DEFAULT CONTAGION MODELLING AND COUNTERPARTY CREDIT RISK

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Contents

Abstract **12**

Declaration **13**

Copyright Statement **14**

Acknowledgements **15**

1 Background **16**

1.1 Introduction 16

1.2 Counterparty Risk 18

1.3 Literature Review 22

1.3.1 Structural models 22

1.3.2 Reduced-form models 28

1.4 Summary and Thesis Layout 40

2 Reduced-Form Modelling **42**

2.1 Preliminary of Stochastic Process 42

2.1.1 Existence and Uniqueness of stochastic differential equation . . 43

2.1.2 The Cauchy Problem and Feynman-Kac representation 44

2.2 Pricing Credit Claims in Reduced-Form Models 45

2.2.1 Default times simulation 48

2.2.2 Recovery methods 49

2.3 The Pricing of Credit Derivatives and PDEs 51

2.3.1 Default bonds 52

2.3.2 Credit default swaps 58

2.3.3 Credit default swaps with unilateral counterparty credit risk . . 63

2.3.4 The PDE of unilateral CVA . . . 67
2.4 Summary . . . 70

3 Pricing CDS and CVA Numerically **72**
3.1 Introduction . . . 72
3.2 Preliminaries of the Finite-Difference Method . . . 73
3.2.1 Crank-Nicolson method for one-dimensional PDEs . . . 74
3.2.2 Numerical schemes for multi-dimensional PDEs . . . 75
3.3 Solving CDS Using Finite Differences . . . 79
3.3.1 Model specification . . . 79
3.3.2 Boundary conditions . . . 81
3.3.3 Numerical results . . . 84
3.4 Solving the CVA Using Finite Differences . . . 92
3.4.1 Boundary conditions . . . 93
3.4.2 Discretisation in the ADI scheme . . . 96
3.4.3 Numerical results . . . 99
3.5 Conclusions . . . 123

4 A New Default Contagion Model **124**
4.1 Introduction . . . 125
4.2 The Proposed Contagion Model . . . 128
4.2.1 Distributions for $\alpha^{\mathbb{Q}}$ . . . 130
4.3 A Two-Firm Model . . . 140
4.4 Numerical Results . . . 146
4.4.1 Survival probability . . . 149
4.4.2 CDS spreads . . . 158
4.4.3 Credit value adjustments . . . 163
4.5 Conclusions . . . 169

5 Default Contagion with Jumps **172**
5.1 Model Specification . . . 174
5.2 A Two Firm Model . . . 176
5.2.1 Survival probability . . . 178

5.2.2 Credit default swap 179
5.2.3 Credit value adjustment 180
5.3 Numerical Schemes 182
5.3.1 Numerical scheme for 1D-PIDE 183
5.3.2 Improved numerical scheme: Extrapolation 186
5.3.3 Numerical scheme for 2D-PIDE 194
5.4 Numerical Results 198
5.4.1 Survival probability 199
5.4.2 Fair swap spread 203
5.4.3 Credit value adjustment 206
5.5 Conclusion 210

6 Bilateral Counterparty Risk **213**
6.1 Principle of Monte-Carlo 214
6.1.1 Law of large numbers 214
6.1.2 Central limit theorem 216
6.1.3 Multi-companies default time simulation 217
6.1.4 Intensity Process Simulation 218
6.2 A Three Firm Model 222
6.3 Implementation 225
6.3.1 Default processes and default times simulation 226
6.3.2 Spread simulation 228
6.3.3 CVA and DVA Simulation 229
6.4 Convergence Analysis 232
6.5 CVA DVA and Fair CDS Spread Analysis 239
6.5.1 CVA, DVA gains and losses 241
6.5.2 CDS spreads and CVA/DVA Charges 248
6.5.3 Counterparty risk with Combinations of $\tilde{\alpha}$ and $\bar{\beta}$ 257
6.6 Conclusion 261

7 Summary and Future Research **264**
7.1 Summary 264
7.2 Future Research 266

A ADI scheme Parameters for CVA **283**

B Lemmas in default contagion model **285**

Word count 53257

List of Tables

3.1 The convergence and computational time of semi-analytic solution . . . 89

3.2 Finite-difference solutions with varying $\Delta\lambda$ and Δt and computational times . . . 90

3.3 Successive difference in finite-difference solutions at λ and t spaces . . . 91

3.4 Global Corporate Average Cumulative Default Rates (%) from 1981 to 2014 and default probabilities given by the CIR intensity given parameters 102

3.5 Convergence of ADI scheme at time space . . . 105

3.6 Convergence of ADI scheme at λ space . . . 105

3.7 Convergence of ADI scheme at time space with correlation . . . 106

3.8 Volatility effects on CVA (Bps) . . . 118

4.1 Statistics of exponential random variable . . . 132

4.2 Statistics of gamma random variable . . . 135

4.3 CVA under default contagions, percentage changes and computational times . . . 167

5.1 Numerical error of OS and IMEX scheme . . . 185

5.2 Numerical solutions and errors of and without approximations to the integral term . . . 190

5.3 Computational times and numerical errors for CDS with and without extrapolation method . . . 195

5.4 Parameters of five cases . . . 205

6.1 Model Parameter for Convergence Test . . . 234

6.2 The CVA (bps) for a one year CDS protection computed by finite-difference scheme . . . 234

6.3 Convergence of CVA Simulation with 95% confidence levels 235
6.4 Intensity Parameters for CVA DVA gains and losses analysis 241
6.5 CVA DVA gains and losses with increasing default contagions $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$ 241
6.6 CVA and DVA with increasing volatility under raising degree of default contagion. σ is applied to all firms' default intensity. 243
6.7 CVA and DVA (bps) with increasing $\bar{\lambda}$ under raising degree of default contagion. 246
6.8 CVA profit and loss against changes in credit risk with and without default contagions 248
6.9 DVA profit and loss against changes in credit risk with and without default contagions 249
6.10 Fair Spread S^* and CVA DVA behaviours with $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$. . . . 250
6.11 The impact of indirect default contagion on CVA and DVA 253

List of Figures

3.1 The ratio of the dominate components of PDE (3.9) at $\lambda \to \infty$ 83

3.2 The Convergence of CDS's semi-analytic solution implemented with Trapezoidal rule 85

3.3 The finite-difference solution of CDS at τ space 86

3.4 The finite-difference CDS solution at λ space compare upper boundary conditions 87

3.5 The finite difference CDS numerical errors at λ space compare boundary conditions 88

3.6 The convergence of numerical error using the heuristic Robin boundary condition at $\lambda \to \infty$ 89

3.7 The first and second derivative with respect to λ_2 of $P(\tau_2 < \tau_1, \tau_2 < 5)$ at large λ_2 96

3.8 A comparison of CDS spreads given the model with a selection of sovereign CDS spreads up to 5 years in EMEA at 6^{th} of July 2016. . . . 101

3.9 A comparison of survival term-structure given the model parameters with the market implied survival term-structure of a selection of sovereign CDS in EMEA at 6^{th} of July 2016. 103

3.10 The finite-difference solution to CVA with respect to time to maturity $(T - t)$ 107

3.11 The finite-difference solution to CVA at λ_1-dimension 108

3.12 The derivative of CVA with respect to λ_1 109

3.13 The finite-difference solution to CVA at λ_2-dimension 110

3.14 The derivative of CVA with respect to λ_2 111

3.15 The impact of correlation on the probability that the counterparty to be the first to default in five years. P($\rho = 0.5$)-P($\rho = 0$) 114

3.16 The change of the CVA of a 5 years CDS protection with 0.5 correlation. CVA($\rho = 0.5$)-CVA($\rho = 0$) 116
3.17 CVA of a CDS with correlated default intensities 117
3.18 Changes to CVA from raising counterparty's volatility. CVA($\sigma_2 = 0.2$)-CVA($\sigma_2 = 0.1$) 121
3.19 Changes to CVA from raising referencing firm's volatility. CVA($\sigma_1 = 0.2$)-CVA($\sigma_1 = 0.1$) 122

4.1 Example of exponential distributions 132
4.2 Example of Weibull distributions 133
4.3 Example of gamma distributions 135
4.4 Example of generalised gamma distributions 137
4.5 Example of skewness of distributions 138
4.6 Example of kurtosis of distributions 139
4.7 Firm 1 Survival Probability under three models and $\bar{\alpha}_{1,2} = 0.02$ 151
4.8 Firm 1 Survival Probability under three models and $\bar{\alpha}_{1,2} = 0.05$ 152
4.9 Firm 1 Survival Probability with different mean reversion speed κ_1 and multiple $\bar{\alpha}_{1,2}$ 152
4.10 Joint Default Probability against reversion speed κ 154
4.11 Joint Default Probability against Default Contagion Strength $\bar{\alpha}_{1,2}$. . . . 154
4.12 Joint Default Probability Premium against Maturity 155
4.13 Increment default probability with default contagion $\bar{\alpha}_{1,2}$ 157
4.14 Marginal Default Probability Premium against Maturity 157
4.15 Fair CDS swap rates of maturity up to 30 years 160
4.16 The swap rate premium against the size of default contagion 161
4.17 The swap rate premium against mean-reverting speed 162
4.18 Post-contagion CDS value distribution 165
4.19 Expected post-contagion CDS value distribution 166
4.20 The change of the CVA of a 5 years CDS protection with $\bar{\alpha}_{1,2} = 0.05$, CVA($\bar{\alpha}_{1,2} = 0.05$)-CVA($\bar{\alpha}_{1,2} = 0$) 170

5.1 Compare extrapolation solutions with ordinary solutions 189
5.2 Compare CDS extrapolation solutions with ordinary solutions 192

5.3 Compare errors of with different boundary 193
5.4 An illustration of available solutions for approximating the integral term in two-dimensional PIDE 197
5.5 Firm A survival probability decrement with default contagion $S^{\mathcal{A}}(\bar{\alpha}_{A,B} = 0.1) - S^{\mathcal{B}}(\bar{\alpha}_{A,B} = 0)$ (black) and exogenous jumps $S^{\mathcal{A}}(\bar{\beta} = 0.1) - S^{\mathcal{B}}(\bar{\beta} = 0)$ (red) 201
5.6 Increment in default probability term-structure in four cases. 202
5.7 The fair CDS spreads S^* of five cases in table 5.4 206
5.8 Trade-off between $\bar{\beta}$ and $\bar{\alpha}_{A,B}$ with same CDS protection spread. 207
5.9 The CVA of five cases in table 5.4 209
5.10 CVA under combinations of external shocks $\bar{\beta}$ and default contagions $\bar{\alpha}$ as shown previously in figure 5.8 210

6.1 Convergence of CVA in Number of simulation 236
6.2 The convergence of CVA in step size 237
6.3 The convergence of fair spread S^* against N 238
6.4 CDS values $V^{\mathcal{C}}$ against referencing firm's intensity λ_{ref} at counterparty's default time $\tau_{cp} = 5$. 245
6.5 The changes in CDS value after adding default contagion and is traded at fair swap rate. 252
6.6 The change of CVA/DVA behaviour against $\bar{\alpha}_{ref,inv}/\bar{\alpha}_{ref,cp}$ with indirect default contagions 255
6.7 CDS spread sensitivity to mean-reverting speed and default contagions 256
6.8 CVA and DVA with default contagions $\bar{\alpha}_{ref,cp}$, $\bar{\alpha}_{ref,cp}$ under different reversion speed κ_{ref} 256
6.9 This figure shows 5 years CDS protection's value against reference firm's default intensity in default state D with different values of mean reversion speed κ_{ref}. Three CDSs are traded at corresponding fair rates as shown in the figure. 257
6.10 The combinations of $\bar{\beta}$, $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$ with the identical CDS spread $S^* = 300$ 258
6.11 The calibration errors in $\bar{\beta}$, $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$ 259

6.12 The CVA of the fair 5-year CDS contract ($S^* = 300$ bps) with different combinations of $\bar{\beta}$, $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$. 260
6.13 The DVA of the fair 5-year CDS contract ($S^* = 300$ bps) with different combinations of $\bar{\beta}$, $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$. 260

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Default Contagion Modelling and Counterparty Credit Risk
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This thesis introduces models for pricing credit default swaps (CDS) and evaluating the counterparty risk when buying a CDS in the over-the-counter (OTC) market from a counterpart subjected to default risk. Rather than assuming that the default of the referencing firm of the CDS is independent of the trading parties in the CDS, this thesis proposes models that capture the default correlation amongst the three parties involved in the trade, namely the referencing firm, the buyer and the seller. We investigate how the counterparty risk that CDS buyers face can be affected by default correlation and how their balance sheet could be influenced by the changes in counterparty risk.

The correlation of corporate default events has been frequently observed in credit markets due to the close business relationships of certain firms in the economy. One of the many mathematical approaches to model that correlation is default contagion. We propose an innovative model of default contagion which provides more flexibility by allowing the affected firm to recover from a default contagion event.

We give a detailed derivation of the partial differential equations (PDE) for valuing both the CDS and the credit value adjustment (CVA). Numerical techniques are exploited to solve these PDEs. We compare our model against other models from the literature when measuring the CVA of an OTC CDS when the default risk of the referencing firm and the CDS seller is correlated.

Further, the model is extended to incorporate economy-wide events that will damage all firms' credit at the same time-this is another kind of default correlation. Advanced numerical techniques are proposed to solve the resulting partial-integro differential equations (PIDE). We focus on investigating the different role of default contagion and economy-wide events have in terms of shaping the default correlation and counterparty risk.

We complete the study by extending the model to include bilateral counterparty risk, which considers the default of the buyer and the correlation among the three parties. Again, our extension leads to a higher-dimensional problem that we must tackle with hybrid numerical schemes. The CVA and debit value adjustment (DVA) are analysed in detail and we are able to value the profit and loss to the investor's balance sheet due to CVA and DVA profit and loss under different market circumstances including default contagion.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Background

1.1 Introduction

Credit risk measurement and credit derivatives pricing is today one of the most intensely studied areas in quantitative finance. The popularity of credit derivatives is due to the fact that they allow market participants to easily trade and manage credit risk. However, some exotic products are very difficult to price and it is also difficult to manage their risk. For instance, the complexity of pricing a Collaterallized Debit Obligation (CDO) tranche is that it involves a risk of multiple defaults during the same time period, which is known as default correlation. Correctly modelling multiple defaults is vitally important in the aftermath of the 2008 crisis. Not only are credit derivatives like a credit default swap (CDS) and CDO at risk from default, but also other derivatives, such as forwards and variety swaps that are traded over-the-counter (OTC) are at risk in the form of counterparty credit risk. This is the risk that the counterparty fails to fulfil their obligation. According to a report from the International Swaps and Derivatives Association (ISDA), the losses incurred in the US banking system due to counterparty defaults on OTC derivatives is reported to be \$2.7 billion from 2007 through the first quarter of 2011, see ISDA (2011a) for more details, while the losses are as much as \$50 billion to non-deposit-taking institutions such as investment banks, ISDA (2011b). Even without defaults, the deterioration of credit quality is enough to cause losses to the credit-adjusted value of trades. During the crisis of 2008, the downward trend of financial markets was accompanied by extensive credit deterioration of financial institutions, which in turn caused damage to credit markets. Following the bankruptcy of Lehman Brothers, massive financial institutions, including Merrill Lynch, AIG, Freddie Mac, Fannie Mae, all came within a whisker of bankruptcy and had to be rescued. This phenomenon has led to further debate about the correlation between companies' default risk and the correlation between default risk and financial asset prices, such as interest rates and equities. From the prospective of OTC trades, correlations may lead to the exposure of a counterparty that is adversely correlated with the credit quality of that counterparty. For instance, imagine if an investor held bonds issued by Washington Mutual and decided to hedge its credit risk by buying CDS protection from Lehman Brothers and then 2008 comes. The panic during the 2008 crisis reminded investors, as well as regulators, of the importance of properly modelling, pricing and managing credit risk, especially when credit correlation exists. The example of buying a CDS referencing to Washington Mutual from Lehman Brothers is a typical example of how default correlation exaggerates the counterparty risk faced by the CDS buyer. The default correlation between Lehman Brothers and Washington Mutual makes the risk-adjusted CDS held by the investor nearly worthless. This is because at the time when the CDS becomes valuable, when Washington Mutual is more likely to default, this is also the time that the CDS seller tends to not fulfil his obligations. The phenomenon of an increase in the derivative's value accompanied by higher default probability of the counterparty is of concern in the financial industry and is frequently referred to as *wrong-way* risk. Developing models to capture *wrong-way* risk while measuring counterparty risk has become the main challenge in credit-risk modelling.

The remainder of this chapter is organised as follows. We will first introduce the concepts of counterparty credit risk and its measures. The measurements of counterparty credit risk are generic and model independent. Next, we review two kinds of credit risk model, namely structural models and reduced-form models. The review of reduced-form models will be looked at in more details, because the majority of research in this thesis is based on this approach.

1.2 Counterparty Risk

Counterparty credit risk arises in OTC derivatives trades, where the counterparty, from whom we buy the derivative contract, may not fulfil its obligations. Counterparty risk is the risk of losses due to the default of a counterparty, which is an aspect of credit risk. If the counterparty defaults at time τ before the portfolio/product expires at time T, the loss to the surviving party is everything but the recovery associated with the portfolio/product if the portfolio/product has positive value to the surviving party. However, if the portfolio has negative value to the surviving party, a liability in other words, then the surviving party does not suffer extra losses. Indeed, the surviving party will have to pay back his liability to the defaulted party in order to close out the transaction. Let $(\Omega, \mathcal{G}_{t\geq 0}, \mathbb{Q})$ represent a filtered space with a finite horizon t, which is the space containing all random events that underlie the stochastic evolution of a financial market. So all our random variables will be $\mathbb{G} = (\mathcal{G})_{t\in\mathbb{R}_+}$ measurable and random times are $\mathbb{R}_+$-valued $\mathbb{G}$-stopping times. Let us denote $\Pi(t,T)$ as a position at time $t \leq T$ with final maturity T, which gives the position holder discounted random cash flows. The credit exposure, according to Brigo et al. (2013), is defined as

$$\max\{\mathbb{E}^{\mathbb{Q}}{}_s\left[\Pi(s,T)|\mathcal{G}_s\right], 0\} \text{ for } \forall s \in [t,T], \tag{1.1}$$

which is the expected value of the portfolio at time s under the risk-neutral measure $\mathbb{E}^{\mathbb{Q}}$ conditional on the filtration $\mathcal{G}_s$. If we assume the counterparty is able to recovery a constant fraction between 0 and 1 of his liability, namely the recovery rate, the loss due to counterparty default associated with the portfolio is

$$(1 - R(\tau)) \times \max\{\mathbb{E}^{\mathbb{Q}}{}_\tau\left[\Pi(\tau,T)|\mathcal{G}_\tau\right], 0\} \text{ for } \forall \tau \in [t,T], \tag{1.2}$$

where $R(\tau)$ is the recovery rate of the defaulting party at the default time.

Credit Value Adjustment (CVA) is the quantitative measurement of counterparty risk. This is defined in Crépey et al. (2014) and Gregory (2012) as the difference between the value of a position traded with a default-free counterparty and the value of the same product when traded with a defaultable party. We denote the investor as I and the counterparty as C and assume the investor themselves is default-free, the CVA associated with a single default-free counterparty is written as

$$\mathrm{CVA}(t,T) = 1_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}{}_t\left[D(t,\tau)\left(1 - R(\tau)\right)1_{\{\tau<T\}}\max\{\Pi(\tau,T),0\}|\mathcal{G}_t\right], \tag{1.3}$$

where $D(t, \tau)$ is the discount factor from default time to present time t. The credit value adjustment is essentially the expected loss if the counterparty defaults prior to the trade ending at T. It is only when the contract has positive value to the investor that the investor suffers from counterparty default loss. Therefore, CVA has some similarity to financial options and could be hedged in terms of options. The similarity between CVA and the price of a contingent credit default swap (CCDS), which pays exposure at default, is discussed by Crépey et al. (2014); Brigo and Pallavicini (2008).

The CVA above assumed that the investor is default-free, which is also referred to as the unilateral CVA (UCVA). However, if two firms are doing a trade, each side will consider the default risk of the other. When considering the default risk of both the counterparty and the investor, the investor only suffers counterparty default risk when the investor's default time is later than that of the counterparty. The CVA computed by the investor should be lower than the UCVA because the set $\{\tau_C < T, \tau_C < \tau_I\} \subseteq \{\tau_C < T\}$. The difference between the two sets is the probability that the investor defaults before the counterparty, where the investor will not have the counterparty default loss. Therefore, the CVA with bilateral default risk is formulated as

$$\mathrm{CVA}(t,T) = 1_{\{\tau_C > t\}} \mathbb{E}^{\mathbb{Q}}{}_t \left[D(t,\tau_C) \left(1 - R_C\left(\tau_C\right)\right) 1_{\{\tau_C < T, \tau_C < \tau_I\}} \max\{\Pi(\tau_C, T), 0\} | \mathcal{G}_t \right]. \tag{1.4}$$

If the investor can default, the investor may benefit from their own default because he only pays back a fraction of his liability rather than in full. The benefits or gains to the investor associated with a portfolio or a product when these are liabilities (negative value) to the investor. Consistent with the definition of CVA, debit value adjustment (DVA) measures the gains due to the investor's own default, which is

$$\mathrm{DVA}(t,T) = -1_{\{\tau_I > t\}} \mathbb{E}^{\mathbb{Q}}{}_t \left[D(t,\tau_I) \left(1 - R_I\left(\tau_I\right)\right) 1_{\{\tau_I < T, \tau_I < \tau_C\}} \min\{\Pi(\tau_I, T), 0\} | \mathcal{G}_t \right]. \tag{1.5}$$

The gains to the investor due to the investor's own default is equivalent to the losses to the counterparty.

The value of a derivative without considering the counterparty party credit risk is usually referred to as the clean price of the derivative, for example, options' prices in the Black-Scholes world. However, if we consider the fair value of the derivative adjusted by the counterparty credit risk, the risk-adjusted value is expressed as

$$\tilde{\Pi}(t,T) = \Pi(t,T) - \mathrm{CVA}(t,T) + \mathrm{DVA}(t,T), \tag{1.6}$$

where $\tilde{\Pi}(t,T)$ is the credit risk adjusted price and $\Pi(t,T)$ is the clean price without counterparty credit risk.

DVA is a controversial quantity as discussed in Brigo et al. (2013). First of all, a firm can benefit from being more risky. All things being equal, increases in the investor's default probability of the investor would make the default indicator

$$1_{\{\tau_I<T,\tau_I<\tau_C\}}$$

in the DVA increase as well and therefore make the DVA term larger. Recall that in equation (1.6) the DVA term can increase the risk-adjusted value of the portfolio. Consequently, the firm can gain in the risk-adjusted-portfolio value as the result of being more likely to default. Secondly, the investor might need to sell protection against themselves in order to hedge the DVA. However, no market participants will buy the CDS with the referencing and the seller being the same firm, because no payments are available to the buyer if the referencing firm defaults. Alternatively, the investor could buy back their own bonds but the corporate bond market is not very liquid.

In above definitions, CVA and DVA are evaluated under the assumption that there exists a single risk-free interest rate and all firms can borrow and lend funds at the risk-free rate. In other words, the funding cost of firms does not account for their individual credit risk. Consequently, cash flows from derivative contracts and cash flows conditional on counterparty's credit event are discounted by the risk-free rate. However, the cost of funding is tightly linked to one's credit risk, which is also known as the funding constraint. For example, a firm who has lower credit worthiness has to borrow money at a higher rate. This implies the firm has to hedge and fund their position at a higher cost. The value of cash flows will not be symmetric relative to two parties with different credit risk because cash flows are discounted at different rates. How to measure counterparty risk under the funding constraint is another important area of counterparty risk research. This is also referred by Crépey et al. (2014) as the multi-curve setup, where the price of a product will be different from the buyer and the seller's prospective. The analysis of counterparty risk and the evaluation of value adjustments under funding constraint are extensively discussed by Crépey (2011); Crépey et al. (2013, 2014); Crépey (2015a,b). Crépey (2011) proves that the value process of a hedged counterparty risky contract with funding constraints can be described in terms of a nonstandard backward stochastic differential equation (BSDE). The position under consideration consists of the OTC derivative contract, the hedging portfolio and the funding portfolio. The BSDE is not driven by Brownian motions and/or Poisson processes but the dividend processes of the derivative contract, funding assets and default close-out. It is nonstandard due to the randomness of default time and the dependence of terminal condition on the portfolio. Crépey et al. (2013) show the value adjustments pricing problems, including CVA, DVA, liquidity value adjustments and replacements cost, can be reduced to Markovian BSDEs. Burgard and Kjaer (2011) also incorporate funding constraint by assuming a bank can fund its derivative position from an internal funding desk. They derive the PDE for an option's value with counterparty risk and funding constraint under the Black-Scholes complete market and the PDE for CVA is obtained. A comparison study is carried out by Crépey (2015a) against the approaches of Burgard and Kjaer (2011) and Crépey (2011). Compared with Burgard and Kjaer (2011), Crépey (2011) allow a firm to default on its funding portfolio, which is implicitly disregarded by Burgard and Kjaer (2011). When a bank is able to default on the funding asset, the nonlinearity of the funding close-out cash-flow leads to extra complexity in the replication strategy thus making the problem more difficult to solve. Using the BSDE approach, Crépey (2015b) analyse the CVA under funding constraint. Given they interpret the CVA as the price of the contingent credit default swap (CCDS), the dynamic of the CVA can also be described by a BSDE and thus the solution of the CVA can be characterised as the unique solution of a semilinear PDE.

In this thesis, we will assume that firms are able to obtain funds at the risk-free interest rate and focus on measuring the CVA and DVA with default correlation models. The concepts of counterparty credit risk is model independent. In order to quantify CVA and DVA, one has to specify what derivative transactions are being traded between the investor and the counterparty as well as the default model being used. In the following sections, we have a literature review of the two commonly used default models, namely structural and reduced-form models.

1.3 Literature Review

1.3.1 Structural models

The philosophy underlying structural models is to assume there is a fundamental process that drives the total value of the firm, which determines the event of default. Merton (1974) assumes that the firm defaults only at maturity when the firm value is lower than the bond principle. Merton's model is extended by Black and Cox (1976), Leland (1994), Longstaff and Schwartz (1995) and Zhou (1997) to allow for more realistic assumptions such as early default, optimal capital structure and jump in firm value.

In Merton's model, the firm's value is composed of equity and liability (i.e. $V_t = E_t + C_t$). By assuming that a firm's liability is composed of a single maturity zero-coupon liability, default happens if the terminal firm value V_T is lower than the liability's face value F_T. At maturity, equity holders can choose to either pay back the principle F_T and receive the remaining firm value $V_T - F_T$ or leave the firm to creditors. Clearly, the equity holder will choose to default when the firm value V_T is less than debt face value F_T. The right to choose makes the equity effectively a European call option and therefore it can be priced as such. We can then define corporation debt as a long position of default-free debt and short a put option to equity holders.

$$E_T = \max\{V_T - F_T, 0\} \tag{1.7}$$

and the liability's value C_T at time T is

$$C_T = \min\{V_T, F_T\} = F_T - \max\{F_T - V_T, 0\}. \tag{1.8}$$

It is assumed that the firm's value follows a Geometric Brownian Motion (GBM),

$$dV_t = rV_t dt + \sigma_v V_t dW_t. \tag{1.9}$$

where r is the risk-free interest rate, σ_v is the volatility of the firm's value and W_t is a standard Brownian motion.

Given these simplifying assumptions, both equity and debt can be priced using the formula derived by Black and Scholes (1973), although the GBM here represents the dynamics of asset value rather than equity. Therefore, V_t and σ_v must to be estimated as V_t cannot be directly observed. However, the model of Merton (1974) is based on weak assumptions including a much simplified capital structure and default time. Empirical tests carried out by Delianedis and Geske (2001) found that Merton's model can only explain a small fraction of credit yield spread when compared to market data; the price of bearing default risk is underestimated within Merton's framework.

Based on Merton's framework, Black and Cox (1976) argued creditors can force the firm into liquidation before maturity if the firm value drops below a certain level, namely a safety threshold. Safety thresholds work as a floor value for a bond to guarantee earlier cash flows to bond holders, avoiding receiving too low recovery at maturity. In this case, the default can happen not only at maturity, but also at any time prior to debt maturity. In other words, the default time τ is the first time when the firm value V_t is less than the threshold L_t, which is

$$\tau = \min\{t > 0 | V_t < L_t\}.$$

By assuming the threshold to be a time-varying function of exponential form related to the debt face value namely, $L_t = \rho F e^{-r(T-t)}$, then the safety threshold is a constant fraction of the present value of the promised final payment. This is also known as a $first\ passage\ model$. Here ρ is the recovery rate if the debt defaults at maturity. The exponential form makes the threshold relatively lower if the time to repayment is long, which allows the firm to possibly recover from bad performance.

Introducing a safety threshold does complicate the model somewhat, since the time at which the firm defaults will not be known a *priori*. The way to derive an analytical price of equity (and any derivative) in this model is similar to the methods we use to solve for barrier options. The probability of default is equivalent to the probability of GBM touching the threshold.

The resulting default probability is always higher than or equal to Merton's model, and we can see that this model will be retrieved as a special case with $L_t = 0$. The result of including the barrier into the option valuation is that it will no longer be monotone with respect to the volatility σ_v. The model directly prices the bond by dividing it into three parts, namely

$$\begin{aligned} C_T &= F_T 1_{\{\tau > T\}} + V_T 1_{\{\tau = T\}} + V_T 1_{\{\tau < T\}} \\ &= F_T - \max\{F_T - V_T, 0\} 1_{\{\tau = T\}} + V_T 1_{\{\tau < T\}}. \end{aligned} \tag{1.10}$$

Clearly, (1.10) is the final payoff of a portfolio comprising a default-free loan with face value F_T maturating at T, a short European put on the firm with strike F_T and a long position of a European down-and-in call on the firm value with zero strike price. Therefore, valuation of the corporate debt C_t is equivalent to valuation of this portfolio. Compared to Merton's model, the first passage model tends to produce lower credit yield spreads. As safety thresholds work as a protection mechanism, bonds with a default barrier have higher values than those without. However, a firm's default probability is higher than Merton's model. The Black and Cox model is the foundation for many extensions, since it is more realistic in terms of default time.

We see that the default threshold feature is popular in the literature and there are many extensions, such as stochastic interest rates (Longstaff and Schwartz, 1995), jump processes (Zhou, 1997, 2001), equity maximization and optimal capital structures (Leland, 1994, 2004) and stochastic volatility (Fouque et al., 2006). All these models are much more complex than the original Merton model and sometimes they may require numerical techniques for solution.

The first passage models come with some desirable features, but they also raise the problem of how to accurately designate a safety threshold. There are several ways in which academics have sought to deal with this. Safety thresholds can be exogenously fixed, as proposed by Black and Cox (1976) who used a time-dependent deterministic function. Alternatively, to make models even simpler, Longstaff and Schwartz (1995) set the barrier as constant and equal to the firm's liability. Longstaff and Schwartz argue that discounting introduces complexity without improving performance, since only the ratio V_t/F is important to credit default. Briys and De Varenne (1997) generalised the safety threshold as a fraction of the firm's liability. Another approach is for the barrier to be modelled as a random process when incorporating incomplete information, as proposed by Duffie and Lando (2001). Without the ability to assess default barriers, investors are uncertain about the distance to default, resulting in high credit yield spreads.

Longstaff and Schwartz (1995) extend the problem to include two types of debt in a simplified maturity structure, by assuming that the firm issues multiple bonds with different maturities. All existing bonds default simultaneously when the firm defaults the bond with the earliest maturity. Longstaff and Schwartz also extend the constant interest rate assumption in Black and Cox (1976) to consider a Gaussian-type stochastic interest rate. This paper derived closed-form solutions for fixed-rate and floating-rate risky bonds. More importantly, the correlation of interest rate and firm value was explored. The relationship between initial short rate and risky bond price is negatively correlated in general. The risk-free rate plays two roles in the value of a risky bond. An increasing risk-free rate will push down the bond price and increase the growth rate of the firm in a risk-neutral world (i.e. the drift in firm value process). Both effects narrow down the risky bond price while the latter lowers the credit yield spread. There is an exception that, for extremely risky bonds, the price can be an increasing function of risk-free rate. Longstaff and Schwartz (1995) also showed that the value of a risky bond depends on the ratio of the firm value to debt face value rather than their absolute values and argued this ratio is sufficient for risky-bond valuation. This property has some important implications for the model. Firstly, the price of a coupon bearing corporate bond can be priced as the sum of corporate zero-coupon bonds conditioning on the ratio V_t/F. In other words, the default status of earlier coupons does not affect the valuation of the latter coupons. On the contrary, in Merton's model, a coupon bond can only be priced as a compound option since the value of the latter coupon payment depends on the default status of the former coupon. This means if the firm defaults at its first coupon, the rest of the coupons are valueless. Secondly, it is sufficient to set the safety threshold to be constant rather than time dependent.

There is a common drawback shared by all of the above *first-passage* models, namely they produce close to zero credit yield spread for short maturity bonds. However, market bond prices have consistently been shown to give a significant yield spread even for short-term bonds. This phenomenon results from the unpredictability of default. When assuming that the asset process follows a diffusion process, or geometric Brownian motion to be precise, the asset process moves along a continuous path that is gradually approaching the critical value before default. This means that investors are able to predict default with increasing precision. As a result, the price for bearing default risk in the near future can be relatively low with known distance to default.

In Zhou (1997) and later Zhou (2001) the authors try to avoid default predictability by introducing a jump process. A firm's default, it is argued, is not only driven by a firm's value continuously going down and crossing a barrier, but it is also affected by unpredicted sudden events. In this model, the firm's value can jump below the barrier causing default without approaching. The firm-value process is given by

$$dV_t = V_t(\mu - \lambda_\pi \nu)\, dt + V_t \sigma_v \, dW_t + (\pi - 1)\, dY_t \tag{1.11}$$

where dY_t is a Poisson process with intensity λ_π and π is the log-normally distributed jump amplitude with expected value $\nu + 1$

$$ln(\pi) \sim N(\mu_\pi, \sigma_\pi), \quad \mathbb{E}[\pi] = \nu + 1 \Rightarrow \nu = e^{\mu_\pi + \frac{1}{2}\sigma_\pi^2} - 1 \tag{1.12}$$

This type of stochastic process is a compound Poisson process such that jump time and jump size are random. Adding the jump will avoid predictability since a Poisson jump is an unpredictable event even in the near future. Under a jump process, default can happen instantaneously because of a sudden drop in firm value. Consequently, this model results in higher credit yield spreads for short-term bonds whilst maintaining appropriate levels for the longer terms. This model shows that the short-term spread is mostly influenced by the jump size volatility σ_π, while diffusion volatility σ_v has more impacts on long-term bonds. Zhou (2001) explains this phenomenon from the property of jump diffusion. For a long-maturity bond, the effect on default probability from jump size volatility σ_π is largely limited by the jump intensity λ_π which is usually small. Consequently, jump size volatility has limited effect on default probability. However, short-term credit-yield spread is increased as a result of recovery value. Jump size volatility determines the firm value after a jump across a default barrier, which is the recoverable value for the creditors. The larger the jump size volatility, the further the firm value may be below the default barrier on average. Consequently, bond holders are expected to receive less after default when the jump size is large. The increased credit yield spread works as compensation for the possibility of a lower recovery.

On the other hand, Leland (1994) presents a model with a totally new default mechanism. He assumes the firm is holding a perpetual debt, which requires continuous coupon payments. This model links the bond value and optimal leverage ratio to not only asset value, asset volatility and risk-free rate, but also tax benefit, bankruptcy cost and leverage ratio. The firm must finance the coupon by issuing new equities. Assuming a firm's asset value as a GBM, Leland derived the corporate bond price, followed by the total firm value, whose difference is the equity's value. In order to set the default threshold V_B to be consistent with positive equity value for any asset value greater than the threshold, the lowest possible value for threshold is

$$\left.\frac{\partial E(V_t, V_B, T)}{\partial V_t}\right|_{V_t=V_B} = 0. \tag{1.13}$$

Intuitively, at the time of default, any movement in the firm's asset value is not related to the equity value. The default condition (1.13) is also known as the endogenous default threshold as compared to previous models, where the threshold is exogenously determined. According to this argument, the default threshold is derived, and is a time-function of coupons but not of the current value of firm's asset.

Leland and Toft (1996) extend Leland (1994) to a much richer debt structure and allow the optimal maturity of debt as well as the optimal amount of debt. The firm can continuously sell new debt to repay old debts and maintain the optimal capital structure. Therefore, the firm defaults when issuing new bonds, in order to raise the firm's value, cannot increase the value of equity. In other words, issuing new bonds does not benefit the shareholders. Therefore, it is optimal not to issue new bonds to repay old ones and close the firm. Leland and Toft show that longer maturity bonds have more tax advantages, but also higher agency costs, which could be substantial. Leland and Toft also show optimal capital structure reflects a trade-off between tax advantages, bankruptcy costs and agent costs. They derived the optimal leverage ratio and corporate bond prices for any maturity. Bankruptcy is determined endogenously and will depend on the maturity of debt as well as its face value. The default condition implies a time-independent default barrier similar to Longstaff and Schwartz (1995). However, the difference here is that the boundary V_B depends on bond maturity, tax advantages and bankruptcy costs. For a capital structure with long-term bonds, the barrier can be less than face value of the liabilities. In contrast, if the bond maturity is extremely short, the default barrier will be greater than the face value due to bankruptcy cost.

In the research of Leland and Toft (1996), their model is compared with that of Longstaff and Schwartz (1995) in terms of default probability prediction. Leland

and Toft found both exogenous and endogenous models have underestimated default probability in the short run, which remains an important research subject.

Leland (1994) can be considered to be a breakthrough for structural models. He considered the dynamics of capital structure as a result of shareholders' behaviour and provides a framework with more business insight than had previously been used. Based on his model, there are many extensions, including but not limited to, Anderson and Sundaresan (1996), Collin-Dufresne and Goldstein (2001), Fouque et al. (2006) and McQuade (2013).

1.3.2 Reduced-form models

Compared to structural models, the default mechanism in reduced-form modelling is chosen to be totally different in order to make the default time is unpredictable. Reduced-form modelling is analogous to term-structure modelling of the risk-free rate. The similarity between term-structure modelling and credit spread modelling is first discussed by Jarrow and Turnbull (1995). Beginning with a single-step discrete time set up, Jarrow and Turnbull show a defaultable zero-coupon bond can be viewed as a default-free zero-coupon bond denominated by a foreign currency, whose value is transformed into the domestic currency by a forward exchange rate. The exchange rate, in this sense, is actually the combination of default probability and default payment. Their approach is then extended to a two-step set up to show how a defaultable zero-coupon bond could be viewed as a fraction of a default-free zero-coupon bonds with some amount of coupons before maturity. They argue that a default bond can be analysed as if it is a default-free bond. The value of coupons and the fraction of principle is determined by the probability of going into the default state and the payment at that state, which is again the default probability and default payment. After generalising their analysis to continuous time, they show there exists a term-structure, in addition to risk-free term-structure, for the return of default bonds with a different maturity from a single issuer. A vulnerable option whose issuer may default can be priced and hedged using bonds, which have the same credit risk. Their work reveals that credit risk can be represented as an additional yield spreads in default bonds and we can bootstrap a firm's credit-yield spreads from the bonds issued by this firm. Following on from their previous work, Jarrow et al. (1997) propose a Markov

Chain model for credit ratings. In the model a firm has some probability of jumping between different credit ratings until finally being absorbed by the default state. The form of the survival probabilities derived from this continuous time model becomes similar to the current reduced-form models. Jarrow et al. (1997) also discuss the estimation of the transition matrix, which may be estimated implicitly from defaultable zero-coupon bonds data or historical data. One drawback of the above Markov Chain model is requiring the independence of default process and interest rate, and there is a strong empirical evidence that default probability of corporate bonds varies with the business cycle. The number of defaulting companies is higher during recessions and is therefore usually accompanied by lower recovery rate and interest rates lower than their long-term mean, see Altman and Kishore (1996).

Lando (1998) present a more coherent framework, which generalises previous methodologies and propose that the credit spread can be modelled using the process from the Cox (1955). Also known as the doubly stochastic process, the *Cox* process is a Poisson process, where both Poisson arrivals events and arrival intensity are stochastic. Defining the first jump as a default event, Lando derived the three building blocks for pricing credit claims, which are payment contingent on default event, payments contingent on survival and continuous payments conditional on survival. Under the *Cox* process formulation, defaultable bonds can be priced as if pricing non-defaultable bonds, so that previous term-structure models can be applied to the default intensity. More importantly, this framework allows a correlation between default spread and the interest rate. As long as the non-negative condition is met, the default intensity process can be generic. For example, the default intensity can be a linear function of other market variables, such as the interest rate, volatility, equity price and so on. Lando showed how the Markov Chains model of Jarrow et al. (1997) can be generalised into his framework. We will use default intensity, default process and credit spread interchangeably to represent the stochastic process driving the jump of the *Cox* process.

Since term-structure models of interest rate can be directly used for modelling default intensity, Duffie (1998) proved the affine jump-diffusion (AJD) term-structure model can also be applied for pricing credit derivatives. The correlation of credit spread and interest rate is achieved by sharing the same AJD process. For instance,

$$r(t) = a_r(t) + b_r(t)X_t,$$
$$\lambda(t) = a_\lambda(t) + b_\lambda(t)X_t,$$

where X_t is a k-dimensional independent AJD process. Duffie showed that pricing the defaultable bond reduces to the computation of a one-dimensional integral of a known function. Further, under a multi-firm economy, Duffie proved the default intensity of the first default event is the sum of all firms' default intensities and provided an algorithm for simulating the first default time. However, one condition in a *Cox* process is that the jumps arrival intensity must be non-negative. Due to the non-negative condition, not all term-structure models are applicable to credit spread, such as the Vasicek (1977) model.

Apart from the correlation of default intensities and interest rates, correlations are observed also among the default times of companies. Default correlation or dependency is the phenomenon that the number of default companies cluster around a period of time, which is usually during an economic crisis, see Das et al. (2007). Duffie and Singleton (1998) first consider correlated defaults in reduced-form modelling. After reviewing several term-structure models applicable to default intensities, Duffie and Singleton consider the losses in a portfolio of 100 corporate bonds, where default correlation may lead to substantial losses. Duffie and Singleton use a mean-reverting log-normal process for default intensities with correlated Brownian Motions. In their numerical examples, correlating the Brownian Motions in obligers' default intensity processes does not make significant changes in the portfolio loss. After discussing the techniques for credit spread modelling in an AJD process and the algorithms for simulating multi-company defaults, Duffie and Garleanu (2001) analysed the systemic and idiosyncratic risk of Collateralized Debt Obligations (CDO) in a multi-factor AJD model, *i.e.*

$$\lambda_i(t) = X_c(t) + X_i(t),$$

where $X_c(t)$ and $X_i(t)$ are independent AJD processes with exponential jump sizes. The $X_c(t)$ is the common factor shared by all companies' intensity to represents systemic risk. Therefore, this model allows common jump times for all intensities, which shows one can have a simple but useful model for simulating correlated defaults with a relatively high degree of default-time correlation. They found that raising the percentage of common factors, or higher default correlations, causes the senior tranche price to reduce in value and the junior tranche to become more expensive.

They also study the distribution of the number of defaults of 100 companies within 10 years. Comparing four cases, where firms are subject to similar default risk but with different jumps arrival frequency, volatilities and jump sizes, they show that if there is low systemic risk, all situations have similar distributions implying defaults are independent. But if the common factor $X_c(t)$ is large, jump volatility and diffusion volatility lead to the similar probability that the number of default firms above 60. However, increases in jump volatility tends to cause a higher probability of a small number of defaults, from 10 to 30, than diffusion volatility. On the other hand, increases in diffusion volatility leads to a higher probability that 30 to 60 firms default than the jump volatility. Compared with large jump sizes, higher jump arrival frequency leads to higher degrees of default dependence.

In reduced-form modelling, the payment at default time, default recovery, has to be specified exogenously, which leaves room for different modelling approaches. In previous research, including Jarrow and Turnbull (1995); Jarrow et al. (1997), when a company defaults at its bonds, the firm recovers an exogenously specified fraction of the treasury bond at its default time, namely *recovery of treasury*. On the other hand, in the Duffie (1998) models the creditor can receive a fraction of the face value, namely *recovery face value*. The choice as to whether to use *recovery of treasury* or *recovery face value* is mainly influenced by the computational burden attributed to *recovery of treasury*, because not only default time but also other market information, such as risk-free interest rate and equity prices, up to default time matter. Duffie and Singleton (1999) propose *recovery market value* such that the recovery is in terms of a fraction of market value. Compared to *recovery of treasury*, if the default asset is a corporate bond, the difference is whether the credit risk of the bond should be taken into the valuation of the defaulted bond. Using *recovery of treasury*, the debtor is expected to recover an amount higher than the current defaultable bond. *Recovery market value* has legal and computational advantages. As argued in Duffie and Singleton (1999), it matches the legal structure of OTC derivatives when considering counterparty credit risk. More importantly, they prove that, when assuming *recovery market value*, it is possible to combine the default payment and non-default payment of a defaultable bond. The valuation of a corporate bond with *recovery market value* reduces to evaluating a non-defaultable bond with risk-adjusted discount rate, $R_t = r_t + \lambda_t L_t$, where L_t is the loss ratio. Consequently, the payment conditional on the corporate defaults can be eliminated. Numerical experiments are carried out to show the difference between using *recovery face value* and *recovery market value*. They correlate interest rate and the mean loss process $h_t = \lambda_t L_t$ in a multi-factor CIR model, *i.e.*

$$\begin{aligned} r_t &= a + Y_t^1 + Y_t^2 - Y_t^0 \\ h_t &= bY_t^0 + Y_t^3, \end{aligned}$$

where Y_t^0, Y_t^1, Y_t^2 and Y_t^3 are independent CIR processes and a and b are constants. They show that the corporate bond price is similar using the *recovery market value* and *recovery face value* if the term structure of credit spread is upward slopping. Otherwise, the bond price with *recovery market value* will be higher because the default event is more likely to occur earlier than later than the default loss with *recovery market value* lower. Guo et al. (2009) propose a new method to model the recovery process. In their models, default works as a trigger for the recovery process. After a firm defaults, its bonds become bonds with random maturities, representing the time to overcome financial distress, and the recovery process starts to affect its value. There are two states after default, the firm may either keeps its solvency and pays a higher fraction K of their debt, like debt reorganization, or go into bankruptcy paying a lower fraction R. The recovery rate depends on the firm's value process (modelled as a regime switching model) after default, using the firm's asset and liability as variables and taking advantage of a structural model.

Since Duffie and Singleton (1998) found correlating Brownian Motions among firms' default intensities cannot produce default correlations that match empirical studies, a branch of research in reduced-form modelling has centred around default correlation modelling and its applications. Default dependence is important; because multi-firms credit derivatives, such as CDO tranches and k^{th}-to-default swaps, the joint distribution of a collection of default times is important. Moreover, in terms of counterparty credit risk, the probability that one party defaults prior to others determines the amount of counterparty risk. As shown by Duffie and Garleanu (2001), correlation can be achieved by modelling intensity processes sharing common state variables, which are also known as *conditional independent* models. Intuitively, the arrivals of firms' Poisson processes are independent, conditional on individual's intensity process, although intensity processes are correlated.

Moreover, some researchers seek other approaches to model default correlation. Copula functions have been used in multivariate statistics for modelling correlated random variables. Li (1999) is the first attempt to use copula functions to model the correlation of default times and to price credit derivatives. Copula functions are multi-variate functions, which are able to transform the marginal default/survival probabilities into a joint distribution. Essentially, a copula is a function that links univariate distribution to joint multivariate distribution. The dependence between the marginal distributions linked by a copula is characterized entirely by the choice of the copula. The joint survival probability of multiple-firms is given by the result of a differentiable copula function, which takes each firm's marginal survival probability as inputs, namely

$$\mathbb{P}^{\mathbb{Q}}(\tau_1 < t_1, ..., \tau_k < t_k, ...\tau_n < t_n) = C(\mathbb{P}^{\mathbb{Q}}(\tau_1 < t_1), ..., \mathbb{P}^{\mathbb{Q}}(\tau_k < t_k), ...\mathbb{P}^{\mathbb{Q}}(\tau_k < t_k)), \tag{1.14}$$

where $C()$ is a copula function. Li (1999) estimates marginal survival probabilities implied by the market using the approach of Jarrow and Turnbull (1995) and shows how the *Bivariate Gaussian* copula produces joint one-year survival probability. The one parameter in the *Bivariate Gaussian* copula describes the degree of correlation is estimated from asset prices. Li (1999) also proposes the default simulation algorithm, which draws a vector of random variables $(u_1, ..., u_k, ...u_n)$ from a $(0, 1)$ correlated uniform distribution to represent survival probabilities of the n firms and are correlated by a copula function. Then, default times can be constructed, accordingly given the market implied survival probabilities. This simulation algorithm is very general and can work with any specification of the default intensities. Using the above algorithm and assuming a flat credit spread, Li evaluates the CDS spread where the seller of the CDS and its referencing firm are correlated. Applying a similar approach, Frey et al. (2001) use different copula functions to produce the loss distribution of a loan portfolio. However, none of the above literature considers the dynamics of default intensities.

Schönbucher and Schubert (2001) extend the above approaches by allowing the default intensity to be stochastic and individually calibrated to their market implied survival term-structure. They show the use of copula functions can be incorporated into the reduced-form modelling framework by Lando (1998) and point out that the advantage of using a copula is being able to independently estimate parameters of the copula function and the firms' default intensities. This independence makes the copula approach especially flexible for modelling. Consequently, it is possible to simulate joint defaults using the algorithm from Duffie and Garleanu (2001) with the vector of uniform variables drawn from copula functions in Li (1999). Rogge and Schönbucher (2003) use *Archimedean* copulas, which admits an explicit formula, to price multi-name products and argue that the *Gaussian* and *student-t* copula do not imply a realistic dynamic process for default intensities. The effects of using copula functions to price multi-name credit derivatives such as Basket Default Swap (BDS) and CDO tranches is considered by Galiani (2003). Galiani presents an analysis of the use of *Gaussian* and *student-t* copula functions to price CDO tranches and find the price for the equity tranche is lower and the price is higher for the senior tranche when using *student-t* copula. Because the *student-t* copula is characterised by fatter left and right tails, which is a higher probability of joint survive and default, it can efficiently capture the risk that a large number of firms default that the *Gaussian* copula cannot. In terms of the copula correlation parameter, both equity and the Mezzanine tranche prices decrease with higher correlation because the probability of the loss is less than 14% of the portfolio is less. On the contrary, the senior tranche, which covers loss beyond 15%, is monotone, increasing with correlation. After the pricing problem of single and multi-name credit derivatives with copula functions as discussed above, some research has investigated the application of copula functions to counterparty credit risk. Brigo and Chourdakis (2009) considered the unilateral counterparty risk of a CDS contract when the referencing firm and the CDS seller's default probabilities are correlated by a bivariate Gaussian copula. They derived the semi-analytical solution to the CDS contract conditional on the CDS seller have defaulting, which is used for evaluating the loss due to the seller defaults. They show that both credit spread and default correlation in the copula have a considerable impact on the counterparty risk. One problem raised by using copula functions, as pointed out by Brigo and Chourdakis

(2009), is that, when default intensity volatilities are low, the CVA drops significantly if the Gaussian copula correlation parameter is above 80% while it was previously monotonically increasing. The reason is that when default processes are not volatile and have high default correlation, the scenarios in simulation are likely to be either joint default or no defaults at all. Therefore, the number of sample paths that count for counterparty risk is lower, leading to a drop in the CVA. This means that, although the default losses are higher, they very rarely happen and therefore the expected value is reduced. Brigo and Capponi (2010) and Brigo et al. (2014) extend the previous model to consider the defaults of both CDS buyer and seller with a multivariate Gaussian copula function and collateral. Crépey et al. (2014) present a dynamic Gaussian copula model to evaluate CDS and CDO tranches and the unilateral CVA without funding constraint.

Apart from using exogenously specified multivariate functions, some approaches focus on modelling the *Cox* processes. Lindskog and McNeil (2003) introduce a common Poisson shock model. A firm's default time is no longer be driven by a single Poisson processes but multiple Poisson processes representing the arrival of firm-specific as well as economy-wide events. Shocks from an economy-wide Poisson process may cause joint default whenever jumps occur. Further, the economy-wide shocks are not necessarily fatal. Even if the Poisson event, which represents economy-wide events, arrives, it only possibly causes the firm to default with a pre-determined probability. For example, for firm i, its Poisson process is

$$N_i(t) = \sum_{k=1}^{m} \gamma_{i,k} N_k(t),$$

where $\gamma_{i,k}$ is the probability the k^{th} Poisson event leads to firm i's default. Lindskog and McNeil prove that the total default intensity to a firm is

$$\lambda_i(t) = \sum_{k=1}^{m} \gamma_{i,k} \lambda_k(t),$$

where $\lambda_k(t)$ is the Poisson jump intensity of Poisson process $N_k(t)$. Applying predetermined probabilities that make the Poisson events fatal is equivalent to lowering the rates that the Poisson events arrive. The degree of default dependence relies on the probabilities that a specific shock leads to those firms' default. For instance, if a Poisson process $N_k(t)$ is shared by all firms and its event arrival is with probability one to make firms default $\gamma_{i,k} = 1$ for all i, then all firms' default time will be exactly the same. However, Lindskog and McNeil (2003) did not model the dynamics of default intensity processes but Liang and Wang (2012) extend the model of Lindskog and McNeil to allow jump intensities to be depend on a stochastic process, namely the common shock model with regime-switching. In this model, there is a Markov Chain driving the changes of economic states. Default intensities remain unchanged until the economy switches from good to bad or the other way round. Regime switches are interpreted as changes in macro-economic conditions and were previously used for other aspects of financial modelling, such as Buffington and Elliott (2002); Elliott et al. (2005). Liang and Wang derived the joint distribution of firms' default times using their proposed model and price basket default swaps with a closed-form solution. Around the same time, Dong et al. (2014a) use the common shock model with a regime switching framework to study the counterparty risk of a CDS contract. They measure the CVA and investigate the parameter sensitivity. Compared to common shock models without regime switching between a good and a bad economy, this model produces a much higher swap rate. Dong et al. found that the initial state, transit rate between regimes as well as correlation level all have a strong impact on CVA. The netting and margining impact on the CVA relative to a portfolio of CDSs under common shock model is considered by Crépey et al. (2014). Since common shock model allow joint defaults to be driven by the Poisson process that are shared by all firms, the unilateral CVA increases significantly. Although netting the long and short CDS positions with the counterparty can greatly reduces CVA, counterparty risk is not likely to be migrated by margining due to joint default events.

The *common shock* approach models the default of a firm that is driven by multiple *Cox* processes whose event arrivals are not necessarily leading to a default. On the other hand, as a *Cox* process is a Poisson process with a stochastic intensity, we can also model the stochastic default intensities of the firms'. Jarrow and Yu (2001) was the first paper to introduce a default contagion model based on reduced-form modelling framework of Lando (1998). In Jarrow and Yu's model, default correlation is incorporated by constructing a direct impact from defaulted firms to the surviving firms. One's default intensity is characterised as the sum of one idiosyncratic factor and the factors that only become a component of the firm's default intensity after others' default events. For instance, the default intensities of firms A and B are modelled as

$$\lambda_A(t) = a_1 + a_2 1_{\{t>\tau_B\}}, \tag{1.15}$$

$$\lambda_B(t) = b_2 + b_2 1_{\{t>\tau_A\}}, \tag{1.16}$$

where a_1, a_2, b_1, b_2 are assumed to be positive constants in their work. This is attempting to capture the case where firm A's defaults on its bond held by firm B, then firm B's default probability may jump higher due to the loss in the firm A's bond, and vice versa. However, deriving firm A's marginal survival probability requires knowledge of B's default times, which in turn depends on A's default time. This leads to a recursive problem also known as *looping default*. We can see the probability of $\tau_A > T$ is depending on τ_A itself

$$\begin{aligned} P(\tau_A > T|\mathcal{F}_t) =& 1_{\tau_A>t} E^{\mathbb{Q}}\left[e^{-\int_t^T a_1 + a_2 1_{\{s>\tau_B\}} ds} \Big| \mathcal{F}_t \right] \\ =& 1_{\tau_A>t} E^{\mathbb{Q}}\left[e^{-\int_t^T a_1 ds} \int_t^T e^{-\int_t^s b_1 + 1_{\{\tau_A>u\}} du} b_1 e^{-\int_s^T a_2 du} ds \Big| \mathcal{F}_t \right]. \end{aligned} \tag{1.17}$$

Jarrow and Yu (2001) do not solve this problem in full but present solutions by defining two categories of firm, namely primary and secondary. Default contagion spreads from primary firms to secondary firms but not the other way round. This assumption will remove the dependence of firm B's default time on A's. Therefore, A's survival probability can be solved after knowing B's default time distribution. However, the restriction to one-way dependency is not sufficient to model the dependence among firms of similar size. Collin-Dufresne et al. (2004) suggested a measure change which eliminates the default indicator in order to solve the *looping default* problem. This absolutely continuous probability measure is defined by

$$\left.\frac{d\bar{\mathbb{P}}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = Z_t := \frac{1_{\{\tau>t\wedge T\}}}{e^{-\int_t^T \lambda(s) ds}}. \tag{1.18}$$

This changes of probability measure is equivalent to attaching zero probability to all events in which a default occurs before maturity. The default indicator in the pricing formula can be absorbed into the Randon-Nikodym density process using a measure change. It is also called survival measure for the reason that it concentrates on the event that a firm survives until maturity. Collin-Dufresne et al. (2004) showed that the marginal survival probabilities can be expressed as simple analytic solutions. Equation

(1.17) can be solved as

$$P(\tau_A > T|\tau_A > t) = E^{\mathbb{Q}}\big[1_{\tau_A>T}\big|\mathcal{F}_t\big]$$
$$=E^{\mathbb{P}_{A,B}}\left[e^{-\int_t^T a_1(s)+b_1(s)ds}\middle|\mathcal{F}_t\right] + E^{\mathbb{P}_A}\left[e^{-\int_t^T a_2(s)ds}\int_t^T e^{-\int_t^s b_1(u)du}b(s)e^{-\int_s^T a_2(u)du}ds\middle|\mathcal{F}_t\right], \tag{1.19}$$

where the first part is expectation under the survival measure of both firms $P_{A,B}$ and the second term is under firm A's survival measure P_A.

Using the framework of Lando (1998) and survival measure, Leung and Kwok (2005) investigate the counterparty risk of CDS contracts with the default contagion model of Jarrow and Yu (2001). Leung and Kwok analysed the excess swap premium required for buying a new CDS after the CDS seller defaults. Apart from CVA, the excess swap premium is another measure of losses to the CDS buyer due to the CDS seller defaults. The results from Leung and Kwok (2005) illustrate that the level of counterparty risk has significant impact on the excess swap premium, which means the CDS buyer has to pay considerable extract amount for buying the same CDS after the seller defaults. Bao et al. (2010) use the survival measure approach to give an analytic solution to survival probabilities and guaranteed debt with CIR-type stochastic default intensities and constant default contagion jumps. In this model, two firms' default intensities are two independent CIR processes before any firm defaults. After a firm has defaulted, the surviving firms' default intensities are the sum of their individual default intensity process plus the defaulting firm's intensity process. For example,

$$\lambda_A(t) = a(t) + 1_{\{\tau_B<t\}}\eta_B b(t)$$
$$\lambda_B(t) = b(t) + 1_{\{\tau_A<t\}}\eta_A a(t),$$

where $a(t)$, $b(t)$ are CIR processes, τ_A, τ_B are default times of firm A and B and η_A and η_B are constants. However, the existence of the default firm's default intensity after the default event is questionable. Bao et al. (2012) extend the model of Bao et al. (2010) to consider stochastic interest rate and the correlation between interest rate and default intensities. Positive correlation is accounted for by formulating interest rates as a linear combination of the intensity processes. Due to the tractability of the model, the market value of counterparty default-free CDS can be derived with the help of the survival measure, leading to a semi-analytic solution for the CVA. Also the bilateral counterparty risk of a CDS contract is studied by Wang and Ye (2013).

In order to study the correlation between interest rate and default intensities, Wang and Ye specify the default intensities of three firms as simple functions of the interest rate process $r(t)$, which is a jump-Vasicek process with constant default contagions. Under a three-firms framework, Wang and Ye study the fair price of CDS contracts under defaultable CDS buyer and seller instead calculating CVA and DVA. Due to the closed-form solution for jump-Vasicek process being available, the fair CDS spread can be derived in closed-form and they found the impact from correlation between interest rate and default risk is significant in CDS pricing. However, the model of Wang and Ye violates the non-negative constraint of default intensity modelling, which we will discuss in the following chapter. Leung and Kwok (2009) model default dependency in terms of sharing external events. In this model, default intensities are modelled as time-varying functions, which switch to a higher level after external events. The arrival of external events are modelled as a Poisson process with affine jump-diffusion arrival intensity, which is the only state-variable in their model. The model restricts the number of external events to be 1, which allows the calibration of the time-varying intensity functions to be matched to the implied survival probabilities exactly. Dong et al. (2014c) combine the model of Leung and Kwok (2009) and the default contagion models of Jarrow and Yu (2001) to allow both the default contagion between firms and an external shock. The arrival intensity of external events is modelled as a Markov chain with finite states. The CDS value process is derived as a semi-analytic solution with or without counterparty risk. Dong and Wang (2014) consider a contagion model with regime-switching. The default intensities are driven by both economic states, which are modelled as a homogeneous Markov Chain, as well as the default state of other firms. Using the survival measure, they derived a closed-form formula for the survival probabilities and the CDS swap rate is obtained by means of Laplace transforms of the integrated intensity process. Wang and Ye (2013) extend the contagion model of Leung and Kwok (2005) to a three-firm framework. Rather than including other firms' default indicator functions in each firms' default intensity process as Jarrow and Yu (2001), these default intensities are described in three different states where all three firms alive, one of them defaulted and two of them defaulted. When a firm defaults, the surviving firms' default intensity jumps to the corresponding state. The technique of survival measure can still be applied and the survival probabilities as well as the fair CDS spread are still tractable.

1.4 Summary and Thesis Layout

In this chapter, we have introduced the concept of counterparty credit risk and its measures CVA and DVA, which are the possible loss due to counterparty default and possible gain due to the default of the investor itself on an OTC transaction. Then we reviewed two credit risk models, the structural and reduced-form models. We are particularly interested in reduced-form modelling due to its ability to match observable market data or market implied survival probability. The calibration problem under reduced-form framework will be no more than a least square minimisation with the set of parameters. As long as market implied survival probabilities are matched, exotic products can be priced with no-arbitrage. On the other hand, the capital structure, a firm's asset value as well as volatility are essential elements to structural models. Unfortunately, all of these are non-observable and the estimation of those inputs is itself a challenging topic.

Default correlation modelling is currently under much discussion, especially their application to counterparty risk. We reviewed four kinds of correlation model, conditional independent default, the Copula method, common shocks and default contagions. Among these models, the mechanism of default contagion modelling is the most intuitive and easy to interpret. However, contagion models commonly have calibration problems because how much damage a firm's default can bring to others is not observable in market. In addition, current contagion models have made many simplifications in modelling default intensities as well as the form of default contagions in order to maintain traceability. In this thesis, we will abandon traceability while introduce a more realistic formulation of default contagion and incorporate another source of default correlation in addition to default contagion.

The layout of this thesis is given as follows. In Chapter 2, we give the background theories of reduced-form modelling, and we show the derivation of Partial Differential Equations (PDE) for pricing default bonds, pricing CDS with and without counterparty risk and measuring the CVA due to buying a CDS protection. Then the numerical schemes for solving the PDEs will be discussed in Chapter 3 including their efficiency and convergence. Then we propose a new default contagion model in Chapter 4. We solve the corresponding PDEs and investigate the properties of a firm's marginal survival probabilities, joint survival probabilities, CDS spread and CVA under the proposed new model. In Chapter 5, the proposed default contagion model is extended to consider the impacts from the arrival of economy-wide events, which we also refer to as external shocks or external events, on firms' default intensities. Under the new model, we extend numerical schemes for solving two-dimensional PDEs to Partial-Integro Differential Equations (PIDEs) to solve the resulting two-dimensional PIDEs in the new model. Further, we propose an extrapolation method to accommodate the boundary problem when solving PIDEs. Our new numerical scheme is shown to be effective and efficient in solving our problems, especially combined with our extrapolation techniques. Bilateral counterparty risk is studied in Chapter 6, where the deafult contagion model in Chapter 5 is extended to a three-firm framework. We propose a hybrid numerical scheme, which combines Monte-Carlo simulation with finite-difference to price CDS spread and solving CVA and DVA under the default contagion model. The behaviour of the CDS spread, CVA and DVA will be studied under our model. Finally, the conclusions for the thesis are given in Chapter 7.

Chapter 2

Reduced-Form Modelling and Pricing Derivatives

In Chapter 1, we gave an introduction to credit risk, counterparty credit risk and structural models, reduced-form models. In this chapter, we first give some key theorems in stochastic calculus that will be used frequently in this thesis. Then we explain the foundations of reduced-form modelling for pricing credit claims in Section 2.2. In Section 2.3, we discuss the PDEs, which we must solve to price defaultable bonds, buyer CDS, buyer CDS with counterparty risk and the CVA due to holding a buyer CDS.

2.1 Preliminary of Stochastic Process

In this thesis, we model stochastic processes of default time in order to capture our desired features. However, there are certain conditions that a stochastic process must satisfy in order to emit a solution, which will put constraints on our model. More importantly, the existence and uniqueness of a stochastic process are essential for the validation of some theorems. In this section, we outline the conditions for a stochastic differential equation to have a solution as well as the Feynman-Kac Theorem, which is frequently used in the derivation of partial differential equations in the rest of this thesis.

2.1.1 Existence and Uniqueness of stochastic differential equation

Suppose we are in the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, a n-dimensional stochastic differential equation is an equation of the form

$$dX_t = \alpha(X_t, t)dt + \beta(X_t, t)dW_t, \text{with } X_0 = x_0 \text{ and } 0 \leq t \leq T \tag{2.1}$$

where $\alpha(X_t, t) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\beta(X_t, t) : [0, t] \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ are given functions called the drift function vector and the diffusion function matrix respectively. W_t is an n-dimensional vector of standard Brownian motions.

In addition to equation (2.1), we must provide an initial condition $X_0 = x$. The filtration $\mathcal{F}_t$ is generated by the sample path of the Brownian motions $0 \leq t$ and the initial position of the process X_0. The problem then is to find, at time t, the solution X_t satisfies,

$$X_t = x_0 + \int_0^t \alpha(X_s, s)ds + \int_0^t \beta(X_s, s)dW_s. \tag{2.2}$$

The existence and uniqueness of a solution X_t satisfying (2.2) requires the drift and volatility functions to satisfy conditions, for which we refer to Steele (2001) Theorem 9.1 and the main outline of the analysis is repeated below.

Theorem 2.1.1 (existence and uniqueness)**.** *If the coefficients of the stochastic differential equation (2.1) $\alpha(,) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\beta(,) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ be measurable functions satisfying the linear growth condition*

$$||\alpha(x, t)||^2 + ||\beta(x, t)||^2 \leq D(1 + ||x||^2), \qquad x \in \mathbb{R}^n, t \in [0, T] \tag{2.3}$$

for some constant C and the Lipschitz condition

$$||\alpha(x, t) - \alpha(y, t)||^2 + ||\beta(x, t) - \beta(y, t)||^2 \leq C||x - y||^2, \qquad x, y \in \mathbb{R}^n, t \in [0, T] \tag{2.4}$$

for some constant D. Then the stochastic differential equation (2.1) exists a unique continuous solution X_t adapted to the filtration $\mathcal{F}_t$ and is uniformly bounded in L^2:

$$\sup_{0 \leq t \leq T} \mathbb{E}[X_t^2] < \infty. \tag{2.5}$$

The first condition (2.3) ensures the process X_t does not tend to infinity so that a solution may exist for any t. Next, the second condition (2.4) will ensure that the solution is unique.

2.1.2 The Cauchy Problem and Feynman-Kac representation

Consider a n-dimensional Markov process X_t and functions $f(x) : \mathbb{R}^n \to \mathbb{R}$, $g(x,t) : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and $k(x,t) : [0,T] \times \mathbb{R}^n \to [0,\infty)$ be Borel-measurable bounded functions and satisfy

$$|f(x)| \leq L(1 + ||x||^{2P}) \quad \text{or} \quad f(x) \geq 0, \text{for } \forall x \in \mathbb{R}^n \tag{2.6}$$

and

$$|g(x,t)| \leq L(1 + ||x||^{2P}) \quad \text{or} \quad g(x,t) \geq 0, \text{for } \forall 0 \leq t \leq T, \forall x \in \mathbb{R}^n, \tag{2.7}$$

for some $L > 0,\ P \geq 1$.

Theorem 2.1.2 (Feynman-Kac theorem)**.** *Under the assumptions the linear growth condition (2.3), suppose that* $v(x,t) : [0,T] \times \mathbb{R}^n$ *is* $C^{1,2}([0,T] \times \mathbb{R}^n)$ *continuous and satisfies the Cauchy problem*

$$\frac{\partial v}{\partial t} + \mathcal{A}(v) + g(X,t) - k(X,t)v = 0, \quad \text{in } [0,T) \times \mathbb{R}^n, \tag{2.8}$$

with terminal condition

$$v(X_T, T) = f(X_T). \tag{2.9}$$

If $v(x,t)$ *also satisfies the polynomial growth condition*

$$\max_{0 \leq t \leq T} |v(x,t)| \leq M(1 + ||x||^{2\nu}),$$

for some $M > 0$ *and* $\nu \geq 1$*. Then* $v(X,t)$ *admits the stochastic representation*

$$v(X_t,t) = \mathbb{E}\left[f(X_T) e^{-\int_t^T k(X_s,s)ds} + \int_t^T g(X_s,s) e^{-\int_0^T k(X_u,u)du} ds \middle| \mathcal{F}_t \right] \tag{2.10}$$

on $[0,T] \times \mathbb{R}^n$ *and such solution is unique.*

The notation $\mathcal{A}()$ is the infinitesimal generator of X_t, which is defined to operate on compactly-supported C^2 functions $h : \mathbb{R}^m \to \mathbb{R}$ by

$$\lim_{t \to 0} \frac{1}{t} \left(\mathbb{E}[h(X_t)] - h(X_0) \right).$$

If the process X_t is defined as n-dimensional Markov process of the form (2.1), the infinitesimal generator of acted on function $v(X_t,t)$ is

$$\mathcal{A}(v) = \sum_i \alpha(X_i,t) \frac{\partial v}{\partial X_i} + \sum_{i,j} (\beta\beta^T)_{i,j}(X_i,t) \frac{1}{2} \frac{\partial^2 v}{\partial X_i \partial X_j}.$$

The Feynman-Kac Theorem relates the expectation (2.10), which depends on the stochastic process (2.2), and a partial differential equation (2.8). In this thesis, the Feynman-Kac theorem is used for obtaining the partial differential equation from an expectation so that the partial differential equation can be solved numerically to obtain $v(X_t, t)$. For a more detailed proof and discussion of the Feynman-Kac Theorem, we refer to Shreve and Karatzas (1998) Chapter 5, Section B.

2.2 Pricing Credit Claims in Reduced-Form Models

Reduced form models were originally introduced by Jarrow and Turnbull (1992), who first illustrate that Cox processes, also known as doubly stochastic Poisson processes, provide a framework for pricing credit derivatives and these are now known as reduced-form models. This paper was followed up by Jarrow and Turnbull (1995) and Lando (1998), who went on to provide a coherent reduced-form modelling framework. This section gives the introduction to reduced-form framework following Bielecki et al. (2009).

In reduced-form or hazard process approach, there are two kinds of information. One is the information regarding to the assets prices and other economic factors, which is denoted as $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$. This filtration $\mathbb{F}$ is generated by market uncertainties, such as interest rate processes and share price processes, but without information on default times. The filtration $\mathcal{F}_t$ is also referred to as the market filtration in this thesis. The other information contains the occurrence of default times. We assume the default time τ a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, satisfying $\mathbb{Q}(\tau = 0) = 0$ and $\mathbb{Q}(\tau > t) > 0$ for $t \in \mathbb{R}_+$. We introduce the right-continuous default indicator process H by setting $H(t) = 1_{\{\tau\leq t\}}$ and $\mathbb{H}$ to denote the filtration generated by the default process H, so that $\mathcal{H} = \sigma(\{\tau \leq u : u \leq t\})$ for any $t \in \mathbb{R}_+$. The total information available at time t is captured by the filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, which is $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ for any time $t \in \mathbb{R}_+$. All filtrations are assumed to satisfy conditions of right-continuous and completeness. The process H is $\mathbb{G}$-adapted but not $\mathbb{F}$-adapted. In other words, the random time τ is $\mathbb{G}$-stopping time but not an $\mathbb{F}$-stopping time.

For any time $t \in \mathbb{R}_+$, we define $F_t = \mathbb{Q}(\tau \leq t|\mathcal{F}_t)$, which satisfies $F_0 = 0$ and $\lim_{t\to\infty} F_t = 1$. On the other hand, we have a survival process $G_t = \mathbb{Q}(\tau > t|\mathcal{F}_t)$, satisfying

$$G_t = 1 - F_t.$$

Obviously, the process F_t and G_t follow a bounded and non-negative $\mathbb{F}$-adapted sub-martingale and super-martingale under $\mathbb{Q}$ measure respectively.

Definition 2.2.1. Assume that $F_t < 1$ for $t \in \mathbb{R}_+$. The $\mathbb{F}$-hazard process of τ under $\mathbb{Q}$, denote by Λ, is defined through the formula $1 - F_t = e^{-\Lambda_t}$. Equivalently, $\Lambda_t = -\ln(G_t) = -\ln(1 - F_t)$ for every $t \in \mathbb{R}_+$.

Throughout this thesis, the inequality $F_t < 1$ is assumed for every $t \in \mathbb{R}_+$ so that the hazard process Λ_t is well defined.

Lemma 2.2.1. *For any $\mathcal{G}-$measurable and $\mathbb{Q}$-integrable random variable X, we have for any $t \in \mathbb{R}_+$,*

$$\mathbb{E}^{\mathbb{Q}}[1_{\{t<\tau\}}X|\mathcal{G}_t] = 1_{\{t<\tau\}}\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}_t] = 1_{\{t<\tau\}}\frac{\mathbb{E}^{\mathbb{Q}}[1_{\{t<\tau\}}X|\mathcal{F}_t]}{\mathbb{Q}(t<\tau|\mathcal{F}_t)}. \tag{2.11}$$

For $X = 1_{\{\tau \leq T\}}$, we have

$$\mathbb{Q}(t < \tau \leq T|\mathcal{G}_t) = 1_{\{t<\tau\}}\frac{\mathbb{Q}(t<\tau\leq T|\mathcal{F}_t)}{\mathbb{Q}(t<\tau|\mathcal{F}_t)} = 1_{\{t<\tau\}}\mathbb{E}^{\mathbb{Q}}\left[1 - e^{\Lambda_t - \Lambda_T}|\mathcal{F}_t\right], \tag{2.12}$$

where (2.12) is the default probability before time T.

For the complete proof of above Lemma 2.2.1, we refer to Bielecki et al. (2009).

Then the corollary to Lemma 2.2.1 follows

Corollary 1. *Let X be a $\mathcal{F}_T$-measurable and $\mathbb{Q}$-integrable random variable. Then for every $t \leq T$,*

$$\mathbb{E}^{\mathbb{Q}}[1_{\{T<\tau\}}X|\mathcal{G}_t] = 1_{\{t<\tau\}}\frac{\mathbb{E}^{\mathbb{Q}}[X1_{\{T<\tau\}}|\mathcal{F}_t]}{\mathbb{E}^{\mathbb{Q}}[1_{\{t<\tau\}}|\mathcal{F}_t]} = 1_{\{t<\tau\}}\mathbb{E}^{\mathbb{Q}}[Xe^{\Lambda_t-\Lambda_T}|\mathcal{F}_t]. \tag{2.13}$$

For the case in which the variable $X = 1$, (2.13) becomes

$$\mathbb{E}^{\mathbb{Q}}\left[1_{\{T<\tau\}}|\mathcal{G}_t\right] = 1_{\{t<\tau\}}\mathbb{E}^{\mathbb{Q}}[e^{\Lambda_t-\Lambda_T}|\mathcal{F}_t], \tag{2.14}$$

which gives the survival probability up to time T. Lemma 2.2.1 and Corollary 1 are crucial important in reduced-form modelling. Because they enable the valuation of $\mathbb{G}$-measurable random variables to be measured conditioning only on filtration $\mathbb{F}$, which implies credit claims can be evaluated with only the asset prices information without default time information. Alternatively, Corollary 1 can be interpreted as how a payment at time T conditional on no prior default can be measured. Then we introduce the following lemmas to be used for the valuation of a recovery payment occurs at default and the continuous payments A_t conditional on no default.

Lemma 2.2.2. *Assume F is a continuous, increasing process so that the equality $dF_t = e^{-\Lambda_t} d\Lambda_t$ is valid and Z is an $\mathbb{F}$-predictable process such that the random variable $Z_\tau 1_{\{\tau \le T\}}$ is $\mathbb{Q}$-integrable. Then we have, for every $t \le T$,*

$$1_{\{t<\tau\}}\mathbb{E}^{\mathbb{Q}}[Z_\tau 1_{\{\tau \le T\}}|\mathcal{G}_t] = 1_{\{t<\tau\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T Z_u dH_u \middle| \mathcal{G}_t\right] = 1_{\{t<\tau\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T Z_u e^{\Lambda_t - \Lambda_u} d\Lambda_u \middle| \mathcal{F}_t\right] \tag{2.15}$$

Lemma 2.2.3. *Assume that A is a bounded, $\mathbb{F}$-predictable process of finite variation. Then for every $t \le T$,*

$$\mathbb{E}^{\mathbb{Q}}\left[\int_t^T (1 - H_u) dA_u \middle| \mathcal{G}_t\right] = 1_{\{t<\tau\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{\Lambda_t - \Lambda_u} dA_u \middle| \mathcal{F}_t\right] \tag{2.16}$$

We begin with defining the zero-coupon bonds, which are driven by the $\mathbb{F}$-adapted interest rate process r_t for every $t \in \mathbb{R}_+$. The price at time t of a zero-coupon bond with maturity T equals

$$D(t,T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r(s)ds}|\mathcal{F}_t]. \tag{2.17}$$

Every defaultable claim can be characterised by three kinds of payment, which is the continuous payments A_t conditional on survival, the recovery payment at default Z_t, the one time payment conditional on survival at maturity X_T. A generic default claim can be represented as

$$V(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T D(t,s)(1 - H_s)dA_s + \int_t^T D(t,s)Z_s dH_s + D(t,T)(1 - H_T)X_T \middle| \mathcal{G}_t\right]. \tag{2.18}$$

Using Corollary 1, Lemma 2.2.2 and Lemma 2.2.3, (2.18) can be further derived as

$$V(t,T) = 1_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T D(t,s)e^{\Lambda_t - \Lambda_s}(dA_s + Z_s d\Lambda_s) + D(t,T)e^{\Lambda_t - \Lambda_T} X_T \middle| \mathcal{F}_t\right]. \tag{2.19}$$

It is important to note default time τ and default process H_t are not involved in (2.19) and the expectation is only conditional on filtration $\mathcal{F}_t$ which contains no default information.

If the process F is absolutely continuous so that $F_t = \int_0^t f_u du$ is valid for some $\mathbb{F}$-progressively measurable non-negative process f. Then the process Λ is also an absolutely continuous and increasing process. Consequently, Λ admits a $\mathbb{F}$-hazard rate λ satisfying

$$\Lambda_t = \int_0^t \lambda_u du, \tag{2.20}$$

which we refer λ as the $\mathbb{F}$-intensity or stochastic intensity of default time τ. Finally, we can further develop (2.19) into

$$V(t,T) = 1_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T D(t,s)e^{-\int_t^s \lambda_u du}(dA_s + \lambda_s Z_s ds) + D(t,T)e^{-\int_t^T \lambda_s ds} X_T \middle| \mathcal{F}_t\right]. \tag{2.21}$$

In terms of counterparty credit risk, we remind ourself of the definition of CVA and DVA as equations (1.3), (1.4) and (1.5). These definitions can be extended under the reduced-form framework. Incorporating the Corollary 1, Lemma 2.2.2 and Lemma 2.2.3 into equation (1.3), we obtain

$$\mathrm{CVA}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r(u)+\lambda_C(u)du}\lambda_C(s)(1-R_C(s))\max\{\Pi(s,T),0\}ds\middle|\mathcal{F}_t\right]. \tag{2.22}$$

Similarly, we can evaluate the CVA of equation (1.4) as

$$\mathrm{CVA}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r(u)+\lambda_I(u)+\lambda_C(u)du}\lambda_C(s)(1-R_C(s))\max\{\Pi(s,T),0\}ds\middle|\mathcal{F}_t\right]. \tag{2.23}$$

Finally, the DVA can be written as

$$\mathrm{DVA}(t,T) = -\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r(u)+\lambda_I(u)+\lambda_C(u)du}\lambda_I(s)(1-R_I(s))\min\{\Pi(s,T),0\}ds\middle|\mathcal{F}_t\right]. \tag{2.24}$$

2.2.1 Default times simulation

Here we briefly describe the most commonly-used algorithm for the construction of a default time associated with the default intensity $\lambda(t)$. First discussed by Norros (1986), the algorithm is used for simulating Poisson processes to represent the life times of technical systems. The algorithm is widely used for simulating default times in reduced-form models for pricing credit claims in Monte Carlo simulation.

We assume that we are given an $\mathcal{F}$-adapted, right-continuous process $\lambda(t)$ defined on a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{Q}})$. After simulating the process up to time T, we have the filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}_T, \tilde{\mathbb{Q}})$, which contains information up to time T. Conditional on $\tilde{\mathcal{F}}_T$, the probability of survival is given by

$$\mathbb{Q}(\tau > T|\tilde{\mathcal{F}}_T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T \lambda(s)ds}\middle|\tilde{\mathcal{F}}_T\right].$$

However, as mentioned earlier, a probability does not explicitly give us a default time. To construct a default time, we have to enlarge the probability space $\tilde{\Omega}$ with a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{Q}})$ defined by a $(0, 1)$ uniform distributed random variable ξ. The uniform variable ξ represents a simulated survival probability, which is compared to $\mathbb{Q}(\tau > T|\tilde{\mathcal{F}}_T)$. Under the enlarged probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, we define the default time as

$$\tau = \inf\{t \in \mathbb{R}^+ : e^{-\Lambda_t} \leq \xi\} = \inf\{t \in \mathbb{R}^+ : \Lambda_t \geq \eta\}, \tag{2.25}$$

where the random variable $\eta = -ln(\xi)$ has a unit exponential law under $\mathbb{Q}$. Intuitively, we compare the survival probability conditional on $\tilde{\mathcal{F}}_T$ and a simulated survival probability. If the survival probability conditional on $\tilde{\mathcal{F}}_T$ is smaller than the one we simulated, the default time is less than T. Specifically, the default time is the first time at which the survival probability calculated by the sample path $\lambda(t)$ is less than the survival probability. For instance, If the sample path of default intensity $\lambda(t)$ simulated increases, this leads to a small survival probability and is more likely to be less than a uniform $(0, 1)$ random variable at an earlier time.

2.2.2 Recovery methods

Let the value Z_t be the payment at the time of default. When the credit claim is a bond, Z_t will be the recovery value $R(t)$ from the default corporate. There are some ways to model recovery value $R(t)$, which are discussed by Jarrow et al. (1997), Duffie (1998) and Duffie and Singleton (1999), which consider *recovery treasury*, *recovery of face value* and *recovery market value* respectively.

When the credit claim is a defaultable bond, we have $X(T) = 1$. If the recovery rate is 0, we have $Z(\tau) = 0$. Therefore, defaultable bonds have the value of

$$V(t, T) = 1_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \lambda(u)+r(u)du}\middle|\mathcal{F}_t\right] \tag{2.26}$$

In this way the bond price is simply the default-free bond multiplied by the probability of the entity surviving beyond maturity.

The simplest case is assuming the default recovery in terms of a fraction of the face value and is payable at the bond's maturity. The valuation of a default bond is straight forward as

$$\begin{aligned} V(t,T) &= 1_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)+\lambda(s)ds} + \left(1-e^{-\int_t^T \lambda(s)ds}\right) Re^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right] \\ &= 1_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} - (1-R)e^{-\int_t^T r(s)ds}\left(1-e^{-\int_t^T \lambda(s)ds}\right) \middle| \mathcal{F}_t \right]. \end{aligned} \tag{2.27}$$

Intuitively, a default bond with recovery R payable at maturity equals to a default default-free bond minus the present value of loss given default times the probability of default prior bond maturity.

If the amount of recovery is seen to be a fraction of the face value paid at the default time, we have $Z(\tau) = R$, which is a number between 0 and 1. As shown by Duffie and Singleton (1999), the bond price becomes

$$V(t,T) = 1_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s \lambda(u)+r(u)du}\lambda(s)Rds + e^{-\int_t^T r(s)+\lambda(s)ds} \middle| \mathcal{F}_t \right].$$

Recovery can be a fraction of the contract value without default-risk. If the claim is a default bond, $Z(\tau) = Re^{-\int_\tau^T r(s)ds}$, which is also referred recovery of treasury bond, which is a risk-free bond, and the bond price is,

$$\begin{aligned} V(t,T) &= 1_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s \lambda(u)+r(u)du}\lambda(s)Re^{-\int_\tau^T r(s)ds}ds + e^{-\int_t^T \lambda(u)+r(u)du} \middle| \mathcal{F}_t \right] \\ &= 1_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left[Re^{-\int_t^T r(u)du}\int_t^T e^{-\int_t^s \lambda(u)du}\lambda(s)ds + e^{-\int_t^T r(u)du}e^{-\int_t^T \lambda(u)du} \middle| \mathcal{F}_t \right]. \end{aligned} \tag{2.28}$$

If the interest rate $r(t)$ is independent of $\lambda(t)$, (2.28) becomes

$$V(t,T) = 1_{\{\tau>t\}}D(t,T)\left(R\mathbb{Q}(t<\tau\leq T|\mathcal{F}_t) + \mathbb{Q}(\tau>T|\mathcal{F}_t)\right), \tag{2.29}$$

which can be seem as the discounted value of a payment R at time T if the firm defaults prior maturity T and a payment 1 at time T without defaults.

