot expect $W(\bar{x})$ to be non-empty. Since $\bar{x}$ is a minimum of the storage function, all the cross-supply rates will be either vanish or be positive there. This is a problem, as, without continuity in v, one is not able to guarantee the existence of solutions for system (6.1), as discussed in Chapter 2. One may work around this point by stabilising a neighborhood of $\bar{x}$ – for instance, a ball of a certain radius around it. This type of property is, in some settings, referred to as practical stability; for instance in [Freeman and Kokotovic, 1996]. Then, continuous extension results (for instance, [Urysohn, 1925]) may be used to extend v continuously into this neighborhood, yielding a feedback function that is continuous everywhere and guarantees stability (and attractivity) for the chosen neighborhood of $\bar{x}$. An alternative approach, which has been used in the literature, involves introducing additional assumptions, to the effect of ensuring that $\bar{W}$ is lower semi-continuous, and $\bar{W}(\bar{x})$ is non-empty. This approach is associated with the so-called small control property, which is discussed at length in [Freeman and Kokotovic, 2008, Section 3.4.1].

The convexity assumption on r_{ik} is another requirement of the theorem, which one may wish to relax. While its role is clear in the setting of the selection problem, where it ensures that the image of every x is convex, it has no intuitive justification, and it appears rather unnatural. It is, then, desirable to find a method to relax this assumption. In the next section, it will be shown, that one may use the core process designed here to design the feedback controller u, even for non-convex cross-supply rates, provided that they have a specific form.

Until now, the results of this section were based on Theorem 6.2. As a final note, it will be shown that, if one of the cross-supply rates is assumed to have some smoothness properties, then similar conclusions to those of Proposition 6.1 can be drawn at a smaller 'cost'. In particular that, at least for some set Q, one may design feedback functions without the need for convex cross-supply rates found in Proposition 6.1. To

achieve this goal, the following Theorem is needed.

Theorem 6.3 (Implicit function theorem [Jost, 2006])**.** *Consider two Banach spaces A and B, and some function $F : \Omega \mapsto \mathbb{R}$, where Ω is an open subset of $A \times B$. Suppose that, for some point $(\check{a}, \check{b}) \in \Omega$, it holds that*

$$F(\check{a}, \check{b}) = 0.$$

If

$$\left. \frac{\partial F(a,b)}{\partial b} \right|_{(a,b)=(\check{a},\check{b})} \neq 0,$$

then there exist open neighborhoods $A_1 \subseteq A$, of $\check{a}$ and $B_1 \subseteq B$, of $\check{b}$, and some function $g : A_1 \mapsto B_1$, such that, for every $a \in A_1$, it holds that

$$F(a, g(a)) = 0.$$

Additionally, g is continuous and unique everywhere in A_1, and it holds that:

$$\frac{\partial g}{\partial a} = -\left(\frac{\partial F}{\partial b}(a, g(a))\right)^{-1} \frac{\partial F}{\partial a}(a, g(a)).$$

■

To be more precise, for the case of a single input system, if some cross-supply rate r_{ik} is known to be differentiable, then, for a differentiable bound ψ, the implicit function theorem can be applied directly on the function $r_{ik} - \psi$, to 'solve' for u, and, therefore, produce a feedback controller that works as required.

Proposition 6.2 (Feedback design – general case II)**.** *Take system (6.1), dissipative according to definition 2.10, with a single input ($m_i = 1$) and an equilibrium point barx from a triplet $(\bar{x}, \bar{u}, i)$. Assume that there exists at least one $k \in \mathcal{N}$, such that r_{ik} is differentiable, and that ψ is $sND(\bar{x})$ $(ND(\bar{x}))$. Suppose that some $(\hat{x}, \hat{u}) \in X \times U$ is a zero of the function $G : X \times U \mapsto \mathbb{R}$:*

$$G(\hat{x}, \hat{u}) = r_{ik}(\hat{x}, \hat{u}) - \psi(\hat{x}) = 0.$$

Assume that, the following holds

$$\frac{\partial r_{ik}}{\partial u}(\hat{x}, \hat{u}) \neq 0.$$

Then, in some neighbourhood of $\hat{x}$, say $Q \subseteq X$, a feedback function $v : X \mapsto U$ exists, such that, for $\mathcal{K}[\bar{x}, v, a]$, it holds that $(K_\emptyset \cap Q)/\{\bar{x}\} = \emptyset$. Additionally, v is continuous everywhere in Q. ∎

Proof. The core part of the result is just an application of Theorem 6.3, once one realizes that $\frac{\partial F}{\partial u} = \frac{\partial r_{ik}}{\partial u}$. It follows, then, that there exists a function v, such that $F(x, v(x)) = 0$ for every $x \in Q$. Then, no x that belongs in Q may be a member of $K_\emptyset$, taken from $\mathcal{K}[\bar{x}, v, \psi]$.

While the exact form of $v(x)$ is not known, it is known that $v(\hat{x}) = \hat{u}$, as well as the fact that following holds everywhere in Q:

$$\frac{\partial v}{\partial x} = \left(\frac{\partial r_{ik}}{\partial u}(x, v(x))\right)^{-1} \frac{\partial r_{ik}}{\partial x}(x, v(x)). \tag{6.7}$$

□

Remark 6.1. The feedback controller produced in the way described in the proof can be continuously and uniquely extended in the neighborhood of every point in Q, by using the implicit function theorem. It is of interest to know when $\bar{x}$ is one of these points. Since $V_i(\bar{x}) = 0$, $r_{ik}(\bar{x}, \bar{u})$ is expected to be non-negative – the theorem may be applied provided that it is non-zero. Unfortunately, in most cases of practical interest, this is not the case. For instance, the quadratic storage function case, where $V_i(x) = (x - \bar{x})^T P_i (x - \bar{x})$ has $\frac{\partial V_i}{\partial x} = 2(x - \bar{x})^T P_i$, and all the cross-supply rates are necessarily zero on $\bar{x}$. Then, one may be forced to use Theorem 6.2 in the manner discussed above, with regard to Theorem 6.1; that is, the Theorem may be used to stabilise neighborhoods of $\bar{x}$.

Proposition 6.2 achieves the goal of providing a feedback controller with similar properties to that of Proposition 6.1, without requiring convexity. In this case, the (a) cross-supply rate is only required to be differentiable; this is considerably easier to check. Many of the storage functions commonly used are continuously differentiable, or even smooth, and the supply rates that are paired with such storage functions usually also have these properties. Further, while it does not give a closed form for the feedback controller, equation (6.7) gives enough information for the purposes of implementation. To be more specific, given (6.7), one may consider the augmented system:

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} f_\sigma(t) \\ \frac{\partial v}{\partial x} f_\sigma(t) \end{bmatrix}.$$

The only requirement for integrating this system is that, for some $\hat{x}$, one is able to find $\hat{u}$, for which both the assumptions of Theorem 6.2 hold; that is $G(\hat{x}, \hat{u}) = 0$ and $\frac{\partial r_{ik}}{\partial u}(\hat{x}, \hat{u}) \neq 0$. Note that the first of these assumptions is, in this case, just a root-finding problem, which one may wish to approach numerically; and, given a solution to this part, checking the second assumption becomes trivial. Finally, note that, while the solution is given for the case where $G(x, u) = 0$, the exact same process may be reproduced for the case where $G(x, u) = -\varepsilon < 0$. In this case, one applies the theorem by considering $G'(x, u) = G(x, u) + \varepsilon$. The conditions, as well as (6.7) remain the same in this case.

Two points need to be made here, with regard to both Proposition 6.1 and Proposition 6.2 and their applicability. Both the results are interesting, from the mathematical point of view, and they both give controllers that are easy to implement. Even better, in the case of Theorem 6.2 the assumptions are easy to check, in the sense discussed above. However, the lack of a closed form for the controller means that it is difficult to obtain an *a priori* estimate about the form of the switching rule. In the case of a given *v*, one may try to plot the cross-supply rates against the state variables, in order to gain some insight about the areas in which some subsystem is allowed to be active – this approach has been shown repeatedly in the previous Chapters, as in Example 3.1 and 3.3. In the theorems of the present section, this is not possible. While this is not a fatal flaw, as one may still implement the rule (this, after all, only requires that one keeps track of the supply rates, and switches accordingly), it may be an impediment; for instance, designing fixed switching surfaces that are in agreement with the switching rule of Theorem 3.1 becomes impossible without a closed form for the controller. Instead, one has to make switching decisions on-the-fly, having Proposition 6.2 as a guarantee that this is possible.

6.2 Results for affine cross-supply rates

In the previous section, the general case of the problem of designing a stabilising controller was discussed, and a solution was proposed, using, amongst others, tools from the area of multi-valued functions and selections. As noted, however, the conditions derived this way are often restrictive and unnatural. This is not entirely unexpected. Multi-valued functions tend to be mathematical objects with weird behaviour, and the available selection theorems are rather restrictive. Besides, all the available results only posit sufficient conditions for the existence/construction of the desired selections. In

order to work around these limitations, in this section an additional assumption will be made on the form that some of the cross-supply rates assume. In particular, it will be assumed that, at least one of the cross-supply rates (the equivalent of r_{ik} of Proposition 6.1) is affine and linear with respect to the input. That is:

$$r_{ik}(x, u) = p(x) + q(x)u, \text{ for some } k \in \mathcal{N}. \tag{6.8}$$

Here, $p : X \mapsto \mathbb{R}$ and $q : X \mapsto \mathbb{R}^{1 \times m_i}$ (m_i is the number of input channels) have to be continuous functions, so that they comply with the assumptions already made for the cross-supply rates. This assumption is not entirely unwarranted. In the relatively common case where the subsystems are affine with respect to their inputs, it is also expected that the cross-supply rates may also have the same property.

An obvious case in which this form may be used to derive stabilisation results is the one were q is some constant function, which is not identically zero. For instance, consider the one-dimensional-input case, where $q = \lambda \in \mathbb{R}$. Then, setting $v(x) = -\lambda^{-1}(p(x) - \psi(x))$. Then, $r_{ik}(x, v(x)) = \psi(x)$, and the condition of Corollary 3.1 (that $r_{ik} - \psi$ is always negative, for some $k \in \mathcal{N}$) is satisified. This approach generalises to the case of mutli-dimensional input, as well as the case of non-constant input as follows.

Proposition 6.3 (Stabilisation - affine cross-supply rates I)**.** *Take system* (6.1)*, which is dissipative in accordance with Definition 2.10. Take an equilibrium $\bar{x}$ from a triplet $(\bar{x}, \bar{u}, i)$ of the system, and some set $Q \subseteq X$. Assume that, for some $k \in \mathcal{N}$, the corresponding cross- supply rate r_{ik} is continuous, and it has the form of equation* (6.8)*. Suppose that for some $j \in \{1, 2, \ldots, m_i\}$ the following holds:*

- *for every $x \in Q$, it holds that $q_j(x) \neq 0$,*

where q_j is the j-th element of q. If this condition holds, there exists a continuous feedback function $v(x)$, such that, for the partition $\mathcal{K}[\bar{x}, v, \psi]$, corresponding to some continuous function ψ, it holds that $(K_\emptyset \cap Q)/\{\bar{x}\} = \emptyset$. That is, the Condition of Corollary 3.1 holds. ∎

Proof. Use u_j to refer to the elements of q and u respectively, seen as vectors. Then r_{ik} takes the following form:

$$r_{ik}(x, u) = p(x) + \sum_{j=1}^{m_i} q_j u_j.$$

Since it is true for at least one j that q_j is alway non-zero, it follows that one may pick $u_j(x) = q_j^{-1}(x)(\psi(x) - max(p(x), 0))$ and $u_{j'}(x) = 0$, for all $j' \neq j$. This results into $r_{ik}(x, u(x)) \leq \psi(x)$. One has, then, that for no member of Q, it is true that every cross-supply rate is over ψ, and, therefore, $(K_{\emptyset} \cap Q)/\{\bar{x}\} = \emptyset$. $\qquad\square$

Remark 6.2. The role of the input in this process is to push the cross-supply rate under the bound ψ. If p is negative, then it contributes in this direction, so it is not eliminated. If, on the other hand, it is positive, then it should be taken into account. This is the role of the max function used in the definition of u_j in the proof.

Example 6.2. Take a linear, switching system, for which the two subsystems have different equilibrium points. Then, each subsystem has the following form:

$$\dot{x} = A_i(x - \bar{x}_i) + B_i u, \tag{6.9}$$

with $i = 1, 2$. In this case, take any Hurwitz matrices $A_1, A_2 \in \mathbb{R}^{n \times n}$, and any B_1, $B_2 \in \mathbb{R}^{n \times m_i}$, chosen so that the corresponding pairs (A_i, B_i) are controllable. Then, both subsystems can be shown to be dissipative with a storage functions $V_i(x) = (x - \bar{x}_i)^T P_i(x - \bar{x}_i)$, for appropriately chosen outputs:

$$y_1(x, u) = (P_1 B_1)^T (x - \bar{x}_1),$$

$$y_2(x, u) = (P_2 B_2)^T (x - \bar{x}_2).$$

In this case, the cross-supply rates take the following form:

$$r_{12}(x, u) = 2(x - \bar{x}_1)^T P_1 A_2(x - \bar{x}_2) + 2(x - \bar{x}_1)^T P_1 B_2 u \tag{6.10}$$

$$r_{21}(x, u) = 2(x - \bar{x}_2)^T P_2 A_1(x - \bar{x}_1) + 2(x - \bar{x}_2)^T P_2 B_1 u. \tag{6.11}$$

When one is interested in stabilising $\bar{x}_1$, only r_{12} is of interest. As a function of u, r_{12} is affine, and, in the notation of Equation (6.8),

$$q(x) = 2(x - \bar{x}_1)^T P_1 B_2.$$

That is, the set of points for which q vanishes form a plane in $\mathbb{R}^n$. When, in particular $n = 2$, it is a line on the state space. Then, away from this line, one may apply Proposition 6.3 to design a feedback function $v(x)$, which guarantees that the condition of Theorem 3.1 relating to $K_{\emptyset}$ holds.

More concretely, consider the following matrices:

$$A_1 = \begin{bmatrix} -4.04551 & -2.28763 \\ -0.516786 & -1.43212 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -2.09875 & -0.53332 \\ -3.36217 & -3.65288 \end{bmatrix},$$

and

$$B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}.$$

The corresponding P matrices are as follows:

$$P_1 = \begin{bmatrix} 0.137164 & -0.106233 \\ -0.106233 & 0.518826 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.451743 & -0.133276 \\ -0.133276 & 0.156337 \end{bmatrix},$$

and, for two equilibria at $\bar{x}_1 = \begin{bmatrix} 2 & 0 \end{bmatrix}^T$ and $\bar{x}_2 \begin{bmatrix} -2 & 0 \end{bmatrix}^T$, one has:

$$p(x) = -0.554388 + 0.138597x_1^2 + 7.34532x_2 - 2.41305x_1x_2 - 3.67711x_2^2, \text{ and}$$

$$q(x) = 0.181533 - 0.0907667x_1 + 0.725123x_2.$$

Figure 6.1 shows a part of the $x_1 - x_2$ plane, along with the line corresponding to $Q_0 = \{x \in X \mid q(x) = 0\}$. Around it, one may take a ribbon of width $2\varepsilon > 0$, which corresponds to the notion of being close to the line; that is, $Q = \{x \in X \mid |q(x)| \geq \varepsilon\}$. This corresponds to the darker, coloured area of Figure 6.1. In accordance with Proposition 6.3, one may use the controller $v(x) = q^{-1}(x)(\psi(x) - max\{p(x),0\})$, for any appropriate ψ bound, say $\psi(x) = -0.1\|x - \bar{x}_1\|^2$, to control the system in Q.

Observe that, for this controller, it always holds that $r_{12}(x, v(x)) \leq \psi(x)$, and, therefore, for the partition $\mathcal{K} = \mathcal{K}[\bar{x}_1, v, \psi]$, one always has $K_0 = \emptyset$, as there is no point in the state space, for which $r_{12}(x, v(x)) > \psi(x)$. The exact form of the partition depends on the value of r_{11}, under u, and, specifically, on the sign of the function $r_{11} - \psi$ corresponding to this controller. In Figure 6.2, this partition is shown for some area around the equilibrium points.

At this point, it pays to consider what information is given by Theorem 6.2, and what information is missing. Because it is known that, for the partition of Figure 6.2, $K_0 \cap Q = \emptyset$, Theorem 3.2 can be invoked, to pick a switching law for this area. For
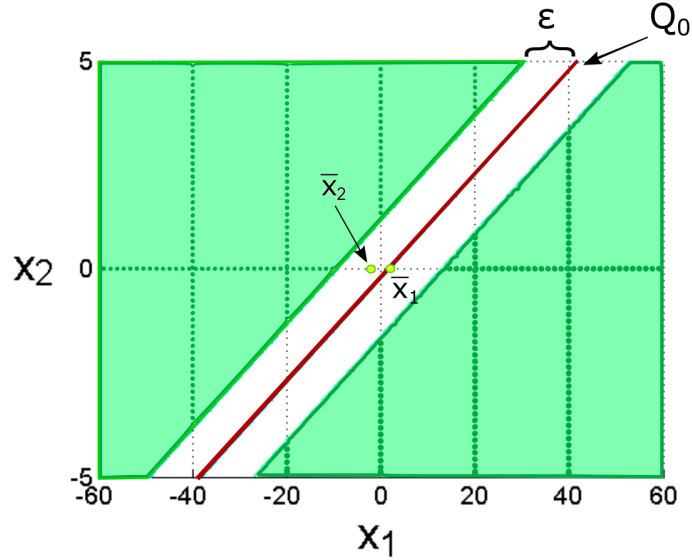
Figure 6.1: Visualisation of the area of applicability Q of Theorem 6.3 – the coloured area. The line $Q_0 = \{x \mid q(x) = 0\}$ is also shown.

this switching law, the stored energy V_1 shall decrease for any trajectory in Q. On the other hand, there is no indication about what should happen in the uncoloured (white) area of Figure 6.2, as, for this, the function v becomes unbounded, and, therefore, it cannot be used. One may, however, use Proposition 5.3, provided that they can find a controller $v' : X \mapsto U$, appropriate for this purpose.

Taking both v' and $\psi' : X \mapsto \mathbb{R}$ (a boundary function), to be identically zero, one has the partition $\mathcal{K}' = \mathcal{K}[\bar{x}, v', \psi']$. The restriction of this partition in the uncoloured section of Figure 6.2 is shown in Figure 6.3. For this input, r_{11} is always non-positive, and K'_0 is empty. Then, the form of the partition is determined by the sign of r_{12}, as usual. One may then apply Proposition 5.3, to conclude that the equilibrium point $\bar{x}$ is stable, provided that the system:

- switches according to $\mathcal{K}$, under the control u in the coloured area of Figure 6.2, and

- switches according to $\mathcal{K}'$, under the control u' in the light area of the same Figure.

A collection of simulated trajectories, generated under this switching/control scheme is shown in Figure 6.4. The stored energy V_1 corresponding to these trajectories is shown in Figure 6.5. The initial conditions are chosen in a uniformly random manner, from the area around the equilibirum point x_1.
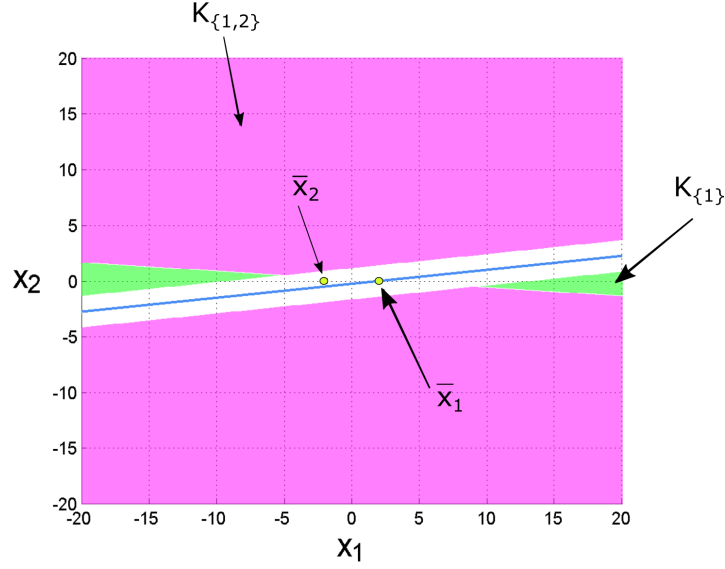
Figure 6.2: Visualisation of the partition $\mathcal{K}[\bar{x}_1, v, \psi]$ from Example 6.2. The equilibrium points and the elements of the partition are also marked on the plot.