The *recovery market value* approach is to set the recovery amount to that of the equivalent market value. Following Duffie and Singleton (1999), we assume $Z(\tau) = R(\tau)V(\tau,T)$, where $R(\tau)$ is a given $\mathcal{F}$-predictable recovery process. Since $V(t,T)$ is the market value of the defaultable claim and $R(t)$ is the recovery rate, so this recovery method is referred to as *recovery market value*. It can be shown that the default bond value is

$$V(t,T) = 1_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)+(1-R(u))\lambda(u)du}\middle|\mathcal{F}_t\right].$$

Note that, a crucial distinction between the structural model and the reduced form model is that the recovery rate process is pre-specified by the knowledge of the liability structure in the structural approach, while it is exogenously supplied in reduced-form models. Because in reduced-form models, the liability structure of a firm is usually assumed to not be observable therefore the recovery rate is not observable either.

2.3 The Pricing of Credit Derivatives and PDEs

Bielecki and Rutkowski (2004) argue that within the framework of the intensity-based approach, a default claim typically cannot be replicated by trading default-free claims. Consequently the standard pricing argument based on replicating strategy does not apply to this setting. However, the market is still arbitrage-free after introducing the credit claim evaluated as (2.18). Since credit claims cannot be replicated or hedged by trading default-free claims, then a partial differential equation cannot be derived by arguing that a hedged credit claim yields a risk-free return. However, we are able to assume the existence of some credit claims without breaking the arbitrage-free market as long as prices are given by (2.18), according to Bielecki and Rutkowski (2004). Since credit claims could be priced in the risk-neutral measure without breaking the no-arbitrage condition, the valuation of credit derivatives under the risk-neutral measure as (2.21) can be directly linked to PDEs through the Feynman-Kac theorem. For example, Duffie and Singleton (1999) and Duffie (2005) use this idea to present the PDE of a defaultable bond with AJD stochastic default intensity. Alternatively, a differential equation for a product can be derived by constructing a risk-free portfolio, which has risk-free return, such as Black and Scholes (1973). The resulting PDEs that are constructed from a risk-free portfolio must align with the one derived from applying the Feynman-Kac theorem to the expectation under the risk-neutral measure.

Due to increasing popularity in counterparty risk research, Burgard and Kjaer (2011) derived a PDE representation for an option contract with counterparty risk and funding cost, which enables them to obtain a PDE for CVA pricing. They describe a replicating strategy for a counterparty risky option using a zero recovery bond issued by the counterparty itself in order to derive an extension to the Black-Scholes PDE, with the presence of bilateral counterparty risk. Burgard and Kjaer (2011) show that counterparty's default risk leads to an extra term in the original PDE for the option's value, which is the default pay-off multiplied by the counterparty's default intensity. They demonstrate how the Feynman-Kac representation for this PDE aligns with the valuation framework described in Section 1.2 when the contract value at default is taken as the counterparty risk-free value. The approach of Burgard and Kjaer (2011) is further discussed by Kromer et al. (2015), who show that the Feynman-Kac theorem can be applied to the CVA with jump-diffusion.

We are interested in PDEs for valuing default bonds, CDS as well as CVA. Because the model we propose does not allow for closed-form solutions of the survival probability, neither CDS nor CVA valuations can be evaluated with semi-analytic solutions. Consequently, we must evaluate these by numerically solving the differential equations in Chapter 3, 4, 5. In this chapter, we consider defaultable bonds and CDS with and without counterparty credit risk. In additional to applying the Feynman-Kac theorem to expectations to obtain these PDEs, we also attempt to derive those PDEs by constructing risk-free portfolios if by assuming that specific products are available for hedging. This is distinct from the work of Burgard and Kjaer (2011), where default intensities are constant, since we also consider stochastic default intensities. Next, we show that the PDE for the CVA from holding a counterparty risky CDS contract can be found following the method of Burgard and Kjaer (2011). This PDE representation of a CVA aligns with the expectation form of CVA as defined by the Feynman-Kac theorem.

2.3.1 Default bonds

Consider an economy with a risk-free bond, which is equivalent to the discount factor $D(t,T)$ that is defined by (2.17). To simplify, we assume there is a flat term-structure for the risk-free rate. This economy is described by a filtered probability space $\left(\Omega, \mathcal{G}_{t\geq 0}, \mathbb{Q}\right)$, where $\mathbb{Q}$ is an equivalent martingale measure under which the discounted asset value processes are all martingale. The filtration $\mathcal{G}_t$ satisfies $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathcal{H}_t$ is generated by the default process $H(t) = 1_{\{\tau \le t\}}$. $\mathcal{F}_t$ is the market filtration generated by any market uncertainties except the information of default times. Assume $F_t = \mathbb{Q}(\tau \le t|\mathcal{F}_t)$ is absolutely continuous so that the default time τ has a $\mathbb{F}$-adapted non-negative stochastic intensity $\lambda(t)$. We assume $\lambda(t)$ satisfies the stochastic differential equation

$$d\lambda^{\mathbb{Q}}(t) = \alpha^{\mathbb{Q}}(\lambda_t, t)dt + \beta^{\mathbb{Q}}(\lambda_t, t)dW_t^{\mathbb{Q}}, \tag{2.30}$$

where $W_t^{\mathbb{Q}}$ is a $\mathcal{F}_t$-measurable one-dimensional Brownian motion. $\alpha^{\mathbb{Q}}(\lambda_t, t)$ and $\beta^{\mathbb{Q}}(\lambda_t, t)$ are Lipschitz real value functions satisfying (2.3) and (2.4) so that the solution to (2.30) exist and unique.

Bonds are the fundamental products in financial markets. Investors of bonds pay an initial payment to the bond issuer and are guaranteed by the issuer that the principle will be paid back at the maturity plus interest. However, if the issuer of a bond is a corporation, it may go into liquidation before the bond maturity and may not have enough assets to pay back its liability. In this case, the investor can only receive a fraction, if any, of the guaranteed value. Denote a defaultable bond at time t that matures at time T as $\tilde{B}(\lambda_t, t, T)$, its value under the risk-neutral measure using reduced-form modelling is

$$\tilde{B}(t,T) = 1_{\{\tau > t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r+\lambda_u du}\lambda_s R_s ds + e^{-\int_t^T r+\lambda_s ds}\middle|\mathcal{F}_t\right], \tag{2.31}$$

which is a special case of (2.21) with $X_T = 1$, $A_s = 0$, $Z_s = R_s$ and flat interest rate.

It is obvious that the terminal condition is $f(\lambda_T, T) = X_T = 1$ and $g(\lambda_t, t) = \lambda(t)R(t)$ satisfies conditions (2.6) and (2.7) respectively given $R(t) \in [0, 1]$. Besides, since $\lambda(t)$ is assumed to satisfy (2.3) and (2.4), then $r + \lambda(t)$ is bounded. Assume the bond value function $\tilde{B}(\lambda_t t, T)$ is $C^{1,2}$ with respect to time and λ_t, then we know from the Feynman-Kac Theorem that (2.31) implies that $\tilde{B}(\lambda_t, t, T)$ solves the backward Kolmogorov partial differential equation

$$\frac{\partial \tilde{B}}{\partial t} + \alpha^{\mathbb{Q}}(\lambda, t)\frac{\partial \tilde{B}}{\partial \lambda} + \frac{1}{2}\beta^{2\mathbb{Q}}(\lambda, t)\frac{\partial^2 \tilde{B}}{\partial \lambda^2} + \lambda R_t - (r + \lambda)\tilde{B} = 0. \tag{2.32}$$

In order to have a better interpretation of pricing derivatives with stochastic default intensity, we show how the risk of credit derivatives can be hedged and lead to a PDE under the risk-neutral measure. Similar to Black and Scholes (1973), we begin with the dynamics of a non-negative default intensity process under physical measure $\mathbb{P}$

$$d\lambda_t^{\mathbb{P}} = \alpha^{\mathbb{P}}(\lambda_t, t)dt + \beta^{\mathbb{P}}(\lambda_t, t)dW_t^{\mathbb{P}}, \tag{2.33}$$

where $W_t^{\mathbb{P}}$ is a one-dimensional Brownian motion under physical measure $\mathbb{P}$, $\alpha^{\mathbb{P}}(\lambda_t, t)$ and $\beta^{\mathbb{P}}(\lambda_t, t)$ are Lipschitz real value functions satisfying usual conditions.

The relationship between the process under physical measure (2.33) and the one under the risk-neutral measure (2.30) lies on changing the Brownian motion's probability measure, which is described by Girsanov's Theorem.

Theorem 2.3.1 (Girsanov's Theorem)**.** *Begin with a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a non-negative random variable* Z *satisfying* $\mathbb{E}^{\mathbb{P}}[Z] = 1$*, We can define a new probability measure* $\mathbb{Q}$ *by the formula*

$$\mathbb{Q}(A) = \int_A Z(\omega)dP(\omega), \ \forall A \in \mathcal{F}.$$

Then any random variable has two expectations under measure $\mathbb{P}$ *and* $\mathbb{Q}$ *given by the formula*

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[ZX].$$

If $\mathbb{P}(Z > 0) = 1$ *then* $\mathbb{P}$ *and* $\mathbb{Q}$ *agree which sets have* 0 *probability, which we call* $\mathbb{Q}$ *a probability measure equivalent to* $\mathbb{P}$*. We say* Z *is the Radon-Nikodým derivative of* $\mathbb{Q}$ *with respect to* $\mathbb{P}$*, which is written as*

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}}. \tag{2.34}$$

In particular, if $W_t^{\mathbb{P}}$*,* $0 \leq t \leq T$*, is a Brownian motion under measure* $\mathbb{P}$ *and let* $\mathcal{F}_t$*,* $0 \leq t \leq T$*, be a filtration for this Brownian motion. let* Θ_t*,* $0 \leq t \leq T$*, be an adapted process, we can define the Radon-Nikodým derivative as*

$$Z_t = \exp\left(-\int_0^t \Theta_s dW_s - \frac{1}{2}\int_0^t \Theta_s^2 ds\right) \tag{2.35}$$

and assume that

$$\mathbb{E}^{\mathbb{P}}[\int_0^T \Theta_s^2 Z_s^2 ds] < \infty. \tag{2.36}$$

Then $\mathbb{E}^{\mathbb{P}}[Z_T] = 1$ *and the process*

$$W_t^{\mathbb{Q}} = W_t + \int_0^t \Theta_s ds \tag{2.37}$$

is a Brownian motion under the probability measure $\mathbb{Q}$*.*

The hedging strategy of defaultable bonds with stochastic default intensity is analogous to the hedging strategy of risk-free bonds discussed in Wilmott et al. (1995). Since there are no other products depending on the intensity process, a defaultable bond needs to be hedged by another defaultable bond with the same issuer but different maturity. Denote $\tilde{B}_1$ and $\tilde{B}_2$ as two defaultable bonds with different maturities. The change in value when holding the two defaultable bonds are

$$\begin{aligned} d\tilde{B}_1 &= \left(\frac{\partial \tilde{B}_1}{\partial t} + \alpha^{\mathbb{P}}(\lambda_t, t)\frac{\partial \tilde{B}_1}{\partial \lambda} + \frac{1}{2}\beta^{2\mathbb{P}}(\lambda_t, t)\frac{\partial \tilde{B}_1^2}{\partial \lambda^2} \right) dt \\ &+ \beta^{\mathbb{P}}(\lambda_t, t)\frac{\partial \tilde{B}_1}{\partial \lambda} dW_t^{\mathbb{P}} + (R(t) - \tilde{B}_1)dH_t \\ d\tilde{B}_2 &= \left(\frac{\partial \tilde{B}_2}{\partial t} + \alpha^{\mathbb{P}}(\lambda_t, t)\frac{\partial \tilde{B}_2}{\partial \lambda} + \frac{1}{2}\beta^{2\mathbb{P}}(\lambda_t, t)\frac{\partial \tilde{B}_2^2}{\partial \lambda^2} \right) dt \\ &+ \beta^{\mathbb{P}}(\lambda_t, t)\frac{\partial \tilde{B}_2}{\partial \lambda} dW_t^{\mathbb{P}} + (R(t) - \tilde{B}_2)dH_t. \end{aligned} \tag{2.38}$$

The problem is that there are two kinds of risk in these bonds, namely the risk of the default intensity changing and the risk of a jump-to-default. The value process of a default bond changes according the changes in the stochastic process, which lead to the Itô's formula term, and the cash flow due to the default process $H(t)$ jumps.

Consider a portfolio which comprises a long position in $\tilde{B}_1$ and Δ amounts of $\tilde{B}_2$

$$\Pi = \tilde{B}_1 - \Delta \tilde{B}_2,$$

then the change in value over a small instant in time can be written as

$$\begin{aligned} d\Pi = &\big((\partial t + \mathcal{L})\tilde{B}_1 - \Delta(\partial t + \mathcal{L})\tilde{B}_2\big)dt + \beta \left(\frac{\partial \tilde{B}_1}{\partial \lambda} - \Delta \frac{\partial \tilde{B}_2}{\partial \lambda} \right) dW_t \\ &+ \big(R(t) - \tilde{B}_1 + \Delta(R(t) - \tilde{B}_2)\big)dH_t, \end{aligned} \tag{2.39}$$

where

$$\mathcal{L} = \alpha^{\mathbb{P}}(\lambda_t, t)\frac{\partial}{\partial \lambda} + \frac{1}{2}\beta^{2\mathbb{P}}(\lambda_t, t)\frac{\partial^2}{\partial \lambda^2}.$$

In order to hedge risk of default intensity moves, the amount Δ should be chosen as

$$\Delta = \frac{\partial \tilde{B}_1}{\partial \lambda} \Big/ \frac{\partial \tilde{B}_2}{\partial \lambda}, \tag{2.40}$$

which is a ratio of the sensitivities of two default bonds' value to the default intensity.

However, a hedging portfolio, which is composed of two risky bonds with different maturities, can hedge the intensity risk but unfortunately the jump risk will still remain. The portfolio is still subject to jump risk which we can see from the following

$$\begin{aligned} d\Pi = &\bigg((\partial t + \mathcal{L})\tilde{B}_1 - \frac{\partial \tilde{B}_1}{\partial \lambda}\Big/\frac{\partial \tilde{B}_2}{\partial \lambda}(\partial t + \mathcal{L})\tilde{B}_2\bigg)dt \\ &+ \bigg(R(t) - \tilde{B}_1 - \frac{\partial \tilde{B}_1}{\partial \lambda}\Big/\frac{\partial \tilde{B}_2}{\partial \lambda}(R(t) - \tilde{B}_2)\bigg)dH_t. \end{aligned} \tag{2.41}$$

The jump risk therefore remains in the portfolio and cannot be hedged away only with bonds. In order to hedge jump risk while avoiding the risk from stochastic intensity, a product that is exposed purely to jump risk is required. Here, we assume that there exists an insurance contract $I(t)$ traded on the market. The insurance contract matures in the next infinite small time dt and a unit amount is paid to the investor if the company defaults. In exchange for the protection, the buyer pays a *variable* rate as insurance premium. The probability of a jump in the limit of small time dt is λdt. So the insurance product is fair and both parties can enter the contract without requiring compensation from the other if the expected insurance payment is a value that is equal to the expected default payment. Therefore, the fair rate at which the protection buyer should pay the seller is $\lambda(t)dt$. The return from the product is then

$$dI_t = rI_t dt - \lambda_t dt + dH_t. \tag{2.42}$$

We know that $\mathbb{E}[-\lambda_t dt + dH_t|\mathcal{G}_0] = 0$ for any $t > 0$ and it follows that the value of the product I_t will stay at 0 if no up-front payment is required such that $I_{t=0} = 0$. Because the contract matures in a very short time, it is safe to argue this product is not subject to intensity risk. However, whether this product exists in the market is under question.

Simple algebra tells us that the amount of this insurance product we need to hold in order to hedge the jump risk in our portfolio is $B_1 + \Delta B_2$. Therefore, the hedging portfolio now becomes

$$\Pi' = \tilde{B}_1 - \Delta\tilde{B}_2 + \big(R(t) - \tilde{B}_1 - \Delta(R(t) - \tilde{B}_2)\big)I, \tag{2.43}$$

and (2.41) becomes

$$\begin{aligned} d\Pi' = &\bigg((\partial t + \mathcal{L})\tilde{B}_1 - \frac{\partial \tilde{B}_1}{\partial \lambda}\Big/\frac{\partial \tilde{B}_2}{\partial \lambda}(\partial t + \mathcal{L})\tilde{B}_2\bigg)dt \\ &+ \lambda_t\bigg(R(t) - \tilde{B}_1 - \frac{\partial \tilde{B}_1}{\partial \lambda}\Big/\frac{\partial \tilde{B}_2}{\partial \lambda}(R(t) - \tilde{B}_2)\bigg)dt = r\Pi' dt. \end{aligned} \tag{2.44}$$

After arranging the terms with B_1 on the left-hand-side and B_2 on the right-hand-side, there follows

$$\frac{(\partial t+\mathcal{L}-\lambda)\tilde{B}_1+\lambda_t R(t)-(r+\lambda_t)\tilde{B}_1}{\partial\tilde{B}_1/\partial\lambda}=\frac{(\partial t+\mathcal{L}-\lambda)\tilde{B}_2+\lambda_t R(t)-(r+\lambda_t)\tilde{B}_2}{\partial\tilde{B}_2/\partial\lambda}. \tag{2.45}$$

Because both sides of (2.45) are a function of their corresponding maturity, the value they are equal to must be a function independent of maturity. Similar to the approach in Wilmott et al. (1995) to derive PDE for risk-free bonds, we assume the function that both sides of (2.45) equal to is $(\alpha^{\mathbb{Q}}(\lambda(t),t)-\omega\beta^{\mathbb{Q}}(\lambda(t),t))$. From this, it is simple to show that

$$\frac{\partial\tilde{B}}{\partial t}+(\alpha^{\mathbb{Q}}(\lambda(t),t)-\omega\beta^{\mathbb{P}}(\lambda(t),t))\frac{\partial\tilde{B}}{\partial\lambda}+\frac{1}{2}\beta^{2\mathbb{P}}(\lambda(t),t)\frac{\partial^2\tilde{B}}{\partial\lambda^2}+\lambda(t)R(t)-(r+\lambda(t))\tilde{B}=0$$

where ω represents the return for bearing the risk of intensity moves, namely the market price of risk. Hence, we are assuming that the intensity process is under the risk-neutral measure where the market price of risk is 0 and discounted value process is a martingale. The dynamics of the default intensity under $\mathbb{Q}-measure$ are (2.30) and the relationship

$$\omega=\frac{\alpha^{\mathbb{P}}(\lambda(t),t)-\alpha^{\mathbb{Q}}(\lambda(t),t)}{\beta^{\mathbb{P}}(\lambda(t),t)} \tag{2.46}$$

$$\beta^{\mathbb{Q}}(\lambda(t),t)=\beta^{\mathbb{P}}(\lambda(t),t) \tag{2.47}$$

holds.

Defining the market price of risk as (2.46), it is essentially to assume

$$\Theta_t=\frac{\alpha^{\mathbb{P}}(\lambda(t),t)-\alpha^{\mathbb{Q}}(\lambda(t),t)}{\beta^{\mathbb{P}}(\lambda(t),t)}. \tag{2.48}$$

Then the relation between the Brownian motion under $\mathbb{Q}$ measure and physical measure $\mathbb{P}$ is described as

$$W_t^{\mathbb{Q}}=W_t^{\mathbb{P}}+\int_0^t\Theta_s ds. \tag{2.49}$$

It is easy to clarify that by substituting $dW_t^{\mathbb{P}}=dW_t^{\mathbb{Q}}-\Theta_t$ into (2.33) then (2.30) follows.

Assuming we are under the risk-neutral measure and discarding the superscripts, the pricing PDE of the default bond is

$$\frac{\partial\tilde{B}}{\partial t}+\alpha^{\mathbb{Q}}(\lambda_t,t)\frac{\partial\tilde{B}}{\partial\lambda}+\frac{1}{2}\beta^{2\mathbb{Q}}(\lambda_t,t)\frac{\partial^2\tilde{B}}{\partial\lambda^2}+\lambda(t)R(t)-(r+\lambda(t))\tilde{B}=0, \tag{2.50}$$

with terminal condition $\tilde{B}(T,T) = 1$ for any maturity, which is equivalent to saying that the value process of a default bond is

$$\frac{d\tilde{B}}{\tilde{B}} = \big(r(t) + \lambda(t)\big)dt - (1 - R(t))dH(t).$$

If $H(t)$ changes from 0 to 1, the bond jumps to 0 and stays unchanged. It is easy to show the discounted price process $\hat{B} = \tilde{B}/B$ is also a martingale.

2.3.2 Credit default swaps

In this section, we look at CDSs and their valuation, and then a PDE representation of the CDS is derived. A credit default swap is an agreement between two parties, namely the protection buyer and the protection seller. The protection seller is required to make a payment to cover the loss of the protection buyer when a particular default event happens to a third party, called the referencing entity. In return, the protection buyer has to make a periodic payment at times $T_1, ..., T_N$, until a credit event happens or the maturity of the contract, whichever comes first. This payment is quoted as an annual rate against the value or notional under protection, namely the swap premium or spread. For simplification, a unit notional amount is used. If the default event happens between two payment dates, an accrual payment is paid to the protection seller. The amount is a fraction of the swap premium representing the protection fee from the last payment date to default time.

Credit Default Swap valuation

We denote the default time of the referencing firm by τ, the swap premium as S, t is the contract initial time and $T = T_N$ is the maturity. The value of a CDS contract has two components, namely default value and premium value. The protection buyer promises to cover the losses of the bond issued by the referencing firm at the time the firm defaults. The present value of the default value is

$$\mathbb{E}^{\mathbb{Q}}\left[LGD(\tau) \times D(t,\tau)1_{\{t<\tau<T\}}\middle|\mathcal{G}_t\right], \tag{2.51}$$

where $LGD(\tau)$ is the loss given default at time τ.

On the other hand, the premium value is

$$\mathbb{E}^{\mathbb{Q}}\left[\sum_{n=1}^{N} 1_{\{\tau>T_n\}}S(T_n - T_{n-1})D(t,T_n) + D(t,\tau)\frac{\tau - T_{n-1}}{T_n - T_{n-1}}S\middle|\mathcal{G}_t\right]. \tag{2.52}$$

From the prospective of a protection buyer, who is paying premium value and receiving default value, the value of the CDS contract is

$$
\begin{aligned}
\mathrm{CDS}(t,T) =& \mathbb{E}^{\mathbb{Q}}\left[LGD(\tau) \times D(t,\tau) 1_{\{t<\tau<T\}} \middle| \mathcal{G}_t \right] \\
&- \mathbb{E}^{\mathbb{Q}}\left[\sum_{n=1}^{N} 1_{\{\tau>T_n\}} S(T_n - T_{n-1}) D(t,T_n) + D(t,\tau)\frac{\tau - T_{n-1}}{T_n - T_{n-1}} S \middle| \mathcal{G}_t \right].
\end{aligned} \tag{2.53}
$$

To be aligned with the notations from Section 2.2, we denote

$$
\begin{aligned}
A(t) &= -\sum_{n=1}^{N} 1_{\{t>T_n\}} S(T_n - T_{n-1}) \\
X(T) &= 0 \\
Z(t) &= LGD(t) - \frac{\tau - T_{n-1}}{T_n - T_{n-1}} S
\end{aligned} \tag{2.54}
$$

from rhe prospective of the CDS protection buyer.

Combined with the results on valuation credit claims with the reduced-form model from Section 2.2, (2.53) can be written as

$$
\begin{aligned}
\mathrm{CDS}(t,T) =& \mathbb{E}^{\mathbb{Q}}\left[\int_t^T D(t,s) e^{-\int_t^s \lambda_u du} \lambda(s) \left(LGD(s) - \frac{s - T_{n-1}}{T_n - T_{n-1}} S \right) ds \middle| \mathcal{F}_t \right] \\
&- \mathbb{E}^{\mathbb{Q}}\left[\sum_{n=1}^{N} e^{-\int_t^{T_n} \lambda_u du} S(T_n - T_{n-1}) D(t,T_n) \middle| \mathcal{F}_t \right].
\end{aligned} \tag{2.55}
$$

If we assume the mutual independence between interest rate, default time, loss given default as well as assuming that the survival probability has a closed-form solution, the CDS value as (2.55) has a semi-analytic solution,

$$
\begin{aligned}
\mathrm{CDS}(t,T) =& \int_t^T D(t,s) \mathbb{Q}(\tau > s) f_\lambda^{\mathbb{Q}}(t,s) \left(LGD(s) - \frac{s - T_{n-1}}{T_n - T_{n-1}} S \right) ds \\
&- \sum_{n=1}^{N} \mathbb{Q}(\tau > T_n) S(T_n - T_{n-1}) D(t,T_n),
\end{aligned} \tag{2.56}
$$

where $f_\lambda^{\mathbb{Q}}(t,s)$ is time s forward default intensity measured at time t. In the remainder of this thesis, (2.56) is used for pricing CDS whenever analytic solutions to survival probability are available.

CDSs are quoted as a spread S^* and are chosen so as to make both the default value and the premium value equal, and as such it is called the fair spread. Once the spread is fixed after initial agreement, the CDS's value can change due to the fluctuation of credit intensity and default events. After rearranging the terms in equation (2.55) with $\mathrm{CDS}(t,T)=0$, the fair spread becomes

$$S^* = \frac{E^{\mathbb{Q}}\left[LGD(\tau)D(\tau,T)1_{\{t<\tau<T\}}|\mathcal{G}_t\right]}{E^{\mathbb{Q}}\left[\sum_{n=1}^{N} 1_{\{\tau>T_n\}}(T_n-T_{n-1})D(t,T_n)+D(t,\tau)\frac{\tau-T_{n-1}}{T_n-T_{n-1}}\bigg|\mathcal{G}_t\right]}, \tag{2.57}$$

or equivalently,

$$S^* = \frac{\mathbb{E}^{\mathbb{Q}}\left[\int_t^T D(t,s)e^{-\int_t^s \lambda_u du}\lambda(s)LGD(s)ds\bigg|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[\sum_{n=1}^{N} e^{-\int_t^{T_n}\lambda_u du}(T_n-T_{n-1})D(t,T_n)\int_t^T D(t,s)e^{-\int_t^s \lambda_u du}\lambda(s)\frac{s-T_{n-1}}{T_n-T_{n-1}}ds\bigg|\mathcal{F}_t\right]}. \tag{2.58}$$

With (2.58), it becomes convenient to numerically evaluate the fair spread. The fair spread can be evaluated from solving (2.55) twice with different parameters. Setting the $LGD(t)$ to be zero solves the denominator of (2.58) and $S=1$ solves the numerator. The fair spread is obtained by dividing the two numerical solutions to the denominator and numerator. In the later chapters, this method is used for evaluating fair spread when analytic solutions are not available for the survival probability.

Differential Equation for CDS

In the last section, we showed that the CDS value can be expressed as the expectation of the present value of the default leg and the premium leg in (2.55). In a constant interest rate environment, as assumed here, (2.55) further reduces to

$$\begin{aligned}\mathrm{CDS}(t,T) =& \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r+\lambda_u du}\lambda(s)\left(LGD(s)-\frac{s-T_{n-1}}{T_n-T_{n-1}}S\right)ds\bigg|\mathcal{F}_t\right]\\ &-\mathbb{E}^{\mathbb{Q}}\left[\sum_{n=1}^{N} e^{-\int_t^{T_n} r+\lambda_u du}S(T_n-T_{n-1})\bigg|\mathcal{F}_t\right],\end{aligned} \tag{2.59}$$

on which we can apply Feynman-Kac Theorem.

It is easy to verify that the functions $\lambda(s)\left(LGD(s)-\frac{t-T_{n-1}}{T_n-T_{n-1}}S\right)$ and $S(T_n-T_{n-1})$ satisfy the conditions (2.7) given the $LGD(t)$ is finite. In a CDS contract, no cash flow occurs due to the expiry of the contract thus the terminal condition is 0, which must satisfy (2.6). However, the value function $\mathrm{CDS}(t,T)$ is not continuous in time everywhere. The value jumps at the time the protection buyer pays the premium. As the Feynman-Kac theorem requires the value process to be $C^{1,2}$, it only applies to the time intervals between two successive payment times, which are $t = T_0$, T_1, ..., T_n, ..., $T_N = T$. Let us denote the CDS value as V whose value is driven by the stochastic process (2.30), by the Feynman-Kac theorem we have the PDE for a CDS contract as

$$\frac{\partial V}{\partial t} + \alpha^{\mathbb{Q}}(\lambda,t)\frac{\partial V}{\partial \lambda} + \frac{1}{2}\beta^{2\mathbb{Q}}(\lambda,t)\frac{\partial^2 V}{\partial \lambda^2} + \lambda\left(LGD(t) - S\frac{t-T_{n-1}}{T_n - T_{n-1}}\right) - (r+\lambda)V = 0 \tag{2.60}$$

for $t \in (T_{n-1}, T_n^-)$ and $n = 1, 2, ...N$ with jump conditions

$$V(T_n^-, T) = V(T_n, T) - S(T_n - T_{n-1}) \text{ for } n = 1, 2, ...N. \tag{2.61}$$

The jump condition appearing here is analogous to that of the PDE of coupon-bearing bonds derived in Wilmott et al. (1995).

Apart from using the Feynman-Kac to obtain the PDE directly, we attempt to derive the PDE using a hedging strategy. For the same reason, Itô's formula only applies to each time intervals (T_{n-1}, T_n). The changes in value due to holding a CDS contract in the limit of small time dt is

$$\begin{aligned} dV = {} & \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial \lambda(t)}d\lambda^{\mathbb{P}}(t) + \frac{1}{2}\beta^{2\mathbb{P}}(\lambda(t),t)\frac{\partial V^2}{\partial \lambda^2}dt \\ & + \left(LGD(t) - S\frac{t-T_{n-1}}{T_n - T_{n-1}} - V\right)dH_t \end{aligned} \tag{2.62}$$

for $t \in (T_{n-1}, T_n)$ and $n = 1, 2, ...N$.

Note that (2.62) is still under physical measure, but it is shown later in this section that the drift in physical measure will be replaced by the drift under the risk-neutral measure. The changes in CDS value also according to two factors. One is the value change due to the stochastic behaviour of the default intensity, where Itô's formula applies, and the default cash flows may receive from holding the contract. If the reference firm defaults, the CDS buyer will receive $LGD(t)$ amount from the seller and the CDS contract ends. There is a jump condition $V(T_n^-, T) = V(T_n, T) - S(T_n - T_{n-1})$ at T_n for $n = 1$, ..., N.

When holding a CDS, the investor is in a short position to the referencing firm's credit risk. In order to hedge, the strategy is to long a portion of a default bond to hedge the intensity risk and some of the insurance products, which we assumed to exist in Section 2.3.1, to hedge the jump risk.

$$\Pi = V + \Delta_1 \tilde{B} + \Delta_2 I.$$

Differentiating the portfolio we have

$$\begin{aligned} d\Pi = & \left(\frac{\partial V}{\partial \lambda} + \Delta_1 \frac{\partial \tilde{B}}{\partial \lambda} \right) d\lambda^{\mathbb{P}}(t) \\ & + (\mathcal{L}V)\, dt + \Delta_1 \left(\mathcal{L}\tilde{B} \right) dt - \Delta_2 \lambda dt \\ & + \left(\left(LGD(t) - S^* \frac{t - T_{n-1}}{T_n - T_{n-1}} - V \right) + \Delta_1 (R(t) - \tilde{B}) + \Delta_2 \right) dH_t, \end{aligned}$$

for $t \in (T_{n-1}, T_n)$ and $n = 1, 2, ...N$, where the operator

$$\mathcal{L} \equiv \frac{\partial}{\partial t} + \frac{1}{2} \beta^{2\mathbb{P}}(\lambda(t), t) \frac{\partial^2}{\partial \lambda^2}.$$

The risk of jump to default and changing default intensity can be hedged out if

$$\begin{aligned} \Delta_1 &= -\frac{\partial V}{\partial \lambda} \Big/ \frac{\partial \tilde{B}}{\partial \lambda} \\ \Delta_2 &= -\Delta_1 (R(t) - \tilde{B}) - \left(LGD(t) - S \frac{t - T_{n-1}}{T_n - T_{n-1}} - V \right). \end{aligned}$$

Then the portfolio should have risk-free return r,

$$\begin{aligned} & \left(\mathcal{L}V + \lambda(t) \left(LGD(t) - S \frac{t - T_{n-1}}{T_n - T_{n-1}} - V \right) \right) - \frac{\partial V}{\partial \lambda} \Big/ \frac{\partial \tilde{B}}{\partial \lambda} (\mathcal{L}\tilde{B} - \lambda(t)\tilde{B}) \\ & = r \left(V - \frac{\partial V}{\partial \lambda} \Big/ \frac{\partial \tilde{B}}{\partial \lambda} \tilde{B} + \Delta_2 I(t) \right), \end{aligned}$$

for $t \in (T_{n-1}, T_n)$, and $n = 1, 2, ...N$. Because $I(t)$ is shown to have zero value any time, arranging the above equation leads to

$$\begin{aligned} & \left(\frac{\partial V}{\partial t} + \frac{1}{2} \beta^{2\mathbb{P}}(\lambda(t), t) \frac{\partial^2 V}{\partial \lambda^2} + \lambda(t)(LGD(t) - S \frac{t - T_{n-1}}{T_n - T_{n-1}} - V) - rV \right) \\ & - \frac{\partial V}{\partial \lambda} \Big/ \frac{\partial \tilde{B}}{\partial \lambda} \left(\frac{\partial \tilde{B}}{\partial t} + \frac{1}{2} \beta^{2\mathbb{P}}(\lambda(t), t) \frac{\partial^2 \tilde{B}}{\partial \lambda^2} + \lambda(t) R(t) - (r + \lambda(t)\tilde{B}) \right) = 0, \end{aligned} \tag{2.63}$$

for $t \in (T_{n-1}, T_n)$, and $n = 1, 2, ...N$. The last important step is to remove terms including the defaultable bond $\tilde{B}$. From Section 2.3.1, we know that default bonds should satisfy (2.50) under the risk-neutral measure. Given the relationship $\beta^{\mathbb{P}}(\lambda_t, t) = \beta^{\mathbb{Q}}(\lambda_t, t)$, we are able to replace $\beta^{\mathbb{P}}(\lambda_t, t)$ with $\beta^{\mathbb{Q}}(\lambda_t, t)$. After substituting (2.50) into the above PDE, we have

$$\begin{aligned}&\left(\frac{\partial V}{\partial t}+\frac{1}{2}\beta^{2\mathbb{Q}}(\lambda(t),t)\frac{\partial^2 V}{\partial\lambda^2}+\lambda(t)\left(LGD(t)-S\frac{t-T_{n-1}}{T_n-T_{n-1}}-V\right)-rV\right)\\&-\frac{\partial V}{\partial\lambda}\Big/\frac{\partial\tilde{B}}{\partial\lambda}\left(\alpha^{\mathbb{Q}}(\lambda_t,t)\frac{\partial\tilde{B}}{\partial\lambda}\right)=0\end{aligned}\tag{2.64}$$

for $t\in(T_{n-1},T_n)$ and $n=1,2,....N$. The risk-neutral drift of default intensity $\alpha^{\mathbb{Q}}(\lambda_t,t)$ is introduced into (2.64) by the hedging strategy and it is equivalent to (2.60), which is obtained by directly applying the Feynman-Kac theorem.

Comparing (2.50) and (2.60), it is noticeable that the difference lies in the cash flows when default occurs. Since both CDSs and default bonds depend on the default intensities, their first three terms are identical given a risk-neutral dynamic of a default intensity. The term λV can be interpreted the extra rate of return λ for bearing default risk.

2.3.3 Credit default swaps with unilateral counterparty credit risk

In this section we study the PDE of the CDS contract with counterparty risk. This arises when buyers and/or sellers involved in the trade may default. Suppose there are three entities in the economy, namely A, B and C. To begin with, we shall only consider the default risk of the referencing firm and the CDS seller, which are firms B and C. Suppose we have a filtered probability space $\left(\Omega,\mathcal{G}_{t\geq 0},\mathbb{Q}\right)$ with two $\mathbb{G}$-stopping times τ_B and τ_C. The filtration $\mathcal{G}_t$ satisfies $\mathcal{G}_t=\mathcal{F}_t\vee\mathcal{H}_t$, where $\mathcal{F}_t$ represents all default free information available in the market. The filtration $\mathcal{H}_t$ is defined by the combination of two sub-filtrations

$$\mathcal{H}_t=\mathcal{H}_t^B\vee\mathcal{H}_t^C,$$

where $\mathcal{H}_t^B$ and $\mathcal{H}_t^C$ are generated by the firm B and C's default process $H_t^B=1_{\{\tau_B\leq t\}}$ and $H_t^C=1_{\{\tau_C\leq t\}}$ respectively. Suppose the $\mathbb{Q}(\tau_B\leq t|\mathcal{F}_t)$ and $\mathbb{Q}(\tau_C\leq t|\mathcal{F}_t)$ are absolutely continuous so that admit $\mathbb{F}$-adapted stochastic intensities $\lambda_B^{\mathbb{Q}}(t)$ and $\lambda_C^{\mathbb{Q}}(t)$, we modelled the intensities as

$$\begin{aligned} d\lambda_B^{\mathbb{Q}}(t) &= \alpha_B^{\mathbb{Q}}(\lambda_B^{\mathbb{Q}}(t),t)dt + \beta_B^{\mathbb{Q}}(\lambda_B^{\mathbb{Q}}(t),t)dW_B^{\mathbb{Q}}(t), \\ d\lambda_C^{\mathbb{Q}}(t) &= \alpha_C^{\mathbb{Q}}(\lambda_C^{\mathbb{Q}}(t),t)dt + \beta_C^{\mathbb{Q}}(\lambda_C^{\mathbb{Q}}(t),t)dW_C^{\mathbb{Q}}(t), \\ d &< W_B^{\mathbb{Q}}(t), W_C^{\mathbb{Q}}(t) >= \rho dt. \end{aligned} \tag{2.65}$$

Suppose there are default bonds of firm B and C, $B_B(t,T)$ and $B_C(t,T)$, and insurance products, $I_B(t)$ and $I_C(t)$, traded in the market. Assuming that the default-free parties A trade a credit default swap written on the entity B's credit risk. The entity A is buying protection reference to the entity B from the entity C. Since entity C can default, the value of CDS depends not only on market variable $\lambda_B^{\mathbb{Q}}(t)$, but $\lambda_C^{\mathbb{Q}}(t)$ as well. In a constant interest rate environment, the value of a CDS contract with counterparty default risk, denoted as $\hat{V}$, is

$$\begin{aligned} \hat{V}(t,T) =& 1_{\{\tau_B>t,\tau_C>t\}} \\ &\times\Bigg(\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r+\lambda_B^{\mathbb{Q}}(u)+\lambda_C^{\mathbb{Q}}(u)du}\lambda_B^{\mathbb{Q}}(s)\left(LGD(s)-\frac{s-T_{n-1}}{T_n-T_{n-1}}S\right)ds\middle|\mathcal{F}_t\right] \\ &-\mathbb{E}^{\mathbb{Q}}\left[\sum_{n=1}^{N} e^{-\int_t^{T_n} r+\lambda_B^{\mathbb{Q}}(u)+\lambda_C^{\mathbb{Q}}(u)du}S(T_n-T_{n-1})\middle|\mathcal{F}_t\right] \\ &+\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r+\lambda_B^{\mathbb{Q}}(u)+\lambda_C^{\mathbb{Q}}(u)du}\lambda_C^{\mathbb{Q}}(s)\big(R_C M^+(s,T)+M^-(s,T)\big)ds\middle|\mathcal{F}_t\right]\Bigg), \end{aligned} \tag{2.66}$$

where $M(t,T)$ denotes the CDS's close-out value at the counterparty's default time. Close-out is what happens when one party defaults on a trade then the residual value of the contract should be determined under the regulation of ISDA (2009). Because it depends on the sign of the contract value, the cash flow at different default times will be different. The legal document ISDA (2009) specifies that the surviving entity receives a recovery value of the contract if the contract has positive value i.e. $M(t,T) > 0$ from the perspective of the alive entity. Otherwise, the full contract value should be paid to the defaulting entity if the value is negative i.e. $M(t,T) < 0$.

The methodology to determine the close-out value $M(t,T)$ is itself a research area studied by Brigo and Morini (2010), Brigo and Morini (2011), Gregory and German (2013) and Burgard and Kjaer (2011). Under the regulation of ISDA (2009), market participants are able to choose whether to take counterparty credit risk into account when determining the close-out value, namely risk-free close-out and risky close-out (or replacement close-out). In other words, $M(t, T)$ could be specified as the CDS value without or without counterparty default risk. In this section, we do not specify the close-out convention to keep the formulation to be general.

As we have previously discussed in Section 2.3.2, the cash flows of a CDS must satisfy the condition (2.7). Now, the cash flow at counterparty's default time

$$\lambda_C^{\mathbb{Q}}(s)\left(R_C M^+(s,T) + M^-(s,T)\right)$$

is just a function of a CDS's value at counterparty's default time, which is bounded by the default payoff. Therefore, it must satisfies (2.7). After assuming the value function $\hat{V}$ to be $C^{1,2}$, we can apply the Feynman-Kac Theorem periodically to (2.66), which leads to

$$\begin{aligned}
&\frac{\partial \hat{V}}{\partial t} + \alpha_B^{\mathbb{Q}}(\lambda_B, t)\frac{\partial \hat{V}}{\partial \lambda_B} + \alpha_C^{\mathbb{Q}}(\lambda_C, t)\frac{\partial \hat{V}}{\partial \lambda_B} \\
&\quad + \frac{1}{2}\beta_B^{2\mathbb{Q}}(\lambda_B, t)\frac{\partial^2 \hat{V}}{\partial \lambda_B^2} + \frac{1}{2}\beta_C^{2\mathbb{Q}}(\lambda_C, t)\frac{\partial^2 \hat{V}}{\partial \lambda_C^2} + \rho\beta_B^{\mathbb{Q}}(\lambda_B, t)\beta_C^{\mathbb{Q}}(\lambda_C, t)\frac{\partial^2 \hat{V}}{\lambda_B \lambda_C} \\
&\quad + \lambda_B^{\mathbb{Q}}\left(LGD(t) - S\frac{t - T_{n-1}}{T_n - T_{n-1}}\right) + \lambda_C(R_C M(t,T)^+ + M(t,T)^-) \\
&\quad - (r + \lambda_B^{\mathbb{Q}} + \lambda_C^{\mathbb{Q}})\hat{V} = 0
\end{aligned} \tag{2.67}$$

for $t \in (T_{n-1}, T_n)$ with $n = 1, 2, ...N$ and the jump condition

$$\hat{V}(T_n^-, T) = \hat{V}(T_n, T) - S(T_n - T_{n-1}) \text{ for } n = 1, 2, ...N, \tag{2.68}$$

applies to (2.67).

In addition to the CDS contract without counterparty risk, the value changes due to holding a CDS contract sold by a defaultable seller will have an extra cash-flow in the case of a counterparty default, which is

$$\begin{aligned}
d\hat{V} = &\Bigg(\frac{\partial \hat{V}}{\partial t} + \alpha_B^{\mathbb{P}}(\lambda_B, t)\frac{\partial \hat{V}}{\partial \lambda_B} + \alpha_C^{\mathbb{P}}(\lambda_C, t)\frac{\partial \hat{V}}{\partial \lambda_C} + \frac{1}{2}\beta_B^{2\mathbb{P}}(\lambda_B, t)\frac{\partial^2 \hat{V}}{\partial \lambda_B^2} + \frac{1}{2}\beta_C^{2\mathbb{P}}(\lambda_C, t)\frac{\partial^2 \hat{V}}{\partial \lambda_C^2} \\
&+ \beta_B^{\mathbb{P}}(\lambda_B, t)\frac{\partial \hat{V}}{\partial \lambda_B}dW_B^{\mathbb{P}}(t) + \beta_C^{\mathbb{P}}(\lambda_C, t)\frac{\partial \hat{V}}{\partial \lambda_C}dW_C^{\mathbb{P}}(t) + \rho\beta_B^{\mathbb{P}}(\lambda_B, t)\beta_C^{\mathbb{P}}(\lambda_C, t)\frac{\partial^2 \hat{V}}{\partial \lambda_B \lambda_C}\Bigg)dt \\
&+ \left(LGD(t) - S\frac{s - T_{n-1}}{T_n - T_{n-1}} - \hat{V}\right)dH_t^B + (R_C M(t,T)^+ + M(t,T)^- - \hat{V})dH_t^C
\end{aligned}$$

for $t \in (T_{n-1}, T_n)$ with $n = 1, 2, ...N$.

We now construct a portfolio, which is composed of five instruments

$$\Pi = \hat{V} + \Delta_1 \tilde{B}_B + \Delta_2 \tilde{B}_C + \Delta_3 I_B + \Delta_4 I_C, \tag{2.69}$$

where two bonds are used to hedge out the intensity risk of party B and C and jump risks are hedged by the insurance products on firm B and C. Following a similar procedure as before when deriving the PDE for a counterparty default-free CDS, we find that in order to eliminate risk we must choose

$$\begin{aligned}
\Delta_1 &= -\frac{\partial \hat{V}}{\partial \lambda_B}\Big/\frac{\partial \tilde{B}_B}{\partial \lambda_B},\\
\Delta_2 &= -\frac{\partial \hat{V}}{\partial \lambda_C}\Big/\frac{\partial \tilde{B}_C}{\partial \lambda_C},\\
\Delta_3 &= \Delta_1(\tilde{B}_B - R_B(t)) - (LGD(t) - \hat{V}),\\
\Delta_4 &= \Delta_2(\tilde{B}_C - R_C(t)) - (R_C M(t,T)^+ + M(t,T)^- - \hat{V}).
\end{aligned}$$

So, if we look at the values changes from holding the entire portfolio, we have

$$d\Pi = \left(\mathcal{L}_{BC}\hat{V} + \Delta_1\mathcal{L}_B\tilde{B}_B + \Delta_2\mathcal{L}_C\tilde{B}_C - \Delta_3\lambda_B^{\mathbb{Q}} - \Delta_4\lambda_C^{\mathbb{Q}}\right)dt = r\Pi dt, \tag{2.70}$$

for $t \in (T_{n-1}, T_n)$ with $n = 1, 2, ...N$, where

$$\begin{aligned}
\mathcal{L}_{BC} &= \frac{\partial}{\partial t} + \frac{1}{2}\beta_B^{2\mathbb{P}}(\lambda_B, t)\frac{\partial^2}{\partial \lambda_B^2} + \frac{1}{2}\beta_C^{2\mathbb{P}}(\lambda_C, t)\frac{\partial^2}{\partial \lambda_C^2} + \rho\beta_B^{\mathbb{P}}(\lambda_B, t)\beta_C^{\mathbb{P}}(\lambda_C, t)\frac{\partial^2}{\lambda_B\lambda_C},\\
\mathcal{L}_B &= \frac{\partial}{\partial t} + \frac{1}{2}\beta_B^{2\mathbb{P}}(\lambda_B, t)\frac{\partial^2}{\partial \lambda_B^2},\\
\mathcal{L}_C &= \frac{\partial}{\partial t} + \frac{1}{2}\beta_C^{2\mathbb{P}}(\lambda_C, t)\frac{\partial^2}{\partial \lambda_C^2}.
\end{aligned}$$

Now since both the risk of changing default intensities and the risk of jump to default are hedged, this portfolio Π must have risk-free return. Note that (2.70) is a PDE containing three products $\tilde{B}_B$, $\tilde{B}_C$ and $\hat{V}$ then because $\tilde{B}_B$ and $\tilde{B}_C$ both satisfy (2.50), the following relationships will hold.

$$\begin{aligned}
\mathcal{L}_B\tilde{B}_B + \alpha^{\mathbb{Q}}\frac{\partial \tilde{B}_B}{\partial \lambda_B} - (r + \lambda_B^{\mathbb{Q}})\tilde{B}_B = 0,\\
\mathcal{L}_C\tilde{B}_C + \alpha^{\mathbb{Q}}\frac{\partial \tilde{B}_C}{\partial \lambda_C} - (r + \lambda_C^{\mathbb{Q}})\tilde{B}_C = 0.
\end{aligned} \tag{2.71}$$

Obviously, substituting (2.71) into (2.70) leads to

$$\begin{aligned}
&\frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\beta_B^{2\mathbb{P}}(\lambda_B,t)\frac{\partial^2 \hat{V}}{\partial \lambda_B^2} + \frac{1}{2}\beta_C^{2\mathbb{P}}(\lambda_C,t)\frac{\partial^2 \hat{V}}{\partial \lambda_C^2} + \rho\beta_B^{\mathbb{P}}(\lambda_B,t)\beta_C^{\mathbb{P}}(\lambda_C,t)\frac{\partial^2 \hat{V}}{\lambda_B \lambda_C} \\
&+ \frac{\partial \hat{V}}{\partial \lambda_B} \Big/ \frac{\partial \tilde{B}_B}{\partial \lambda_B}\left(\alpha_B^{\mathbb{Q}}(\lambda_B,t)\frac{\partial \tilde{B}_B}{\partial \lambda_B}\right) + \frac{\partial \hat{V}}{\partial \lambda_C} \Big/ \frac{\partial \tilde{B}_C}{\partial \lambda_C}\left(\alpha_C^{\mathbb{Q}}(\lambda_C,t)\frac{\partial \tilde{B}_C}{\partial \lambda_C}\right) \\
&+ \lambda_B^{\mathbb{Q}}\left(LGD(t) - S\frac{t-T_{n-1}}{T_n - T_{n-1}} - \hat{V}\right) + \lambda_C^{\mathbb{Q}}(R_C M(t,T)^+ + M(t,T)^- - V) - r\hat{V} = 0, \\
&\text{for } t \in [T_{n-1}^+, T_n^-], \text{ and } n = 1,2,....N.
\end{aligned}$$

We may note that (2.67) follows after the volatility terms in physical measure $\beta_B^{\mathbb{P}}(\lambda_B(t),t)$ and $\beta_C^{\mathbb{P}}(\lambda_C(t),t)$ are replaced by those under the risk-neutral measure $\beta_B^{\mathbb{Q}}(\lambda_B(t),t)$, $\beta_C^{\mathbb{Q}}(\lambda_C(t),t)$ respectively.

2.3.4 The PDE of unilateral CVA

In Section 1.2, we gave a generic definition of the CVA as presented in equation (1.3). However, the CVA is specific to a particular product, which means, that given the nature of a derivative, there may be slight differences in the definition of the CVA. For example, in a CDS contract, the counterparty risk occurs only when the first default event is the counterparty rather than the referencing firm. Therefore, (1.3) for a CDS contract should be rewritten as

$$\mathrm{CVA}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[D(t,\tau_C)(1-R(\tau_C))1_{\{\tau_C<T,\tau_C<\tau_B\}}\max\{M(\tau_C,T),0\}|\mathcal{G}_t\right], \quad (2.72)$$

where τ_C is the default time of counterparty, τ_B is the default time of the reference firm as to whom the CDS is written on and $M(\tau_C,T)$ is the CDS's value, using either the risky or risk-free close-out convention, at counterparty's default time. According to Lemma 2.2.2, (2.72) can be further rewritten as

$$\mathrm{CVA}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r(u)+\lambda_B(u)+\lambda_C(u)du}(1-R(\tau_C))\lambda_C(s)\max\{M(\tau_C,T),0\}|\mathcal{F}_t\right]. \quad (2.73)$$

In the previous sections, we have shown how the PDE for a payer CDS can be derived with and without counterparty risk. In this section, the PDE for the unilateral CVA of a the CDS contract is constructed following a similar method to Burgard and Kjaer (2011).

If we denote the CDS contract with counterparty risk as $\hat{V}$ and the one without counterparty risk as V then PDE (2.67) holds for $\hat{V}$. We know that the CVA is a value adjustment to the counterparty default-free contract, adjusted by the expected loss due to counterparty defaults. The relationship between the contract value with counterparty risk $\hat{V}$, the contract value without counterparty risk V and the CVA U is, by definition, $V - U = \hat{V}$ where U represents the CVA. By substituting $V - U = \hat{V}$ into PDE (2.67) we have

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \alpha_B(\lambda_B, t)\frac{\partial V}{\partial \lambda_B} + \alpha_C(\lambda_C, t)\frac{\partial V}{\partial \lambda_C} + \frac{1}{2}\beta_B^2(\lambda_B, t)\frac{\partial^2 V}{\partial \lambda_B^2} + \frac{1}{2}\beta_C^2(\lambda_C, t)\frac{\partial^2 V}{\partial \lambda_C^2} \\
&\quad + \rho\beta_B(\lambda_B, t)\beta_C(\lambda_C, t)\frac{\partial^2 V}{\lambda_B\lambda_C} + \lambda_B\left(LGD(t) - S\frac{t - T_{n-1}}{T_n - T_{n-1}}\right) \\
&\quad - (r + \lambda_B + \lambda_C)V \\
= &\frac{\partial U}{\partial t} + \alpha_B(\lambda_B, t)\frac{\partial U}{\partial \lambda_B} + \alpha_C(\lambda_C, t)\frac{\partial U}{\partial \lambda_C} + \frac{1}{2}\beta_B^2(\lambda_B, t)\frac{\partial^2 U}{\partial \lambda_B^2} + \frac{1}{2}\beta_C^2(\lambda_C, t)\frac{\partial^2 U}{\partial \lambda_C^2} \\
&\quad + \rho\beta_B(\lambda_B, t)\beta_C(\lambda_C, t)\frac{\partial^2 U}{\lambda_B\lambda_C} - \lambda_C(R_C M(t,T)^+ + M(t,T)^-) \\
&\quad - (r + \lambda_B + \lambda_C)U,
\end{aligned} \tag{2.74}$$

for $t \in (T_{n-1}, T_n)$ with $n = 1, 2, ...N$.

The value of V is independent of party C's default intensity since it is assumed that C is to be default free and so partial derivative terms of V with respect to λ_C are zero. Then PDE (2.74) changes to

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \alpha_B(\lambda_B, t)\frac{\partial V}{\partial \lambda_B} + \frac{1}{2}\beta_B^2(\lambda_B, t)\frac{\partial^2 V}{\partial \lambda_B^2} + \lambda_B\left(LGD(t) - S\frac{t - T_{n-1}}{T_n - T_{n-1}}\right) \\
&\quad - (r + \lambda_B + \lambda_C)V \\
= &\frac{\partial U}{\partial t} + \alpha_B(\lambda_B, t)\frac{\partial U}{\partial \lambda_B} + \alpha_C(\lambda_C, t)\frac{\partial U}{\partial \lambda_C} + \frac{1}{2}\beta_B^2(\lambda_B, t)\frac{\partial^2 U}{\partial \lambda_B^2} + \frac{1}{2}\beta_C^2(\lambda_C, t)\frac{\partial^2 U}{\partial \lambda_C^2} \\
&\quad + \rho\beta_B(\lambda_B, t)\beta_C(\lambda_C, t)\frac{\partial^2 U}{\lambda_B\lambda_C} - \lambda_C(R_C M(t,T)^+ + M(t,T)^-) \\
&\quad - (r + \lambda_B + \lambda_C)U,
\end{aligned} \tag{2.75}$$

for $t \in (T_{n-1}, T_n)$, and $n = 1, 2, ...N$. Because V satisfies the PDE (2.60), then substituting PDE (2.60) into the left hand side of equation (2.75), we obtain

$$\begin{aligned} &-\lambda_C V = \\ &\frac{\partial U}{\partial t} + \alpha_B(\lambda_B,t)\frac{\partial U}{\partial \lambda_B} + \alpha_C(\lambda_C,t)\frac{\partial U}{\partial \lambda_C} + \frac{1}{2}\beta_B^2(\lambda_B,t)\frac{\partial^2 U}{\partial \lambda_B^2} + \frac{1}{2}\beta_C^2(\lambda_C,t)\frac{\partial^2 U}{\partial \lambda_C^2} \\ &+ \rho\beta_B(\lambda_B,t)\beta_C(\lambda_C,t)\frac{\partial^2 U}{\lambda_B\lambda_C} - \lambda_C(R_C M(t,T)^+ + M(t,T)^-) \\ &-(r+\lambda_B+\lambda_C)U = -\lambda_C V, \end{aligned} \tag{2.76}$$

for $t \in (T_{n-1}, T_n)$, and $n = 1, 2, ...N$. In (2.76), there are two terms not associated with U, therefore, combining those terms yields

$$\begin{aligned} &\frac{\partial U}{\partial t} + \alpha_B(\lambda_B,t)\frac{\partial U}{\partial \lambda_B} + \alpha_C(\lambda_C,t)\frac{\partial U}{\partial \lambda_B} \\ &+ \frac{1}{2}\beta_B^2(\lambda_B,t)\frac{\partial^2 U}{\partial \lambda_B^2} + \frac{1}{2}\beta_C^2(\lambda_C,t)\frac{\partial^2 U}{\partial \lambda_C^2} + \rho\beta_B(\lambda_B,t)\beta_C(\lambda_C,t)\frac{\partial^2 U}{\lambda_B\lambda_C} \\ &+ \lambda_C\big(V - R_C M(t,T)^+ - M(t,T)^-\big) - (r+\lambda_B+\lambda_C)U = 0. \end{aligned} \tag{2.77}$$

Unlike (2.60) and (2.67), where discrete payments lead to jump conditions at payment time, (2.77) applies during the entire time span. When performing the subtraction $U = V - \hat{V}$, the discrete payments are cancelled.

Equation (2.77) is still an extremely generic equation since $M(t,T)$ has not been specified. In this thesis, risk-free close-out will be used for two reasons. First of all, the opinions that show support to a risky close-out, such as Brigo and Morini (2011); Gregory and German (2013), argue that when acquiring enquiring quotes for the contract value from other institutions, their quotes will reflect the creditworthiness of the firm, which is the DVA. However, DVA is not included in here since we assume the CDS buyer is default-free. More importantly, as shown by Burgard and Kjaer (2011), using risky close-out leads to a PDE for the CVA whose Feynman-Kac representation becomes a recursive formula, different from the classical definition of CVA in (1.3) and (1.4), which has appeared in the majority of the literature such as Gregory (2012) and Brigo et al. (2013).

If the contract close-out value at counterparty default has no reference to counterparty risk, we should specify that

$$M(t,T) = V(t,T), \tag{2.78}$$

where $V(t,T)$ satisfies PDE (2.60). Substituting equation (2.78) into the second to the last term of PDE (2.77), we have

$$\begin{aligned}
&\lambda_C\big(V - R_C M(t,T)^+ - M(t,T)^-\big)\\
=&\lambda_C\big(V - R_C V^+ - V^-\big)\\
=&\lambda_C\big(V^+ + V^- - R_C V^+ - V^-\big)\\
=&\lambda_C\big(V^+ - R_C V^+\big)\\
=&\lambda_C(1 - R_C)V^+.
\end{aligned}$$

Then (2.77) becomes,

$$\begin{aligned}
&\frac{\partial U}{\partial t} + \alpha_B(\lambda_B,t)\frac{\partial U}{\partial \lambda_B} + \alpha_C(\lambda_C,t)\frac{\partial U}{\partial \lambda_B}\\
&+\frac{1}{2}\beta_B^2(\lambda_B,t)\frac{\partial^2 U}{\partial \lambda_B^2} + \frac{1}{2}\beta_C^2(\lambda_C,t)\frac{\partial^2 U}{\partial \lambda_C^2} + \rho\beta_B(\lambda_B,t)\beta_C(\lambda_C,t)\frac{\partial^2 U}{\lambda_B\lambda_C}\\
&+\lambda_C(1-R_C)V^+ - (r+\lambda_B+\lambda_C)V = 0.
\end{aligned} \tag{2.79}$$

Finally, if we use the risk-free close-out, $M(t,T) = V(t,T)$, and apply to equation (2.73), the equation becomes

$$U(t,T,\lambda_C,\lambda_B) = \mathbb{E}^Q\left[(1-R_C)\int_t^T e^{-\int_t^s r(s)+\lambda_B(s)+\lambda_C(s)}\lambda_C(s)V(s,T,\lambda_B(s))^+ ds\middle|\mathcal{F}_t\right]. \tag{2.80}$$

Obviously, the terminal condition $U(T,T,\lambda_C(t),\lambda_B(t)) = 0$ satisfies condition (2.6) and $\lambda_C(s)V(s,T,\lambda_B(s))^+$ satisfies condition (2.7) because a CDS's value $V(t,T,\lambda_B(t))$ is bounded by the default payoff. After assuming the value function U to be $C^{1,2}$, we can apply Feynman-Kac Theorem to (2.80) and arrive at the exact same PDE as (2.79).

2.4 Summary

In this chapter we discuss and give detailed derivations of the PDEs for the valuation of default bonds, CDS with or without counterparty risk and the CVA under reduced-form modelling with counterparty risk. We show the PDEs for credit claims as well as CVA can be constructed using a risk-free portfolio, which are identical to the ones constructed by applying Feynman-Kac theorem. However, these PDEs are still generic because particular stochastic processes are not specified for the default intensities.

In the next chapter, we will specify a popular stochastic processes for firms' default intensity and the PDEs are ready to be solved. The numerical issues, such as the finite-difference schemes, boundary conditions and the rate of convergence, will be discussed.

Chapter 3

Pricing CDS and CVA Numerically

3.1 Introduction

The pricing of financial derivatives forms the majority of problems in mathematical finance. Most of the time, with no fully analytic solutions available, academics and participants have to resort to approximations or alternatively efficient numerical methods. Here, we know that the CDS could be priced in a semi-analytical form with some conditions satisfied according to Section 2.3.2. However, it is not normally the case that an analytic solution can be found for derivative pricing problems, especially when the model specification is more complicated or the problem itself is complex, for example the CVA. So in order that we will be able to solve more complex models and gain insight into the solutions, we will be generally required to use a numerical method.

One common approach to solving numerically complex PDEs in applied mathematics is that of finite differences. All of the relevant derivatives pricing problems in this thesis can be shown to be described by a PDE with relevant boundary conditions, for example (2.60) for a CDS contract. One advantage of finite-difference methods is that they can be very flexible for pricing certain types of exotic derivatives such as those with early exercise options. The fact that solving a PDE stores the values of the derivative of the underlying asset's all possible levels is a big advantage which can help us to identify the exposure at default (1.1) and the valuation of CVA. For example, pricing a simple European vanilla option with finite differences will give us a matrix of the option's values corresponding to different stock price level and time to maturity. Therefore, it is easy for us to quantify the option buyer's losses due to the option seller defaults at any time before the option's maturity. Once we have decided to apply finite differences there are a number of different schemes such as explicit, implicit, Crank-Nicolson and alternating direction implicit (ADI) whose convergence rates, stability conditions and behaviours are well studied, see Wilmott et al. (1995) and Duffy (2006). Here we choose to use the Crank-Nicolson scheme for solving the one-dimensional PDEs and an ADI scheme for two-dimensional PDEs because they are second-order accurate in time and space. In addition, the Crank-Nicolson scheme is unconditionally stable and the ADI scheme (without correlation) is also unconditionally stable.

The value of a derivative is a function of time to maturity and underlying assets' level. The PDEs will describe how value changes according to movements in time and asset level. If the conditions at boundaries are given, the derivative value can be known at any time and asset level according to the PDE. Typically finite-difference methods require a truncation of the domain and discretisation in both space and time. Truncating the domain fixes a finite region to approximate an infinite domain. The discretisation usually divides the domain into an equally spaced grid in spatial dimension, although it is possible to apply non-standard grids for special cases such as In't Hout and Foulon (2010).

In this chapter, we begin with a background introduction of finite-difference methods. As a benchmark to check the accuracy of the schemes a simple CDS contract will be valued using a finite-difference scheme. We will show that the numerical solution converges to the semi-analytical solution if appropriate boundary conditions are applied. Later, we will introduce the second firm, who is selling the CDS and defaultable, and solving the CVA. This requires a finite-difference method for two-dimensional PDEs; in this particular case we implement an ADI scheme. The details for implementing ADI scheme and numerical solutions for CVAs are given.

3.2 Preliminaries of the Finite-Difference Method

Let us consider a generic differential equation generated by SDE (2.30) which takes the form,

$$\frac{\partial V}{\partial t} + \alpha(\lambda, t)\frac{\partial V}{\partial \lambda} + \frac{1}{2}\beta^2(\lambda, t)\frac{\partial^2 V}{\partial \lambda^2} + f(\lambda, t) - g(\lambda, t)V = 0, \tag{3.1}$$

defined in $t \in [0, T] \times \lambda \in [\lambda_{min}, \lambda_{max}]$.

Typically, the solution space is discretised with $M + 1$ points in asset level, or space, and $N + 1$ points in time. We may write

$$\Delta t = \frac{T}{N}, \qquad t = n\Delta t$$
$$\Delta\lambda = \frac{\lambda_{max} - \lambda_{min}}{M}, \qquad \lambda = m\Delta\lambda$$

and then

$$V(\lambda, t) = V_m^n.$$

3.2.1 Crank-Nicolson method for one-dimensional PDEs

The Crank-Nicolson method has been discussed in many finance literature of its efficiency and stability for solving PDEs for evaluating derivatives, such as Wilmott et al. (1995) and Duffy (2006). Thus it will be used throughout this thesis for solving PDE with one asset and here we take some time to discuss it in detail. The advantage of a Crank-Nicolson scheme is that it is both unconditionally stable and exhibits a second-order rate of convergence in space and time for sufficiently smooth boundary conditions. The approximations for V and its derivatives are as follows,

$$\frac{\partial V}{\partial t} \approx \frac{V_m^n - V_m^{n+1}}{\Delta t}$$
$$\frac{\partial V}{\partial \lambda} \approx \frac{1}{2}\left(\frac{V_{m+1}^{n+1} - V_{m-1}^{n+1}}{2\Delta\lambda} + \frac{V_{m+1}^n - V_{m-1}^n}{2\Delta\lambda}\right)$$
$$\frac{\partial^2 V}{\partial \lambda^2} \approx \frac{1}{2}\left(\frac{V_{m+1}^{n+1} - 2V_m^{n+1} + V_{m-1}^{n+1}}{\Delta\lambda^2} + \frac{V_{m+1}^n - 2V_m^n + V_{m-1}^n}{\Delta\lambda^2}\right)$$
$$V(\lambda, t) \approx \frac{1}{2}\left(V_m^{n+1} + V_m^n\right).$$

After substituting into (3.1) and performing simple algebra, collecting terms we can write

$$a_m V_{m-1}^n + b_m V_m^n + c_m V_{m+1}^{n+1} = d_m,$$

where

$$
\begin{aligned}
a_m &= -\alpha_m \frac{1}{4\Delta\lambda} + \frac{1}{4\Delta\lambda^2}\beta_m^2 \\
b_m &= -\frac{1}{\Delta t} - \beta_m^2 \frac{1}{2\Delta\lambda^2} - \frac{1}{2} g_m^n \\
c_m &= +\alpha_m \frac{1}{4\Delta\lambda} + \frac{1}{4\Delta\lambda^2}\beta_m^2 \\
d_m &= -a_m V_{m-1}^{n+1} + \left(\frac{1}{\Delta t} + \beta_m^2 \frac{1}{2\Delta\lambda^2} + \frac{1}{2} g_m^{n+1}\right) V_m^{n+1} - c_m V_{m+1}^{n+1} - \frac{1}{2}(f_m^n + f_m^{n+1}).
\end{aligned}
$$

The result is a tridiagonal system of linear equations to solve at each time step from maturity T backwards to the present. The system is simple to solve and we use the popular Thomas algorithm when coding the problems in this thesis.

3.2.2 Numerical schemes for multi-dimensional PDEs

The main difficulty when it comes to evaluating the CVA is coping with the extra dimension. Finite-difference solutions for high dimensional PDE are frequently mentioned in the option-pricing literature, which has been well developed. There is however a dearth of literature in finance on lattice based numerical schemes when evaluating counterparty risk, as most authors choose simulation and semi-analytic solution with simplified model. If we consider a standard option-pricing problem, there are two situations that can lead to a high dimensional problem. Options could be written on multi-assets such as maxi-minimum options, exchange options and convertible bonds. Alternatively, complicated models may occur in option pricing with stochastic volatility and interest rates, see In't Hout and Foulon (2010) and Haentjens and In't Hout (2012) for more details. Johnson (2008) discussed extensively finite-difference schemes for pricing multi-asset options and improves the numerical efficiency with knowledge of the solution topology, using advanced matrix solvers and an appropriate transformation of the PDE.

Before solving the PDE, we have a choice such as between the alternating direction implicit scheme and the Crank-Nicolson scheme. In the two dimensional case, a general PDE discretised by Crank-Nicolson scheme can be written in the following form,

$$
\begin{aligned}
&\alpha_{i,j} u_{i-1,j}^k + \left(\frac{1}{\Delta t} + \beta_{i,j}\right) u_{i,j}^k + \gamma_{i,j} u_{i+1,j}^k + \delta_{i,j} u_{i,j-1}^k + \epsilon_{i,j} u_{i,j+1}^k \\
&\qquad + \zeta_{i,j}^k \left(u_{i-1,j-1}^k - u_{i-1,j+1}^k - u_{i+1,j-1}^k + u_{i+1,j+1}^k\right) = \eta_{i,j}^{k+1}.
\end{aligned}
\tag{3.2}
$$

For the one-dimensional case, we have already discussed how the PDE discretised by Crank-Nicolson can be solved directly by Thomas algorithm. However, in the two-dimensional case, the above equation leads to a problem of solving a sparse matrix system of the form,

$$\mathbf{A}u^k = \mathbf{B}^{k+1}, \tag{3.3}$$

where $\mathbf{A}$ is the sparse matrix. Johnson (2008) compares the direct and iteration solvers for the sparse system in European option pricing problems. The direct solver is more difficult to code, which involves inverting a large matrix, using a Big-Banded Solver (BBS). Alternatively, the system might be solved by iteration schemes such as point SOR or line SOR. However, there are circumstances where the rate of convergence can be slow such as at high volatilities. The solution matrix is a I by J matrix, where I, J are the number of nodes in each dimension. Instead of solving the $I \times J$ matrix simultaneously like BBS, point SOR solves

$$\begin{aligned} u_{i,j}^{k,q+1} = \frac{1}{\frac{1}{\Delta t} + \beta_{i,j}} \Bigg(& \eta_{i,j}^{k+1} - \alpha_{i,j} u_{i-1,j}^{k,q+1} + \gamma_{i,j} u_{i+1,j}^{k,q} + \delta_{i,j} u_{i,j-1}^{k,q+1} + \epsilon_{i,j} u_{i,j+1}^{k,q} \\ & + \zeta_{i,j}^{k} \left(u_{i-1,j-1}^{k,q+1} - u_{i-1,j+1}^{k,q} - u_{i+1,j-1}^{k,q+1} + u_{i+1,j+1}^{k,q} \right) \Bigg), \end{aligned} \tag{3.4}$$

where q denotes the number of the iteration. Alternatively, line SOR solves

$$\begin{aligned} & \alpha_{i,j} u_{i-1,j}^{k,q+1} + \left(\frac{1}{\Delta t} + \beta_{i,j} \right) u_{i,j}^{k,q+1} + \gamma_{i,j} u_{i+1,j}^{k,q+1} = \\ & \eta_{i,j}^{k+1} - \delta_{i,j} u_{i,j-1}^{k,q+1} - \epsilon_{i,j} u_{i,j+1}^{k,q} - \zeta_{i,j}^{k} \left(u_{i-1,j-1}^{k,q+1} - u_{i-1,j+1}^{k,q} - u_{i+1,j-1}^{k,q+1} + u_{i+1,j+1}^{k,q} \right), \end{aligned} \tag{3.5}$$

Note that, (3.5) suggests, that beginning with $j = 0$, we find the values over all i in the line with a fixed j using $\mathbf{LU}$ decomposition or Gaussian elimination. Then we can go through all values of j to solve the entire matrix. This procedure is repeated until the change in the solution at the next iteration is lower than a given tolerance. We may interchange the directions for better accuracy. In addition, Saad and Schultz (1986) also propose an iteration method named GMRES for solving non-symmetric linear systems. This algorithm approximates the solution u^k using the order-d^{th} *Krylov* subspace generated by the sparse matrix A and $\mathbf{B}^{k+1}$ in linear system (3.3). The solution u^k will converge to the exact solution after the number of iteration equals to the dimension of $\mathbf{A}$ but the algorithm may stop when the error is lower than tolerance. If the sparse matrix $\mathbf{A}$ is not too far from normality, the GMRES algorithm has fast convergence. This means after a small number of iteration the order-d^{th} *Krylov* subspace is a good approximation to the exact solution.

When a two or higher dimensional PDE is discretised according to Crank-Nicolson, it inevitably leads to a large sparse matrix equation requiring iteration schemes or a direct solver. There are alternative way to discretise the problem and Peaceman and Rachford (1955) propose the Alternating Direction Implicit (ADI) scheme to avoid solving a complicated system while maintaining both fast convergence in time and (asset level) spaces and is unconditional stability. The ADI scheme works particularly well when solving two-dimensional diffusion equations. Douglas and Rachford (1956) extend the original ADI scheme to solve two-dimensional PDEs with a correlation term. The scheme shows unconditional stability with first order rate of convergence whenever the correlation term is not zero. Craig and Sneyd (1988) improve the previous scheme to arrive at second order convergence in space even with the correlation term, namely the CS scheme. In't Hout and Welfert (2009) generalise the scheme from Craig and Sneyd (1988) to allow a free choice for step size parameter θ, which is similar to how the Crank-Nicolson scheme is a special case of a θ-scheme with $\theta = \frac{1}{2}$.

Lipton (2001) applies an ADI scheme for solving a two-dimensional PDE arising from pricing foreign exchange options. In't Hout and Foulon (2010) compare three different ADI schemes with the two-dimensional PDEs derived from option pricing with Hestons stochastic volatility model. The unconditional stability and convergence properties are observed in an experiment involving pricing European options and also down-and-out options. In addition, they also introduce a non-uniform grid. More grid nodes are placed to the region where the stock price and the volatility will likely to be. Consequently, accuracy increases in those regions and we can save on computational effort. Haentjens and In't Hout (2012) extend the pricing problem to a Hull-White stochastic interest rate and Heston volatility, where the resulting PDE is now extended to three dimensions. They test a variety of scenarios including a full range of correlation, time-dependent interest and satisfaction of the *Feller* condition. Their tests have shown that all ADI schemes perform very well in terms of stability, accuracy and efficiency. In particular they always maintain an unconditionally stable behaviour.

In the original ADI scheme, one backward step in time is split into two steps. At the first step, one dimension, such as λ_1, is discretised with an implicit scheme while the other dimension is discretised with an explicit scheme. At the next time step, we treat λ_2 as implicit and λ_1 as explicit. In the two-dimensional case, a general PDE discretised as an ADI scheme can be written into the following form,

$$\alpha_{i,j}\bar{u}_{i-1,j} + \left(\frac{1}{\frac{1}{2}\Delta t} + \beta_{i,j}\right)\bar{u}_{i,j} + \gamma_{i,j}\bar{u}_{i+1,j} = \eta_{i,j}^{k+1} \tag{3.6a}$$

$$\delta_{i,j}u_{i,j-1}^{k} + \left(\frac{1}{\frac{1}{2}\Delta t} + \epsilon_{i,j}\right)u_{i,j}^{k} + \epsilon_{i,j}u_{i,j+1}^{k} = \bar{\eta}_{i,j}^{k+1}, \tag{3.6b}$$

for all i with a fixed j, where $\bar{u}$ are intermediate solutions and

$$\begin{aligned}
\eta_{i,j}^{k+1} =& -\delta_{i,j}u_{i,j-1}^{k+1} + \left(\frac{1}{\frac{1}{2}\Delta t} - \epsilon_{i,j}\right)u_{i,j}^{k+1} - \zeta_{i,j}u_{i,j+1}^{k+1} \\
& - \mu(u_{i-1,j-1}^{k+1} - u_{i-1,j+1}^{k+1} - u_{i+1,j-1}^{k+1} + u_{i+1,j+1}^{k+1}) \\
\bar{\eta}_{i,j}^{k+1} =& -\alpha_{i,j}\bar{u}_{i-1,j} + \left(\frac{1}{\frac{1}{2}\Delta t} - \beta_{i,j}\right)\bar{u}_{i,j} - \gamma_{i,j}\bar{u}_{i,j+1} \\
& - \mu\left(\bar{u}_{i-1,j-1} - \bar{u}_{i-1,j+1} - \bar{u}_{i+1,j-1} + \bar{u}_{i+1,j+1}\right).
\end{aligned} \tag{3.7}$$

When discretising in the ADI scheme, the problem of solving a $I \times J$ matrix is broken down into solving I vectors, each with J elements, at one step and J vectors, each with I elements, at the other step. For each column or row, the problem can be written in matrix form as,

$$\mathbf{A}u^k = \mathbf{B}^{k+1},$$

where $\mathbf{A}$ is a tridiagonal matrix and can be solved directly. Details of solving equation system (3.20) are shown in Section 3.4.2.

Throughout this thesis, we choose the ADI scheme from Douglas and Rachford (1956) for two-dimensional problems. We believe that this is the best suited to our problem. Johnson (2008) shows by use of the dual-strike European option that the computational times of the SOR solver and the Big-Banded solver are similar. At the same time, the ADI scheme does not require the inversion of a large sparse matrix and the times of solution estimation at each time step is less than iterative solvers. For example it has been demonstrated by Villeneuve and Zanette (2002) that solving for an American option with a modified ADI scheme is around five times faster than the corresponding iterative solver PSOR.

We note that the development of ADI schemes has been driven by the approach to accommodate mixed spatial derivatives due to correlation, such as Douglas and

Rachford (1956), Craig and Sneyd (1988) and In't Hout and Welfert (2009). These all introduce more intermediate steps when solving each step backward in time, which require more computational effort and can be difficult to implement. Since the latter chapters in this thesis focus on modelling default correlation in terms of default contagions, correlating the Brownian motions of default intensities is not necessary so we need not worry about accommodating the correlation term. The most efficient way for us to proceed is to apply the finite-difference scheme of Douglas and Rachford (1956), which always treat the spatial derivative term explicitly. This scheme is efficient and accurate for almost all of the important scenarios of our research in this thesis, where the correlation between Brownian motions is not considered.

3.3 Solving CDS Using Finite Differences

In Section 2.3.2, we derived the PDE for a CDS contract which also has analytic solutions. In order to benchmark the numerical scheme and check on the appropriateness of our boundary conditions we will present numerical solutions for this problem, and show how it converges to the analytic solution. It is common in the literature on new reduced-form models for the authors to put much of their emphasis on deriving the default-time distribution and pricing credit claims analytically. To the best of my knowledge, there is little research on the numerical issues of solving this type of derivative contract under the reduced-form framework. However, we will see that the finite-difference method has the advantage of providing exposures at default, which is defined by (1.1), for the computation of CVA, where it may be computationally expensive to recalculate the analytic solution repeatedly. Moreover, later on in this thesis we propose complex models for which numerical solutions are required for pricing CDSs and evaluating CVAs. Thus, the limitation and effectiveness in solving the PDE (2.60) is important and is therefore discussed in detail here and improvements are considered in chapters 5, 6.

3.3.1 Model specification

In the original derivation of PDE (2.60) for evaluating CDS, we gave a general model, where $\alpha(\lambda, t)$ and $\beta(\lambda, t)$ are not specified. In order to obtain some meaningful results, we employ CIR process for (2.30) as is standard in the literature including but not limited to Duffie and Singleton (1999), Duffie (2005) and Brigo and Chourdakis (2009),

$$d\lambda(t) = \kappa(\theta - \lambda(t))dt + \sigma\sqrt{\lambda(t)}dW(t). \tag{3.8}$$

Equation (3.8) is a mean-revering process or mean-reverting square-root process, where θ is the mean rate of default, κ is the speed of mean reversion, σ is the volatility of default intensity and $W(t)$ is a standard Brownian Motion. When the intensity $\lambda(t)$ is above/below its mean rate θ, the deterministic term will push the intensity down towards the mean at the rate of κ. It is the Brownian motion component that introduces the stochastic feature into the process. In addition, it is very easy to verify (3.8) satisfies (2.3), which is essential for apply Feynman-Kac theorem. Due to the fact that the variance of the process is related to the square-root of asset level, the process is non-negative and its distribution follows a non-central χ^2 distribution.

The popularity of using this process to model rates is because of its strictly positive property, which is a requirement in reduced-form models for the non-decreasing condition of the hazard process $\Lambda(t) = \int_0^t \lambda(s)ds$. Intuitively, the process $\lambda(t)$ represents the rate of an event happening, which should not be negative. Although for convenience when deriving a closed form solution, some previous research uses Gaussian type default intensities, which violate the non-negative condition, for example Höcht and Zagst (2010), Filipović and Trolle (2013) and Wang and Ye (2013).

Modelling default intensity as a CIR process omits close-form solutions to survival probability, see Lando (1998) for more details, therefore a semi-analytic solution exists to CDS, which we will benchmark against. In terms of the PDE, (2.60) changes to

$$\frac{\partial V}{\partial t} + \kappa(\theta - \lambda)\frac{\partial V}{\partial \lambda} + \frac{1}{2}\sigma\lambda\frac{\partial^2 V}{\partial \lambda^2} + \lambda\left((1-R)e^{-r(T-t)} - S\frac{t - T_{n-1}}{T_n - T_{n-1}}\right) - (r+\lambda)V = 0, \tag{3.9}$$

for $t \in (T_{n-1}, T_n)$ with $n = 1, ..., N$ and subject to the terminal condition

$$V(T) = 0,$$

and jump conditions

$$V(T_n^-) = V(T_n) - S(T_n - T_{n-1}) \quad \text{for} \quad n = 1, 2, ..., N.$$

Unlike options where just the terminal condition provides the payoff, the CDS contract has no cash flows at maturity; hence the condition at maturity is zero. However, we can see from the PDE that the cash flows that occur in the CDS are represented as both jump conditions at pre-specified times and the term $\lambda\left((1-R)e^{-r(T-t)} - S\frac{t-T_{n-1}}{T_n - T_{n-1}}\right)$ in the PDE rather than a terminal condition as they include both the swap premium and the default payoff.

3.3.2 Boundary conditions

Boundary conditions specify either the value or the behaviour of the value when the independent variable goes to its lower limit and upper limit. In our case, the lower limit is $\lambda \to 0$ and the upper limit is $\lambda \to \infty$. The infinite domain of $[0, \infty] \times [0, T]$ should be truncated into a finite domain $[0, \lambda_{max}] \times [0, T]$, where λ_{max} should be sufficiently large so as not to disturb the valuation in the region of interest. Although the CIR process does not allow the intensity to be zero, a condition has to be specified at $\lambda_{min} = 0$. There are three common types of boundary conditions, namely Dirichlet, Neumann or mixed (Robin) type conditions. A boundary condition that specifies the value of the function itself is a Dirichlet boundary condition, or a first-type boundary condition. A boundary condition, which specifies the value of the derivative of the function, is a Neumann boundary condition, or a second-type boundary condition. Finally, we have mixed or Robin-type conditions which contain both the value and its first derivative. In this section, we test a Dirichlet, a Robin and a heuristic Robin condition, which we will numerically justify.

The determination of lower boundary condition at $\lambda \to 0$ is straightforward. By substituting $\lambda = 0$ to the PDE (3.9) directly, this give us a Robin-type boundary condition at $\lambda \to 0$, which is

$$\frac{\partial V}{\partial t} + \kappa\theta\frac{\partial V}{\partial \lambda} - rV = 0, \quad \text{for} \quad \lambda \to 0. \tag{3.10}$$

This boundary condition shows the CDS value changes over time depend not only on the risk-free interest rate but also the fact that the default intensity returns to its mean value θ at the mean-reverting speed κ. This boundary condition at $\lambda \to 0$ is applied the PDE (3.9) no matter which upper boundary condition will be applied at $\lambda \to \infty$.

For the upper boundary at $\lambda \to \infty$, on the other hand, a condition cannot be so easily obtained and we are required to further analyse the PDE (3.9). When $\lambda \to \infty$, the referencing firm is very likely to default shortly so the CDS value V is less sensitive to the time to maturity, so we should find that $\frac{\partial V}{\partial t}$ is relatively small compared to other terms. If we ignore the time derivative term, PDE (3.9) reduces to an inhomogeneous ordinary differential equation (ODE),

$$\kappa(\theta - \lambda)\frac{\partial V}{\partial \lambda} + \frac{1}{2}\sigma\lambda\frac{\partial^2 V}{\partial \lambda^2} + \lambda\left((1-R)e^{-r(T-t)} - S\frac{t - T_{n-1}}{T_n - T_{n-1}}\right) - (r+\lambda)V = 0. \quad (3.11)$$

However, solution of (3.11) is still very complicated with a Hyper-geometric function and a Laguerre function, which makes it quite difficult to analyse its behaviour when $\lambda \to \infty$. More simplifications have to be made to (3.11) in order to derive a Dirichlet or Neumann condition for the upper boundary. If we look at the limit when $\lambda \to \infty$, (3.11) will be dominated by the terms associated with λ, so we consider only those terms leading to an inhomogeneous ODE

$$\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial \lambda^2} - \kappa\frac{\partial V}{\partial \lambda} + \left((1-R)e^{-r(T-t)} - S\frac{t - T_{n-1}}{T_n - T_{n-1}}\right) - V = 0, \quad (3.12)$$

whose solution V has the form $V = A_1 e^{\alpha\lambda} + A_2 e^{-\alpha\lambda} + f(t)$, where

$$\alpha = \frac{\kappa - \sqrt{\kappa^2 + 2\sigma^2}}{\sigma^2}.$$

Dirichlet condition

If we only consider the solution to be a time function $f(t)$, it is easy to verify that

$$V_p = f(t) = (1-R)e^{-r(T-t)} - S\frac{t - T_{n-1}}{T_n - T_{n-1}} \quad (3.13)$$

can be a particular solution to (3.12). Clearly, in the limit $\lambda \to \infty$, then it implies we have an instant default and the protection's value is then the default payoff equals the $Z(t)$ in (2.54).

Robin condition

As mentioned earlier, the ODE (3.12) has a solution of the form $V = A_1 e^{\alpha\lambda} + A_2 e^{-\alpha\lambda} + f(t)$ and we only look for the solution with exponential decay, which is the part $V = A_1 e^{\alpha\lambda} + f(t)$. The former part $A_1 e^{\alpha\lambda}$ can be viewed as the general solution V_g and the latter part $f(t)$ can be viewed as the particular solution V_p to the ODE (3.12). Therefore, we are able to apply the Robin-type condition

$$\frac{\partial V}{\partial \lambda} = \alpha(V - V_p) \quad (3.14)$$

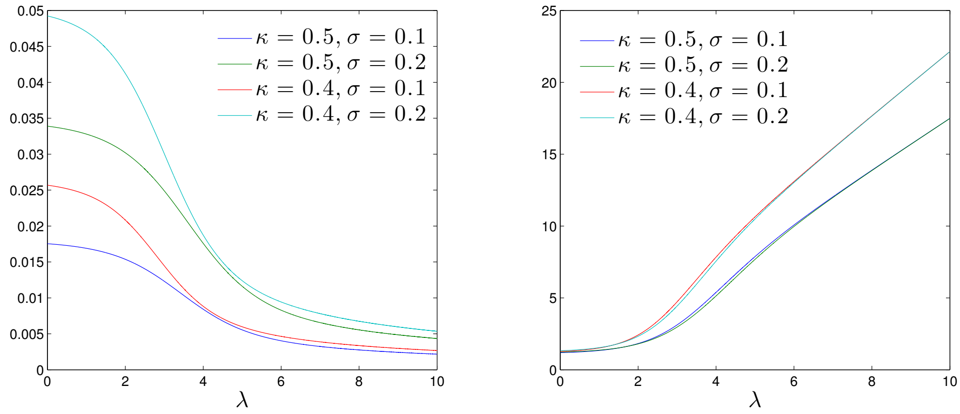

Figure 3.1
The ratio of the dominate components of PDE (3.9) at $\lambda \to \infty$
The left panel shows the ratio of $-\kappa\lambda\frac{\partial V_g}{\partial\lambda}$ to $\frac{1}{2}\sigma^2\lambda\frac{\partial^2 V_g}{\partial\lambda_2^2}$. The right panel shows the ratio of $-\lambda V_g$ to $-\kappa\lambda\frac{\partial V_g}{\partial\lambda}$. Both subject to different volatilities and mean-reverting rate.

for $\lambda \to \infty$.

Heuristic Robin Condition

We apply a heuristic Robin-type condition commonly used in finance which is for discarding the second derivative.

$$\frac{\partial V}{\partial t} + \kappa(\theta - \lambda)\frac{\partial V}{\partial \lambda} + \lambda\left((1-R)e^{-r(T-t)} - S\frac{t - T_{n-1}}{T_n - T_{n-1}}\right) - (r+\lambda)V = 0, \quad \text{for} \quad \lambda \to \infty. \tag{3.15}$$

Although this is not rigorous, we can present some numerical justification. At the upper bound $\lambda \to \infty$, we notice that the second derivative term is far less than the first derivative term, which is

$$\kappa(\theta - \lambda)\frac{\partial V}{\partial \lambda} >> \frac{1}{2}\sigma^2\lambda\frac{\partial^2 V}{\partial \lambda^2}. \tag{3.16}$$

As discussed earlier, the CDS solution V can be viewed as the sum of general solution V_g and the particular solution V_p, which is a time function irrelevant to $\lambda(t)$. The CDS value decays to the particular solution exponentially when $\lambda \to \infty$. Therefore, the behaviour of the solution V with $\lambda(t)$ is driven by the general solution, which satisfies

$$\frac{\partial V_g}{\partial t} + \kappa(\theta - \lambda)\frac{\partial V_g}{\partial \lambda} + \frac{1}{2}\sigma^2\lambda\frac{\partial^2 V_g}{\partial \lambda_2^2} - (r+\lambda)V_g = 0, \tag{3.17}$$

which solves the difference between the CDS value V and the default payoff V_p.

Figure (3.1) shows the ratios among the three terms associated with $\lambda(t)$ in (3.17), which are $-\kappa\lambda\dfrac{\partial V_g}{\partial \lambda}$, $\dfrac{1}{2}\sigma^2\lambda\dfrac{\partial^2 V_g}{\partial \lambda_2^2}$ and $-\lambda V_g$. The left panel of 3.1 plots $\dfrac{\frac{1}{2}\sigma^2\lambda\frac{\partial^2 V_g}{\partial \lambda_2^2}}{-\kappa\lambda\frac{\partial V_g}{\partial \lambda}}$ and the right panel plots $\dfrac{-\lambda V_g}{-\kappa\lambda\frac{\partial V_g}{\partial \lambda}}$. Clearly, the second derivative term $\dfrac{1}{2}\sigma^2\lambda\dfrac{\partial^2 V_g}{\partial \lambda_2^2}$ is far less than the first derivative term $-\kappa\lambda\dfrac{\partial V_g}{\partial \lambda}$ as the ratio is always in two decimal places. Because the second derivative term is scaled down by $\dfrac{1}{2}\sigma^2\lambda$, which will be far less than the scale $-\lambda\kappa$ given the *Feller* condition is satisfied. In addition, the first derivative term is far less than the term $-\lambda V_g$ as the ratio much greater than 1. Consequently, (3.16) can be satisfied with large enough λ if the *Feller* condition holds. Therefore, the second derivative term can be omitted and the PDE (3.9) reduces to (3.15). Imposing the heuristic Robin conditions (3.15) is seeking a solution with exponential decay behaviour at the boundaries.

3.3.3 Numerical results

First we will present results for the semi-analytic solution of CDS (2.56), which is an integral over the time to maturity of the CDS. We choose to solve the integral with a trapezoidal rule. Figure 3.2 shows the convergence of the CDS value with reducing the step-size. The semi-analytic result converges to -0.0244470233253265 or 224.47 basis points (*bps*) with the error lower than 10^{-9}. This result is used as the benchmark solution to compare against fully numeric results.

The PDE (2.60) contains jump-style conditions, which are due to the swap premiums are paid at discrete times. When the time period $[t, T]$ is equally divided into $N+1$ nodes, N must be divided by the premium payment period $T_i - T_{i-1}$ to ensure we can implement the jump conditions at the correct swap premium settlement times. Figure 3.3 presents a set of CDS contract values solved with finite-difference over varying time to maturity. After the settlement dates, the CDS value exhibits an upward jump because a payment is made and the future payments the buyer is expected to pay lessen, which leads to the saw shape of the solution. Between two settlement times, the value will decrease because of discounting.

There are two reasons why the value to the CDS buyer increases as time approaches maturity. As the number of future payments decreases, the protection is more valuable. In addition, since we apply *recovery of treasury*, the default payoff will be higher with

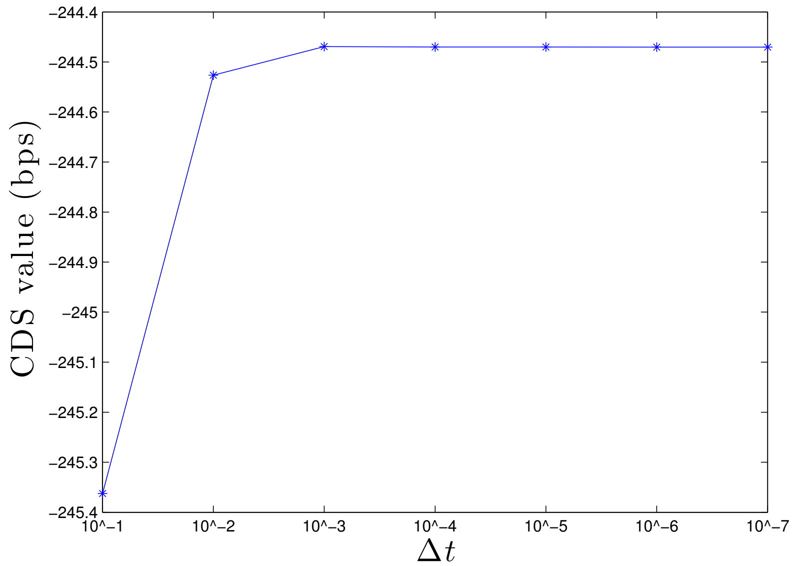

Figure 3.2
The Convergence of CDS's semi-analytic solution implemented with Trapezoidal rule
Convergence of a CDS protection $w.r.t$ the size of time step $\Delta t = \frac{T}{N}$. Parameter choose are $T = 10$, $S = 0.03$, $R = 0.4$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$, $\lambda(0) = 0.05$ and swap premiums are paid quarterly, which is $T_i - T_{i-1} = 0.25$ for all i.

a shorter time to maturity. On the other hand, the value of a protection also depends on the probability that the entity defaults before maturity. This causes the value to slump if there is little time before the protection ends. For instance, the light blue curve in figure 3.3 remains at a high level when the maturity is less than three years because the intensity of default is high and default within 10 years is very likely. But the value slumps when there are three years to maturity remaining. The reduction in default probability is so significant that it dominates other effects that make the CDS value increases.

In addition to the variations in τ space, the solutions in λ space are shown in figure 3.4, which gives a graphical representation of how the Dirichlet, Robin and heuristic Robin conditions affects the numerical solution in λ space. The values of $\Delta\lambda$ and Δt and all other parameters are the same, apart from the three boundary conditions suggesting that the difference can be solely attributed to these. Due to the nature of reduced-form modelling, although the intensity is infinity high, this does not imply that the firm will default instantly. Consequently, since the infinite domain has to be truncated into a finite one, the Dirichlet condition will only provide satisfactory results with a sufficiently large λ_{max}, which in turn requires extra computational time if we

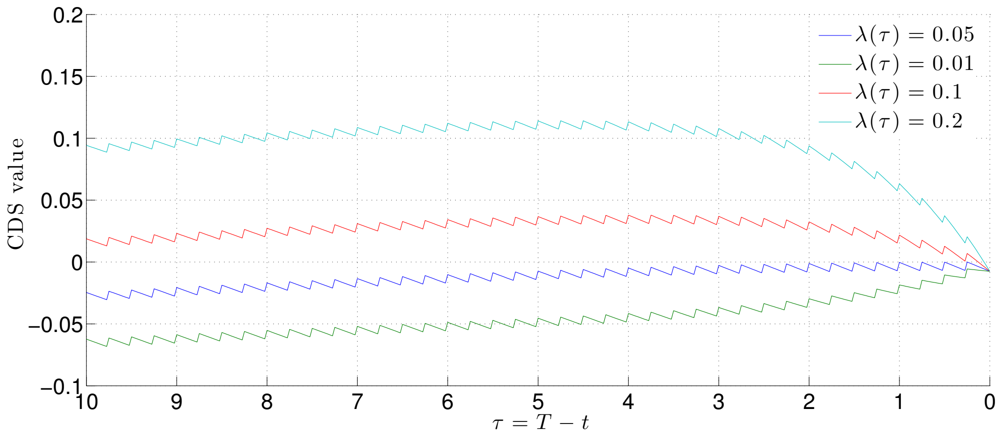

Figure 3.3
The finite-difference solution of CDS at τ space
The finite-difference solution of CDS with different time to maturity and default intensity. Parameter choose are $T = 10$, $S = 0.03$, $R = 0.4$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$, $\lambda(0) = 0.05$ and swap premiums are paid quarterly, which is $T_i - T_{i-1} = 0.25$ for all i. $\Delta t = 0.0025$, $\Delta\lambda = 0.001$.

choose sufficiently small $\Delta\lambda$ to maintain accuracy. If λ_{max} is not wide enough, the solutions are not accurate near the upper boundary and it can even effect the solution in the centre. The numerical solution we obtain with a Dirichlet boundary condition will be lifted up to the default payoff, according to the left panel of figure 3.4.

According to figure 3.4, the Robin and heuristic Robin conditions have better performance at the upper boundary, as their solutions are much closer to the semi-analytic one. The analytic solution tends to the default payoff asymptotically with the speed of that increase tending to zero. The Robin type condition and the heuristic Robin condition, which enable exponential decays at the boundaries, yields solutions that align with the analytic solutions at $\lambda \to \infty$. At the lower boundary, where $\lambda \to 0$, the different choice of boundary conditions at $\lambda \to 0$ has no influence on the solutions around $\lambda(0) = 0$ and the numerical solutions are identical to 16 decimal places. The numerical solutions with different boundary conditions have no observable difference for $\lambda < 0.9$. The indifference of numerical solution to the choice of upper boundary condition can attribute to using mean-reverting default intensity. When we solve the CDS value backward from time step t to time step $t - \Delta t$, the CDS value $V(\lambda(t - \Delta t), t - \Delta t)$ depends on $V(\lambda(t), t)$, which in turn depends on the dynamic of $\lambda(t)$ from t to $t - \Delta t$. Because $\lambda(t)$ is a mean-reverting process we can show that the drift of λ from t to $t - \Delta t$ is

$$\lambda(t - \Delta t) = \lambda(t) - \kappa(\theta - \lambda(t))\Delta t. \tag{3.18}$$

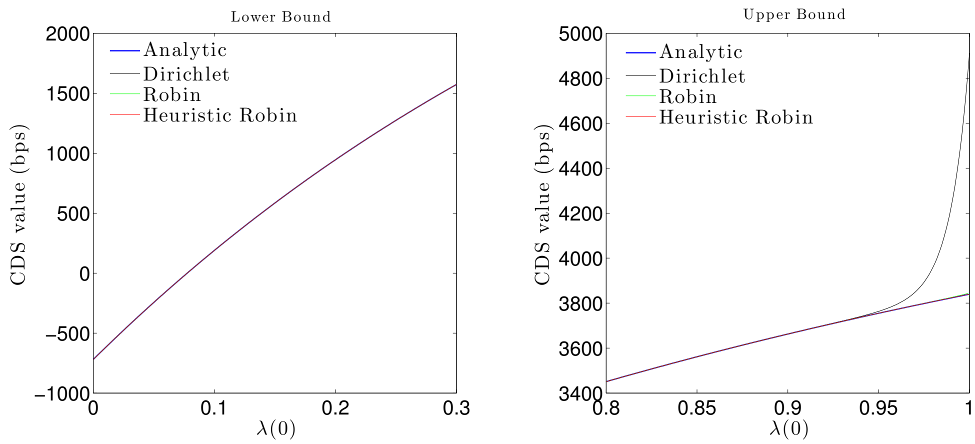

Figure 3.4
The finite-difference CDS solution at λ space compare upper boundary conditions
The finite-difference solution of CDS with different time to maturity and default intensity. Parameter choose are $T = 10$, $S = 0.03$, $R = 0.4$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$, $\lambda(0) = 0.05$, $r = 0.02$ and swap premiums are paid quarterly, which is $T_i - T_{i-1} = 0.25$ for all i. $\Delta t = 0.0025$, $\Delta\lambda = 0.001$

It is well known that a mean-reverting process's movement, if we are to only consider drift, will at the next time step forward return towards the mean value, which means that λ is moving from outside to inside. When we consider the dynamics going backwards in time, λ will move from inside to outside. Therefore, the value of CDS $V(\lambda(t-\Delta t), t-\Delta t)$ relies more heavily on the values $V(\lambda(t), t)$ such that $\lambda(t-\Delta t) > \lambda(t)$ if $\lambda(t-\Delta t) > \theta$ or $\lambda(t-\Delta t) < \lambda(t)$ if $\lambda(t-\Delta t) < \theta$. Therefore, the impacts from boundary conditions at $\lambda \to \infty$ decays as $\lambda(t)$ is far from the boundaries.

In addition, the numerical errors are sensitive to the width of the finite domain $[0, \lambda_{max}]$, given Δt and $\Delta\lambda$ the same, which is shown in figure 3.5. When $\lambda_{max} = 1$, in figure 3.5, the error using heuristic Robin condition is around 10^{-5} and the Robin condition is around 10^{-4}. On the other hand, the Dirichlet boundary leads to errors of around 10^{-1}. While with $\lambda_{max} = 10$, we observe a reduction in numerical errors because the assumptions that underlie the different types of conditions will be less satisfactory. But as discussed earlier, the influence of different boundary conditions decays very fast.

Although the choice of different upper boundary conditions make no significant difference to the solutions, we choose the heuristic Robin boundary condition (3.15) at $\lambda \to \infty$ because it apparently leads to the smallest error around the upper boundary. Figure 3.6 shows the pattern of the CDS value convergence at $\lambda(0) = 0.05$ using the heuristic Robin boundary conditions. Obviously, both Δt and $\Delta\lambda$ have to be small

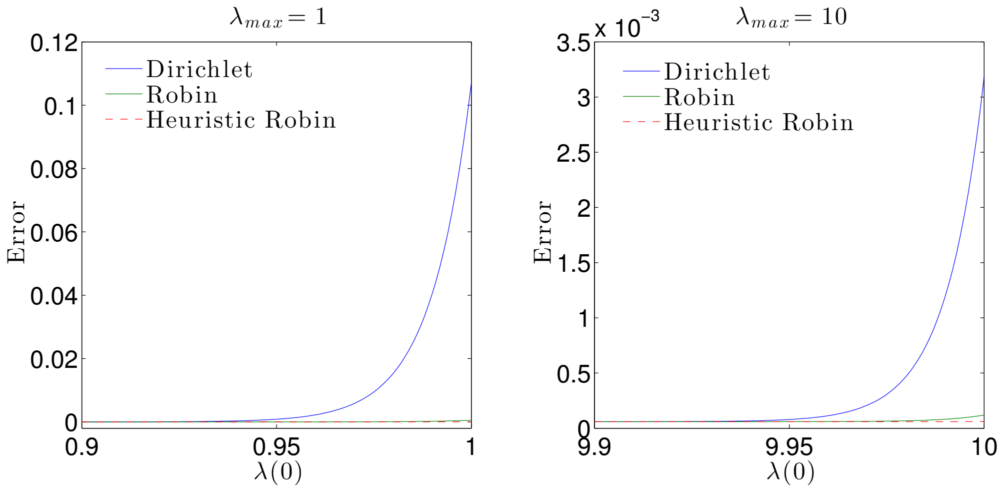

Figure 3.5
The finite difference CDS numerical errors at λ space compare boundary conditions
The finite-difference numerical errors in CDS value with three types of upper boundary conditions and larger λ_{max}. Parameter choose are $T = 10$, $S = 0.03$, $R = 0.4$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$, $\lambda(0) = 0.05$, $r = 0.02$ and swap premiums are paid quarterly, which is $T_i - T_{i-1} = 0.25$ for all i. $\Delta t = 0.0025$, $\Delta\lambda = 0.001$

enough for the numerical error to reduce towards zero. Otherwise, the numerical solution converges than a value different to the exact solution. Also we notice that the Crank-Nicolson scheme is stable even with large time step size $\Delta t = \frac{1}{4}$, which is just one step between two swap payment dates.

Table 3.1 summarises the convergence and computational time of the semi-analytic solution evaluated using the trapezoidal rule. Clearly, the semi-analytic formula requires a numerical implementation and therefore bears numerical errors. Since we have no real analytic solution to compare with, we take the difference of two numerical solutions in order to see to which decimal place the numerical solution would stop changing with even smaller step size in time. For example, in table 3.1, the numerical solution with $\Delta t = 10^{-7}$ only differ from the solution with $\Delta t = 10^{-6}$ at the 9^{th} digit. The fourth column of table 3.1 shows the numerical solution difference when Δt is ten times lower. The most accurate solution we obtained is stable to the 8^{th} digits and this requires 119 seconds for computing in a 2GHz Intel Core i7 machine.

To compare with the computational time and accuracy using semi-analytic solution, tables 3.2 and 3.3 summarises the convergence and computational time, in bracket, using finite-difference with the Robin condition. The finite difference method shows convergences in both time and space dimension, table 3.3 to show the difference

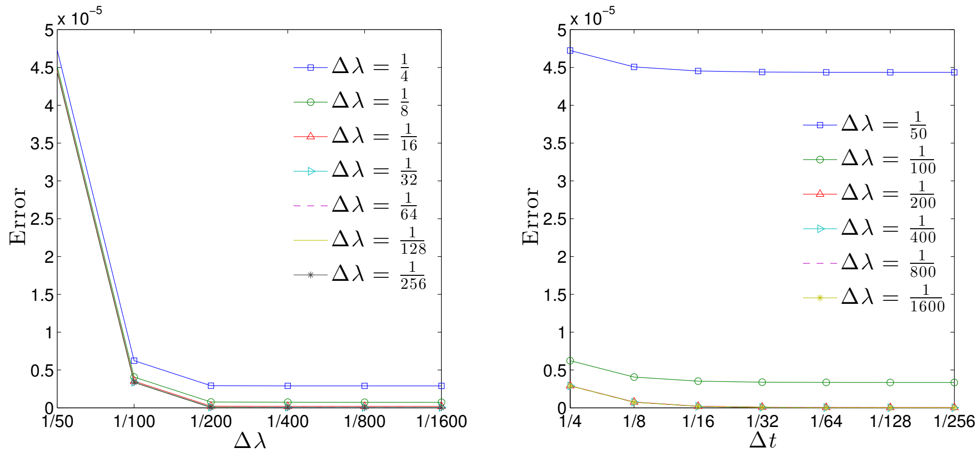

Figure 3.6
The convergence of numerical error using the heuristic Robin boundary condition at $\lambda \to \infty$
Convergence of finite-difference numerical errors of CDS value using heuristic Robin conditions. Parameter choose are $T = 10$, $S = 0.03$, $R = 0.4$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$, $\lambda(0) = 0.05$ and swap premiums are paid quarterly, which is $T_i - T_{i-1} = 0.25$ for all i. The semi-analytic solution -0.0244470233253265 or 244.470233253265 (bps).

Δt	Value	Computation Time (Sec)	Successive Differ
10^{-1}	-0.024536235448	0.0002	
10^{-2}	-0.024452637972	0.0014	8.359×10^{-5}
10^{-3}	-0.024446933941	0.0131	5.704×10^{-6}
10^{-4}	-0.024447014396	0.1254	8.045×10^{-8}
10^{-5}	-0.024447022440	1.4059	8.044×10^{-9}
10^{-6}	-0.024447025259	12.755	2.818×10^{-9}
10^{-7}	-0.024447023325	119.55	1.934×10^{-9}

Table 3.1
The convergence and computational time of semi-analytic solution
Semi-analytic formula is implemented using Trapezoidal rule. Parameter choose are $T = 10$, $S = 0.03$, $R = 0.4$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$, $\lambda(0) = 0.05$ and swap premiums are paid quarterly, which is $T_i - T_{i-1} = 0.25$ for all i.

between solutions with reduced Δt or Δx. According to table 3.3, the most accurate solution obtained from finite difference is -0.024447023304 with $\Delta t = \frac{1}{6400}$ and $\Delta\lambda = \frac{1}{6400}$ and computational time 104 seconds. This solution has stabilised number to the 10^{th} decimal place compared to the solutions with $\Delta t = \frac{1}{3200}$ and $\Delta\lambda = \frac{1}{6400}$ and $\Delta t = \frac{1}{6400}$ and $\Delta\lambda = \frac{1}{3200}$.

Given the computational times of the most accurate semi-analytic and finite difference solution are similar, the finite difference method out-perform the semi-analytical solution with two more digits stabilised number.

$\Delta\lambda$ \ Δt	$\frac{1}{100}$	$\frac{1}{200}$	$\frac{1}{400}$	$\frac{1}{800}$
$\frac{1}{100}$	-0.0244436810929748 (0.03)	-0.024443684558787 (0.07)	-0.0244436854252462 (0.13)	-0.024443685641861 (0.33)
$\frac{1}{200}$	-0.0244469883795621 (0.06)	-0.0244469918456643 (0.16)	-0.024446992712193 (0.25)	-0.0244469929288226 (0.48)
$\frac{1}{400}$	-0.0244470126864719 (0.11)	-0.0244470161525799 (0.25)	-0.0244470170191076 (0.44)	-0.0244470172357406 (0.86)
$\frac{1}{800}$	-0.0244470168547387 (0.22)	-0.0244470203208489 (0.42)	-0.0244470211873774 (0.84)	-0.0244470214040101 (1.62)
$\frac{1}{1600}$	-0.024447018239472 (0.42)	-0.024447021705582 (0.82)	-0.0244470225721098 (1.71)	-0.0244470227887439 (3.22)
$\frac{1}{3200}$	-0.0244470185942909 (0.79)	-0.0244470220604014 (1.62)	-0.0244470229269304 (3.35)	-0.0244470231435597 (6.25)
$\frac{1}{6400}$	-0.0244470186831502 (1.59)	-0.0244470221492553 (3.15)	-0.0244470230157897 (6.32)	-0.0244470232324218 (12.5)

$\Delta\lambda$ \ Δt	$\frac{1}{1600}$	$\frac{1}{3200}$	$\frac{1}{6400}$
$\frac{1}{100}$	-0.0244436856960041 (0.65)	-0.0244436857095454 (0.99)	-0.0244436857129541 (1.87)
$\frac{1}{200}$	-0.0244469929829787 (0.9)	-0.02444699299651 (1.75)	-0.02444699299898 (3.49)
$\frac{1}{400}$	-0.0244470172898963 (1.78)	-0.0244470173034394 (3.32)	-0.0244470173068367 (6.59)
$\frac{1}{800}$	-0.0244470214581675 (3.34)	-0.0244470214717076 (6.63)	-0.0244470214750848 (12.9)
$\frac{1}{1600}$	-0.0244470228429049 (6.43)	-0.0244470228564459 (13)	-0.0244470228598311 (25.3)
$\frac{1}{3200}$	-0.0244470231977233 (12.8)	-0.0244470232112683 (26)	-0.0244470232146657 (49.6)
$\frac{1}{6400}$	-0.0244470232865798 (25.8)	-0.0244470233001214 (51.5)	-0.024447023303505 (104)

Table 3.2
Finite-difference solutions with varying $\Delta\lambda$ and Δt and computational times
The convergence of solution using Crank-Nicolson finite-difference scheme with heuristic Robin condition at $\lambda \to \infty$. Parameter choose are $T = 10$, $S = 0.03$, $R = 0.4$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$, $\lambda(0) = 0.05$ and swap premiums are paid quarterly.

	Difference in Δt space						
$\Delta\lambda$ \ Δt	$\frac{1}{100}$	$\frac{1}{200}$	$\frac{1}{400}$	$\frac{1}{800}$	$\frac{1}{1600}$	$\frac{1}{3200}$	$\frac{1}{6400}$
$\frac{1}{100}$	NA	-3.4658×10^{-9}	-8.6646×10^{-10}	-2.1661×10^{-10}	-5.4143×10^{-11}	-1.3541×10^{-11}	-3.4087×10^{-12}
$\frac{1}{200}$	NA	-3.4661×10^{-9}	-8.6653×10^{-10}	-2.1663×10^{-10}	-5.4156×10^{-11}	-1.3531×10^{-11}	-3.3880×10^{-12}
$\frac{1}{400}$	NA	-3.4661×10^{-9}	-8.6653×10^{-10}	-2.1663×10^{-10}	-5.4156×10^{-11}	-1.3543×10^{-11}	-3.3973×10^{-12}
$\frac{1}{800}$	NA	-3.4661×10^{-9}	-8.6653×10^{-10}	-2.1663×10^{-10}	-5.4157×10^{-11}	-1.3540×10^{-11}	-3.3772×10^{-12}
$\frac{1}{1600}$	NA	-3.4661×10^{-9}	-8.6653×10^{-10}	-2.1663×10^{-10}	-5.4161×10^{-11}	-1.3541×10^{-11}	-3.3852×10^{-12}
$\frac{1}{3200}$	NA	-3.4661×10^{-9}	-8.6653×10^{-10}	-2.1663×10^{-10}	-5.4164×10^{-11}	-1.3545×10^{-11}	-3.3974×10^{-12}
$\frac{1}{6400}$	NA	-3.4661×10^{-9}	-8.6653×10^{-10}	-2.1663×10^{-10}	-5.4158×10^{-11}	-1.3542×10^{-11}	-3.3836×10^{-12}
	Difference in $\Delta\lambda$ space						
$\Delta\lambda$ \ Δt	$\frac{1}{100}$	$\frac{1}{200}$	$\frac{1}{400}$	$\frac{1}{800}$	$\frac{1}{1600}$	$\frac{1}{3200}$	$\frac{1}{6400}$
$\frac{1}{100}$	NA	NA	NA	NA	NA	NA	NA
$\frac{1}{200}$	-3.3073×10^{-6}	-3.3073×10^{-6}	-3.3073×10^{-6}	-3.3073×10^{-6}	-3.3073×10^{-6}	-3.3073×10^{-6}	-3.3073×10^{-6}
$\frac{1}{400}$	-2.4307×10^{-8}	-2.4307×10^{-8}	-2.4307×10^{-8}	-2.4307×10^{-8}	-2.4307×10^{-8}	-2.4307×10^{-8}	-2.4307×10^{-8}
$\frac{1}{800}$	-4.1683×10^{-9}	-4.1683×10^{-9}	-4.1683×10^{-9}	-4.1683×10^{-9}	-4.1683×10^{-9}	-4.1683×10^{-9}	-4.1682×10^{-9}
$\frac{1}{1600}$	-1.3847×10^{-9}	-1.3847×10^{-9}	-1.3847×10^{-9}	-1.3847×10^{-9}	-1.3847×10^{-9}	-1.3847×10^{-9}	-1.3847×10^{-9}
$\frac{1}{3200}$	-3.5482×10^{-10}	-3.5482×10^{-10}	-3.5482×10^{-10}	-3.5482×10^{-10}	-3.5482×10^{-10}	-3.5482×10^{-10}	-3.5483×10^{-10}
$\frac{1}{6400}$	-8.8859×10^{-11}	-8.8854×10^{-11}	-8.8859×10^{-11}	-8.8862×10^{-11}	-8.8856×10^{-11}	-8.8853×10^{-11}	-8.8839×10^{-11}

Table 3.3
Successive difference in finite-difference solutions at λ and t spaces

3.4 Solving the CVA Using Finite Differences

The previous section discusses the finite-difference solution to CDS with the CIR model comparing with the semi-analytic solution. In this section, we present the PDE and boundary conditions used for valuing the CVA. Pricing a CDS contract is a preliminary step before pricing the CVA, where the value of the CDS will be used as a known source of loss given default. The specification of the close-out convention affects the numerical procedure that we use to solve for the CVA valuation. Recall that the close-out value $M(t,T)$ in (2.77) can be specified as the CDS value with or without counterparty risk. If we assume the counterparty recovery is a fraction of the CDS value with counterparty risk, we may solve PDE (2.67) for the CDS value with counterparty risk and subtract it from the solution of PDE (2.60) to obtain the CVA. This comes directly from the definition of the CVA as the gap between the contract value with defaultable and default-free counterparty. Alternatively, Section 2.3.4 gives a PDE representation for the CVA given the close-out value $M(t,T)$ as the CDS value without counterparty risk, which can be solved by the PDE (2.60). Then the PDE (2.79) will be solved directly to obtain the CVA with the CDS values V in (2.79) from the solutions of (2.60). From the numerical aspect, both approaches involve solving a two-dimensional PDE. Close-out is similar to the recovery methods discussed in Section 2.2 when pricing default bonds. In our previous pricing of CDS, *recovery of treasury* is used where the recovery amount is in terms of a bond without default risk. In addition, as we discussed in Section 2.3.3, the close-out value as the CDS value without counterparty risk is closer to the conventional definition of a CVA. Therefore, we use the CDS value without counterparty risk, which is the risk-free close-out method, in valuing CVAs throughout this thesis.

In this section, we use the simplest CIR model and discuss the numerical scheme for pricing the CVA, which will be used in later chapters when we propose new models. Let us assume that we are a default-free CDS buyer, and then the value adjustment will only consider the default risk of the counterparty, or unilateral CVA. Let the default intensities of the reference firm and the counterparty follow the SDEs

$$\begin{aligned}
&d\lambda_1(t) = \kappa_1(\theta_1 - \lambda_1(t))dt + \sigma_1\sqrt{\lambda_1(t)}dW_1(t) \text{ referencing firm} \\
&d\lambda_2(t) = \kappa_2(\theta_2 - \lambda_2(t))dt + \sigma_2\sqrt{\lambda_2(t)}dW_2(t) \text{ counterparty firm} \\
&d < W_1(t), W_2(t) >= \rho dt.
\end{aligned} \tag{3.19}$$

It is shown in Section 2.3.4 that in order to calculate the value of the CVA we must solve

$$\begin{aligned}
&\frac{\partial U}{\partial t} + \kappa_1(\theta_1 - \lambda_1)\frac{\partial U}{\partial \lambda_1} + \kappa_2(\theta_2 - \lambda_2)\frac{\partial U}{\partial \lambda_2} \\
&+ \frac{1}{2}\sigma_1^2\lambda_1\frac{\partial^2 U}{\partial \lambda_1^2} + \frac{1}{2}\sigma_2^2\lambda_2\frac{\partial^2 U}{\partial \lambda_2^2} + \rho\sigma_1\sigma_2\sqrt{\lambda_1\lambda_2}\frac{\partial^2 U}{\lambda_1\lambda_2} \\
&+ \lambda_2(1 - R_2)V^+ - (r + \lambda_1 + \lambda_2)U = 0,
\end{aligned} \tag{3.20a}$$

with

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \kappa_1(\theta_1 - \lambda_1)\frac{\partial V}{\partial \lambda_1} + \frac{1}{2}\sigma_1^2\lambda_1\frac{\partial^2 V}{\partial \lambda_1} \\
&+ \lambda_1\left((1 - R_1)e^{-r(T-t)} - S\frac{t - T_{n-1}}{T_n - T_{n-1}}\right) - (r + \lambda_1)V = 0.
\end{aligned} \tag{3.20b}$$

3.4.1 Boundary conditions

The PDE system (3.20) involves a one-dimension PDE and a two-dimension PDE. The numerical scheme for the one-dimensional PDE has been discussed with numerical results in Section 3.3. In the two-dimension case, boundary conditions are imposed along edges rather than points. The PDE must be solved subject to one terminal and four boundary conditions in which the two default intensities go to zero and infinity.

When the CDS expires, the CVA will no longer exist. Therefore, we have the following terminal condition,

$$U(\lambda_1, \lambda_2, T) = 0.$$

The boundary conditions are rather more subtle for CVA. On the $\lambda_1 = 0$ boundary, the referencing firm's default risk is the lowest and the CDS value is the lowest as a consequence. If the protection buyer pays a reasonable premium, the CDS may have negative value, where the CDS buyer will not suffer losses if the counterparty defaults then the CVA should be zero. However, as discussed in Section 3.3, $\lambda = 0$ does not imply default-free. Therefore, it is still possible that the counterparty defaults when the CDS has positive value. Thus, it is inappropriate to apply zero value at λ_1 because the λ_1 will return back to normal level. Therefore, we use the similar heuristic Robin condition and substitute $\lambda_1 = 0$ or $\lambda_2 = 0$ into the PDE (3.20a) at the boundaries where λ_1 or $\lambda_2 = 0$, namely (3.25a).

We still require boundary conditions as the intensities become large. However, this is not straightforward due to the nature of CVAs. Clearly, the CVA is an increasing function of both the referencing firm's and the counterparty's default intensities. If the referencing firm is very likely to default, $\lambda_1 \to \infty$, one may think the CVA should be zero because the probability that the counterparty defaults earlier is not likely. However, this always underestimates the CVA no matter how large λ_1 is. One feature in reduced-form modelling is that a large default intensity does not imply instant default. Although we have $\lambda_1 \to \infty$, there is still counterparty default risk due to the fact that we are under reduced-form modelling. On the other hand, if the counterparty has defaulted, the loss given default will be very large because the referencing firm is very risky. This is a situation where the probability of loss occurring is very small but the loss will be substantial if the counterparty defaults. Therefore, the CVA should have a positive value. On the other hand, consider $\lambda_1 >> \lambda_2$, the probability that the counterparty default first will be a small value compared to the loss given default. Therefore, the behaviour of CVA will be dominated by the loss given default, which is just a linear function of the CDS's value $(1 - R_C)\max\{V, 0\}$ given the CDS is positive and this is true for large λ_1. As a result, we are able to safely apply the same heuristic Robin condition as we did for the CDS to the CVA.

Now for the other boundary as $\lambda_2 \to \infty$, the CVA should be bounded by the counterparty default loss, which is again $(1 - R_C)\max\{V, 0\}$. However, as discussed above, it is difficult to apply default payoff in reduced-form modelling. Unlike λ_1, λ_2 only influences the CVA in terms of the probability that the counterparty defaults earlier than the referencing, for which we have an analytic solution. Therefore, we investigate the behaviour of this probability in the limit $\lambda_2 \to \infty$ in order to infer the CVA's behaviour at $\lambda_2 \to \infty$.

Denote the probability that the counterparty defaults earlier than the referencing firm as $P(\tau_1 > \tau_2, \tau_2 < T)$, which is the expectation

$$\mathbb{E}\left[\int_t^T e^{-\int_t^s \lambda_1 + \lambda_2} \lambda_2 ds \middle| \mathcal{F}_t\right]. \tag{3.21}$$

If we set $\rho = 0$, the expectation (3.21), according to Brigo and Mercurio (2006), has an analytic solution, which is

$$\int_t^T P(\lambda_1(t), t, s) P(\lambda_2(t), t, s) f(\lambda_2(s), s, T) ds. \tag{3.22}$$

The probabilities $P(\lambda_1(t), t, s)$ and $P(\lambda_2(t), ts)$ are calculated as

$$P(\lambda(t), t, T)) = A(t, T) e^{-B(t,T)\lambda(t)}, \tag{3.23}$$

where

$$\begin{aligned} A(t, T) &= \left[\frac{2h \exp((\kappa + h)(T - t)/2)}{2h + (\kappa + h)(exp(h(T - t)) - 1)} \right]^{\frac{2\theta\kappa}{\sigma^2}} \\ B(t, T) &= \frac{2(\exp((\kappa + h)(T - t)) - 1)}{2h + (\kappa + h)(exp(h(T - t)) - 1)} \\ h &= \sqrt{\kappa^2 + 2\sigma^2}. \end{aligned}$$

Then the term $f(\lambda_2(s), s, T)$ is the forward default intensity, which is calculated by

$$\begin{aligned} f(\lambda(t), t, T) = & \frac{2\kappa\theta(\exp((T - t)h) - 1)}{2h + (\kappa + h)(\exp((T - t)h) - 1)} \\ & + \lambda(t) \frac{4h^2 \exp((T - t)h)}{[2h + (\kappa + h)(\exp((T - t)h) - 1)]^2}. \end{aligned} \tag{3.24}$$

Figure 3.7 shows the first derivative term and the second derivative term of this probability with different positions of λ_1 and λ_2. Similar to Section 3.3.2, we observe the second derivative term is far less than the first derivative term at all values of λ_1 since the *Feller* condition is satisfied. Therefore, we can safely eliminate the second derivative term $\dfrac{\partial^2 U}{\partial \lambda_2^2}$ in order to obtain a condition for the upper boundary of the CVA at $\lambda_2 \to \infty$. The boundary conditions of the CVA are summarised in (3.25).

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_2} + \kappa_1 \theta_1 \frac{\partial}{\partial \lambda_1} \right) U + \lambda_2 (1 - R_2) V^+ - (r + \lambda_2) U = 0 \quad \text{for} \quad \lambda_1 \to 0, \\ &\left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_1} + \kappa_2 \theta_2 \frac{\partial}{\partial \lambda_2} \right) U - (r + \lambda_1) U = 0 \quad \text{for} \quad \lambda_2 \to 0. \end{aligned} \tag{3.25a}$$

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_2} + \kappa_1 (\theta_1 - \lambda_1) \frac{\partial}{\partial \lambda_1} \right) U + \lambda_2 (1 - R_2) V^+ - (r + \lambda_2 + \lambda_1) U = 0 \quad \text{for } \lambda_1 \to \infty, \\ &\left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_1} + \kappa_2 (\theta_2 - \lambda_2) \frac{\partial}{\partial \lambda_2} \right) U + \lambda_2 (1 - R_2) V^+ - (r + \lambda_1 + \lambda_2) U = 0 \quad \text{for } \lambda_2 \to \infty. \end{aligned} \tag{3.25b}$$

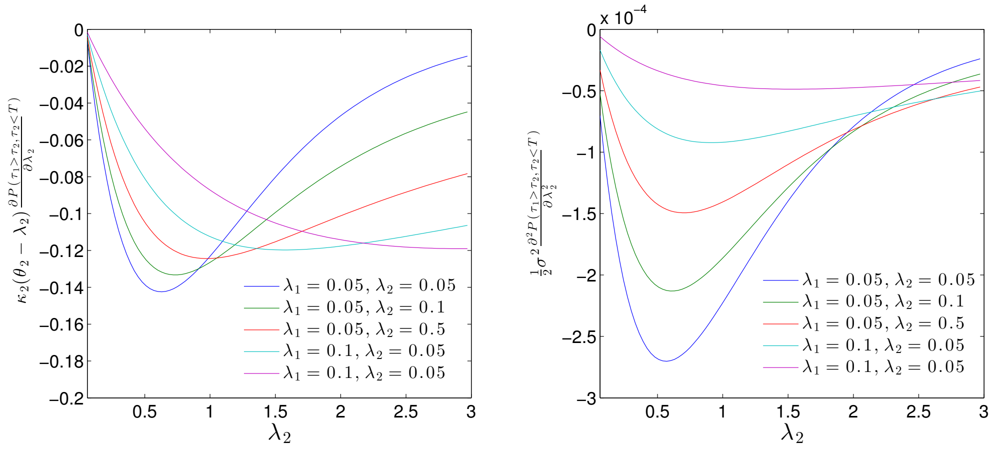

Figure 3.7
The first and second derivative with respect to λ_2 of $P(\tau_2 < \tau_1, \tau_2 < 5)$ at large λ_2
The first and second derivative terms are shown with small and large λ_1. Other parameters $\kappa_1 = \kappa_2 = 0.5$, $\sigma_1 = \sigma_2 = \frac{\sqrt{\kappa_2\theta_2}}{2}$ and $\theta_1 = \theta_2 = 0.05$, $T = 5$.

Because both firms' default intensity processes are mean-reverting processes, we can expect the boundary conditions have small influence on the numerical solutions. Imposing the boundary conditions (3.25a) and (3.25b), we end up with reduced PDEs at the corners, which are

$$\lambda_1 \to 0 \;\; \lambda_2 \to 0:$$
$$\left(\frac{\partial}{\partial t} + \kappa_2\theta_2\frac{\partial}{\partial\lambda_2} + \kappa_1\theta_1\frac{\partial}{\partial\lambda_1}\right)U - rU = 0,$$
$$\lambda_1 \to \infty \;\; \lambda_2 \to \infty:$$
$$\left(\frac{\partial}{\partial t} + \kappa_2(\theta_2-\lambda_2)\frac{\partial}{\partial\lambda_2} + \kappa_1(\theta_1-\lambda_1)\frac{\partial}{\partial\lambda_1}\right)U + \lambda_2(1-R_2)V^+ - (r+\lambda_1+\lambda_2)U = 0,$$
$$\lambda_1 \to \infty \;\; \lambda_2 \to 0:$$
$$\left(\frac{\partial}{\partial t} + \kappa_2\theta_2\frac{\partial}{\partial\lambda_2} + \kappa_1(\theta_1-\lambda_1)\frac{\partial}{\partial\lambda_1}\right)U - (r+\lambda_1)U = 0,$$
$$\lambda_1 \to 0 \;\; \lambda_2 \to \infty:$$
$$\left(\frac{\partial}{\partial t} + \kappa_2(\theta_2-\lambda_2)\frac{\partial}{\partial\lambda_2} + \kappa_1\theta_1\frac{\partial}{\partial\lambda_1}\right)U + \lambda_2(1-R_2)V^+ - (r+\lambda_2)U = 0.$$

3.4.2 Discretisation in the ADI scheme

In this section we present a detailed implementation of the ADI scheme as mentioned earlier. As a preliminary step towards the numerical solution, we truncate the infinite region to a finite domain of $[0, \lambda_{1max}] \times [0, \lambda_{2max}]$ and we apply a uniform grid in space and time, where $\Delta t = \frac{T}{N}$, $\Delta\lambda_1 = \frac{\lambda_{1max}}{I}$ and $\Delta\lambda_2 = \frac{\lambda_{2max}}{J}$. Because the ADI introduces an intermediate step when time moves backward from $n+1$ to n, we use notation $n+0.5$ as the time index to the middle step.

At the first step, we have the time discretisation

$$\frac{\partial U}{\partial t} \approx -\frac{U_{i,j}^{n+0.5} - U_{i,j}^{n+1}}{0.5\Delta t} \quad \text{for} \quad \forall i, j \in [0, I] \times [0, J]. \tag{3.26}$$

To approximate the first derivative $\frac{\partial U}{\partial \lambda_1}$ and $\frac{\partial U}{\partial \lambda_2}$ at the first step, we consider three approximations,

$$\begin{aligned}
\frac{\partial U}{\partial \lambda_1} &\approx \frac{U_{i+1}^{n+0.5} - U_{i-1}^{n+0.5}}{2\Delta\lambda_1} \text{ for } i \in [1, I-1], \\
\frac{\partial U}{\partial \lambda_1} &\approx \frac{U_{1}^{n+0.5} - U_{0}^{n+0.5}}{\Delta\lambda_1} \text{ for } i = 0, \\
\frac{\partial U}{\partial \lambda_1} &\approx \frac{U_{I}^{n+0.5} - U_{I-1}^{n+0.5}}{\Delta\lambda_1} \text{ for } i = I \\
\frac{\partial U}{\partial \lambda_2} &\approx \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta\lambda_2} \text{ for } j \in [1, J-1], \\
\frac{\partial U}{\partial \lambda_2} &\approx \frac{U_{1}^{n+1} - U_{0}^{n+1}}{\Delta\lambda_2} \text{ for } j = 0, \\
\frac{\partial U}{\partial \lambda_2} &\approx \frac{U_{J}^{n+1} - U_{J-1}^{n+1}}{\Delta\lambda_2} \text{ for } j = J.
\end{aligned} \tag{3.27}$$

At the boundary where one of the intensities is zero or infinite, we use one-sided difference for the first derivatives. To approximate the second derivative, we deal with the finite-difference approximation,

$$\begin{aligned}
\frac{\partial^2 U}{\partial \lambda_1^2} &\approx \frac{U_{i+1}^{n+0.5} - 2U_{i}^{n+0.5} + U_{i-1}^{n+0.5}}{\Delta\lambda_1^2} \quad \text{for} \quad \forall j \text{ and } i \in [1, I-1] \\
\frac{\partial^2 U}{\partial \lambda_2^2} &\approx \frac{U_{j+1}^{n+1} - 2U_{j}^{n+1} + U_{j-1}^{n+1}}{\Delta\lambda_2^2} \quad \text{for} \quad \forall i \text{ and } j \in [1, J-1].
\end{aligned} \tag{3.28}$$

At the first step, we treat the terms associated with λ_1 at the intermediate time step $n+0.5$ and the terms associated with λ_2 at the (known) time step $n+1$. So at the second step, we treat the terms associated with λ_1 at the (known) time step $n+0.5$ and the terms associated with λ_1 at the next time step n. Similarly, at the second step, we have the time discretisation

$$\frac{\partial U}{\partial t} \approx -\frac{U_{i,j}^{n} - U_{i,j}^{n+0.5}}{0.5\Delta t} \quad \text{for} \quad \forall i, j \in [0, I] \times [0, J]. \tag{3.29}$$

The approximations to the first derivative terms are

$$
\begin{aligned}
\frac{\partial U}{\partial \lambda_1} &\approx \frac{U_{i+1}^{n+0.5} - U_{i-1}^{n+0.5}}{2\Delta\lambda_1} \text{ for } i \in [1, I-1], \\
\frac{\partial U}{\partial \lambda_1} &\approx \frac{U_{1}^{n+0.5} - U_{0}^{n+0.5}}{\Delta\lambda_1} \text{ for } i = 0, \\
\frac{\partial U}{\partial \lambda_1} &\approx \frac{U_{I}^{n+0.5} - U_{I-1}^{n+0.5}}{\Delta\lambda_1} \text{ for } i = I \\
\frac{\partial U}{\partial \lambda_2} &\approx \frac{U_{j+1}^{n} - U_{j-1}^{n}}{2\Delta\lambda_2} \text{ for } j \in [1, J-1], \\
\frac{\partial U}{\partial \lambda_2} &\approx \frac{U_{1}^{n} - U_{0}^{n}}{\Delta\lambda_2} \text{ for } j = 0, \\
\frac{\partial U}{\partial \lambda_2} &\approx \frac{U_{J}^{n} - U_{J-1}^{n}}{\Delta\lambda_2} \text{ for } j = J
\end{aligned}
\tag{3.30}
$$

and the approximations to the second derivative terms are

$$
\begin{aligned}
\frac{\partial^2 U}{\partial \lambda_1^2} &\approx \frac{U_{i+1}^{n+0.5} - 2U_{i}^{n+0.5} + U_{i-1}^{n+0.5}}{\Delta\lambda_1^2} \quad \text{for} \quad \forall j \text{ and } i \in [1, I-1] \\
\frac{\partial^2 U}{\partial \lambda_2^2} &\approx \frac{U_{j+1}^{n} - 2U_{j}^{n} + U_{j-1}^{n}}{\Delta\lambda_2^2} \quad \text{for} \quad \forall i \text{ and } j \in [1, J-1].
\end{aligned}
\tag{3.31}
$$

As mentioned earlier, ADI schemes were proposed to solve the heat equation thus the mixed spatial derivative term was not considered. The ADI schemes such as In't Hout and Foulon (2010) and Haentjens and In't Hout (2012) are able to deal with these by adding in more intermediate steps in order to achieve second-order convergence in space with the mixed spatial derivative term.

In In't Hout and Foulon (2010), the mixed spatial derivative term is taking central difference to both direction and one-sided difference at boundaries. However, due to different boundary conditions (3.25b) that are applied in our case, the cross-derivatives at the upper boundaries does not vanish. To approximate the cross derivative, we deal with the four finite-difference approximations

$$
\begin{aligned}
\frac{\partial^2 U}{\partial \lambda_1 \lambda_2} &\approx \frac{U_{i+1,j+1}^{d} - U_{i+1,j-1}^{d} - U_{i-1,j+1}^{d} + U_{i-1,j-1}^{d}}{4\Delta\lambda_1\Delta\lambda_2} \quad \text{for} \quad i, j \in [1, I-1] \times [1, J-1], \\
\frac{\partial^2 U}{\partial \lambda_1 \lambda_2} &\approx \frac{U_{I,j+1}^{d} - U_{I,j-1}^{d} - U_{I-1,j+1}^{d} + U_{I-1,j-1}^{d}}{2\Delta\lambda_1\Delta\lambda_2} \quad \text{for} \quad i = I, j \in [1, J-1], \\
\frac{\partial^2 U}{\partial \lambda_1 \lambda_2} &\approx \frac{U_{i+1,J}^{d} - U_{i+1,J-1}^{d} - U_{i-1,J-1}^{d} + U_{i-1,J-1}^{d}}{2\Delta\lambda_1\Delta\lambda_2} \quad \text{for} \quad j = J, i \in [1, I-1], \\
\frac{\partial^2 U}{\partial \lambda_1 \lambda_2} &\approx \frac{U_{I,J}^{d} - U_{I-1,J}^{d} - U_{I,J-1}^{d} + U_{I-1,J-1}^{d}}{\Delta\lambda_1\Delta\lambda_2} \quad \text{for} \quad i = I, j = J,
\end{aligned}
\tag{3.32}
$$

for $d = n + 1$ and $n + 0.5$, depending on where we may be in the grid. Clearly, the approximation (3.32) can be obtained by an application of approximation (3.27) successively in the λ_1 and λ_2 directions. In the interior of the grid, the first equation of (3.32) is used because solutions for $i \pm 1$ and $j \pm 1$ are available. The second and third approximations of (3.32) are used when either i or j reaches the upper boundaries and the last one is used if both reach boundaries. There are no concerns if i or j reaches lower boundaries since the cross derivatives term vanishes.

Applying the approximations (3.27), (3.28) and (3.32) to a two-dimensional PDE leads to a linear-algebra system (3.6) that must be solved, which is no more than solving tridiagonal systems. In order to solve the CVA, we apply these approximations to the first equation of the PDE system (3.20) and the details of the resulting parameters in (3.6) are listed in Appendix A.

3.4.3 Numerical results

In this section we begin with a discussion of the choice of CIR intensity parameters. The fair CDS spread and default probability generated by the set of parameters we choose will be compared to the market CDS spread of sovereign bonds and Standard & Poor's credit rating in order to show what degree of default risk the model represents in the market. In the second part, we present the convergence of the ADI scheme to show its effectiveness and efficiency. We choose an example of joint survival probability with zero correlation, which enables us to use a closed-form solution for comparison. The second part of this section analyses the solution of PDE system (3.20) for the CVA. We will focus on some properties of the CVA solution such as the behaviour for different λ_1 and λ_2 and the impacts from correlation and volatilities.

CIR intensity parameters

The default risk of a CIR type default intensity is characterised by the quadruple of parameters (θ, κ, σ, $\lambda(0)$). Here θ determines the long-term or mean default risk. The mean-reverting speed κ and volatility σ jointly determine the variance of default risk. Finally, the initial position $\lambda(0)$ tells us the current instantaneous risk of default. Since this thesis focuses on modelling default contagion and proposing numerical solutions rather than calibrating the model to market data, the parameter we choose will reflect to which category of default firm that we would like our parameters to represent. In Brigo and Chourdakis (2009), θ and κ are chosen to be 0.05 and 0.5 respectively as representative of a risky referencing firm in the long term and we will use the same θ and κ in this thesis. We set the volatility level σ to be 0.1 rather than the 0.5 in Brigo and Chourdakis (2009) for two reasons. Given $\theta = 0.05$ and $\kappa = 0.5$, the maximum volatility satisfies *feller* condition is $\sqrt{0.05} \approx 0.22$ so $\sigma = 0.5$ cannot guarantee positive intensity. Secondly, since we will later vary the volatility to investigate the volatility impact on counterparty risk, choosing $\sigma = 0.1$ gives us more flexibility to change volatility. Similarly, because we have to change the initial default rate to be higher and lower than the benchmark so that we can study the impacts from intensity movements, we set the initial default rate $\lambda(0) = 0.05$, which equal to the long-term default rate θ.

Given the parameters as described above ($\theta = 0.05$, $\kappa = 0.5$, $\sigma = 0.1$, $\lambda(0) = 0.05$) to the CIR default intensity, we compare the term-structure of CDS spread S^* given by the model and some selected sovereign CDS market quotes in Europe, the Middle East and Africa (EMEA) in figure 3.8.

Before comparing the CDS spread term-structure between the model we use and the market, we give a discussion about the market term-structure of CDS spreads. Similar to the term-structure of yield-curve, the term-structure of CDS spread has three typical shapes as shown in the top panel in figure 3.8, which are upward-sloping, downward-sloping and hump shape. The term-structure of CDS spread reflects the market expectation of default risk during each period of time. For example, if the annualised CDS spread for a two year protection is higher than the one year protection, this may imply that the default probability in the second year conditional on no default in the first year is higher than its one year default probability. In other words, an upward-sloping structure implies a firm/country's default probability increases more slowly in the near future than in the longer time horizon. Upward-sloping structure is the most common term-structure, which is observed from high rating sovereign CDSs such as UK, France and Germany. For some of the higher risk sovereign CDSs, such as Egypt and Cyprus, we observed steeper upward-sloping term-structures, which implies a significantly faster increase in default probability in the long term. Downward-sloping and hump shape term-structure are observed for Ukraine and Greece CDS, which are

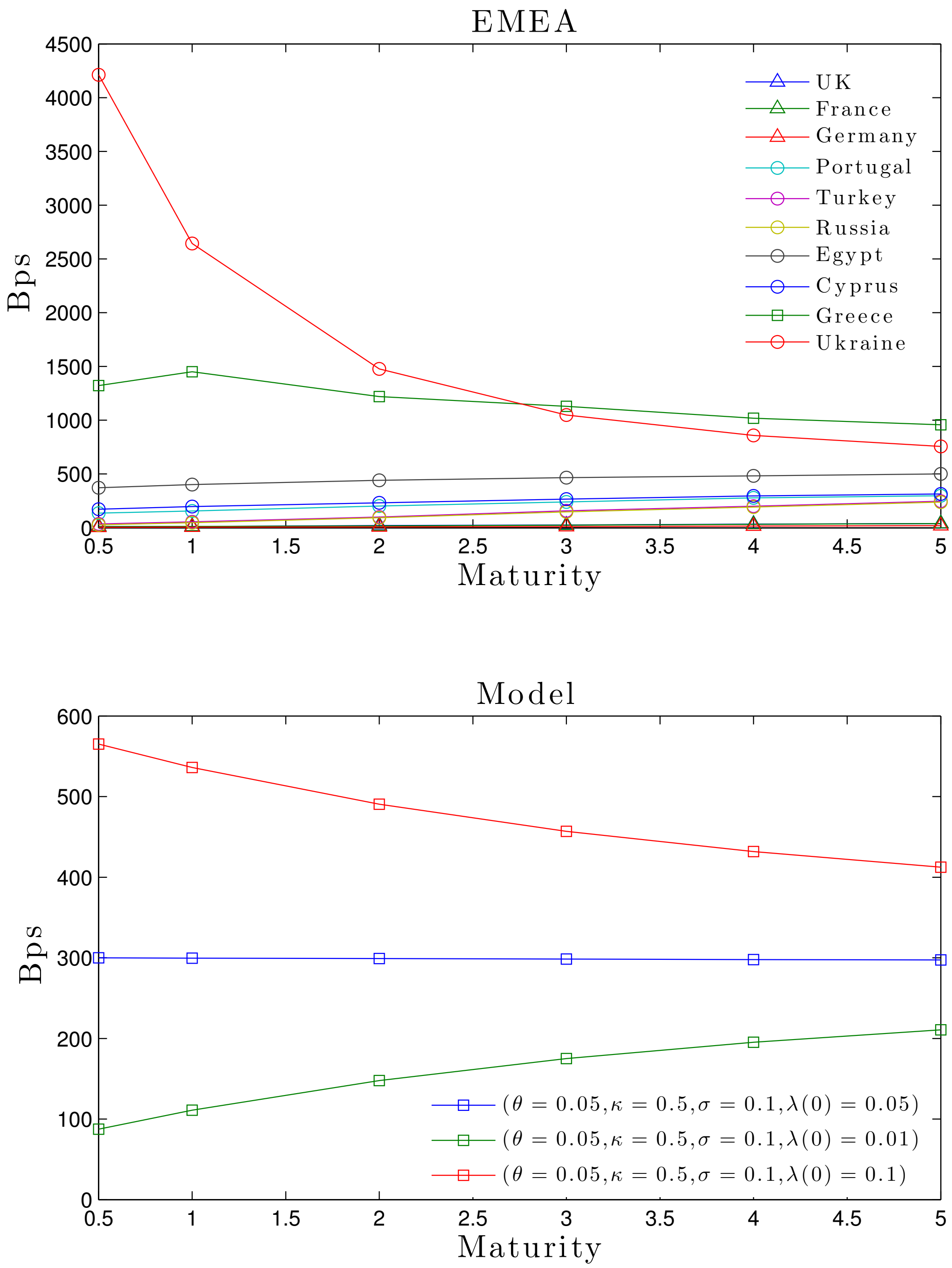

Figure 3.8
A comparison of CDS spreads given the model with a selection of sovereign CDS spreads up to 5 years in EMEA at 6^{th} of July 2016.
Model CDS spread is calculated with assumed 0 interest rate and spread is paid semi-annually.

	Time horizon									
Rating	1	2	3	4	5	6	7	8	9	10
AAA	0	0.03	0.14	0.24	0.36	0.47	0.53	0.61	0.67	0.74
AA	0.02	0.07	0.13	0.24	0.35	0.46	0.56	0.65	0.73	0.82
A	0.07	0.16	0.27	0.41	0.57	0.75	0.95	1.13	1.32	1.51
BBB	0.2	0.57	0.96	1.46	1.95	2.43	2.84	3.26	3.66	4.06
BB	0.76	2.35	4.23	6.06	7.71	9.28	10.59	11.75	12.8	13.74
B	3.88	8.8	12.97	16.22	18.7	20.72	22.37	23.69	24.82	25.91
CCC/C	26.38	35.58	40.67	43.77	46.28	47.24	48.27	49.06	50.03	50.73
Model	4.87	9.49	13.86	18.00	21.94	25.69	29.25	32.64	35.86	38.93

Table 3.4
Global Corporate Average Cumulative Default Rates (%) from 1981 to 2014 and default probabilities given by the CIR intensity given parameters
Source: S& Ps 2014 Annual Global Corporate Default Study And Rating Transitions report, page 56.

very high risk entities. Ukraine's decreasing CDS term-structure means it is currently very likely to default shortly. However, if it does not default in half year, the market expected that its credit quality will recover and its default probability in the second half year is lower. In other words, the default probability during the second half year conditional on the survive the first half year is lower. Finally, the observed "hump shape" is a situation in between the previous two cases, which indicates its default probability grows faster in a particular period. For example, the default probability of Greece in the second half of the first one year.

According to the lower panel in figure 3.8, the term-structure of CDS spread computed with our parameters ($\theta = 0.05$, $\kappa = 0.5$, $\sigma = 0.1$, $\lambda(0) = 0.05$) has a flat term-structure. While raising or reducing the initial default rate $\lambda(0)$ to 0.1 or 0.01 can lead to upward- or downward-sloping structure. In terms of the level of the CDS spread, the spreads produced by this set of parameter are around $299bps$, which are closer to Cyprus's and Portugal's spreads level. In figure 3.9, we compare the selected countries' survival term-structure of implied from their CDS spread with the term-structure produced by a CIR intensity with selected parameters and it suggests that the default term-structure we produced has a similar credit quality to Portugal and Cyprus.

Apart from comparing with market CDS spreads and implied survival term-structure of sovereign CDSs, we also justify to which category of firms that our parametrised

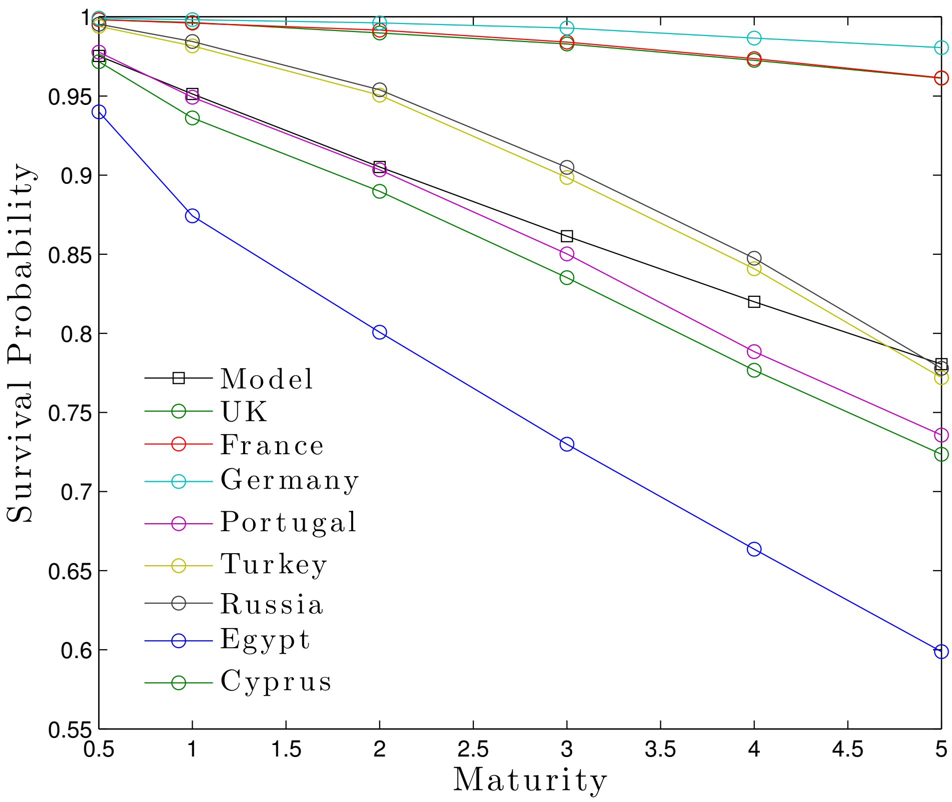

Figure 3.9
A comparison of survival term-structure given the model parameters with the market implied survival term-structure of a selection of sovereign CDS in EMEA at 6^{th} of July 2016.
The market implied survival term-structure is bootstrapped with assumptions of 0 flat interest rate, recovery rate 0.4 and default can happen in the middle of each period. The assumption of 0 interest rate is valid given current low or negative interest rate environment in US and Europe.

CIR intensity represents by investigating the historical rate of defaults. Table 3.4 summarises the global corporate default rate break down by Standard and Poor's credit rating and time to default from 1981 to 2014. In addition, the default term-structure using the CIR intensity is listed in the last column. According to Table 3.4, using a CIR intensity with ($\theta = 0.05$, $\kappa = 0.5$, $\sigma = 0.1$, $\lambda(0) = 0.05$) leads to default probabilities higher than the B rate corporates but lower than CCC rate companies. Therefore, our benchmarking default intensity, which is a CIR process with ($\theta = 0.05$, $\kappa = 0.5$, $\sigma = 0.1$, $\lambda(0) = 0.05$), can be seem as a representative of a B^- rating firm.

Convergence of the scheme

Setting the correlation ρ and the interest rate r to be zero and the recovery rate R_2 to be unity, the PDE (3.20a) describes the joint survival probability of the two firms if we apply the terminal condition $U(\lambda_1(T), \lambda_2(T), T) = 1$. If there is no correlation, the analytic solution to the joint survival probability must be the product of the two individual's marginal survival probability, which is calculated by (3.23).

Assuming the numerical errors follow a functional form of grid sizes, the rate of convergence can be estimated by three successive numerical solutions. Here, we assume the difference between the exact solution and numerical solutions, which is subjected to numerical error, is of the following form

$$\frac{A}{I^{x_i}} + \frac{B}{J^{x_j}} + \frac{C}{N^{x_n}} + o(\Delta\lambda_1, \Delta\lambda_2, \Delta t), \tag{3.33}$$

where A, B, C are constants and x_i, x_j and x_n are the rates of convergence in λ_1, λ_2 and time direction. $x_i = 1$ implies if the grid size in the I-dimension increases from I to $2I$, the errors in this direction reduces by half. The rate of convergence can be approximated with three successive numerical solutions doubling grid sizes. For example, x_n is approximated by

$$x_n = log_2(\frac{U(4N) - U(2N)}{U(2N) - U(N)}). \tag{3.34}$$

We choose the parameters $\lambda_1(0) = \lambda_2(0) = 0.05$, $\theta = 0.05$, $\sigma = 0.2$ and $\kappa = 0.5$ for default intensities, the joint survival probability in one year is 0.9052564, which is the product of the two firm's one year survival probability given by (3.23). Defining our grid on the three dimensional cube $[0, I] \times [0, J] \times [0, N]$ and taking I and J sufficiently

Δt	Numerical Solution	Error	x_n
1×10^{-01}	0.905255872	-6.26×10^{-07}	
5×10^{-02}	0.905256342	-1.56×10^{-08}	
2.5×10^{-02}	0.905256459	-3.81×10^{-08}	2.00
1.25×10^{-02}	0.905256489	-8.73×10^{-09}	2.00
6.25×10^{-03}	0.905256496	-1.39×10^{-09}	2.00

Table 3.5
Convergence of ADI scheme at time space
Convergence of joint survival probability numerical solution at τ spaces with ADI scheme. Parameter choose are $T = 1$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.2$, $\lambda_1(0) = \lambda_2(0) = 0.05$. $\Delta\lambda_1 = \Delta\lambda_2 = 10^{-4}$, which is sufficiently large. The analytical solution is taken out from the solutions to transform into errors.

$\Delta\lambda_2$	Numerical Solution	Error	x_j
1×10^{-1}	0.90526793485	1.14×10^{-5}	
5×10^{-2}	0.905258001066	1.50×10^{-7}	
2.5×10^{-2}	0.905256684760	1.87×10^{-7}	2.92
1.3×10^{-2}	0.905256524634	2.71×10^{-8}	3.04
6.3×10^{-3}	0.905256502373	4.87×10^{-9}	2.85
3.1×10^{-3}	0.905256498649	1.15×10^{-9}	2.58
1.6×10^{-3}	0.905256497941	4.38×10^{-10}	2.39
7.8×10^{-4}	0.905256497796	2.94×10^{-10}	2.30
3.9×10^{-4}	0.905256497766	2.64×10^{-10}	2.27

Table 3.6
Convergence of ADI scheme at λ space
Convergence of joint survival probability numerical solution at λ spaces with ADI scheme. Parameter choose are $T = 1$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.2$, $\lambda_1(0) = \lambda_2(0) = 0.05$. $\Delta\lambda_1 = \Delta t = 10^{-4}$, which is sufficiently large. The analytical solution is taken out from the solutions to transform into errors.

large at 1000, the numerical solutions, errors and rate of convergence at time space are displayed in table 3.5. Following in a similar manner, we fix N and I to show the convergence in λ_2 space and the results are presented in table 3.6. Due to symmetries in the λ_1 and λ_2 directions, we would expect the same pattern of convergence in λ_2-direction as we see in the λ_1-direction.

We can see now that the ADI scheme appears to be unconditionally stable. A solution with 5 digits error is obtained with $\Delta t = 10^{-1}$ while the step size at both λ_1 and λ_2-direction is very small at 10^{-4}. The rate of convergence at time x_n is 2 and the most accurate solution is to the 9^{th} decimal place. The convergence rate at space x_j is higher than 2 with large $\Delta\lambda_2$ and the convergence rate tends to around 2 as the $\Delta\lambda_2$ getting smaller.

Δt	Solution $\rho = 1$	x_n	Solution $\rho = 0.5$	x_n	Solution $\rho = 0$	x_n
1×10^{-1}	0.905667872	NA	0.905458153	NA	0.905256521	NA
5×10^{-2}	0.905669252	NA	0.905458812	NA	0.905256526	NA
2.5×10^{-2}	0.905669941	1.00	0.90545914	1.00	0.905256527	2.00
1.3×10^{-2}	0.905670285	1.00	0.905459304	1.00	0.905256527	2.00
6.3×10^{-3}	0.905670457	1.00	0.905459386	1.00	0.905256527	2.00
	Solution $\rho = -0.5$	x_n	Solution $\rho = -1$	x_n		
1×10^{-1}	0.90506296	NA	0.904877463	NA		
5×10^{-2}	0.905062378	NA	0.904876362	NA		
2.5×10^{-2}	0.905062086	0.99	0.90487581	1.00		
1.3×10^{-2}	0.90506194	1.00	0.904875534	1.00		
6.3×10^{-3}	0.905061867	1.00	0.904875396	1.00		

Table 3.7
Convergence of ADI scheme at time space with correlation
Convergence of joint survival probability numerical solution at τ spaces with correlation and ADI scheme. Parameter choose are $T = 1$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.2$, $\lambda_1(0) = \lambda_2(0) = 0.05$. $\Delta\lambda_1 = \Delta\lambda_2 = 10^{-2}$. Correlation takes value -1, -0.5, 0, 0.5 and 1.

The correlation effects are shown in table 3.7. Since the correlation term is discretised in a fully explicit manner, the rate of convergence at τ space reduces to first-order whenever correlation is non-zero as mentioned by Douglas and Rachford (1956).

CVA numerical results

After solving the PDE system (3.20), the behaviour of the CVA in both time and the two firms' default intensity are detailed in tables 3.10-3.14. Here we are evaluating the CVAs due to holding a five-year CDS. The buyer pays 3%, or 300*bps*, swap premium of the notional quarterly, i.e. $t_i - t_{i-1} = 0.25$, for a five-year protection period. Recovery rates, R_1 and R_2, are assumed to be always 0.4 and the risk-free interest rate is $r = 0.02$. We assume that the referencing firm and the counterparty are equally risky such that the intensity parameters of reference and counterparty are identical, and are $\kappa_1 = \kappa_2 = 0.5$, $\theta_1 = \theta_2 = 0.05$, $\sigma_1 = \sigma_2 = 0.1$ and $\lambda_1(0) = \lambda_2(0) = 0.05$.

The CVA, in general, is an increasing function of time to maturity because there is a higher probability that the counterparty credit event will occur and cause a loss to the CDS buyer. However, the CVA is very sensitive to changes in the CDS value since this determines the loss when the counterparty defaults. Therefore, we observe there to be a saw shape behaviour for CVA (which is more obvious in the black curve) due to the saw shape behaviour for the CDS value at premium settle dates, which

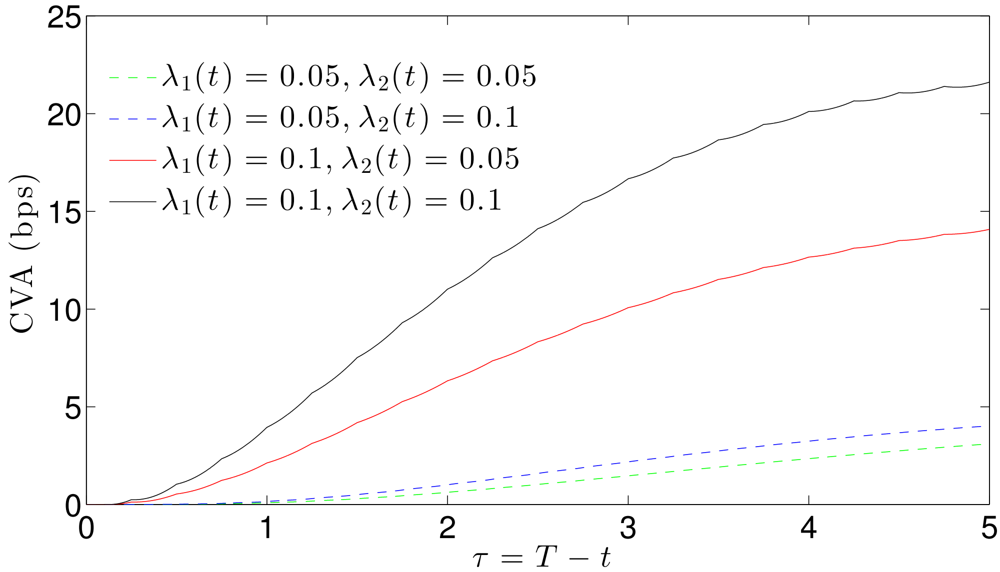

Figure 3.10
The finite-difference solution to CVA with respect to time to maturity $(T - t)$
An illustration of the CVA from time 0 to maturity, with different positions of both firms default intensities. CVA is more sensitive to the increments in reference firm's default risk than the counterparty. CVA tends to have saw shape as time proceeds, given other factors unchanged. Parameters are $S = 0.03$, $t_i - t_{i-1} = 0.25$, $R_1 = R_2 = 0.4$, $r = 0.02$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$, $\lambda(0) = 0.05$.

was shown earlier in figure 3.3 for the saw shape of CDS value. Compared with the counterparty, the CVA is more sensitive to the referencing firm's credit, or λ_1 equivalently. If the CDS protection has very low value, or the referencing firm has low default risk, doubling the counterparty's credit risk hardly raises the CVA at all. However, a deterioration in the referencing firm's credit leads to significant changes in the CVA. In addition, a more risky referencing firm also leads to a situation where the CVA is sensitive to the counterparty's credit because when the loss at default is higher, a rise in the counterparty's default probability will increase the CVA considerably.

Although deteriorations in the referencing firm's credit will have the effect of increasing the loss given default, this does not necessarily correspond to higher CVA. Apart from the loss given default, the CVA will also depend on the relative credit risk between the two firms, or, more precisely, the probability that the counterparty defaults earlier than the referencing firm. Therefore, CVA is not a monotone function of referencing firm's default intensity, but rather it has a hump shape, for which we see the evidence in figure 3.11. Further in figure 3.11, we see that, when λ_1 is small, the CVA is close to zero no matter how risky the counterparty is because the CDS has

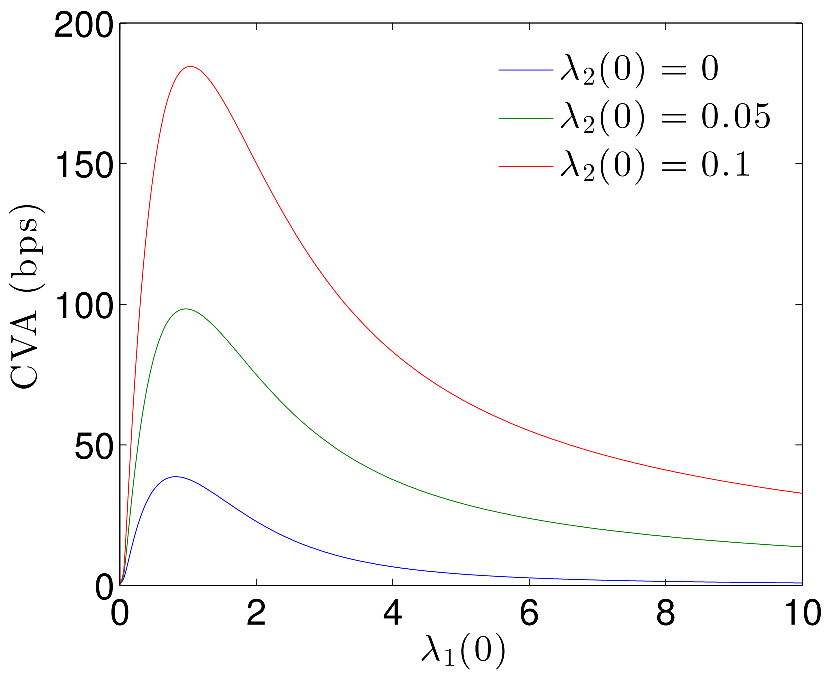

Figure 3.11
The finite-difference solution to CVA at λ_1-dimension
The CVA at λ_1 space with different value of λ_2 at time 0. Parameters are $S = 0.03$, $t_i - t_{i-1} = 0.25$, $R_1 = R_2 = 0.4$, $r = 0.02$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$, $\lambda(0) = 0.05$.

negative value with low λ_1. In this region close to $\lambda_1 = 0$, the CVA increases rapidly with the referencing firm's default intensity because the underlying protection's value changes from negative to positive, which means that a counterparty's default triggers a loss to the investor. However, if the referencing firm is extremely risky compared to the counterparty, it is very likely that the referencing firm will default shortly so the CVA declines. Under these circumstances, CVA is also decreasing to zero meaning a counterparty default event is rare. From a numerical prospective, as discussed in Section 3.4.1, applying the Dirichlet condition $U(\lambda_1, \lambda_2, t) = 0$ for $\lambda_1 \to \infty$ is only satisfied when λ_1 is considerably greater than λ_2. Figure 3.11 also suggests, when λ_2 is higher, there is considerable CVA even for large λ_1.

Figure 3.12 shows the sensitivity of the CVA with respect to referencing firm's credit λ_1. At first, the CVA grows increasingly fast and is followed by slowing down and becomes decreasing against λ_1. This situation implies the difficulty in dynamically hedging the CVA from the prospective of a holder of the payer CDS, especially when the referencing firm is very risky. Since the CVA's movements with respect to the reference firm's credit may change sign when λ_1 large, the direction of trading hedging

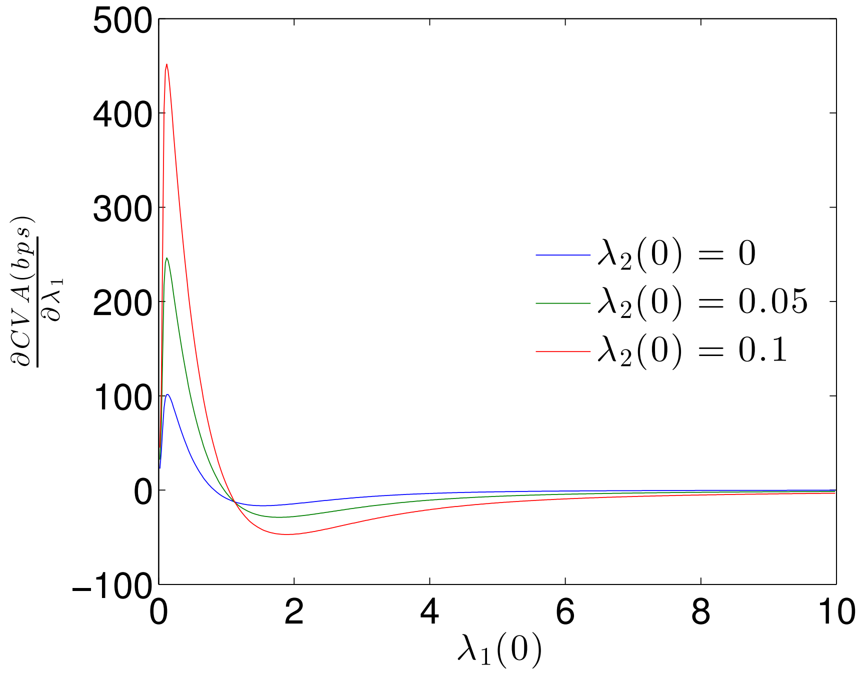

Figure 3.12
The derivative of CVA with respect to λ_1
An approximation of the CVA sensitivity to referencing firm's default intensity. The derivatives are approximated by central difference. Parameters are $S = 0.03$, $t_i - t_{i-1} = 0.25$, $R_1 = R_2 = 0.4$, $r = 0.02$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$.

securities can be different and could be expensive because the reference firm's credit risk is high. In addition, a risky counterparty will enlarge the CVA sensitivity to the referencing firm's credit, because this means the loss is more likely to happen. Therefore, any increments in the CDS's value imply higher losses to the investor.

CVA has a very diversified behaviour corresponding to the counterparty's default intensity, depending on the value of the underlying CDS. Figure 3.13 shows the CVA can be a hump shape or monotonically increasing with the counterparty's credit. If the CDS is expensive, or has high positive value, the CVA is always increasing with the counterparty's default intensity λ_2. The CVA will grow very fast when the counterparty's default probability changes from negligible to moderate. Then the speed at which it rises will slow down and the CVA convergences to a value lower than the CDS value. The reason for the CVA does not converges to the CDS value is that a high CDS value also implies a non-negligible probability that the referencing firm defaults earlier. In addition, the CVA is a concave function of λ_2, which is also evidenced by the CVA sensitivity of λ_2 in figure 3.14. On the other hand, if the CDS protection has

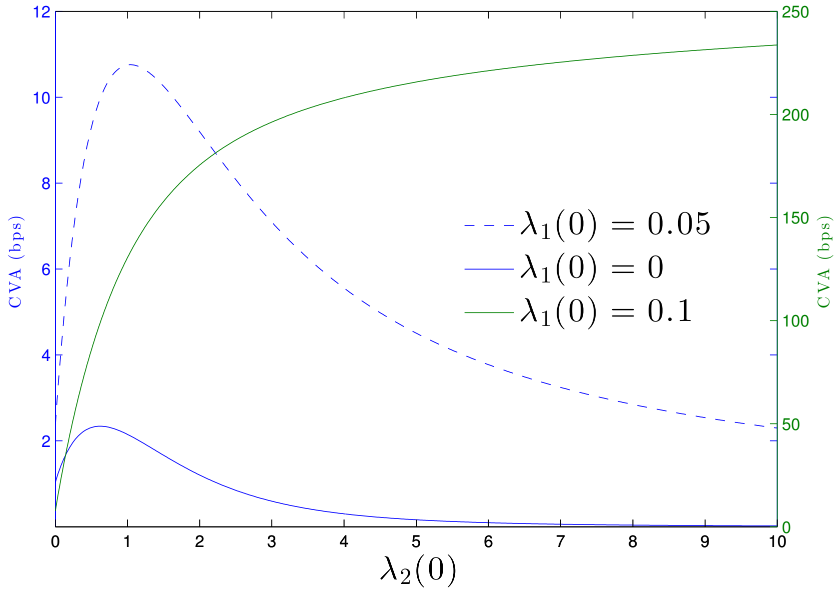

Figure 3.13
The finite-difference solution to CVA at λ_2-dimension
The CVA at λ_2 space with different value of λ_1 at time 0. Parameters are $S = 0.03$, $t_i - t_{i-1} = 0.25$, $R_1 = R_2 = 0.4$, $r = 0.02$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$, $\lambda(0) = 0.05$.

a very low value, the protection buyer is subject to smaller counterparty risk when the counterparty is too risky or too safe. Even if the counterparty is very likely to default soon but the CDS has negative value, then there is no loss to the investor. The CVA will be higher if the counterparty is very likely to default before maturity, whilst still allowing enough time for the CDS value to revert to a normal level. Under this situation, when the CDS has a low value, the CVA sensitivity to λ_2 changes sign indicating a potential difficulty in the hedging strategies. But since the magnitude of the changes in CVA with respect to the changes in λ_2 is significantly lower than the case where the CDS is expensive, this implies there will not be huge gain/loss due to a mistake in hedging CVA against λ_2.