□

The main idea in Proposition 6.3 is that, if some cross-supply rate r_{ik} has the from (6.8), and if q is known to be non-zero, then some feedback can be designed, shaping r_{ik} so that the resulting partition $\mathcal{K}$ is well-behaved (that is, it may be used for the purpose of stabilisation). From a specific point of view, this is a rather conservative outlook. Evidently, if q does vanish, then the input cannot influence the chosen cross-supply rate. However, this is an issue, only if there was a need for such influence; that is, if, for these values (which make q equal to zero), the cross-supply rate is over the bound. More precisely, when, on $Q_0 = \{x \in X \mid q(x) = 0\}$, it holds that $r_{ik}(x, u) = p(x) > \psi$. If this is not the case, then the system is already in the right direction, and no additional control effort is needed to steer it. Unfortunately, the simple controller used in Proposition 6.3 is no longer appropriate, since, on Q_0, it tends to infinity. Even though any finite value would be good for the described area of Q_0, u described there does become infinite everywhere on Q_0. A solution to this problem can be derived by adapting the apporach used in [Sontag, 1989], for the case of feedback stabilisation in smooth systems. The outcome of this adaptation is a controller that is continuous, and finite in the set under consideration.

Proposition 6.4 (Stabilisation - affine cross-supply rates II). *Take system* (6.1)*, dissipative under definition 2.10, and with a single input. Consider the set-up of Proposition*
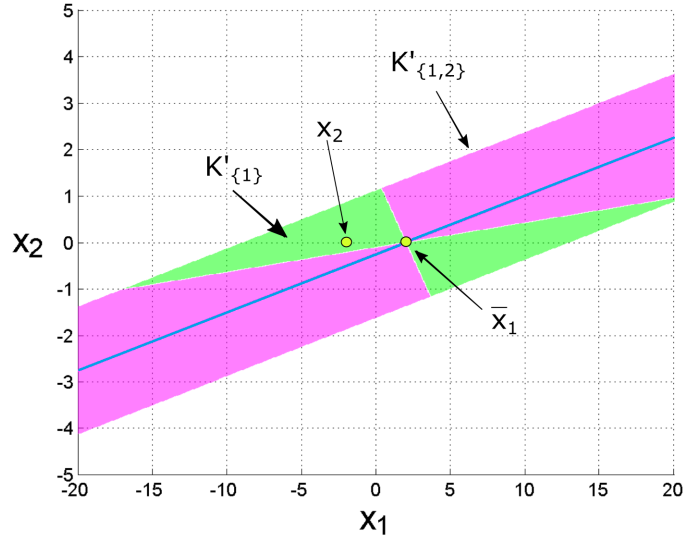
Figure 6.3: Visualisation of the second partition $\mathcal{K}[\bar{x}_1, v', \psi]$ from Example 6.2. Zoom in the uncoloured area of Figure 6.2.

6.3. Assume that there exists at least one $k \in \mathcal{N}$, such that r_{ik} has the form (6.8), and that ψ has the $ND(\bar{x})$ property. Consider the following condition:

- *For every $x \in Q$, it holds that, if $q(x) = 0$, then $p(x) < \psi(x)$.*

If this condition holds, there exists a function $v(x)$, such that, for $\mathcal{K}[\bar{x}, v, a]$, it holds that $(K_\emptyset \cap Q)/\{\bar{x}\} = \emptyset$. Additionally, $v(x)$ is continuous everywhere on Q, with the possible exception of $\bar{x}$. ∎

Proof. The whole process rests upon the idea developed in [Sontag, 1989], concerning the stabilisation of non-switching systems. First, consider a function $F : \mathbb{R}^3 \mapsto \mathbb{R}$, given by

$$F(a, b, k) = bk^2 - 2ak - b.$$

F is a differentiable function, and for it, it holds that

$$\frac{1}{2}\frac{\partial F}{\partial k} = qk - p.$$

Take the set $S = \{(a, b) \in \mathbb{R}^2 \mid b \neq 0\} \cup \{(a, b) \in \mathbb{R}^2 \mid a < 0, b = 0\}$. Consider also the following function:
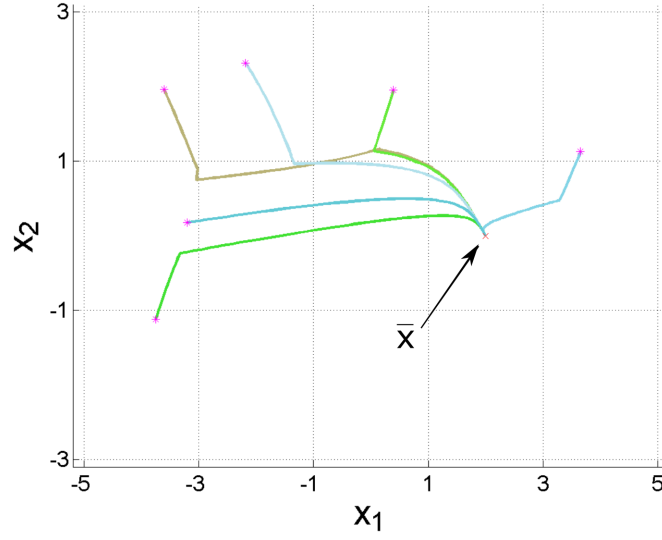
Figure 6.4: Trajectories for the system of Example 6.2 – asymptotic stability around x_1. Randomly generated initial conditions, marked with stars.

$$k(a,0) = 0, \quad \forall a \in \mathbb{R}, \tag{6.12}$$

$$k(a,b) = \frac{a + \sqrt{a^2 + b^2}}{b}, \quad \forall a \in \mathbb{R}, \quad \forall b \in \mathbb{R}/\{0\}. \tag{6.13}$$

For this choice of k, it always holds that $F(a,b,k(a,b)) = 0$. Also, $k(a,b)$ yields:

$$\frac{1}{2}\frac{\partial F}{\partial k} = -a, \text{ when } b = 0 \,,$$

$$\frac{1}{2}\frac{\partial F}{\partial k} = \sqrt{a^2 + b^2}, \text{ otherwise.}$$

Therefore, on S, $\frac{\partial F}{\partial k} \neq 0$, and, by Theorem 6.3, it follows that k is continuously differentiable everywhere on S. The assumption stated above guarantees that, if one takes $a = p(x) - \psi(x)$ and $b = q(x)$, then $(p(x) - \psi(x), q(x)) \in S$, for all $x \in Q/\{\bar{x}\}$. It follows that

$$v(x) = -k(p(x) - \psi(x), q(x)) \tag{6.14}$$

is a feedback function which is continuous everywhere in Q, except possibly at $\bar{x}$. By substituting (6.14) into (6.8), one has that $r_{ik}(x,u(x)) \leq \psi(x)$. This is because:

- $r_{ik}(x,v(x)) = p(x) < \psi(x)$, by assumption, for all x such that $q(x) = 0$, and
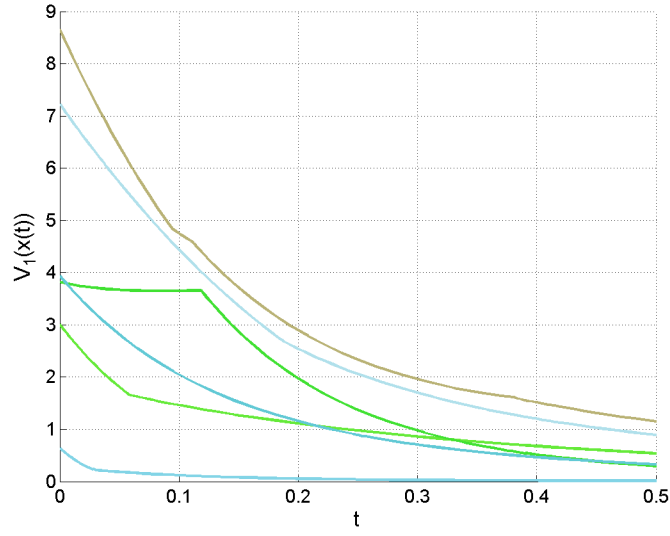
Figure 6.5: Stored energy V_1 for the trajectories shown in Figure 6.4. The energies decrease with time.

- $r_{ik}(x, v(x)) = \psi(x) - \sqrt{(p(x) - \psi(x))^2 + q^2(x)} \leq \psi(x)$, for the rest.

Hence, r_{ik} is always under the bound ψ, and for the element $K_\emptyset$ of the partition $\mathcal{K}[\bar{x}, v, \psi]$, it holds that $K_\emptyset \cap Q = \emptyset$, as required. $\qquad\square$

Example 6.3. Consider, again, the system of Example 6.3. As stated there, the controller of Proposition 6.3 can be used away from the area close to the set $Q_0 = \{x \in X \mid q(x) = 0\}$, shown in Figure 6.2. While this may be enough for some cases, it is also possible that it becomes too restrictive. For instance, for a simple requirement, such as having subsystem 2 available everywhere on the left half-plane the controller of Proposition 6.3 becomes unusable, since, there, the condition that the function q is non-zero does not hold. Proposition 6.4, on the other hand can be used.

To see that, consider the condition of Proposition 6.4 on q. To locate the area where this condition holds, one must find where the following (in)equalities hold together:

$$q(x) = 0,$$

$$p(x) \leq \psi(x).$$

The first of these represents the straight line shown in Figure 6.2. Solving it for x_1 and substituting in the function $p - \psi$, one gets the plot of Figure 6.6; doing the same for x_2 yields Figure 6.7. From those, it can be seen that $p - \psi$ is non-positive everywhere
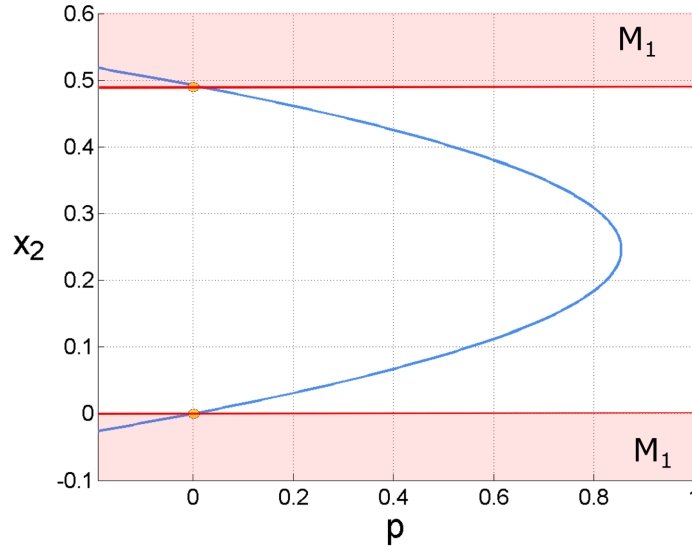
Figure 6.6: Values of the function p, from (6.8), for some range of values of x_1. Roots at marked on the Figure. $M_1 = \{(x_1, x_2) \in X \mid x_2$ in the coloured area defined by the red lines$\}$.

on Q_0, outside an area approximately bounded by the lines $x_1 = 2$, $x_1 = 9.2$, $x_2 = 0$ and $x_1 = 0.92$ – the intersection of M_1 and M_2, shown in Figures 6.6 and 6.7. Then, in accordance with Proposition 6.4, one may use (6.14) to control the system outside this area ($M_1 \cap M_2$), in a way that, under the switching rule induced by the partition $\mathcal{K}[\bar{x}, v, \psi]$, the equilibrium $\bar{x}$ is stabilised asymptotically. Figure 6.8 shows this partition, as well as the area in which the theorem does not guarantee that it can be used.