Impacts from correlation

In order to understand how default correlation can affect the CVA of a CDS contract, we will now carry out a detailed analysis on the CVA for a simple correlation model whose referencing firm and protection seller (the counterparty) are specified by (3.19). It is commonly believed that positive correlations of the referencing and counterparty will lead to a higher CVA, the reason being that it implies the CDS will be more

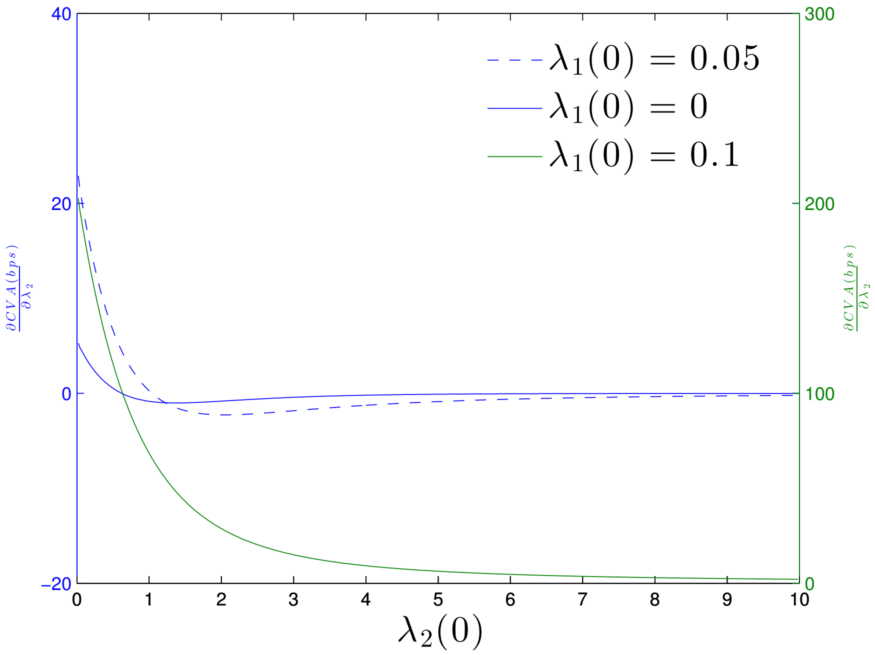

Figure 3.14
The derivative of CVA with respect to λ_2
An approximation of the CVA sensitivity to referencing firm's default intensity. The derivatives are approximated by central difference. Parameters are $S = 0.03$, $t_i - t_{i-1} = 0.25$, $R_1 = R_2 = 0.4$, $r = 0.02$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$.

valuable while the counterparty is more likely to default, which is also referred to as *wrong-way risk*. This argument holds for the most situations, however, our numerical results show that this is not true in extreme situations.

Defined by (2.72), CVA is a combination of two key components, the probability of the counterparty defaulting earlier than the referencing firm (and the maturity of the CDS) and that the CDS value which determines the loss given a counterparty default. The correlation between default intensities λ_1 and λ_2 affects on CVA by increasing the probability that the CDS value and the counterparty default probability are both high at the same time. The CDS value itself is not affected by the correlation, because it is solely dependent on the marginal default probability of the referencing firm so it can be described by the PDE (2.60). The probability $\mathbb{Q}(\tau_2 < T, \tau_2 < \tau_1|\mathcal{F}_t)$, on the other hand, is dependent on both of the entities, as well as the correlation between the two entities.

We define $P(\lambda_1, \lambda_2, t, T, \rho) := \mathbb{Q}(\tau_2 < T, \tau_2 < \tau_1 | \mathcal{F}_t)$. Under reduced-form framework, this probability is given by the expectation

$$\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s \lambda_1(s)+\lambda_2(s)} \lambda_2(s) ds \middle| \mathcal{F}_t\right]. \tag{3.35}$$

which is the expectation at time t of the probability that the counterparty defaults before time T and earlier than the referencing firm's default time. According to Feynman-Kac theorem, (3.35) satisfies the PDE

$$\begin{aligned} &\frac{\partial P}{\partial t} + \kappa_1(\theta_1 - \lambda_1)\frac{\partial P}{\partial \lambda_1} + \kappa_2(\theta_2 - \lambda_2)\frac{\partial P}{\partial \lambda_2} \\ &+ \frac{1}{2}\sigma_1^2 \lambda_1 \frac{\partial^2 P}{\partial \lambda_1^2} + \frac{1}{2}\sigma_2^2 \lambda_2 \frac{\partial^2 P}{\partial \lambda_2^2} + \rho\sigma_1\sigma_2\sqrt{\lambda_1\lambda_2}\frac{\partial^2 P}{\lambda_1 \lambda_2} + \lambda_2 - (\lambda_1 + \lambda_2)P = 0, \end{aligned} \tag{3.36}$$

subject to the terminal condition $P(\lambda_1, \lambda_2, T, T) = 0$ and we will use boundary conditions (3.25a) and (3.25b) to solve PDE (3.36) numerically.

In figure 3.15 we show how the probability, $\mathbb{Q}(\tau_2 < T, \tau_2 < \tau_1 | \mathcal{F}_t)$ is affected by different values for the correlation parameter $\rho = 0.5$ and $\rho = 0$ by plotting the change in value $P(\lambda_1, \lambda_2, t, T, \rho = 0.5) - P(\lambda_1, \lambda_2, t, T, \rho = 0)$. The probability behaves differently in each of the two directions, λ_1 and λ_2. In part (a) of the figure we see that the correlation pushes down the probability that the counterparty defaults earlier and the reduction in the probability becomes deeper when λ_2 increases from 0 to 0.1. On the other hand, the probability tends to be insensitive to the correlation when the reference firm's intensity λ_1 becomes extremely large. The behaviour however changes when the counterparty's intensity λ_2 is very large, which is shown in part (b), in the region where λ_2 is between 3 and 5. At first, when the reference's intensity is relatively low, the probability is raised by correlation. Again, when λ_1 is large enough, the probability will be lowered by the correlation. In the other direction, which is shown in parts (c) and (d), the probability is lower with smaller λ_2 compared to λ_1 and the probability becomes raised by correlation when λ_2 is greater than λ_1. The amount that the probability increases slows down when λ_2 becomes extremely large but it is still higher than the probability without positive correlation as long as λ_1 is far less than λ_2. When λ_2 is very large or the referencing firm is very risky in part (d), we observe similar behaviour as that which was seen in part (c) but the difference is that it converges to be slightly lower than zero when both firms' intensity approaches 5.

We now attempt to explain the behaviour of the probability seen in figure 3.15. The probability that the counterparty defaults first can be viewed as the sum of two probabilities. One is the probability that the counterparty defaults before time T while the referencing firm survives. The other probability is that both firms default before time T with the counterparty's default time is the earlier one. In other words,

$$\mathbb{Q}(\tau_2 < T, \tau_2 < \tau_1 | \mathcal{F}_t) = \mathbb{Q}(\tau_2 < T, \tau_1 > T | \mathcal{F}_t) + \mathbb{Q}(\tau_2 < \tau_1 < T | \mathcal{F}_t). \tag{3.37}$$

The first term on the right-hand-side represents the probability of only one firm (the counterparty) defaults and the second term is the probability that of both firms default. Clearly, the probability that only one firm defaults will be lower with a positive correlation. For the other probability, however, the response to a change in correlation is highly dependent on the relative riskiness of the two firms. First of all, $\mathbb{Q}(\tau_1 < T, \tau_2 < T | \mathcal{F}_t)$ grows with correlation, which is the sum of $\mathbb{Q}(\tau_2 < \tau_1 < T | \mathcal{F}_t)$ and $\mathbb{Q}(\tau_1 < \tau_2 < T | \mathcal{F}_t)$. When the referencing firm and the counterparty are just as risky as each other, we can expect that they both grow by the same amount due to symmetric arguments. However, if one firm's intensity is significantly greater than the other, for instance the counterparty's intensity is far beyond the referencing firm's, then $\mathbb{Q}(\tau_2 < \tau_1 < T | \mathcal{F}_t)$ will be raised by correlation greatly. This is because the probability that the referencing firm defaults before T is raised by the correlation to the counterparty's default, which is very likely with large λ_2. Consequently, we see the probability is lower with correlation when λ_1 and λ_2 are comparable mainly due to the smaller probability of just one of the firms defaulting. While one firm's default intensity is much higher than the other's, we observe that the probability is raised by correlation and we can see it in both parts (b) and (c) of figure 3.15, which is due to the rise in $\mathbb{Q}(\tau_2 < \tau_1 < T | \mathcal{F}_t)$ overcoming the reduction in $\mathbb{Q}(\tau_2 < T, \tau_1 > T | \mathcal{F}_t)$. At the right hand side of part (a), where the referencing firm's intensity is far beyond the counterparty, both $\mathbb{Q}(\tau_2 < T, \tau_1 > T | \mathcal{F}_t)$ and $\mathbb{Q}(\tau_2 < \tau_1 < T | \mathcal{F}_t)$ are less sensitive to correlation. The right hand side of part (b) is crossing the zero point when λ_2 increases from 3 to 5. It shows that the reduction in $\mathbb{Q}(\tau_2 < T, \tau_1 > T | \mathcal{F}_t)$ cannot be compensated by increases in $\mathbb{Q}(\tau_2 < \tau_1 < T | \mathcal{F}_t)$ when $\lambda_1 = \lambda_2 \approx 5$. The right hand side of part (d) shows the same effect but in the opposite direction.

Figure 3.16 demonstrates the CVA change in basis points due to the introduction of a positive correlation of 0.5 over a range of values in λ_1 and λ_2. Apart from the

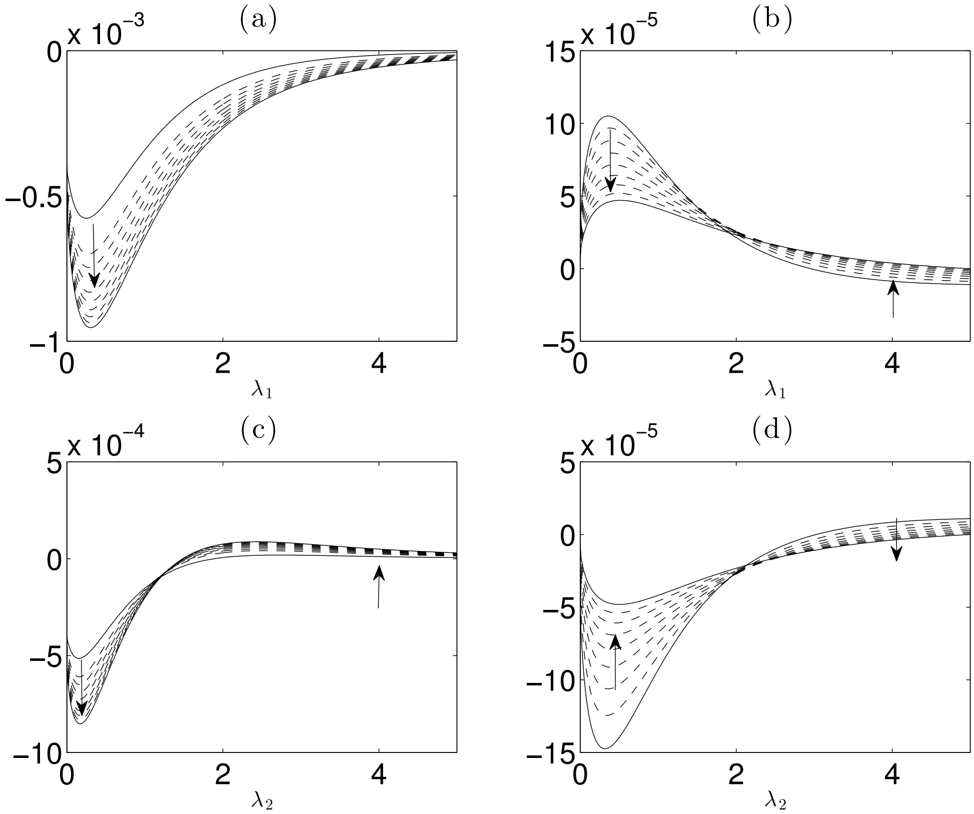

Figure 3.15
The impact of correlation on the probability that the counterparty to be the first to default in five years. P($\rho = 0.5$)-P($\rho = 0$)
A illustration of the probability changes due to raising the correlation coefficient between counterparty and referencing firm with parameters are $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$. (a) The probability change against λ_1 and λ_2 raises from 0 to 0.1. (b) The probability change against λ_1 and λ_2 raises from 3 to 5. (c) The probability change against λ_2 and λ_1 raises from 0 to 0.1. (d) The probability change against λ_2 and λ_1 raises from 3 to 5. The pointer points out the direction of moving when λ_1 or λ_2 raises.

probability (3.37), the value of the CDS also has very strong impacts on the CVA, which are strongly linked to λ_1. Positive correlation usually leads to higher CVA as shown in most regions in parts (a), (b), (c) and (d). Although the positive correlation rarely results in a higher probability of counterparty's early default, the CDS value or the loss given default is more likely to be high, which leads to a larger CVA. Since we know that only when there is a positive CDS value will there be a loss to the protection buyer, CVA is especially sensitive to the correlation when λ_1 is around the break-even point of the CDS contract, which is shown by the peak in value in parts (a) and (b) of figure 3.16. However, it is found that the CVA could be lower due to a positive correlation if λ_1 is far greater than λ_2. The value of the CDS, as shown in figure 3.4, is a concave function and converges to its upper bound as $\lambda_1 \to \infty$. When the referencing firm is already very risky and the CDS value is high enough such that the increments in λ_1 will not increase the CDS value, the positive correlation can no longer lead to higher loss given default. On the other hand, as discussed, the probability of having a counterparty credit event is lower when $\lambda_1 >> \lambda_2$. As a result, CVA is decreasing against positive correlation, although it should be noted here that the actual changes are incredibly small (fractions of a basis point).

Lastly, we analyse the way that CVA changes against correlations from a perfectly negative to a perfectly positive one at current default intensities' level $\lambda_1 = \lambda_2 = 0.05$ in figure 3.17. Taking our base parameters, we see that the CVA is increasing against correlation. Part (a) of figure 3.17 shows the CVA of three different CDS contracts whose swap premium are paid at 250 *bps*, 282 *bps* and 300 *bps*, which are currently have a positive value, break-even and one has a negative value and part (b) shows the relative percentage changes as compared to the case where $\rho = 0$. Although the CDS is paying low spreads it has higher CVA, the one contract that pays the high spread has a higher percentage change due to correlation changes. However, in terms of the CVA changes in basis points, the CDS, which has positive value, increases more with the correlation. Although it is well documented that correlated Brownian motions in reduced-form modelling leads to a low degree of default correlation, the percentage change in CVA due to correlated Brownian motions is considerably large. But it is worth noticing that the absolute changes in CVA are not significant, namely within a few basis points.

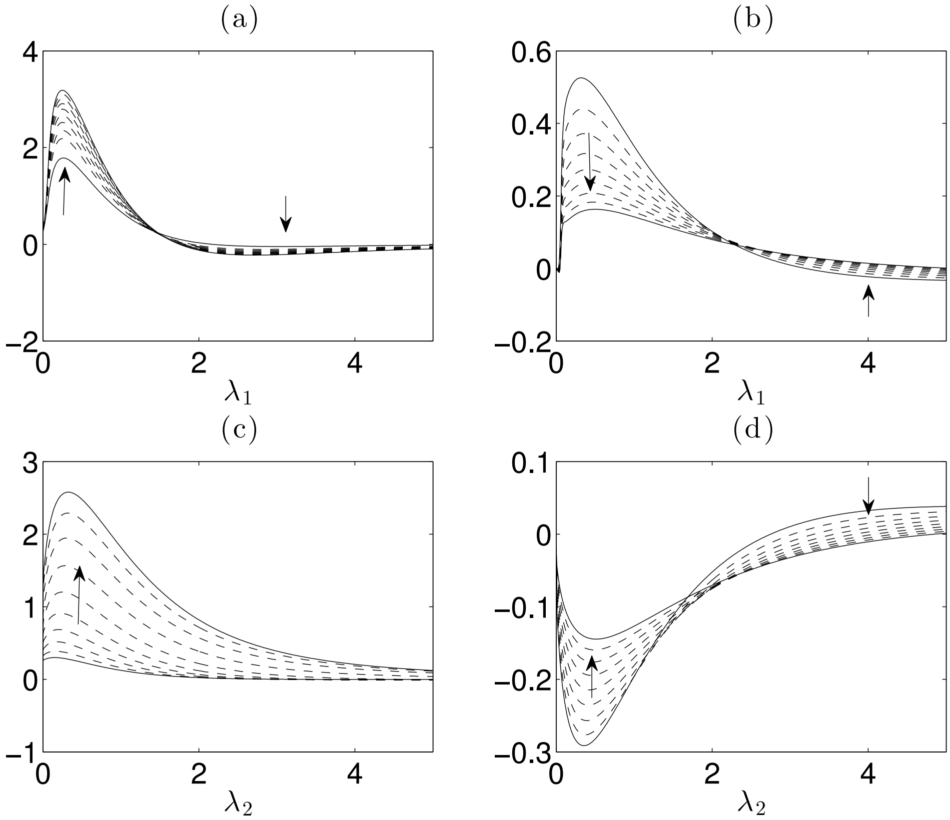

Figure 3.16
The change of the CVA of a 5 years CDS protection with 0.5 correlation. CVA($\rho = 0.5$)-CVA($\rho = 0$)
A illustration of the CVA changes due to raising the correlation coefficient between counterparty and referencing firm with parameters are $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$. (a) The CVA change against λ_1 and λ_2 raises from 0 to 0.1. (b) The CVA change against λ_1 and λ_2 raises from 3 to 5. (c) The CVA change against λ_2 and λ_1 raises from 0 to 0.1. (d) The CVA change against λ_2 and λ_1 raises from 3 to 5.

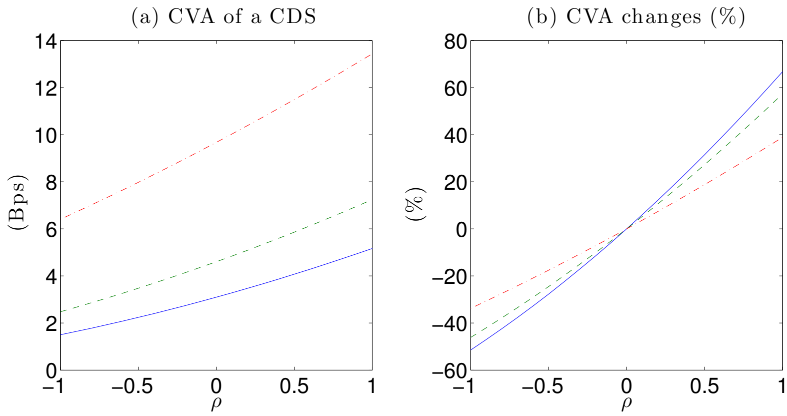

Figure 3.17
CVA of a CDS with correlated default intensities
A illustration of the CVA due to holding a CDS protection when referencing and counterparty firms' default intensities are correlated, with spreads equal 250 bps(solid), 282 bps(dash) and 300 bps(dash-dot), where 282 bps is the fair swap spread. The CVA increases as correlation coefficient raises from -1 to 1 (left). And the figure on the rights shows the percentage of CVA changes compared with non-correlated intensities. Parameters are $t_i - t_{i-1} = 0.25$, $R_1 = R_2 = 0.4$, $r = 0.02$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$, $\lambda_1(0) = \lambda_2(0) = 0.05$

Impacts from volatilities

Brigo and Chourdakis (2009) were one of the first to investigate the impacts of stochastic default intensities' volatilities on the CVA for a CDS with a Gaussian copula function. They found perfect correlation in Gaussian copula with small volatilities lead to strange behaviour in CVA, which will disappear with high volatilities. However, in Brigo and Chourdakis (2009), the volatility impact on CVA is only investigated under the influence of a specific copula function and the volatility impacts on the CVA is not discussed thoroughly.

In this section, we do not specify a correlation method and discuss the volatility impact on the CVA in terms of the probability that the counterparty to be the first default firm and the loss given default, which are the two components of the CVA. Due to the advantage of finite-difference methods, CVA solutions are available at a wide range of the two firms' default intensities' position λ_1 and λ_2. So we attempt to give a more detailed investigation into the impact of volatility on CVA and in particular the different impacts given a range of the two firms' possible intensity positions.

Table 3.8 shows the CVA due to holding a 5-year CDS whose counterparty and the referencing firm's volatility increases from 0.01, which is almost deterministic, up to and including 0.25, which is almost the maximum volatility for which the *Feller*

	σ_1					
σ_2	0.01	0.05	0.1	0.15	0.2	0.25
0.01	0.282	2.205	4.639	6.787	8.563	9.944
0.05	0.281	2.201	4.630	6.775	8.547	9.926
0.10	0.280	2.189	4.605	6.738	8.501	9.872
0.15	0.277	2.170	4.565	6.679	8.426	9.785
0.20	0.274	2.144	4.509	6.598	8.324	9.667
0.25	0.270	2.112	4.442	6.498	8.198	9.521

Table 3.8
Volatility effects on CVA (Bps)
An illustration of how the CVA for CDS changes due to counterparty and referencing firm's spread volatility. σ_1 is the volatility parameter of referencing firm and σ_2 is the one of the counterparty. Parameters are $t_i - t_{i-1} = 0.25$, $S = 282$ *bps*, $R_1 = R_2 = 0.4$, $r = 0.02$, $\kappa = 0.5$, $\theta = 0.05$, $\lambda_1(0) = \lambda_2(0) = 0.05$.

condition will hold, given other parameters in the model. It is well known that, when modelling default intensity as a CIR process, the probability of survival is a decreasing function of its volatility. If the counterparty has higher volatility, the probability of having a counterparty credit event is lowered which should therefore lead to a lower CVA. This pattern applies regardless of the referencing firm's volatility. On the other hand, CVA can be raised significantly by the volatility of the referencing firm's default intensity. Since only when the CDS value is positive then the default of the counterparty will cause a loss to the CDS buyer, CVA is in some sense analogous to options that are more expensive with higher volatility in general. A higher volatility of the referencing firm's credit intensity means that the value distribution of the CDS has fatter tails. In other words, the CDS value is more likely to be high or low. However, since only a positive value will trigger a loss to the investor, the CVA is raised by the volatility of the referencing firm's default intensity. However, we will show later in this section this does not always hold for the CIR-type default intensity.

Figures 3.18 and 3.19 illustrate the results of CVA when raising the volatility of the counterparty and the referencing firm separately. We undertook this by doubling the volatility parameter σ_1 or σ_2 in turn and then taking the difference to the original CVA solution at $\lambda_1 \times \lambda_2 \in [0, 10] \times [0, 10]$.

According to figure 3.18, since we see that all sections in the figure are negative this implies that higher counterparty's spread volatility will always lower the CVA by reducing the probability that the counterparty credit event occurs. On the other hand, CVA has a very disjointed behaviour against the referencing firm's volatility depending on the position of its own default intensity λ_1, as well as the relative position of two firms' default intensities. In other words, both the riskiness of the referencing firm and the relative riskiness of the referencing and the counterparty changes CVA's behaviour with the volatility of the referencing firm's default intensity.

Part (a) in figure 3.19 plots the CVA changes in the λ_1-direction. We see that when λ_1 is relatively low, the CVA is increased by the referencing firm's default intensity volatility. However, we also see that the CVA could be lower with the referencing firm's higher volatility when the referencing firm's default intensity is very high. Similar to options that are more expensive with higher volatility, CVA can be raised by the referencing firm's spread volatility as well. The reason lies in the fatter tailed distribution of the CDS at the positive side. However, unlike an option whose underlying is a stock with unbounded value, a CDS's value is bounded by the instant default payoff. Therefore, if λ_1 is large, this means the CDS is already expensive and the CDS's value distribution in the future will be centralised near the upper bound. Therefore, higher volatility σ_1 will not make the value distribution of the CDS even fatter. In contrast, since we model the credit intensity as a CIR process, higher volatility always corresponds to lower probability of default, which corresponds to lower CDS values than before. Therefore, we can see the CVA is lowered by the referencing firm's spread volatility. Further, we can see that when λ_1 is extremely high then it is very unlikely that the counterparty will default earlier, volatility does not have effect on CVA, which tends to zero.

The CVA changes in the λ_2-direction is apparently more interesting depending on the relative value of λ_1 and λ_2, which we can see in parts (b), (c) and (d) of figure 3.19. We know that the CVA is dependent on the probability that the counterparty defaults first and also what the loss given default is, which in this case is the CDS value. As discussed earlier, raising the referencing firm's volatility has two effects. One is to fatten the tails of the CDS's value distribution and the other is to reduce the referencing firm's default probability, which is equivalent to lowering the CDS value. Also, the probability that the counterparty defaults before the referencing firm is higher due to the fact that the referencing firm is less likely to default. The red curve in part (b) of figure 3.19 indicates the case that CVA changes given $\lambda_1 = 0.04$. This shows how the CVA is raised by the referencing firm's volatility when λ_2 is low, which is the combination of both higher probability of the counterparty credit events and the CDS value distribution becoming fatter in the tails. However, when the counterparty is very risky with $\lambda_2 = 10$, the volatility effects on the counterparty default probability will not be significant as the counterparty is very likely to default first compared to the referencing firm, who is with $\lambda_1 = 0.04$. Therefore, the effect of an increase in the referencing firm's volatility is reduced. The green line in part (b) represents the case where the volatility effect on fatter tailed the CDS's value distribution is not sufficient to compensate the effect which lowers the CDS value. Therefore, the CVA increases with low λ_2 here, is mainly due to the counterparty default probability being higher. However, the CVA reduces when the increment in the probability that the counterparty defaults earlier than reference is negligible and the loss given default is lowered by the increment of the referencing firm's volatility σ_1. Finally, the blue curve in part (b) shows the case that $\lambda_1 = 0.2$ where higher referencing firm's volatility can only reduces loss given default significantly. In this case, the increase in the counterparty defaults first probability is not sufficient to compensate the lower loss given default, so the CVA only reduces against the volatility for all values of λ_2.

Although increasing the referencing firm's volatility σ_1 can reduce the referencing firm's default probability and the corresponding CDS value, the difference between the CDS values tend to zero as $\lambda_1 \to \infty$. In other words, if λ_1 is high enough, increasing volatility will have negligible effect on the loss given default. In figure 3.19 part (c), λ_1 increases from 3 to 5 and the reduction of CVA narrows down because the higher referencing firm's volatility σ_1 has negligible effect on loss given default with large λ_1. Finally the CVA slightly increases when $\lambda_1 = 5$ and λ_2 is around 1. This is due to the increase in the probability that the counterparty defaults first.

Finally, in part (d) of figure 3.19 shows that the impacts from the higher volatility σ_1 on the CVA is slowing down to slightly above 0 when λ_1 rises from 6 to 10. The changes in CVA seen in part (d) of this figure are solely due to the fact that the probability of the counterparty defaults earlier than the referencing firm is higher given that the referencing firm's default intensity is more volatile. The actual amount that higher the referencing firm's volatility will increase this probability is dependent on the relative riskiness of the two firms. The increment will be negligible when λ_2

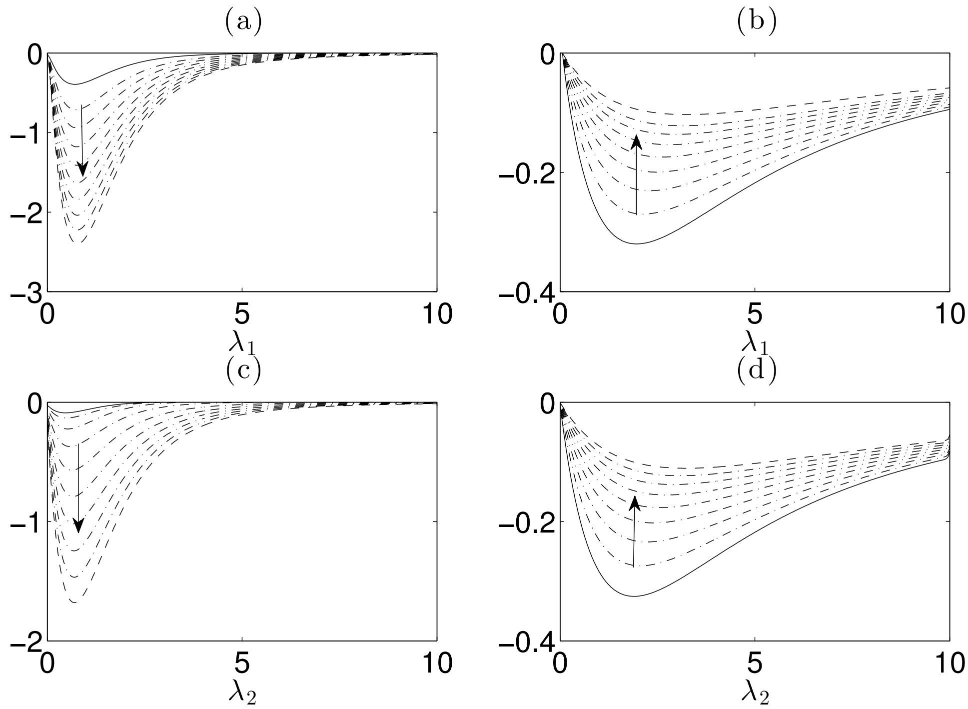

Figure 3.18
Changes to CVA from raising counterparty's volatility. CVA($\sigma_2 = 0.2$)-CVA($\sigma_2 = 0.1$)
A illustration of the CVA difference due to raising counterparty's volatility from 0.1 to 0.2. Parameters are $\kappa = 0.5$, $\theta = 0.05$. (a) The CVA change against λ_1 with λ_2 raises from 0 to 0.1. (b) The CVA change against λ_1 with λ_2 raises from 3 to 5. (c) The CVA change against λ_2 with λ_1 raises from 0 to 0.1. (d) The CVA change against λ_2 with λ_1 raises from 3 to 5.

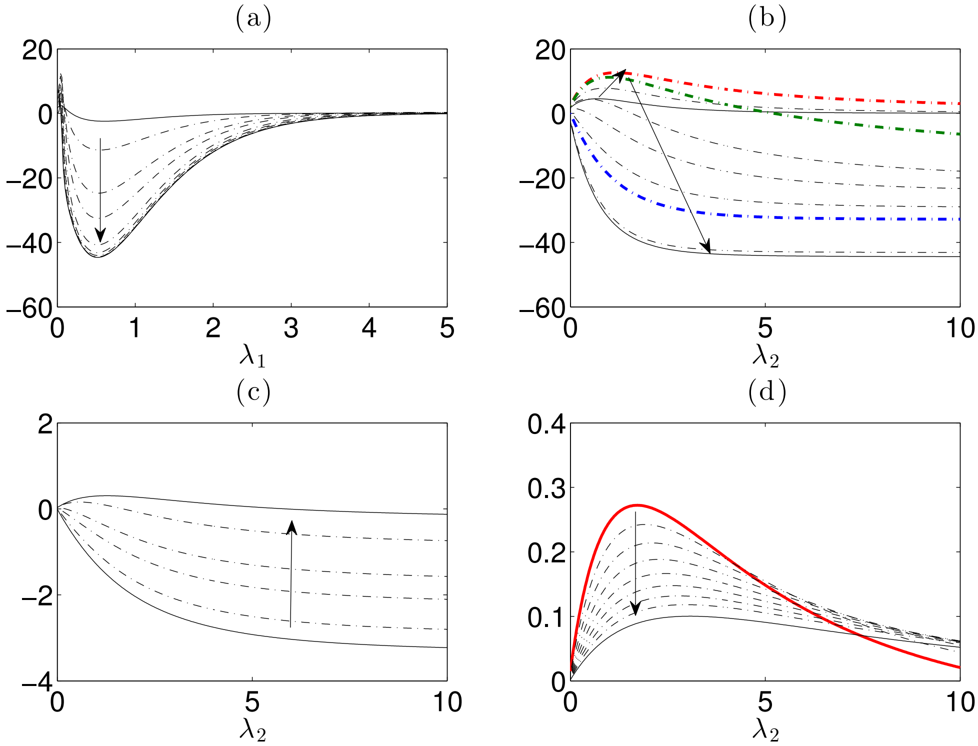

Figure 3.19
Changes to CVA from raising referencing firm's volatility. CVA($\sigma_1 = 0.2$)-CVA($\sigma_1 = 0.1$) A illustration of the CVA difference due to raising referencing firm's volatility from 0.1 to 0.2. Parameters are $\kappa = 0.5$, $\theta = 0.05$. (a) The CVA change against λ_1 with λ_2 raises from 0 to 10. (b) The CVA change against λ_2 with λ_2 raises from 0 to 0.6. (c) The CVA change against λ_2 with λ_1 raises from 3 to 5. (d) The CVA change against λ_2 with λ_1 raises from 6 to 10.

is small because the counterparty's default probability very low so that whether the referencing firm's default probability will be lowered by the referencing firm's higher volatility or not does not matter. Another situation is when $\lambda_1 << \lambda_2$, because it is almost surely that the first firm to default will be the referencing firm. For the same reason, the volatility effect is the most significant when λ_2 is not small and is similar to λ_1. This is evidenced by the red curve in the plot of part (d) whose $\lambda_1 = 3$ and the CVA has the maximum increment of 0.3 *bps* at λ_2 around 2. As λ_1 increases to 10, we observe that the increases in CVA is dropping slowly with λ_2. We observe this because it is more difficult to achieve $\lambda_1 << \lambda_2$ with a large λ_1 in a finite domain.

3.5 Conclusions

This chapter discusses the boundary conditions for pricing CDS contracts and the CVA using a finite-difference approximation. In reduced-form modelling, we discuss the difficulties we face in setting boundary conditions due to credit events that are not guaranteed even with very large intensity levels. We investigate the effect of applying a Dirichlet, Robin and heuristic Robin type conditions when pricing a CDS contract. Numerical solutions with these three types of condition are compared against the CDS's semi-analytical solution. We found that the heuristic Robin conditions are the most suitable among the Dirichlet and Robin conditions for describing the boundary behaviour and it is both fast and accurate. Using Crank-Nicolson scheme with Robin conditions, our numerical solution achieve 8^{th} digit accuracy with computational time four times lower than the semi-analytic solution with numerical integration. In terms of the CVA, we address the difficulty in imposing boundary conditions for solving the PDE of CVA as the behaviour at boundaries depend on the relative riskiness of each of the counterparty and the referencing firm. We also explore the finite-difference approach for solving the CVA of a CDS using the similar heuristic Robin type conditions. Compared to previous simulation based numerical schemes, the finite-difference scheme enables us to observe the CVA's behaviour over a larger range of default intensities' position and to estimate the CVA sensitivities to default intensities of the referencing firm and the counterparty. We also investigate how the correlation between the two firms' intensities and their volatilities can change the CVA. It is found that CVA can have very different behaviours with volatilities and correlation depending on the firms' default intensity or the relative riskiness between the two firms.

Chapter 4

A New Default Contagion Model

One major concern in credit risk management is the presence of default correlation. It has frequently been observed that the bankruptcy of a large company is likely to cause difficulties for other firms who have a close business relationship with them, such as their partners, suppliers or customers but sometimes just being part of the same sector can cause problems. Another possibility is that a creditor suffers a loss when its debtor defaults which introduces financial instability to the creditor or, in the worst scenario, bankruptcy. We call the situation in which the default events of some firms cause losses and potentially defaults to other firms as *default-contagion*. Default contagion modelling is firstly considered by Jarrow and Yu (2001) as the mechanism of default dependency between firms. This default dependence is crucially important when analysing a portfolio of credit derivatives or a product that involves more than one firm, such as CDO and CDS with counterparty risk.

In this chapter we adapt the idea of default contagion as described by Yu (2007), in which the default of one firm triggers a jump in the other alive firms' default intensities, in order to propose our own new default contagion model. Combined with existing numerical techniques, we show how the problem of solving for the survival probabilities, pricing a CDS and its CVA can be solved efficiently with finite-difference schemes, as we have discussed in Chapter 3.

4.1 Introduction

The term correlation in financial markets is usually used to describe the co-movement in the market price of assets, and this means that it is a key element in financial modelling. Tracing back to portfolio theory, a higher correlation of assets in a portfolio raises the risk that the investors face whilst a negative correlation will lower the risk to the investor. The same argument will also apply to credit-risk management and measurement. The importance of capturing default correlation was brought to light in the aftermath of the crisis of 2008. In credit risk modelling as opposed to fund management, the correlations are more complicated to model because credit correlation includes not just small movements in the default intensity, but also the CDS spread return and the arrival of default times.

Default dependence modelling is a major research area of credit-risk modelling as we have previously reviewed in Section 1.3. Many models have been developed in order to produce an appropriate level of default clustering that matches empirical studies, one such example of an empirical study is that Altman et al. (2005) who study the number of defaults of companies throughout history. Their research found the number of defaulting firms will be substantially higher in some period than others, which indicates default events are correlated. In addition, the amount of recoverable from defaulting firms are negatively correlated to the number of default firms during the same period. In other words, higher default probability is coupled with low default recovery. The simplest correlation approach is to add in correlation between the Brownian motions of default intensities. This has been used for some time, for in example Duffie and Singleton (1998), they study the losses distribution in a bond portfolio using stochastic default intensities with correlated Brownian motions. But unfortunately as they showed the portfolio loss is not sensitive to the correlation in Brownian motions. In addition, Meissner et al. (2013) apply a Libor Market Model (LMM) for the referencing firm and the counterparty's intensities and include correlation in the Brownian motions. Rather than computing CVA, they investigate the impact on the fair CDS spread with probability that the counterparty defaults. Tested via simulation, they find that just correlating the intensity processes leads to a rather low impact on the CDS spread. As reviewed in Section 1.3, there have been four approaches developed for modelling default dependency, namely *conditional-independent defaults*, *Copula*, *common-shock* and *default contagion*. Among these default dependence models, we believe that the *default contagion* model is the most intuitive in terms of describing the interaction among companies' defaults. As such, we attempt to extend *default contagion* model and propose our new *default contagion* model in this chapter.

We build out model upon the general structure of a default contagion model first proposed by Jarrow and Yu (2001), in that model, the default intensity of the i^{th} firm can be generalised as

$$\lambda_i(t) = a_i(t) + \sum_{j \neq i} b_{i,j} 1_{\{\tau_j < t\}}, \quad \text{for} \quad i = 1, 2, ... \tag{4.1}$$

where $b_{i,j}$ $\forall i \neq j$ are the processes that determine the i^{th} firm's default intensity process after the j^{th} firm defaults and $a_i(t)$ for all i are the processes of a firm's idiosyncratic factor, which determine both pre-contagion and post-contagion default intensities. Different default contagion models distinguish themselves by the specification of processes $a_i(t)$ and $b_{i,j}(t)$. For example, the parameters $a_i(t)$ and $b_{i,j}(t)$ are constants in Jarrow and Yu (2001); Leung and Kwok (2005); Herbertsson and Rootzén (2007); Yu (2007); Collin-Dufresne et al. (2004). Extensions to the default contagion modelling attempt to introduce stochastic behaviour to default intensities whilst maintaining tractability. Leung and Kwok (2007) use Markov Chain formulations to study the default contagion modelling and the counterparty risk in CDS with constant $a_i(t)$ and $b_{i,j}(t)$. They discard the default contagions between the referencing firm and the counterparty but let both the counterparty's and the referencing firm's default intensity depends on an external firm's default. In other words, the external firm's default triggers default contagion to the counterparty and the referencing firm. In the later work by the same authors, Leung and Kwok (2009) extend the Leung and Kwok (2007) model to allow the external firm's intensity process to be stochastic. Because the default contagion is only from the external firm to the referencing firm and the counterparty in Leung and Kwok (2007) and Leung and Kwok (2009), the problem of *looping default*, which is faced by most default contagion models, is avoided. In order to study the pricing of basket default swaps (BDS) under contagion model, Zheng and Jiang (2009) extend the model of Yu (2007) to allow for stochastic idiosyncratic factors $a_i(t)$. The new contagion model is with $a_i = x(t)$ for all i, where

$x(t)$ is AJD process and the $b_{i,j}$ are identical constants. However, in order to have a closed-form solution to the joint default time distribution for pricing BDS, constant $b_{j,i}$ are necessary. Gu et al. (2013) extend the model of Zheng and Jiang (2009) to separate the obliger into two groups, where default events in different groups have different strength of default contagion to others. To put it simply, this means that the $b_{i,j}$ are different for each group of firms. However, this extension introduces complexities into the solution of the default time distribution. Bao et al. (2010) use CIR type stochastic intensities for a firm's idiosyncratic factor $a_i(t)$ and let the default contagion factor $b_{i,j}$ be related to the defaulting firm's intensity, which is written in the paper as $b_{i,j} = \varpi a_j(t)$ with ϖ a constant. With the help of *survival measure* developed by Collin-Dufresne et al. (2004), a firm's marginal survival probabilities can be derived in closed-form. Therefore, the CDS can be priced with semi-analytic solutions and the unilateral CVA can be approximated with further assumptions made, see Bao et al. (2012) for more details. However, in order to have analytic solutions for marginal survival probabilities, ϖ must be a simple function, which highly restricts the form that default contagion effects may take. In addition, as mentioned in Section 1.3, it is difficult to assume the default intensity process $a_j(t)$ still exists after the default of firm j. In Wang and Ye (2013), the idiosyncratic part of firms' default intensity is specified as $a_i(t) = c_i + d_i r(t)$, where the interest rate process $r(t)$ is a Vasicek process with deterministic jumps and c_i, d_i are constants and the contagion part $b_i(t)$ is also constant as well. Although closed-form solutions to marginal survival probability and default times distribution are available, the model made very strong assumptions that all firms' credit risks are based on one factor, which is the interest rate. Moreover, the Vasicek-type process violates the non-negative requirement for default intensities.

The core idea underlies default contagion models is the upward shifts to the surviving firms' default intensity at other firms' default times. In order to gain flexibility in modelling default contagions, we propose an alternative formulation for the default contagion effect rather than as (4.1). Instead of insisting on deriving analytic solutions at the cost of a simplified model, we seek efficient numerical solutions to survival probabilities, pricing CDS as well as measuring CVA.

4.2 The Proposed Contagion Model

In previous models, such as those of Jarrow and Yu (2001), Leung and Kwok (2005), Bao et al. (2012) and Wang and Ye (2013), the default contagion is included as an additional part of the default intensity. This means that the dynamics of the default intensity changes forever after default contagion events. This property adds complexity to the dynamic of firms' credit risk after credit events and is somehow counter intuitive. We believe that although a firm's credit risk may be undermined by others' default in the near future, as long as the firm manages to keep a stable cash-flow and remain profitable, the firm is likely to recover from default contagions. In the longer term, the firm's credit risk should rely on its idiosyncratic factors rather than that from a default contagion. Therefore, the conventional formulation of default contagion may exaggerate a firm's default risk over a longer time horizon. In this section, we shall introduce a new formulation of default contagion following the framework described by Jarrow and Yu (2001), Yu (2007), Bao et al. (2010) and Bao et al. (2012).

Consider now a filtered probability space $\left(\Omega^W, \mathcal{F}_t^W, \mathbb{Q}^W\right)$ up to the finite time horizon t, satisfying the conditions of right continuous and completeness. On this probability space, there are Brownian motions $(W_t^i)_{i=0}^I$ representing the uncertainties in the economy. A filtration $\mathcal{F}_t^W$ is defined as

$$\mathcal{F}_t^W = \sigma\left((W_s^i)_{i=0}^I, 0 \leq s \leq t\right),$$

which is a Brownian filtration. In order to construct stochastic default contagion, we enlarge the probability space Ω^W with a probability space $(\Omega^\alpha, \mathcal{F}^\alpha, \mathbb{Q}^\alpha)$ defined by $I \times (I-1)$ non-negative bounded real-value random variables $\alpha_{i,j}$ with a known joint density function $\bar{\eta}()$ for $i \in I$, $j \neq i$. We denote the enlarged filtration as $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\alpha$, which is the filtration contains all market information except the default information.

Further, we assume there are I non-negative random variables $(\tau_i)_{i=0}^I$ defined on the filtered probability space $(\Omega, \mathcal{G}_t, \mathbb{Q})$, satisfying $\mathbb{Q}(\tau^i = 0) = 0$ and $\mathbb{Q}(\tau^i > t) > 0$ for $t \in \mathbb{R}_+$ and $i \in I$. Let the right-continuous default indicator processes $H_t^i = 1_{\{\tau^i \leq t\}}$ and $\mathbb{H}^i$ denote the filtration generated by the default process H_t^i, so that $\mathcal{H}_t^i = \sigma(\{\tau^i \leq u : u \leq t\})$ for any $t \in \mathbb{R}_+$. The total default information is captured by the filtration $\mathcal{H}_t = \vee_{i \in I} \mathcal{H}_t^i$. Then the total information, including market information and default information, is captured by the filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ and $\mathbb{H} = (\mathcal{H}_t)_{t \in \mathbb{R}_+}$. All filtrations are assumed to satisfy conditions of right-continuous and completeness.

In the reduced-form model discussed in Section 2.2, the existence of a default intensity process λ_t associated with the default time τ relies on the $\mathbb{F}$-adapted default process $F_t = \mathbb{Q}(\tau \leq t | \mathcal{F}_t)$ being an absolutely continuous submartingale under $\mathbb{Q}$. However, in a multiple default framework, assuming the default process F_t^i of a default time τ^i for all $i \in I$ to be $\mathbb{F}$-adapted excludes the dependence between default times and the intensity process, which is desired in default contagion models. Similar to Bao et al. (2010), Bao et al. (2012) and Yu (2007), we define $F_t^i = \mathbb{Q}(\tau^i \leq t | \mathcal{G}_t^{-i})$ to be a $\mathbb{G}^{-i} = \left(\mathcal{G}_t^{-i}\right)_{t \in \mathbb{R}_+}$-adapted default process, where $\mathcal{G}_t^{-i} = \vee_{j \in I, j \neq i} \mathcal{H}_t^j \vee \mathcal{F}_t$. The filtration $\mathcal{G}_t^{-i}$ contains all information except the i^{th} default time τ^i. With this definition of $\mathcal{G}_t^{-i}$, the filtration $\mathcal{G}_t$ can be construct by any two filtrations $\mathcal{G}_t^{-i}$ and $\mathcal{G}_t^{-j}$ as long as $i \neq j$.

Let us assume that F_t^i is an absolutely continuous submartingale under $\mathbb{Q}$ for all $i \in I$. Then the $\mathbb{G}^{-i}$-hazard process Γ_t^i of τ^i under $\mathbb{Q}$, defined as $1 - F_t^i = e^{-\Gamma_t^i}$, admits the $\mathbb{G}^{-i}$-adapted intensity $\lambda_i^{\mathbb{Q}}(t)$ of default time τ^i under $\mathbb{Q}$. In this Chapter, we model the non-negative Markovian default intensity processes as

$$d\lambda_i^{\mathbb{Q}}(t) = \mu^{\mathbb{Q}}(\lambda_i(t), t)dt + \sigma^{\mathbb{Q}}(\lambda(t), t)dW_t^i + \sum_{j \in I, j \neq i}^{I} \alpha_{i,j}^{\mathbb{Q}} dH_t^j, \forall j \in I \tag{4.2}$$

where $\mu^{\mathbb{Q}}(\lambda_i(t), t)$ and $\sigma^{\mathbb{Q}}(\lambda_i(t), t)$ being $\mathbb{F}$-adapted Lipschitz real value functions satisfying (2.3) and (2.4). The processes $\lambda_i^{\mathbb{Q}}(t)$ are assumed to be non-negative for all $i \in I$ and are supposed to be the $\mathbb{G}^{-i}$-adapted stochastic intensity process. The stochastic jump $\alpha_{i,j}$ has a marginal probability density function $\eta_{i,j}(\alpha_{i,j})$.

The $\alpha_{i,j}^{\mathbb{Q}}$ represents the default contagion effect on the i^{th} firm, which is caused by the default of the j^{th} firm for all $j \in I$ and $j \neq i$. The summation term in (4.2) indicates that every firm can be affected by the defaults of any of the other firms and every default event will trigger default contagion jumps in all other firms that haven't yet defaulted. Therefore, all firms are interacting with each other.

Similar to previous contagion models, in this new model, a default event triggers a contagion to all alive firms, which is represented by a random upward movement $\alpha_{i,j}^{\mathbb{Q}}$ in their default intensities. It is summation term of default processes H_t^j in (4.2) that makes each firm's default intensity process depend on the default times of all other firms. Therefore, all firms are interdependent. We note here that there are some clear advantages of our default contagion modelling approach. First of all, pricing problems can be solved by solving a system of two simple PDEs. Next, as we will see latter, there are some desirable properties that arise when we model the default intensities as mean-reverting processes. Finally, there is a lot of flexibility when it comes to modelling the random contagion jump size $\alpha^{\mathbb{Q}}$, which we will now discuss in the next section.

4.2.1 Distributions for $\alpha^{\mathbb{Q}}$

A major contribution of this thesis is to model default contagion in a new approach and to propose the corresponding numerical schemes for measuring counterparty risk, which we discuss in later chapters. In previous default contagion models, such as Yu (2007), Bao et al. (2010) and Zheng and Jiang (2009), the default contagion jumps are modelled to be constant because there is no analytical method available for solving if the jumps are random. However, due to the fact that we are able to solve our model numerically, we do not have to place as many restrictions on our way to model contagions and choice of random default contagion jumps.

For modelling the default contagion jump $\alpha^{\mathbb{Q}}_{i,j}$, for all $i \in I$ and $j \neq i$, we always have to ensure that the size of jump is non-negative. Mathematically, this guarantees the default intensity process to be non-negative. Besides, the economic meaning of $\alpha^{\mathbb{Q}}_{i,j}$ is the credit deterioration to firm i due to firm j's default thus $\alpha^{\mathbb{Q}}_{i,j}$ must be non-negative, which means that firm i cannot have a better credit standard after another firms' default. More importantly, the random jump $\alpha^{\mathbb{Q}}_{i,j}$ has to be $\int_{\mathbb{R}_+} \alpha^{\mathbb{Q}}_{i,j}\eta(\alpha^{\mathbb{Q}}_{i,j})d\alpha^{\mathbb{Q}}_{i,j} < \infty$, which means the expected jump size cannot be unbounded. This condition makes sure intensity processes are still bounded after default contagion jumps occurs. Fortunately, there are plenty of well-known distributions satisfy these conditions and are thus available for modelling the default contagion jumps.

We will first discuss univariate distribution used for model each $\alpha^{\mathbb{Q}}_{i,j}$ individually. Then we will discuss the possibility of modelling $\alpha^{\mathbb{Q}}$ with multivariate distribution to introduce correlation among $\alpha^{\mathbb{Q}}_{i,j}$s. In order to avoid notational ambiguity, we remind reader that the μ and σ represent the mean and standard derivation of the distributions discussed in this section. But μ and σ may represent other parameters in the rest of the thesis.

Univariate distributions

The easiest non-negative random variable one may think of is exponential variable. An exponential distribution describes the time to the next event arrival of a homogeneous Poisson process, whose events happen independently at a constant rate $1/\beta$. Exponentially distributed random variables have memoryless property. This means the waiting time for next event arrival conditional on the time past over the last event occurrence is the same as the unconditional distribution. The probability density function (*pdf*) is defined as

$$y(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

So the cumulative distribution function is

$$Y(x) = \begin{cases} 1 - e^{-x/\beta} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

The mean and standard derivation of an exponential variable equal to β. β is also called the scale parameter of the distribution, which controls the shape of distribution. Figure 4.1 gives examples of the density function with a different mean or scale parameter β. Obviously, the probability density function is exponentially monotone decreasing to 0. When x approaches 0 from above, the probability tends to $1/\beta$. We also notice that, the higher the values of β, the higher the probability that the random variable x takes large values.

Defined as $\mathbb{E}\left[\left((X - \text{mean})/\text{standard derivation}\right)^3\right]$, skewness is the measure of the symmetry of the distribution between its mean. A normal distribution has 0 skewness because the normal distribution is symmetric about its mean value. On the other hand, exponential variable has a positive skewness equal to 2. Having a skewness of 2 implies that the exponential distribution is concentrated on the right tail. But we cannot tell from skewness whether a distribution has a long and thin or fat or a short tail in general.

The exceeds kurtosis of exponential variables is equal to 6. Kurtosis is defined as $\mathbb{E}\left[\left((X - \text{mean})/\text{standard derivation}\right)^4\right]$. On the other hand, Exceeds kurtosis is

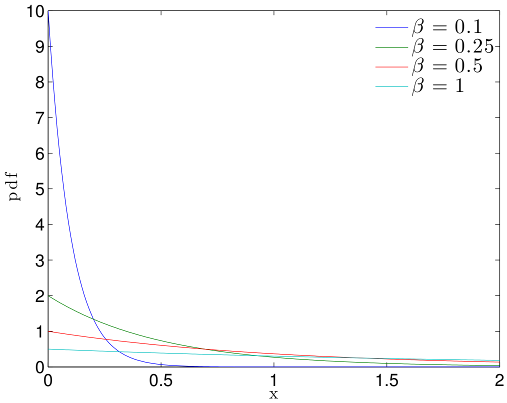

Figure 4.1
Example of exponential distributions

Mean	Standard Derivation	Skewness	Exceed kurtosis	Median	Mode
β	β	2	6	$\beta ln(2)$	0

Table 4.1
Statistics of exponential random variable

defined as kurtosis minus 3, which is the kurtosis of normal distribution. Having exceed kurtosis over 0 means the distribution has a heavier tail and is more likely to produce outliers than a normal distribution. The key statistics are summarised in Table 4.1. It is worth noting that if modelling default contagion shocks $\alpha^{\mathbb{Q}}_{i,j}$ by i.i.d. exponentially distributed jumps, we can only control the jump mean and standard derivation by a single scale parameter β. However, we can not control the jump size skewness and kurtosis.

Another choice for modelling default contagion shocks $\alpha^{\mathbb{Q}}_{i,j}$ is the two-parameter Weibull distribution. Weibull distribution is a continuous probability distribution whose density function is defined as

$$y(x) = \begin{cases} \frac{\gamma}{\beta}\left(\frac{x}{\beta}\right)^{\gamma-1} e^{-\left(\frac{x}{\beta}\right)^{\gamma}} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

So the cumulative distribution function is

$$Y(x) = \begin{cases} 1 - e^{-(x/\beta)^{\gamma}} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

Compared with exponential distribution, Weibull distribution has another parameter γ defined on $(0, +\infty)$, which is called the shape parameter. It is easy to see that

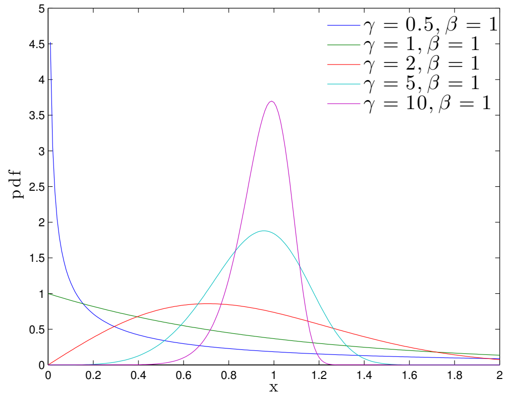

Figure 4.2
Example of Weibull distributions

exponential distribution is a special case of Weibull distribution with $\gamma = 1$. The mean and standard derivation of Weibull distribution are

$$\beta\Gamma(1 + \frac{1}{\gamma})$$

and

$$\beta\sqrt{\Gamma\left(1 + \frac{2}{\gamma}\right) - \Gamma^2\left(1 + \frac{1}{\gamma}\right)}$$

respectively, where $\Gamma()$ is gamma function. With the additional shape parameter, we can control the shape of the density function, see figure 4.2.

When $\gamma < 1$, the density function tends to positive infinity as x approaches 0 from above and the probability is monotone decreasing. When $\gamma = 1$, Weibull distribution is identical to exponential distribution with scale parameter β. Finally, when $\gamma > 1$, Weibull distribution has 0 probability density at $x = 0$ and is increasing until its mode

$$\beta\left(\frac{\gamma - 1}{\gamma}\right)^{1/\gamma}$$

. Especially, when $\gamma \to \infty$, Weibull distribution is concentrating at β.

The advantage of modelling default contagion shocks with Weibull distributions over exponential distribution is being able to control the skewness and exceed kurtosis. The skewness of Weibull distribution in Rinne (2008) is computed as

$$Skewness = \frac{\Gamma_3 - 3\Gamma_1\Gamma_2 + 2\Gamma_1^3}{(\Gamma_2 - \Gamma_1^2)^{3/2}},$$

where $\Gamma_i = \Gamma\left(1 + \frac{i}{\gamma}\right)$. The two-parameter Weibull distribution is positively skewed or right-tailed when γ is less than γ_0, where $\gamma_0 \approx 3.60$. On the other hand, the distribution is negatively skewed or left-tailed when $\gamma > \gamma_0$ and the skewness tends to -1.139 when $\gamma \to \infty$ according to Rinne (2008).

Apart from skewness, the kurtosis can be computed as

$$Kurtosis = \frac{\Gamma_4 - 4\Gamma_3\Gamma_1 + 6\Gamma_2\Gamma_1^2 - 3\Gamma_1^4}{(\Gamma_2 - \Gamma_1^2)^2}.$$

The kurtosis is decreasing until its minimum approximately to 2.71 at $\gamma \approx 3.36$. After that, the kurtosis is increasing to 5.4 with increasing γ. Compared with the normal distribution, the Weibull distribution has kurtosis lower than 3 with $2.25 < \gamma < 5.75$ and greater kurtosis when γ is outside this interval.

Note that the skewness and kurtosis rely only on the shape parameter γ but not β. If we model default contagion jumps $\alpha_{i,j}^{\mathbb{Q}}$ with two-parameter Weibull distribution, we can control the expected strength of default contagion and the probability of having unexpected weak or strong shocks separately. More precisely, we can calibrate β and γ in the Weibull distribution to achieve the desired default contagion jumps' expected size, skewness and kurtosis. As a result of having high Kurtosis, default contagion jumps to alive firms' default intensity are more likely to be unexpectedly strong or weak.

Another possible statistical distribution for the jumps is the Gamma distribution, whose probability density function is defined as

$$y(x) = \begin{cases} \frac{1}{\beta\Gamma(\epsilon)} \left(\frac{x}{\beta}\right)^{\epsilon-1} e^{-\frac{1}{\beta}x} & x \geq 0, \\ 0 & x < 0, \end{cases} \tag{4.3}$$

where $\epsilon \in \mathbb{R}_+$ is known as the shape parameter and $\beta \in \mathbb{R}_+$ the scale parameter of Gamma distributions. The cumulative distribution function is

$$Y(x) = \begin{cases} \frac{\Gamma_L(\epsilon, \frac{x}{\beta})}{\Gamma(\epsilon)} & x \geq 0, \\ 0 & x < 0, \end{cases}$$

where $\Gamma_L(\epsilon, \frac{x}{\beta})$ is the lower incomplete Gamma function defined as

$$\Gamma_L(\epsilon, \frac{x}{\beta}) = \int_0^{\frac{x}{\beta}} t^{\epsilon-1} e^{-t} dt.$$

It is easy to see that a Gamma distribution reduces to an exponential distribution with $\epsilon = 1$. Figure 4.3 gives an example of Gamma distributions with different shape

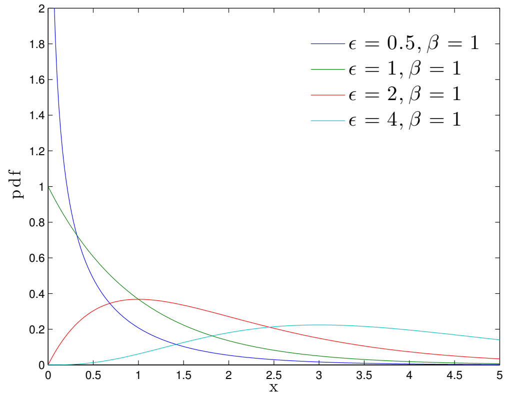

Figure 4.3
Example of gamma distributions

Mean	Standard Derivation	Skewness	Exceed kurtosis
$\beta\epsilon$	$\beta\sqrt{\epsilon}$	$\frac{2}{\sqrt{\epsilon}}$	$\frac{6}{\epsilon}$

Table 4.2
Statistics of gamma random variable

parameters. Table 4.2 summarise the key statistics of Gamma distributions. Gamma distributions have statistics similar to exponential distribution but it is scaled by the shape parameter ϵ, whose intuition is the number of events.

Modelling default contagion shocks $\alpha_{i,j}^{\mathbb{Q}}$ with a Gamma distribution would allow us to control the skewness and kurtosis by shape parameter ϵ. However, the disadvantage of Gamma distribution compared with Weibull is that its skewness cannot be negative. In other words, we cannot model default contagion jumps to be left-tailed since both skewness and exceed kurtosis of Gamma distribution are monotone decreasing with ϵ. The default contagion jumps can only become more even and thin tailed with increasing ϵ. It is less likely to produce jumps far away from its expected value and therefore not likely to see unexpected weak/strong jumps.

An even more flexible version of Gamma distribution is created by adding a third parameter to give the generalized Gamma distribution, whose density function is defined as

$$y(x) = \begin{cases} \frac{1}{(\beta)^{\epsilon\gamma}\Gamma(\epsilon)}\psi\left(\frac{x}{\beta}\right)^{\epsilon\gamma-1} e^{-\left(\frac{1}{\beta}x\right)^{\gamma}} & x \geq 0, \\ 0 & x < 0, \end{cases}$$

where $\gamma \in \mathbb{R}_+$ is the shape parameter in addition to Gamma distribution. The cumulative distribution function is

$$Y(x) = \begin{cases} \frac{\Gamma_L\left(\epsilon,\left(\frac{x}{\beta}\right)^{\gamma}\right)}{\Gamma(\epsilon)} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

The generalised Gamma distribution has a close relationship with other distributions. For example, with $\psi = 1$ and $\epsilon = 1$, a generalised Gamma distribution reduces to an exponential distribution. It also include Weibull distributions with $\epsilon = 1$ and Gamma distributions with $\gamma = 1$. In addition, generalised Gamma distributions can represent half-normal, log-normal and χ^2 distribution, Johnson et al. (1995). The mean and variance of a generalised Gamma distribution are

$$E[x] = \beta \frac{\Gamma(\epsilon + \frac{1}{\gamma})}{\Gamma(\epsilon)}$$
$$Var[x] = \beta^2 \left(\frac{\Gamma(\epsilon + \frac{2}{\gamma})}{\Gamma(\epsilon)} - \left(\frac{\Gamma(\epsilon + \frac{1}{\gamma})}{\Gamma(\epsilon)} \right)^2 \right).$$

Further, Johnson et al. (1995) provide the r^{th} moment can be computed as

$$\beta^r \frac{\Gamma(\epsilon + \frac{r}{\gamma})}{\Gamma(\epsilon)},$$

then its skewness and kurtosis can be computed according to their definitions.

The two parameters γ and ϵ define the shape of the distribution, which at least combines the properties of both Weibull and Gamma distribution, see figure 4.4. Consequently, the generalised Gamma distribution provide more flexibility to model the default contagion jumps.

Figures 4.5 and 4.6 provide more details on the skewness and kurtosis of Weibull, Gamma and generalised Gamma distribution. These figures tell us how the default contagion jumps are distributed if they are modelled by the distributions discussed. Compared with Gamma distributed jumps, generalised Gamma distributed jumps can be more left-skewed and more tailed or right-skewed and less tailed than Gamma distributed jumps. For example, according to the left panel of figure 4.5 and 4.6, the default contagion jumps are more likely to be stronger/weaker than expected than a Gamma distribution when with $\gamma < 1$ and vice versa. In addition, a generalised

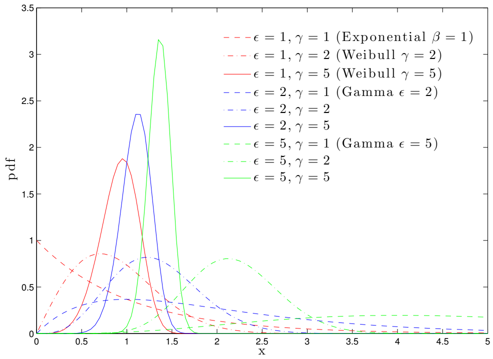

Figure 4.4
Example of generalised gamma distributions

Gamma distribution can be negative skewed, which is not possible for Gamma distributions. With a very large ψ and small ψ, the default contagion jumps are more likely to be strong and less likely to have unexpected jump sizes.

Similarly, a generalised Gamma distributed default contagion jumps can be more left/right skewed and more/less tailed than Weibull distributed jumps. However, the cases are more complicated when compared with Weibull distributed jumps. When γ is smaller than approximately 3.60, both Weibull and generalised Gamma distributed jumps are right-skewed. In this case, if the shape parameter ϵ in the generalised Gamma distribution is less than 1, the jump sizes are more right-skewed. In other words, the probability density is more heavily concentrated on the left. At the same time, the kurtosis of generalised Gamma distributed jumps can also be less or more tailed than Weibull distributed jumps. Because the kurtosis of generalised Gamma distribution with $\epsilon < 1$ can be more or less tailed than Weibull depending on whether the other shape parameter γ exceeds a threshold. The opposite behaviour is observed in the right panel of figures 4.5 and 4.6, where γ is approximately greater than around 8 and less than 14. In this region, generalised Gamma jumps are less tailed with $\epsilon < 1$ but more skewed to the left than Weibull jumps. However, if γ greater than approximately 14, the generalised Gamma distributed jumps become more tailed and left-skewed. Consequently, it is more likely to have unexpected small jump sizes although the probability density concentrates on having strong jumps. Also, this means the impacts from a firm's default to another firm is usually expected to be strong but there can be small impacts occasionally.

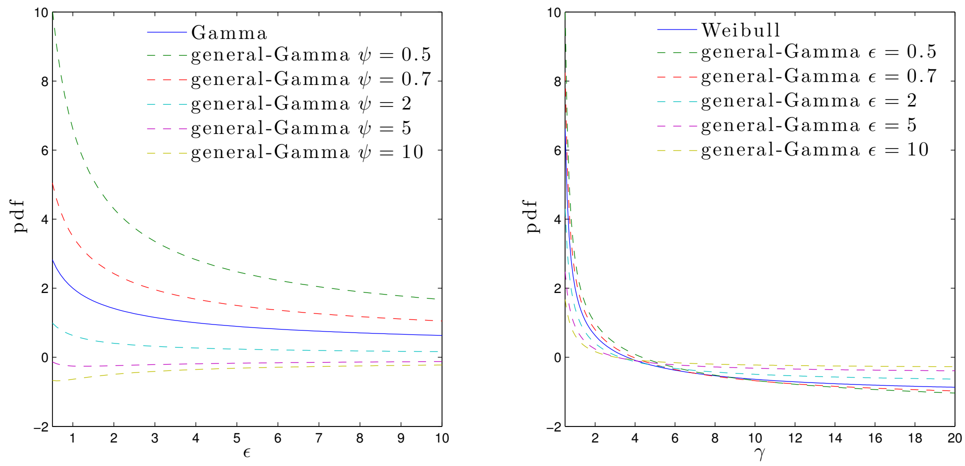

Figure 4.5
Example of skewness of distributions

Gamma, Weibull and generalised Gamma distributions provide flexibility to control the skewness and kurtosis of default contagion jumps' distributions. If we model default intensities (4.2) to be mean-reverting processes, the distribution choice of default contagion jumps $\alpha^{\mathbb{Q}}$ may affect the importance of mean-reverting speed. We will see in the subsequent chapters, where we model default intensities (4.2) as mean-reverting processes, that the capacity of alive firms' recovery, which is modelled as the speed of mean-reverting, is critical to the survival probability of a firm after the impacts from other defaulted firms. The speed of mean-reverting and the strength of default contagion jointly determine how severe alive firms' credit could be damaged by another firm's default. For example, if default contagions have weak impacts, whether a firm can recover fast is not so important compared to the situation where default contagions have strong impacts. Consequently, whether default contagion jump sizes are more likely to be large/small and whether the jump sizes are likely to be unexpectedly weak/strong also influence the importance of fast mean-reverting. For example, when the distribution of default contagion shocks is right-skewed with high kurtosis, the default contagion jumps are more likely to be stronger than expected. Consequently, we should be able to see the survival probability of a firm and the value of its credit derivatives become more sensitive to the capacity of recovery or mean-reverting speed of the firm.

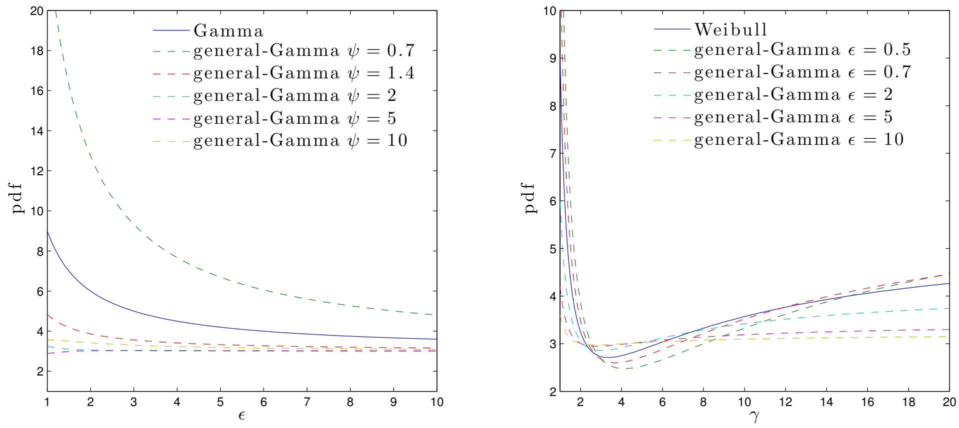

Figure 4.6
Example of kurtosis of distributions

Multivariate distributions

Apart from the unilateral distributions discussed, it is also possible to model jointly dependent default contagion jumps $\alpha^{\mathbb{Q}}_{i,j}$ for all $i, j \in I$ and $i \neq j$ utilising multivariate distributions. The majority of distributions discussed in Section 4.2.1 have their multivariate extensions and could be used for modelling dependent default contagion jumps. Since the main focus of this thesis is the default contagion mechanism and numerical methods, we only give the multivariate Gamma distribution of Mathai and Moschopoulos (1991) as an example. Mathai and Moschopoulos (1991) introduce a multivariate Gamma distribution using univariate Gamma distributions as building blocks. Consider the $d+1$ dimensional random variables g_k, $k = 0, 1, ..., d$, be mutually independent and Gamma distributed with density function (4.3), shape parameter ϵ_k and scale parameter β_k. Multivariate Gamma distribution can be constructed by

$$x_k = \frac{\epsilon_k}{\epsilon_0} g_0 + g_k.$$

The random variable $\mathbf{X} = (x_1, ..., x_d)^T$ is a multivariate Gamma distribution. Mathai and Moschopoulos (1991) show that x_k are Gamma distributions with scale parameter $\beta_0 + \beta_k$ and shape parameter ϵ_k, $k = 1, ..., d$. As a result, the mean and variance are

$$E[x_k] = (\beta_0 + \beta_k)\epsilon_k,$$

$$Var[x_i] = (\beta_0 + \beta_k)\epsilon_k^2.$$

More importantly, the covariance of x_k and x_m for $m \neq k$ is

$$Cov(x_k, x_m) = \beta_0 \epsilon_k \epsilon_m > 0.$$

With the approach of Mathai and Moschopoulos (1991), we can construct positively correlated Gamma distributed default contagion jumps by independent Gamma distributed random variables. Although we could model any collections of $\alpha^{\mathbb{Q}}_{i,j}$ for all i and $j \in I$, $i \neq j$, to be jointly dependent, it is more intuitive to model the joint behaviour of $\alpha^{\mathbb{Q}}_{i,j}$ for all $i \neq j$ with a given j. That is to say, the default contagion impacts on alive firms, due to a particular firm's default, are positively correlated. Apart from the benefits of modelling default contagion jumps as Gamma distributed variables, positively correlated jumps introduce another source of default correlation among the I firms in the economy. Default dependence among firms are due to both the default contagion mechanism introduced by our model and the size of default contagion jumps. For example, when default contagion jumps applied to all alive firms are jointly large, we are more likely to observe multiple defaults compared to independent default contagion jumps.

4.3 A Two-Firm Model

First, we shall study the properties of the new default contagion model in a two firm framework, since it is simple to carry out mathematical analysis. In this section, we show the valuation of survival/default probabilities and credit derivatives under this model. In addition, when the two firms are the referencing firm and the seller of a CDS contract, we are able to measure the unilateral CVA (2.22) under the default contagion model with two firms. Finally, PDEs are derived using Feynman-Kac theorem and solved numerically.

Under the framework introduced in Section 4.2, we further assume that there are two firms, indexed by 1 and 2, in the economy, whose default intensities are,

$$\begin{aligned} d\lambda_1(t) &= \mu_1(\lambda_1(t), t)dt + \sigma_1(\lambda_1(t), t)dW_1(t) + \alpha_{1,2} dH^2_t, \\ d\lambda_2(t) &= \mu_2(\lambda_2(t), t)dt + \sigma_2(\lambda_2(t), t)dW_2(t) + \alpha_{2,1} dH^1_t, \\ d &< W_1(t), W_2(t) >= \rho dt. \end{aligned} \tag{4.4}$$

In order to simplify notations, we drop the superscript $\mathbb{Q}$ in the parameters. Then the default status of two firms implies that we must take account of 4 states in order to capture all possibilities. We denote these states as:

$$\mathcal{A} = \{H_1(t) = 0, H_2(t) = 0\},\ \mathcal{B} = \{H_1(t) = 0, H_2(t) = 1\}$$
$$\mathcal{C} = \{H_1(t) = 1, H_2(t) = 0\},\ \mathcal{D} = \{H_1(t) = 1, H_2(t) = 1\},$$

where $H_i(t) = 1$ indicates that the i^{th} firm has defaulted. Since $\mu_1^{\mathbb{Q}}(\lambda_1(t), t)$, $\mu_2^{\mathbb{Q}}(\lambda_2(t), t)$, $\sigma_1^{\mathbb{Q}}(\lambda_1(t), t)$ and $\sigma_2^{\mathbb{Q}}(\lambda_2(t), t)$ are assumed to be $\mathbb{F}$-adapted processes, both intensity processes $\lambda_1(t)$ and $\lambda_2(t)$ are $\mathcal{F}_t$-measurable in state $\mathcal{A}$. While in state $\mathcal{B}$, $\lambda_1(t)$ will be $\mathcal{G}_t^{-1}$-measurable and $\lambda_2(t)$ does not exist since firm 2 has defaulted. Similarly, $\lambda_2(t)$ is $\mathcal{G}_t^{-2}$-measurable in state $\mathcal{C}$ and both intensities do not exist in state $\mathcal{D}$. Note that, when separating the economy into four states, the default processes $H_1(t)$ and $H_2(t)$ are constants in a specific state, either 1 or 0. So the dynamic of the two default intensities $d\lambda_i(t)$ are only driven by $\mu_i(\lambda_i(t), t)dt$ and $\sigma_i(\lambda_i(t), t)dW_i(t)$ for $i \in \{1, 2\}$. The value change of $H_i(t)$ happens at the time of state change.

Any credit claims can be characterised by three building blocks, which are the present values of the payment conditional on the firm survive to time T, denoted by X_T, the payment conditional on the firm defaults before T, denoted by Z_t and a continuous payment A_t conditional on the firm survive. In order to price a generic credit claim, we assume an $\mathcal{F}_t$-adapted stochastic interest rate $r(t)$ and we introduce the following lemmas.

Lemma 4.3.1. *Consider an bounded $\mathbb{F}$-predictable process of finite variation A_t, $\mathcal{G}_t-$ measurable and $\mathbb{Q}$-integrable random variable X_T and $\mathbb{F}$-predictable process Z_t such that the random variable $Z_{\tau_1} 1_{\{\tau_1 \leq T\}}$ is $\mathbb{Q}$-integrable, which represent the continuous dividend payment conditional on firm 1 survive, the payment conditional on survive up to time T and the recovery payment at time tau_1 conditional on firm 1 defaults at time tau_1 respectively. The present value $u^{\mathcal{A}}$ of any default claims referencing to firm*

1 *under the default contagion model at state* $\mathcal{A}$ *can be evaluated as*

$$
\begin{aligned}
&u^{\mathcal{A}}(\lambda_1(t), \lambda_2(t), t, T) \\
&= \mathbb{E}\left[1_{\{\tau_1 > t, \tau_2 > t\}} 1_{\{\tau_1 > T\}} X_T + 1_{\{t < \tau_1 \le T\}} Z_{\tau_1} + \int_t^T 1_{\{\tau_1 > s\}} dA_s \middle| \mathcal{G}_t \right] \\
&= 1_{\{\tau_1 > t, \tau_2 > t\}} \mathbb{E}\Bigg[e^{-\int_t^T r(u) + \lambda_1(u) + \lambda_2(u) du} X_T + \int_t^T e^{-\int_t^s r(u) + \lambda_1(u) + \lambda_2(u) du} (\lambda_1(s) Z_s ds + dA_s) \\
&+ \int_t^T e^{-\int_t^{s^-} r(u) + \lambda_1(u) + \lambda_2(u) du} \lambda_2(s^-) \int_0^\infty u^{\mathcal{B}}(\lambda_1(s^-) + \alpha_{1,2}, s, T) \eta(\alpha_{1,2}) d\alpha_{1,2} ds^- \Bigg| \mathcal{F}_t \Bigg],
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
&u^{\mathcal{B}}(\lambda_1(t), t, T) \\
=&\mathbb{E}\left[1_{\{\tau_1 > t, \tau_2 \le t\}} 1_{\{\tau_1 > T\}} X_T + 1_{\{t < \tau_1 \le T\}} Z_{\tau_1} + \int_t^T 1_{\{\tau_1 > s\}} dA_s | \mathcal{G}_t \right] \\
=&1_{\{\tau_1 > t, \tau_2 \le t\}} \mathbb{E}\left[e^{-\int_t^T r(u) + \lambda_1(u) du} X_T + \int_t^T e^{-\int_t^s r(u) + \lambda_1(u) du} \lambda_1(s) \left(Z_s ds + dA_s \right) | \mathcal{G}_t^{-1} \right].
\end{aligned}
\tag{4.6}
$$

Equation (4.5) is a combination of Corollary 3, Lemmas B.0.4 and B.0.6 with interest rate $r(t)$ in addition. Equation (4.6) is the combination of Corollary 2, Lemmas B.0.3 and B.0.5 with interest rate as well. More details are provided in Appendix B.

Usually, a default claim referencing to firm 1 exists in state $\mathcal{A}$ and $\mathcal{B}$, where firm 1 has not defaulted yet. So, the valuation of the credit claim in state $\mathcal{A}$ should reflect the possibility of economic state changes to state $\mathcal{B}$ thus the claim's value changes. This is reflected by the last integral term in (4.5) when firm 2 defaults at time s^-, economic state changes to state $\mathcal{B}$ and the claim's value changes to the value corresponding to a higher firms $1's$ default intensity $\lambda_1(s^-) + \alpha_{1,2}$.

Now, it is clear that the cash flows X_T, A_t and Z_τ specified in survival probability, CDS and CVA satisfy the conditions (2.6) and (2.7) as discussed in Section 2.3. Therefore, Feynman-Kac theorem mentioned in Section 2.1 will be applicable to (4.5) and (4.6) with additional conditions on $u^{\mathcal{A}}(\lambda_1(t), \lambda_2(t), t, T)$ and $u^{\mathcal{B}}(\lambda_1(t), t, T)$. In the following sections, we will apply Feynman-Kac theorem to derive the PDEs for survival probability, CDS and CVA under the contagion model.

Survival probability

In default correlation modelling, it is important to investigate both the marginal as well as the joint default probability. Studying these probabilities can give us an insight into how the default model affects a firm's default probability as well as the degree to which default correlation is created. We now outline how to use our model to calculate these probabilities.

Consider that we are in the state $\mathcal{B}$ after firm 2 defaulted and denote $S^{\mathcal{B}}(\lambda_1(t), t, T)$ as the survival probability of firm 1 in the state $\mathcal{B}$. Firm 1's survival probability equals (4.6) with $X_T = 1$, $A_t = 0$, $Z_{\tau_1} = 0$ and $r(t) = 0$, which is

$$S^{\mathcal{B}}(\lambda_1(t), t, T) = \mathbb{E}\left[e^{-\int_t^T \lambda_1(s)ds} \middle| \mathcal{G}_t^{-1} \right]. \tag{4.7}$$

Assume $S^{\mathcal{B}}(\lambda_1(t), t, T)$ to be $\mathcal{C}^{1,2}$ with respect to time t and $\lambda_1(t)$, apply Feynman-Kac theorem to $S^{\mathcal{B}}(\lambda_1(t), t, T)$ we will have

$$\frac{\partial S^{\mathcal{B}}}{\partial t} + \mu_1(\lambda_1(t), t)\frac{\partial S^{\mathcal{B}}}{\partial \lambda_1} + \frac{1}{2}\sigma_1^2(\lambda_1(t), t)\frac{\partial^2 S^{\mathcal{B}}}{\partial \lambda_1^2} - \lambda_1 S^{\mathcal{B}} = 0, \tag{4.8}$$

subject to the terminal condition $S^{\mathcal{B}}(\lambda_1(T), t, T) = 1$.

Now assume that we are in state $\mathcal{A}$, then firm 1's survival probability will be affected by the possibility of firm 2's default event. The survival probability of firm 1 at state $\mathcal{A}$, denoted as $S^{\mathcal{A}}(\lambda_1(t), t, T)$, is equation (4.5) with $X_T = 1$, $A_t = 0$, $Z_{\tau_1} = 0$ and $r(t) = 0$, which is

$$\begin{aligned} S^{\mathcal{A}}(\lambda_1(t), \lambda_2(t), t, T) = \mathbb{E}\Bigg[& e^{-\int_t^T \lambda_1(u)+\lambda_2(u)du} \\ + \int_t^T & e^{-\int_t^{s^-} \lambda_1(u)+\lambda_2(u)du} \lambda_2(s^-) \left(\int_0^\infty S^{\mathcal{B}}(\lambda_1(s^-) + \alpha_{1,2}, s, T)\eta(\alpha_{1,2})d\alpha_{1,2} \right) ds^- \Bigg| \mathcal{F}_t \Bigg]. \end{aligned} \tag{4.9}$$

According to Feynman-Kac theorem, assumed $S^{\mathcal{A}}(\lambda_1(t), \lambda_2(t), t, T)$ satisfies $\mathcal{C}^{1,2,2}$ with respect to t, $\lambda_1(t)$ and $\lambda_2(t)$, equation (4.9) satisfies the differential equation

$$\begin{aligned} & \frac{\partial S^{\mathcal{A}}}{\partial t^-} + \mu_1(\lambda_1, t)\frac{\partial S^{\mathcal{A}}}{\partial \lambda_1} + \mu_2(\lambda_2, t)\frac{\partial S^{\mathcal{A}}}{\partial \lambda_2} \\ & + \frac{1}{2}\sigma_1^2(\lambda_1, t)\frac{\partial^2 S^{\mathcal{A}}}{\partial \lambda_1^2} + \frac{1}{2}\sigma_2^2(\lambda_2, t)\frac{\partial^2 S^{\mathcal{A}}}{\partial \lambda_2^2} + \rho\sigma_1(\lambda_1, t)\sigma_1(\lambda_2, t)\frac{\partial^2 S^{\mathcal{A}}}{\lambda_1\lambda_2} \\ & + \lambda_2\bigg(\int_0^\infty S^{\mathcal{B}}(\lambda_1 + \alpha_{1,2}, t, T)\eta(\alpha_{1,2})d\alpha_{1,2} \bigg) - (\lambda_1 + \lambda_2)S^{\mathcal{A}} = 0, \end{aligned} \tag{4.10}$$

subject to terminal condition $S^{\mathcal{A}}(\lambda_1(T), \lambda_2(t), t, T) = 1$.