At this point, two routes are open. First, one might try using the controller (6.14) everywhere, including the white area M of Figure 6.8. Proposition 6.4 makes no claims about the behaviour of the partition in this area; however, the form of the partition in M may be approximated and plotted. This is done in Figure 6.9, where it is shown that $M \cap K_{\emptyset} = \emptyset$. Then, Theorem 3.2 may be applied, and the asymptotic stability of $\bar{x}$ is concluded. Of course, this is only an approximate method, and it should be taken with caution; no amount of precision in plotting the partition can ensure one that, at a finer level, the constraints of the theorem are not violated.

Alternatively, one may look for another controller, say v' to use within M, such that the conditions of Proposition 5.3 hold, and, again, stability can be concluded under v'. In the present case, v' and ψ' may be taken to be identically zero, since subsystem 1 is passive. Then, partition $\mathcal{K}[\bar{x}, v', \psi']$ may be formed. In accordance with Proposition 5.3, if switching is ruled by $\mathcal{K}'$ in M, and by $\mathcal{K}$ in X/M, then $\bar{x}$ is rendered Lyapunov
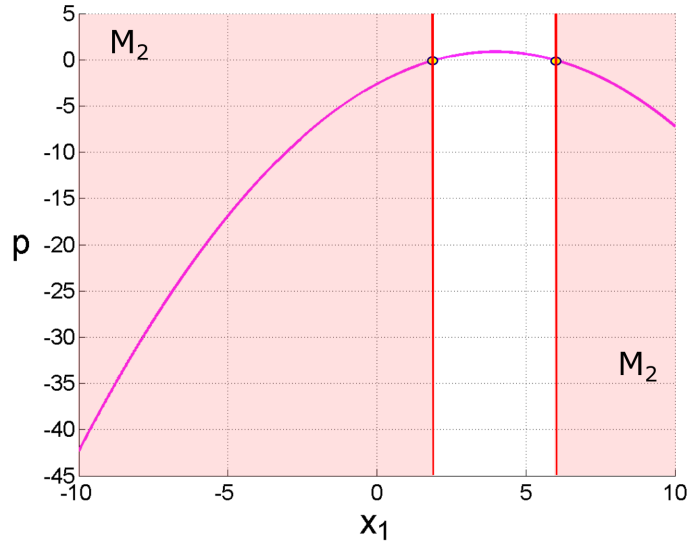
Figure 6.7:   Values   of   the   function p,   from   (6.8),   for   some   range   of values   of x_2.   Roots   marked   on   the   Figure. $M_2 = \{(x_1, x_2) \in X \mid$ x_1 in the coloured area defined by the red lines$\}$.

(but not asymptotically) stable. Figure 6.10 shows the form of $\mathcal{K}'$ within M. Note that, while this partition is also (necessarily) approximate, in this case, the guarantee that $K_\emptyset = \emptyset$ is formal, rather than approximate.

Note that, in both cases, the conlusion is the same. Namely, that stability of x_1 may be guaranteed by using v everywhere, and switching in accordance with the combination of Figures 6.8 and 6.9. Or, alternatively, switch according to the rule induced by the partitions shown in Figures 6.8 and 6.10, using the controllers v and v', respectively, for each of them.   □

While the results of this section were stated for the case of systems with affine cross-supply rates, such as (6.8), they are applicable for a wider class of systems, namely, those of the form

$$r_{ik}(x, u) = p(x) + q(x)\xi(u),$$

where $\xi : U \mapsto U$ is some bijective function. In this case, one is able to calculate an input $v(x)$ from the previous theorems, and then apply the control $u(x) = \xi^{-1}(v(x))$ to the actual system, where the -1 superscript denotes the inverse function.
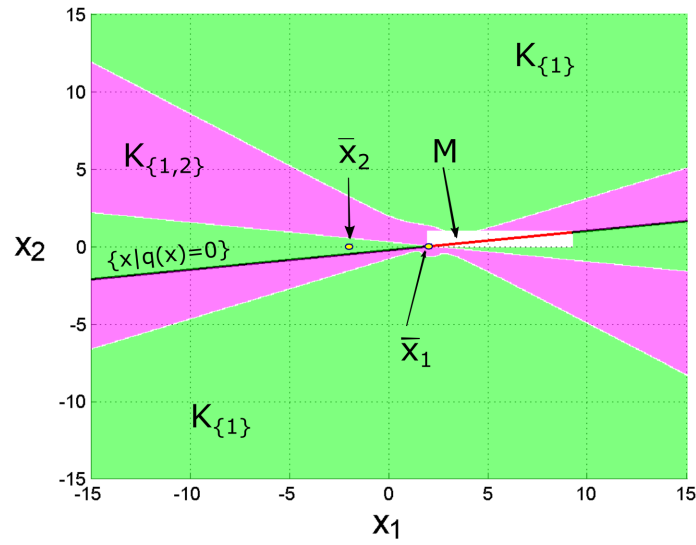
Figure 6.8: Visualisation of the partition for Example 6.3. The system is controlled by (6.14) in the coloured area. Elements of the partition are marked on the diagram.

Overview

In this chapter, a collection of feedback stabilisation design results were proposed, for non-linear, dissipative, switching systems. These results make use of the infrastructure developed in Chapters 3 and 5, and, in particular, the partition of Definition 3.1 and the switching rule described in Theorem 3.1. For the results of this chapter, the approach of Chapter 3 is turned on its head. In Chapter 3, one starts with a given input, and they determine conditions (for instance, the Condition of Theorem 3.1) for which this input, along with a switching rule, stabilises the system. Here, the inputs are constructed with the objective of satisfying those conditions. To that effect, techniques are used from the areas of analysis (Proposition 6.2), multi-valued functions (Proposition 6.1) and non-switching system feedback stabilisation (Propositions 6.3 and 6.4). It is shown that, besides the value they have on their own, the results of this section extend the applicability of the results of the previous Chapters (see Example 6.3).
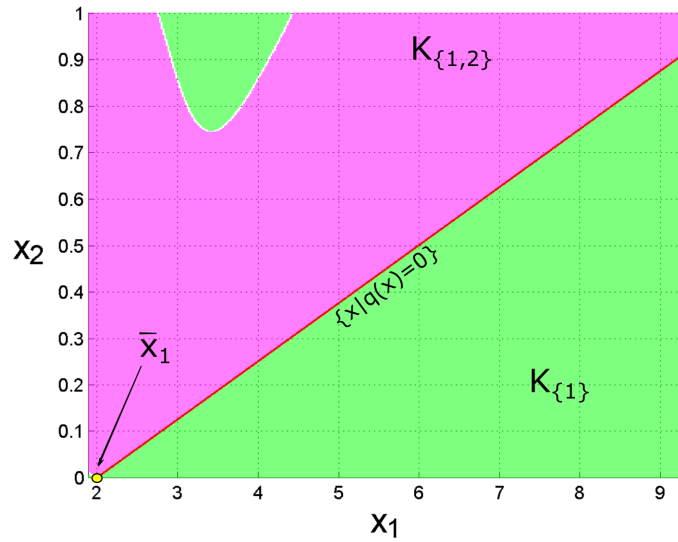
Figure 6.9: Visualisation of the partition for Example 6.3, in the set M of Figure 6.8. The system is controlled by (6.14). Elements of the partition are marked on the diagram.
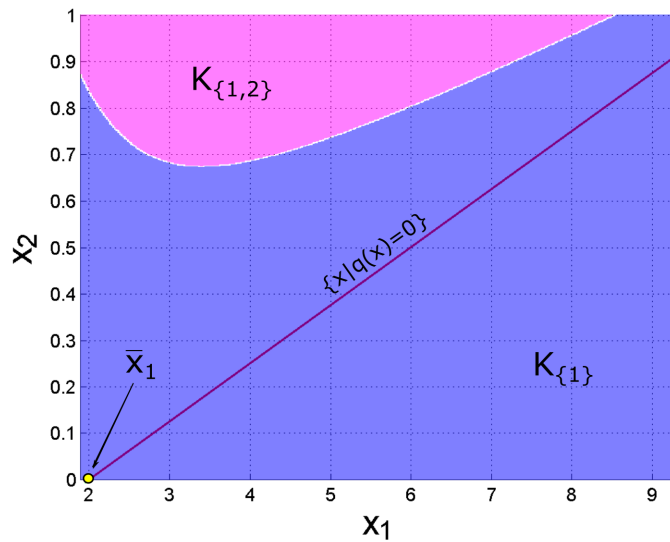

Figure 6.10: Visualisation of the partition for Example 6.3, in the set M of of Figure 6.8. The system is controlled by $v' \equiv 0$. Elements of the partition are marked on the diagram.

Chapter 7

Zeno behaviours and Stabilisation

Up to this point, stabilisation issues were the exclusive focus of this work. The partition that has been introduced in Definition 3.1 was used to derive rules for the switching law σ, and it was shown that, under certain conditions, following the rules ensures stability under a given v. Another issue that is of interest concerns the 'timing' characteristics of the switching laws that are admissible by the rules. In particular, it is useful to know if, among the admissible switching laws, one my find some that do not exhibit the type of pathological behaviour, which is known as Zeno behaviour. Here, a short study of this question is conducted. It is shown that, in the general case, one should expect that the switching rule will allow some Zeno switching laws. Subsequently, an approach to pick non-Zeno signal is described, and conditions are established, under which this approach is applicable.