Credit default swaps

Denoting $V^{\mathcal{B}}(\lambda_1(t), t, T)$ as the CDS value in state $\mathcal{B}$ with swap settlement dates $t = T_0$, T_1, ..., T_n, ..., $T_N = T$ for swap rate S. The CDS's value can be expressed as (4.6) with multiple payments $X_n = S(T_n - T_{n-1})$ at time T_n and $Z_t = (1-R)e^{-r(T-t)} - S\frac{t-T_{k-1}}{T_k-T_{k-1}}$ with $T_{k-1} \le t < T_k$ and $A_t = 0$, which is

$$\begin{aligned} &V^{\mathcal{B}}(\lambda_1(t), t, T) \\ =&\mathbb{E}\Bigg[\int_t^T e^{-\int_t^s r(u)+\lambda_1(u)+\lambda_s(u)du}\lambda_1(s)\left((1-R_1)\,e^{-r(T-s)} - \frac{s-T_{n-1}}{T_n-T_{n-1}}S\right)ds \\ &-\sum_{n=1}^{N} e^{-\int_t^{T_n} r+\lambda_1(s)ds}S(T_n-T_{n-1})\Bigg|\mathcal{G}_t^{-1}\Bigg]. \end{aligned} \tag{4.11}$$

However, $V^{\mathcal{B}}(\lambda_1(t), t, T)$ is not differentiable with respect to time at $t = T_n$ for $n = 1, ..., N$ then Feynman-Kac must not apply directly to (4.11) everywhere at $[t, T]$. Assuming $V^{\mathcal{B}}(\lambda_1(t), t, T)$ satisfies smooth condition $\mathcal{C}^{1,2}$ between (T_{n-1}, T_n), then according to Feynman-Kac theorem, the CDS value $V^{\mathcal{B}}(\lambda_1(t), t, T)$ with swap premium S satisfies the PDE

$$\begin{aligned} &\frac{\partial V^{\mathcal{B}}}{\partial t} + \mu_1(\lambda_1, t)\frac{\partial V^{\mathcal{B}}}{\partial \lambda_1} + \frac{1}{2}\sigma_1^2(\lambda_1, t)\frac{\partial^2 V^{\mathcal{B}}}{\partial \lambda_1^2} \\ &+\lambda_1\left((1-R)e^{-r(T-t)} - S\frac{t-T_{n-1}}{T_n-T_{n-1}}\right) - (r+\lambda_1)V^{\mathcal{B}} = 0, \end{aligned} \tag{4.12}$$

for $t \in (T_{n-1}, T_n)$ and $n = 1, 2, ...N$. Further, (4.12) subjects to terminal condition $V^{\mathcal{B}}(\lambda_1(T), T, T) = 0$ and jump conditions

$$V^{\mathcal{B}}(T_n^-, T) = V^{\mathcal{B}}(T_n, T) - S(T_n - T_{n-1}) \text{ for } n = 1, 2, ...N,$$

at each payment date.

While in state $\mathcal{A}$, the CDS has the same cash-flows but now the only difference lies in the fact that firm 2's default will cause firm 1's default intensity to jump to a higher value, which corresponds to higher CDS value and the brings the CDS value from state $\mathcal{A}$ to $\mathcal{B}$. Denoting the CDS value in this state $\mathcal{A}$ as $V^{\mathcal{A}}(\lambda_1(t), \lambda_2(t), t, T)$, it satisfies (4.9) with $X_n = S(T_n - T_{n-1})$ at time T_n and $Z_t = (1-R)e^{-r(T-t)} - S\frac{t-T_{k-1}}{T_k-T_{k-1}}$

with $T_{k-1} \leq t < T_k$ and $A_t = 0$, which is

$$\begin{aligned}
&V^{\mathcal{A}}(\lambda_1(t), \lambda_2(t), t, T) \\
=&\mathbb{E}\Bigg[\int_t^T e^{-\int_t^{s^-} r(u)+\lambda_1(u)+\lambda_2(u)du}\Bigg(\lambda_1(s^-)\left((1-R_1)e^{-r(T-s^-)} - S\frac{s^- - T_{n-1}}{T_n - T_{n-1}}\right) \\
&+\lambda_2(s^-)\left(\int_0^\infty V^{\mathcal{B}}(\lambda_1(s^-)+\alpha_{1,2}, s, T)\eta(\alpha_{1,2})d\alpha_{1,2}\right)\Bigg)ds^- \\
&-\sum_{n=1}^N e^{-\int_t^{T_n} r+\lambda_1(s)+\lambda_2(s)ds} S(T_n - T_{n-1})\Bigg|\mathcal{F}_t\Bigg].
\end{aligned} \tag{4.13}$$

Similarly to $V^{\mathcal{B}}(\lambda_1(t), t, T)$, we assume $V^{\mathcal{A}}(\lambda_1(t), \lambda_2(t), t, T)$ satisfies $\mathcal{C}^{1,2,2}$ with respect to t, $\lambda_1(t)$ and $\lambda_2(t)$ between (T_{n-1}, T_n) for $n = 1, ..., N$, then according to Feynman-Kac theorem, we have the following PDE

$$\begin{aligned}
&\frac{\partial V^{\mathcal{A}}}{\partial t^-} + \mu_1(\lambda_1, t)\frac{\partial V^{\mathcal{A}}}{\partial \lambda_1} + \mu_2(\lambda_2, t)\frac{\partial V^{\mathcal{A}}}{\partial \lambda_2} + \frac{1}{2}\sigma_1^2(\lambda_1, t)\frac{\partial^2 V^{\mathcal{A}}}{\partial \lambda_1^2} + \frac{1}{2}\sigma_2^2(\lambda_2, t)\frac{\partial^2 V^{\mathcal{A}}}{\partial \lambda_2^2} \\
&+ \rho\sigma_1(\lambda_1, t)\sigma_2(\lambda_2, t)\frac{\partial^2 V^{\mathcal{A}}}{\lambda_1\lambda_2} + \lambda_1\left((1-R)e^{-r(T-t)} - S\frac{t - T_{n-1}}{T_n - T_{n-1}}\right) \\
&+ \lambda_2\left(\int_0^\infty V^{\mathcal{B}}(\lambda_1 + \alpha, t, T)\eta(\alpha_{1,2})d\alpha_{1,2}\right) - (r + \lambda_1 + \lambda_2)V^{\mathcal{A}} = 0
\end{aligned} \tag{4.14}$$

for $t \in (T_{n-1}, T_n)$ with $n = 1, 2, ...N$ and it is subject to the same terminal condition $V^{\mathcal{A}}(\lambda_1(T), \lambda_2(T), T, T) = 0$ and jump conditions as before, namely

$$V^{\mathcal{A}}(T_n, T) = V^{\mathcal{A}}(T_n, T) - S(T_n - T_{n-1}) \text{ for } n = 1, 2, ...N. \tag{4.15}$$

Credit value adjustment

In Section 2.3.4, we have shown that CVA can be derived as (2.73). However, in our default contagion model, we also consider the situation in which the counterparty and the referencing firm are correlated. The default of the counterparty will now instantly raise the default intensity of the referencing firm $\lambda_1(\tau_2)$ at time τ_2. Then the CVA, denoted as $U^{\mathcal{A}}(\lambda_1(t), \lambda_2(t), t, T)$, becomes

$$\begin{aligned}
U^{\mathcal{A}}(\lambda_1(t), \lambda_2(t), t, T) = \mathbb{E}\Bigg[\int_t^T & e^{-\int_t^{s^-} r+\lambda_1(u)+\lambda_2(u)du}(1-R_2)\lambda_2(s^-)\times \\
&\left(\int_0^\infty \max\{V^{\mathcal{B}}(\lambda_1(s^-)+\alpha_{1,2}, s, T), 0\}\eta(\alpha_{1,2})d\alpha_{1,2}\right) ds^-\Bigg|\mathcal{F}_t\Bigg],
\end{aligned} \tag{4.16}$$

where $V^{\mathcal{B}}(\lambda_1(s^-) + \alpha_{1,2}, s, T)$ is the CDS's value at time s after the counterparty defaults at time s^- and satisfies (4.12). Equation (4.16) is also a special case of (4.5) with $X_T = 0$, $A_t = 0$ and $Z_t = 0$ but with

$$u^{\mathcal{B}}(\lambda_1(s^-) + \alpha_{1,2}, s, T) = \max\{V^{\mathcal{B}}(\lambda_1(s^-) + \alpha_{1,2}, s, T), 0\}, \tag{4.17}$$

which is the a CDS buyer's loss given the counterparty default.

Assume $U^{\mathcal{A}}(\lambda_1(t), \lambda_2(t), t, T)$ satisfies $\mathcal{C}^{1,2,2}$ with respect to t, $\lambda_1(t)$ and $\lambda_2(t)$. Applying Feynman-Kac theorem to (4.16) leads to PDE

$$\begin{aligned}
&\frac{\partial U^{\mathcal{A}}}{\partial t} + \mu_1(\lambda_1(t), t)\frac{\partial U^{\mathcal{A}}}{\partial \lambda_1} + \mu_2(\lambda_2(t), t)\frac{\partial U^{\mathcal{A}}}{\partial \lambda_2} \\
&+ \frac{1}{2}\sigma_1^2(\lambda_1(t), t)\frac{\partial^2 U^{\mathcal{A}}}{\partial \lambda_1^2} + \frac{1}{2}\sigma_2^2(\lambda_2(t), t)\frac{\partial^2 U^{\mathcal{A}}}{\partial \lambda_2^2} + \rho\sigma_1(\lambda_1(t), t)\sigma_2(\lambda_2(t), t)\frac{\partial^2 U^{\mathcal{A}}}{\lambda_1 \lambda_2} \\
&+ \lambda_2\left((1 - R_2)\int_0^\infty \max\{V^{\mathcal{B}}(\lambda_1(t) + \alpha_{1,2}, t, T), 0\}\eta(\alpha_{1,2})d\alpha_{1,2}\right) \\
&- (r + \lambda_1 + \lambda_2)U^{\mathcal{A}} = 0,
\end{aligned} \tag{4.18}$$

subject to the terminal condition $U^{\mathcal{A}}(\lambda_1(T), \lambda_2(T), T, T) = 0$.

4.4 Numerical Results

In order that we can calculate the survival probabilities, price CDS contracts and measure the CVA, we will need to solve the PDEs (4.10), (4.14) and (4.18). In order to implement the model, we first specify the functions $\mu_i(\lambda_i(t), t)$ and $\sigma_i(\lambda_i(t), t)$ for $i = 1, 2$. We will use CIR processes for $\mu_i(\lambda_i(t), t)$ and $\sigma_i(\lambda_i(t), t)$ which is

$$\begin{aligned}
\mu(\lambda_i(t), t) &= \kappa_i(\theta_i - \lambda_i(t)) \\
\sigma(\lambda(t), t) &= \sigma_i\sqrt{\lambda_i(t)}.
\end{aligned} \tag{4.19}$$

Because CIR process can ensure default intensities to be non-negative and satisfy conditions (2.3) and (2.4), which is essential for Feynman-Kac theorem to be applicable.

Further, we choose default contagion jump sizes $\alpha_{i,j}$ $\forall i \neq j$ and to be $\mathbb{F}$-adapted independent exponential variables whose density function is defined as,

$$\eta(\alpha_{i,j}) = \frac{1}{\bar{\alpha}_{i,j}} e^{-\frac{1}{\bar{\alpha}_{i,j}}\alpha_{i,j}},$$

where $\bar{\alpha}_{i,j}$ is the mean value. We choose exponential random variables for the jump sizes because these guarantee positive jumps and easy to implement so that we can focus on studying the default contagion effects. However, there are variety distributions applicable for modelling default contagion jumps as discussed in Section 4.2.1.

When the j^{th} firm defaults, this will trigger a jump in the default intensity of the i^{th} firm which we can write as $\lambda_i(t) = \lambda_i(t^-) + \alpha_{i,j}$. If one firm's default intensity process is characterised by a large value of the parameter κ, which is the speed that the intensity level reverts back to its long-term mean θ, then the intensity variable is able to recover faster after a default contagion event. The time horizon over which a credit event stays relevant is therefore dependent on the mean reversion parameter.

The integral terms in the equations (4.10), (4.14) and (4.18) have limits over zero to infinity. To implement this alongside a finite-difference scheme, these have to be truncated over a finite domain and discretised accordingly. For instance, if we assume the PDE in state $\mathcal{B}$ is truncated to $[0, \lambda^{\mathcal{B}}_{1,max}]$ and the PDE in state $\mathcal{A}$ is truncated to $[0, \lambda^{\mathcal{A}}_{1,max}] \times [0, \lambda^{\mathcal{A}}_{2,max}]$, then the integral term of equation (4.10) is discretised as

$$\begin{aligned}
&\int_0^\infty V^{\mathcal{B}}(\lambda(t) + \alpha_{1,2}, t, T)\eta(\alpha_{1,2})d\alpha_{1,2} \\
\approx &\sum_{k=0}^{K} \omega_k V^{\mathcal{B}}(i\Delta\lambda_1 + k\Delta\lambda_1, t, T)\eta(k\Delta\lambda_1)\Delta\lambda_1, \text{ for } \forall i \in [0, I]
\end{aligned}$$

where $\omega_0 = \omega_K = \frac{1}{2}$ and 1 otherwise, and $K = \frac{\lambda^{\mathcal{B}}_{1,max} - \lambda_1(t)}{\Delta\lambda_1}$. In this way we can discretise the integral term to match the discretisation of λ_1 in the normal finite difference scheme for a PDE as described in Section 4.3.

In Chapter 3, we solve the two PDEs in system (3.20) with the same truncation in both λ-dimensions or $\lambda^{\mathcal{A}}_{1,max} = \lambda^{\mathcal{B}}_{1,max} = \lambda^{\mathcal{A}}_{2,max}$. However, this decision makes it difficult as we will have to estimate the value for large λ_1 in state $\mathcal{A}$ because there are insufficient available values in state $\mathcal{B}$ to approximate the post-contagion values. Consider evaluating the integral term at the point $(\lambda^{\mathcal{A}}_1(t^-), \lambda^{\mathcal{A}}_2(t^-))$, then the integral term is

$$\begin{aligned}
&\lambda^{\mathcal{A}}_2(t^-) \int_0^\infty u^{\mathcal{B}}(\lambda^{\mathcal{A}}_1(t^-) + \alpha_{1,2}, t, T)\eta(\alpha_{1,2})d\alpha_{1,2} \\
=&\lambda^{\mathcal{A}}_2(t^-)\Bigg(\underbrace{\int_0^{\lambda^{\mathcal{B}}_{1,max} - \lambda^{\mathcal{A}}_1(t^-)} u^{\mathcal{B}}(\lambda_A(t^-) + \alpha_{1,2}, t, T)\eta(\alpha_{1,2})d\alpha_{1,2}}_{\text{post-jump values available}} \\
&+ \underbrace{\int_{\lambda^{\mathcal{B}}_{1,max} - \lambda^{\mathcal{A}}_1(t^-)}^\infty u^{\mathcal{B}}(\lambda^{\mathcal{A}}_1(t^-) + \alpha_{1,2}, t, T)\eta(\alpha_{1,2})d\alpha_{1,2}}_{\text{post-jump values not available}} \Bigg) \\
=&\lambda^{\mathcal{A}}_2(t^-) \int_0^{\lambda^{\mathcal{B}}_{1,max} - \lambda^{\mathcal{A}}_1(t^-)} u^{\mathcal{B}}(\lambda_A(t^-) + \alpha, t, T)\eta(\alpha_{1,2})d\alpha_{1,2} + \text{error},
\end{aligned}$$

where $u^{\mathcal{A}}$ denotes the quantity we are solving for. We can see that the error here comes from when the post-jump intensity, $\lambda_1^{\mathcal{B}}(t) = \lambda_1^{\mathcal{A}}(t^-) + \alpha_{1,2}$, exceeds the upper bound $\lambda_{1,max}^{\mathcal{B}}$, taking us into a region where solutions are not available for the computation of the integral. As a result the largest error will occur when the pre-contagion intensity is at its highest $\lambda_1^{\mathcal{A}}(t^-) = \lambda_{1,max}^{\mathcal{A}}$, where the number of the available solutions in state $\mathcal{B}$ is at its least. So we can say that the error is increasing with the pre-jump firm 1's intensity $\lambda_1^{\mathcal{A}}(t^-)$.

In order to limit numerical errors, we require the probability that the contagion jump size is greater than $\lambda_{1,max}^{\mathcal{B}} - \lambda_{1,max}^{\mathcal{A}}$ to be less than some small number ϵ. Consequently, we impose the following condition

$$\begin{aligned} \text{Highest Error} &= \int_{\lambda_{1,max}^{\mathcal{B}}-\lambda_{1,max}^{\mathcal{A}}}^{\infty} u^{\mathcal{B}}(\lambda_{1,max}^{\mathcal{A}} + \alpha_{1,2}, t)\eta(\alpha_{1,2})d\alpha_{1,2} \\ &< \int_{\lambda_{1,max}^{\mathcal{B}}-\lambda_{1,max}^{\mathcal{A}}}^{\infty} \eta(\alpha_{1,2})d\alpha_{1,2} < \epsilon, \end{aligned} \tag{4.20}$$

for some small enough ϵ. The first inequality comes from the fact that all of our survival probabilities, CDS value and CVA solutions are less than 1 given that the CDS notional is 1. For all of the numerical solutions that we present in this and the later chapters of the thesis, ϵ will be set to be 10^{-5}, which means the probability that the post-contagion default intensity exceeds the limit will be less than 10^{-5}. Since the distribution function of an exponential random variable is known, we can compute the range of $\lambda_{1,max}^{\mathcal{B}}$ given $\lambda_{1,max}^{\mathcal{A}}$ as

$$\begin{aligned} &\int_{\lambda_{1,max}^{\mathcal{B}}-\lambda_{1,max}^{\mathcal{A}}}^{\infty} \eta(\alpha_{1,2})d\alpha_{1,2} < \epsilon \\ &e^{-\frac{1}{\bar{\alpha}_{1,2}}(\lambda_{1,max}^{\mathcal{B}}-\lambda_{1,max}^{\mathcal{A}})} < \epsilon \\ &\lambda_{1,max}^{\mathcal{B}} - \lambda_{1,max}^{\mathcal{A}} > -\bar{\alpha}_{1,2} ln(\epsilon) \\ &\lambda_{1,max}^{\mathcal{B}} > \lambda_{1,max}^{\mathcal{A}} - \bar{\alpha}_{1,2} ln(\epsilon). \end{aligned} \tag{4.21}$$

Finally, we note that it is the best to impose the following condition in order to avoid unnecessary interpolation

$$\frac{\lambda_{1,max}^{\mathcal{B}}}{I^{\mathcal{B}}} = \frac{\lambda_{1,max}^{\mathcal{A}}}{I^{\mathcal{A}}} = \Delta\lambda_1, \tag{4.22}$$

where $\lambda_{1,max}^{\mathcal{B}}$ is set to be the smallest number satisfies both (4.21) and (4.22). Imposing the condition 4.22 is to make sure that the grid sizes in the finite difference scheme in state $\mathcal{A}$ and $\mathcal{B}$ are the same, which makes it convenient to compute the integral terms in the PDE in state $\mathcal{A}$ using the solutions from the PDE in state $\mathcal{B}$.

4.4.1 Survival probability

In this section, a range of results to show how the survival probabilities of a firm is effected by different strength of contagion. The insights gained here will help the reader to understand the results of more complex contracts we price later on in the chapter.

In order to solve (4.8) and (4.10) to find the survival probability of firm 1, we have to specify the behaviour at the boundaries where $\lambda_1(t)$ and $\lambda_2(t)$ tends to either 0 or infinity. While in state $\mathcal{B}$, $\lambda_1(t) = 0$ is substituted into (4.8) and the reduced PDE is solved at the lower boundary.

At the upper boundary where $\lambda_1(t) \to \infty$ or $\lambda_2(t) \to \infty$, we refer to the analysis of the heuristic Robin conditions in Section 3.3.2. A series of grid checks were performed in Section 3.3.2 to convince ourselves that the second derivative is of secondary importance in the calculations meaning that we can simply remove the second derivative terms from our original PDEs leading to the heuristic Robin conditions. In addition, according to Windcliff et al. (2004) and the analysis in Section 3.3.2, the solutions of our PDEs are less sensitive to the choice of upper boundary because the processes are with large negative drift when they go to infinity, which is always true when using (4.19). Therefore, the same argument is applied to derive the upper boundary conditions for the PDEs of survival probability, CDS value and CVA.

As a result, the reduced PDEs (4.23) are satisfied at the lower and upper boundary.

$$\begin{aligned}
&\frac{\partial S^{\mathcal{B}}}{\partial t} + \kappa_1\theta_1\frac{\partial S^{\mathcal{B}}}{\partial \lambda_1} = 0 \quad \text{for} \quad \lambda_1(t) \to 0 \\
&\frac{\partial S^{\mathcal{B}}}{\partial t} + \kappa_1(\theta_1 - \lambda_1)\frac{\partial S^{\mathcal{B}}}{\partial \lambda_1} - \lambda_1 S^{\mathcal{B}} = 0 \quad \text{for} \quad \lambda_1(t) \to \infty.
\end{aligned} \tag{4.23}$$

While in state $\mathcal{A}$, the same argument is applied to the boundaries of λ_1 and λ_2. The only difference is the reduced PDE at boundaries has an extra term as the value changes if firm 2 defaults.

$$
\begin{aligned}
\lambda_1 \to 0 \ : & \\
& \left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_2} + \kappa_1\theta_1\frac{\partial}{\partial \lambda_1}\right) S^{\mathcal{A}} - \lambda_2 S^{\mathcal{A}} \\
& + \lambda_2\left(\int_0^\infty S^{\mathcal{B}}(\lambda_1 + \alpha_{1,2}, t, T)\eta(\alpha_{1,2})d\alpha_{1,2}\right) = 0 \\
\lambda_1 \to \infty : & \\
& \left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_2} + \kappa_1\theta_1\frac{\partial}{\partial \lambda_1}\right) S^{\mathcal{A}} - (\lambda_1 + \lambda_2) S^{\mathcal{A}} \\
& + \lambda_2\left(\int_0^\infty S^{\mathcal{B}}(\lambda_1 + \alpha_{1,2}, t, T)\eta(\alpha_{1,2})d\alpha_{1,2}\right) = 0 \qquad (4.24) \\
\lambda_2 \to 0 : & \\
& \left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_1} + \kappa_2\theta_2\frac{\partial}{\partial \lambda_2}\right) S^{\mathcal{A}} - \lambda_1 S^{\mathcal{A}} = 0, \\
\lambda_2 \to \infty : & \\
& \left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_1} + \kappa_2\theta_2\frac{\partial}{\partial \lambda_2}\right) S^{\mathcal{A}} - (\lambda_1 + \lambda_2) S^{\mathcal{A}} \\
& + \lambda_2\left(\int_0^\infty S^{\mathcal{B}}(\lambda_1 + \alpha_{1,2}, t, T)\eta(\alpha_{1,2})d\alpha_{1,2}\right) = 0.
\end{aligned}
$$

Our model of contagion, now we call the *temporary contagion model* is compared alongside two benchmark models, which we choose to be a model with no contagion effects at all, which we call the *independent default model*, and the model of Zheng and Jiang (2009), which we call *permanent contagion model*. Note that the *independent default model* is essentially the CIR-intensity model used in Chapter 3. Therefore, we can compare the numerical results produced by the default contagion model with the previous chapter. To show the difference between the temporary and permanent contagion models, we have coded up the model of Zheng and Jiang (2009) with $a_i(t)$ to be a CIR process $\lambda_i(t)$ and $b_i(t)$ as a constant equal to $\bar{\alpha}_{1,2}$.

Under previous contagion model, an additional part is added to firm 1's intensity after firm 2 defaults, which is $\lambda_2^{\mathcal{B}}(t) = \lambda_1^{\mathcal{A}}(t) + \bar{\alpha}_{1,2}$. Firm 1's default intensity remains $\bar{\alpha}_{1,2}$ higher than its normal level for the rest of time. However our approach allows the firm's intensity to follow its own fixed dynamics, and it therefore allows the intensity to revert back after the default contagion event. In order to demonstrate this effect figures 4.7 and 4.8 show the survival probabilities with different values of mean-reverting speed for each of the three models. We see that the survival probability of firm 1 behaves

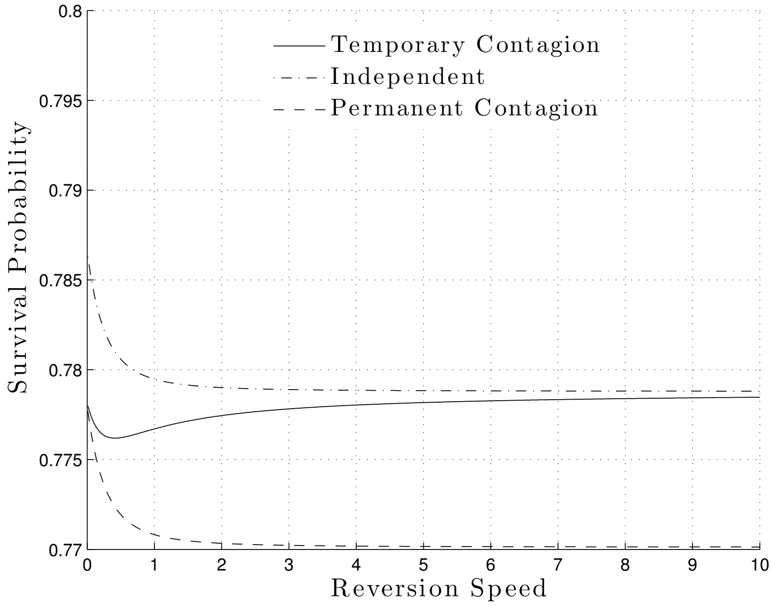

Figure 4.7
Firm 1 Survival Probability under three models and $\bar{\alpha}_{1,2} = 0.02$
An illustration of 5 years survival probability under independent default model, permanent contagion and temporary contagion. Parameters are $\theta_1 = \theta_2 = 0.05$, $\sigma_1 = \sigma_2 = 0.1$ and $\lambda_1(0) = \lambda_2(0) = 0.05$.

differently in different models with κ_1. For the independent default model, survival probability decreases with faster mean-reverting speed κ_1, and this general trend is also seen in the permanent contagion model. However, in our model, the size of default contagion $\bar{\alpha}_{1,2}$ leads to different behaviour for the survival probability with κ_1. When the default contagion effect is strong, a faster mean reversion speed can significantly raise survival probability because it makes the firm's default intensity return back to normal faster. When the default contagion has only temporary impacts on its default intensity, firm 1's surviving probability lies in between the other two models, depending on the reversion speed. Our model tends to the permanent contagion model when reversion speed tends to zero and to the independent one when mean reversion tends to infinity. Intuitively, one's reversion speed represents its ability to recover from default contagion shocks, so a faster mean reversion forces the intensity process to drop quicker after a sudden jump leading to a smaller compensator or higher survival probability equivalently.

In a temporary contagion model, the mean-reverting speed has two effects on a firm's survival probability. Firstly, faster mean reversion will reduce the total variance

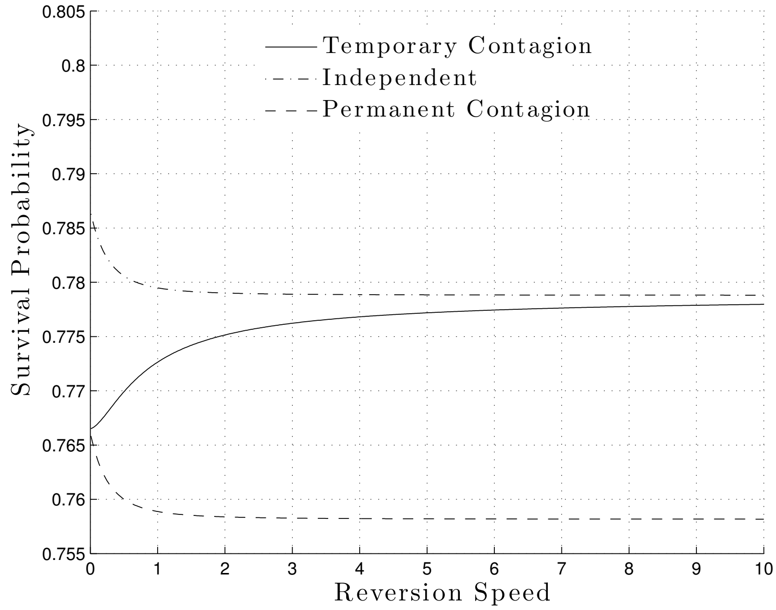

Figure 4.8
Firm 1 Survival Probability under three models and $\bar{\alpha}_{1,2} = 0.05$
An illustration of 5 years survival probability under independent default model, permanent contagion and temporary contagion. Parameters are $\theta_1 = \theta_2 = 0.05$, $\sigma_1 = \sigma_2 = 0.1$ and $\lambda_1(0) = \lambda_2(0) = 0.05$.

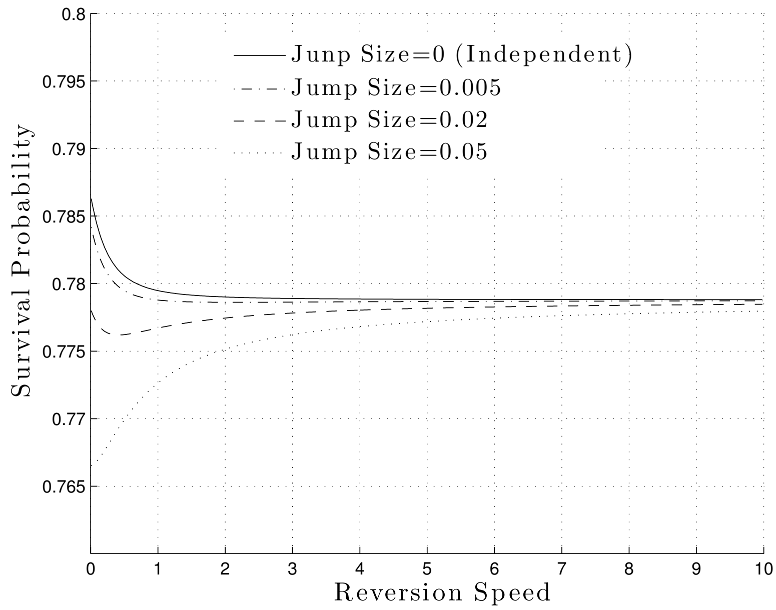

Figure 4.9
Firm 1 Survival Probability with different mean reversion speed κ_1 **and multiple** $\bar{\alpha}_{1,2}$

of the process and thus have the effect on lowering survival probability. Secondly, the default intensity can recover from default contagions quicker so survival probability is higher. Therefore, the strength of default contagion makes survival probability behave differently with reversion speed, and we see this in figure 4.9. When the size of default contagion is relatively small, say $\bar{\alpha}_{1,2} = 0.005$, the variance reduction effect dominates and the survival probability is a decreasing function of reversion speed similar to the other two models. On the other hand, if the jump size is higher, the curve drops at the beginning then increases. When default contagion strength is strong, for instance $\bar{\alpha}_{1,2} = 0.05$, the latter effect dominates and survival probability is higher with faster mean reversion. Intuitively, if the default events of other firms have significant impacts on the surviving firms, their capacities to recover from default contagions are crucially important to their survival in the future.

In default correlation modelling, the joint default probability, $\mathbb{Q}(\tau_1 < T, \tau_2 < T)$, indicates how the model creates correlated defaults. We can calculate the joint default probability according to the relationship

$$\mathbb{Q}(\tau_1 < T, \tau_2 < T) = 1 - \big(\mathbb{Q}(\tau_1 > T) + \mathbb{Q}(\tau_2 > T) - \mathbb{Q}(\tau_1 > T, \tau_2 > T)\big), \qquad (4.25)$$

where $\mathbb{Q}(\tau_1 > T)$ is solved numerically and $\mathbb{Q}(\tau_1 > T, \tau_2 > T)$ has an analytic solution, which is the product of individual's survival probability conditional on that the other firm is still alive.

We compare the joint default probability of two firms under our model and independent default model of Chapter 3 in figure 4.10. As expected, there will be higher joint default probability with slow mean-reverting speed since the alive firm's default intensity remains high after the other defaults. Intuitively, a firm that is weaker at recovering from bad events is more susceptible to go bankrupt due to another's default.

In addition, the effect of increasing the default contagion strength $\bar{\alpha}_{1,2}$ and $\bar{\alpha}_{2,1}$ on joint default probability is shown in figure 4.11. The joint default probability increases rapidly with stronger default contagion. Again, if two firms have slower mean-reverting speed, the joint default probability increases faster with default contagion. We define the joint default probability *premium* to be the joint default probability with default correlation models minus the value from the independent default model.

In figure 4.13, we present the marginal default probability premium, which is firm 1's marginal default probability with our *temporary contagion model* minus the default

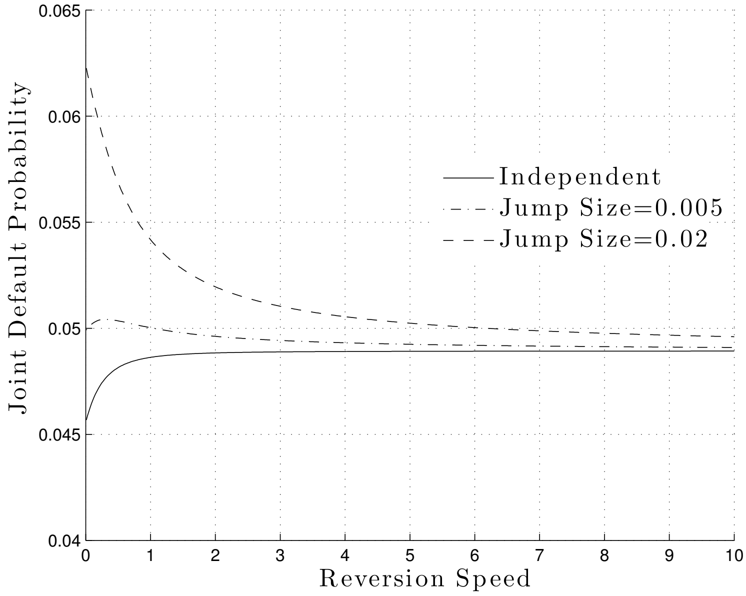

Figure 4.10
Joint Default Probability against reversion speed κ
An illustration of 5 years Joint Default Probability with Contagion Jump Size $\bar{\alpha}_{1,2}$ of 0, 0.005, 0.02. Parameters are $\theta_1 = \theta_2 = 0.05$, $\sigma_1 = \sigma_2 = 0.1$ and $\lambda_1(0) = \lambda_2(0) = 0.05$.

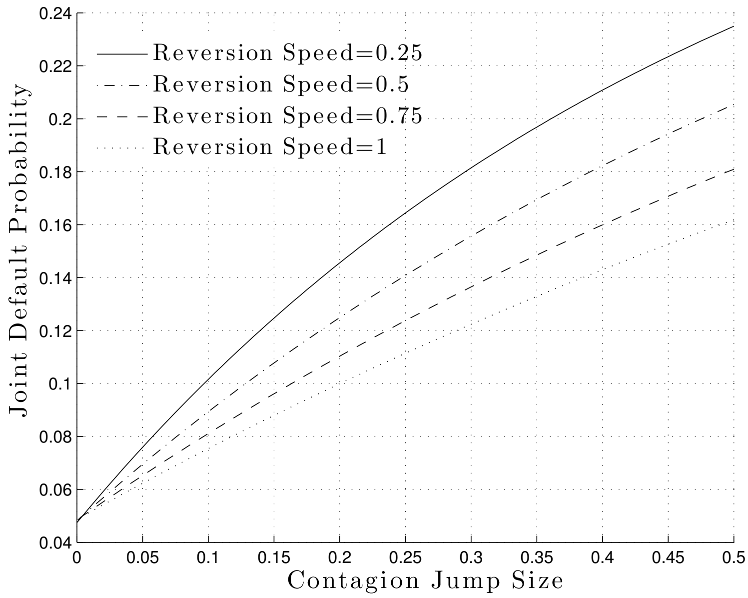

Figure 4.11
Joint Default Probability against Default Contagion Strength $\bar{\alpha}_{1,2}$.
Parameters are $\theta_1 = \theta_2 = 0.05$, $\sigma_1 = \sigma_2 = 0.1$ and $\lambda_1(0) = \lambda_2(0) = 0.05$.

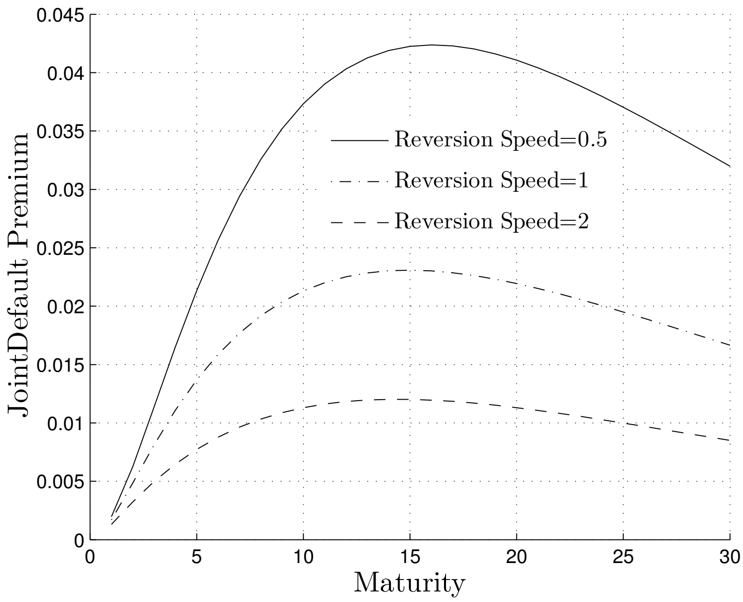

Figure 4.12
Joint Default Probability Premium against Maturity
The premium of joint default probability with default contagion over independent default. Parameters are $\theta_1 = \theta_2 = 0.05$, $\sigma_1 = \sigma_2 = 0.1$, $\lambda_1(0) = \lambda_2(0) = 0.05$, $\kappa_1 = \kappa_2 = 0.5$ and $\bar{\alpha}_{1,2} = \bar{\alpha}_{2,1} = 0.02$.

probability with *independent default model.* The mean-reverting speeds κ_1 and κ_2 are set to be 0.5, 1 and 2 and we see that, even with the same contagion jump size of 0.05, there can be large differences in firm 1's default probabilities premium. The increment to default probabilities with slow mean-reverting speed $\kappa_1 = 0.5$ are around three times in absolute term as much as a faster reversion of $\kappa_1 = 2$. Also, we see that the reversion speed determines the time-horizon to which the contagion has the highest impact. If two firms both have a mean-reverting speed of $\kappa_1 = \kappa_2 = 0.5$, the default probability premium takes its highest value at around 17 years time. In the case of $\kappa_1 = \kappa_2 = 2$, the peak value takes at 14 years time. The reason is that the mean-reverting speed κ_1 determines firm 1's capacity to recover from an abnormal level to its mean value θ_1. Therefore, with a smaller κ_1 implies a weaker ability to recover from the default contagion due to firm 2's default. So firm 1's default intensity will stays at a high level for longer time. Consequently, we see the default contagion will have its strongest impact on the longer-term default probability with small κ_1. Then with large κ_1, which means strong recovery capacity, the default contagion has its strongest impact on a shorter-term default probability.

The *premium* is shown in figure 4.14 with maturities up to 30 years. Over a short time horizon, there is little difference between the two models because the probability that one firm defaults in one year is very limited. Even there is a default in one year, it is not likely the other one also defaults due to contagion in such a short time. However, over a very long time horizon, both firms are likely to default no matter what the value of correlation, therefore the premium of both models drop to zero with maturity T. Figure 4.14 also compare *temporary contagion model* and *permanent contagion model*. Obviously, the *permanent contagion model* of Zheng and Jiang (2009) has a stronger and longer-term impact on the default probability compared to our default contagion model. The increment in default probability is higher with *permanent contagion model* because the default contagion in previous models will not vanish but we allow the affected firm to recover from contagions. Because we allow the alive firm to recover from contagions, the time horizon to which a default contagion still have influence on a firm's default probability can be very different. In the permanent default contagion model, the impact starts to decrease after 25 years. With our model, the impact starts to vanish at around 15 years due to the fact that the firm is recovering, which is what our model tries to capture. In other words, our model has short-term impact compared with the permanent contagion model.

To conclude, we are able to obtain accurate numerical solutions of survival probabilities under our proposed default contagion model by solving the corresponding PDE. Given hypothetical parameters, we found that the *temporary contagion model* gives rise to a different behaviour of the survival probability against mean-reverting speed not seen in the other models. At first, *temporary contagion model* tends to the same values as the *independent default model* when $\kappa \to 0$ or alternatively the *permanent contagion model* when $\kappa \to \infty$. Secondly, the default probability could have a different behaviour against reversion speed. The reason lies in the fact that the reversion speed κ can reduce the intensity variance whilst at the same time increase the ability to recover from contagion. When we calculate the joint default probabilities, we see that the two effects from increasing κ that were observed in survival probability also appear in the joint default probability. We observed that a firm's ability to recover is crucially important when determining joint default probability. With just 0.02 default contagion, the increment in joint default probability with $\kappa_1 = \kappa_2 = 0.5$ could be more than three times compared to the one with $\kappa_2 = \kappa_2 = 2$. In terms of the

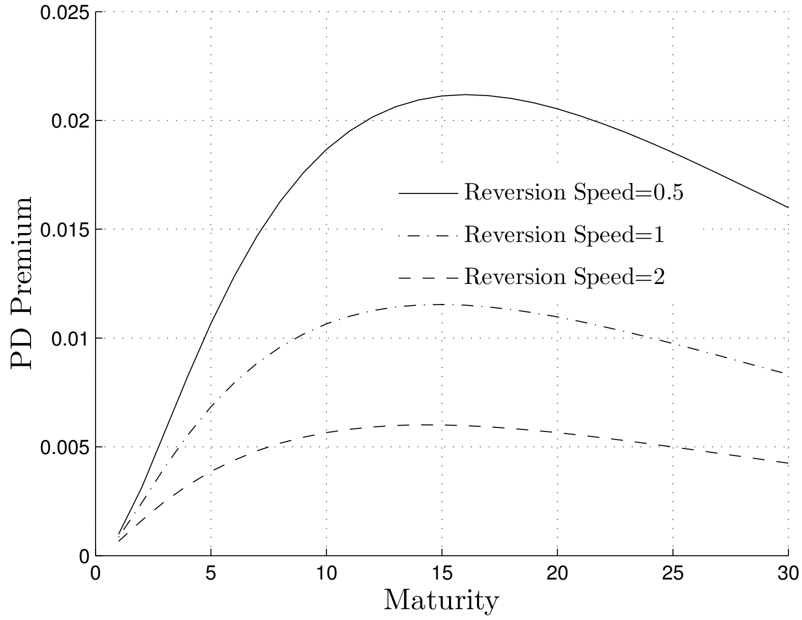

Figure 4.13
Increment default probability with default contagion $\bar{\alpha}_{1,2}$
The increment to marginal default probability if a temporary contagion size of 0.05. The results compared with different value of mean reversion speed.

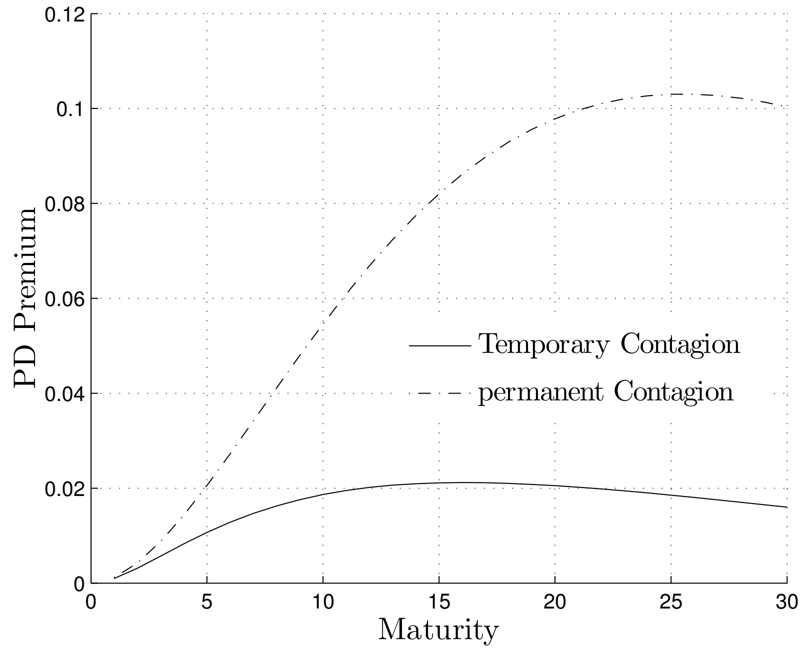

Figure 4.14
Marginal Default Probability Premium against Maturity
The increases in marginal default probability with default contagion models over the independent default model. Parameters are $\theta_1 = \theta_2 = 0.05$, $\sigma_1 = \sigma_2 = 0.1$, $\lambda_1(0) = \lambda_2(0) = 0.05$, $\kappa_1 = \kappa_2 = 0.5$ and $\bar{\alpha}_{1,2} = \bar{\alpha}_{2,1} = 0.02$.

term-structure of marginal default probabilities, reversion speed has two impacts on the term-structure. The marginal default probability premium, which is additional to default probability without default contagion, are influenced by reversion speed in terms of the maximum value and the time horizon to which it reaches the maximum.

4.4.2 CDS spreads

In this section, we will compare the CDS spreads under the three different models and we will further demonstrate how the mean-reverting speed and the default contagions influence the CDS premium. We showed in Section 4.3 that we can evaluate a CDS's fair spread by solving PDEs (4.12) and (4.14) simultaneously. The CDS spread or the swap rate is one of the input parameters for the PDE solver. According to (2.58), CDS spread is the ratio of default value and the premium value. Intuitively, default value is the CDS value when the protection buyer pays 0 swap rate, which is $S = 0$, and the premium value is the CDS value with recovery $R = 1$. Therefore, we can solve PDEs (4.12) and (4.14) with appropriate input parameters to evaluate the default and the premium values separately then taking their ratio to obtain the fair swap rate S^* according to (2.58).

Numerically solving (4.12) and (4.14) requires boundary conditions for both PDEs. For the one-dimensional PDE (4.12), we apply the heuristic Robin boundary conditions (3.15), which we have already shown to be the superior over other types of conditions. Then for the two-dimensional PDE (4.14), the heuristic Robin conditions are used, which we refer our analysis to Section 3.3.2. The resulting PDEs are as follow

$$
\begin{aligned}
&\lambda_1 \to 0\ : \\
&\qquad \left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_2} + \kappa_1\theta_1\frac{\partial}{\partial \lambda_1}\right)V^{\mathcal{A}} + \lambda_2\left(\int_0^\infty V^{\mathcal{B}}(\lambda_1+\alpha_{1,2},t,T)\eta(\alpha_{1,2})d\alpha_{1,2}\right) \\
&\qquad - (r+\lambda_2)V^{\mathcal{A}} = 0, \\
&\lambda_1 \to \infty : \\
&\qquad \left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_2} + \kappa_1\theta_1\frac{\partial}{\partial \lambda_1}\right)V^{\mathcal{A}} + \lambda_2\left(\int_0^\infty V^{\mathcal{B}}(\lambda_1+\alpha_{1,2},t,T)\eta(\alpha_{1,2})d\alpha_{1,2}\right) \\
&\qquad + \lambda_1\left((1-R)e^{-r(T-t)} - S\frac{t-T_{n-1}}{T_n-T_{n-1}}\right) - (r+\lambda_2+\lambda_1)V^{\mathcal{A}} = 0, \\
&\lambda_2 \to 0 : \\
&\qquad \left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_1} + \kappa_2\theta_2\frac{\partial}{\partial \lambda_2}\right)V^{\mathcal{A}} + \lambda_1\left((1-R)e^{-r(T-t)} - S\frac{t-T_{n-1}}{T_n-T_{n-1}}\right) \\
&\qquad - (r+\lambda_1)V^{\mathcal{A}} = 0, \\
&\lambda_2 \to \infty : \\
&\qquad \left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_1} + \kappa_2\theta_2\frac{\partial}{\partial \lambda_2}\right)V^{\mathcal{A}} + \lambda_2\left(\int_0^\infty V^{\mathcal{B}}(\lambda_1+\alpha_{1,2},t,T)\eta(\alpha_{1,2})d\alpha_{1,2}\right) \\
&\qquad + \lambda_1\left((1-R)e^{-r(T-t)} - S\frac{t-T_{n-1}}{T_n-T_{n-1}}\right) - (r+\lambda_2+\lambda_1)V^{\mathcal{A}} = 0.
\end{aligned} \tag{4.26}
$$

Figure 4.15 shows the swap rates of CDS contracts up to 30 years maturity. Since the amount of the protection buyer has to pay reflects the referencing firms' default probabilities, it has a similar behaviour to that we saw for the default probability. Compared with the permanent contagion model, the rates are similar for a one year CDS but the gap is widening with longer maturities due to our model allowing for recovery from default contagion.

Taking the difference between swap rates using our *temporary contagion model* and the *independent defaults model*, we can define the *swap rate premium* due to default contagion, which is the extra fee due to introducing default contagion into the model. Figures 4.17 and 4.16 show the extra fees that a protection buyer should pay for firm 1's protection due to the amount that its default probability may be raised by the possibility of contagion from firm 2's default.

Given the results that were observed for firm 1's default probabilities, we should expect that stronger default contagion means the protection buyer has to pay more.

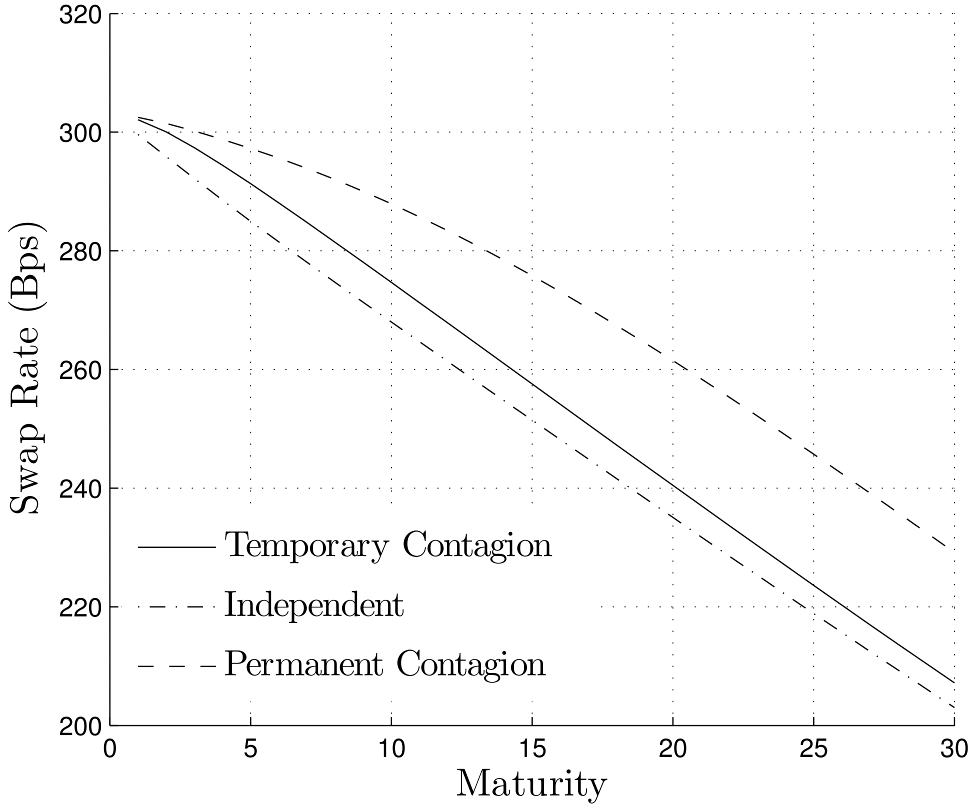

Figure 4.15
Fair CDS swap rates of maturity up to 30 years
CDS premium Fair Swap Rates with our default contagion model, previous default contagion model and independent. Default contagion size $\bar{\alpha}_{1,2}$ is 0.02, settlement period $T_i - T_{i-1} = 1$, recovery $R = 0.4$, interest rate $r = 0.02$.

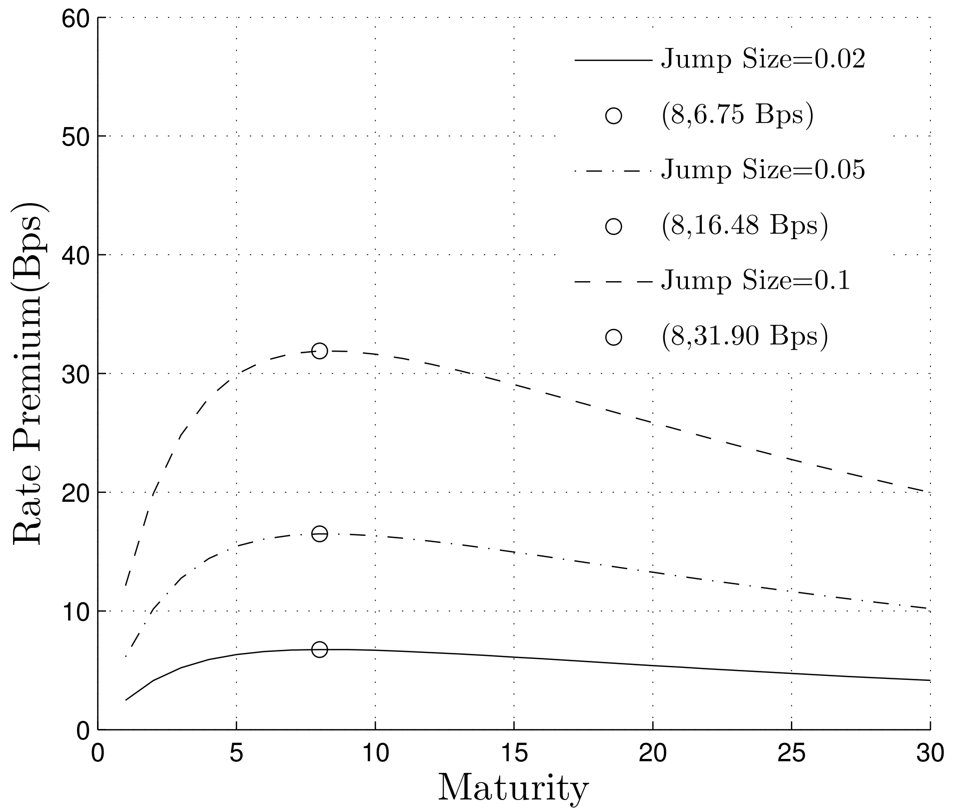

Figure 4.16
The swap rate premium against the size of default contagion
The swap rate is the increases in fair swap rate due to using temporary contagion model over independent model. With the speed of reversion $\kappa_1 = 0.5$ and different strengths of default contagion $\bar{\alpha}_{1,2}$.

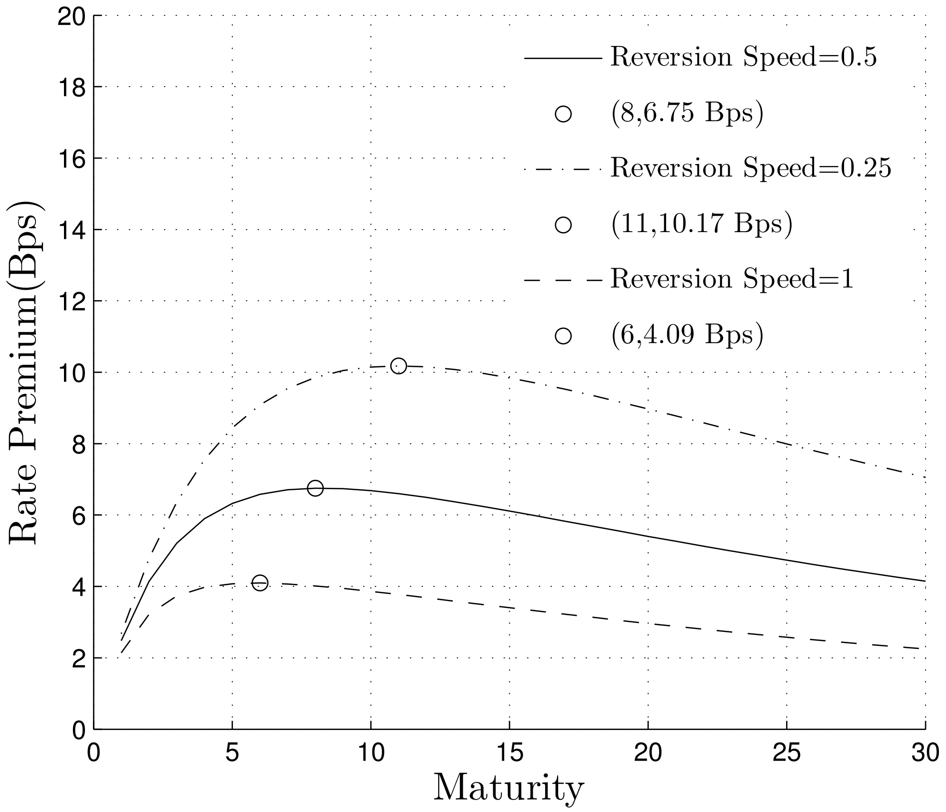

Figure 4.17
The swap rate premium against mean-reverting speed
The swap rate is the increases in fair swap rate due to using temporary contagion model over independent model. With the strength of default contagion 0.02 and different speed of mean-reverting.

We see again that the premiums for one and thirty years protections are less than those for five to ten years. If we only raise the default contagion jump size $\bar{\alpha}_{1,2}$, we notice that the maturity where the default contagion has the highest impact will not change. For instance, in figure 4.16, the eight-year protection has the highest premium compared to all other maturities. On the other hand, if we slow down firm 1's and firm 2's mean-reverting speeds κ_1 and κ_2, as we do in figure 4.17, then the premium will be higher and the highest premium happens to longer maturity contracts. For example, with $\kappa_1 = \kappa_2 = 0.25$, the 11 years protection has the highest increment, which is 10.17 *bps*. When $\kappa_1 = \kappa_2 = 1$, the price increment is the highest at 6 years protection. This can be explained by noticing that firm 2 becomes safer in state $\mathcal{A}$ so the probability it defaults is lower. Consequently, default contagion to firm 1 is more likely to occur later rather than earlier. In addition, lower κ_1 also make firm 1 less likely to recover thus higher default probability, which implies higher premium for the protection.

4.4.3 Credit value adjustments

In this section, we let firms 1 and 2 be the referencing firm and the CDS protection seller. We investigate the default contagion impact on the CVA defined when buying a CDS contract from a counterparty whose default leads to a default contagion jump in the referencing firm's default intensity.

First of all, we specify the boundary conditions for solving (4.18). Note that the boundary conditions for solving (4.12), which determines the loss given the counterparty defaults, has been addressed in Section 4.4.2. Also, we demonstrated in Section 3.4.1 that the heuristic Robin boundary conditions (3.25) can be used for solving PDEs to CVA. So we modify the heuristic Robin boundary conditions (3.25) to account for the default contagion effects for solving (4.18), which are

$$
\begin{aligned}
\lambda_1 \to 0 \ :& \\
&\left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_2} + \kappa_1\theta_1\frac{\partial}{\partial \lambda_1}\right)U - (r+\lambda_2)U \\
&+ \lambda_2\bigg((1-R_2)\int_0^\infty \max\{V^{\mathcal{B}}(\lambda_1(t)+\alpha_{1,2},t,T),0\}\eta(\alpha_{1,2})d\alpha_{1,2}\bigg) = 0, \\
\lambda_1 \to \infty :& \\
&\left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_2} + \kappa_1\theta_1\frac{\partial}{\partial \lambda_1}\right)U - (r+\lambda_1+\lambda_2)U \\
&+ \lambda_2\bigg((1-R_2)\int_0^\infty \max\{V^{\mathcal{B}}(\lambda_1(t)+\alpha_{1,2},t,T),0\}\eta(\alpha_{1,2})d\alpha_{1,2}\bigg) = 0, \\
\lambda_2 \to 0 :& \\
&\left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_1} + \kappa_2\theta_2\frac{\partial}{\partial \lambda_2}\right)U - (r+\lambda_1)U = 0, \\
\lambda_2 \to \infty :& \\
&\left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_1} + \kappa_2\theta_2\frac{\partial}{\partial \lambda_2}\right)U - (r+\lambda_1+\lambda_2)U \\
&+ \lambda_2\bigg((1-R_2)\int_0^\infty \max\{V^{\mathcal{B}}(\lambda_1(t)+\alpha_{1,2},t,T),0\}\eta(\alpha_{1,2})d\alpha_{1,2}\bigg) = 0.
\end{aligned}
\tag{4.27}
$$

The lower boundaries are simply substituting $\lambda_1 = 0$ or $\lambda_2 = 0$ into the PDE. The reason for choosing these upper boundaries to be similar as we did for CDS and Survival probabilities is that the analysis in Chapter 3 suggests the upper boundary conditions have negligible effects on the solution when using mean-reverting type process and the heuristic Robin boundary conditions have the best performance over other boundary conditions.

We note that the probability that the counterparty defaults earlier than the referencing firm does not change after introducing the default contagion. Because both firms' default intensity are in no way different as compared to the independent default model or permanent default model before a default event happens. A default contagion jump only directly impacts on the distribution of the CDS's value *after* contagion and therefore the losses at default. Before the default contagion occurs, a referencing firm's default intensity follows a non-central χ^2 distribution since we are using CIR processes. But after the counterparty defaults, a default contagion jump, which is

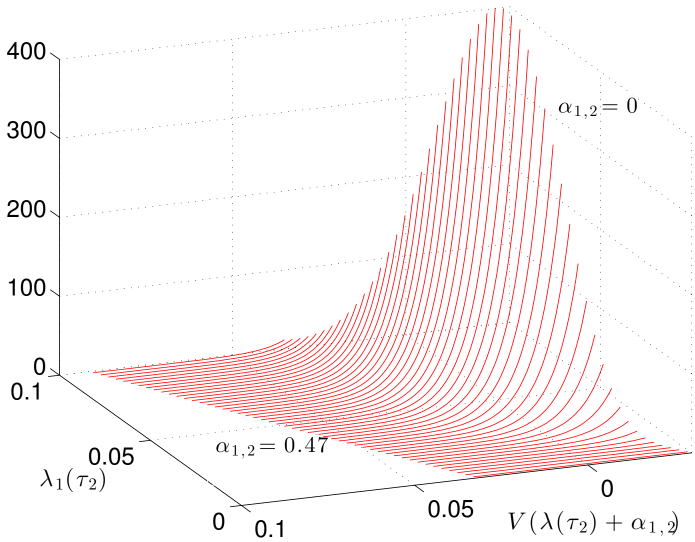

Figure 4.18
Post-contagion CDS value distribution
The one year CDS's value distribution after default contagion. Assumed the counterparty defaults at $\tau_2 = 0.75$. The default contagion size $\alpha_{1,2}$ ranges from 0 to 0.47. Parameters are $T = 1$, $T_i - T_{i-1} = 1$, $S = 0.03$.

exponentially distributed with mean $\bar{\alpha}_{1,2}$, is added to the referencing firm's default intensity $\lambda_1(\tau_2) = \lambda_1(\tau_2^-) + \alpha_{1,2}$. Consequently, the distribution of the referencing firm's default intensity $\lambda_1(\tau_2)$ is changed by default contagion and therefore so is the value distribution of the CDS. We try to give insight to this by showing the post-contagion value distribution $V^{\mathcal{B}}(\lambda_1(\tau_2))$ of a one year CDS contract given that the counterparty has defaulted at time $\tau_2 = 0.75$, in figure 4.18.

At each $\lambda_1 \in [0, \lambda_{1,max}]$, we must compute the post-contagion CDS value as the loss given default, if any, which is the integral term in PDE (4.18). Figure 4.19 compares the CDS value distributions of a one year standard CDS contract with and without default contagion, conditional that the counterparty defaults at $t = 0.002$, 0.25, 0.5 and 0.75 in the future. In the case of $\mu(\alpha_{1,2}) = 0$, it means default contagion does not exist thus this is the independent default model we discussed in previous two section and in Chapter 3. The CDS value is a distribution depending on pre-contagion intensity position $\lambda_1(\tau_2^-)$. The black bar on each plot divides these distributions into two areas, where on the right-hand-side the value of the CDS is positive and there is a loss to the protection buyer if the counterparty defaults. The role of a default contagion jump is to create a rightward shift to the CDS value distribution at time

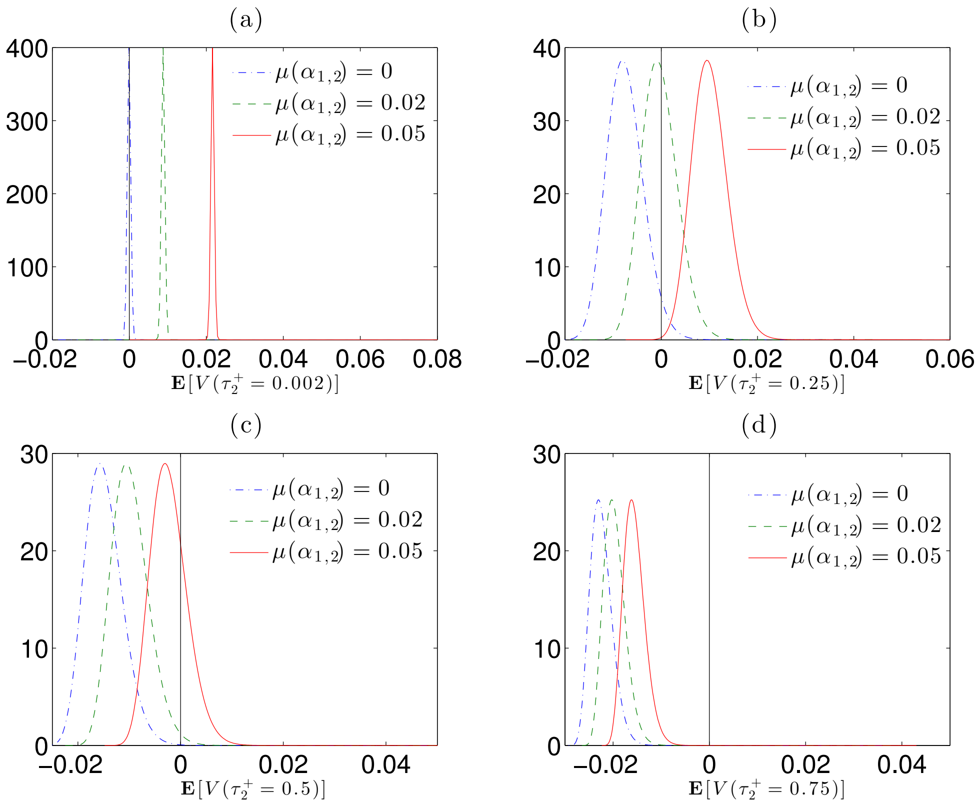

Figure 4.19
Expected post-contagion CDS value distribution
An illustration of a one year CDS value distribution assumed the counterparty defaults at $t = 0.002$, 0.25, 0.5 and 0.75. Default contagion strength are $\bar{\alpha}_{1,2} = 0.$, $\bar{\alpha}_{1,2} = 0.02$ and $\bar{\alpha}_{1,2} = 0.05$.

τ_2^-. Consequently, when compared to the independent default model, there is a higher probability that the CDS buyer will suffer a loss from a counterparty default and the losses are also likely to be higher. For instance, if the counterparty's default time is 0.5 in panel (c), the CDS values are so low without contagion (blue dot-dash line) that the CDS buyer will not suffer from a counterparty default loss in any but the very worst case situations. However, with default contagion $\bar{\alpha}_{1,2} = 0.05$, there is a sufficiently large probability that the buyer loses out and the CVA will be higher as a consequence.

Maturity	$\bar{\alpha}_{a,2} = 0$	$\bar{\alpha}_{a,2} = 0.025$		$\bar{\alpha}_{a,2} = 0.05$		$\bar{\alpha}_{a,2} = 0.1$	
Computational Time (Sec)	CVA (bps)	CVA (bps)	CVA change (%)	CVA (bps)	CVA change (%)	CVA (bps)	CVA change (%)
1	0.09	1.02	1009%	2.58	2708%	5.82	6240%
	(34)	(61)		(85)		(153)	
2	0.72	4.14	477%	9.23	1187%	19.24	2584%
	(68)	(122)		(171)		(305)	
3	1.82	8.24	353%	17.46	860%	35.22	1836%
	(68)	(183)		(256)		(458)	
4	3.16	12.63	299%	26.02	723%	51.40	1525%
	(136)	(250)		(341)		(608)	
5	4.60	17.00	270%	34.29	645%	66.73	1351%
	(171)	(305)		(427)		(759)	
6	6.06	21.19	250%	42.02	594%	80.78	1234%
	(203)	(364)		(512)		(907)	
7	7.50	25.16	236%	49.15	556%	93.43	1146%
	(234)	(419)		(611)		(1061)	
8	8.91	28.91	224%	55.65	524%	104.71	1074%
	(268)	(478)		(712)		(1235)	
9	10.32	32.46	215%	61.58	497%	114.69	1012%
	(304)	(539)		(787)		(1393)	
10	11.71	35.84	206%	66.98	472%	123.50	954%
	(336)	(598)		(869)		(1478)	

Table 4.3
CVA under default contagions, percentage changes and computational times
The CDS are traded with the fair spread without default contagions. Parameters are $\theta_1 = \theta_2 = 0.05$, $\sigma_1 = \sigma_2 = 0.1$, $\lambda_1(0) = \lambda_2(0) = 0.05$, $\kappa_1 = \kappa_2 = 0.5$, interest rate =0.02, recovery rate=0.4 . Numerical parameters: $\Delta t = 0.001$, $\Delta\lambda_1 = \Delta\lambda_2 = 0.005$.

Table 4.3 shows the CVA movements with default the contagion $\bar{\alpha}_{1,2}$ of the CDS contracts up to 10 year maturities and the corresponding computational time. We find the CVA is very sensitive to default contagions and it is especially so for short-term contracts at least if we look at the relative effects. However, the short-term contracts are originally subject to very low CVA due to low counterparty default probability. Although the relative increment in CVA is extremely high for short-term contracts, the increases in absolute terms are very small. The long-term contracts are subject to significantly higher counterparty default risk. For example, in a 10-year contract, the CVA is 35.84 *bps* and it rises to 66.98 *bps* and 123.5 *bps* as the default contagion strength increases from 0.025 to 0.1, which means it is almost doubled every time. According to table 4.3, the amount of CVA that the investor will charge the CDS seller due to default contagion is in general increasing with longer maturity contracts. This is because counterparty is more likely to default when the time horizon is longer thus the default contagion is more likely to influence CVA. In addition, we see the computational time is increasing with the default contagion strength $\bar{\alpha}_{1,2}$. Because the conditions, which we discussed in Section 4.4, require the number of grid points for $\lambda_2^{\mathcal{B}}$ to be sufficiently large in order to minimise the computational errors so the computational time increases accordingly.

Introducing a default contagion from counterparty to the referencing firm will always increase the CVA charged to the CDS seller. However, the amount the CVA changes due to default contagion varies significantly depending on, again, the relative riskiness of the two firms. Figure 4.20 shows the difference in CVA resulting from the inclusion of default contagion. We construct figure 4.20 by subtracting the CVA with default contagion $\bar{\alpha}_{1,2} = 0.05$ to the CVA without default contagion. The amount of the CVA changes is usually increasing with the counterparty's default intensity λ_2, which can be seen in part (a). However, if the referencing firm is very safe and the counterparty is risky, the CVA change due to default contagion could be small, because when the referencing firm is very safe, the CDS's value is so deeply out of the money that whether counterparty's default will deteriorate the referencing firm's credit does not change the CDS value from negative to positive therefore there may be no counterparty default loss. In the situation that the referencing firm's default risk is lower than normal level, default contagion has greater impact when the counterparty is more risky than the referencing firm but the counterparty is not so extremely risky that the counterparty's default is going to happen instantly, which we can see from the red curve in part (b). Because we model the intensities as mean-reverting processes, we know that the firm's intensity will return to its mean value. Therefore, having a slightly more risky counterparty means it is more likely to default earlier than the referencing firm whilst also allowing enough time for the CDS value to increase to a normal level.

However, if the referencing firm $\lambda_1(t)$ is risky enough, so that the CDS value at the counterparty's default time is almost surely positive, then the default contagion could have a significant impact on the CVA and the amount of change in the CVA is always increasing with larger λ_2, which is the more risky counterparty. The maximum CVA increment is almost 250 *bps* when the counterparty is about to default. In parts (b) and (c), we see that the CVA increment tends to the expected post-contagion CDS value given $\lambda_1(\tau_2)$. When the referencing firm is very risky, whether there is default contagion does not change the CVA a lot, see parts (a) and (c). There are two reasons for this phenomenon. Firstly, a risky referencing firm reduces the probability that counterparty defaults earlier so there will be no default contagion effects. Secondly, CDS value is a concave function of λ_1. Consequently, given $\bar{\alpha}_{1,2}$ unchanged, the higher λ_1 implies that the amount of loss given default raised by default contagion is lower.

4.5 Conclusions

In this chapter, we propose a new approach for modelling default contagion, which default contagion is modelled into the SDE of firms' default intensity process. We discuss valuation problems of survival probability, CDS and CVA under this model and discuss the various approaches to model the random default contagion jumps. While consider only two firms, indexed by 1 and 2, in the economy, we derive and discuss the PDE systems to describe firm 1's marginal survival probability, the CDS value referencing to the firm 1 and the CVA due to buying the CDS from firm 2.

By further specifying the default intensity processes to be mean-reverting type, our model allows firms to recover from default contagions depending on their own mean-reverting speed. One advantage of this model over previous default contagion

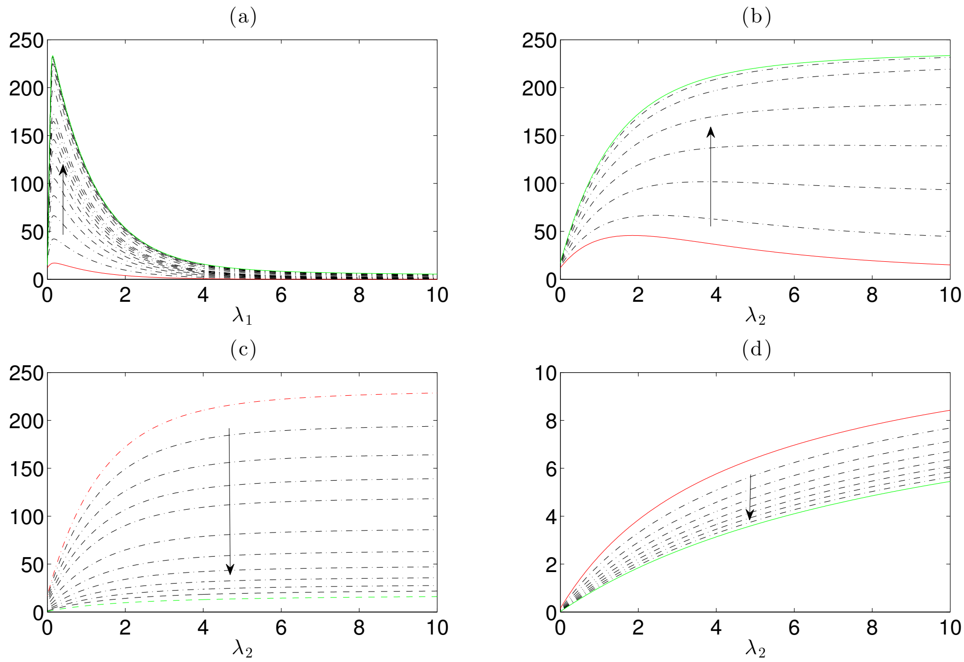

Figure 4.20
The change of the CVA of a 5 years CDS protection with $\bar{\alpha}_{1,2} = 0.05$, CVA($\bar{\alpha}_{1,2} = 0.05$)-CVA($\bar{\alpha}_{1,2} = 0$)
A illustration of the CVA changes due to raising default contagion from counterparty to referencing firm with parameters are $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$. (a) The CVA change against λ_1 and λ_2 raises from 0 to 5. (b) The CVA change against λ_2 and λ_1 raises from 0 to 0.08. (c) The CVA change against λ_2 and λ_1 raises from 0.1 to 2. (d) The CVA change against λ_2 and λ_1 raises from 3 to 5.

models is that we can easily to obtain numerical solutions for the PDE systems using finite-difference and we discuss the corresponding finite-difference scheme in detail.

After solving the PDE systems, which contain a two-dimensional and a one-dimensional PDE, we show the marginal and joint default probabilities with default contagion. We see that firms' mean-reverting parameter is critical important to one's marginal default probability and joint default probability. While with the default contagion, firms 1's as well as firm 2's mean-reverting speed will determine the time when the default contagion has the strongest impact on the marginal and joint default probability. Also, slow mean-reverting speed of the alive firm implies default contagion will have stronger impacts on the firm because the firm is less likely to recover from the default contagion. In terms of fair CDS spreads, since they are directly linked to the referencing firm's default probability, similar behaviours are observed. As to the CVA, the default contagion's role is to create a random upward jump in the CDS value, or the loss given default in other words, at the counterparty's default time. But default contagion does not changing the probability that the counterparty credit event occurs. The CVA change due to default contagion is very significant for maturities with up to 10 year CDS contracts. It is found that the default contagion's impacts on CVA vary according to the CDS's value and the relative riskiness of the referencing versus the counterparty. The default contagion will have the strongest impact on CVA when the CDS has positive value but far from its upper limit. Also, the counterparty is relatively more risky than the referencing firm so that the probability of the counterparty defaults earlier is higher.

Chapter 5

Default Contagion Model with Jump-Diffusion process

In the previous chapter to this we described a novel default contagion framework in which we model default contagion through a jump to the stochastic differential equation of surviving firms' default intensity at other firms' default times. Within that framework, we have only considered the default correlations due to the interactions among a closed set of firms. However, there are other external factors that may affect all firms at the same time. For instance, the changing of macroeconomic factors, such as the interest rate set by central banks or some new regulations, may lead to a situation where all firms face financial difficulty or higher default risk. We would like to take into account the impacts on firms' default risk caused by changing macroeconomic factors and we will refer to them as external or economy-wide shocks, which are modelled as jump processes. For the motivations to model asset prices with jump diffusion processes, we refer to Kou (2007).

In this chapter, we consider default correlation that are driven by both interactions among companies and macroeconomic factors, namely default contagion shocks and external shocks respectively. In a similar way to how we have modelled the impact from default contagion events, the impacts from macroeconomic factors are modelled by an inhomogeneous Poisson process added into the default intensities and this Poisson process will be shared by all firms' default intensities. In this way we can generalise the occurrence of all the unexpected external shocks from a variety of economy-wide sources into event arrivals of a single Poisson process. Because the Poisson process is shared by all the firms' default intensity SDEs so that all the intensities jump up at the same time. Under this framework, the default contagion from Chapter 4 is still included so that the default event of a firm can trigger jumps to alive ones.

We are not the first to try and include a jump component into default intensity processes, for example Duffie (1998) Duffie et al. (1999), Brigo and El-Bachir (2010). Duffie (1998) is one of the first papers to mention how affine-jump-diffusion (AJD) processes could be used for modelling the default intensity process and provide closed-form solutions for default bonds. The application of AJD processes is further studied by Duffie (2005), who derive closed-form solutions to the three building blocks for pricing credit derivatives introduced by Lando (1998), which we discussed in Section 2.2. Consequently, closed-form solutions are available for pricing credit derivatives including CDS and credit guarantees. The pricing of more complicated credit derivatives with an AJD process, such as the credit default swaption, is discussed by Brigo and El-Bachir (2006). Brigo and El-Bachir also describe how to calibrate their AJD process to the implied swaption volatility. A semi-analytic solution of credit swaption has now been provided by Brigo and El-Bachir (2010), which is based on a Fourier transform. These papers have focused on pricing single name credit derivatives using jump processes but did not treat the jump term as a shared component of multiple firms. In our model the jump term is contributing as a source of default correlation and we will therefore be able to analyse the impact on default correlation. The jump component as a source of default risk shared by all companies, in order to represent economy-wide risk, has been studied by Duffie and Garleanu (2001), who investigate the pricing of CDO and the probability distribution of the number of defaulted firms. Using the default contagion model of Yu (2007), Hao and Ye (2011) consider a model where the CDS seller and the referencing firm's default intensities share a Vasicek process with exponential jumps. Due to the normality of the model, Hao and Ye manage to derive the closed-form solution to the fair CDSs spreads. The model of Hao and Ye (2011) is extended by Wang and Ye (2013) to derive the CDS spread when the CDS buyer is also defaultable. However, neither Hao and Ye (2011) nor Wang and Ye (2013) did quantitative analysis of the shared jump's impact on default correlation and CVA.

The main contribution we make here is the combination of jump diffusion default intensities with default contagion model as well as also presenting efficient numerical schemes that can solve these complex derivatives. Unlike Hao and Ye (2011) and Wang and Ye (2013), our model does not break the non-negative restriction for default intensities. Moreover, we explore the shared jumps' impact on joint default probabilities, CDS spreads and CVA. We are able to compare the different roles of external shocks and default contagion shocks play in the derivation of the CVA.

In this chapter, model specifications are discussed in Section 5.1, where we combine external shocks into our proposed default contagion model. The extended model leads us to a problem that involves solving one-dimensional and two-dimensional PIDEs, and we will revisit the previous numerical scheme and show how to handle the extended model in Section 5.3. Numerical results of survival probabilities, CDS prices and CVA will be shown in Section 5.4 to illustrate the external shock effects as well as default contagion on default correlation, CDS spread and CVA.

5.1 Model Specification

Consider a filtered probability space $\left(\Omega^W, \mathcal{F}_t^W, \mathbb{Q}^W\right)$ up to the finite time horizon t, satisfying the conditions of right continuous and completeness. On this probability space, there are Brownian motions $(W_t^i)_{i=0}^I$ representing the uncertainties in the economy. A filtration $\mathcal{F}_t^W$ is defined as

$$\mathcal{F}_t^W = \sigma\left((W_s^i)_{i=0}^I, 0 \leq s \leq t\right),$$

which is the filtration generated by Brownian motions. In order to construct stochastic default contagion and external shocks, we enlarge the probability space Ω^W with a probability space $\left(\Omega^{\alpha,\beta}, \mathcal{F}^{\alpha,\beta}, \mathbb{Q}^{\alpha,\beta}\right)$ defined by $I \times (I-1)$ non-negative bounded real-value random variables $\alpha_{i,j}$ with a known joint density function $\bar{\eta}()$ for $i \in I$, $j \neq i$.

Compared to Chapter 4, we introduce an extra $\mathbb{F}$-adapted homogeneous Poisson process $J^{\mathbb{Q}}(t)$ with a constant arrival rate $\bar{\lambda} \in \mathbb{R}_+$ and jump size as a random variable $\beta \in \mathbb{R}_+$ with density function $\nu(\beta)$ such that $\int_{\mathbb{R}_+} \bar{\lambda}\nu(\beta)d\beta < \infty$. According to Tankov (2003), the process $J_t^{\mathbb{Q}}$ is of finite variate and has finite first moment, which is essential for modelling default intensities using $J^{\mathbb{Q}}(t)$. The homogeneous Poisson process $J_t^{\mathbb{Q}}$ is shared by all firms, hence it will not be indexed by i. Event arrivals of $J_t^{\mathbb{Q}}$ represent the occurrence of external shocks that do harm to every firm's credit quality. We denote the filtration generated by the homogeneous Poisson process as

$$\mathcal{F}_t^J = \sigma\left((J_s^{\mathbb{Q}}), 0 \leq s \leq t\right).$$

Then the entire market filtration contain all market information except default is defined as

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}^{\alpha,\beta} \vee \mathcal{F}_t^J. \tag{5.1}$$

Further, we assume there are I non-negative random variables $(\tau_i)_{i=0}^I$ defined on the filtered probability space $(\Omega, \mathcal{G}_t, \mathbb{Q})$, satisfying $\mathbb{Q}(\tau^i = 0) = 0$ and $\mathbb{Q}(\tau^i > t) > 0$ for $t \in \mathbb{R}_+$ and $i \in I$. Let the right-continuous default indicator processes $H_t^i = 1_{\{\tau^i \leq t\}}$ and $\mathbb{H}^i$ to denote the filtration generated by the default process H_t^i, so that $\mathcal{H}_t^i = \sigma(\{\tau^i \leq u : u \leq t\})$ for any $t \in \mathbb{R}_+$. The total default information is captured by the filtration $\mathcal{H}_t = \vee_{i \in I} \mathcal{H}_t^i$. Then the total information, including market information and default information, is captured by the filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ and $\mathbb{H} = (\mathcal{H}_t)_{t \in \mathbb{R}_+}$. All filtrations are assumed to satisfy conditions of right-continuous and completeness.

Due to the nature of default contagion modelling discussed in Section 4.2, we define $F_t^i = \mathbb{Q}(\tau^i \leq t | \mathcal{G}_t^{-i})$ to be a $\mathbb{G}^{-i} = \left(\mathcal{G}_t^{-i}\right)_{t \in \mathbb{R}_+}$-adapted default process, where $\mathcal{G}_t^{-i} = \vee_{j \in I, j \neq i} \mathcal{H}_t^j \vee \mathcal{F}_t$. The filtration $\mathcal{G}_t^{-i}$ contains all information except the i^{th} default time τ^i.

Let us F_t^i is an absolutely continuous submartingale under $\mathbb{Q}$ for all $i \in I$. Then the $\mathbb{G}^{-i}$-hazard process Γ_t^i of τ^i under $\mathbb{Q}$, defined as $1 - F_t^i = e^{-\Gamma_t^i}$, admits the $\mathbb{G}^{-i}$-adapted intensity $\lambda_i^{\mathbb{Q}}(t)$ of default time τ^i under $\mathbb{Q}$. In this Chapter, we model the non-negative default intensity processes as

$$\begin{aligned} d\lambda_i^{\mathbb{Q}}(t) =& \mu^{\mathbb{Q}}(\lambda_i(t), t)dt + \sigma^{\mathbb{Q}}(\lambda(t), t)dW_t^i + \beta dJ^{\mathbb{Q}}(t) + \sum_{j \in I, j \neq i}^{I} \alpha_{i,j}^{\mathbb{Q}} dH_t^j \\ & d < W_i, W_j >= \rho_{i,j} dt, \text{ for } \forall i \neq j. \end{aligned} \tag{5.2}$$

where $\mu^{\mathbb{Q}}(\lambda_i(t), t)$ and $\sigma^{\mathbb{Q}}(\lambda_i(t), t)$ being $\mathbb{F}$-adapted Lipschitz real value functions satisfying (2.3) and (2.4). The processes $\lambda_i^{\mathbb{Q}}(t)$ are assumed to be non-negative and for all $i \in I$ and are supposed to be the $\mathbb{G}^{-i}$-adapted stochastic intensity process. The stochastic jump $\alpha_{i,j}$ has a marginal probability density function $\eta_{i,j}(\alpha_{i,j})$.