7.1 Motivation

To repeat some of the discussion of Chapter 2, a system is said to be Zeno, when the switching law exhibits 'too many' points of discontinuity, in a short period of time. The intuitive basis for its introduction is that any natural system must have a limit in the frequency with which it may switch. More precisely:

Definition 7.1 (Zenoness). A switching signal $\sigma : \mathbb{R} \mapsto \mathcal{N}$, corresponding to system (2.3), is said to be Zeno, if there exists a finite time interval, in which the signal has infinite points of discontinuity. ∎

Figure 7.1 gives an intuitive sense of the meaning of this definition. The L_1. L_2 lines in the figure correspond to some switching rules – when they are crossed, a switch
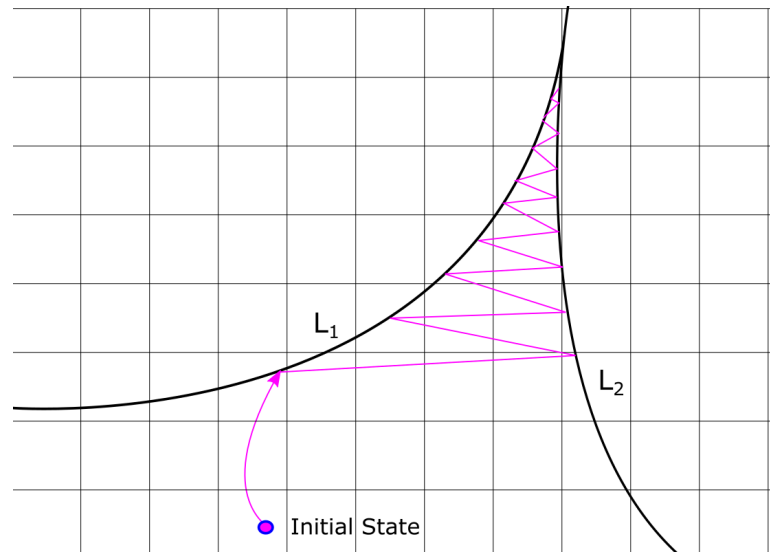
Figure 7.1: Zenoness – sample trajectory (magenta line) under state-dependent switching (L_1, L_2).

is forced. As they approach, the frequency of switching increases, and the swithcing law resembles the one shown in Figure 7.2. Evidently, switching laws that behave this way are not easily applicable in practical settings; and it would be useful to have a way to ensure that they do not come up.
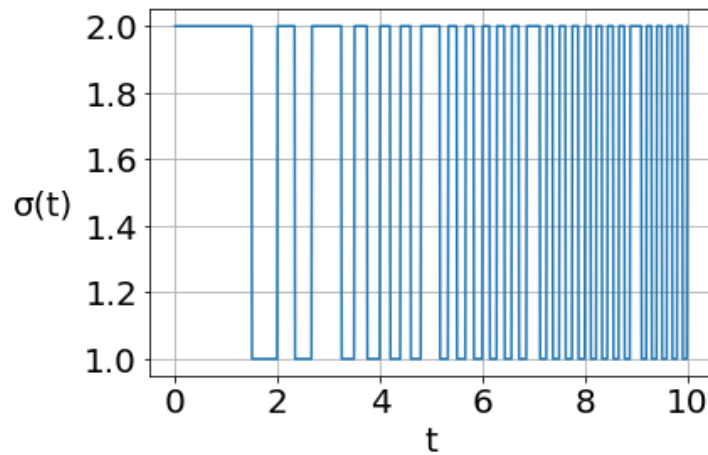
Figure 7.2: Zenoness – sample switching law.

Given Definition 7.1, one may wonder if a specific set of restrictions on switching (such as the switching rule of Theorem 3.1) causes the switching law of the system to be Zeno. Ideally, in a control engineering setting, one would want to identify such situations, in order to avoid them. This is because, in a realistic scenario, one cannot

expect to implement a Zeno switching signal – real systems cannot switch arbitrarily fast, and, therefore, trying to use a Zeno switching law is bound to cause malfunctions. Zenoness is also a problem from a mathematical point of view, as it is not clear how to continue the solutions of the system after point of Zenoness (see the work in [Dashkovskiy and Feketa, 2017])

By nature, this line of enquiry is meaningful only *a posteriori*. The pre-condition for it is that one has shown that a given v stabilises the system, using one of the results of Chapter 3. In the exposition given here, it is assumed that some system is given in the form of Equation (2.3), along with some feedback function v, which gives rise to a partition $\mathcal{K}[\bar{x}, v, \psi]$, corresponding to some equilibrium point $\bar{x}$ and some continuous function ψ, as described in Definition 3.1. It is additionally assumed that the results are applied in some set $Q \subseteq X$, for which $(K_{\emptyset} \cap Q)/\{\bar{x}\}$ is empty, meaning that one may apply Lemma 3.1 (or Theorems 3.1 and 3.2 , if Q is a neighborhood of the equilibrium point $\bar{x}$) to select some switching law σ that leads to (possibly asymptotic) stability.

There are specific cases, where answering this type of question is easy. For instance, if Corollary 3.1 can be applied, meaning that for a specific $j \in \mathcal{N}$, it holds that $r_{ij} - \psi \leq 0$, everywhere in the area of interest (the set E of the Theorem), and, trivially, the non-Zeno switching law $\sigma(t) = j$ for all $t \leq t_0$. However, for the general case, it is not clear how one would determine if such a switching law actually exists. In fact, it will be shown below that trying to enforce switching in accordance with partition $\mathcal{K}$ might cause Zeno behaviour to emerge. More than any practical application, the purpose of the results of this section is to offer assurance that, under general conditions, one may find non-Zeno switching laws that ensure stabilisation, as promised by the theorems of Chapter 3.

One may identify two distinct sources of Zenoness for the set-up under study. The first is external: when the state is in some element of the partition, say $K_{\{1,2\}}$, the rules allow for any switching between subsystems 1 and 2, including 'pathological' ones, like Zeno switchings. This source of Zenoness is of less interest – one may assume that the designer of the switching signal does not purposefully cause the system to behave in this fashion. The second source is internal. It is possible that the partition itself is the cause of Zenoness, by demanding that infinite switches happen in finite time. This source is more difficult to spot, and, therefore, more interesting.

To get a better idea about how the partition may cause the appearance of infinite switches, consider an element K_J, belonging to the partition of the state space of some system, induced by some equilibrium point, some input and some bound ψ, with J

some subset of $\mathcal{N}$. K_J consists of two parts, the interior K_J^o and the boundary ∂K_J. In order to represent the boundary, consider, for some set $E \subseteq \mathcal{P}(\mathcal{N})$, the following function $\hat{Q} : \mathcal{P}(\mathcal{N}) \times \mathcal{P}(\mathcal{N}) \mapsto \mathcal{P}(X)$: simplicity):

$$\begin{aligned}\hat{Q}(J,E) = &\{x \in X \mid r_{ij}(x,v(x)) = \psi(x), j \in E\} \cap \\ &\{x \in X \mid r_{ij}(x,v(x)) < \psi(x), j \in J/E\} \cap \\ &\{x \in X \mid r_{ij}(x,v(x)) > \psi(x), j \in (\mathcal{N}/J)/E\}.\end{aligned}$$

Using this notation, one may write:

$$\partial K_J \subseteq \bigcup_{E \in \mathcal{P}(\mathcal{N})} \{\hat{Q}(E,J) \mid E \neq \emptyset\}.$$

The notation is rather obscure, but the core idea is relatively simple. The boundary consists of points for which at least one of the non-strict inequalities that form the set K_J (which have the form $r_{ij} - \psi \leq 0$, for $j \in J$) is 'active'; that is, an equality. However, not every point for which this happens actually belongs to the boundary – to see this, consider the function of Figure 7.3. This observation also follows by the fact that the boundary of a set is always closed, while the sets $\hat{Q}$ are not. While no rigorous proof of the assertions made in this paragraph will be offered, the following should suffice, instead of a proof sketch: due to the continuity of the cross-supply rates and the function ψ, the points for which all the inequalities are strict must belong to the interior of the set K_J, as they are contained in neighborhoods for which the inequalities still hold.

Under the assumptions stated here, take some initial point for system (2.3), and assume that it belongs to the interior of K_J. Then, if the condition of Theorem 3.1 holds, it is known that $J \neq \emptyset$, and one may pick some index $j \in J$, for which the inequality $r_{ij} - \psi \leq 0$ (used to define K_J) is strict. Clearly, for such a choice, one may always construct a non-Zeno switching signal, at least up to the boundary of K_J, by just keeping σ constant, and equal to j. However, upon reaching the boundary, this may become impossible. In particular, the trajectory may cross the boundary of K_J at some point $\hat{x}$, for which $r_{ij}(\hat{x}, v(\hat{x})) = \psi(\hat{x})$. At this point, depending on the direction of motion (that is, f_j), the system might enter some other element of the partition, say K_L, for $L \subseteq \mathcal{N}$, such that $j \notin L$. In this case, some other subsystem has to be picked. This forced switch is the possible source of Zenoness mentioned above. In particular, when $J \cap L = \emptyset$, one may conceive of a scenario where a forced switch is followed
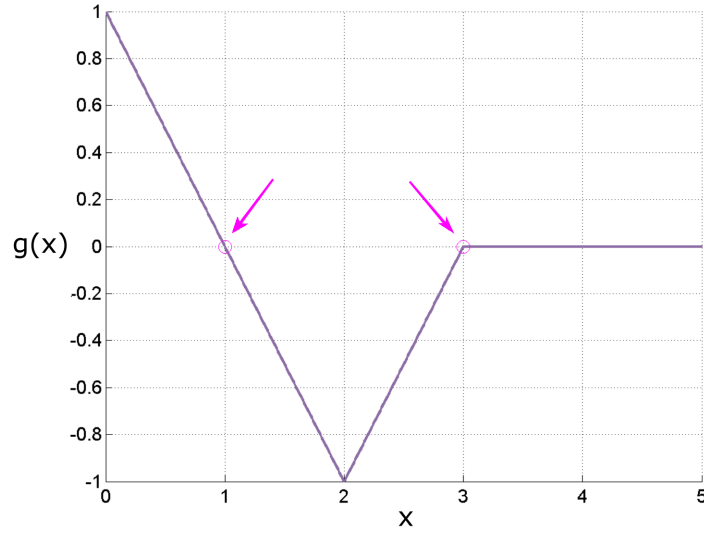
Figure 7.3: For the pictured function $g : \mathbb{R} \mapsto \mathbb{R}$, the circled points are members of the closure of $\{x \mid g(x) < 0\}$, but the points to the right of $x = 3$ are not.

immediately by another forced switch, causing faster and faster switching around the boundary, and, ultimately, Zeno behaviour.

Under these circumstances, one may think that it is possible to avoid Zenoness altogether by positing a geometric condition that makes the specific configuration ($J \cap L = \emptyset$) impossible. This would entail working with partitions for which neighboring regions share at least one common index, which can be active in both. Such partitions would look like the one in Figure 7.4 (compare with Figure 3.2). However, this would not work. Even if the common boundary of ∂K_J and ∂K_L does not give rise to pathological behaviours, there is no guarantee that such behaviours do not appear. It might be the case that, for the same J, L, and for some other $R \subseteq \mathcal{N}$, for which $J \cap R = \emptyset$ and $L \cap R \neq \emptyset$, the distance between $\partial K_J \cap \partial K_L$ and $\partial K_L \cap \partial K_R$ may become infinitely small. If this happens for a sequence of subsets, it is possible that infinite switches are forced, even though no particular pair of partition elements forces the change.

7.2 Rules for non-Zeno Switching

In order to obtain a working approach, one may adapt the method introduced in [Liu et al., 2010]. In that work, the authors introduce two nested sets of switching rules, under which the origin – a common equilibrium of all the subsystems – is stable. The condition that enables this construction is akin to the dissipativity conditions discussed
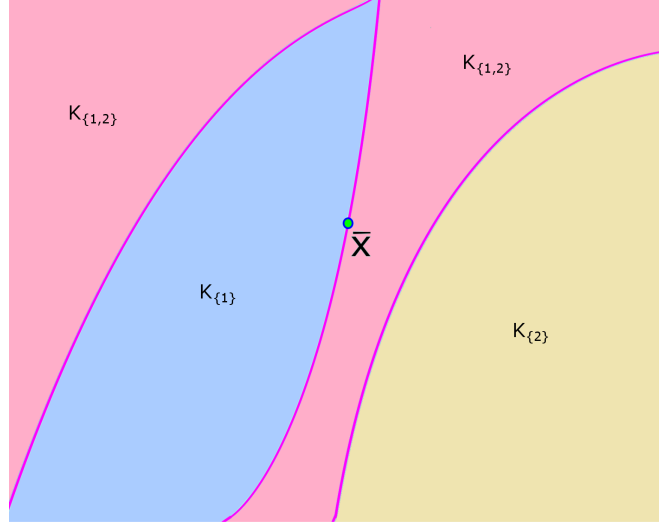
Figure 7.4: Part of a generic partition for a system with two subsystems. Neighbouring elements of the partition have at least one common index.

in this work. The difference is that, in [Liu et al., 2010], this condition is required to hold only for some convex combination of the subsystem dynamics, rather than for each subsystem separately.

The process that will be described here has two steps. First, a set of switching rules will be introduced (rule-set I), which, under some conditions, may guarantee an energy decrease behaviour, similar to that of Lemma 3.1. To that effect, they make choices using the elements of a partition $\mathcal{K}$. At the same time, these rules keep the dwell-time of the system non-zero; to do that, choices of arguments based on the relative placement of the boundaries are made, similar to the ones discussed previously in the present section. Then a second set of rules (rule-set II) will be proposed. These rules subsume rule-set I in a larger process, the purpose of which is to ensure that the system has stable, non-Zeno behaviour.

The first set of rules takes the following form:

Algorithm 7.1 (Switching rule-set I)**.**

1. *For the initial point $x(\tau_0) = x_0 \in K_J$, pick some $\sigma(\tau_0) = j \in J$, such that $r_{ij}(x_0, v(x_0))$*
   $< \psi(x_0)$.

2. *Keep σ constant until τ_1, when $x(\tau_1) = x_1 \in \partial K_L$, for some L such that $j \notin L$.*

3. *Pick another (possibly the same) subsystem $\sigma(\tau_1^+) = k$, such that $r_{ik}(x_1, v(x_1)) <$*
   $\psi(x_1)$.

    4. *set $\tau_0 = \tau_1$ and return to step 2.*

∎

The process works in time steps. For each time step, a subsystem is chosen, and it is kept active for the whole time the system state spends within the current element of the partition. This subsystem is chosen so that the switching rule of Theorem 3.1 is obeyed, and, therefore, the stored energy decreases. When the boundary of the partition element is reached, the next time step begins; another subsystem is picked (bearing in mind the switching rule), kept active, etc. Because the energy decrease argument holds for every step, then the system is expected to behave in accordance with Lemma 3.1, for as long as the rules of Algorithm 7.1 are followed.

Before examining any advantages that this set of rules might have, it is useful to know if it is actually feasible. In particular, it is of interest to determine if the rules can be applied everywhere in (a chosen subset of) the state space. In the general case, the answer to this question is negative. If the set-up and the condition of Theorem 3.1 are known to hold, then, in the area of interest, it holds that $(K_0 \cap Q)/\{\bar{x}\} = \emptyset$. This means that, for every point in Q, a subsystem can be chosen, for which the non-strict form of the inequality in steps 1 and 3 is satisfied. The strict form, however, is rather more demanding. That is because one may not preclude the possibility that the trajectory reaches the component of the boundary for which all the cross-supply rates are either over or equal to the bound ψ. That is, it may be that $x_1 \in \hat{Q}(J, J)$ (using the notation introduced earlier in this section). Given this discussion, consider the following result:

Lemma 7.1 (Positive interval between switches)**.** *Consider System* (2.3)*, dissipative in accordance with Algorithm 2.10. Suppose that it has some weak equilibrium point $\bar{x}$, from a triplet $(\bar{x}, \bar{u}, i)$, and that V_i is differentiable and it has the PD($\bar{x}$) property. Assume, additionally, that, for some $Q \subseteq X$, it holds that $\bar{K}_0 \cap Q = \emptyset$. Then, under the switching scheme described by rule-set I (Algorithm 7.1), for each switching instant t_z ($z \in \mathbb{N}$) there exists some $\varepsilon_z > 0$, such that no other switching happens in the interval $[t_z, t_z + \varepsilon_z)$. Additionally, for any two time instants τ_a, τ_b, such that $\tau_a \leq \tau_b$, it holds that*

$$V_i(x(\tau_b)) \leq V_i(x(\tau_a)).$$

∎

Proof. The energy decrease that is asserted by the second part of the lemma follows by the fact that the switching scheme used here is in accordance with the rule of Theorem

3.1. For the first assertion, note that, right after the switching, the function $r_{ij} - \psi$ is continuous and strictly negative. Then, there is a neighborhood of x, for which the sign of $r_{ij} - \psi$ will retain the same sign. Further, all the functions in $\mathcal{F}$ are continuous everywhere in X (a superset of Q). It follows that a positive amount of time is required for the trajectory to leave this neighborhood, and the result follows. $\square$

While the result states that the ε_z is positive, there is no guarantee that, as $z \to \infty$, $\varepsilon_z \not\to 0$. However, if it happens that Q is a compact set, such that $\bar{x} \notin Q$, then it can be shown that, as long as the trajectory remains within Q, the dwell time, that is, the minimal difference between consecutive switching times, is positive.

Lemma 7.2 (Non-Zeno switching). *Consider the same set-up as in Lemma 7.1. Assume, additionally, that Q is compact and that it does not contain $\bar{x}$. Then, there exists some δ, possibly dependent on Q, such that, for any consecutive switching times t_j, t_{j+1}, it holds that $t_{j+1} - t_j \geq \delta$; since the lower bound of the switching times is positive, no Zeno switching is forced, as long as the state remains within Q.* ∎

Proof. Assume that, somewhere on ∂K_L, subsystem j gets activated. By the switching scheme, it will remain active until the trajectory reaches some point on a subset of ∂K_F, with K_F another element of the partition, for which $r_{ij}(x, v(x)) = \psi(x)$. The set of all elements of $\partial K_L \cap Q$ for which j may be activated is compact and disjoint from ∂K_F. It follows that for any of those points in ∂K_L, a positive amount of time is needed to reach ∂K_F. Further, because of compactness of Q, all these positive amounts of time have a (positive) minimal. Since there is a finite number of (pairs of) elements in the partition, one may find some δ_j, the minimal amount of time between the activation of subsystem j and its next deactivation, while the trajectory remains in Q. Similarly, one may find δ_k for every other $k \in \mathcal{N}$, and, since the number of subsystems is also finite, one may also find the required $\delta = \min_k \delta_k$.

The requirement that the equilibrium point is not in Q cannot be omitted. Since $(\partial K_{\emptyset} \cap Q)/\{\bar{x}\} = \emptyset$, $\bar{x}$ is the only point that may belong to the closure of every one of the partition elements. Including it to the set would mean that ∂K_L and ∂K_F may not be disjoint, in which case the δ described here may be zero. $\square$

Lemma 7.2, provides a partial solution to the problem of determining if the restrictions on switching, introduced in Chapter 3, force Zenoness. In particular, it is shown that, at least within some compact set, the switching rule of Theorem 3.1 allows for (at least one way) of non-Zeno switching. This is already quite useful. For instance,

by picking $Q = \{x \in X \mid \rho_1 \leq \|x\| \leq \rho_2\}$, for some real numbers $\rho_1 \leq \rho_2$, one may apply Lemma 7.2 to construct a non-Zeno switching law that guarantees the stability of a neighborhood of the equilibrium point (the ball around it, with radius equal to ρ_1). One may also use Lemma 7.2 in conjunction with the results of Chapter 5, to produce a 'jigsaw' approach to the design of non-Zeno, stabilising switching law/controller pairs. This would involve splitting the system to different sets Q (say, Q_j), finding controllers v for which Proposition 5.3 may be applied, and using Lemma 7.2 to obtain non-Zeno switching laws. In this case, however, one would still need to show that Zenoness is not caused by the co-existence of the different pieces of the jigsaw (Q_i).

In [Liu et al., 2010], a second set of rules is produced, with the purpose of extending the result presented here, so that the requirement that $\bar{x}$ does not belong to the set Q is not needed. The idea behind the approach is to split Q into energy levels, in based on V_i (the storage function corresponding to the partition, whose elements K_J are used in the exposition above). These energy levels are take the form of 'rings' of stored energy (say, from 1 to 2 units, from 2 to 4 etc), and they are, by definition, compact sets (to be precise, their closures are compact). Then, in every one of these 'rings' one may apply Lemma 7.2 to conclude that, within each one of them, the energy decreases and Zenoness does not arise. If some extra care is taken, to ensure that, while moving between the rings, Zenoness is not caused, one may also conclude stability under the given feedback function v. In this case, the following type of rings will be used:

$$D_k = \{x \in X \mid 2^k < V_i(x) < 2^{k+1}, \quad k \in \mathbb{N}\}.$$

Then, the following rules may be used:

Algorithm 7.2 (Switching rule set II)**.**

1. *Pick some positive constant $\tau > 0$.*

2. *If the initial state, $x(t_a) \in D_{k_0}$, belongs to some element K_J of the partition, set $\sigma(t) = j$, for some $j \in J$.*

3. *Switch according to Switching rule set I (Algorithm 7.1), until $x(t_b) \in \partial D_{k_0-2}$.*

4. *If $t_b - t_a \geq \tau$, then set $t_a = t_b$ and return to the step 2.*

5. *If $t_b - t_a < \tau$, then keep σ unchanged until some time t_c, for which $t_c - t_a \geq \tau$, or until $x(\tau) \in \partial D_{k_0}$.*

6. If the first happens, then set $t_a = t_c$ and return to step 2.

7. If the second happens, then start switching according to Switching rule set I, until some t_d for which $x(t_d) \in \partial D_{k_0-2}$. Set $t_a = t_d$ and go to step 5.