When there is an arrival of the Poisson process $J^{\mathbb{Q}}(t)$, the external shock β is applied to all surviving firms' credit intensities. In other words, all firms' default risk is driven up at the same time and by the same amount. Introducing the homogeneous Poisson process $J^{\mathbb{Q}}(t)$ will lead to the default intensity processes $\lambda_i^{\mathbb{Q}}(t)$, excluding default contagion, to be a jump-diffusion process. We name this new type of default intensity process as jump-diffusion-contagion process.

5.2 A Two Firm Model

In a similar way to Section 4.3, we will first present the new default contagion model in a two-firm framework. In the two-firm framework, we present the derivation of the partial integro-differential equations to survival probability, CDS price and the CVA. Now, let us now consider a simple case where there are just two firms A and B and we model the default intensities as

$$\begin{aligned} d\lambda_A(t) &= \mu_A(\lambda_A(t),t)dt + \sigma_A(\lambda_A(t),t)dW_A(t) + \alpha_{A,B}dH_B(t) + \beta dJ(t), \\ d\lambda_B(t) &= \mu_B(\lambda_B(t),t)dt + \sigma_B(\lambda_B(t),t)dW_B(t) + \alpha_{B,A}dH_A(t) + \beta dJ(t), \\ &d < W_A(t), W_B(t) >= \rho_{A,B}dt. \end{aligned} \tag{5.3}$$

We drop the superscript $\mathbb{Q}$ in parameters for simplicity. Then the surviving or default state of these two firms means that there are 4 states in the economy, and we will denote them by the following,

$$\mathcal{A} = \{H_A(t) = 0, H_B(t) = 0\},\ \mathcal{B} = \{H_A(t) = 0, H_B(t) = 1\}$$
$$\mathcal{C} = \{H_A(t) = 1, H_B(t) = 0\},\ \mathcal{D} = \{H_A(t) = 1, H_B(t) = 1\}.$$

Note that, since $\mu_A(\lambda_A(t),t)$, $\mu_B(\lambda_B(t),t)$, $\sigma_A(\lambda_A(t),t)$, $\sigma_B(\lambda_B(t),t)$ and the process J_t are $\mathbb{F}$-adapted, then both intensity processes $\lambda_A(t)$ and $\lambda_B(t)$ are $\mathcal{F}_t$-measurable in state $\mathcal{A}$. While in state $\mathcal{B}$, $\lambda_A(t)$ will be $\mathcal{G}_t^{-A}$-measurable and $\lambda_B(t)$ does not exist since firm B defaulted. Similarly, $\lambda_B(t)$ is $\mathcal{G}_t^{-B}$-measurable in state $\mathcal{C}$ and both intensities do not exist in state $\mathcal{D}$. We remind readers that the default intensities remain to be jump-diffusion processes in the corresponding states they exist. Default contagion jumps happen at the time of state changes due to a firm defaults.

The derivation of (4.5) and (4.6) does not rely on the specific dynamic of default intensities. Therefore, (4.5) and (4.6) still valid under the jump-diffusion-contagion processes. However, the Feynman-Kac theorem in Section 2.1 can be applied only with the additional condition that $\int_{\mathbb{R}_+} \bar{\lambda}\nu(\beta)d\beta < \infty$ and the infinitesimal generators below, which we refer to Tankov (2003). In addition, the infinitesimal generator in a two-dimensional framework is

$$\begin{aligned}(\mathcal{A}\{u^{\mathcal{A}}\})(\lambda_A, \lambda_B, t) = &\sum_{i\in\{A,B\}} \mu_i(\lambda_i, t)\frac{\partial u^{\mathcal{A}}}{\partial \lambda_i} + \sum_{i,j\in\{A,B\}} \frac{1}{2}(\sigma\sigma^T)_{i,j}(\lambda_i, \lambda_i, t)\frac{\partial^2 u^{\mathcal{A}}}{\partial\lambda_i\partial\lambda_j} \\ &+ \bar{\lambda}\int_{\mathbb{R}} \left(u^{\mathcal{A}}(\lambda_A+\beta, \lambda_B+\beta, t)\nu(\beta)d\beta - u^{\mathcal{A}}(\lambda_A, \lambda_B, t^-)\right)\end{aligned} \tag{5.4}$$

and for the one-dimensional case is

$$\begin{aligned}\left(\mathcal{A}\{u^{\mathcal{B}}\}\right)(\lambda_i, t) = &\mu_i(\lambda_i, t)\frac{\partial u^{\mathcal{B}}}{\partial \lambda_i} + \frac{1}{2}\sigma_i^2(\lambda_i, t)\frac{\partial^2 u^{\mathcal{B}}}{\partial \lambda_i^2} \\ &+ \bar{\lambda}\int_{\mathbb{R}} \left(u^{\mathcal{B}}(\lambda_i+\beta, t)\nu(\beta)d\beta - u^{\mathcal{B}}(\lambda_i, t^-)\right), \text{for } i = \text{A or B}\end{aligned} \tag{5.5}$$

where $u^{\mathcal{A}}(\lambda_A, \lambda_B, t, T)$ and $u^{\mathcal{B}}(\lambda_A, t, T)$ are the generic credit claims (4.5) and (4.6) respectively. $u^{\mathcal{A}}(\lambda_A(t^-)+\beta, \lambda_B(t^-)+\beta, t, T)$ is the post-jump value after an arrival of $J(t)$ and the pre-jump value is $u^{\mathcal{A}}(\lambda_A(t^-), \lambda_B(t^-), t^-, T)$. It is worth noting that because we model the strength of external shocks to the firms' default intensity as identical then the post-jump value uses the same β for each firms' default intensity. In this situation, we have the greatest level of correlation because both the external jump times and sizes are identical to all firm. For the validation of (5.4), (5.5) the applicability of the Feynman-Kac theorem to credit derivatives with jump-diffusion default intensities, we refer to Duffie and Singleton (1999), Duffie et al. (1999) and Tankov (2003).

Here we follow a similar procedure as laid out in above literature and apply it to our default contagion model. First, a credit claim in state $\mathcal{A}$, denoted by $u^{\mathcal{A}}(\lambda_A(t), \lambda_B(t), t, T)$, with default contagions can be expressed as (4.5) if cash flows X_T, A_t and Z_{τ_A} satisfy the appropriate technical conditions. When the default intensities are modelled as (5.3), applying the Feynman-Kac theorem to (4.5) leads to the PIDE

$$\frac{\partial u^{\mathcal{A}}}{\partial t^-} + \mathcal{L}\{u^{\mathcal{A}}\} + \mathcal{J}\{u^{\mathcal{A}}\} + f^{\mathcal{A}}(\lambda_A, \lambda_B, t) - g^{\mathcal{A}}(\lambda_A, \lambda_B, t)u^{\mathcal{A}} = 0, \tag{5.6}$$

subject to terminal condition $\psi(\lambda_A, \lambda_B, T) = X(T)$, where

$$\mathcal{A}\{u^{\mathcal{A}}\} = \mathcal{L}\{u^{\mathcal{A}}\} + \mathcal{J}\{u^{\mathcal{A}}\} \tag{5.7}$$

and

$$\mathcal{J}\{u^{\mathcal{A}}\} = \bar{\lambda}\left(\int_0^\infty u(\lambda_A(t^-)+\beta, \lambda_B(t^-)+\beta, t)\nu(\beta)d\beta - u(\lambda_A(t^-), \lambda_B(t^-), t^-)\right). \tag{5.8}$$

In order to simplify notations and make it easy to introduce numerical methods in Section 5.3, we use functions $f(\lambda_A, \lambda_B, t)$ and $g(\lambda_A, \lambda_B, t)$ to summarise the terms after applying Feynman-Kac theorem to (4.5), which are

$$\begin{aligned} f^{\mathcal{A}}(\lambda_A, \lambda_B, t) &= \lambda_A Z(t) + dA(t) + \lambda_B \int_0^\infty u^{\mathcal{B}}(\lambda_1 + \alpha_{A,B}, t, T)\eta(\alpha_{A,B})d\alpha_{A,B} \\ g^{\mathcal{A}}(\lambda_A, \lambda_B, t) &= r(t) + \lambda_A + \lambda_B \\ \psi^{\mathcal{A}}(\lambda_A, \lambda_B, T) &= X_T, \end{aligned} \tag{5.9}$$

where $r(t)$ is an $\mathbb{F}$-adapted interest rate process and the function $\psi(\lambda_A, \lambda_B, T)$ represents the terminal condition at time T state $\mathcal{A}$.

Similarly, for the credit claim referencing to firm A in state $\mathcal{B}$, we apply Feynman-Kac theorem to (4.6) and it leads to

$$\frac{\partial u^{\mathcal{B}}}{\partial t^-} + \mathcal{L}\{u^{\mathcal{B}}\} + \mathcal{J}\{u^{\mathcal{B}}\} + f^{\mathcal{B}}(\lambda_A, t) - g^{\mathcal{B}}(\lambda_A, t)u^{\mathcal{B}} = 0, \tag{5.10}$$

subject to terminal condition $\psi(\lambda_A, T) = X(T)$, where

$$\begin{aligned} f^{\mathcal{B}}(\lambda_A, t) &= \lambda_A Z(t) + dA(t) \\ g^{\mathcal{B}}(\lambda_A, t) &= r(t) + \lambda_A \\ \psi^{\mathcal{B}}(\lambda_A, T) &= X_T. \end{aligned} \tag{5.11}$$

Similar to Section 4.3, the value of any default claim referencing to firm A with the jump-diffusion-contagion model is the solution of the two PIDE system (5.6) and (5.10) with specifications of X_T, A_t and Z_{τ_A} in (4.5) and (4.6). For the rest of this section, we introduce the PIDEs to survival probability, CDS value and the CVA as special cases of (5.6) and (5.10).

5.2.1 Survival probability

Similar to the survival probability in Chapter 4, the firm A's marginal survival probability within our model should be considered in two states, $\mathcal{A}$ and $\mathcal{B}$. If no firm has defaulted, the default intensities are described as (5.3). On the other hand, if B firm has defaulted, in state $\mathcal{B}$, the default intensity is simply a jump-diffusion process. Denote firm A's default probability from time t to T in state $\mathcal{B}$ as $P_{\mathcal{B}}(\lambda_A(t), t, T)$. As discussed in Section 4.3, we know that it must satisfy (4.6) with $X(T) = 1$, $A_s = 0$, $Z_s = 0$ and $r(t) = 0$ so we have the PIDE (5.10) with

$$\begin{aligned} f^{\mathcal{B}}(\lambda_A, t) &= 0 \\ g^{\mathcal{B}}(\lambda_A, t) &= \lambda_A \\ \psi^{\mathcal{B}}(\lambda_A, T) &= 1, \end{aligned}$$

which is

$$\begin{aligned} &\frac{\partial P_{\mathcal{B}}}{\partial t^-} + \mu_A(\lambda_A, t)\frac{\partial P_{\mathcal{B}}}{\partial \lambda_A} + \frac{1}{2}\sigma_A^2(\lambda_A, t)\frac{\partial^2 P_{\mathcal{B}}}{\partial \lambda_A^2} - \lambda_A P_{\mathcal{B}} \\ &+ \bar{\lambda}\left(\int_0^\infty P_{\mathcal{B}}(\lambda_A(t^-) + \beta, t, T)\nu(\beta)d\beta - P_{\mathcal{B}}(\lambda_A(t^-), t^-, T)\right) = 0. \end{aligned} \tag{5.12}$$

Similarly, in state $\mathcal{A}$, the firm A's survival probability $P_{\mathcal{A}}(\lambda_A(t), \lambda_B(t), t, T)$ is a special case of (4.6) with $X(T) = 1$, $A_s = 0$, $Z_s = 0$ and $r(t) = 0$. Therefore, $P_{\mathcal{A}}(\lambda_A(t), \lambda_B(t), t, T)$ satisfies the PIDE (5.6) with

$$\begin{aligned} f^{\mathcal{A}}(\lambda_A, \lambda_B, t) = \lambda_B \int_0^\infty P_{\mathcal{B}}(\lambda_A + \alpha_{A,B}, t, T)\eta(\alpha_{A,B})d\alpha_{A,B} \\ g^{\mathcal{A}}(\lambda_A, \lambda_B, t) = \lambda_A + \lambda_B, \\ \psi^{\mathcal{A}}(\lambda_A, \lambda_B, T) = 1, \end{aligned}$$

which gives us

$$\begin{aligned} &\frac{\partial P_{\mathcal{A}}}{\partial t^-} + \mathcal{L}\{P_{\mathcal{A}}\} + \mathcal{J}\{P_{\mathcal{A}}\} + \lambda_B\left(\int_0^\infty P_{\mathcal{B}}(\lambda_A + \alpha_{A,B}, t, T)\eta(\alpha_{A,B})d\alpha_{A,B}\right) \\ &- (\lambda_A + \lambda_B)P_{\mathcal{A}} = 0. \end{aligned} \tag{5.13}$$

5.2.2 Credit default swap

In Section 4.3, we show that the CDS's value can be expressed as (4.6) with multiple payments $X_n = S(T_n - T_{n-1})$ at time T_n and $Z_{\tau_A} = (1-R)e^{-r(T-\tau_A)} - S\frac{\tau_A - T_{k-1}}{T_k - T_{k-1}}$ with $T_{k-1} \le \tau_1 < T_k$ and $A_t = 0$. Denote $V_{\mathcal{B}}(\lambda_A(t), t, T)$ as the CDS value in state $\mathcal{B}$ and we remind the reader here that S is the swap spreads to be paid at $T_1, T_2, ..., T_N$, which we treat as jump conditions on the PIDE at the end of each time period. Then we know from Section 5.2 that applying Feynman-Kac theorem to (4.6) leads $V_{\mathcal{B}}(\lambda_A(t), t, T)$ to be the solution of PIDE

$$\begin{aligned}
&\frac{\partial V_{\mathcal{B}}}{\partial t^-} + \mathcal{L}\{V_{\mathcal{B}}\} + \mathcal{J}\{V_{\mathcal{B}}\} + \lambda_A \left((1-R_A)e^{-r(T-t^-)} - S\frac{t^- - T_{n-1}}{T_n - T_{n-1}} \right) \\
&\qquad - (r+\lambda_A)V_{\mathcal{B}} = 0
\end{aligned} \tag{5.14}$$

for $t \in (T_{n-1}, T_n)$ with $n = 1, 2, ...N$. It is subject to the terminal condition $V_{\mathcal{B}}(\lambda_A, T, T) = 0$ and jump conditions

$$V_{\mathcal{B}}(T_n^-, T) = V_{\mathcal{B}}(T_n, T) - S(T_n - T_{n-1}) \text{ for } n = 1, 2, ...N.$$

Or we could say that (5.14) satisfies (5.10) with

$$\begin{aligned}
f^{\mathcal{B}}(\lambda_A, t) &= \lambda_A \left((1-R_A)e^{-r(T-t)} - S\frac{t - T_{n-1}}{T_n - T_{n-1}} \right) \\
g^{\mathcal{B}}(\lambda_A, t) &= r + \lambda_A, \\
\psi^{\mathcal{B}}(\lambda_A, T) &= 0.
\end{aligned}$$

Similarly, we denote the CDS value in state $\mathcal{A}$ as $V_{\mathcal{A}}(\lambda_A(t), \lambda_B(t), t, T)$, then a CDS referencing to firm A will satisfy the generic PIDE (5.6) with

$$\begin{aligned}
f^{\mathcal{A}}(\lambda_A, \lambda_B, t) &= \lambda_B \int_0^\infty V_{\mathcal{B}}(\lambda_A + \alpha_{A,B}, t^-)\eta(\alpha_{A,B})d\alpha_{A,B} \\
&\quad + \lambda_A \left((1-R_A)e^{-r(T-t)} - S\frac{t - T_{n-1}}{T_n - T_{n-1}} \right) \\
g^{\mathcal{A}}(\lambda_A, \lambda_B, t) &= r + \lambda_A + \lambda_B \\
\psi^{\mathcal{A}}(\lambda_A, \lambda_B, T) &= 0.
\end{aligned} \tag{5.15}$$

Then we can write the equation for the CDS as

$$\begin{aligned}
&\frac{\partial V_{\mathcal{A}}}{\partial t^-} + \mathcal{L}\{V_{\mathcal{A}}\} + \mathcal{J}\{V_{\mathcal{A}}\} + \lambda_A \left((1-R_A)e^{-r(T-t)} - S\frac{t - T_{n-1}}{T_n - T_{n-1}} \right) \\
&+ \lambda_B \int_0^\infty V_{\mathcal{B}}\big(\lambda_A + \alpha_{A,B}, t^-\big)\eta(\alpha_{A,B})d\alpha_{A,B} - \big(r + \lambda_A + \lambda_B\big)V_{\mathcal{A}} = 0
\end{aligned} \tag{5.16}$$

for $t \in (T_{n-1}, T_n)$ with $n = 1, 2, ...N$ with terminal condition $V_{\mathcal{A}}(\lambda_A, \lambda_B, T, T) = 0$ and jump conditions

$$V_{\mathcal{A}}(T_n^-, T) = V_{\mathcal{A}}(T_n, T) - S(T_n - T_{n-1}) \text{ for } n = 1, 2, ...N.$$

5.2.3 Credit value adjustment

We measure the CVA here of buying a CDS protection from a CDS seller whose default has a default contagion effect on the referencing firm. In addition to default contagion shocks, we are interested in how the CVA will also be affected by external shocks.

Introducing a jump process $J(t)$ into our model makes the default intensity model falls into the class of jump-diffusion process. Duffie (2005) shows the PIDE for pricing credit claims in jump-diffusion process and Burgard and Kjaer (2011) show the PDE representation of the CVA when the underlying asset process is a diffusion process. It is not until recently that the PIDE representation of the CVA when the underlying asset process is a jump-diffusion was proven by Kromer et al. (2015). Kromer et al. (2015) prove the PIDE for CVA in very generic settings such that the conclusion holds for a wide range of jump-diffusion processes with random jump sizes. Therefore, we can use their results to obtain the PIDE for the CVA in our model.

Unlike a CDS written on firm A, which exists in both states $\mathcal{A}$ and $\mathcal{B}$, the CVA due to buying a CDS contract referencing to firm A from firm B only exists in state $\mathcal{A}$. This is because if either firm A or B defaults, this implies the termination of the CDS contract and the CVA does not exist either. By definition, when the contract expires, there is no counterparty risk therefore the CVA at terminal time equals 0. If the counterparty defaults earlier than the referencing firm, the values from PIDE (5.14), which is the CDS's value in the counterparty's default state, will determine whether or not the CDS buyer suffers from a loss. The losses, as discussed in Chapters 3 and 4, is the non-recoverable amount of the CDS's price if it is positive, $(1-R_B)\max\{V_{\mathcal{B}}(\tau_B, T), 0\}$. The difference between the CVA in this chapter and the one in Section 4.18 is that for this jump-diffusion contagion model the CVA will jump to the CVA associated with a higher referencing firm's and the counterparty's default intensities after external shocks.

As discussed in Section 4.3, the CVA to a default-free entity due to buying a CDS, referencing to firm A, from firm B is a special case of (4.5) with $A_t = 0$, $X_T = 0$ and $Z_{\tau_B} = (1 - R_B)\displaystyle\int_0^\infty \max\{V^{\mathcal{B}}(\lambda_A(\tau_B^-) + \alpha_{A,B}, \tau_B, T), 0\}\eta(\alpha_{A,B})d\alpha_{A,B}$. Applying Feynman-Kac theorem to (4.5) with the infinitesimal generator (5.4), the CVA, denoted as $U(\lambda_A(t), \lambda_B(t), t, T)$, with our jump-diffusion-contagion model satisfies PIDE

$$\begin{aligned}&\frac{\partial U}{\partial t^-} + \mathcal{L}\{U\} + \mathcal{J}\{U\} - (r + \lambda_A + \lambda_B)U \\ &\qquad + \lambda_B \int_0^\infty (1 - R_B)\max\{V_{\mathcal{B}}(\lambda_A + \alpha_{A,B}, t^-, T)\}\eta(\alpha_{A,B})d\alpha_{A,B} = 0,\end{aligned} \tag{5.17}$$

subject to the terminal condition $U(\lambda_A, \lambda_B, T, T) = 0$, which is a special case of (5.6) with

$$\begin{aligned} f^{\mathcal{A}}(\lambda_A, \lambda_B, t) &= \lambda_B \int_0^{\infty} (1 - R_B) \max\{V_{\mathcal{B}}(\lambda_A + \alpha_{A,B}, t, T)\} \eta(\alpha_{A,B}) d\alpha_{A,B}, \\ \psi^{\mathcal{A}}(\lambda_A, \lambda_B, T) &= 0 \\ g^{\mathcal{A}}(\lambda_A, \lambda_B, t) &= r + \lambda_A + \lambda_B. \end{aligned} \tag{5.18}$$

5.3 Numerical Schemes

In order to study the numerical implementation of our jump-diffusion-contagion model, we cannot avoid explicitly specify the functions of drift $\mu()$, volatility $\sigma()$ and the distribution of jumps $\alpha_{A,B}$, $\alpha_{B,A}$ and β, which are $\eta_{A,B}()$, $\eta_{B,A}()$ and $\nu()$.

In order to make our numerical result comparable with previous chapters, we still choose the drift and volatility functions $\mu()$ and $\sigma()$ to be mean-reverting square-root type, which are

$$\begin{aligned} \mu(\lambda_i(t), t) &= \kappa_i(\theta_i - \lambda_i(t)) \\ \sigma(\lambda_i(t), t) &= \sigma_i \sqrt{\lambda_i(t)}, \end{aligned} \tag{5.19}$$

for $i \in \{A, B\}$. The reason to choose (5.19) is to ensure non-negative and satisfy conditions (2.3) and (2.4), which is essential for Feynman-Kac theorem to be applicable.

Further, we choose $\alpha_{A,B}$, $\alpha_{B,A}$ and β to be $\mathbb{F}$-adapted independent exponential random variables with density functions,

$$\begin{aligned} \eta(\alpha_{A,B}) &= \frac{1}{\bar{\alpha}_{A,B}} e^{-\frac{1}{\bar{\alpha}_{A,B}} \alpha_{A,B}} \\ \eta(\alpha_{B,A}) &= \frac{1}{\bar{\alpha}_{B,A}} e^{-\frac{1}{\bar{\alpha}_{B,A}} \alpha_{B,A}}, \end{aligned} \tag{5.20}$$

and

$$\nu(\beta) = \frac{1}{\bar{\beta}} e^{-\frac{1}{\bar{\beta}} \beta}. \tag{5.21}$$

Although variety of random jumps are available for modelling as discussed in Section 4.2.1, there are a couple of reasons to choose independent exponential random variables for jump sizes β, $\alpha_{A,B}$ and $\alpha_{B,A}$. As discussed in Section 4.4, it is easy to implement without influencing the analysis the impacts from default contagion. Besides, letting β, $\alpha_{A,B}$ and $\alpha_{B,A}$ to have same distribution helps to identify the different impacts from external shocks and default contagion shocks on survival probability, CDS value and the CVA. Finally, our numerical results are comparable with previous chapters.

Solving the PIDE in our model requires numerical schemes for both one- and two-dimensional PIDEs. Given the fact that we are solving a problem with a jump process we must be careful how the truncation of the grid is affecting our solutions. We propose that using approximate solutions outside the truncated domain can reduce the localisation error due to computing the integral term in a truncated domain. In this section, we specify the finite difference scheme for the one-dimensional PIDE and the way we deal with a truncated grid for both the survival probability and the CDS, our results will highlight the efficiency of our methods. Later on, the scheme for the two-dimensional PIDE is both described and analysed.

5.3.1 Numerical scheme for 1D-PIDE

In Section 3.3, we discussed the Crank-Nicolson scheme for solving one-dimensional PDEs. However, in this chapter, because we model external shocks into the default intensity processes which leads to PIDEs even in the one dimensional case that must be solved and therefore we need a different approach for our finite difference schemes. We propose some improvements on the existing finite-difference scheme, which are specifically suitable for solving survival probabilities and pricing CDS.

PIDEs are discussed in much of the option pricing literature where the stock return is modelled as a jump-diffusion process, such as Merton (1976); Kou (2002). Andersen and Andreasen (2000) indicate that, when solving a PIDE, standard implicit or semi-implicit discretisations of the PIDE will result in a dense matrix system, which requires a $O(N^3)$ algorithm to be solved at every time step. Due to this unusually difficult matrix problem, the standard second-order in time unconditionally stable implicit schemes are not in any way feasible. Some paper have sought improvements in the computation of this dense matrix system, for example see Almendra and Oosterlee (2005) who suggest that an ADI algorithm can be used if the implicit integral term is solved by fast Fourier transform. Alternatively, to avoid these complications, one can abandon any hopes of second order convergence altogether and use an implicit-explicit (IMEX) scheme or an operator splitting (OS) scheme, which are discussed by Duffy (2006). In order to avoid solving a dense matrix system, which leads to computational burden, we choose to adapt the IMEX and OS schemes to our problem and we will present the numerical results showing their accuracy for solving our particular PIDEs.

The IMEX method treats the differential terms as implicit while the integral term is always treated explicitly, whereas the OS method splits the differential equation into two parts, where one contains only the differential term and the other contains only the integral term. At each step then, with the IMEX scheme we solve in one shot where the OS will require us to solve two reduced problems separately.

Remain that (5.10) is a generic form of one-dimensional PIDE, which we can solve it for the credit claim referencing to firm A in state $\mathcal{B}$. We drop the super-script $\mathcal{B}$ as we only interest in the numerical aspect of solving a PIDE of the form

$$\frac{\partial u}{\partial t^-} + \mathcal{L}\{u\} + \mathcal{J}\{u\} + f(\lambda, t) - g(\lambda, t)u = 0, \tag{5.22}$$

where λ represents the state variable.

For the IMEX method, if we apply the so called θ-scheme to the differential terms and an explicit scheme to integral terms, then equation (5.22) can be discretised as

$$\frac{u^n - u^{n+1}}{\Delta\tau^-} = \theta\mathcal{L}\{u^{n+1}\} + (1-\theta)\mathcal{L}\{u^n\} + \mathcal{J}\{u^{n+1}\} + f(\lambda, t) - g(\lambda, t)(\theta u^{n+1} + (1-\theta)u^n). \tag{5.23}$$

The integral term $\mathcal{J}\{u\}$, as we can see, is always treated at time step $n+1$. This scheme itself has been studied many times and as such many improvements have been made to adapt it to a particular option pricing problem, for examples of this we refer to Cont and Voltchkova (2005b); Feng and Linetsky (2008); Pindza et al. (2014).

The OS scheme, as discussed by Duffy (2006), will tackle the problem in a slightly different way. Here the idea is to split the equation into two,

$$\frac{\partial u}{\partial \tau^-} = \mathcal{L}\{u\} + f(\lambda, t) - g(\lambda, t)u \quad \text{and} \quad \frac{\partial u}{\partial \tau^-} = \mathcal{J}\{u\}. \tag{5.24}$$

If we now discretise those two equations, we have a system of two equations

$$\frac{u^n - Y}{\Delta\tau^-} = \mathcal{L}\{u^n\} + f(\lambda, t) - g(\lambda, t)u^n \tag{5.25a}$$

$$\frac{Y - u^{n+1}}{\Delta\tau^-} = \mathcal{J}\{u^{n+1}\}, \tag{5.25b}$$

where Y is the intermediate solution when we solve (5.25a) and (5.25b) one by one.

Next we present some results of our implementation of these two schemes in order to solve the survival probability of firm A in state $\mathcal{B}$, which is (5.12). We choose this simple problem for the fact that it has an analytic solution available that we can

$\lambda_{max} = 0.5$	OS	IMEX	$\lambda_{max} = 1$	OS	IMEX
$dx = \frac{1}{2000}$	1.37×10^{-4}	1.42×10^{-4}	$dx = \frac{1}{2000}$	1.69×10^{-4}	1.74×10^{-4}
$dx = \frac{1}{4000}$	9.24×10^{-6}	1.40×10^{-5}	$dx = \frac{1}{4000}$	4.18×10^{-5}	4.65×10^{-5}
$dx = \frac{1}{8000}$	-2.27×10^{-5}	-1.80×10^{-5}	$dx = \frac{1}{8000}$	9.85×10^{-6}	1.46×10^{-5}

Table 5.1
Numerical error of OS and IMEX scheme
one year survival probability with JCIR process. Parameters are ($\kappa = 0.5, \theta = 0.05, \sigma = 0.1, \lambda(0) = 0.05, \bar{\lambda} = 0.2, \beta \sim exp(0.05)$)

benchmark against. We will use the notation $u(\lambda, t)$ rather than $P_{\mathcal{B}}(\lambda_A(t), t, T)$ in order to align with (5.22) when analysing numerical schemes.

The state variable λ is restricted to a bounded domain $\lambda \in [0, \lambda_{max}]$ with λ_{max} taken sufficiently large. Similarly, the integral term is also truncated to $[0, \lambda_{max}]$. The issue of localization error is addressed in the next section. For the boundary conditions in IMEX scheme, we use the heuristic Robin conditions

$$\begin{aligned} &\frac{\partial u}{\partial t^-} + \kappa\theta \frac{\partial u}{\partial \lambda} + \bar{\lambda}\left(\int_0^\infty u(\lambda+\beta, t^-, T)\nu(\beta)d\beta - u(\lambda, t^-, T)\right) = 0 \quad \text{for} \quad \lambda \to 0 \\ &\frac{\partial u}{\partial t^-} + \kappa(\theta - \lambda)\frac{\partial u}{\partial \lambda} - \lambda u = 0 \quad \text{for} \quad \lambda \to \infty. \end{aligned} \tag{5.26}$$

to solve this problem. For the OS scheme, boundary conditions (4.23) can be used directly. This set of boundary conditions have already been tested and found to be superior than other conditions in Section 3.3. Equation (5.12) is discretised on $[0, \lambda_{max}]$ with a uniform mesh $0 = \lambda_0 < \lambda_1 ... < \lambda_i < ... < \lambda_I = \lambda_{max}$. The integral term is implemented using the trapezoidal numerical integration scheme.

Table 5.1 shows the relative error of the one year probability with both IMEX and OS schemes. When the infinite domain is bounded to a relative low value, say $\lambda_{max} = 0.5$, both schemes converge to a value lower than the analytic solution. This phenomenon is due to the computational error from the integral term, which will be discussed in the next section. This problem is not observed when λ_{max} is taken larger, such as $\lambda_{max} = 1$. We observed that two schemes show no real advantage over each other when solving for the survival probability, although the OS scheme has slightly smaller errors.

However, in the later sections and chapters of this thesis we will favour the use of the OS scheme when solving our PIDEs. This is because the OS scheme is much easier to be adapted into the ADI scheme for solving two-dimensional PIDEs, which is a requirement for solving survival probabilities, CDSs and CVA in state $\mathcal{A}$. The

OS scheme split the PIDE into a PDE term and an integral term, which are solved individually at each step backward. This feature enables us to only solve the two-dimensional PDE part using the ADI scheme, which is a well known stable and efficient method. However, given that the IMEX scheme will solve the entire PIDE in a single step then how incorporate ADI scheme with IMEX is not straightforward.

5.3.2 Improved numerical scheme: Extrapolation

Models that include jump processes with exponentially distributed jump size will yield to a PIDE with an integral term over zero to infinity. When solving the PIDE, this integral term has to be truncated to a finite domain, which we term here as the upper limit λ_{max}. For example,

$$\begin{aligned}
\bar{\lambda}\int_0^{\infty} u(\lambda+\beta,t^-)\nu(\beta)d\beta &= \bar{\lambda}\int_0^{\lambda_{max}-\lambda} u(\lambda+\beta,t^-)\nu(\beta)d\beta + \text{error} \\
&> \bar{\lambda}\int_0^{\lambda_{max}-\lambda} u(\lambda+\beta,t^-)\nu(\beta)d\beta \\
&= \bar{\lambda}\int_0^{\lambda_{max}-i\Delta\lambda} u(i\Delta\lambda+\beta,t^-)\nu(\beta)d\beta \\
&\approx \sum_{k=i}^{I} w_k u_k \frac{1}{\beta} e^{-\frac{1}{\beta}(k-i)\Delta\lambda}\Delta\lambda, \quad \text{for} \quad 0 \le i \le I
\end{aligned} \tag{5.27}$$

where $\Delta\lambda = \dfrac{\lambda_{max}}{I}$ and $w_k = 0.5, 1, ...1, 0.5$ defines the trapezoidal rule. As we can see from (5.27), the integral term is always be underestimated given solutions are positive, which leads to the results in table 5.1. In general, a wide enough lower and/or upper bounds can be fixed so that the default intensity process $\lambda(t)$ has very low probability of reaching the bounds even with jumps then the difference between infinite integral and its truncated approximation can be ignored. However, the result is a scheme, which is not very efficient. More importantly, the error in calculating the integral term will increase with large λ. Since we have assumed that the upper limit is fixed at λ_{max}, evaluating the integration at a large i means there is less available solutions to approximate the post-jump values $u(\lambda+\beta,t)$ for larger sizes of jump β.

Solving the one-dimensional PIDE both accurately and efficiently is important in our model. As our default contagion modelling yields to a PIDE system that combines both a two-dimensional PIDE and a one-dimensional PIDE, we need accurate values from the one-dimensional PIDE to feed into the two-dimensional one. As we draw values from the one-dimensional PIDE to estimate the values after a default contagion, errors from the one-dimensional PIDE have the potential to contaminate the two-dimensional PIDE solutions. This is especially true for the CVA, as the post-contagion CDS value plays a crucial role in determining the loss given the counterparty defaults.

In this section, we propose a new approximation method for values outside the grid, $u(\lambda, t)$ with $\lambda > \lambda_{max}$, in order to improve the accuracy of the numerical integration term in the differential equations. The method used will have to be tailored to the problem and we show how to do it for survival probabilities and CDS prices. This technique can yield remarkable accuracy without hardly any extra computational burden. This is based on the properties of the solution when the default intensity tends to infinity.

First, we must define another term $y(\lambda, t)$ that adds into the approximation for the integral term as an adjustment so that the approximation to the integral term becomes

$$\bar{\lambda} \int_0^\infty u(\lambda+\beta, t^-)\nu(\beta)d\beta \approx \bar{\lambda}\left(\int_0^{\lambda_{max}-\lambda} u(\lambda+\beta, t^-)\nu(\beta)d\beta + y(\lambda, t^-)\right). \tag{5.28}$$

Now we can discuss how to derive the appropriate form of $y(\lambda, t)$ for survival probabilities and CDS prices separately in the following two sections.

Survival Probability

At each time step n, we know the survival probability under AJD processes is an affine function tending towards zero with default intensity λ. When the heuristic Robin type boundary conditions (4.23) are applied, the solutions decay exponentially with large λ. Therefore, assuming its solution declines exponentially seems appropriate. Denoting λ_i, $i = 0, 1, ..., I$ where $I \times \Delta\lambda = \lambda_{max}$ as the default intensity level and t_n, $n = 0, 1, ..., N$ where $N \times \Delta t = T$ as the time level. Let u_i^n be $u(i\Delta\lambda, n\Delta t)$, then we assume

$$u(\lambda, t) = A(t)e^{-B(t)\lambda}, \quad \forall \lambda \in [\lambda_{max}, \infty) \tag{5.29}$$

The two time-varying parameters $A(t)$ and $B(t)$ must be estimated at each time t_n. It follows that the value at the boundary and its slope must satisfy:

$$\begin{aligned} u(\lambda_{max}, t) &= A(t)e^{-B(t)\lambda_{max}} \\ \frac{\partial u}{\partial \lambda}(\lambda_{max}, t) &= -B(t)A(t)e^{-B(t)\lambda_{max}}, \text{ for } \forall t \in [0, T], \end{aligned} \tag{5.30}$$

We approximate the derivative term in (5.30) using first-order one sided difference. The solution of $A(t)$ and $B(t)$ can be obtained by solving the following equations at the boundary

$$A(t) = \frac{u(\lambda_{max}, t)}{e^{-B(t)\lambda_{max}}}$$
$$B(t) = -\frac{u'(\lambda_{max}, t)}{u(\lambda_{max}, t)} \approx -\frac{u_I^n - u_{I-1}^n}{u_I^n \Delta\lambda} \text{ for } \forall t.$$

Now we can extrapolate to find solutions for $\lambda > \lambda_{max}$ according to (5.29). Further, given (5.28), an analytic expression for the term $y(\lambda, t)$ can be derived as

$$\begin{aligned} y(\lambda, t) &= \int_{\lambda_{max}-\lambda}^{\infty} u(\lambda + \beta, t)\nu(\beta)d\beta \\ &= \int_{\lambda_{max}-\lambda}^{\infty} A(t)e^{-B(t)(\lambda+\beta)}\frac{1}{\bar{\beta}}e^{-\frac{1}{\bar{\beta}}\beta}d\beta \\ &= \frac{u_I^n}{1 - \bar{\beta}\frac{u_I^n - u_{I-1}^n}{u_I^n \Delta\lambda}} e^{-\bar{\beta}(\lambda_{max}-\lambda)}, \text{ for } \forall n \end{aligned} \tag{5.31}$$

Intuitively, the error term in (5.27), which due to truncating the infinite domain of the integral term to a finite domain, is compensated by $y(\lambda, t)$, which is obtained by assuming the solutions behaves as (5.29) outside the boundary. Although the method here has not be formally analysed, there are some properties of the method which lead us to think that it performance should be stable. For instance, so long as $B(t) > 0$ the result

$$y(\lambda, t) < u(\lambda, t)\int_{\lambda_{max}-\lambda}^{\infty} \nu(\beta)d\beta$$

will hold so that $y(\lambda, t)$ is always bounded and finite.

Figure 5.1 shows the solution of the five year survival probability. Here the results with and without the extrapolation method are compared with the analytic solution. Without using the approximation method, which is given by the green line, there is an obvious slump near the upper boundary and we can see the survival probability is underestimated. This is because when we compute the integration part in the PIDE (5.12) with large λ_A, the integration ends at $\lambda_A + \beta = \lambda_{max}$. That is equivalent to assuming those solutions with $\lambda_A > \lambda_{max}$ are zeros. Therefore, the survival probabilities in this region are underestimated. This problem could be solved by taking the domain $[0, \lambda_{max}]$ large enough so that the probability that λ_A reaches the upper bound is very tiny then whether or not the integration has been computed accurately at large λ_A has negligible impact on the solution. However, in order to maintain small $\Delta\lambda_A$, taking a

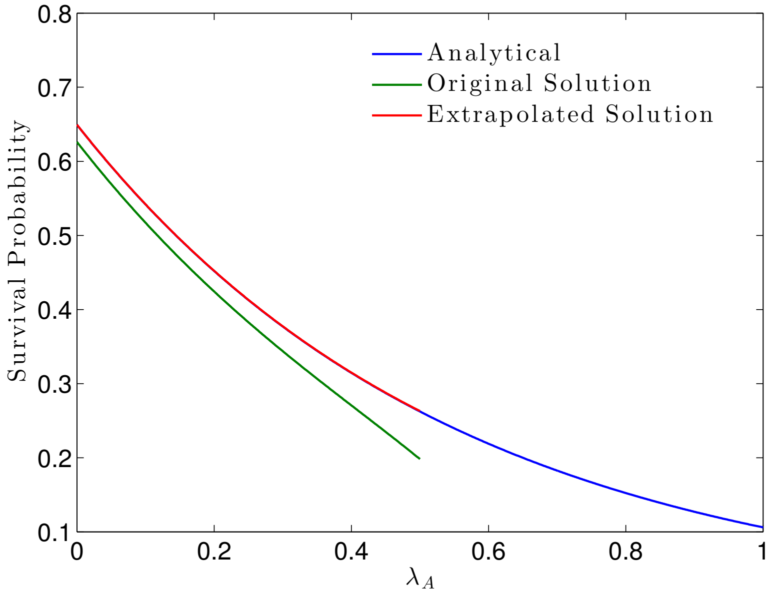

Figure 5.1
Compare extrapolation solutions with ordinary solutions
An illustration the effectiveness of extrapolation approximation. Here is the five year survival probability solutions of firm A with $\lambda_A \in [0, 0.5]$. The green line is the solution without extrapolation method with $\lambda_A \in [0, 0.5]$. The red line is the solution with extrapolation method with $\lambda_A \in [0, 0.5]$. The blue line is the analytic solutions.

larger λ_{max} implies that there will be a longer computational time. On the other hand, the red line is the solution using the extrapolation method. The slump that appeared before now disappears in our improved scheme as it becomes aligned with the analytic solution. This implies that, without taking λ_{max} very large, the integration term in PIDE (5.12) can be approximated accurately using our extrapolation method outside the grid. The capacity to reduce the truncation domain $[0, \lambda_{max}]$ means the scheme can be more efficient without loss of accuracy.

To further analyse the results, in table 5.2 we compare the errors in the numerical results when using extrapolation compared to the original scheme for the one year survival probability. We then increase the λ_{max} gradually from 0.5 to 0.8 to show the effect of changing the domain size. When the domain is taken to be large enough so that the default intensity $\lambda_A(t)$ is never likely to reach the λ_{max} given the starting point, then the values outside this region will not matter. However, it is at the cost of computational power as we must use more points to remain accurate. We can see from the results in table 5.2 that the scheme without extrapolation approximation tends to underestimate the survival probability. This problem disappears when the infinite

		$\lambda_A \in [0, 0.5]$		
N	Original	Error	Extension	Error
100	0.947468052	1.25E-04	0.947498735	1.58E-04
200	0.94735296	3.81E-06	0.94738371	3.63E-05
400	0.947325662	-2.50E-05	0.947356445	7.49E-06
800	0.947319575	-3.14E-05	0.947350374	1.08E-06
1600	0.947318421	-3.26E-05	0.947349228	-1.24E-07
		$\lambda_A \in [0, 0.6]$		
N	Original	Error	Extension	Error
100	0.947565343	2.28E-04	0.947569634	2.33E-04
200	0.947397117	5.04E-05	0.947401421	5.50E-05
400	0.947356555	7.61E-06	0.947360865	1.22E-05
800	0.947347163	-2.30E-06	0.947351476	2.25E-06
1600	0.947345189	-4.39E-06	0.947349504	1.66E-07
		$\lambda_A \in [0, 0.7]$		
N	Original	Error	Extension	Error
100	0.947652873	3.20E-04	0.947653469	3.21E-04
200	0.947421782	7.65E-05	0.94742238	7.71E-05
400	0.947365506	1.71E-05	0.947366104	1.77E-05
800	0.947352186	3.00E-06	0.947352785	3.63E-06
1600	0.947349232	-1.21E-07	0.947349831	5.12E-07
		$\lambda_A \in [0, 0.8]$		
N	Original	Error	Extension	Error
100	0.947750114	4.23E-04	0.947750196	4.23E-04
200	0.947446485	1.03E-04	0.947446568	1.03E-04
400	0.947372068	2.40E-05	0.947372151	2.41E-05
800	0.947354214	5.14E-06	0.947354297	5.23E-06
1600	0.947350126	8.23E-07	0.947350209	9.11E-07

Table 5.2
Numerical solutions and errors of and without approximations to the integral term
The 1-year survival probability of firm A with original OS scheme and the extended scheme with extrapolation. The limit of λ_{max} is increasing 0.5 to 0.8 and N is the number of grids in time direction.

domain is truncated with a sufficiently wide finite domain, but for smaller domains the extrapolation has achieved obvious improvement. With a finite domain of $\lambda_{max} = 0.5$, the original errors are around 10^{-5} whereas we are able to achieve results that are around a hundred times more accurate using the approximation. The advantage we gain from using the extrapolation vanishes as λ_{max} grows since the region outside the boundary has less influence on the solution. Clearly the performance will be problem specific but since the computational cost is minimal it seems unwise not to implement the scheme in all cases.

Credit Default Swaps

The solution of a CDS is a concave function with default intensity. It is an increasing function and should be bounded by the default pay-off $(1 - R_A)e^{-r(T-t)} - S\frac{t-T_{n-1}}{T_n-T_{n-1}}$ when default intensity tends to infinity. We remind the reader of the analysis in Section 3.3 that the solution of a CDS can be viewed as the sum of a particular solution and a general solution, $V = V_p + V_g$. The particular solution is the default pay-off and the general solution is a function with exponential decay behaviour against large default intensity. Therefore, we assume a functional form,

$$u(\lambda_A, t) = C(t) - A(t)e^{-B(t)\lambda_A}, \text{ for } \forall\lambda_A \geq \lambda_{max}. \tag{5.32}$$

This function is concave and bounded by $A(t)$ when λ_A tends to infinity if $B(t) > 0$. Now, given the specification of a CDS contract we know that

$$C(t) = (1 - R)e^{-r(T-t)} - S\frac{t - T_{n-1}}{T_n - T_{n-1}}, \tag{5.33}$$

which is the default payoff. Proceeding as in the previous section, parameters $A(t)$ and $B(t)$ can be estimated using a first order one-sided finite difference approximation at the boundary λ_{max}. The analytic expression for the term $y(\lambda_A, t)$ can then be written as

$$\begin{aligned} y(\lambda_A, t) &= \int_{\lambda_{max}-\lambda_A}^{\infty} u(\lambda_A + \beta, t)\nu(\beta)d\beta \\ &= \int_{\lambda_{max}-\lambda_A}^{\infty} \left(C(t) - A(t)e^{-B(t)(\lambda_A+\beta)}\right)\frac{1}{\bar{\beta}}e^{-\frac{1}{\bar{\beta}}\beta}d\beta \\ &= e^{-\frac{1}{\bar{\beta}}(\lambda_{max}-\lambda_A)}\left(C(t) - \frac{\frac{1}{\bar{\beta}}\left(C(t) - u(\lambda_{max}, t)\right)}{\frac{1}{\bar{\beta}} + B(t)}\right), \text{ for } \forall t. \end{aligned} \tag{5.34}$$

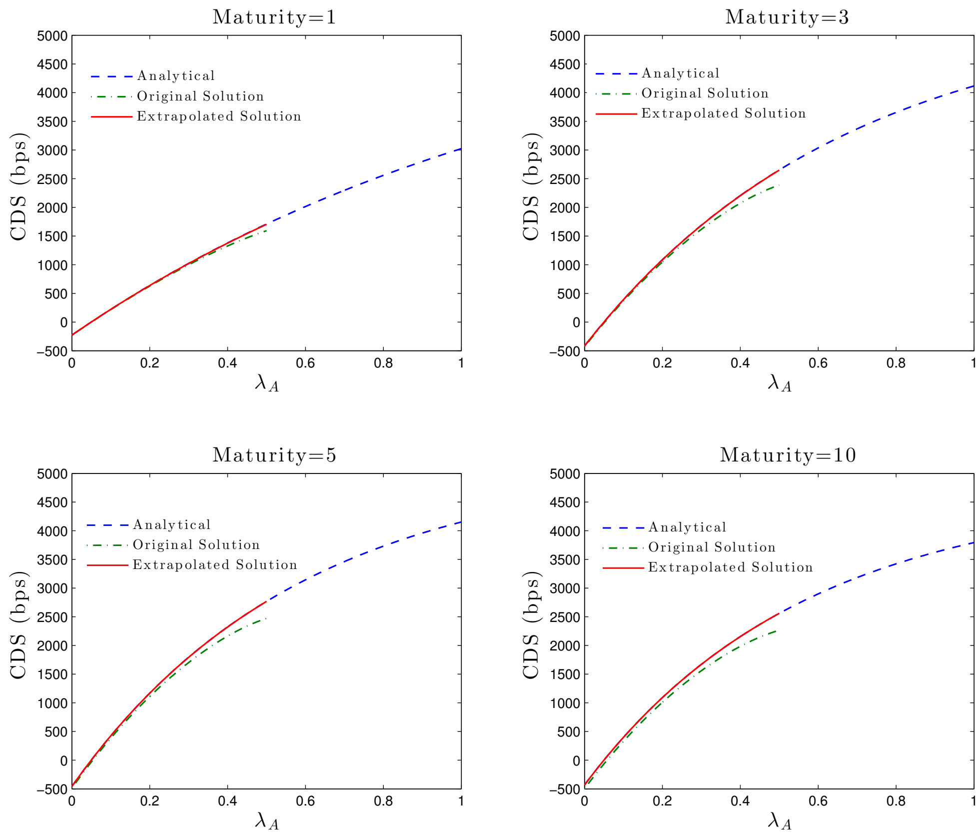

Figure 5.2
Compare CDS extrapolation solutions with ordinary solutions
Parameters are $\kappa_A = 0.5$, $\theta_A = 0.05$, $\sigma_A = 0.1$, $T_n - T_{n-1} = 0.25$, $S = S^*$,$r = 0.02$,$\bar{\lambda} = 0.2$, $\bar{\beta} = 0.1$. Numerical Scheme Parameters are $\Delta t = \frac{1}{100}$, $\Delta\lambda = \frac{1}{1000}$, $\lambda_{max} = 0.5$.

Now we demonstrate this by solving PIDE (5.14) for some CDS contracts using the OS scheme with extrapolations. The boundary conditions for solving (5.14) are described in (3.15). Solutions are compared with ordinary OS scheme without extrapolation and the semi-analytic solution. The fair swap spread S^* is paid quarterly.

Figure 5.2 shows the CDS solutions against intensity with maturities 1, 3, 5 and 10 year. Similar to what we saw in the survival probability, the solution without extrapolation will suffer from a slump near the upper boundary and the slump will be deeper for the longer maturity contracts. This problem is especially severe for CDS contracts as compared to those seen for the survival probability in figure 5.1. As default intensity tends to infinity, survival probability tends to 0 while the CDS value tends to the default payoff, which is a value significantly above zero. Therefore, ignoring the post-jump values above λ_{max} is going to underestimate by a greater amount in the case

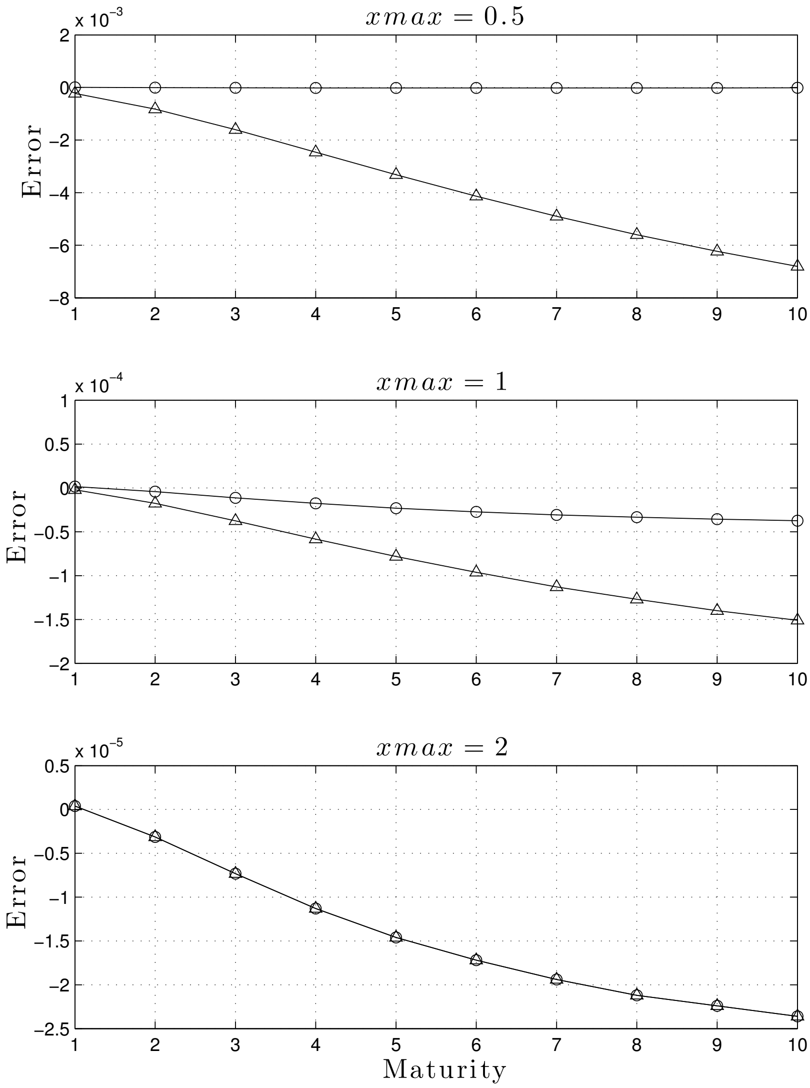

Figure 5.3
Compare errors of with different boundary
An illustration of numerical error with and without extrapolation with different maturities and $xmax$. Triangle lines are numerical solutions without extrapolation and circles are the ones with. Parameters are $\kappa_A = 0.5$, $\theta_A = 0.05$, $\sigma_A = 0.1$, $T_n - T_{n-1} = 0.25$, $S = S^*$,$r = 0.02$,$\bar{\lambda} = 0.2$, $\bar{\beta} = 0.1$. Numerical Scheme Parameters are $\Delta t = \frac{1}{100}$, $\Delta\lambda = \frac{1}{1000}$, $\lambda_{max} = 0.5$, $T = 10$.

of pricing CDSs. Again though, we see that this problem can be fixed by using our extrapolation method, giving a result where the improved numerical solutions more closely align with the semi-analytic solutions.

In order to further analyse the extrapolation method for CDSs, we look at the numerical errors with different contract maturities and λ_{max}, which are shown in figure 5.3. We observed that without the extrapolation, the original OS scheme always underestimates the CDS value for the reason mentioned earlier, and those errors are increasing with maturity. It is obvious that this will happen because when the time horizon is longer, there is higher probability the default intensity will reach a higher level and more numerical errors will be made to the integral term. On the other hand, if we implement the extrapolation method, the numerical errors are significantly lower for all maturities in the cases of $\lambda_{max} = 0.5$ and $\lambda_{max} = 1$. The numerical errors are around 5 decimal places compared to the original one giving only 3 decimals with $\lambda_{max} = 0.5$. In the case of $\lambda_{max} = 2$, the advantage vanishes now as the post-jump intensity greater than 2 is too tiny to make any real difference to the solutions. Finally, table 5.3 compares the computational times and the numerical errors at $\lambda_A = 0.05$ with and without our extrapolation method for solving CDSs up to 10 year maturity. Using our extrapolation method does not lead to computational burden as the computational times do not show any obvious differences. We find that the computational time grows linearly with maturity because the same time step size is maintained. In terms of the numerical errors, the extrapolation method leads to results that are around 100 times more accurate especially for long maturity contracts.

5.3.3 Numerical scheme for 2D-PIDE

Solving for the survival probability, CDS price and CVA with our default contagion model requires us to keep track of the values in two different states $\mathcal{A}$ and $\mathcal{B}$. In the state $\mathcal{A}$, the stochastic variables $(\lambda_A(t), \lambda_B(t))$ must be restricted to a bounded domain $[0, \lambda^{\mathcal{A}}_{A,max}][0, \lambda^{\mathcal{A}}_{B,max}]$. Whilst in the state $\mathcal{B}$, the only variable left is $\lambda^{\mathcal{B}}_{A}(t)$ which must be bounded in $[0, \lambda^{\mathcal{B}}_{A,max}]$ determined by the approach discussed in Section 4.4. Let $(I^{\mathcal{A}}, J^{\mathcal{A}}, N)$ denote the grid size of the three variables $(\lambda_A(t), \lambda_B(t), t)$ in state $\mathcal{A}$. Lastly, we use $(I^{\mathcal{B}}, N)$ grid points for the $(\lambda_A(t), t)$ space in state $\mathcal{B}$.

In the last section, we discussed the numerical solution to a one-dimensional PIDEs

Numerical Error and Computational Time (Mini-Sec)				
Maturity	With Extrapolation		No Extrapolation	
1	1.67×10^{-5}	515.165	-2.74×10^{-4}	495.497
2	7.07×10^{-6}	985.38	-9.51×10^{-4}	979.585
3	-5.30×10^{-6}	1467.974	-1.78×10^{-3}	1417.914
4	-1.50×10^{-6}	1906.661	-2.65×10^{-3}	1914.639
5	-2.14×10^{-6}	2412.062	-3.48×10^{-3}	2423.837
6	-2.50×10^{-6}	2812.037	-4.25×10^{-3}	2856.451
7	-2.67×10^{-6}	3393.944	-4.95×10^{-3}	3316.693
8	-2.73×10^{-6}	3741.818	-5.57×10^{-3}	3855.703
9	-2.70×10^{-6}	4176.838	-6.12×10^{-3}	4312.418
10	-2.65×10^{-6}	4749.687	-6.60×10^{-3}	4736.573

Table 5.3
Computational times and numerical errors for CDS with and without extrapolation method
Parameters are $\kappa_A = 0.5$, $\theta_A = 0.05$, $\sigma_A = 0.1$, $T_n - T_{n-1} = 0.25$, $S = S^*$,$r = 0.02$,$\bar{\lambda} = 0.2$, $\bar{\beta} = 0.1$. Numerical Scheme Parameters are $\Delta t = \frac{1}{100}$, $\Delta\lambda = \frac{1}{1000}$, $\lambda_{max} = 0.5$.

and how the extrapolation technique can be used to improve accuracy. In this section, the finite difference scheme for the two-dimensional PIDE problem for survival probability, CDS value and CVA will be discussed. We apply the ADI scheme, a detailed discussion of which is presented in Section 3.2.2.

The ADI scheme and its variations has been applied to various multi-dimensional problems, such as In't Hout and Foulon (2010); In't Hout and Welfert (2009); Haentjens and In't Hout (2012). However, it is not originally designed for high-dimensional PDEs that include integral terms, which is what we want to solve in this section. Inspired by the OS scheme, here we propose a scheme which will combine the ADI and the OS for our two-dimensional PIDEs, which we term the ADIOS scheme.

The general problem that we wish to solve in this section is described in (5.6), and to solve it we are going to use the following three-step procedure. We denote $u(\lambda_A, \lambda_B, t)$ as the solution to the PIDE (5.6), which can be determined by the specification of functions $f(\lambda_A, \lambda_B, t)$, $g(\lambda_A, \lambda_B, t)$ and the terminal condition $\psi(\lambda_A, \lambda_B, t)$. Again, we drop the super-script $\mathcal{A}$ because we only analyse the numerical schemes. The ADIOS scheme on this equation is given by,

$$\begin{aligned}
\frac{Y_1 - u^{n+1}}{\frac{1}{2}\Delta t} &= \mathcal{L}_{\lambda_A}\{Y_1\} + \mathcal{L}_{\lambda_B}\{u^{n+1}\} + f(\lambda_A, \lambda_B, t) - g(\lambda_A, \lambda_B, t)Y_1 \\
\frac{Y_2 - Y_1}{\frac{1}{2}\Delta t} &= \mathcal{L}_{\lambda_A}\{Y_1\} + \mathcal{L}_{\lambda_B}\{Y_2\} + f(\lambda_A, \lambda_B, t) - g(\lambda_A, \lambda_B, t)Y_1 \\
\frac{u^n - Y_2}{\Delta t} &= \mathcal{J}\{Y_2\},
\end{aligned} \tag{5.35}$$

where Y_1 and Y_2 are both intermediate solutions. The PIDE is split into a pure PDE part and the integral part in a similar way a standard OS scheme in one-dimension. The bit we add in here is to solve the PDE part in the first two steps like a typical ADI scheme. After the PDE terms have been taken care of, the last step is to solve the integral term.

When solving (5.13), (5.16) and (5.17) with the ADIOS scheme, boundary conditions are required for solving the PDE terms. Because the integral term $\mathcal{J}\{u\}$ is temporarily ignored, we only consider boundary conditions for the PDE term. In other words, the heuristic Robin boundary conditions (4.24), (4.26) and (4.27) can be applied for survival probability, CDS price and CVA respectively.

In PIDEs (5.13), (5.16) and (5.17), the integral term associated with λ_B can be generalised by including in the term $f(\lambda_A, \lambda_B, t)$ in (5.6). There are now two integral terms $\mathcal{J}\{u\}$ and $f(\lambda_A, \lambda_B, t)$ in our PIDEs as oppose to just one in the models from Chapter 4. The former one $\mathcal{J}\{u\}$ comes from the external shocks and the latter one $f(\lambda_A, \lambda_B, t)$ is due to the default contagion jumps. The numerical issues that we must consider when computing the integral $f(\lambda_A, \lambda_B, t)$ have been discussed in detail in Section 4.4 and the same methodology applies here. Note that the default contagion can be included in the PDE term and be solved by the ADI scheme because this integral term uses solutions that come from other equations. In other words, computing the integral term $f(\lambda_A, \lambda_B, t)$ does not use $u(\lambda_A, \lambda_B, t)$. The last step in (5.35) is to compute the integral term, which is

$$\begin{aligned}
\mathcal{J}\{Y_2\} &= \bar{\lambda}\int_0^\infty Y_2\big(i\Delta\lambda_A + \beta, j\Delta\lambda_B + \beta, (n+1)\Delta\tau\big)\nu(\beta)d\beta \\
&= \underbrace{\bar{\lambda}\int_0^{(\lambda^{\mathcal{A}}_{A,max} - i\Delta\lambda_A)\wedge(\lambda^{\mathcal{A}}_{B,max} - j\Delta\lambda_B)} Y_2\big(i\Delta\lambda_A + \beta, j\Delta\lambda_B + \beta, (n+1)\Delta\tau\big)\nu(\beta)d\beta}_{\text{Available}} \\
&+ \underbrace{\bar{\lambda}\int_{(\lambda^{\mathcal{A}}_{A,max} - i\Delta\lambda_A)\wedge(\lambda^{\mathcal{A}}_{B,max} - j\Delta\lambda_B)}^\infty Y_2\big(i\Delta\lambda_A + \beta, j\Delta\lambda_B + \beta, (n+1)\Delta\tau\big)\nu(\beta)d\beta}_{\text{Not available}}\,.
\end{aligned} \tag{5.36}$$

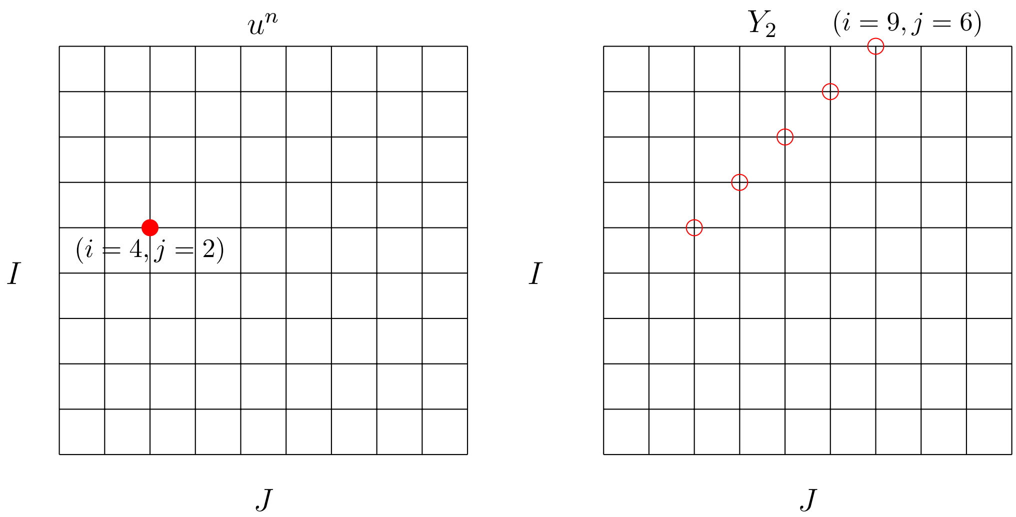

Figure 5.4
An illustration of available solutions for approximating the integral term in two-dimensional PIDE

The upper limit of (5.36), $(\lambda^{\mathcal{A}}_{A,max} - i\Delta\lambda_A) \wedge (\lambda^{\mathcal{A}}_{B,max} - j\Delta\lambda_B)$, is an expression which tells us where the integral term will be truncated given the current position (i, j) in the grid according to the domain of available solutions in Y_2. We illustrates this point in figure 5.4 with an example, where $I = 9$ and $J = 9$ are the number of nodes in the grids in the λ_A and λ_B directions. Assume we are going to calculate the final solution at time $t = n\Delta t$ with current location $(i = 4, j = 2)$, which is $u^n(i, j)$ shown as the red dot in the left panel of figure 5.4. When computing $u^n(i, j)$ by the last step in (5.35), because we assume the jump sizes β are identical to both firms A and B then we only have the solutions in the Y_2 grid to compute $J\{Y_2\}$ term, which are circled in the right panel. The furthest available temporary solution in Y_2 grid is $Y_2(i = 9, j = 6)$ because the current A firm's default intensity λ_A is standing at $i = 4$, which is closer to its upper bound $I = 9$ than the other direction, where λ_B is standing at $j = 2$.

Previously, when computing the integral term $\mathcal{J}\{u\}$ in the one-dimensional PIDE in Section 5.3.1, we meet the same problem of an increasing error when we evaluate the integral for large default intensities λ_A and/or λ_B. In Section 5.3.2, we solved this problem by assuming a functional form for the solution outside the boundary. However, this approach cannot be so easily applied on the integral term $\mathcal{J}\{u\}$ in the two-dimensional PIDEs. Our previous approach relies on knowing a particular form of the solution for the survival probabilities and the CDS value, which is that the solutions have exponentially decaying behaviour. However, in the two-dimensional case, particular solutions are not clear so we cannot know the behaviour of survival probabilities, CDS value or CVA outside the grid. This is because the solutions outside the grid rely on both $\lambda_A(t)$ and $\lambda_B(t)$. Consequently, our only option is to take $\lambda^{\mathcal{A}}_{A,max}$ and $\lambda^{\mathcal{A}}_{B,max}$ to be sufficiently large to capture all effects.

Therefore, equation (5.36) is approximated by

$$\mathcal{J}\{Y_2\} \approx \bar{\lambda} \sum_{k=0}^{K} Y_2\big(i+k, j+k, n+1\big)\nu(k\Delta\beta)\Delta\beta, \text{ for } \forall i, j \in [0, \lambda^{\mathcal{A}}_{A,max}] \times [0, \lambda^{\mathcal{A}}_{B,max}] \tag{5.37}$$

where

$$K = \frac{(\lambda^{\mathcal{A}}_{A,max} - i\Delta\lambda_A) \wedge (\lambda^{\mathcal{A}}_{B,max} - j\Delta\lambda_B)}{\Delta\beta}.$$

5.4 Numerical Results

Now that we have described the numerical schemes and the approaches to accommodate the integral term, we are ready to solve the PIDEs. This section performs some numerical analysis of the survival probabilities, the fair prices of CDSs as well as the CVAs. Firstly, we will look at how external shocks β changes A firm's survival probability compared with the default contagion $\alpha_{A,B}$ at different time horizons and default risk of the two firms. We will show how the survival probabilities can be affected different by the two different kinds of shock. Then we will price CDS contracts referencing to the firm A in five different situations. From the fair CDS spreads, we will discuss the role of external shocks and default contagion shocks on CDS protections with maturities ranging from one to ten years. In addition, the counterparty risk from buying those CDS from firm B will be measured. We attempt to manipulate the strength of external shocks $\bar{\beta}$ and the default contagion $\bar{\alpha}_{A,B}$ so that the price of the CDS stays constant. This result will show us the trade-off between two kinds of risk and the corresponding CVA will be shown to illustrate their influences on CVA for a variety of different values of $\bar{\beta}$ and $\bar{\alpha}_{A,B}$.

5.4.1 Survival probability

Note that, in Section 5.3, we have specified the drift $\mu()$ and volatility $\sigma()$ functions to be (5.19) for the default intensities (5.3). Then the generic PIDE system (5.12) and (5.13) becomes

$$\frac{\partial P_{\mathcal{A}}}{\partial t^-} + \sum_{i\in\{A,B\}} \kappa_i(\theta_i - \lambda_i)\frac{\partial P_{\mathcal{A}}}{\partial \lambda_i} + \frac{1}{2}\sigma_A^2\lambda_A\frac{\partial^2 P_{\mathcal{A}}}{\partial \lambda_A^2} + \frac{1}{2}\sigma_B^2\lambda_B\frac{\partial^2 P_{\mathcal{A}}}{\partial \lambda_B^2} + \rho_{A,B}\sigma_A\sigma_B\sqrt{\lambda_A\lambda_B}\frac{\partial^2 P_{\mathcal{A}}}{\partial \lambda_A\lambda_B}$$
$$+ \mathcal{J}\{P_{\mathcal{A}}\} + \lambda_B\left(\int_0^\infty P_{\mathcal{B}}(\lambda_A + \alpha_{A,B}, t, T)\eta(\alpha_{A,B})d\alpha_{A,B}\right) - \big(\lambda_A + \lambda_B\big)P_{\mathcal{A}} = 0, \tag{5.38a}$$

$$\frac{\partial P_{\mathcal{B}}}{\partial t^-} + \kappa_A(\theta_A - \lambda_A)\frac{\partial P_{\mathcal{B}}}{\partial \lambda_A} + \frac{1}{2}\sigma_A^2\lambda_A\frac{\partial^2 P_{\mathcal{B}}}{\partial \lambda_A^2} + \mathcal{J}\{P_{\mathcal{B}}\} - \lambda_A P_{\mathcal{B}} = 0. \tag{5.38b}$$

The distributions of external shocks β and default contagion jumps $\alpha_{A,B}$ and $\alpha_{B,A}$ are modelled as (5.20) and (5.21).

The numerical scheme and corresponding boundary conditions for solving (5.38b) are discussed in Section 5.3.2 with an extrapolation method to improve numerical accuracy. For solving the PIDE (5.38a), we use the ADIOS scheme introduced in Section 5.3.3. As mentioned in Section 5.3.3, ADIOS scheme solves a PIDE by solving PDE part and integral part separately, this enable us to apply the boundary conditions used for the PDE of survival probability in Section 4.4.1, which is (4.24). After adjusting the notations in (4.24), we have the following boundary conditions for (5.38a).

$$
\begin{aligned}
\lambda_A \to 0 \; : & \\
& \left(\frac{\partial}{\partial t^-} + \mathcal{L}_{\lambda_B} + \kappa_A \theta_A \frac{\partial}{\partial \lambda_1}\right) P_{\mathcal{A}} - \lambda_B P_{\mathcal{A}} \\
& + \lambda_B \left(\int_0^\infty P_{\mathcal{B}}(\lambda_A + \alpha_{A,B}, t, T) \eta(\alpha_{A,B}) d\alpha_{A,B} \right) = 0 \\
\lambda_A \to \infty : & \\
& \left(\frac{\partial}{\partial t^-} + \mathcal{L}_{\lambda_B} + \kappa_A \theta_A \frac{\partial P_{\mathcal{A}}}{\partial \lambda_A}\right) P_{\mathcal{A}} - (\lambda_A + \lambda_B) P_{\mathcal{A}} \\
& + \lambda_B \left(\int_0^\infty P_{\mathcal{B}}(\lambda_A + \alpha_{A,B}, t, T) \eta(\alpha_{A,B}) d\alpha_{A,B} \right) = 0 \qquad (5.39) \\
\lambda_B \to 0 : & \\
& \left(\frac{\partial}{\partial t^-} + \mathcal{L}_{\lambda_A} + \kappa_B \theta_B \frac{\partial P_{\mathcal{A}}}{\partial \lambda_B}\right) P_{\mathcal{A}} - \lambda_A P_{\mathcal{A}} = 0, \\
\lambda_B \to \infty : & \\
& \left(\frac{\partial}{\partial t^-} + \mathcal{L}_{\lambda_A} + \kappa_B \theta_B \frac{\partial P_{\mathcal{A}}}{\partial \lambda_2}\right) P_{\mathcal{A}} - (\lambda_A + \lambda_B) P_{\mathcal{A}} \\
& + \lambda_B \left(\int_0^\infty P_{\mathcal{B}}(\lambda_A + \alpha_{A,B}, t, T) \eta(\alpha_{A,B}) d\alpha_{A,B} \right) = 0.
\end{aligned}
$$

In our model, default correlation among firms come from two different sources. One is the external shocks driven by $J(t)$, which affects both firms A and B with the same random size β. The other is the default contagion between firms A and B with strength $\alpha_{A,B}$ and $\alpha_{B,A}$. If we raises the expected strength of default contagion $\bar{\alpha}_{A,B}$, or the external shocks through its shock arrival frequency $\bar{\lambda}$ or the strength $\bar{\beta}$, firm A must be less likely to survive. Figure 5.5 shows how default contagion $\bar{\alpha}_{A,B}$ and the external shocks reduce firm A's survival probability. The reduction in firm A's survival probability is shown against its intensity $\lambda_A^{\mathcal{A}}$ and firm B's intensity $\lambda_B^{\mathcal{A}}$. According to figure 5.5, how much amount of firm A's default probability will be influenced by the default contagion or the external shocks is depending on how risky firm A is. If firm A is very risky, this implies neither the default contagion nor the external shocks are likely to happen before it defaults. Therefore, when firm A's intensity is increasing, both shocks make no difference at all to A's survival probability, which shown in left panel of figure 5.5. Clearly, whether there is an occurrence of external shock relies on $\bar{\lambda}$ but not firm B's credit. So the red line in the right panel is independent of $\lambda_B^{\mathcal{A}}$. On the other hand, if firm B is more risky it will obviously make it more likely for

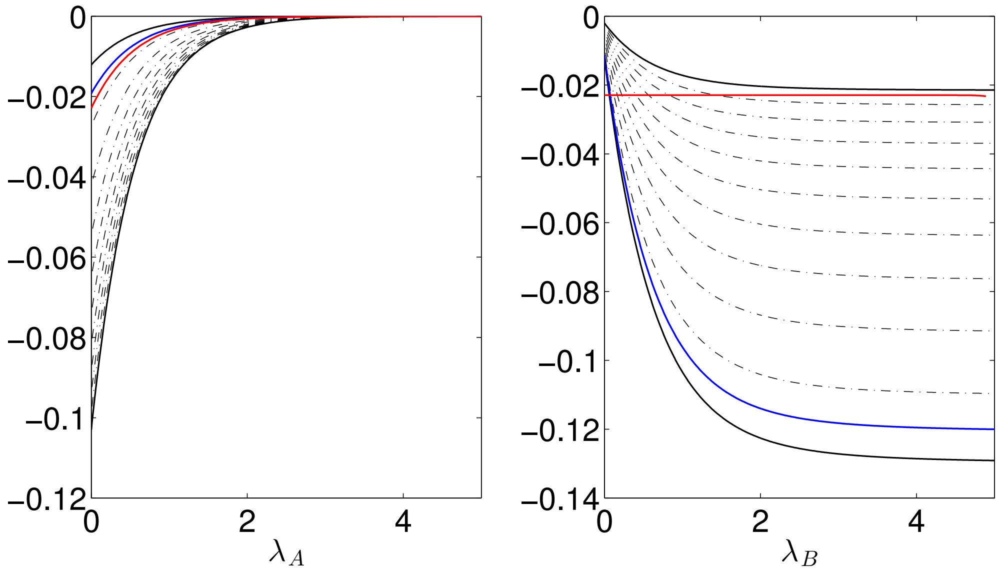

Figure 5.5
Firm A survival probability decrement with default contagion $S^{\mathcal{A}}(\bar{\alpha}_{A,B} = 0.1) - S^{\mathcal{B}}(\bar{\alpha}_{A,B} = 0)$ (black) and exogenous jumps $S^{\mathcal{A}}(\bar{\beta} = 0.1) - S^{\mathcal{B}}(\bar{\beta} = 0)$ (red)
An illustration of how survival probability decreases with only either default contagion (black lines) or exogenous jumps (red). The left panel shows solutions in $\lambda_A^{\mathcal{A}}$-direction with $\lambda_B^{\mathcal{A}}$ raising from 0 to 3 (dashed) and the right panel is $\lambda_B^{\mathcal{A}}$-direction with $\lambda_A^{\mathcal{A}}$ raising from 0 to 3 (dashed). Default correlation parameters are $\bar{\alpha}_{A,B} = 0.1$, $\bar{\beta} = 0$, $\bar{\lambda} = 0$ or $\bar{\alpha}_{A,B} = 0$, $\bar{\beta} = 0.1$, $\bar{\lambda} = 0.05$. Other parameters are $\kappa_A = \kappa_B = 0.5$, $\theta_A = \theta_B = 0.05$, $\sigma_A = \sigma_B = 0.1$.

the default contagion to affect firm A so that firm A is more likely to default. In our numerical example, although the likelihood of external shock and default contagion could happen is similar, which is θ_B equals to $\bar{\lambda}$, firm A's survival probability reduces more due to default contagion than to external shocks when firm B is currently more risky.

In our model, there is a similarity between the default contagion and external shocks which is that their jump sizes are both modelled to be exponentially distributed. One question to ask is whether simply increasing the pure jump intensity in a jump-CIR model, $\bar{\alpha}_{A,B} = 0$, can achieve the same effects on one's default term structure as we see by including default contagion. If so, it may not be worth the computational effort to incorporate the default contagion at least for survival probabilities. Figure 5.6 shows the trade-off between of both kinds of jump on firm A's default term-structure. Compared to the benchmark, where $\bar{\alpha}_{A,B} = 0$, $\bar{\beta} = 0.05$, $\bar{\lambda} = 0.5$, we either raise default contagion strength $\bar{\alpha}_{A,B}$ from 0 to 0.05 or the external shocks arrival rate $\bar{\lambda}$ from 0.5 to 0.55, which are two comparable cases since the increment of external

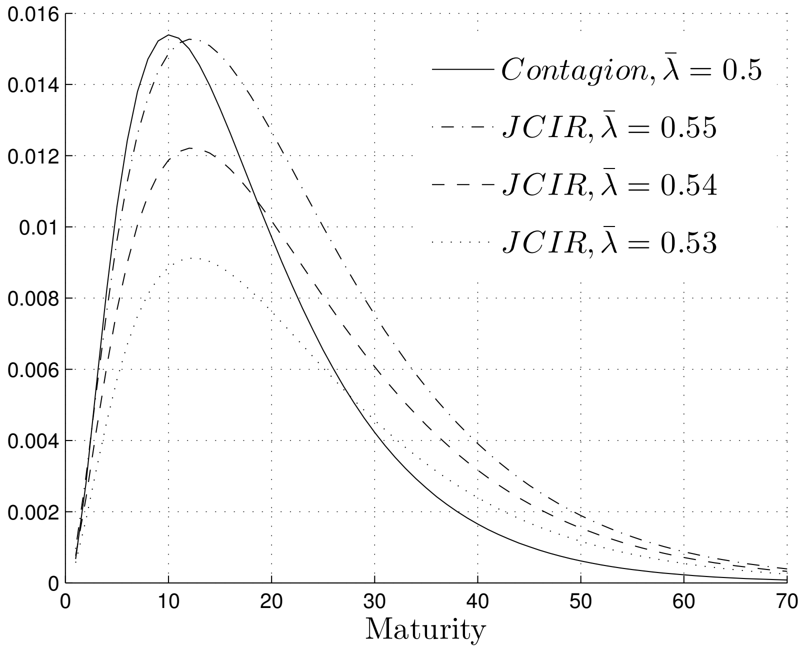

Figure 5.6
Increment in default probability term-structure in four cases.
Increment of firm A's default probability term structure against benchmark case $\bar{\alpha}_{A,B} = 0$, $\bar{\beta} = 0.05$, $\bar{\lambda} = 0.5$. Case 1: $\bar{\lambda} = 0.5$ $\bar{\alpha}_{A,B} = 0.05$ (Solid line). Case 2: $\bar{\lambda} = 0.55$, $\bar{\alpha}_{A,B} = 0$ (Solid line). Case 3: $\bar{\lambda} = 0.54$ $\bar{\alpha}_{A,B} = 0$. Case 4: $\bar{\lambda} = 0.53$ $\bar{\alpha}_{A,B} = 0$

jump's arrival rate equals to firm B's long-term default rate $\theta_B = 0.05$. Besides this, we also show more cases for different external shocks arrival rate to fully develop the picture. Obviously, in the long term, the firm A is more likely to default with higher external shocks arrival rate than with a single default contagion even if we choose a lower arrival rate $\bar{\lambda} = 0.53$. This phenomenon can be explained by the fact that the external shock can occur multiple times whilst there is only one chance that firm B can default and cause a default contagion to firm A. Therefore, in the long run, external shocks have stronger influences on firm A than the default contagion.

On the other hand, from 1 to 12 years horizon, having default contagion risk is different from simply raising the external shocks arrival rate $\bar{\lambda}$. Incorporating the default contagion makes firm A to be more risky in the in 12 years horizon compared to more frequent external shock arrivals. The reason is the probability, which firm B defaults earlier than firm A, is rising faster against maturity with external shocks. External shocks make both firms A and B more likely to default and so is the probability that B defaults earlier than A. For all three cases with only external shocks we see that their maximum impact are all on the 12 year horizon. However, to which time horizon that default contagions have the strongest effect is depending on both firms' risk profile as well as the external shocks. This characteristic also implies the effects from default contagion cannot be simply replicated by more frequent external shocks rate.

5.4.2 Fair swap spread

When specifying the drift $\mu()$ and volatility $\sigma()$ functions as (5.19), the PIDE system (5.14) and (5.16) becomes

$$\begin{aligned}&\frac{\partial V_{\mathcal{A}}}{\partial t^-}+\sum_{i\in\{A,B\}}\kappa_i(\theta_i-\lambda_i)\frac{\partial V_{\mathcal{A}}}{\partial \lambda_i}+\frac{1}{2}\sigma_A^2\lambda_A\frac{\partial^2 V_{\mathcal{A}}}{\partial \lambda_A^2}+\frac{1}{2}\sigma_B^2\lambda_B\frac{\partial^2 V_{\mathcal{A}}}{\partial \lambda_B^2}+\rho_{A,B}\sigma_A\sigma_B\sqrt{\lambda_A\lambda_B}\frac{\partial^2 V_{\mathcal{A}}}{\partial \lambda_A\lambda_B}\\&+\lambda_A\left((1-R_A)e^{-r(T-t^-)}-S\frac{t^--T_{n-1}}{T_n-T_{n-1}}\right)+\lambda_B\int_0^\infty V_{\mathcal{B}}\big(\lambda_A+\alpha_{A,B},t^-\big)\eta(\alpha_{A,B})d\alpha_{A,B}\\&+\mathcal{J}\{V_{\mathcal{A}}\}-\big(r+\lambda_A+\lambda_B\big)V_{\mathcal{A}}=0\end{aligned} \tag{5.40a}$$

$$\begin{aligned}&\frac{\partial V_{\mathcal{B}}}{\partial t^-}+\kappa_A(\theta_A-\lambda_A)\frac{\partial V_{\mathcal{B}}}{\partial \lambda_A}+\frac{1}{2}\sigma_A^2\lambda_A\frac{\partial^2 V_{\mathcal{B}}}{\partial \lambda_A^2}+\lambda_A\left((1-R_A)e^{-r(T-t^-)}-S\frac{t^--T_{n-1}}{T_n-T_{n-1}}\right)\\&+\mathcal{J}\{V_{\mathcal{B}}\}-(r+\lambda_A)V_{\mathcal{B}}=0\end{aligned} \tag{5.40b}$$

for $t\in(T_{n-1},T_n)$ with $n=1,2,....N$ and terminal condition $V_{\mathcal{A}}(\lambda_A,\lambda_B,T,T)=0$, $V_{\mathcal{B}}(\lambda_A,T,T)=0$ and jump conditions

$$\begin{aligned}V_{\mathcal{A}}(T_n^-,T)&=V_{\mathcal{A}}(T_n^+,T)-S(T_n-T_{n-1})\\V_{\mathcal{B}}(T_n^-,T)&=V_{\mathcal{B}}(T_n^+,T)-S(T_n-T_{n-1}),\end{aligned}$$

at $n=1,2,....N$. The distributions of external jumps β and default contagion jumps $\alpha_{A,B}$ and $\alpha_{B,A}$ are modelled as (5.20) and (5.21).

The numerical scheme and corresponding boundary conditions for solving (5.40b) is illustrated in Section 5.3.2. The two-dimensional PIDE (5.40a) is solved using ADIOS scheme with the heuristic Robin boundary conditions

$$
\begin{aligned}
\lambda_A \to 0\ :&\\
&\left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_B} + \kappa_A\theta_A\frac{\partial}{\partial \lambda_A}\right)V^{\mathcal{A}} + \lambda_B\left(\int_0^\infty V^{\mathcal{B}}(\lambda_A+\alpha_{A,B},t,T)\eta(\alpha_{A,B})d\alpha_{A,B}\right)\\
&-(r+\lambda_B)V^{\mathcal{A}} = 0,\\
\lambda_A \to \infty :&\\
&\left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_B} + \kappa_A\theta_A\frac{\partial}{\partial \lambda_A}\right)V^{\mathcal{A}} + \lambda_B\left(\int_0^\infty V^{\mathcal{B}}(\lambda_A+\alpha_{A,B},t,T)\eta(\alpha_{A,B})d\alpha_{A,B}\right)\\
&+\lambda_A\left((1-R_A)e^{-r(T-t)} - S\frac{t-T_{n-1}}{T_n-T_{n-1}}\right) - (r+\lambda_B+\lambda_A)V^{\mathcal{A}} = 0,\\
\lambda_B \to 0 :&\\
&\left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_A} + \kappa_B\theta_B\frac{\partial}{\partial \lambda_B}\right)V^{\mathcal{A}} + \lambda_A\left((1-R_A)e^{-r(T-t)} - S\frac{t-T_{n-1}}{T_n-T_{n-1}}\right)\\
&-(r+\lambda_A)V^{\mathcal{A}} = 0,\\
\lambda_B \to \infty :&\\
&\left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_A} + \kappa_B\theta_B\frac{\partial}{\partial \lambda_B}\right)V^{\mathcal{A}} + \lambda_B\left(\int_0^\infty V^{\mathcal{B}}(\lambda_A+\alpha_{A,B},t,T)\eta(\alpha_{A,B})d\alpha_{A,B}\right)\\
&+\lambda_A\left((1-R_A)e^{-r(T-t)} - S\frac{t-T_{n-1}}{T_n-T_{n-1}}\right) - (r+\lambda_B+\lambda_A)V^{\mathcal{A}} = 0.
\end{aligned}
\tag{5.41}
$$

The condition (5.41) is the same as (4.26) except the notation difference because the ADIOS scheme enable us to use the (4.26) as boundary conditions for the PDE part component of the PIDE. For the validation of using the heuristic type Robin boundary conditions we refer to Section 3.3.2.