∎

While the description of the rules is rather complex, their meaning is intuitively clear. The time is split into segments of length τ, and rule-set I is used in each of them, until the system moves to a lower 'energy level'. If this happens before τ, however, the switching stops, and the system is allowed to move back to higher energy levels. In the end of every segment, two things have happened: first, the system has moved to lower energy, and, second, τ units of time have passed. It is clear that, in this way, the system will eventually move to the equilibrium point. Further, since the initial energy level is never escaped, the system is also stable in the Lyapunov sense. Considering Zenoness, one only needs to prove that within every τ segment, only finite switches happen. This can be done by the argument used in Lemma 7.2, as the closure of every set D_k is a compact set.

The rules given here still depend on the assumption that $\partial K_{\emptyset}$, since, otherwise, one may not be able to argue that moving between elements of the partition (or, rather, their boundaries) takes a positive amount of time. Further, is should not go amiss that, in contrast with the first set of rules, the second set does not keep the switching rule of Theorem 3.1 active everywhere. Indeed, in step 5, the rule may be broken. The scheme, however, still guarantees that the energy decreases in a macro- level (in consecutive 'steps' of length τ).

While the results presented here were described as being parallel to those in [Liu et al., 2010], this does not mean that the discussion here is just a repetition of the results there. The authors of [Liu et al., 2010] use a different fundamental switching rule, and the partition on which they base their switching scheme is entirely different. To say more, the authors in that work base the partition, as well as the switching rule, to a minimum rule, which is a proper subset of the switching signals which are allowed in the present work. Still, it would be interesting to consider the possibility of applying the idea of an 'expanded' partition, which is introduce there, and its applicability in the case of the partition of Definition 3.1.

Overview

In this chapter, Zenoness properties were discussed, for the class of non-linear, dissipative, switching systems of (2.3). It was shown that, if one uses Theorem 3.1 or the other results of Chapter 3 to conclude that a given function v stabilises such systems, it is possible that the resulting restrictions cause Zeno behaviours to appear. Subsequently, rule-sets where introduced, in the form of Algorithm 7.1 and Algorithm 7.2, and it was proved (with Lemma 7.2) that, with the rules therein, one may, under quite permissive conditions, pick switching laws that are in accordance with the switching rule of Theorem 3.1, and non-Zeno.

Chapter 8

Discussion and Conclusions

In this Chapter, a summary of the results of the present work will be given, followed by some critical discussion. In particular, some discussion of the strengths and limitations of the proposed approach, as well as its placement within the area will be provided. Further, a brief outline will be given, of the possible directions of future work, which have been identified until now.

8.1   Summary of the results

In one sentence, the endeavour of this work was the following: to use the dissipativity properties of the system to establish criteria and design methods for the stabilisation of switching and hybrid systems. This problem was treated from two different directions. First, as an analysis problem, in which case the input is taken as a given, and second, as a design problem, in which case an input is to be constructed, with the objective of rendering the system stable.

To say more, in Chapter 3, the main tool of this work is introduced. This is a partition of the state space, introduced in Definition 3.1. This partition encodes information about the energy behaviour of the system, and, under some conditions, it can be used to 'filter-out' switching signals which, if chosen, would make the system unstable. For the results of Chapter 3, the partition is used to formulate conditions for stability under a given input signal, which is given in the form of a state feedback function. These results come in both local and global versions, and the energy decrease argument that underpins them is shown to hold for arbitrary subsets of the state space, regardless if they contain the equilibrium point. It is argued that for the chosen class of systems, these results are simpler to apply, both in the conceptual level and in the practical level.

In particular, it is shown that they offer a better method to establish stability in comparison to the result in [Zhao and Hill, 2008a], which is one of the most widely cited in the literature.

In Chapters 5 and 7, these results were explored further. In Chapter 5, it was shown that, instead of using just one partition, one may construct multiple partitions, corresponding to different inputs. Then, these partitions may be used to determine the family of admissible switching laws, either by using different partitions in different areas of the state space, or by using them interchangeably, everywhere. In Chapter 7, it was shown that, under some conditions, it may be certified that the families of switching laws allowed by the Theorems of Chapter 3 (and, by extension, Chapter 5) always contain elements that are non-Zeno; that is, they do not force infinite switchings in finite time intervals. These results further validate the claims of Chapter 3 on the usefulness of the results presented there, as they show that the relevant infrastructure is not tied the existence of a sole feedback function, which would correspond to a rather restrictive scenario. Further, they show that the families of admissible switching produced by the results of this work are useful, even in the general case, where the existence of non-Zeno, stabilizing signals may not be concluded in advance.

In Chapter 6, the perspective was flipped. Instead of taking a given control input, designing a partition based on that input, and asking if the partition has properties that render the system stable, the opposite way was taken. That is, starting with a system, and having a specific equilibrium and bound in mind, the question was posed if one may find an input, such that the corresponding partition has properties that guarantee stability. It was shown that, besides their own merit, these results further extend the applicability of the results in the previous Chapters, in the sense that they allow their application in a wider class of systems.

Finally, in Chapter 4, these results were extended to a class of hybrid systems. In particular, it was shown that, if the jumping is also available for control, then the partition of Definition 3.1 can be used in conjunction with a similar construct, governing the jumping. Using these constructs one may derive two types of conditions for the stabilisation of hybrid systems. One in the spirit of Theorem 3.1, and one which gives a lower bound to switching frequency, in line with the result presented in [Liu et al., 2007]. As part of the process, a new property was introduced, called flow dissipativity. It is argued that for the specific class of systems under study, the new property corresponds to a sensible compromise between traditional dissipativity concepts and applicability.

8.2 Discussion and position in the literature

Here, some general comments will be made, on the assumptions that underlie the results, as well as their position in the literature, in relation to other similar works.

The most significant assumption, which underlies this whole work has to do with the conception used for the systems under study. It is not uncommon in the literature to assume that hybrid and switching systems are characterised by domains; that is, subsets of the state space, associated with specific subsystems. This approach is taken by some works in the area of hybrid automata [Branicky et al., 1998], as well as in the framework developed in [Johansson and Rantzer, 1998]. Here, a different route is taken. From the viewpoint assumed here, a switching system is just a collection of 'normal' systems, which may be activated on demand, everywhere in the state space. A hybrid system is similar, with the additional characteristic that it also includes a jumping mechanism, which is also available on demand everywhere on the state space. In this description, then, the articulation of the elements (the subsystems) is weak, and, in principle, most behaviours are allowed. Further, it is assumed that the input, the switching law and the jumping instants are (potentially) selected by a user. That is, for instance, a subsystem may be activated or deactivated, if the user wants it too.

This viewpoint places the work presented here in the same space as the results in [Zhao and Hill, 2008a], discussed at length in Chapter 3, as well as the whole space of results based on dwell-time restrictions (for example, [Zhang and Gao, 2010; Hespanha and Morse, 1999]). The present work differs from the dwell-time results, because it gives state space, rather than time-based restrictions on the switching. And, it differs from the approach of Zhao and Hill, because it uses an entirely different type of bound-function (state, rather than time-dependent) for the cross-supply rates. It is argued (in Chapter 3) that this difference overcomes some significant difficulties, found in the application of the main result of [Zhao and Hill, 2008a], and, more generally, it allows for a simpler-to-check test for stability.

The results presented here depart from much of the existing literature, in the sense that they propose a co-design process for the system input and the hybrid components of the system (switching, jumping). In previous work, it is usually assumed that, either the switching law (and, where it exists, the jumping) is 'hard-coded' in the system (say, when the switching happens in accordance to pre-decided domains), or that the input is given, and only the switching is the subject of the design process. The work in [Johansson and Rantzer, 1998] is an instance of the first category, while the dwell-time results mentioned above exemplify the second one. The co-design process is implied

in the results of [Zhao and Hill, 2008a], and, in the wider sense, in works that employ state space partitions, such as [Peleties and DeCarlo, 1992] and [Pogromsky et al., 1998]. However, no explicit exposition exists in the literature.

Here, the design problem is treated from two angles. First, the possibility of obtaining families of switching rules for which the system is stable, when controlled by a given feedback function, is explored; this is the work presented in Chapter 3 and Chapter 5. Second, the reverse process, that is picking a feedback function for which such families are known to exist, is studied; this is the work in Chapter 6. By nature, this approach is more flexible, in two senses. First, since the design happens over a range of 'parameters', it is reasonable to expected that it yields results in more cases. To see that consider that, in the 'hard-coded' switching case, one might ask if, for a given input, an equilibrium is stable. In the methodology developed here, the equivalent question is if, for the same input, there are some switchings for which the system is stabilised. Of course, these results are not necessary (multiple storage function results seldom are), but, for 'well-behaved' cases, one may hope that the second approach will give the same information as the first, and more.

Another point of departure of the results presented here has to do with the conception of dissipativity it is based on. As explained in Chapter 2, a very permissive definition of dissipativity is considered for the case of switching systems, and the same is done in the hybrid case, introduced in Chapter 4. The idea behind this approach is that, instead of using the energies of the subsystems to build a system-wide energy, as it is proposed in [Johansson and Rantzer, 1998], or requiring that all the subsystems behave in a similar way, with respect to their energies, as it is proposed in [Zhao and Hill, 2008a], one may assert that the system is characterized, at the same time, by all the energies. Then, if only one is found to exhibit desirable behaviour, it can be concluded that the system itself also has this behaviour. This bottom up approach is related, but not identical to the one taken in [McCourt and Antsaklis, 2010a]. A positive outcome of this approach is that the results proposed here may be applied independently to different equilibrium points, and there is no assumption that a common equilibrium exists. As noted in Chapter 2, only a small subset of the available literature deals with the case of switching or hybrid systems for which the equilibrium points of the subsystems do not coincide. Therefore, this should be considered as another advantage for the method proposed here.

8.3 Limitations of the approach

In the previous section, and throughout this work, it has been argued that the results presented here are novel, and, to a degree, useful. This is not to say that there are no limitations to their application. The purpose of this section is to draw attention to some of those limitations, and try to set them into context.