Remind that the value of a payer CDS contract under our jump-diffusion-contagion model is the solution to the PIDE system (5.14) and (5.16). The fair swap rate or spread can be obtained according to (2.58), which is the ratio of the default payment's and swap premiums' present value. This requires us to solve the PIDE system with $S = 0$ and $(R_A = 1,\ S = 1)$ respectively then taking the ratio. The fair swap spread S^* is computed following this procedure.

In this section, we compare the default correlation effects from external shocks and default contagion on the fair CDS spread S^*. Due to the default contagion effects among companies, the price of the CDS protection not only relies solely on referencing firm's default risk, but also on other interacting firms, which is the firm B in our case.

Cases	$\bar{\lambda}$	$\bar{\beta}$	$\bar{\alpha}_{A,B}, \bar{\alpha}_{B,A}$
1	0.05	0.1	0
2	0.05	0.05	0.05
3	0	0	0.1
4	0.05	0.1	0.05
5	0	0	0

Table 5.4
Parameters of five cases

We specify five cases as in table 5.4 that have been chosen to highlight the different effects.

We have chosen case 1 to be without default contagion but it has a higher external shocks strength $\bar{\beta}$. For case 2, the external shock strength is reduced to 0.05 but we introduce the default contagion shocks as 0.05. In case 3, we eliminate external shocks but increase the strength of default contagion shocks. These three cases each have a relatively similar strength of jumps but they come from different sources, so they are able to show the different effects of external shocks and default contagion have on the CDS prices. In addition, case 4 has stronger default contagion shock $\bar{\alpha}_{A,B}$ than case 1 and stronger external shocks $\bar{\beta}$ than case 2, which shows the impact from only raising default contagion shock or external shocks. Finally, the case 5, there are no external shocks or default contagion shocks, which is the simplest model. Note that, case 5 is equivalent to the model we implemented in Chapter 3 and the case 3 is the default contagion model in Chapter 4. By comparing the difference cases, we are enable to identify the difference between default contagion and external shocks on CDS spread and CVA.

Figure 5.7 displays the fair CDS spread S^* of the five cases in table 5.4. The CDS spreads in case 5 are significantly lower than other cases because the firm A is with the lowest default risk in this case. The difference between case 5 and others are the most significant for the long-term contracts because both external shocks and default contagion shocks are more likely to occur with longer time. The CDS spreads in cases 1 and 2 are similar for short maturities but the spread is higher in case 1 than case 2 for the long-term contracts. This phenomenon coincides with our observation in figure 5.6 that default contagions have much weaker impacts on long-term default probability than external shocks. At last, case 4 has the highest CDS spreads due to the fact its

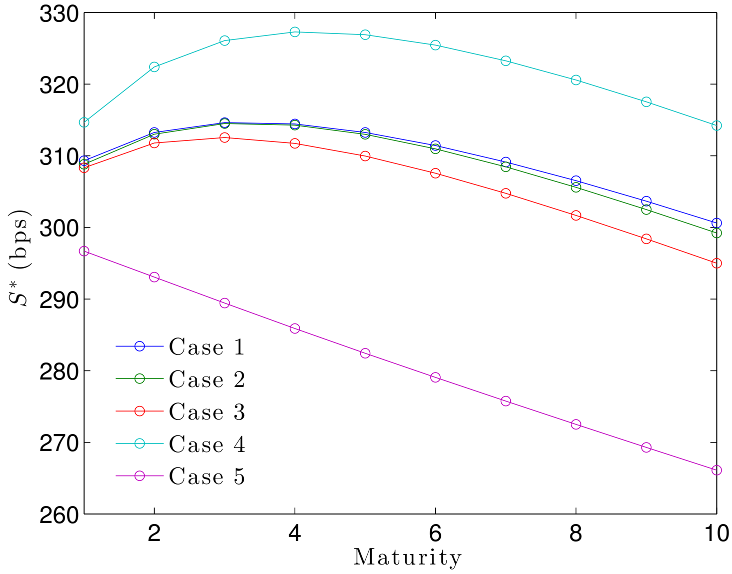

Figure 5.7
The fair CDS spreads S^* of five cases in table 5.4
Other parameters are $\kappa_A = 0.5$, $\theta_A = 0.05$, $\lambda_A(0) = 0.05$, $T_n - T_{n-1} = 0.25$, $r = 0.02$.

external shocks and default contagion shocks are stronger than the others.

In our model, the default risk of a company can be represented by the different jump sizes $\bar{\alpha}$ and $\bar{\beta}$. So a CDS referencing to the firm A may have the same price but it could be due to different risk components. In figure 5.8, we show the trade-off between the two kinds of jump risk using a five-year contract as an example. By reducing $\bar{\beta}$ while raising $\bar{\alpha}$ at the same time, the default of firm A becomes more likely due to default contagions rather than external shocks. In the two extreme cases where only one kind of jump is used, we notice that the expected size of default contagion jump $\bar{\alpha}_{A,B}$ has to be higher than the external shock strength $\bar{\beta}$ in order to achieve the same CDS spread. This is due to the fact that the default contagion is a one-time only event in our two-firm model, but there could be many more than one external shocks arriving.

5.4.3 Credit value adjustment

In the previous section, we priced CDS contracts referencing to the firm A whose credit risk could be affected by firm B's default. The CDS contracts are priced without consideration of the CDS seller's default risk. We now consider the default risk of the CDS seller, which is also the firm B, and compute the CVAs of those CDSs. Note that

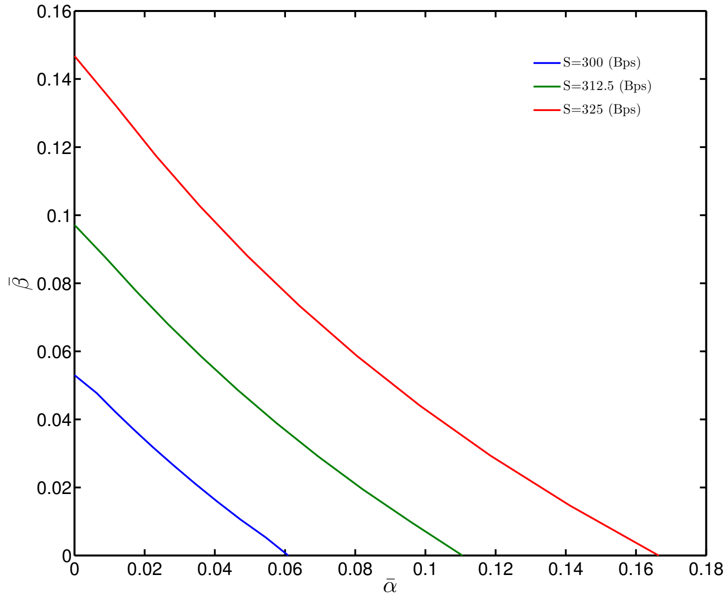

Figure 5.8
Trade-off between $\bar{\beta}$ and $\bar{\alpha}_{A,B}$ with same CDS protection spread.
This figure shows the combinations of external shocks $\bar{\beta}$ and default contagion $\bar{\alpha}_{A,B}$ such that a 5 year CDS contract is fair at the annual spread of $300bps$, $312.5bps$ and $325bps$. Other parameters are $\kappa_A = \kappa_B = 0.5$, $\theta_A = \theta_B = 0.05$, $\lambda_A(0) = \lambda_B(0) = 0.05$, $\sigma_A = \sigma_B = 0.1$, $T_n - T_{n-1} = 0.25$, $r = 0.02$.

we calculate CVA based on the spreads given in the last section.

With the jump-diffusion-contagion model, we showed the CVA is a solution of the generic PIDE (5.17). With the specifications of (5.19), (5.17) becomes

$$\begin{aligned}&\frac{\partial U}{\partial t^-} + \sum_{i\in\{A,B\}} \kappa_i(\theta_i - \lambda_i)\frac{\partial V_{\mathcal{A}}}{\partial \lambda_i} + \frac{1}{2}\sigma_A^2\lambda_A\frac{\partial^2 V_{\mathcal{A}}}{\partial \lambda_A^2} + \frac{1}{2}\sigma_B^2\lambda_B\frac{\partial^2 V_{\mathcal{A}}}{\partial \lambda_B^2} + \rho_{A,B}\sigma_A\sigma_B\sqrt{\lambda_A\lambda_B}\frac{\partial^2 V_{\mathcal{A}}}{\partial \lambda_A\lambda_B}\\ &+ \lambda_B\int_0^\infty (1-R_B)\max\{V_{\mathcal{B}}(\lambda_A + \alpha_{A,B}, t, T)\}\frac{1}{\bar{\alpha}_{A,B}}e^{-\left(\alpha_{A,B}/\bar{\alpha}_{A,B}\right)}d\alpha_{A,B}\\ &+ \mathcal{J}\{U\} - (r + \lambda_A + \lambda_B)U = 0,\end{aligned} \tag{5.42}$$

where $V_{\mathcal{B}}(\lambda_A + \alpha_{A,B}, t, T)$ is the solution of (5.40b) and the external shocks β and contagion shocks $\alpha_{A,B}$ and $\alpha_{B,A}$ are described as (5.20) and (5.21).

We solve (5.42) for the CVAs using ADIOS scheme with the five cases in Table 5.4. Following the same reason as survival probability and CDS value in Sections 5.4.1 and

5.4.2, we have the boundary conditions for PIDE (5.42) as

$$
\begin{aligned}
&\lambda_A \to 0 \ : \\
&\qquad \left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_B} + \kappa_A\theta_A\frac{\partial}{\partial\lambda_A}\right)U - (r+\lambda_B)U \\
&\qquad + \lambda_B\left((1-R_B)\int_0^\infty \max\{V^{\mathcal{B}}(\lambda_A(t)+\alpha,t,T),0\}\eta(\alpha_{A,B})d\alpha_{A,B}\right) = 0, \\
&\lambda_A \to \infty : \\
&\qquad \left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_B} + \kappa_A\theta_A\frac{\partial}{\partial\lambda_A}\right)U - (r+\lambda_A+\lambda_B)U \\
&\qquad + \lambda_B\left((1-R_B)\int_0^\infty \max\{V^{\mathcal{B}}(\lambda_A(t)+\alpha_{A,B},t,T),0\}\eta(\alpha_{A,B})d\alpha_{A,B}\right) = 0, \\
&\lambda_B \to 0 : \\
&\qquad \left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_A} + \kappa_B\theta_B\frac{\partial}{\partial\lambda_B}\right)U - (r+\lambda_A)U = 0, \\
&\lambda_B \to \infty : \\
&\qquad \left(\frac{\partial}{\partial t} + \mathcal{L}_{\lambda_A} + \kappa_B\theta_B\frac{\partial}{\partial\lambda_B}\right)U - (r+\lambda_A+\lambda_B)U \\
&\qquad + \lambda_B\left((1-R_B)\int_0^\infty \max\{V^{\mathcal{B}}(\lambda_A(t)+\alpha_{A,B},t,T),0\}\eta(\alpha_{A,B})d\alpha_{A,B}\right) = 0.
\end{aligned}
\tag{5.43}
$$

Figure 5.9 displays the CVA of the five cases in table 5.4. The CVA is computed based on the protection buyer is paying the spreads in figure 5.7. Compared to external shocks, default contagion shocks have significantly stronger impacts on the CVA in all five cases. The CDS buyer will have to pay higher CDS spreads when the external shocks are stronger but the counterparty risk faced by the buyer is far less sensitive to external shocks compared with default contagion shocks. In the last section, figure 5.7 shows that the CDS prices S^* are very close in case 1 and 2 for short-term CDSs and the difference are tiny in long-term CDSs. However, the CVA in case 2, which is with the default contagion shock, is two times higher than in case 1, which is without default contagion shock. Although the CDS spreads are lower in the case 3 than cases 1 and 2, the CVA in the case 3 is the highest among all cases. These results indicate that CVA is dominated by the default contagions rather than external shocks. Remember that the CVA considers both the probability of counterparty defaults earlier than the referencing firm and the loss given the counterparty defaulted. We discussed in Section 4.4.3 how the default contagion shocks only influence CVA in terms of loss

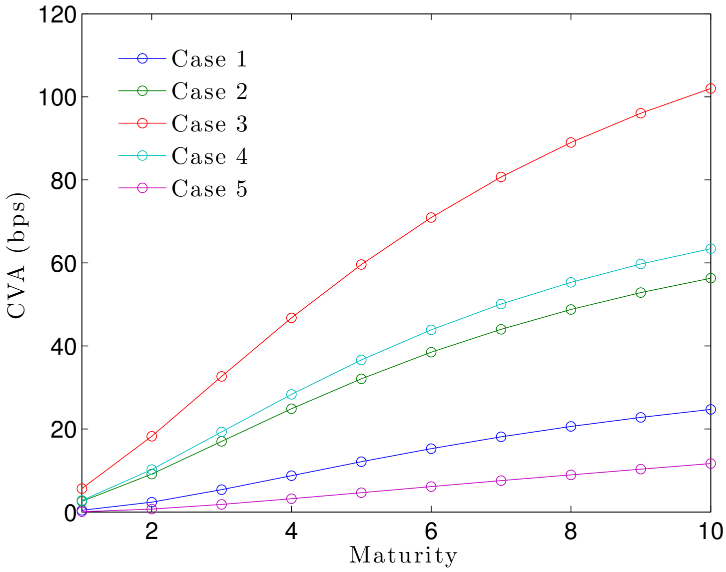

Figure 5.9
The CVA of five cases in table 5.4
The CVA are evaluated using CDS spreads in figure 5.7. Other parameters are $\kappa_A = 0.5$, $\theta_A = 0.05$, $\lambda_A(0) = 0.05$, $T_n - T_{n-1} = 0.25$, $r = 0.02$.

given counterparty defaults, which is raising the CDS value at the CDS seller's default time. External shocks have no impacts on the CDS value at the default time because external shocks are not necessarily happen at the time of counterparty defaults. On the other hand, the stronger external shocks leads to all firms more likely to default so they only effect the probability that the counterparty credit event will happen. In case 4 we have increase $\bar{\beta}$ by 0.05 compared to that in case 2, which implies there will be higher probability of counterparty default event. However, the CVA to the longest maturity contract in case 4 only has around 6 *bps* higher than in the case 2. In other words, of the five cases, cases 1 and 4 have the same probability that the counterparty defaults, which is the highest among the five cases, and the case 3 has the lowest probability. But our numerical results suggest that the CVA is mainly driven by the losses given default, which will be affected by default contagion risk, rather than the counterparty default probability, which is controlled by the external shocks.

In order to further illustrate the difference effects between default contagion shocks and external shocks on CVA, figure 5.10 displays the CVAs of some 5 year CDS contracts, where the prices are identical but subject to different strength of external shocks $\bar{\beta}$ and default contagion shocks $\bar{\alpha}_{A,B}$, the relationship between them is shown

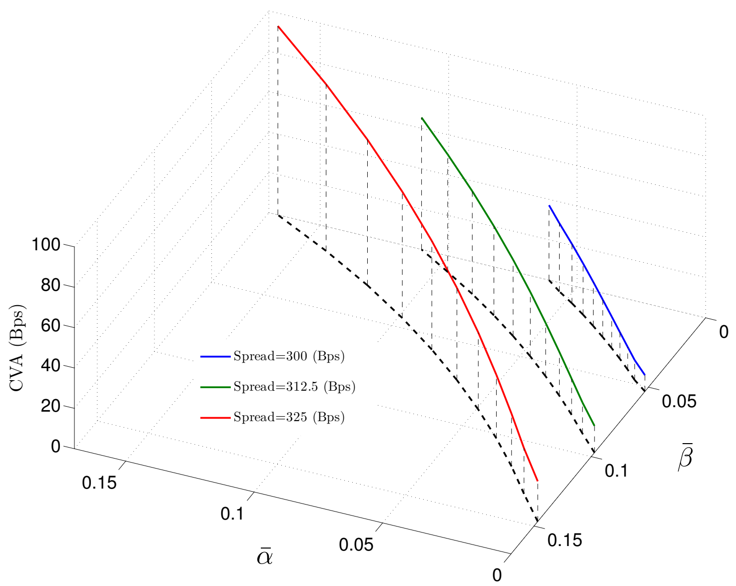

Figure 5.10
CVA under combinations of external shocks $\bar{\beta}$ and default contagions $\bar{\alpha}$ as shown previously in figure 5.8

in figure 5.8. According to figure 5.10, even though the all of the protections along the line have the same CDS prices, the amount of the counterparty risk the buyer faces varies a lot according to its source of default risk. In our examples, if the referencing firm's 5-year CDS is selling for 325 *bps* (paid quarterly) and the referencing firm's default risk is largely due to the default contagion from the CDS seller, the amount of CVA is around 93 *bps*, compared to just 20 *bps* when the referencing firm's default risk is mainly due to external shocks. The reason for this is that when the referencing firm's credit worthiness is deteriorated by the counterparty's default, the referencing firm's default intensity surges to a very high level at that moment. The consequence is to make the cost of replacing the protection very high. In other words, there is an enormous amount of loss for the protection buyer if the protection seller defaults.

5.5 Conclusion

We propose a reduced-form framework with default contagion and exogenous shocks in order to model firms that are shocked by other firms' credit events and economy-wide events. The model is applied to the valuation of default probabilities, CDS and CVA.

By modelling in this way we end up with one- and two-dimensional PIDEs to solve for the valuation problems. The two-dimensional PIDEs are tackled using a combination of ADI and the operator splitting finite-difference scheme, which we term the ADIOS scheme. In addition, extrapolation methods are invented when solving the one-dimensional PIDEs for survival probabilities and CDS values. The underestimation problems in computing the integral terms are resolved efficiently leading to results that are up to 100 times more accurate with the extrapolation method with hardly any additional computational burden.

In term of a firm's default/survival term-structure, modelling default contagions is different from simply including external shocks. The default contagion shocks have stronger impacts on the short-term survival probability and their effects depend on the relative riskiness of the firm itself and the firms whose defaults have contagion effects on the firm. On the other hand, how likely it is that external shocks arrive is steady at rate $\bar{\lambda}$. Given that the external shocks can arrive at multiple times, this will have a stronger impact on a firm's long-term survival probability. Consequently, the variations to one's default term-structure could be more diverse by including both default contagion and external shocks.

We price CDSs referencing to firm A whose default risk may be deteriorated by firm B's default. Comparing five cases with a variety of default contagion shocks and external shocks, we show that the protection prices are more sensitive to changes in external shocks, especially for the longer maturity contracts. External shocks make all the firms more likely to default thus enhance the probability of default contagion happening. Consequently, the CDS prices become more sensitive to default contagions as well if there are stronger and more frequent external shocks. The CVA of the five different cases show that the CVA is dominated by default contagion risk, which determines losses given default at counterparty's default time, rather than external shocks.

A comparison study is carried out on the CVA of three different 5-year CDS protections where each has the same fair spread but the referencing firm's default intensity is subject to different strengths of default contagion and external shocks. Given the tremendous difference in the CVA under our scenarios, numerical results suggest that appropriately modelling default contagions between the referencing firm and the counterparty is crucial important in the calculation of CVA. Even though the default risk of the referencing firm is identical, the CDS buyer is facing substantially more counterparty risk if the default risk of the referencing firm is mainly attributed to the default contagion from the counterparty. Although the external shocks shared by the two firms also create default correlations between them, the CVA with only default correlations created by external shocks can be up to five times less than the case that with only default contagion.

Chapter 6

Bilateral Counterparty Risk with Default Contagion Model

In Chapters 4 and 5 we discussed the counterparty risk that a CDS investor will face when buying a CDS protection from a seller who can default and whose default will increase the CDS value. However, those investigations were based on the assumption that the investor to be default-free. This assumption may not be appropriate when the CDS buyer has a similar credit rating compared to the seller so that the scenario where the buyer defaults earlier than the seller cannot be ignored. When the two companies are negotiating an OTC trade, it is essential that both sides consider the counterparty's default probability as well as their own. Otherwise, it is difficult to arrive at a price that both sides will agree on.

In this chapter, the default of the CDS buyer is allowed and our default contagion model is extended to consider the interactions amongst the three firms, namely the referencing firm, the CDS seller and the CDS buyer. We are able to analyse the fair spread of a CDS in this new environment as well as the corresponding counterparty credit risk measures, which are CVA and DVA. The cost of introducing another default party is to increase the dimensionality of our problem to three dimensions. For example, the value of a CDS is now dependent on three stochastic processes, so a numerical scheme for a high-dimensional problem is required. Although finite-difference has the advantage of higher order convergence, the computational time grows exponentially with the dimension of the problem. So in order to overcome this numerical obstacle, we propose a novel scheme in which we break down the CVA and DVA valuation problem into two components, which can be solved by simulation and finite-difference respectively. The resulting scheme will be a hybrid simulation and finite-difference scheme that is shown to be particularly useful for the valuation problems presented in this chapter.

The chapter will be arranged as follows. First, we address the preliminary principles of Monte-Carlo simulation in Section 6.1 including the methodology to simulate multi-company default times and the stochastic intensity process. A formal model formulation is given in Section 6.2 followed by the descriptions of pricing algorithms in Section 6.3. Finally, we present a full set of numerical results in order to analyse the CDS spreads, CVA and DVA. We also discuss the convergence of the numerical scheme as well as the impacts on counterparty risk by introducing CDS buyer's default event and parameter sensitivities of the CVA and DVA.

6.1 Principle of Monte-Carlo

Before we move onto describing the full hybrid scheme, we present some fundamental theorems of Monte-Carlo to illustrate the principles that support the numerical method.

6.1.1 Law of large numbers

Consider a random variable X with unknown expectation $\mathbb{E}[X] = \mu$ and variance $Var[X] = \sigma^2$. If we have a sequence of samples X_1, X_1,..., X_n, which are independent variables with the same distribution as X, then we expect the estimator $\hat{\mu}$

$$\hat{\mu} := \frac{1}{n}\sum_{i=1}^{n} X_i \tag{6.1}$$

to be the unbiased estimator of mean, that is, its expectation aligns with the original mean value, which is

$$\begin{aligned}
\mathbb{E}[\hat{\mu}] &= \frac{1}{n}\{\mathbb{E}[X_1] + \mathbb{E}[X_2]+, ..., +\mathbb{E}[X_n]\} \\
&= \frac{1}{n}\{\mu + \mu+, ..., +\mu\} \\
&= \mu.
\end{aligned}$$

Because the average of a list *i.i.d.* random variables is an unbiased estimator of the unknown expectation, an accurate estimation of the mean with a required error tolerance can be computed by drawing an increasing number of samples.

Let X_1,..., X_i,..., X_n be an independent trials process, with finite expectation value $\mu = \mathbb{E}[X_i]$ and finite variance $\sigma^2 = Var[X_i]$. Now, let $S_n = \sum_{i=1}^{n} X_i$, then for any $\epsilon > 0$,

$$\mathbb{P}\left(\left|\frac{S_n}{n}\right| \leq \epsilon\right) \to 1 \; as \; n \to \infty. \tag{6.2}$$

In other words, this will ensure that the estimator $\hat{\mu}$ converges to the correct value μ, as the number of random variables n increase. The idea of a Monte-Carlo approximation is based on the fact that true expectation is the average of each sample generated by simulations. As long as the number of simulation is taken large enough, the difference between the estimated mean and the true mean can be lower than any small given tolerance.

Additionally, the variance of the random variable σ can be estimated using the sample variance $\hat{\sigma}$, which is computed as

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu})^2. \tag{6.3}$$

Then $\hat{\sigma}$ is also the unbiased estimator of σ, because we know that

$$\begin{aligned}
\mathbb{E}\left[\hat{\sigma}^2\right] &= \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \hat{\mu}\right)^2\right] \\
&= \frac{1}{n-1}\mathbb{E}\left[\sum_{i=1}^{n}\left(X_i^2 - 2X_i\hat{\mu} + \hat{\mu}^2\right)\right] \\
&= \frac{1}{n-1}\left[\left(\mathbb{E}\left[\sum_{i=1}^{n}X_i^2\right] - \mathbb{E}\left[\sum_{i=1}^{n}2X_i\hat{\mu}\right] + \mathbb{E}\left[\sum_{i=1}^{n}\hat{\mu}^2\right]\right)\right] \\
&= \frac{1}{n-1}\left[\left(\mathbb{E}\left[\sum_{i=1}^{n}X_i^2\right] - \mathbb{E}\left[n\hat{\mu}^2\right]\right)\right] \\
&= \frac{1}{n-1}\left[\left(\sum_{i=1}^{n}\mathbb{E}\left[X_i^2\right] - n\mathbb{E}\left[\hat{\mu}^2\right]\right)\right],
\end{aligned}$$

since

$$\mathbb{E}[\hat{\mu}^2] = \frac{\sigma^2}{n} + \mu^2$$

therefore,

$$\begin{aligned}\mathbb{E}[\hat{\sigma}^2] &= \frac{1}{n-1}\left[n\left(\sigma^2+\mu^2\right)-n\left(\frac{\sigma^2}{n}+\mu^2\right)\right]\\ &= \sigma^2.\end{aligned}$$

6.1.2 Central limit theorem

Let X_1,..., X_i,..., X_m be a sequence of *i.i.d.* random variables with expectation μ and variance σ^2, then the distribution of their sum

$$\frac{1}{m}\sum_{i=1}^{m} X_i \sim N(\mu,\sigma^2) \quad as \quad m\to\infty. \tag{6.4}$$

Equivalently,

$$\frac{\hat{\mu}-\mu}{\frac{\sigma}{\sqrt{m}}} \sim N(0,1) \quad as \quad m\to\infty. \tag{6.5}$$

The implication is that, whatever the distribution of X_i is, their average is a normal distributed random variable centralised at μ with error $O(\frac{1}{\sqrt{m}})$. This will enable us to make statistical inference on the simulation results using confidence levels. By applying the unbiased estimators $\hat{\mu}$ in (6.1) and $\hat{\sigma}^2$ in (6.3), the confidence interval for the estimate $\hat{\mu}$ with probability 0.95 can be gained from

$$\begin{aligned}&\mathbb{P}\left(\left|\frac{\hat{\mu}-\mu}{\frac{\sigma}{\sqrt{m}}}\right|\right) = 0.95\\ &\mathbb{P}\left(\hat{\mu}-\frac{1.96\sigma}{\sqrt{m}} \le \mu \le \hat{\mu}+\frac{1.96\sigma}{\sqrt{m}}\right) = 0.95,\end{aligned}$$

that is

$$\left[\hat{\mu}-\frac{1.96\sigma}{\sqrt{m}},\hat{\mu}+\frac{1.96\sigma}{\sqrt{m}}\right]. \tag{6.6}$$

If the true solution, assuming that it is known, does not lie inside the 95% confidential level, then the simulated result should be rejected as the correct estimator of the true solution. In most cases where the true value is not known, then a statement could be made that we are 95% confident that the true value is within a certain interval.

The point to note here is that the notation m is used to differentiate from n in Section 6.1.1, which itself is the number of simulations to obtain one estimation of the original variable. The variable m is the number of estimations where each one is the average of n simulations. In order to obtain a confidence interval for a solution $\hat{\mu}$ with

n simulations, we will have to simulate $\hat{\mu}$ for m times to estimate its variance. We will see the confidence interval of $\hat{\mu}$ converges with increasing m. The central limit theorem implies that to obtain one more digit accuracy, the number of simulation m should be increased 100 times. Equivalently, the rate of convergence is a half or the estimation error will be $O\left(\frac{1}{\sqrt{m}}\right)$.

6.1.3 Multi-companies default time simulation

Now that we have introduced some of the concepts of calculating expectations with simulation, we are going to describe how we can pricing credit derivatives using Monte-Carlo. In order to use Monte-Carlo for these derivative, it will require us to simulate the default events of multiple companies. The algorithm for multiple defaults simulation is developed by Duffie and Singleton (1998) and here we outline the algorithm in more details.

Suppose that the default event times τ_1, ..., τ_n of n firms each have a respective intensity process $\lambda_1(t)$, ..., $\lambda_n(t)$, which can be correlated. In order to simplify things somewhat, we assume that there are no simultaneous defaults and so the probability $\tau_i = \tau_j$ is null for $i \neq j$. Duffie and Singleton (1998) have proven that we can simulate the times τ_1, ..., τ_n with the correct joint distribution using the *Multi-Compensator Simulation* algorithm. If we let T be the time horizon of interest in which a default time τ must fall to be of interest, then default time must be in the set $(0, T]$. The *Multi-Compensator Simulation* algorithm can be described as follows:

Algorithm 1 Multi-Compensator Simulation

1: Simulate $\lambda_1(t)$, ..., $\lambda_n(t)$ up to time T
2: Compute compensator $\Lambda_1(T)$,..., $\Lambda_n(T)$ according to (2.20)
3: Simulate n independent unit-mean exponential distributed random variables Z_1,...,Z_n.
4: For each company i, if $\Lambda_i(T) < Z_i$, then $\tau_i > T$
5: Otherwise $\tau_i = \inf\{t > 0, \Lambda_i(t) = Z_i\}$.

When the intensity processes are stochastic, they are simulated through time T at equally spaced discrete time points $t_0 = 0$, ..., $t_m = m * \Delta t$,...,$t_M = T$. Then we can compute the compensator $\Lambda_1(t)$,...,$\Lambda_n(t)$ through simple numerical integration. Now we will have a discrete sample path of the compensator $\Lambda_i(t_0)$,..$\Lambda_i(t_m)$,...$\Lambda_i(t_M)$. However, the last step in Algorithm 1 cannot be done with discrete points of the compensator process. Instead we will need to locate a time slot such that

$$\Lambda_i(m\Delta t) < Z \leq \Lambda_i((m+1)\Delta t).$$

Unless the above equality holds, the default time in which $\tau_i = \inf\{t > 0, \Lambda_i(t) = Z_i\}$ lies between $m\Delta$ and $(m+1)\Delta t$. Once we have this region, we can linearly interpolation between $\Lambda_i(m\Delta t)$ and $\Lambda_i((m+1)\Delta t)$ to allocate the correct default time, which is

$$\tau_i = m\Delta t + \Delta t \frac{Z_i - \Lambda_i(m\Delta t)}{\Lambda_i((m+1)\Delta t) - \Lambda_i(m\Delta t)}. \tag{6.7}$$

6.1.4 Intensity Process Simulation

As we have discussed in the Algorithm 1, a sample path of default intensity $\lambda(t)$ has to be simulated. In the framework built up in chapters 4 and 5, we have to also add in a Poisson process $J(t)$ shared by two companies to model the external shocks and default contagions among firms. This means that each firm's default intensity is composed of a CIR process, a Poisson process and jumps due to default contagion. Then when it comes to simulating the default intensity sample paths it will be a three step procedure as we must simulate the external shock arrival times throughout the entire time horizon T at the same time as going through the CIR process step by step. For those external shock arrival times, we must simultaneously check whether the current time slot will be an external shock or if there is a firm defaults. If so, an exponential random variable is simulated which can then be added to the alive firms' default intensity. In the next section we provide some more details on the exact procedure.

Simulating CIR process

There are two approaches to simulate stochastic processes. When the distribution of a process in a future time is known in closed-form, we can draw a random number from the distribution to be a future position of the process. For instance, the return of a Geometric Brownian motion (GBM) is known to be normally distributed under a known transformation so that we can sample from the normal distribution to compute the location of the GBM at the next time step. However, no easy closed-form solution is available for the CIR process, and if we look to the literature there have been several proposed discretisation schemes for a CIR stochastic differential equation, which maintain the distribution of the underlying process (non-central χ^2) whilst making sure that the process does not fall below zero during simulation.

We point the reader to the following papers for a detailed description of those schemes: Deelstra and Delbaen (1998), Brigo and Alfonsi (2005), Glasserman (2003) and Berkaoui et al. (2008). Here we are just going to give a short overview of those simulation schemes and it is the *implicit Milstein* scheme from Brigo and Alfonsi (2005) that we will use for simulating CIR process in this thesis. The *implicit Milstein* scheme has higher order accuracy and it guarantees the sample path will be strictly positive as long as the *Feller* condition is satisfied.

Assume the process is simulated up to time T in discrete time intervals $t_0 = 0$, ..., $t_m = m * \Delta t$,...,$t_M = T$, then the simplest adaptation will be based on an Euler method, where the value of the CIR process at the next step is computed from

$$y(t + \Delta t) = y(t) + \kappa \left(\theta - y(t)\right) \Delta t + \sigma \sqrt{y(t)\Delta t} W_1, \tag{6.8}$$

where W_1 is a standard normal distribution.

At each step forward, a single normal random variable is drawn before applying (6.8). The problem in simulating the CIR process with (6.8) is that the process can fall below zero. No matter how small Δt is taken, there is non-zero probability that the sample from normal distribution will be large and negative so that the random term forces the process to go below zero. If this occurs, the process cannot be simulated further due to the fact that the square-root term is undefined for negative numbers.

To deal with this problem, some modifications are proposed. Deelstra and Delbaen (1998) provide a scheme in which they set the process to 0 if it is negative. We can write this as

$$y(t + \Delta t) = y(t) + \kappa \left(\theta - \max\{y(t), 0\}\right) \Delta t + \sigma \sqrt{\max\{y(t), 0\}} \sqrt{\Delta t} (W_1).$$

Similarly, Berkaoui et al. (2008) proposed the reflection scheme, in which strict positiveness is guaranteed. The process is simulated according to

$$y(t + \Delta t) = |y(t) + \kappa \left(\theta - y(t)\right) \Delta t + \sigma \sqrt{y(t)} \sqrt{\Delta t} (W_1)|.$$

Basically, the reflection scheme takes an absolute value to the simulated process at each time step in order to avoid negative values.

Each of those schemes outlined above have first-order weak convergence, in the sense that expectations of functions of $y(t)$ will approach their true values as $\mathcal{O}(\Delta t)$. In order to improve the order of accuracy, the *Milstein scheme* is described in Glasserman (2003) for generic stochastic process, where the accuracy is increased by considering higher order expansions of the drift and diffusion coefficient using Itô's lemma. In this case, the CIR process is simulated via

$$y(t+\Delta t) = y(t) + \kappa(\theta - y(t))\Delta t + \sigma\sqrt{y(t)\Delta t}W_1 + \frac{1}{4}\sigma^2\Delta t(W_1^2 - 1). \tag{6.9}$$

This is similar to the Euler scheme (6.8), in that the *Milstein scheme* cannot avoid having negative $y(t)$. But the probability that the process falls below zero is greatly reduced by the extra term. Kahl and Jäckel (2006) review some *implicit Milstein* schemes and propose a scheme with even lower probability of dropping below zero, which is written as

$$y(t+\Delta t) = \frac{y(t) + \kappa\theta\Delta t + \sigma\sqrt{y(t)}\sqrt{\Delta t}(W_1) + \frac{1}{4}\sigma^2\Delta t(W_1^2 - 1)}{(1+\kappa\Delta t)}. \tag{6.10}$$

The idea of the *implicit Milstein scheme* is to treat the term $-\kappa y(t)\Delta t$ in the forward time step $t+\Delta t$ as $-\kappa y(t+\Delta t)\Delta t$ so that the numerator of (6.10) can be shown to be strictly positive as long as *Feller* condition is satisfied.

Another *implicit Milstein* scheme is proposed by Brigo and Alfonsi (2005). In this version not only is the drift term $-\kappa y(t)\Delta t$ but also the diffusion term $\sqrt{y(t)}$ is treated at the unknown time step $t+\Delta t$. Consequently, $\sqrt{y(t+\Delta t)}$ is the positive root of a second-degree polynomial when *Feller* condition is satisfied. It follows that

$$y(t+\Delta t) = \left(\frac{\sigma\sqrt{\Delta t}W_1 + \sqrt{\sigma^2\Delta t {W_1}^2 + 4(y(t) + (\Delta t)(\kappa\theta - \frac{\sigma^2}{2}))(1+\kappa(\Delta t))}}{2(1+\kappa\Delta t)}\right)^2. \tag{6.11}$$

The advantage of this scheme that it is $\mathcal{O}(\Delta t^2)$ as well as ensuring that the strictly positive requirement of $y(t)$ is guaranteed.

Apart from the approaches that use a discretisation in time, another branch of research attempts to simulate the probability density function of the CIR process. It is actually mentioned in Cox's original paper Cox et al. (1985) that $y(T)$ given $y(t)$ follows a non-central chi-square distribution given by

$$y(\tau) \backsim c(\tau)\chi_d^2(ncp(\tau))$$

where

$$c(\tau) = \frac{\sigma^2(1 - 4e^{-\kappa\tau})}{4\kappa}, \quad d = \frac{4\theta\kappa}{\sigma^2}, \quad ncp(\tau) = \frac{4\kappa e^{-\kappa\tau}}{\sigma^2(1 - e^{-\kappa\tau})}, \quad \tau = T - t.$$

Here $y(T)$ exactly follows $c(\tau)$ times of a non-central χ^2 distribution with non-central parameter $ncp(\tau)$ and degree of freedom d. Although the distribution of CIR process is known to have a non-central χ^2 distribution, sampling from a non-central χ^2 distribution is computationally intensive, mainly as a result of the need to evaluate Γ functions. In this thesis, the implicit scheme (6.11) is chosen whilst at the same time we make sure in our parameter choices that the *Feller* condition satisfied.

Simulating Exponential Variables

The exponential distribution with mean value $\bar{\beta}$ is a probability distribution that describes the waiting time between events in a homogeneous Poisson process and whose density function is

$$f(x) = \frac{1}{\bar{\beta}} e^{-\frac{1}{\bar{\beta}}x}. \tag{6.12}$$

The cumulative distribution function is then given by

$$F(x) = 1 - e^{-\frac{1}{\bar{\beta}}x}. \tag{6.13}$$

To simulate the exponential variable, Glasserman (2003) describes the inverse transform method. Suppose we want to generate a random variable X with the property that $\mathbb{P}(X \leq x) = F(x)$ for all x, then the inverse transform method sets

$$X = F^{-1}(u), \ u \sim U(0,1),$$

where F^{-1} is the inverse of the function F and $U(0,1)$ denotes the uniform distribution on $(0,1)$.

Using Inverse Transform Method, an exponential variable is generated in the procedure Algorithm 2.

Algorithm 2 Generate Exponential Variable

1: $u \leftarrow U(0,1)$
2: $x \leftarrow \bar{\beta} \times ln(1-u)$ **return** x

Simulating Poisson Processes

Next we describe the way in which to generate the event times of a homogeneous Poisson process with rate $\bar{\lambda}$, which is based on the *Inverse Transformation* method. One important property of homogeneous Poisson processes is that the waiting time between events is exponentially distributed with rate $\bar{\lambda}$. Therefore, the cumulative distribution function of the time between two successive events is

$$F(t) = \int_0^t \bar{\lambda} e^{-\bar{\lambda} t} = 1 - e^{-\bar{\lambda} t}, \tag{6.14}$$

and the inverse is

$$F^{-1}(u) = -\frac{\ln(1-u)}{\bar{\lambda}},\ 0 \le u < 1. \tag{6.15}$$

Therefore, if $u \sim U(0,1)$ is a uniform random variable in $[0,1)$, the next event occurs at time

$$t = -\frac{\ln(1-u)}{\bar{\lambda}}. \tag{6.16}$$

So Poisson event arrivals with homogeneous arrival rate $\bar{\lambda}$ within a time horizon T can be simulated by repeating the above procedure until the cumulative time exceeds T. We describe such a the procedure in Algorithm 3.

Algorithm 3 Simulate Poisson Arrival Times

[This function take two arguments. The expire time T and the arrival intensity $\bar{\lambda}$]

$N \leftarrow 0$ (The number of Jumps)
$Jt \leftarrow 0$ (The time of jumps arrive)
$CurrentTime \leftarrow 0$
while $CurrentTime < T$ **do**
    $u \leftarrow U(0,1)$
    $t \leftarrow \frac{1}{\bar{\lambda}} ln(1-u)$ (waiting time)
    $N+=1$
    $Jt[N] = CurrentTime + t$
    $CurrentTime+=t$

6.2 A Three Firm Model

The modelling framework is exactly the same as described in Section 5.1. For the detailed model specifications such as the definition of filtrations, we refer to Section 5.1. The contribution in this chapter is to extend the two firm framework in Section 5.2 to include the default of the CDS buyer. So we have three mutually interacting firms. This will enable us to analyse the fair CDS spreads S^* on multiple interacting firms with external shocks and default contagions. More importantly, we can analyse the CVA and DVA under our default contagion framework.

To provide more specifications to the model in Section 5.1, we assume that the economy contains three companies, namely the investor, counterparty and referencing firm. For simplicity, the referencing firm, the counterparty and the investor will be indexed by 1, 2 and 3 or by using the abbreviations ref, cp and inv respectively. There are 3 non-negative random variables (τ_i) for $i \in \{ref, cp, inv\}$ defined on the filtered probability space $(\Omega, \mathcal{G}_t, \mathbb{Q})$, satisfying $\mathbb{Q}(\tau_i = 0) = 0$ and $\mathbb{Q}(\tau_i > t) > 0$ for $t \in \mathbb{R}_+$. Then we have three right-continuous default indicator processes

$$\begin{aligned} H_{ref}(t) &= 1_{\{\tau_{ref} \le t\}}, \\ H_{cp}(t) &= 1_{\{\tau_{cp} \le t\}}, \\ H_{inv}(t) &= 1_{\{\tau_{inv} \le t\}}. \end{aligned}$$

The filtration $\mathcal{H}_t$ is generated by the default processes, which is

$$\begin{aligned} \mathcal{H}(t) &= \vee \mathcal{H}_i(t) \\ \mathcal{H}_i(t) &= \sigma(\{\tau_i \le u : u \le t\}), \text{ for } i \in \{ref, cp, inv\}. \end{aligned}$$

The filtration $\mathcal{H}(t)$ contains default information.

After assuming the $\mathbb{G}^{-i}$-adapted default processes $F_i(t) = \mathbb{Q}(\tau_i \le t | \mathcal{G}_t^{-i})$ to be absolutely continuous submartingale under $\mathbb{Q}$ for all $i \in \{ref, cp, inv\}$, there exists $\mathbb{G}^{-i}$-adapted intensities of default times τ_i, which we specify as

$$\begin{aligned} d\lambda_{ref}(t) &= \kappa_{ref}\left(\theta_{ref} - \lambda_{ref}(t)\right) dt + \sigma_{ref}\sqrt{\lambda_{ref}(t)} dW_{ref}(t) \\ &\quad + \alpha_{ref,cp} dH_{cp}(t) + \alpha_{ref,inv} dH_{inv}(t) + \beta dJ(t), \\ d\lambda_{cp}(t) &= \kappa_{cp}\left(\theta_{cp} - \lambda_{cp}(t)\right) dt + \sigma_{cp}\sqrt{\lambda_{cp}(t)} dW_{cp}(t) \\ &\quad + \alpha_{cp,ref} dH_{ref}(t) + \alpha_{cp,inv} dH_{inv}(t) + \beta dJ(t), \\ d\lambda_{inv}(t) &= \kappa_{inv}\left(\theta_{inv} - \lambda_{inv}(t)\right) dt + \sigma_{inv}\sqrt{\lambda_{inv}(t)} dW_{inv}(t) \\ &\quad + \alpha_{inv,ref} dH_{ref}(t) + \alpha_{inv,cp} dH_{cp}(t) + \beta dJ(t), \end{aligned} \tag{6.17}$$

for $i \in \{ref, cp, inv\}$. The definition of filtration $\mathbb{G}^{-i} = (\mathcal{G}_t)_{t \in \mathbb{R}_+}$, where $\mathcal{G}_t^{-i} = \vee_{j \in \{ref, cp, inv\}, j \neq i} \mathcal{H}_j(t) \vee \mathcal{F}_t$. Also, the market filtration $\mathcal{F}_t$ is defined as (5.1). The filtration $\mathbb{G}^{-i}$ contains all information except the firm i's default time τ^i. (6.17) is a special case of (5.2) with the drift and volatility functions $\mu()$ and $\sigma()$ to be (5.19). The correlations of these Brownian motions are described by the correlation matrix Σ_W, which is

$$\Sigma_W = \begin{bmatrix} 1 & \rho_{ref,cp} & \rho_{ref,inv} \\ \rho_{cp,ref} & 1 & \rho_{cp,inv} \\ \rho_{inv,ref} & \rho_{inv,cp} & 1 \end{bmatrix}.$$

Again, the $J(t)$ term in (6.17) is the $\mathbb{F}$-adapted homogeneous Poisson process assumed to be shared by all firms' default intensities that represents the external or economy-wide shocks as outlined in Chapter 5. To keep consistent with previous chapters, those external shocks β are again modelled as exponential random variables defined by (5.21). Note that there are still other chooses for modelling β discussed in Section 4.2.1.

When one of the companies defaults, its default process $H_i(t)$, $i \in \{ref, cp, inv\}$, changes from 0 to 1, which triggers a jump to alive firms' default intensities $\lambda_j(t)$, such that $j \in \{ref, cp, inv\}$ and $j \neq i$. In order to describe the default contagion impacts amongst multiple firms, a default contagion matrix $\tilde{\alpha}$ is defined as

$$\tilde{\alpha} = \begin{bmatrix} 0 & \bar{\alpha}_{ref,cp} & \bar{\alpha}_{ref,inv} \\ \bar{\alpha}_{cp,ref} & 0 & \bar{\alpha}_{cp,inv} \\ \bar{\alpha}_{inv,ref} & \bar{\alpha}_{inv,cp} & 0 \end{bmatrix}.$$

In this matrix each entry $\bar{\alpha}_{i,j}$ for a given $i \neq j$ represents the expected size of jump that firm j's default will trigger on firm i's default intensity. Again, we specify the random jump size $\alpha_{i,j}$ follows independent exponential distributions, which are

$$\eta(\alpha_{i,j}) = \frac{1}{\bar{\alpha}_{i,j}} e^{-\frac{\alpha_{i,j}}{\bar{\alpha}_{i,j}}}.$$

In fact the default contagion matrix does not necessary need to be symmetric since the impact the firm's default has on a small firm is not necessarily going to be the same as the effect a small firm's default will have on a big firm. Later, we will denote $\tilde{\alpha} = A$ to represent a scenario where every entry in the matrix is the same constant A except for the diagonals which are always zero.

Given that there are now three firms in the economy the survival and default states of each firm divides the state space into 8 possibilities. We denote each of those possible states as:

$$\mathcal{A} = \{H_{ref}(t) = 0, H_{cp}(t) = 0, H_{inv}(t) = 0\},\ \mathcal{B} = \{H_{ref}(t) = 0, H_{cp}(t) = 0, H_{inv}(t) = 1\}$$
$$\mathcal{C} = \{H_{ref}(t) = 0, H_{cp}(t) = 1, H_{inv}(t) = 0\},\ \mathcal{D} = \{H_{ref}(t) = 0, H_{cp}(t) = 1, H_{inv}(t) = 1\}$$
$$\mathcal{E} = \{H_{ref}(t) = 1, H_{cp}(t) = 0, H_{inv}(t) = 0\},\ \mathcal{F} = \{H_{ref}(t) = 1, H_{cp}(t) = 0, H_{inv}(t) = 1\}$$
$$\mathcal{G} = \{H_{ref}(t) = 1, H_{cp}(t) = 1, H_{inv}(t) = 0\},\ \mathcal{H} = \{H_{ref}(t) = 1, H_{cp}(t) = 1, H_{inv}(t) = 1\}$$

The default risk of a firm in each of the different states will not be the same given the different possible ways that default contagions could affect them. For instance, the referencing firm in state $\mathcal{A}$ has higher default risk than in state $\mathcal{D}$ due to the fact that the counterparty and investor are still alive and their defaults will cause default contagions on the referencing firm. Therefore, a CDS written on the referencing firm has different values or fair spread in different states. Similar to Chapters 4 and 5, we will refer the contract value under a specific state.

6.3 Implementation

We remind the reader that we are primarily interested in evaluating the counterparty risk for a situation in which a defaultable investor is buying a CDS contract from a defaultable counterparty and there are default contagions amongst the investor, the counterparty and the referencing firm. At the beginning of the trade, the investor buys a CDS protection on the referencing firm's default from the counterparty at the cost of an annual fair swap rate S^* paid quarterly according to (2.57). So the annual fair swap rate S^* is evaluated in state $\mathcal{A}$, where all the firms are still alive. Similarly, CVA and DVA, which are the value adjustments in (1.4) and (1.5), are evaluated in state $\mathcal{A}$. However, we need to price the CDS in state $\mathcal{A}$, $\mathcal{B}$ as well as $\mathcal{C}$. If the counterparty/investor defaults, the economic state changes from $\mathcal{A}$ to $\mathcal{C}$ / $\mathcal{B}$, the CDS value in the new state can be the loss, if any, given default.

According to (1.4) and (1.5), there are three components which determine the CVA and the DVA. They are: the discount factor; the event in which the counterparty or the investor is the first to default; and the post-default contract value at default time. In our model, the discount factor is assumed constant.

Taking account of the event in which the first firm to default is the counterparty or the investor is not in any way trivial. Because if we are in state $\mathcal{A}$, there are three firms whose default intensities are all stochastic and are all interacting with each other. Therefore, Monte-Carlo simulation will be used to simulate those credit events using the default times simulation Algorithm 1. And for the last part, as mentioned earlier, when the counterparty or the investor defaults before the CDS expires, the economic state changes from $\mathcal{A}$ to $\mathcal{B}$ or $\mathcal{C}$, and the post-default value of the CDS must be calculated in order to determine the loss due to the counterparty or the investor defaulting. With default contagions in our model, the CDS value is dependent on two state variables in both $\mathcal{B}$ and $\mathcal{C}$. The two state variables are the referencing firm's default intensity and the default intensity of either the counterparty's or the investor's. In Section 5.2.2, we showed that such a CDS with default contagion and shared external shocks can be numerically evaluated by solving the two-dimensional PIDE system (5.14) and (5.16). Consequently, we see that the three building blocks of the CVA and DVA can be computed with different numerical schemes. In this section, we first detail the approach required to simulate the three default intensities $\lambda_{ref}(t)$, $\lambda_{cp}(t)$ and $\lambda_{inv}(t)$. Our approach will combine: the *implicit Milstein* scheme (6.11); the multi-companies default times simulation; the default contagion jumps $H_i(t)$ for $i = \{ref, cp, inv\}$ and external shocks $J(t)$. This default intensities simulation algorithm is then used later on to produce the fair CDS spread in state $\mathcal{A}$. Finally, we describe the hybrid numerical scheme, which combines Monte-Carlo simulation with finite-difference in order to compute the CVA and DVA.

6.3.1 Default processes and default times simulation

If we were to have a model without default contagion, then the default times of multiple firms can be simulated following Algorithm 1, in which we identify the default times after all of the default intensity processes have been simulated through maturity. However, when there are default contagion events happening, identification of a default event must be considered at each and every time step so the default intensity process can only be moved on step by step. We cannot decouple the processes and simulate all the way to maturity because each sample path of a firm's default intensity $\lambda(t)$ is dependent on when one of other firms defaults. If a firm defaults at the current time step, we must add in a jump to other alive firms' default intensity.

So to describe this approach we first divide the time horizon T into equally M steps, *i.e.* $t_0 = t$, ..., $t_m = m\Delta t$, ..., $t_M = T$ as before. Then the simulation procedure is described in Algorithm 4.

Algorithm 4 Contagion Model Processes Simulation

[This function take arguments, T, M,$\bar{\lambda}$, $\bar{\beta}$, $\tilde{\alpha}$ and Σ_w]

Simulating a vector of J_t arrival times with Algorithm 3.
$K \leftarrow$ the number of J_t jump times.
$k \leftarrow 1$ (the counter of exogenous jumps)
$\Delta t \leftarrow \frac{T}{M}$
Simulate 3 independent unit-mean exponentially distributed random variables Z_1, Z_2, Z_3.
for `m=1;m<=M;m++` **do**
    Current time $t \leftarrow m \times \Delta t$
    Simulate three standard Brownian motions $\big(W_{ref}, W_{cp}, W_{inv}\big)$subject to Σ_w
    Simulate $\big(\lambda_{ref}(t), \lambda_{cp}(t), \lambda_{inv}(t)\big)$ according to (6.11) using $\big(W_{ref}, W_{cp}, W_{inv}\big)$
    if $J_t[k] < t$ and $k < K$ **then**
        Simulate an exponential random variable β with mean $\bar{\beta}$ according to Algorithm 2
        $\big(\lambda_{ref}(t) += \beta, \lambda_{cp}(t) += \beta, \lambda_{inv}(t) += \beta\big)$
        $k++$
    for `Every surviving firm` $i \in (ref, cp, inv)$ **do**
        Calculate compensator $\Lambda_i(t) += 0.5\big(\lambda_i(t) + \lambda_i(t - \Delta t)\big)\Delta t$
    for `Every surviving firm` $i \in (ref, cp, inv)$ **do**
        if $\Lambda_i(t) \geq Z_i$ **then**
            Determine default time τ_i according to (6.7)
            for $j \in \{ref, cp, inv\}$ and $j \neq i$ **do**
                Simulate α_{ji} with mean $\bar{\alpha}_{ji}$
                $\lambda_j(t) += \alpha_{ji}$
Return $\big(\tau_{ref}, \tau_{cp}, \tau_{inv}\big)$ and Sample Paths of λ_{ref}, λ_{cp} and λ_{inv}

The arrivals of the homogeneous Poisson process $J(t)$ represent external events whose arrivals are independent of the sample paths of three firms' default intensity. As a result, $J(t)$ can be simulated in advance independently of any default intensity calculations. At the start, a vector of Poisson arrivals are simulated up to time T with arrival intensity $\bar{\lambda}$. Then, the intensity processes are simulated step by step simultaneously. If there are external event arrivals between two successive time $[(m-1)\Delta t, m\Delta t]$, an exponential jump with mean $\bar{\beta}$ is drawn and added to all of the alive companies' intensity at time $m\Delta t$. The compensator $\Lambda_t(m\Delta t)$ of each firm is then calculated in order to identify whether there are defaults at that time interval. Once a firm defaults, its default time is determined by (6.7) and exponential jumps are calculated according to $\tilde{\alpha}$ and then added to the surviving firms' intensity processes $\lambda_i(m\Delta t)$ for $\forall\ i \in (ref, cp, inv)$.

6.3.2 Spread simulation

Every CDS contract is characterised by an annualised fair swap spread S^* in (2.57), where the CDS investor is required to pay the amount $S^*\Delta T$ at successive time slots $T_1 = \Delta T$, ..., $T_k = k\Delta t$, ..., $T_K = K\Delta T$ before the referencing firm defaults.

Measuring the fair swap spread S^* requires us to evaluate the expectations in (2.57). The numerator is the expected value of protection buyers' receivable when the referencing firm has defaulted before expiry, which is called the *Default Payment*. The denominator is composed of two components. One is the present value of the quarterly ($T_k - T_{k-1} = \frac{1}{4}$) payment that the buyers have to pay until the referencing firm defaults or the CDS expires, namely the *Premium*. The rest is the payment that the investor should pay if the referencing firm defaults between two settlement dates T_{k-1}, T_k, namely *Accrual*. Because the insurance payment is paid at the end of each time slot, the insurance fee for the period $(T_{k-1}, T_k]$ is not paid if the referencing defaults between $(T_{k-1}, T_k]$. Instead, the *Accrual* should be paid for the protection period from the last payment time, say T_{k-1}, to the default time. For each scenario of the referencing firm's default time, the value of those three payments can be determined. The expected values of *Default Payment*, *Premium* and *Accrual* are computed by simulation and their ratio is taken to obtain the fair swap spread S^*. It is important to be aware here that the fair swap spread S^* simulated reflects the default risk of the referencing firm but not the counterparty risk from trading the CDS. This is because the defaults of the counterparty and the investor will only trigger default contagion jumps $\alpha_{ref,cp}$ and $\alpha_{ref,inv}$ to the referencing firm but are not regarded as a termination condition on the CDS. This is similar to the situation that there is a regulator independent from the three firms system and it is evaluating the fair price of the CDS on the referencing firm.

In order to determine the values of *Default Payment*, *Premium* and *Accrual*, what we need to do is to simulate the default time of the referencing company on whom the CDS is written. In our default contagion model, everybody's default time is linked to the other two companies due to possible default contagions. So Algorithm 4 is used to simulate the referencing firm's default time. The simulation procedure for the fair swap spread S^* is given by algorithm 5. Here the discount factor $D(t, T)$ is calculated as $e^{-r(T-t)}$ with constant risk-free interest rate r.

Algorithm 5 CDS spread simulation

[This function take arguments: Maturity T, $J(t)$ arrival rate $\bar{\lambda}$, External jump's expected value $\bar{\beta}$, Number of steps M, Swap payment period ΔT, Default contagion matrix $\tilde{\alpha}$, Correlation matrix Σ_w, Number of Simulation N]

Default Payment $\leftarrow 0$
Premium Payment $\leftarrow 0$
Accrual Payment $\leftarrow 0$
$K \leftarrow \frac{T}{\Delta T}$ the number of total payment
for `n=1;n<=N;n++` **do**
    Simulate the default times $\{\tau_{ref}, \tau_{cp}, \tau_{inv}\}$ with Algorithm 4.
    if $\tau_{ref} \leq T$ **then**
        Default Payment$+= D(0, \tau_{ref}) LGD(\tau_{ref})$
        Premium Payment$+= \sum_{k=1}^{\bar{K}} \Delta T D(0, k\Delta T)$, such that $\bar{K}\Delta T < \tau_{ref} < (\bar{K}+1)\Delta T$
        Accrual Payment$+= D(0, \tau_{ref}) \frac{\tau_{ref} - \bar{K}\Delta t}{\Delta T}$
    else
        Premium Payment$+= \sum_{k=1}^{K} \Delta t D(0, k\Delta T)$
return $S^* = \frac{\text{Default Payment}}{\text{Premium Payment+Accrual Payment}}$

6.3.3 CVA and DVA Simulation

In this section, we describe the algorithm for simulating CVA and DVA, which integrates Algorithm 4 and the ADIOS finite-difference scheme proposed in Chapter 5. CVA and DVA need to be calculated when the two parties trading the CDS are defaultable. In the situation that our defaultable investor buys the CDS from the defaultable counterparty, any credit event from the investor, the counterparty or the referencing firm can lead to the termination of the contract. But the investor /counterparty will suffer losses only in the scenario that the counterparty/investor is the first to default as this is the only way that losses/gains can occur.

As discussed in Section 6.2, CVA and DVA are computed in state $\mathcal{A}$, where three firms are still alive and CVA, DVA are simulated simultaneously in a two step procedure. The first step is to simulate a scenario of the three firms' default time through Algorithm 4 and then identify whether this scenario is subject to a counterparty credit event, which can be evaluated as the cases when

$$\{\tau_{inv} < T, \tau_{inv} < \tau_{ref}, \tau_{inv} < \tau_{cp}\}, \{\tau_{cp} < T, \tau_{cp} < \tau_{ref}, \tau_{cp<\tau_{inv}}\}.$$

The second step is to quantify how much the CDS contract is worth after the investor or the counterparty defaults. If the counterparty or the investor defaults, the economic state changes from $\mathcal{A}$ to $\mathcal{B}$ or $\mathcal{C}$, therefore the value of the CDS will be taken from the appropriate state with the positions of the two default intensities at the default time. We can evaluate the CDS in both states $\mathcal{B}$ and $\mathcal{C}$.

While in states $\mathcal{B}$ and $\mathcal{C}$, we are facing a two-firm model, which we discussed in Chapter 5. In Chapter 5, we prove a CDS's value with our model is the solution to PIDE (5.40). A subtle difference is that the two firms are the referencing firm and the counterparty in state $\mathcal{B}$ but the two firms are the referencing firm and the investor when in state $\mathcal{C}$. Applying the two-firm model in $\mathcal{B}$ and $\mathcal{C}$ to (5.40) leads to the following PIDE system

$$\begin{aligned}\frac{\partial V_{\mathcal{D}}}{\partial t^-} &+ \mathcal{L}^{\mathcal{D}}\{V_{\mathcal{D}}\} + \mathcal{J}\{V_{\mathcal{D}}\} - (r + \lambda_{ref})\, V_{\mathcal{D}} \\ &+ \lambda_{ref}\left((1 - R_{ref})\, e^{-r(T-t)} - S\frac{t - T_{n-1}}{T_n - T_{n-1}}\right) = 0,\end{aligned} \tag{6.18a}$$

$$\begin{aligned}&\frac{\partial V_{\mathcal{B}}}{\partial t^-} + \mathcal{L}^{\mathcal{B}}\{V_{\mathcal{B}}\} + \mathcal{J}\{V_{\mathcal{B}}\} + \lambda_{ref}\left((1 - R_{ref})e^{-r(T-t)} - S\frac{t - T_{n-1}}{T_n - T_{n-1}}\right) \\ &+ \lambda_{cp}\int_0^\infty V_{\mathcal{D}}\left(\lambda_{ref} + \alpha_{ref,cp}, t^-, T\right)\eta(\alpha_{ref,cp})d\alpha_{ref,cp} - (r + \lambda_{ref} + \lambda_{cp})\, V_{\mathcal{B}} = 0,\end{aligned} \tag{6.18b}$$

$$\begin{aligned}&\frac{\partial V_{\mathcal{C}}}{\partial t^-} + \mathcal{L}^{\mathcal{C}}\{V_{\mathcal{C}}\} + \mathcal{J}\{V_{\mathcal{C}}\} + \lambda_{ref}\left((1 - R_{ref})e^{-r(T-t)} - S\frac{t - T_{n-1}}{T_n - T_{n-1}}\right) \\ &+ \lambda_{inv}\int_0^\infty V_{\mathcal{D}}\left(\lambda_{ref} + \alpha_{ref,inv}, t^-, T\right)\eta(\alpha_{ref,inv})d\alpha_{ref,inv} - \big(r + \lambda_{ref} + \lambda_{inv}\big)V_{\mathcal{C}} = 0,\end{aligned} \tag{6.18c}$$

where

$$\begin{aligned}
\mathcal{L}^{\mathcal{D}} &= \kappa_{ref}(\theta_{ref} - \lambda_{ref})\frac{\partial}{\partial \lambda_{ref}} + \frac{1}{2}\sigma_{ref}^2 \lambda_{ref}\frac{\partial^2}{\partial \lambda_{ref}^2}, \\
\mathcal{L}^{\mathcal{B}} &= \kappa_{ref}(\theta_{ref} - \lambda_{ref})\frac{\partial}{\partial \lambda_{ref}} + \frac{1}{2}\sigma_{ref}^2 \lambda_{ref}\frac{\partial^2}{\partial \lambda_{ref}^2} \\
&+ \kappa_{cp}(\theta_{cp} - \lambda_{cp})\frac{\partial}{\partial \lambda_{cp}} + \frac{1}{2}\sigma_{cp}^2 \lambda_{cp}\frac{\partial^2}{\partial \lambda_{cp}^2} \\
&+ \rho_{ref,cp}\sigma_{ref}\sigma_{cp}\sqrt{\lambda_{ref}\lambda_{cp}}\frac{\partial^2}{\partial \lambda_{ref}\lambda_{cp}}, \\
\mathcal{L}^{\mathcal{C}} &= \kappa_{ref}(\theta_{ref} - \lambda_{ref})\frac{\partial}{\partial \lambda_{ref}} + \frac{1}{2}\sigma_{ref}^2 \lambda_{ref}\frac{\partial^2}{\partial \lambda_{ref}^2} \\
&+ \kappa_{inv}(\theta_{inv} - \lambda_{inv})\frac{\partial}{\partial \lambda_{inv}} + \frac{1}{2}\sigma_{inv}^2 \lambda_{inv}\frac{\partial^2}{\partial \lambda_{inv}^2} \\
&+ \rho_{ref,inv}\sigma_{ref}\sigma_{inv}\sqrt{\lambda_{ref}\lambda_{inv}}\frac{\partial^2}{\partial \lambda_{ref}\lambda_{inv}}.
\end{aligned}$$

Due to the discontinuous swap spread payments that occur at T_1, ..., T_k, ..., T_K, there are jump conditions

$$\begin{aligned}
V_{\mathcal{D}}(\lambda_{ref}, T_k^-, T) &= V_{\mathcal{D}}(\lambda_{ref}, T_k, T) - S(T_k - T_{k-1}) \\
V_{\mathcal{B}}(\lambda_{ref}, \lambda_{cp}, T_k^-, T) &= V_{\mathcal{B}}(\lambda_{ref}, \lambda_{cp}, T_k, T) - S(T_k - T_{k-1}) \\
V_{\mathcal{C}}(\lambda_{ref}, \lambda_{inv}, T_k^-, T) &= V_{\mathcal{C}}(\lambda_{ref}, \lambda_{inv}, T_k, T) - S(T_k - T_{k-1})
\end{aligned}$$

for $\forall k = 1, ..., K$ and terminal conditions given by

$$\begin{aligned}
&V_{\mathcal{D}}(\lambda_{ref}, T, T) = 0, \\
&V_{\mathcal{B}}(\lambda_{ref}, \lambda_{cp}, T, T) = 0, \\
&V_{\mathcal{C}}(\lambda_{ref}, \lambda_{inv}, T, T) = 0.
\end{aligned}$$

Equations (6.18b) and (6.18c) are same as (5.40a) except the notations of the state and firms. (6.18a) is identical to (5.40b) except the notation of the state. Equation (6.18a) solves the CDS value $V_{\mathcal{D}}$ in state $\mathcal{D}$, where only the referencing firm is left alive so that there is no default contagion risk involved. The solutions of (6.18a) can then be fed into (6.18b) and (6.18c) to solve for the CDS value in states $\mathcal{B}$ and $\mathcal{C}$, *i.e.* $V_{\mathcal{B}}$ and $V_{\mathcal{C}}$. Due to the nature of solving PIDEs, our solutions $V_{\mathcal{B}}$ and $V_{\mathcal{C}}$ contain the CDS values for $V_{\mathcal{B}}$ at $\lambda_{ref} \times \lambda_{cp} \in [0, \lambda_{ref,max}] \times [0, \lambda_{cp,max}]$ and $V_{\mathcal{C}}$ at $\lambda_{ref} \times \lambda_{inv} \in [0, \lambda_{ref,max}] \times [0, \lambda_{inv,max}]$ for all $t \in [0, T]$, given $\lambda_{ref,max}$, $\lambda_{cp,max}$ and $\lambda_{inv,max}$ are the upper bounds of the truncated domains. For the details of solving these PIDEs, we refer to Section 5.3.

After simulating the default times using Algorithm 4, we are given the default times and sample paths, and we can select the scenarios that the investor or the counterparty is the first to default. Further, we have the positions of the default intensities' $\big(\lambda_{ref}(\tau_{inv}), \lambda_{cp}(\tau_{inv})\big)$ or $\big(\lambda_{ref}(\tau_{cp}), \lambda_{inv}(\tau_{cp})\big)$ which are used to locate the CDS value, $V^{\mathcal{B}}$ or $V^{\mathcal{C}}$, in the PIDE solutions at the default time, τ_{inv} or τ_{cp}. It is necessary to ensure that the maximum value of default intensity $\lambda_{ref,max}$, $\lambda_{cp,max}$ and $\lambda_{inv,max}$ in the finite-difference grid is not less than the simulated default intensities. Failure to satisfy this condition will make us unable to determine losses given default and will result in an underestimation of the CVA. Therefore, we will take the maximum level of the finite-difference domains $\lambda_{ref,max}$, $\lambda_{cp,max}$ and $\lambda_{inv,max}$ to be sufficiently large when solving (6.18b) and (6.18c) in ADIOS scheme.

After determined the scenario in which the investor or the counterparty has defaulted and the corresponding CDS values at that time, we are now able to know the losses/gains, if any, to the surviving party. Hence the discounted value of the losses/gains is one simulation of the CVA or DVA and the average over a sufficiently large number of simulations will give us the expected values of the CVA or DVA. The entire procedure for simulating CVA and DVA is described in Algorithm 6.

6.4 Convergence Analysis

In Section 6.3 we describe our numerical implementations of the contagion model for pricing CDS and measuring CVA, DVA. In this section, we demonstrate the effectiveness of the numerical schemes by showing the convergence of CVA in terms of the number of simulation N and the number of steps in time M.

Since there is no analytic solution to compare against our numerical solution, we use the finite-difference solution of the CVA from Chapter 5 as a benchmark for the Monte Carlo version. The CVA computed in this chapter with a defaultable investor, counterparty and referencing is the CVA defined in (1.4). Compared with the CVA computed in Chapter 5, which is defined as (1.3), the CVA we consider in this chapter includes the possibility that the investor defaults. However, the difference between the CVA in Chapter 5 and the one we will see in this chapter will narrow down if there is an extremely safe investor, everything else being equal. In other words, given

Algorithm 6 CVA and DVA Simulation

[This function take arguments. The expire time T, exogenous jump arrival intensity $\bar{\lambda}$, exogenous jump sizes mean value $\bar{\beta}$, number of steps M, default contagion matrix $\tilde{\alpha}$ and the correlation matrix Σ_w]

Solve (6.18c) for the CDS value $V_{\mathcal{C}}(\lambda_{ref}, \lambda_{inv}, t, T)$ in state $\mathcal{C}$
Solve (6.18b) for the CDS value $V_{\mathcal{B}}(\lambda_{ref}, \lambda_{cp}, t, T)$ in state $\mathcal{B}$
CVA $\leftarrow 0$, DVA $\leftarrow 0$
for `n=1;n<=N;n++` **do**
    Simulate the default times $(\tau_{ref}, \tau_{cp}, \tau_{inv})$ and sample paths of $(\lambda_{ref},\ \lambda_{cp},\ \lambda_{inv})$ for $t \in (0, T]$ with Algorithm 4
    if $\tau_{cp} \leq T$ and $\tau_{cp} < \tau_{ref}$ and $\tau_{cp} < \tau_{inv}$ **then**
        Find the post-default intensities' position $\lambda_{ref}(\tau_{cp}^{+})$ and $\lambda_{inv}(\tau_{cp}^{+})$ from sample paths.
        Obtain post-default CDS value in state $\mathcal{C}$ by interpolating $V_{\mathcal{C}}$ at the point $\big(\lambda_{ref}(\tau_{cp}^{+}), \lambda_{inv}(\tau_{cp}^{+}), \tau_{cp}^{+}\big)$
        CVA $+= (1 - R_{cp}) D(0, \tau_{cp}) \max\{V_{\mathcal{C}}(\lambda_{ref}(\tau_{cp}^{+}), \lambda_{inv}(\tau_{cp}^{+}), \tau_{cp}^{+}, T), 0\}$
    if $\tau_{inv} \leq T$ and $\tau_{inv} < \tau_{ref}$ and $\tau_{inv} < \tau_{vp}$ **then**
        Find the post-default intensities' position $\lambda_{ref}(\tau_{inv}^{+})$ and $\lambda_{cp}(\tau_{inv}^{+})$ from sample paths.
        Obtain post-default CDS value in state $\mathcal{B}$ by interpolating $V_{\mathcal{B}}$ at the point $\big(\lambda_{ref}(\tau_{inv}^{+}), \lambda_{cp}(\tau_{inv}^{+}), \tau_{inv}^{+}\big)$
        DVA$- = (1 - R_{inv}) D(0, \tau_{inv}) \min\{CDS(\lambda_{ref}(\tau_{inv}), \lambda_{cp}(\tau_{inv}), \tau_{inv}), 0\}$
Return CVA$=\frac{\text{CVA}}{N}$ and DVA$=\frac{\text{DVA}}{N}$

	ref	cp	inv
κ	0.5	0.5	10^{10}
θ	0.05	0.05	10^{-10}
σ	0.1	0.1	10^{-10}
$\lambda(0)$	0.05	0.05	10^{-10}
	$\bar{\lambda} = 0.05$	$\bar{\beta} = 0.05$	

Table 6.1
Model Parameter for Convergence Test

	$\Delta t = \frac{1}{252}$	$\Delta t = \frac{1}{252\times 2}$	$\Delta t = \frac{1}{252\times 4}$
$\Delta x = \frac{1}{100}$	30.454	30.4406	30.4339
$\Delta x = \frac{1}{200}$	30.4243	30.411	30.4043
$\Delta x = \frac{1}{400}$	30.4166	30.4033	30.3967
$\Delta x = \frac{1}{800}$	30.4146	30.4013	30.3946
$\Delta x = \frac{1}{1600}$	30.414	30.4007	30.394

Table 6.2
The CVA (bps) for a one year CDS protection computed by finite-difference scheme
Protection buyer is paying 0 bps spread quarterly in rear and interest rate is 0%. Recovery method is recovering a risk-free bond and recovery rate is fixed in $R = 0.4$. Δx is the step size of both reference and counterparty's default intensities

an extremely safe investor, the CVA computed by Algorithm 6 should align to the CVA solved by finite difference in Chapter 5. We will now present some test results calculated using Algorithm 6 with a base set of parameters as given in table 6.1, where the investor is set to be extremely safe so that the results should be comparable with Chapter 5.

The default contagion matrix $\tilde{\alpha}$ is given as below

$$\tilde{\alpha} = \begin{bmatrix} 0 & 0.05 & 0 \\ 0.05 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which implies there are only default contagions between the referencing firm and the counterparty, which also coincides with Chapter 5.

Assume there is a one-year CDS and the investor is paying zero spread and the risk-free interest rate is zero. The convergence of the CVA of this CDS, which is solved by finite difference, with non-defaultable investor is given by table 6.2, where the CVA converges to 30.394 *bps*

The simulation results in table 6.3, figures 6.1 and 6.2 are showing the convergence of the Algorithm 6 with respect to the number of simulation N and the number of

N	$\hat{\mu}$	$\hat{\sigma}_{\bar{\mu}}$	$\hat{\mu}_{95\%}$
100	31.1751	15.76	(30.1969, 32.1533)
1000	30.4057	4.89	(30.1020, 30.7093)
10000	30.3328	1.52	(30.2380, 30.4276)
100000	30.3797	0.51	(30.3478, 30.4117)

Table 6.3
Convergence of CVA Simulation with 95% confidence levels
The convergence of the CVA (bps) for a one year CDS contract. Protection buyer is paying 0 bps spread quarterly in rear and interest rate is 0%. Recovery method is recovering risk-free bond and recovery rate is fixed in 0.4. The step number of the simulation path (M) is fixed at 4×252. Statistics are estimated from a group of 1000 samples with each sample is the average of N simulations.

steps M. The number of steps in time M is chosen to be multiples of 252, which is the number of trading days in a year. According to central limit theorem, simulation results should converge to the accurate results and the variance of the result should reduce against increasing number of simulations N. Table 6.3 and figure 6.1 show the pattern of convergence in N with the corresponding estimated variance and 95% confidence interval. With the number of steps in time is fixed at $M = 4 \times 252$, each estimation of CVA $\hat{\mu}$ is run for N scenarios. The variance is estimated from 1000 $\hat{\mu}$ with (6.3) and the 95% confidence interval is calculated according to (6.6).

In table 6.3, the volatility of our estimator reduces by three times when N increases by ten times, which aligns with the theoretical value $10^{\frac{1}{2}} \approx 3.16$. In addition, when we visualise the solutions of table 6.3 in figure 6.1 and plot them alongside the *best* finite-difference solution with $\Delta t = \frac{1}{4\times 252}$ and $\Delta x = \frac{1}{1600}$, which is 30.394 shown by the dotted red line. We can see that the finite-difference solution always lies within the 95% confidential intervals (green bars) of the simulated solution. We can also see how the confidence intervals narrow down with an increasing number of simulations N.

On the other hand, figure 6.2 shows the pattern of convergence in M with a fixed number of simulations $N = 10^6$. In the case of the sample paths of $\lambda(t)$ have to be simulated throughout time to obtain a solution, we cannot guarantee the accuracy without small enough time step size $\frac{T}{M}$. This problem is common in pricing path-dependent derivatives, for example Asian options. When the number of steps in time $M = (1 \text{ or } 2) \times 252$, the finite-difference solutions stand outside the 95% confidence interval, by which we reject the Monte-Carlo results to be the unbiased estimator of the finite-difference solution. If sufficiently small step size is taken, the finite-difference

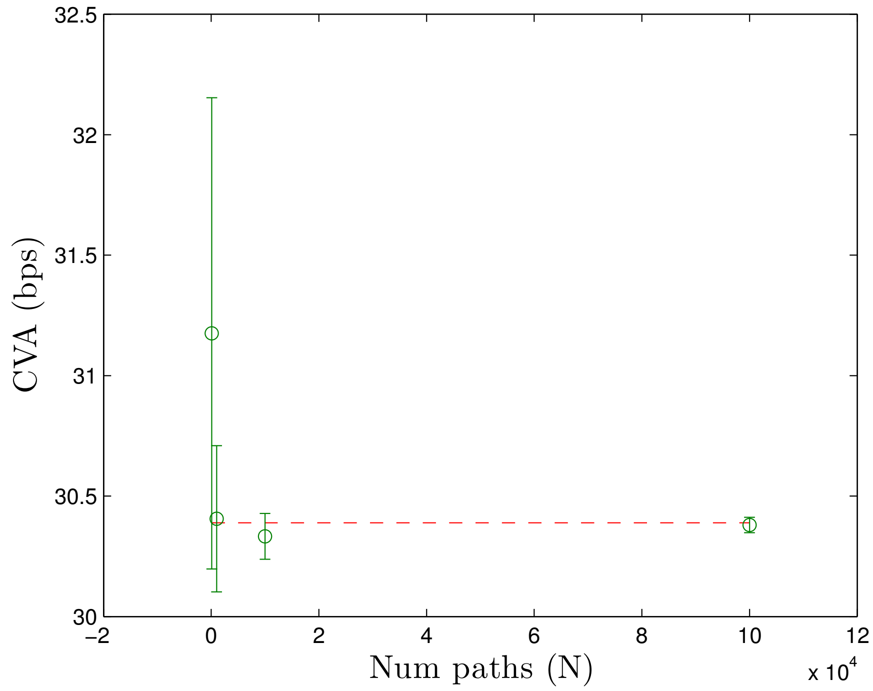

Figure 6.1
Convergence of CVA in Number of simulation
This figure shows the convergence of the CVA to the one-year CDS protection mentioned against different N. The green dot is the sample mean and vertical lines showing the 95% confidence interval. And the red dash is the finite-difference result with $\Delta t = \frac{1}{4\times 252}$ and $\Delta x = \frac{1}{1600}$.

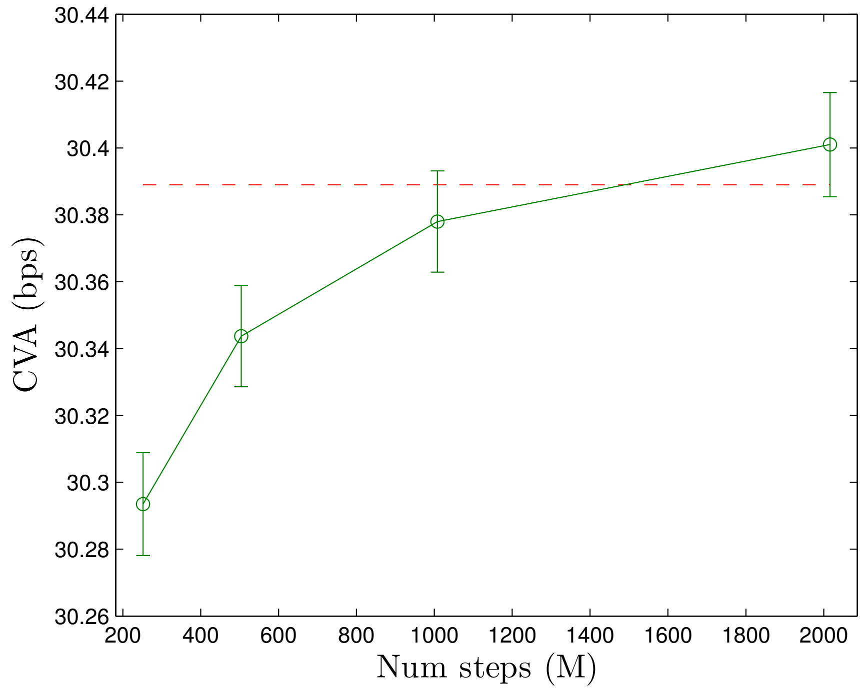

Figure 6.2
The convergence of CVA in step size
This figure shows the convergence of the CVA to the one-year CDS protection mentioned against different M. The green dot is the sample mean and vertical lines showing the 95% confidence interval. And the red dash is the finite-difference result with $\Delta t = \frac{1}{4 \times 252}$ and $\Delta x = \frac{1}{1600}$

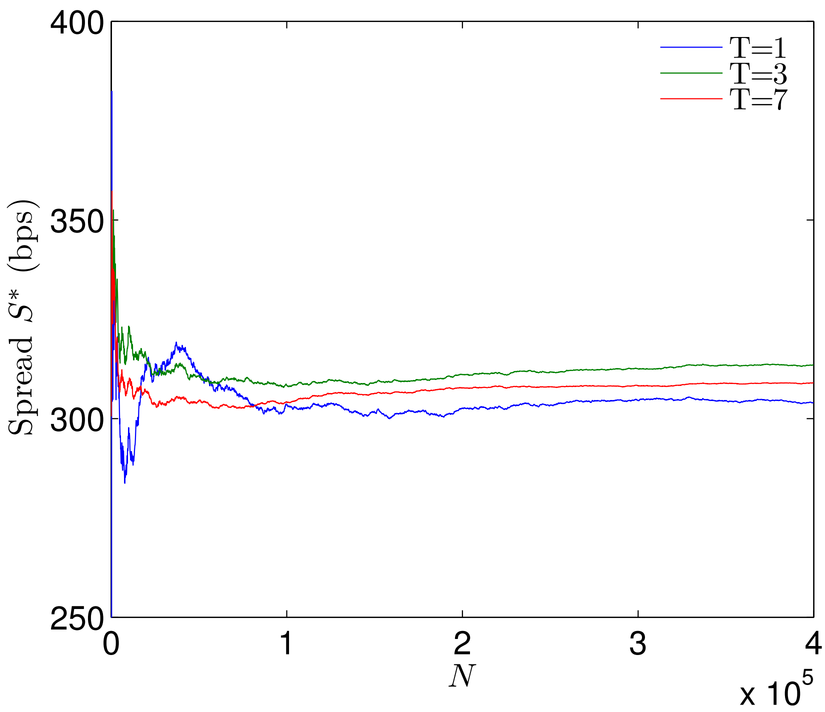

Figure 6.3
The convergence of fair spread S^* against N
This figure shows the convergence of the CVA to the one-year CDS protection mentioned against different M. The green dot is the sample mean and vertical lines showing the 95% confidence interval. And the red dash is the finite-difference result with $\Delta t = \frac{1}{4\times 252}$ and $\Delta x = \frac{1}{1600}$

result starts to lies in between the 95% confidence interval, so we can confirm the simulation is valid. This pattern is due to the accuracy in default times and arrivals of external shock $J(t)$. Remember that the default times are linearly interpolated by (6.7), so when the default intensities are simulated with a large time step $\frac{T}{M}$ the default times will be located very roughly in a time interval leading to large numerical errors. In addition, there might be multiple external shocks that happen to arrive within one time step Δt. If we simulate two exogenous jumps βs to the intensity at one time point using Algorithm 4, the exact timing of jump arrival times are ignored and the accuracy of the sample path won't be very good. Therefore, the wider that Δt is, the more errors that are made to the allocation of jump arrival times. The approach to avoid these two problems is to take a small enough time step size in order to have an accurate estimation of sample paths and default times.

The convergence of fair swap spreads S^* can pose a problem, which is that shorter maturity spreads require larger number of simulations N than the longer maturity contracts to obtain accurate solutions. Figure 6.3 shows the convergence of swap spreads S^* with three different maturities. Obviously, the spread with $T = 1$ is more volatile than the cases with $T = 3$ and $T = 7$. This phenomenon is similar to pricing deep out-of-money options with simulation. The scenarios that the simulated stock value exceeds the strike price are very rare, so we find that simulating out-of-money options requires higher numbers of simulation to converge. Similarly, the referencing firm is more likely to default in longer rather than shorter time horizons. So simulating a one years CDS spread S^* requires sufficient scenarios that the firm defaults within one year and this can be rare. Therefore, it is more difficult for the shorter maturity spreads to converge.

6.5 CVA DVA and Fair CDS Spread Analysis

Traditionally, the CVA and DVA are charged on an upfront basis. In other words, at the time just before the transaction takes place, the investor will have to measure how much CVA should be charged to the counterparty and also the DVA amount they should pay to the counterparty, which is simulated by Algorithm 6 under our contagion framework. Ideally, both parties will have sufficient amount of money to cover the possible losses due to the other party defaults. However, in the real world the investor and the counterparty are still facing potential risk even after settling on the CVA and DVA at the date the transaction begins.