The first limitation has to do with the lack of real-world applications. This is, in principle, an issue with the presentation, but it also indicative of the difficulty of applying the theory – finding real scenarios in which the results are applicable is not a trivial task. The largest part of the blame for this deficiency lays with the author, who is always fascinated by high theory, at the expense of usefulness. However, at least partially, this situation points to endemic problem in the area of hybrid systems. This claim is supported by a quick review of the latest results published in prestigious journals, such as Automatica ([Küsters and Trenn, 2018], [Lai et al., 2018] [Liu et al., 2018], all from 2018), and Nonlinear Analysis: Hybrid Systems ([Pang and Zhao, 2018], [Yuan and Wu, 2016], also from 2018); as well as many of the seminal publications of the area, including [Goebel et al., 2012] and [Branicky, 1998b]. These publications present deep and interesting theoretical results, but they showcase their applications with numerical examples, or occasionally, toy examples like the bouncing ball, and they give little guidance about any realistic scenario where the results could be put into use.

The implication here seems to be that the study area of hybrid systems produces more theory than what the current state of engineering practice is able to consume. There are, of course, exceptions. Reachability results, for instance, have been applied to a variety of real-world problems, including the case of track platooning and collision avoidance for aircraft. And, in the area of stability and dissipativity, one may find real examples, such as the drill string described in [Carter, 2013] and [Navarro-López and Laila, 2013]. Other works which include examples include [Denman, 2017] and [Navarro-López and O'Toole, 2018], in which mechanical systems with friction are discussed. A common thread, connecting all these examples is that they model systems by splitting their state space into domains, and assigning unique dynamics to each of these domains. While this approach is interesting and fruitful, it corresponds to a simplified case. That is, it makes use of only a subset of potential of the mathematical infrastructure of hybrid systems.

As explained in the previous section, this 'splitting' approach is not taken here. Instead, it is assumed that all the subsystems are available for activation on demand,

regardless of the current position of the state in the state space. It is argued that this approach is natural, at the conceptual level, and in line with much of the available literature, and that it offers some flexibility, lacking more restrictive formulations. Then, deriving the results presented here is a worthwhile effort, not just as an intellectual exercise, but also on a practical level. One may hope that, as the theory becomes more refined, and the available results more powerful, the appeal of the modelling frameworks on which they are based will increase, and, ultimately, more realistic examples will follow.

Another possible limitation, more specific to the work presented here, has to do with scaling. While it is easy to envision how the results may be applied in the case of two or three-dimensional systems, using visual aids such as those given in Examples 3.1 and 3.3, it is not readily apparent if high-dimensional systems may also be treated in the same manner. In particular, the problem that may arise is that the verification of conditions such as the one posed in Theorem 3.1 and its sister results may not be trivial in the general case. Note that, in essence, the problem is that of certifying that a collection of inequalities (of the form $r_{ij} - \psi > 0$) never hold together. In the lower-dimensional cases, this problem may be tackled by brute force. The area of interest may be filled with a grid of points and the value of each of the cross-supply rates may be calculated, and checked against the chosen ψ, assuming that the cross-supply rates and ψ are smooth (otherwise, one may be unable to find a grid that is dense enough to meet the required tolerance levels). This method quickly hits complexity constraints. For instance, taking 100 points in each direction yields 10^4 points in two dimensions and 10^6 in three.

While this is a real limitation of the proposed approach, its importance should not be overestimated. First, the results of Chapter 6 may be applied to obtain a controller for which the condition is known to hold. Then, the brute force method only needs to be applied to a small (possibly empty) subset of the initial area of interest. Second, in the relatively common case where the cross-supply rates and bound function ψ are polynomial functions in the elements of the state, formal methods may be used to verify that the inequalities do not hold together, using techniques such as the one presented in [Vorobjov and Grigoriev, 1988]. Finally, while verifying that the condition on the cross-supply rates holds is a necessary part of the process, it should be noted that it constitutes a form of pre-processing. The question if $K_\emptyset$ has the required form needs to be answered only once, in the beginning of the process, while, for the actual application of the method one only needs to keep track of the current values of the cross-supply

rates. For instance, one may start a simulation and implement the behaviour implied by the theorems, using only local information (the current values of the cross-supply rates and ψ), and without any additional information about the form of the partition.

Another point to be made also has to do with the model that was assumed for the behaviour of hybrid and switching systems, as described in this section; namely, that all the subsystems are always available. The results of the previous chapters are geared toward this type of systems, and, therefore, they are less appropriate in the alternative case, for which each subsystem has its own domain. This does not imply that systems of the second variety are out of the scope of the results presented here. They are not, and the way the partition is described in Definition 3.1 specifically allows (via *L*) for this possibility. However, in this case, some of the requirements of the theorem may be redundant or overly restrictive. To see that, observe that, in the model behaviour discussed in this work, it is necessary to have information on the energy behaviour of every one of the subsystems, everywhere in the space – any subsystem is a candidate for activation, so it is necessary to know how it will affect the stored energy. In the alternative model, however, it is redundant to consider the energy behaviour of a subsystem in an area where it will never be active. For such subsystems, the reader is advised to consider other methods to ensure stability, more along the lines of, for example, [Johansson and Rantzer, 1998].

Another point that deserves some mention is that the methods developed here may be less easy to apply, when the problem under consideration is the feedback interconnection of switching systems. This, however, is less a drawback of the specific method, and more a characteristic of the whole multiple storage (and also Lyapunov) function space. In general, the feedback interconnection of non-switching, dissipative systems are not themselves dissipative. Therefore, if one wanted to apply any multiple storage function result to an interconnection of switching systems, they would have to guarantee, first of all, that, for any possible combination of active subsystems, the interconnection is dissipative; evidently, such a step is not without complications. [Zhao and Hill, 2008a] and [Zhao and Hill, 2008b] offer an approach to that effect, while [McCourt and Antsaklis, 2010a] gives another.

A final pair of observations has to do with the use of dissipativity made in this work. First, one may claim the use of dissipativity made here is not true to the spirit of the multiple storage function framework. That is, since the conditions are mostly placed on one of the available storage functions, one may as well consider this a common storage function problem. This is true, to the extend that one can obtain such a function,

without the dissipativity definition used here. However, finding a function like this, for a switching or hybrid system is, in general, a much more demanding task than finding a storage function for one of the subsystems. That is to say that, without knowledge about the dissipativity of the subsystems, involved and unintuitive conditions would be necessary to ensure that a function of the required form is available.

The second observation is that, in the present work, dissipativity is only for the purpose of establishing stability, which makes it quire redundant. Indeed, instead of looking for storage functions, one could look directly for Lyapunov functions. This critique is also valid, for the present work, as well as the space of dissipativity in general. As a counter-point, one should consider the possibility of generalising the results presented here. For instance, picking different types of ψ functions in the formation of the partition of Definition 3.1, one may explore different types of behaviour (finite gain and input to state stability, for instance). This approach is a variation of the use of the partition made in the case of Theorem 4.2, and it indicates that, while the present work focuses on stability, the tools and techniques used may be re-purposed to establish different properties.

8.4 Future Work

From the preceding discussion, some directions, along which the results may be extended, have become apparent. Here, some discussion of the four main directions, which have been identified, will be offered. However, the reader should keep in mind that, by nature, any remark made here is speculative in nature, and subject to revision.

A first approach one may take in extending these results has to do with basic theory. In the Theorems presented here, observability and detectability properties of the systems were not taken into account. Such properties were included implicitly, by the requirement that the bound functions (ψ) are negative definite, but the exact repercussions of this step were not discussed at any length. Similarly, controllability properties were not discussed. Such properties have been the subject of extensive study in the literature, as discussed in Chapter 2, and it would be interesting to use the available results in the treatment of the present work.

An obvious side-effect of such an effort would be a possible strengthening of the results presented here. In particular, in the present work, it is assumed that, for asymptotic stability to ensue, the cross-supply rates have to be bound away from zero. This implies, in the case where the storage function of interest is differentiable, that the

derivative of the stored energy is always strictly negative. From the non-hybrid case, however, it is known that one may obtain LaSalle-style results, which allow one to conclude that an equilibrium is asymptotically stable, even in the case of where the bounds in that derivative are only non-positive. It is known since the early days of dissipativity theory, for instance in [Hill and Moylan, 1976] that such results are connected with the detectability properties of the system, further supporting the thesis that extending the present research in the direction of incorporating observability-like properties would be a worthwhile goal.

Another extension involves the possibility of automating the application of the results presented here. In Section 8.3, it was commented that, if the conditions of the results presented here (for instance, Theorem 3.1) are known to hold, one may design both simulated and real switching signals, without significant numerical overhead – the only requirement is that the evolution of the supply rate and the cross-supply rates is tracked. However, checking whether the conditions hold may be trickier, especially as the system dimensions and the number of subsystems grow. It is of interest, then, to explore if the validation of the conditions may be done programmatically. While such an endeavour may be impossible in the general case (more precisely, undecidable [Henzinger et al., 1995]), for particular types of cross-supply rates, it may be possible to solve this. In particular, if one had polynomial cross-supply rates, as is the case when the systems are linear, one could pose the problem as a satisfiability problem, where it is asked if it is possible to find some *x* for which a collection of (polynomial) inequalities are found to hold together. It is known that, in some cases, such problems can be addressed; for example, one may make use of the results in [Vorobjov and Grigoriev, 1988]. Evidently, implementing such a solution would significantly extend the scope of application of the results presented in this work.

Another direction of interest has to do with the possibility of re-purposing the infrastructure developed here, in order to accommodate more complex scenarios. Such a process would initially take two forms. First, one may try to determine if the partition of Definition 3.1 may be used (possibly, after some adaptation), to establish different properties. This would presumably look like the use of the partition shown in Theorem 4.2. It is clear that this may be done in the case of finite gain, and input-to-state stability. Similarly, it seems possible in the case where the property of interest is instability. Then, exploring these possibilities, along with others that may be incorporated with in this framework would be a potentially fruitful area of study. Second, the results of Chapter 4 have some potential for extension. It was pointed out that the results of that

chapter assume a relatively simple form for the systems under consideration – a form for which the jumping scheme is constant, regardless of the active subsystem. It would be relatively straightforward to extend the application of Theorems 4.1 and 4.2 to a richer kind of systems, for which each subsystem is associated with its own jumping scheme (in the notation of (4.2a), one would have multiple g_i, instead of g). Such an extension would only require slight changes in the notation, and the phrasing of the arguments. It would be equally interesting to explore if, for these cases, an augmented partition, or a new jumping partition (an extension of the set Λ of Theorem 4.1) may be used.

A final direction of future research, which is possibly the most important of those discussed here, is the search for real-world applications of the results and framework proposed here. As discussed in the previous section, the lack of such results is not a particularity of the work presented here, but rather a common characteristic in some areas of hybrid and switching systems. Regardless, finding such examples would significantly strengthen the claims made here, regarding applicability and ease of use. Particular directions for this search, which have not been examined here due to time and space constraints, include chaos synchronisation systems, as well as network systems (such as power networks), for which the network topology may change on demand, but other possibilities should also be examined.

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