The investors of OTC derivative transactions will not only have risk exposure to the derivatives but also CVA and DVA. According to (1.6), the risk-adjusted derivative's value $\tilde{\Pi}$, held by the investor, could be higher or lower if market conditions change suddenly. From the modelling prospective, this implies certain parameters in the model have changed (or were under or over estimated initially), which will lead to losses and gains to the investor even without actual defaults. From the investor's prospective, if there is a movement in either party's credit risk and/or the derivative's value after the measurement day, the initial amount of CVA and DVA settled might be not enough or it could be more than enough to cover the possible loss due to the counterparty or the investor default. For example, if the counterparty's default probability becomes higher, then the probability that the counterparty defaults earlier than the investor is higher and the probability that the investor defaults earlier than the counterparty is lower. Consequently, the CVA received is not enough to cover the possible loss due to the counterparty's default and the DVA paid is more than enough to cover the possible gains due to the investor's default, which imply losses in both CVA and DVA to the investor.

More importantly, under the current accounting regulation International Financial Reporting Standard (IFRS), firms are required to report their OTC transactions in terms of the risk-adjusted value, which is defined in (1.6). In simple terms, the CVA and DVA under current market circumstance will have to be reported as components of the fair value of the investor's derivative transaction. Any profits and losses in CVA and DVA will directly reflect on the investor's balance sheet, which should be managed. During the financial crisis in 2008, roughly two-thirds losses attributed to counterparty credit risk were due to CVA losses and only about one-third were due to actual default, according to Brigo et al. (2013).

This section will be subdivided into three parts. In Section 6.5.1, we imagine that the investor has bought the CDS contract from the counterparty, and we will investigate the profit and losses in the CVA and the DVA under various changes in market circumstances, including the default risk of firms are more correlated than was first thought, the firms' default intensities are more volatile and credit quality deteriorates to the investor and the counterparty. Then in Section 6.5.2, we consider an investor with full knowledge of default contagion on the day they are going to buy a CDS contract from the counterparty. Therefore, the investor has to measure the fair CDS spread S^*, CVA and DVA using the contagion model proposed. Again we will investigate the sensitivities of the fair spreads, CVA and DVA to the changes of market circumstances. Finally, in Section 6.5.3, we will investigate how the CVA and the DVA may be different under different combinations of the external shocks' strength $\bar{\beta}$ and default contagion strength $\tilde{\alpha}$ while keeping the CDS with the same price S^* or the referencing firm has the same default risk in other words. Although the referencing firm's default risk remains the same, how much of the default risk attributes to the idiosyncratic risk, external shocks and default contagions has different implications for the counterparty risk.

	ref	cp	inv
κ	0.5	0.5	0.5
θ	0.05	0.05	0.05
σ	0.1	0.1	0.1
$\lambda(0)$	0.05	0.05	0.05
$\bar{\lambda} = 0.05$		$\bar{\beta} = 0.05$	

Table 6.4
Intensity Parameters for CVA DVA gains and losses analysis

$\bar{\alpha}_{ref,cp}$	cva (bps)	dva (bps)	$\bar{\alpha}_{ref,inv}$	cva (bps)	dva (bps)
0.	6.706	6.873	0.	6.706	6.873
0.01	11.019	6.559	0.01	6.940	4.026
0.02	16.707	6.240	0.02	7.362	2.749
0.03	23.262	5.844	0.03	7.495	2.038
0.04	29.241	5.598	0.04	8.083	1.652
0.05	35.396	5.314	0.05	8.207	1.469
0.06	42.159	4.922	0.06	8.585	1.224
0.07	48.213	4.642	0.07	9.222	1.048
0.08	54.142	4.401	0.08	9.142	0.949
0.09	60.458	4.253	0.09	9.548	0.855
0.1	64.916	4.032	0.1	10.179	0.809

Table 6.5
CVA DVA gains and losses with increasing default contagions $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$

6.5.1 CVA, DVA gains and losses

Assume the investor bought a 5-year CDS protection from the counterparty written on the referencing firm. Initially, the CDS is traded at the fair rate $S^* = 299$ *bps*, which is an annual rate paid quarterly under the market environment with 2% interest rate and external shocks are happening at the rate $\bar{\lambda} = 0.05$ with shock strength $\bar{\beta} = 0.05$. At the initial time, no default contagion risk is considered by the investor and the counterparty, *i.e.* $\tilde{\alpha} = 0$. All three firms have credit risk parameters listed in table 6.4.

When there is no consideration of default contagion risk among the three firms, both the protection seller and buyer are facing very close counterparty credit risk, which are CVA=6.706 *bps* and DVA=6.873 *bps* respectively. In this numerical example, the probabilities of the investor or the counterparty being the first default firm are identical due to the same risk parameters in table 6.4. Further, when there is no default contagion, the CDS has identical value in states $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ because whether the counterparty or the investor has defaulted or not does not influence the credit risk of the referencing firm.

However, if the market environment changes and we include a default contagion from the counterparty to the referencing firm $\bar{\alpha}_{ref,cp}$, the investor suffers from significant CVA loss and relatively small amount of DVA loss, which are shown by the second and third column in table 6.5. According to table 6.5, if the default contagion from the counterparty to the referencing firm $\bar{\alpha}_{ref,cp}$ increases to 0.1, the CVA increases from 6.706 *bps* to 64.9 *bps*, which implies a loss of 58 *bps* in the risk-adjust value $\tilde{\Pi}$ due to the CVA loss according to (1.6). On the other hand, the maximum DVA loss is 2.89 *bps*, which is almost insignificant.

The reasons for the CVA and DVA losses are as follows. Since default contagions only occurs when a firm defaults, the probability that the investor or the counterparty is the first one to default remains unchanged, as discussed in Section 4.4.3. The CVA and DVA changes are only due to the changes in loss given default. With an increase in the default contagion $\bar{\alpha}_{ref,cp}$, the referencing firm has higher default risk than before. Therefore, the CDS values in states $\mathcal{A}$ and $\mathcal{B}$ will slightly increase and they have positive value due to the possible default contagion from the counterparty. As a result, the CDS is less likely to be a liability to the investor, from which the investor can benefit from defaulting. Consequently, DVA is slightly lower than at the measurement day given current market conditions. In terms of CVA, although the referencing firm is no more risky in state $\mathcal{C}$, the CDS value at counterparty's default time $V^{\mathcal{C}}(\tau_{cp}^{+})$ will be immediately driven up by the default contagion that has been applied to the referencing firm's intensity, which is $V^{\mathcal{C}}(\lambda_{ref}(\tau_{cp})) = V^{\mathcal{C}}(\lambda_{ref}(\tau_{cp}^{-}) + \alpha_{ref,cp})$. Therefore, the loss given counterparty defaults is significantly greater than the case without default contagion.

In addition, we look at the situation where the default contagion is from the investor to the referencing firm $\bar{\alpha}_{ref,inv}$. The investor also suffers losses in both CVA and DVA. There is a less than 5 *bps* CVA losses and around 6 *bps* DVA losses to the investor when $\bar{\alpha}_{ref,inv}$ is raised to 0.1. The CVA losses are attribute to the referencing firm being more risky in state $\mathcal{C}$, which gives slightly more counterparty default losses to the investor. But the reason for DVA losses is that there is default contagion from the investor to the referencing firm. At the investor's default time, the CDS's value

$\tilde{\alpha}$	0	0	0.0175	0.0175	0.025	0.025	0.05	0.05	0.075	0.075
σ	cva	dva	cva	dva	cva	dva	cva	dva	cva	dva
0	3.106	2.699	14.817	0.368	20.963	0.255	40.541	0.142	58.508	0.095
0.05	4.749	4.394	15.362	1.102	21.898	0.761	40.589	0.345	59.239	0.177
0.1	6.706	6.873	16.099	2.627	21.841	1.917	39.668	0.849	57.907	0.480
0.15	8.612	9.783	17.025	4.474	21.659	3.376	39.360	1.66	56.890	0.897
0.2	9.945	12.733	17.483	6.429	21.373	5.103	37.219	2.563	55.486	1.552

Table 6.6
CVA and DVA with increasing volatility under raising degree of default contagion. σ is applied to all firms' default intensity.

$V^{\mathcal{C}}(\tau_{inv})$ jumps to a higher value as a result of the default contagion jump $\alpha_{ref,inv}$ so then the CDS value is unlikely to be negative. As a result the investor is not likely to benefit from their default.

In summary, in our default contagion model, if the default risk of the referencing firm becomes dependent on either the counterparty or the investor, the investor will suffer losses in both CVA and DVA. But the CVA loss due to the default contagion from the counterparty to the referencing firm is significantly greater than the other situation. This is because the CDS will be very valuable after the default contagion is applied to the referencing firm after the counterparty's default, which leads to a significant loss for the investor. The losses in DVA are limited by the DVA amount that the investor has paid to the counterparty at the initial time, although we do find that the DVA losses caused by default contagion from the investor $\alpha_{ref,inv}$ is greater than the one caused by the counterparty $\alpha_{ref,cp}$. We can explain this by noting that in the former case, the default contagion jump occurred to the referencing firm at the investor default time while in the latter case it only increases slightly the CDS value by possible default contagion in the future. Therefore, the impacts on the CDS value at the investor's default time $V^{\mathcal{B}}(\tau_{inv})$ are significantly different.

Now when the economy is in a recession, the tightening dependence of firms' default risk can be accompanied by a rise in default intensity, higher intensity volatility and more frequently arrival of external shocks. The interactions among different market variables may lead to a complex profit/loss profile for the CDS investor. Table 6.6 shows the profit and loss in CVA and DVA to the investor when there are default contagions among all firms $\tilde{\alpha} > 0$ and high intensity volatilities σ_{ref}, σ_{cp}, and σ_{inv}, which we denote as σ.

Unlike stronger default contagions $\tilde{\alpha}$, which will make substantial DVA losses, the investor will always have DVA profits with higher default intensity volatilities. On the other hand, the profit/loss in CVA shows quite complicated behaviour against volatilities, which is dependent on the strength of default contagions. If the strength of default contagion $\tilde{\alpha}$ is relatively small, such as $\tilde{\alpha} = 0.0175$, higher volatilities lead to CVA losses of 1.4 *bps*. But the investor may have CVA gains with higher volatilities when $\tilde{\alpha}$ is larger than 0.025 in our numerical examples.

We discussed in Chapter 3, the first effect that higher volatilities have on counterparty risk is to lower the probability that the investor or the counterparty to be the first firm to default. Therefore, it is less likely that the counterparty credit event occurs. The reason for this phenomenon is that we model default intensities as mean-reverting square-root processes whose default probabilities will be lower with high volatilities. In terms of the loss given default, higher volatilities are fattening the CDS value distribution at any time $t > 0$ but the CDS value for every $\lambda_{ref}(t)$ is lower than before. In other words, the CDS is more likely to take extreme values with higher volatilities but the mean value of the CDS will be lower because the referencing firm is safer. In the case without default contagions $\tilde{\alpha} = 0$, which are the first two columns in table 6.6, there shows a CVA loss and DVA profit with higher volatilities. In terms of the CVA, although the mean value of CDS is lower, the fattening of the CDS value distribution leads to higher chance that a big loss occurs to the investor if the counterparty defaults. On the other hand, both lower mean value of the CDS and the fatter value distribution tend to make the CDS contract more likely to be a liability to the investor therefore DVA increases against volatilities, which means a DVA gain to the investor.

Another interesting phenomenon is that the CVA decreases from 58.51 *bps* to 55.49 *bps* when volatility increases from 0 to 0.2 given the default contagion strength $\tilde{\alpha} = 0.075$. The behaviour against volatility reverses while there are strong default contagions among firms. We will illustrate this phenomenon in figure 6.4. We remind ourselves that higher volatilities make the referencing firm's default intensity $\lambda_{ref}(\tau_{cp}^-)$ have a higher probability that it will take large values, which results in greater CDS values at the counterparty's default time. However, if there is a strong default contagion link from the counterparty to the referencing, the post-contagion default intensity

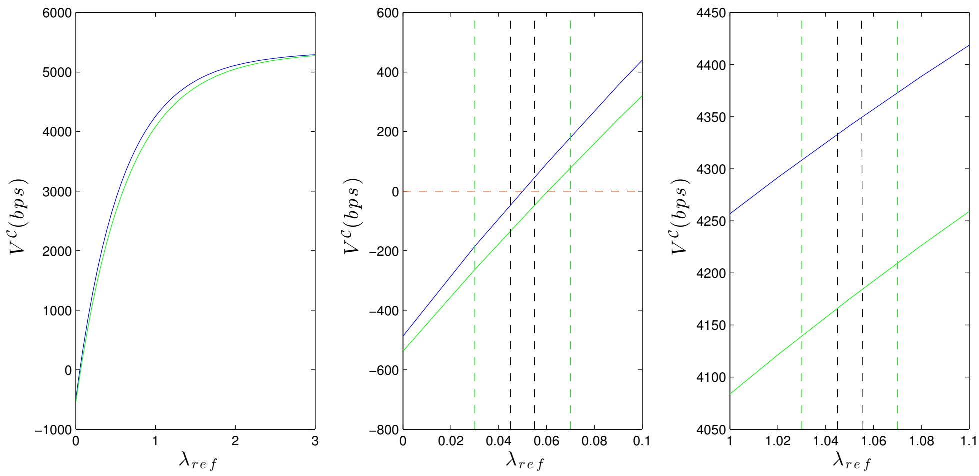

Figure 6.4
CDS values $V^{\mathcal{C}}$ against referencing firm's intensity λ_{ref} at counterparty's default time $\tau_{cp} = 5$.
An illustration of the volatility effects on the CDS value $V^{\mathcal{C}}(\tau_{cp}^{+})$ with low (blue line) and high (green line) volatility. The left panel shows the CDS value against $\lambda_{ref} \in [0, 3]$. The middle panel assumes the post-contagion default intensity of referencing firm $\lambda_{ref}(\tau_{cp}^{+})$ is around 0.05 and shows the 95% confidential interval of $\lambda_{ref}(\tau_{cp}^{+})$ in low (blue dash lines) and high (green dash lines) volatility. The right panel assumes the post-contagion default intensity of referencing firm $\lambda_{ref}(\tau_{cp}^{+})$ is around 1.05 and shows the 95% confidential interval of $\lambda_{ref}(\tau_{cp}^{+})$ in low (blue dash lines) and high (green dash lines) volatility.

of the referencing firm $\lambda_{ref}(\tau_{cp})$ will be large already due to the default contagion. Besides, the value function of CDS against default intensity $\lambda_{ref}(t)$ is a concave function so the CDS value will not increase a lot against $\lambda_{ref}(t)$ if $\lambda_{ref}(t)$ is already large, see the left panel of figure 6.4. Consequently, if there is a strong default contagion applied to the referencing firm, raising volatilities will not have as much of an effect on the CDS value at the counterparty's default time as it did when there were no default contagions. But higher volatilities will still make the referencing firm safer so that the CDS value should be lower than it will be with low volatilities. For example, in the middle and right panels of figure 6.4, we plot the CDS value against $\lambda_{ref}(t)$ at $t = 0$ with $\sigma = 0.05$ (blue) and $\sigma = 0.2$ (green) and assume the corresponding 95% confidence level are given by the dash lines. We see the middle panel for the situation without default contagion to the referencing firm. Although the CDS value function is lower with $\sigma = 0.2$ than with $\sigma = 0.05$, there is higher probability the CDS can take larger values. On the other hand, the right panel indicates the CDS value at the counterparty's default time will be significantly lower given $\alpha_{ref,cp} = 1$. As a result, higher volatilities lead to the CDS value being lower, which undermines the CVA.

$\tilde{\alpha}$	0	0	0.0125	0.0125	0.025	0.025	0.0375	0.0375	0.05	0.05
$\bar{\lambda}$	cva	dva	cva	dva	cva	dva	cva	dva	cva	dva
0	2.859	9.832	7.670	4.883	15.431	2.943	24.114	1.917	33.617	1.268
0.05	6.706	6.873	12.959	3.304	21.481	1.918	30.976	1.218	39.746	0.852
0.1	11.437	4.842	18.835	2.224	28.012	1.258	37.302	0.859	47.518	0.560
0.15	16.641	3.327	25.497	1.476	35.643	0.871	45.468	0.608	55.825	0.396
0.2	23.178	2.257	32.136	0.964	42.262	0.605	52.770	0.412	61.405	0.288

Table 6.7
CVA and DVA (bps) with increasing $\bar{\lambda}$ under raising degree of default contagion.

In addition to higher intensity volatilities, an economy in recession or a stressful market can also mean *bad news* might happen more frequently and if it comes it will do harm to all firms' credit worthiness. In our model, the occurrence of economy-wide events or what we call external shocks are represented by the Poisson process $J(t)$ with an arrival rate $\bar{\lambda}$ and the Poisson arrivals cause exponential jumps β to all alive firms simultaneously so we can model an increase in bad news by simply increasing $\bar{\lambda}$.

The profits and losses to the investor, when external shocks are arriving more frequently, are given in table 6.7. The investor will suffer from CVA and DVA losses if the arrival rate of external shocks becomes more frequent. When there is no default contagions and the external shock arrival rate raises from $\bar{\lambda} = 0.05$ to $\bar{\lambda} = 0.1$, CVA raises from 6.7 *bps* to 11.4 *bps* and DVA reduces from 6.87 *bps* to 4.84 *bps*, which corresponds to around 4.7 *bps* losses in CVA and 2 *bps* losses in DVA. There could be slightly higher CVA losses with default contagions, where CVA increases from 39.668 *bps* to 49.325 leading to almost 10 *bps* CVA losses. But the DVA losses due to an increasing external shock arrival rate are immaterial because the DVA is so tiny when we include default contagion.

Compared to what we have seem when we change the volatility, the way that external shocks affect the CVA/DVA profit and loss is very different. Firstly, a higher external shock arrival rate will always deteriorate all firms' credit risk whereas higher volatilities make firms safer. Therefore, the CDS contract will be more valuable in any economic states and the probabilities that the counterparty or the investor to be the first to default firm will be higher. Secondly, in terms of the CDS value distribution, higher volatilities makes the value distributions become fatter in both tails because the default intensities are equally likely to take lower as well as higher values. However, since external shocks only cause positive jumps to default intensities, the CDS value will have a thinner left tail and a fatter right tail. Consequently, the investor's losses at counterparty's default time will be greater but the gains at its own default time will be less. Clearly, both effects make the CVA higher in the current market environment so the investor will see losses in CVA when the external shocks happen more often. It is easy to notice that higher $\bar{\lambda}$ has the opposite effect on DVA. On one hand, higher $\bar{\lambda}$ increases the probability that the investor can be the first to default, and at the same time higher $\bar{\lambda}$ makes the CDS value $V^{\mathcal{B}}(\tau_{inv})$ less likely to be a liability, or negative value equivalently, to the investor. However, according to our numerical example table 6.7, the latter effect will overcome the former one causing the investor losses in DVA.

Next, we are going to see the profit/loss if either the counterparty or the investor deteriorates or improves in their credit, which we show in tables 6.8 and 6.9. The firms' default intensities λ_{cp} and λ_{inv} at time $t = 0$ are moved down or up to represent an improvement or deterioration in credit risk. Clearly, the investor will have CVA losses if the counterparty deteriorates in its credit risk, which improves the probability of the counterparty is the first firm to default. On the other hand, the investor will have profit in CVA if the investor becomes more risky as this will lower the probability that the counterparty is the first firm to default. When we set default contagions $\tilde{\alpha} > 0$, the CVA gains and losses are significantly more sensitive to the two firms' credit risk. This is because the loss given default will be higher with default contagions if the counterparty defaults. Consequently, the CVA will be more sensitive to counterparty's default probability. For instance, the maximum CVA loss is 45 *bps* compared with 5.2 *bps* without any default contagion.

Similar to how they did with CVA, the investor can have profits in their DVA when their credit deteriorates, according to table 6.9. The reason is that the investor is more likely to be the first firm to default after λ_{inv} increased. So the DVA amount paid to the counterparty at measurement day is lower in those market circumstances. Similarly, the loss will be applied to the DVA if the counterparty is more risky. However, it is clear that the DVA gain and losses are not significant with default contagion, which is the opposite of what we found with the CVA. We knew that only when the CDS has negative values to the investor is it possible for the investor to gain from their own default, and this is where DVA occurs. However, with default contagions, whenever the

CVA gain/loss	$\tilde{\alpha} = 0$				
$\lambda_{cp}(0)$ / $\lambda_{inv}(0)$	0	0.05	0.1	0.15	0.2
0	0.882	-0.649	-2.362	-3.709	-5.240
0.05	1.469	0.000	-1.738	-3.504	-4.420
0.1	1.563	0.077	-1.638	-2.863	-4.162
0.15	2.065	0.284	-1.030	-2.349	-3.324
0.2	2.219	0.490	-0.593	-1.793	-3.188
CVA gain/loss	$\tilde{\alpha} = 0.05$				
0	16.066	-0.879	-16.657	-31.213	-45.384
0.05	17.055	0.000	-15.707	-29.470	-42.736
0.1	18.225	3.104	-13.543	-28.304	-41.441
0.15	19.367	2.974	-12.615	-26.860	-40.711
0.2	20.733	3.988	-11.828	-25.468	-38.632

Table 6.8
CVA profit and loss against changes in credit risk with and without default contagions

investor defaults, the CDS becomes more valuable since the referencing firm becomes affected by the contagion from investor's default. Therefore, the DVA is so tiny that where there is a movement in both firms' default risk it does not change the DVA by a large amount. In other words, the DVA is less sensitive to firms' credit status when there is default contagion.

6.5.2 CDS spreads and CVA/DVA Charges

The previous section analyses the CVA and DVA profit and loss due to changing market circumstance after the CDS transaction has been set up. In this section, we analyse the fair swap spread S^* and the corresponding CVA and DVA that the investor has to compute **before** they buy the CDS from the counterparty. We analyse these three measures, S^*, CVA and DVA, under a variety of market circumstances, especially their response to the degree of default contagion $\tilde{\alpha}$. We are primarily interested in whether the referencing firm has weak capacity to recover from default contagion and the impact from direct and indirect default contagion. We expect that the CVA and DVA will have different behaviours compared with Section 6.5.1, because the credit risk of the referencing firm under different market circumstances has already been priced into S^* by the investor. Therefore how the CVA and DVA respond to changing market circumstances will be more complicated compared to the last section.

Initially we set the swap spread S^* to be the fair spread in state $\mathcal{A}$ but it may no

DVA gain/loss	$\tilde{\alpha} = 0$				
$\lambda_{inv}(0)$ \ $\lambda_{cp}(0)$	0	0.05	0.1	0.15	0.2
0	-2.307	-2.613	-2.975	-3.218	-3.383
0.05	0.547	0.000	-0.178	-0.720	-0.941
0.1	3.155	2.620	2.241	1.713	1.289
0.15	5.573	5.059	4.431	3.914	3.528
0.2	7.744	7.257	6.342	5.883	5.498
DVA gain/loss	$\tilde{\alpha} = 0.05$				
0	-0.176	-0.194	-0.292	-0.360	-0.413
0.05	0.111	0.000	-0.115	-0.215	-0.280
0.1	0.399	0.182	0.064	-0.092	-0.197
0.15	0.583	0.395	0.196	0.039	-0.086
0.2	0.744	0.526	0.279	0.158	-0.009

Table 6.9
DVA profit and loss against changes in credit risk with and without default contagions

longer be fair once we move into states $\mathcal{B}$ and $\mathcal{C}$, after which either the investor or the counterparty has defaulted. Therefore, this will have implications for evaluating the loss given default in states $\mathcal{B}$ and $\mathcal{C}$. In this section, the terms CVA or DVA *gain and loss* are no longer appropriate since the investor is evaluating the position at the initial time, so the contract has not been initiated. The spreads in this section state the fair price the investor has to pay for the CDS protection and the CVA and DVA are the value adjustment that the investor and the counterparty are going to charge each other. So we will use the terms CVA charges and DVA costs where appropriate.

We construct table 6.10 in a similar way to table 6.5 to show the resulting CVA and DVA when either the default contagion is from counterparty to the referencing firm $\bar{\alpha}_{ref,cp}$ is strong or the default contagion from the investor to the referencing firm $\bar{\alpha}_{ref,inv}$ is strong. Note that since there is now a strong default contagion, the referencing firm is now more risky. Therefore, the investor will be asked by the counterparty for a higher price to compensate this risk, and we see this in the corresponding fair spreads S^* that are shown in the first column of table 6.10. Again, the parameters of the intensity processes are in table 6.4. When either the counterparty or the investor's default could cause a default contagion jump to the referencing firm, the fair spread S^* is increased from 299 *bps* to 330 *bps* which is around 10% increase in spread. This must be considered against the fact that we have set the mean value of jump size to 0.1 which is around two times of referencing firm's long-term default rate θ_{ref}.

S^* (*bps*)	$\bar{\alpha}_{ref,cp}$	CVA (bps)	DVA (bps)	$\bar{\alpha}_{ref,inv}$	CVA (bps)	DVA (bps)
299	0	6.706	6.873	0	6.706	6.873
301	0.01	11.051	6.740	0.01	6.758	4.448
304	0.02	16.237	6.803	0.02	6.731	3.150
309	0.03	21.169	7.055	0.03	6.564	2.683
311	0.04	27.492	7.255	0.04	6.551	2.392
313	0.05	32.743	7.455	0.05	6.546	1.972
316	0.06	38.465	7.598	0.06	6.548	1.996
319	0.07	43.557	7.732	0.07	6.484	1.953
321	0.08	48.835	7.774	0.08	6.502	1.717
326	0.09	53.488	7.950	0.09	6.373	1.556
328	0.1	59.147	8.202	0.1	6.362	1.666

Table 6.10
Fair Spread S^* and CVA DVA behaviours with $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$.

If the counterparty's default event can cause a strong jump to the reference firm's default intensity, the investor should charge the counterparty a large amount of CVA as expected because the CDS is very valuable thus high loss to the investor at the time counterparty defaults. It is also noticed that the increment in CVA in the third column of table 6.10 is slightly less than those in the second column of table 6.5, although the increments in $\bar{\alpha}_{ref,cp}$ are the same. Because of the situation here, the investor is paying the fair rate S^*, which is fair in state $\mathcal{A}$. If the counterparty defaults, the fair rate S^* in state $\mathcal{A}$ is too expensive in state $\mathcal{C}$ and the losses to the investor are lower because there is no default contagion in this state. However, in the cases of table 6.5, the investor is paying on a lower fair rate at the measurement time when no default contagion is added in. For the same reason, the DVA in table 6.10 is not decreasing as fast as it is in table 6.5 against increasing $\bar{\alpha}_{ref,inv}$. We can explain this because the CDS in state $\mathcal{B}$, $V^{\mathcal{B}}$, is more likely to be a liability to the investor so a higher spread S^* should be paid so that the investor can benefit more from their own default.

Compared to the previous cases in table 6.5, the CVA and DVA react very differently with respect to the default contagion from the investor to the referencing firm $\bar{\alpha}_{ref,cp}$ and the contagion from the counterparty to the referencing firm respectively, the results of which are shown in the 4^{th} and 6^{th} columns in table 6.10. In table 6.10, when the default contagion from the counterparty to the referencing firm $\bar{\alpha}_{ref,cp}$ increases to 0.1, the amount of DVA cost for the investor is increasing to 8.2 *bps*. However, in table 6.5, the DVA is decreasing to 4.032 *bps*. Similarly, in table 6.10, when the default contagion from the investor to the referencing firm $\bar{\alpha}_{ref,inv}$ increases to 0.1, the amount of CVA charge to the investor is decreasing to 6.362 *bps*. However, in table 6.5, the CVA is increasing to 10.179 *bps*. As we explained for table 6.5, the changes in CVA and DVA due to the CDS will be more valuable in the investor's or the counterparty's default state. The reasons for the reverse behaviour in table 6.10 are detailed as follows.

The above two cases have one thing in common and that is that neither the investor nor the counterparty has any influence on the referencing firm's default risk so the CDS value in state $\mathcal{B}$ or $\mathcal{C}$ after the investor or the counterparty defaults might still be deemed to be fair with the spread S^* in state $\mathcal{A}$. Given default contagions do not change the probability of the counterparty or the investor to be the first default firm, the DVA increases in the 4^{th} column and the CVA decreases in the 6^{th} column is solely due to how the CDS value behaves differently after introducing a default contagion from the counterparty or from the investor.

We first analyse why CVA will be decreasing with $\bar{\alpha}_{ref,inv}$ with figure 6.5, then the reason for the DVA to be increasing with $\bar{\alpha}_{ref,cp}$ will follow. If the investor's default event can cause a default contagion to the referencing firm, the value of the CDS in state $\mathcal{C}$, denoted as $\hat{V}^{\mathcal{C}}$, is a function depending on both $\lambda_{ref}(t)$ and $\lambda_{inv}(t)$. In terms of CVA, we knew that the investor suffers a counterparty default loss only when the CDS value is positive, which corresponds to a relatively more risky referencing firm. The left panel of figure 6.5 tells us, the CDS value, with a very risky referencing firm, tends to be the same regardless how risky the investor is. One way to explain this is to say that when the referencing firm is very risky then the default contagion from the investor's default is less likely to happen. Besides, if the referencing firm is deemed to be very risky, even if the investor defaults and causes the default contagion to the referencing firm, the CDS will not become much more valuable since the CDS value is a concave function of λ_{ref} and the value of λ_{ref} is already large. In addition, according to the right panel of figure 6.5, if the referencing firm is very risky, the CDS becomes more likely to have a lower value as compared to the case without default contagion, which we denoted as $V^{\mathcal{C}}$. This is because the investor has to pay a higher spread to compensate the default contagion, which is not likely to happen with a very risky referencing firm. Consequently, the CVA will be lower as a result of the fact that the

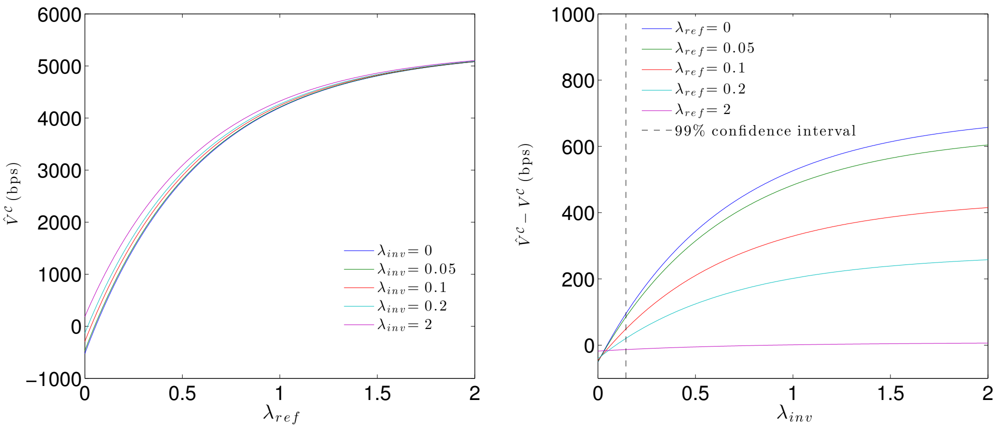

Figure 6.5
The changes in CDS value after adding default contagion and is traded at fair swap rate. An illustration of the difference of two CDS value. One CDS $\hat{V}^{\mathcal{C}}$ is with default contagion $\bar{\alpha}_{ref,inv} = 0.1$ in state $\mathcal{C}$ traded at fair spread $S^* = 328$ bps. The other $V^{\mathcal{C}}$ is without default contagion in state $\mathcal{C}$ traded at fair spread $S^* = 299$ bps.

CDS value $\hat{V}^{\mathcal{C}}$ is more likely to be lower with default contagion.

In the case of $\bar{\alpha}_{ref,inv} = 0$ and $\bar{\alpha}_{ref,cp} = 0$, the CDS value $\hat{V}^{\mathcal{B}}$ in state $\mathcal{B}$ is symmetric to the one value $\hat{V}^{\mathcal{C}}$ in state $\mathcal{C}$ as shown in figure 6.5. We know that the DVA only considers the situations in which the CDS has negative value to the investor, where the referencing firm is safe at the investor's default time. If the referencing firm is safe, whether the default contagion can occur depends on the default risk of the counterparty, so CDS value $\hat{V}^{\mathcal{B}}$ may be higher or lower than $V^{\mathcal{B}}$ depending on the how risky the counterparty is. This is similar to the way that an increase in the volatility of the CDS value cause the DVA to be higher in the 4^{th} column of table 6.10.

Default dependence risk does not have to be just a direct influence on a firm, rather it can happen in a roundabout way. For example, firm C is customer of firm A and it is highly dependent on another firm B. Once firm B goes into bankruptcy, firm C will be in a dangerous situation as well and if it defaults it will have implications for firm A. So we see that the firm A is impacted by the firm B's default even though there is no direct link. Indirect contagion risk to the referencing firm can also be considered in pricing the CDS spread S^*, CVA and DVA. In our model, the indirect contagion effect to the referencing firm is characterised by setting high values in both $\bar{\alpha}_{cp,inv}$ and $\bar{\alpha}_{inv,cp}$, which are the default contagions between the counterparty and the investor. As long as there is a default contagion from either the counterparty or the investor to the referencing firm, there will be indirect default contagion to the referencing firm.

		$\bar{\alpha}_{ref,cp} = 0.1$		$\bar{\alpha}_{ref,inv} = 0.1$	
S^* (bps)	$\bar{\alpha}_{cp,inv}$ and $\bar{\alpha}_{inv,cp}$	CVA (bps)	DVA (bps)	CVA (bps)	DVA (bps)
328	0.00	59.147	8.202	6.362	1.666
329	0.02	59.276	7.157	6.818	1.663
330	0.04	59.259	6.656	7.251	1.714
330	0.06	58.470	6.279	7.786	1.709
329	0.08	58.963	5.745	8.615	1.672
330	0.10	59.260	5.335	9.186	1.733
330	0.12	58.872	5.140	9.832	1.725
331	0.14	59.059	4.897	10.185	1.778
332	0.16	58.593	4.713	10.854	1.791
331	0.18	58.373	4.368	11.454	1.765
332	0.20	58.215	4.332	12.152	1.771

Table 6.11
The impact of indirect default contagion on CVA and DVA
A table for illustrating the CDS spread S^* CVA and DVA with increasing indirect default contagions $\bar{\alpha}_{cp,inv}$ and $\bar{\alpha}_{inv,cp}$ from 0 to 0.2 in tow occasions. One is there exists default contagion from the counterparty to the referencing $\bar{\alpha}_{ref,cp} = 0.1$ and the other is from the investor to the referencing $\bar{\alpha}_{ref,inv} = 0.1$. The fair spreads are the identical to two cases as referencing firm's default risk is the same.

For example, if the investor's default causes a default contagion jump $\alpha_{cp,inv}$ to the counterparty, this increases the chance of a default in the counterparty which can trigger $\alpha_{ref,cp}$ to the referencing firm.

We show in table 6.11 that the fair spreads S^* are only a few basis points more expensive than before, so the default risk of the referencing firm is not really affected in a significant way by the indirect default contagions as compared to direct default contagions. The fair spread S^* is only raised by a maximum 4 bps.

Table 6.11 shows, in the case with only $\bar{\alpha}_{ref,cp} = 0.1$, CVA slightly reduces with indirect default contagions. This is caused by the fact that only the counterparty's default can result in default contagion on the referencing firm. The indirect contagion can only affect the referencing firm when the investor defaults earlier than the counterparty. However, CVA measures the investor's possible losses due to the counterparty being the first to default. Therefore, in the state $\mathcal{C}$, the referencing firm's credit risk is indifferent to the investor's credit risk. Since the investor is required to pay slightly higher spreads S^*, the CDS value at counterparty's default time $V^{\mathcal{C}}(\tau_{cp})$ will be slightly lower meaning a lower CVA. On the other hand, the investor has to pay less DVA to the counterparty and this decreasing DVA can be seen in the fourth column. Because at the time the investor defaults, a default contagion jump is triggered to the counterparty $\lambda_{cp}(\tau_{inv}) = \lambda_{cp}(\tau_{inv}^{-}) + \alpha_{cp,inv}$. In state $\mathcal{B}$, the CDS is more valuable with a more risky counterparty because their default can directly impact on the referencing firm. Therefore, the CDS value $V^{\mathcal{B}}(\tau_{inv})$ is raised by the indirect default contagion $\bar{\alpha}_{cp,inv}$ so any gains to the investor are lessened.

In the other case with $\bar{\alpha}_{ref,inv} = 0.1$, a similar reasoning as detailed above can be applied but in a reverse way. We know that the indirect default contagion can only influence the referencing firm if the counterparty is the first to default. Therefore, we can see the CVA is increasing slightly by the indirect default contagion from the counterparty to the investor, which raises the investor's loss at the counterparty's default time. Also the DVA increases because the CDS value $V^{\mathcal{B}}$ in state $\mathcal{B}$ has a lower value, which is a larger gain to the investor.

In table 6.10, we observed that, when there are increasing default contagions from the investor to the referencing firm $\bar{\alpha}_{ref,inv}$, the CDS spread should be higher to compensate the default contagion risk and the corresponding CVA charge to the counterparty will be lower. On the other hand, the DVA charge to the investor will be higher when we raise the default contagion from the counterparty to the referencing firm $\bar{\alpha}_{ref,cp}$. The above two phenomena are presented in figure 6.5. However, the above patterns will revert after indirect contagions are considered, which is shown in figure 6.6. At the time the counterparty defaults, the CDS value after the default contagion $V^{\mathcal{C}}(\tau_{cp})$ becomes more valuable because the investor will become more risky when there are indirect contagions, thus the referencing firm will become more likely to default. Therefore, the CVA we charge is higher due to the fact that the investor's loss given default will be higher. We also see that the strength of indirect default contagion from the counterparty to the referencing firm will be stronger with greater $\bar{\alpha}_{ref,inv}$ so CVA is increasing against $\bar{\alpha}_{ref,inv}$. Similarly, with default contagion from investor to counterparty $\bar{\alpha}_{cp,inv}$, the CDS value at investor's default time $V^{\mathcal{B}}(\tau_{inv})$ is more valuable. However, this means the investor is less likely to benefit from their own default, which leads to a decrease in the DVA.

In terms of the fair CDS spread S^*, we show it is very sensitive to the mean-reverting speed, especially with default contagions. The mean reversion speed κ_{ref} measures how fast the referencing firm's default intensity λ_{ref} can return back to the

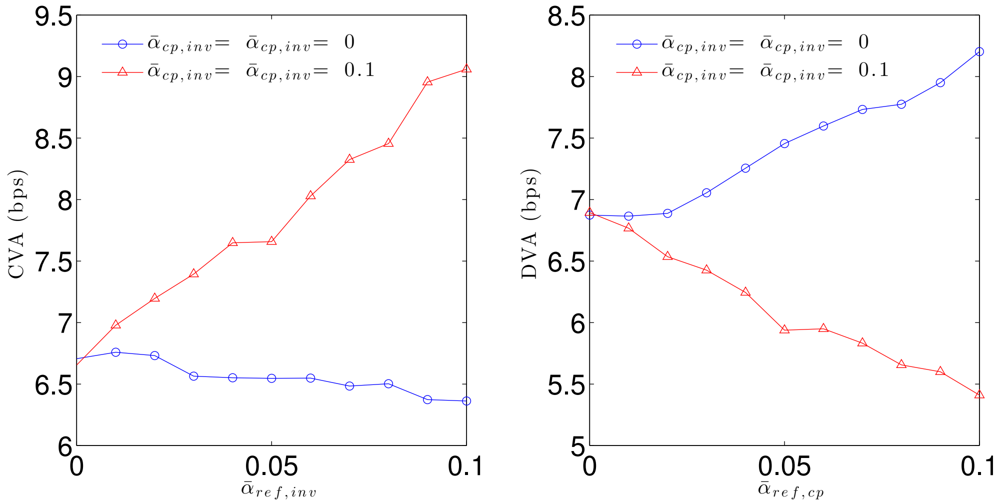

Figure 6.6
The change of CVA/DVA behaviour against $\bar{\alpha}_{ref,inv}/\bar{\alpha}_{ref,cp}$ with indirect default contagions
The left panel illustrates of CVA changes from decreasing to increasing against $\bar{\alpha}_{ref,inv}$ after adding indirect default contagions. The right panel illustrates of DVA changes from decreasing to increasing against $\bar{\alpha}_{ref,cp}$ after adding indirect default contagions.

normal level θ_{ref}. So, in our model, the mean-reverting speed also determines how severely the referencing firm can be affected by default contagions or recover from default contagions, which we discussed in Section 4.4.1. Figure 6.7 shows the fair swap spreads S^* of the 5-year CDS contract under increasing default contagion from both parties to the referencing firm. Suppose the referencing firm has a different mean-reverting speed κ_{ref}, then the fair CDS spread that the investor has to pay can vary a lot especially if we include default contagions. Even without default contagions, the fair spread will be slightly higher with a slower mean-reverting speed. When the referencing firm has slow mean-reverting speed, the referencing firm becomes much more likely to be affected by default contagions. Consequently, the CDS spread becomes more sensitive to default contagions. The spread increases from 303 *bps* to 550 *bps* with $\kappa_{ref} = 0.1$ compared to increases from 295 *bps* to 450 *bps* with $\kappa_{ref} = 0.9$.

In figure 6.8 we show the CVA charge and DVA cost given the spreads in figure 6.7. Obviously, the CVA charge and DVA cost will be much higher with a slow mean-reverting speed. The CVA with a slow mean-reverting speed shows faster growth against default contagion regardless the fact that the investor has to a pay significantly higher spread. On the other hand, the right panel of figure 6.8 also suggests that the slower the mean-reverting speed is, the higher the DVA cost will be. This is because

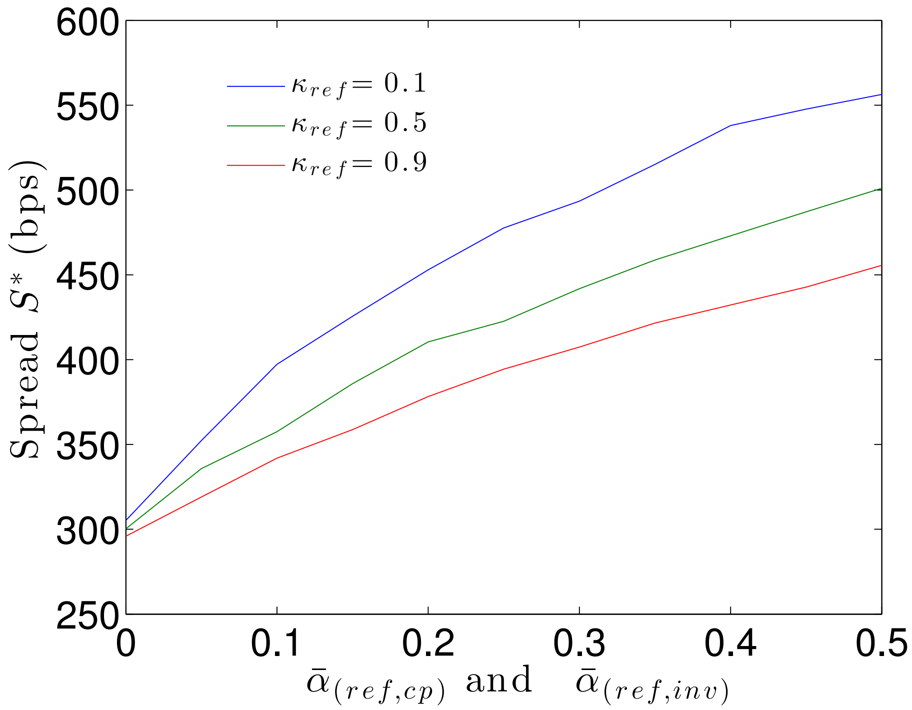

Figure 6.7
CDS spread sensitivity to mean-reverting speed and default contagions

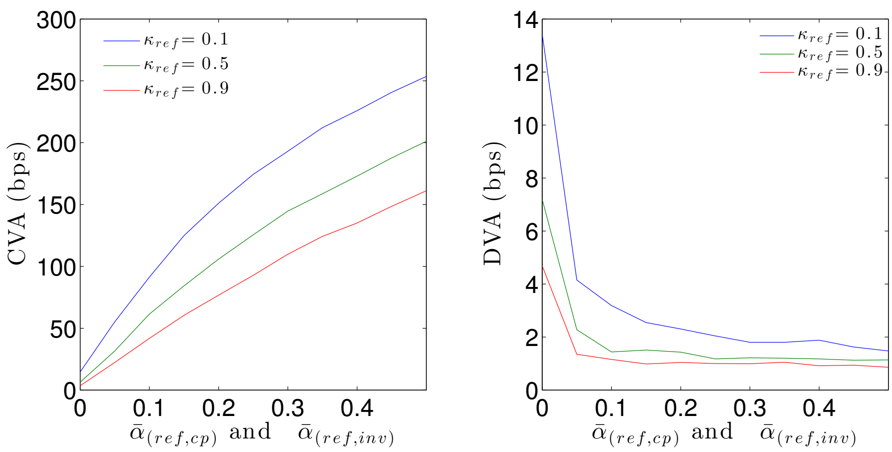

Figure 6.8
CVA and DVA with default contagions $\bar{\alpha}_{ref,cp}$, $\bar{\alpha}_{ref,cp}$ under different reversion speed κ_{ref}

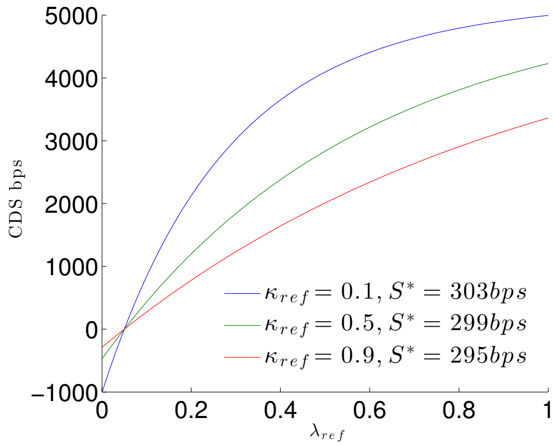

Figure 6.9
This figure shows 5 years CDS protection's value against reference firm's default intensity in default state D with different values of mean reversion speed κ_{ref}. Three CDSs are traded at corresponding fair rates as shown in the figure.

with slow mean reversion, the shape of the CDS value tends to be steeper against referencing firm's default intensity $\lambda_{ref}(t)$, for which we give an example of in figure 6.9. We know that if the intensity process is not likely to return to a normal level, then the default risk of the firm will be much higher with a large $\lambda_{ref}(t)$ or lower with a small $\lambda_{ref}(t)$. As a consequence, the corresponding CDS value will be much more valuable or in the opposite case much less valuable. Although the probabilities that the investor or the counterparty are the first to default is reduced by the more risky referencing firm, the investor's losses due to counterparty defaults or gains due to their own default will be higher, which lead to higher CVA charge and DVA cost.

6.5.3 Counterparty risk with Combinations of $\tilde{\alpha}$ and $\bar{\beta}$

In our model, the default risk of the referencing firm comes from three components. Apart from the referencing firm's idiosyncratic default risk, which is characterised by the CIR process, there is also systemic risk, which is modelled as external shocks with strength $\bar{\beta}$, and also default contagion risk. The total risk of default can be kept the same with different combinations of systemic risk $\bar{\beta}$ and default contagion risk $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$. In this section, we investigate the CVA and DVA under different combinations of default contagion risk and systemic risk while keeping the total default risk of the referencing firm consistent. We will show that, even though the CDS has the same price, the CVA and DVA can be very different depending on where the default

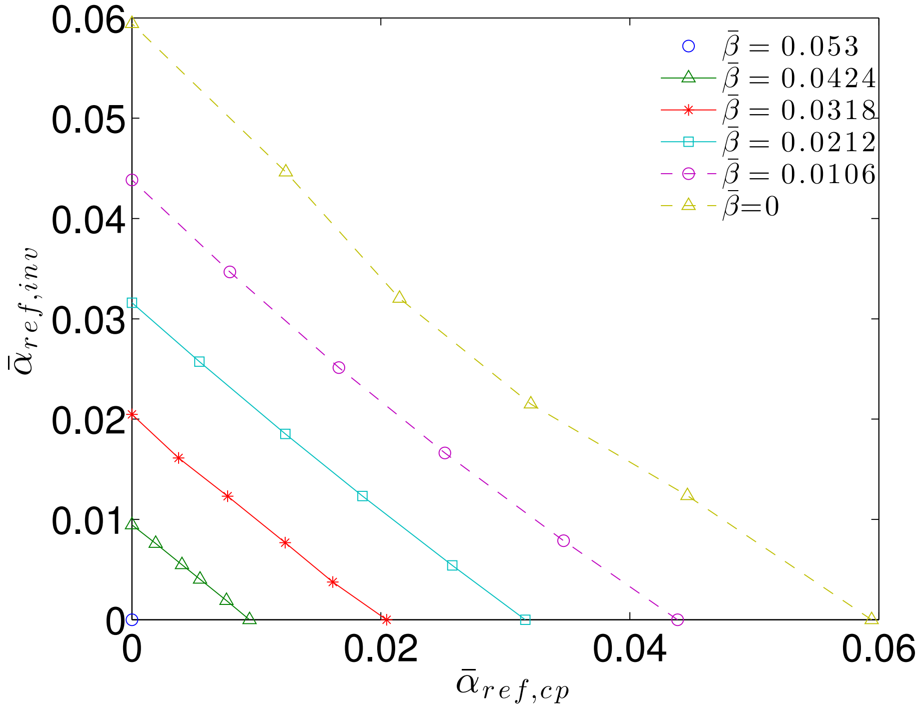

Figure 6.10
The combinations of $\bar{\beta}$, $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$ with the identical CDS spread $S^* = 300$

risk comes from.

Suppose the investor is going to buy a 5 year CDS from the counterparty at the fair swap rate $S^* = 300\ bps$. At the time the investor measures the referencing firm's credit risk, only systemic risk is taken into consideration with external shocks arrival rate $\bar{\lambda} = 0.05$ and shock strength $\bar{\beta} = 0.053$. Then we reduce the strength of external shocks $\bar{\beta}$ and search for the default contagion strengths $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$ so that the fair swap spread S^* remains constant in state $\mathcal{A}$. Therefore, the referencing firm has the same default risk and the CDS has the same price but the risk components are more weighted towards default contagion.

Figure 6.10 shows the combinations of external shock strength $\bar{\beta}$ and default contagion strengths $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$. Each line in figure 6.10 is the combination of default contagion jumps $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$ at a given $\bar{\beta}$ level. External shocks and default contagions are similar in the sense that both jumps are exponentially distributed. The difference lies on the exogenous jumps have a constant arrival rate with multiple arrivals while the default contagions have stochastic arrival rate, which are the counterparty's and the investor's default intensity. As the default contagion from the

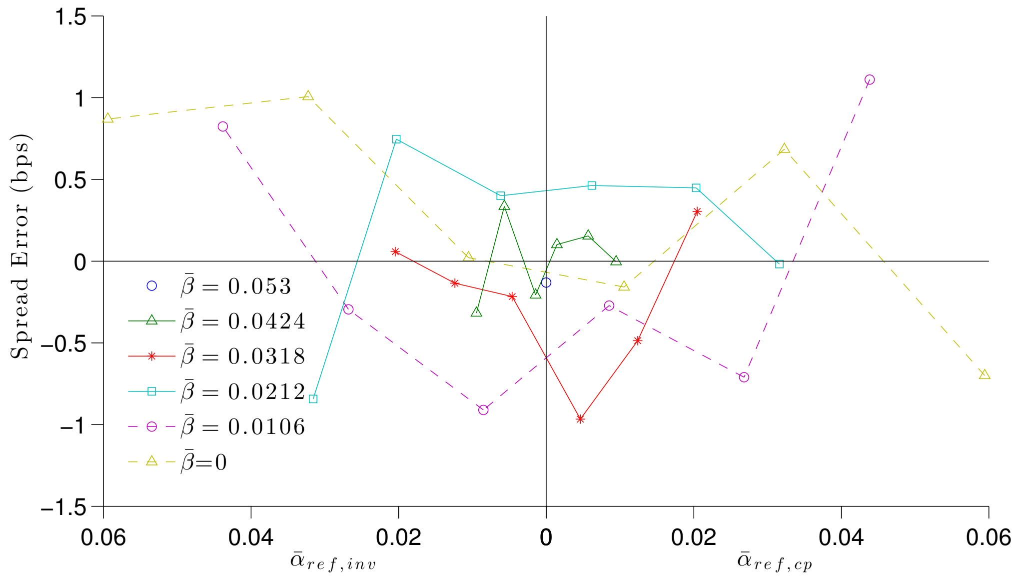

Figure 6.11
The calibration errors in $\bar{\beta}$, $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$

investor or the counterparty can only occur one time, we find that reductions on $\bar{\beta}$ requires a higher increment in default contagions to compensate. Foe example, the case that with only external shocks has $\bar{\beta} = 0.053$ and the case that with only the default contagion from the counterparty to the referencing firm is $\bar{\alpha}_{ref,cp} = 0.059$.

While trying to find the set of parameters $\bar{\beta}$, $\bar{\alpha}_{ref,cp}$, $\bar{\alpha}_{ref,inv}$ shown in figure 6.10, we minimise the target function, which is the difference between the fair CDS spread S^* given by $\bar{\beta}$, $\bar{\alpha}_{ref,cp}$, $\bar{\alpha}_{ref,inv}$ and the spread 300 *bps* we chose in the beginning. The calibration errors are shown in figure 6.11. Each spread with the new set of parameters is simulated with one million simulations so that the maximum error is just 1.11 *bps*.

Next we are able to measure the counterparty risk of the 5 year CDS contract given the parameters in figure 6.13. The CVA and DVA are given in figures 6.12 and 6.13. Each line in figures 6.12 and 6.13 is the CVA or DVA with the same external shocks $\bar{\beta}$ but different $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$. Apart from the referencing firm's idiosyncratic factors, if the default risk only comes from external shocks, the CVA and DVA are around 7.02 *bps* and 6.89 *bps* respectively.

However, if the referencing firm's default risk comes from default contagion, the CVA charge raises significantly against $\bar{\alpha}_{ref,cp}$. In the scenario, where the default risk only due to the default contagion from the counterparty, $\bar{\beta} = 0$, $\bar{\alpha}_{ref,inv} = 0$, the CVA is significantly greater than 7.02 *bps*, reaching a maximum of 34 *bps*. This implies the

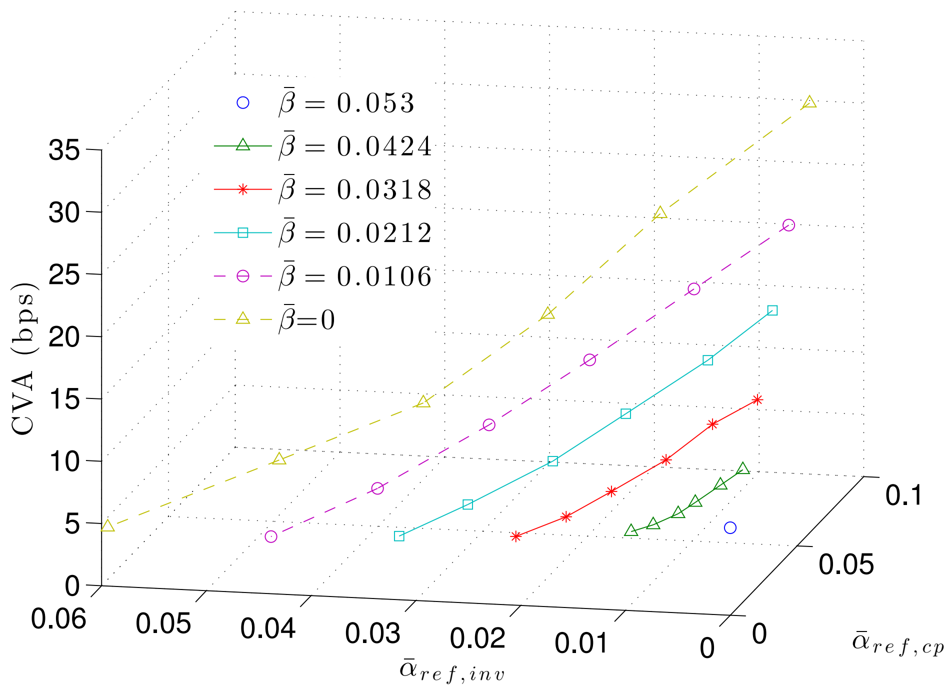

Figure 6.12
The CVA of the fair 5-year CDS contract ($S^* = 300$ bps) with different combinations of $\bar{\beta}$, $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$.

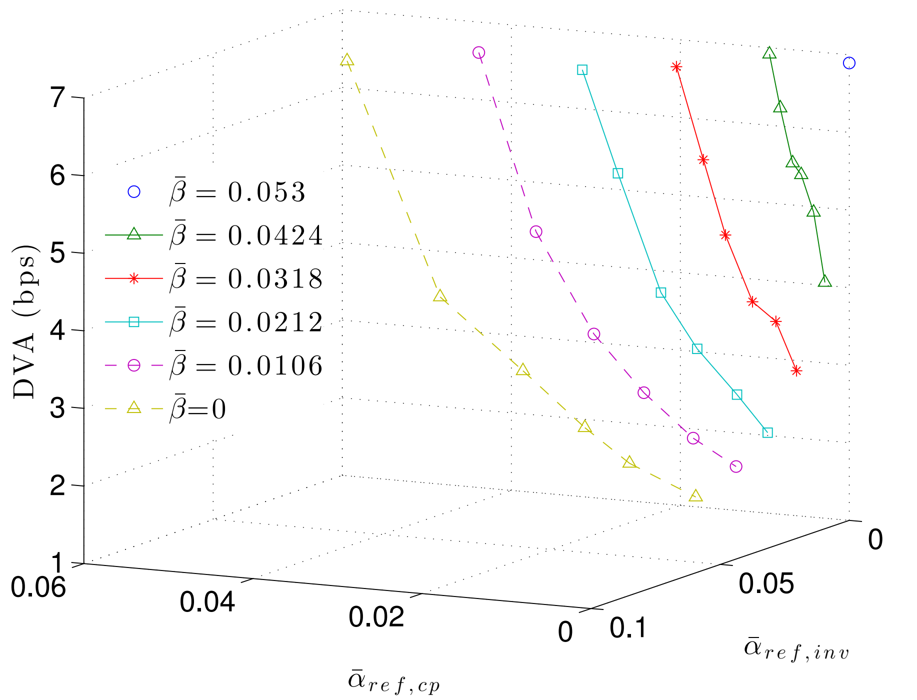

Figure 6.13
The DVA of the fair 5-year CDS contract ($S^* = 300$ bps) with different combinations of $\bar{\beta}$, $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$.

investor has not charged enough CVA from the counterparty to cover the possible loss due to a counterparty default.

On the other hand, it is obvious that if the investor underestimates the default contagion from the investor to the referencing $\bar{\alpha}_{ref,inv}$, then the investor will significantly overestimate the possible gains from their own default and give the counterparty too much DVA. Although the spread $S^* = 300bps$ is too high in state $\mathcal{B}$ in this case, where the investor already defaulted, the default contagion from the investor to the referencing firm $\bar{\alpha}_{ref,inv}$ will increase the CDS value significantly, which will not lead to any gains for the investor.

Another important result that we notice is that the CVA charge decreases towards a minimum for cases when $\bar{\alpha}_{ref,cp} = 0$, as these are the cases where the reductions in external shocks are only compensated by the default contagion from the investor's default. Similarly, the DVA decreases towards a minimum as we approach $\bar{\alpha}_{ref,inv} = 0$, as now the reductions in external shocks are only compensated by the default contagion from the counterparty's default. The reason is that all the three firms become safer when we reduce $\bar{\beta}$. Therefore, the probability that the counterparty or the investor will be the first firm to default will be lower. As well as this, we know that the CDS value will not be raised by a default contagion after the counterparty defaults with $\bar{\alpha}_{ref,cp} = 0$ or after the investor defaults with $\bar{\alpha}_{ref,inv} = 0$. In other words, the loss given default is not influenced by the default event but the probability that the default event will happen is lower. The CVA and DVA decreases as a consequence.

6.6 Conclusion

In this chapter, we extend our proposed default contagion model in Chapter 5 to include the default risk of the CDS buyer, namely the investor. The counterparty credit risk faced by the CDS buyer is thus extended to bilateral counterparty risk, including CVA and DVA.

The extension leads our valuation problem to have three stochastic default processes, which means that we need high-dimensional numerical schemes for pricing. A hybrid numerical scheme is proposed, in which we combine Monte-Carlo simulation and finite-difference schemes in order to simulate the CVA and DVA with default contagions amongst the three firms. By solving the CVA in a special case, we can demonstrate that our proposed numerical scheme to converge to the exact solutions as expected given a small enough time step size in the default intensities' sample path simulation.

From the investor's prospective, we analyse the profit and loss in the CVA and the DVA due to changing market conditions, which are different from the initial time at which the contract is agreed. Four scenarios are studied, where the counterparty's and the investor's default events are able to cause default contagions to the referencing firm, where the firms' intensity volatilities are higher, the arrival rate of external shock becomes more frequent and a case in which either of the counterparty or the investor becomes more risky. Among those changing in market conditions, the investor needs to be more wary of changes in default contagion because the investor will suffer substantially losses in the CDS's risk-adjusted value from both the CVA and the DVA. In particular, the CVA losses can be substantially more than those from the DVA. Increase in intensity volatilities mostly lead to CVA loss and DVA gain, however, we should notice the exceptions that the investor will have a CVA gain rather than loss under higher volatilities when there is a strong default contagion from the counterparty to the referencing firm. Unlike intensity volatilities, increasing the frequency at which external shocks arrive leads to both CVA and DVA losses. Finally, if the counterparty/investor deteriorates in their credit risk, the investor losses/gains in the CVA and DVA due to the counterparty/investor being more likely to default earlier. When there are default contagions, the CVA profit and losses are significantly more sensitive to the changes in two firms' credit risk but DVA will be less sensitive.

Next, we price the fair CDS spread and calculate the corresponding CVA and DVA when the transaction is about to initiate, under various market circumstances. Since the CDS spread has taken all components of the referencing firm's default risk into account, the differences we see in the CVA/DVA are much less pronounced as compared to the previous case where the spread does not reflect the default contagion risk. We found that the DVA can actually increase against increasing $\bar{\alpha}_{ref,cp}$ and the CVA will reduce with increasing $\bar{\alpha}_{ref,inv}$, which are opposite trends from the previous experiment, where market circumstances change after the CDS be initiated. Although indirect default contagions do not contribute significantly to a referencing firm's default risk, the existence of indirect default contagions to the referencing firm, $\bar{\alpha}_{inv,cp}$ and $\bar{\alpha}_{cp,inv}$, reverses the behaviour of CVA and DVA against direct contagions $\bar{\alpha}_{ref,cp}$ and $\bar{\alpha}_{ref,inv}$. We found that the referencing firm's low capacity to recover from default contagion will lead to a situation in which the investor pays a higher spread and charges the counterparty a higher CVA. Although the investor needs to pay a higher DVA the increase in DVA is tiny while the increases in CDS spread and CVA are significant.

Finally, we study the CVA and DVA under different contributions from the external shock risk and default contagion risk given the 5 year default probability of the referencing firm is fixed. Our numerical examples show the dangers if the investor neglect default contagion risk. Although the CDS has the same price, the counterparty risk faced by the investor can be substantially different. To the investor, ignoring the default contagion risk from the counterparty to the referencing firm means that they may undervalue the CVA charge by as little as a fifth of what it should be. On the other hand, underestimating the default contagion risk from the investor to the referencing firm leads to them paying too much DVA although the difference is relatively small compared to the CVA.

Chapter 7

Summary and Future Research

7.1 Summary

This thesis proposes a new approach for modelling default contagions under a reduced-form framework. Our approach provides us with added flexibility to model the form of default contagion shocks and we are able to show that the valuation of credit claims can be easily solved to finite-difference or hybrid numerical schemes, which have been the restrictions of previous contagion models. Further, by specifying a mean-reverting intensity process, we can enable firms to recover from default contagion according to their capacity to recover. We develop fast and efficient numerical schemes for pricing credit derivatives and measuring counterparty risk using our proposed model, which tries to take into account both default contagion risk and systemic risk.

Before proposing the new model, we are able to derive the PDEs for corporate bonds, CDS and the CVA using the techniques provided by Wilmott et al. (1995) and Burgard and Kjaer (2011). Pricing credit derivatives and measuring CVA with finite-difference methods has not been widely discussed in the literature, so we review the one- and two-dimensional finite-difference schemes and boundary conditions to solve the PDEs of a CDS and the CVA of a CDS. Testing different types of boundary conditions for CDS contracts against the semi-analytic solution, we found that the heuristic Robin condition is superior to other boundary conditions in terms of numerical errors at boundaries. The resulting finite-difference scheme is surprisingly four times faster than the benchmark numerical integration (Trapezoidal rule) of the semi-analytic formula to reach the same level of accuracy. Due to the convenience of finite-difference method, we are able to examine the sensitivity of the CVA to changes in model parameters such as volatility of credit risk. We found there are extreme situations where CVA can reduce with higher correlation between default intensities.

Next, we propose the new default contagion model and demonstrate how the valuation of a generic credit claim under our model can be linked to the solutions of a system of PDEs. Further, we show a variety of uni- and multi-variate distributions for modelling default contagion shocks. In order to obtain numerical solutions, we specify the default intensities to be of mean-reverting type so that the firms can recover from default contagion. Our numerical scheme is shown to be accurate and efficient for pricing credit claims under the model. Since our default contagion model enables firms to recover from default contagions, the marginal as well as joint default term structure show high sensitivity to the capacity of recovering and lead to an unique default probability term-structure, which is very different from a previous default contagion model and an independent default model. We also reveal the circumstances under which the default contagion will have the strongest impact on the CVA.

The default contagion model proposed is extended to consider both default contagions and systemic risk, which is represented by a jump process affecting all firms. New approximation methods are proposed to improve the OS finite-difference method for solving the PIDEs resulting from modelling systemic risk. Without additional computational time, our improved boundary condition is able to achieve up to 100 times more accurate solutions and the problem that solutions *slump* near the upper boundary is resolved. Evaluating credit claims with our model requires two-dimensional finite-difference scheme for PIDE, which we tackle by combining the ADI scheme and the OS scheme. We show the difference between modelling systemic risk and default contagion risk and their implications for pricing CDSs and CVAs. We found that the default contagions have a much stronger impact on the CVA as compared to systemic risk. However, the CDS spreads are more sensitive to systemic risk. This is because the firms' default probabilities are tied to systemic risk but default contagions directly influence losses at the counterparty's default time.

Finally, we extended the model to also consider the default risk of the CDS investor so that we may study the bilateral counterparty risk. This extension leads to multi-dimensional problems and a hybrid Monte-Carlo and finite-difference scheme is proposed to simulate CDS spreads, CVAs and DVAs. The hybrid numerical scheme shows convergence to the benchmark solution. We investigate the behaviours of the CVA and DVA of a 5-year OTC CDS under two situations. In the first situation we present the CVA, DVA gains and losses due to changing market circumstances, such as increasing intensity volatilities, more frequent arrival of systemic shocks and credit deterioration of either party, all this after the investor has agreed the price of the CDS. We found that higher intensity volatilities usually lead to larger losses at default thus higher CVA and DVA however this pattern reverts in CVA when there are strong default contagions. It appears that default contagion raises the sensitivity of CVA and DVA to changes in the systemic risk and default intensities. For the second situation we price the fair CDS spread and the corresponding CVA charge and DVA cost under different market conditions. We show how the CVA and DVA react very differently to default contagions compared to the former case because the default risk of the referencing firm has been priced into the CDS spread. Indirect default contagions are also analysed and we show the CVA and DVA can have different behaviour compared to direct default contagion with or without indirect default contagions. Finally, we show that even if the referencing firm's CDS has the same price, the different contribution of systemic risk and default contagion risk can lead to the investor facing very different counterparty risk. The investor may charge substantially lower CVA and pay higher DVA as the result of underestimating the default contagion risk from the counterparty and the investor itself to the referencing firm.

7.2 Future Research

In this thesis, we measure the counterparty risk of credit derivatives with simplifications to the features in the derivative transaction. In practice, a collateral provision may apply to the derivative transaction, which requires both trading parties to post collateral to each other according to the value movement of the trade. The collateral requirement will partially reduce the counterparty risk because the collateral posted may compensate the loss at the default time. With collateralisation, the counterparty risk faced by the two parties, who trade CDS contracts, is related to how often the two parties settle collateral, namely *margining frequency*, the minimum transfer amount and more importantly the default correlation, as indicated by Brigo et al. (2014). Using our proposed model, we would expect that the CVA and DVA will be reduced with the collateral requirement but there will still be loss at the default due to default contagion risk that can not be compensated by the collateral. It would be interesting to investigate the CVA and DVA behaviour with both default contagions and collateral.

Future research could also incorporate our default contagion model into a framework with funding constraints, such as Crépey (2015a). When entering into a derivative position, the investor may have to obtain funds for operations such as hedging the position, posting collateral, paying coupons and so on. However, there is a cost for obtaining funds, namely the funding cost, as it is unlikely even for big financial institutions that they will be able to borrow and lend at the risk-free rate. Alternatively, if the investor can receive funds from the derivative position, the investor may use those funds for other activities, which is often called funding benefits. A firm's funding cost and benefit are, of course, related to its default risk, and this is where our default contagion model can come into the picture. The funding cost or benefit, as is argued frequently in the literature, for example Laughton and Vaisbrot (2012); Castagna (2012); Burgard and Kjaer (2012); Crépey (2015a), should be considered in the valuation of the derivative. It would be interesting to investigate what implications does our default contagion framework have for the CVA and DVA after the funding constraints are considered.

Incorporating a stochastic interest rate is another possible direction to complete our model. It is important to investigate the correlation between the interest rate and the default intensities and its implication for CVA and DVA because empirical studies indicate the time period with large number of default events is likely to be accompanied by a low interest rate environment. However, the extended model may become computationally more difficult as we are under a four-dimensional framework.

Alternatively, we may pay attention to the efficiency of the Monte Carlo simulation. The hybrid Monte-Carlo and finite-difference scheme has been shown to be a feasible tool for measuring the CVA to CDS contracts under our default contagion framework. In Section 6.4, we notice that default simulation in a short time-horizon is a small probability event so we observe that the convergence will be slower. This problem is similar to pricing deep out-of-money options by simulation, where the accuracy can be improved by *importance sampling* according to Glasserman (2003). Therefore, it might be worth considering how to apply importance sampling when simulating default times in order to improve the CVA and DVA computational efficiency.

Similar to probably every other default contagion model, the calibration problem is a major challenge, which is neglected in this thesis. It is an open question as to how we can quantify the impact to a firm's default intensity from other firms' default events, which are the expected default contagion jump sizes $\tilde{\alpha}$. In addition, the estimation of other parameters (e.g. external shock size, external shock arrival frequency, etc.) must make the model produce survival probabilities which match the market implied survival probabilities of a firm. We give numerical examples of how we can lower the external shocks' strength and raise default contagion shock strength while maintaining the CDS price or the survival probability equivalently. This suggest we may first calibrate two jump-CIR processes to two firms' market implied survival probabilities then incorporate default contagion between the two firms by the trade-off between external shock strength and default contagion. We hope that by providing the accurate and efficient methods in this thesis we have moved a step closer towards a feasible calibration of our default contagion model.

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Appendix A

ADI scheme Parameters for CVA

$$-\alpha_{i,j}Y^n_{i-1,j} + (\frac{1}{0.5\Delta t} - \beta_{i,j})Y^n_{i,j} - \gamma_{i,j}Y^n_{i+1,j} = Z_{i,j} \tag{A.1}$$

where

$$Z_{i,j} = \delta_j U_{i,j-1} + \frac{1}{0.5\Delta t} + \epsilon_j U_{i,j} + \varepsilon_j U_{i,j+1} + \eta_{i,j} + f_{i,j}.$$

Solutions $Y_{i,j}$ are the intermediate solutions. At the second half step, we estimate J-direction at the unknown time step and I-direction at the known time step, which leads to

$$-\delta_{i,j}U^n_{i,j-1} + (\frac{1}{0.5\Delta t} - \epsilon_{i,j})U^n_{i,j} - \varepsilon_{i,j}U^n_{i,j+1} = \tilde{Z}_{i,j}, \tag{A.2}$$

where

$$\tilde{Z}_{i,j} = \alpha_i U_{i-1,j} + (\frac{1}{0.5\Delta t} + \beta_j)U_{i,j} + \gamma_j U_{i+1,j} + \eta_{i,j} + f_{i,j}.$$

The coefficients given by

$$\alpha_{i=0} = 0\ , \qquad \alpha_i = -\frac{\kappa_1(\theta_1 - i\Delta\lambda_1)}{2\Delta\lambda_1} + \frac{0.5\sigma_1^2 i\Delta\lambda_1}{\Delta\lambda_1^2}\ ,\ \alpha_{i=I} = -\frac{\kappa_1(\theta_1 - i\Delta\lambda_1)}{2\Delta\lambda_1}$$

$$\gamma_{i=0} = \frac{\kappa_1\theta_1}{2\Delta\lambda_1}\ , \qquad \gamma_i = -\frac{\kappa_1(\theta_1 - i\Delta\lambda_1)}{2\Delta\lambda_1} + \frac{0.5\sigma_1^2 i\Delta\lambda_1}{\Delta\lambda_1^2}\ ,\ \gamma_{i=I} = 0$$

$$\beta_{i=0} = -\frac{\kappa_1\theta_1}{\Delta\lambda_1} - \frac{(r + j\Delta\lambda_2)}{2}\ ,\ \epsilon_{j=0} = -\frac{\kappa_2\theta_2}{\Delta\lambda_2} - \frac{(r + i\Delta\lambda_1)}{2}$$

$$\beta_i = -2\frac{0.5\sigma_1^2 i\Delta\lambda_1}{\Delta\lambda_1^2} - \frac{(r + i\Delta\lambda_1 + j\Delta\lambda_2)}{2}\ ,\ \epsilon_j = -2\frac{0.5\sigma_2^2 j\Delta\lambda_2}{\Delta\lambda_2^2} - \frac{(r + i\Delta\lambda_1 + j\Delta\lambda_2)}{2}$$

$$\beta_I = \frac{\kappa_1(\theta_1 - i\Delta\lambda_1)}{\Delta\lambda_1} - \frac{(r + i\Delta\lambda_1 + j\Delta\lambda_2)}{2}\ ,\ \epsilon_{j=J} = \frac{\kappa_2(\theta_2 - j\Delta\lambda_2)}{\Delta\lambda_2} - \frac{(r + i\Delta\lambda_1 + j\Delta\lambda_2)}{2}$$

$$\delta_{j=0} = 0 \ , \qquad \delta_j = -\frac{\kappa_2(\theta_2 - j\Delta\lambda_2)}{2\Delta\lambda_1} + \frac{0.5\sigma_2^2 j\Delta\lambda_2}{\Delta\lambda_1^2} \ , \ \delta_{j=J} = -\kappa_2(\theta_2 - j\Delta\lambda_2)\frac{1}{2\Delta\lambda_1}$$

$$\varepsilon_{j=0} = -\frac{\kappa_2\theta_2}{2\Delta\lambda_2} \ , \qquad \varepsilon_j = \frac{\kappa_2(\theta_2 - j\Delta\lambda_2)}{2\Delta\lambda_2} - \frac{0.5\sigma_2^2 j\Delta\lambda_2}{\Delta\lambda_2^2} \ , \ \varepsilon_{j=J} = 0,$$

$$\begin{aligned}
\eta_{i,j} =& \frac{\rho\sigma_1\sigma_2\sqrt{\lambda_1\lambda_2}}{4\Delta\lambda_1\Delta\lambda_2}(U_{i+1,j+1} - U_{i-1,j+1} - U_{i+1,j-1} + U_{i-1,j-1}) \\
\eta_{i=I,j} =& \frac{\rho\sigma_1\sigma_2\sqrt{\lambda_1\lambda_2}}{2\Delta\lambda_1\Delta\lambda_2}(U_{I,j+1} - U_{i-1,j+1} - U_{I,j-1} + U_{i-1,j-1}) \\
\eta_{i,j=J} =& \frac{\rho\sigma_1\sigma_2\sqrt{\lambda_1\lambda_2}}{2\Delta\lambda_1\Delta\lambda_2}(U_{i+1,J} - U_{i-1,J} - U_{i+1,J-1} + U_{i-1,J-1}) \\
\eta_{i=I,j=J} =& \frac{\rho\sigma_1\sigma_2\sqrt{\lambda_1\lambda_2}}{\Delta\lambda_1\Delta\lambda_2}(U_{I,J} - U_{I-1,J} - U_{I,J-1} + U_{I-1,J-1}) \\
f_{i,j} =& j\Delta\lambda_2 \max\{\frac{V_i^{n+1} + V_i^n}{2}, 0\}.
\end{aligned}$$

The function $f_{i,j}$ corresponding to the CDS value solved by the second equation of the PDE system 3.20, which is solved by Crank-Nicolson. In Crank-Nicolson scheme, there is no intermediate steps between time n and $n+1$. Therefore, linear interpolation is taken to approximate the CDS value at the middle. For a fixed $j \in [0, J]$ or $i \in [0, I]$, equation A.1 or A.2 is corresponding to a tridiagonal linear system, which we solves by Gaussian elimination.

Appendix B

Lemmas in default contagion model

In default contagion modelling, pricing derivatives at time t referencing to firm 1 under two events $\{\tau_1 > t, \tau_2 > t\}$ and $\{\tau_1 > t, \tau_2 \leq t\}$. Because when $\{\tau_1 > t, \tau_2 \leq t\}$, we have to price the derivatives with the possibility that firm 2 defaults prior to firm 1 and causes a default contagion to firm 1. Note that we model the default contagion jump $\alpha_{1,2}$ as a non-negative random variable with density function $\eta(\alpha_{1,2})$. Therefore, given firm 2's default time t^-, a well-behaved $C^{1,2}$ value function $L()$ of time t and λ_1 satisfies

$$\mathbb{E}\left[L(\lambda_1(t^-), t^-, T)|\mathcal{G}_{t^-}^{-1}\right] = \int_0^\infty \mathbb{E}\left[L(\lambda_1(t^-) + \alpha_{1,2}, t, T)|\mathcal{G}_t^{-1}\right] \eta(\alpha_{1,2}) d\alpha_{1,2}. \tag{B.1}$$

Given the default intensity processes (4.4), if $\tau_2 \leq t$, then $\lambda_1(t)$ is $\mathcal{G}_t^{-1}$-measurable. It means that the default time τ_1 depends on the default time of firm 2, which is $\mathcal{H}_t^2$-measurable. On the other hand, on the event $\{\tau_1 > t, \tau_2 > t\}$, both $\lambda_1(t)$ and $\lambda_2(t)$ are $\mathcal{F}_t$-measurable. The processes (4.4) are driven by Brownian motions, which are $\mathcal{F}_t$-measurable, in state $\mathcal{A}$. In addition, $\mu_i(\lambda_i(t), t)$ and $\sigma_i(\lambda(t), t)$ for $i = 1, 2$ are assumed $\mathbb{F}$-adapted Lipschitz real value functions.

Here we assumed the existence of default intensities $\lambda_1(t)$ and $\lambda_2(t)$ with respect to default times τ_1 and τ_2, which implies the default processes $F_t^1 = \mathbb{Q}(\tau_1 \leq t|\mathcal{G}_t^{-1})$ and $F_t^2 = \mathbb{Q}(\tau_2 \leq t|\mathcal{G}_t^{-2})$ are absolutely continuous. Consequently, $dF_t^i = e^{-\int_0^t \lambda_i(u)du}\lambda_1(t)$ for $i = 1, 2$.

In order to prove Lemma 4.3.1, we prove the pricing formulas of cash flows X_T conditional on $\tau_1 > T$, Z_{τ_1} at default time τ_1 and continuous payments A_s conditional on $\tau_1 > s$ for $t < s \leq T$ using our default contagion model.

At first, we will discuss how random variables $1_{\{\tau_1>t,\tau_2>t\}}X$ and $1_{\{\tau_1>t,\tau_2\leq t\}}X$ are measured using our default contagion model.

Lemma B.0.1. *For any $\mathcal{G}-$ measurable and $\mathbb{Q}$-integrable random variable X, we have for any $t \in \mathbb{R}_+$,*

$$\begin{aligned}
&\mathbb{E}\left[1_{\{\tau_1>t,\tau_2\leq t\}}X|\mathcal{G}_t\right]\\
=&1_{\{\tau_1>t,\tau_2\leq t\}}\mathbb{E}\left[X|\mathcal{G}_t\right]\\
=&1_{\{\tau_1>t,\tau_2\leq t\}}\mathbb{E}\left[X|\mathcal{F}_t\vee\mathcal{H}_t^1\vee\mathcal{H}_t^2\right]\\
=&1_{\{\tau_1>t,\tau_2\leq t\}}\frac{\mathbb{E}\left[X1_{\{\tau_1>t\}}|\mathcal{G}_t^{-1}\right]}{\mathbb{Q}(\tau_1>t|\mathcal{G}_t^{-1})}
\end{aligned} \tag{B.2}$$

Lemma (B.0.1) is a direct extension to Lemma (2.2.1), except we consider the event $\{\tau_1 > t, \tau_2 \leq t\}$ in a two firms environment. In the forth equality, the expectation must be conditional on $\mathcal{G}_t^{-1} = \mathcal{F}_t \vee \mathcal{H}_t^2$ because the default time τ_1 depends on the default time tau_2 on the event $\{\tau_1 > t, \tau_2 \leq t\}$ under our default contagion model.

Then the corollary to Lemma B.0.1 follows

Corollary 2. *Let X_T be a $\mathcal{F}_T$-measurable and $\mathbb{Q}$-integrable random variable. Then for every $t \leq T$,*

$$\begin{aligned}
&\mathbb{E}\left[1_{\{\tau_1>T,\tau_2\leq t\}}X_T|\mathcal{G}_t\right]\\
=&1_{\{\tau_1>t,\tau_2\leq t\}}\mathbb{E}\left[X_T1_{\{\tau_1>T\}}|\mathcal{G}_t^{-1}\right]\\
=&1_{\{\tau_1>t,\tau_2\leq t\}}\frac{\mathbb{E}\left[X_T1_{\{\tau_1>T\}}|\mathcal{G}_t^{-1}\right]}{\mathbb{Q}(\tau_1>t|\mathcal{G}_t^{-1})}\\
=&1_{\{\tau_1>t,\tau_2\leq t\}}\frac{\mathbb{E}\left[e^{-\int_0^{\tau_2^-}\lambda_1(u)du}e^{-\int_{\tau_2}^T\lambda_1(u)du}X_T|\mathcal{G}_t^{-1}\right]}{e^{-\int_0^{\tau_2^-}\lambda_1(u)du}e^{-\int_{\tau_2}^t\lambda_1(u)du}}\\
=&1_{\{\tau_1>t,\tau_2\leq t\}}\mathbb{E}\left[e^{-\int_t^T\lambda_1(u)du}X_T|\mathcal{G}_t^{-1}\right]
\end{aligned} \tag{B.3}$$

The third equality holds because we assumed the default process F_t^1 to be absolutely continuous thus default intensity exist. We remind ourselves that $\lambda_1(\tau_2) = \lambda_1(\tau_2^-) + \alpha_{1,2}$ at default time τ_2, where $\alpha_{1,2}$ is a $\mathcal{F}$-measurable variable. Therefore, the integrals in the third equality should indicate the difference.

Corollary 2 tells how to evaluate a payment X conditional on firm 1 survive to time T on the event that $\{\tau_1 > t, \tau_2 \leq t\}$. The most important step is to show the evaluation of this payment on the event that $\{\tau_1 > t, \tau_2 > t\}$.

Lemma B.0.2. *For any $\mathcal{G}-$ measurable and $\mathbb{Q}$-integrable random variable X, we have for any $t \in \mathbb{R}_+$,*

$$\begin{aligned}
&\mathbb{E}\left[1_{\{\tau_1>t,\tau_2>t\}}X|\mathcal{G}_t\right] \\
=&1_{\{\tau_1>t,\tau_2>t\}}\mathbb{E}\left[X|\mathcal{G}_t\right] \\
=&1_{\{\tau_1>t,\tau_2>t\}}\mathbb{E}\left[X|\mathcal{F}_t \vee \mathcal{H}_t^1 \vee \mathcal{H}_t^2\right] \\
=&1_{\{\tau_1>t,\tau_2>t\}}\frac{\mathbb{E}\left[X1_{\{\tau_1>t,\tau_2>t\}}|\mathcal{F}_t\right]}{\mathbb{Q}(\tau_1>t,\tau_2>t|\mathcal{F}_t)}
\end{aligned} \tag{B.4}$$

The last equality is conditional on filtration $\mathcal{F}_t$ because the default times of firm 1 and 2 depends only on the history of the Brownian motions, which are $\mathcal{F}_t$-measurable.

The following corollary to Lemma B.0.1 tells us how a cash flow X_T conditional on firm 1 survival beyond time T in state $\mathcal{A}$.

Corollary 3. *Let X_T be a $\mathcal{F}_T$-measurable and $\mathbb{Q}$-integrable random variable. Then for every $t \leq T$,*

$$\begin{aligned}
\mathbb{E}\left[1_{\{\tau_1>T,\tau_2>t\}}X_T|\mathcal{G}_t\right] = 1_{\{\tau_1>t,\tau_2>t\}}\Bigg(&\mathbb{E}\left[X_T e^{-\int_t^T \lambda_1(u)+\lambda_2(u)du}\Big|\mathcal{F}_t\right] \\
+\mathbb{E}\left[\int_t^T e^{-\int_t^{s^-}\lambda_1(u)+\lambda_2(u)du}\lambda_2(s^-)\mathbb{E}\left[e^{-\int_s^T \lambda_1(u)du}X_T\middle|\mathcal{G}_{s^-}^{-1}\right]ds^-\middle|\mathcal{F}_t\right]&\Bigg)
\end{aligned} \tag{B.5}$$

Proof.

$$
\begin{aligned}
&\mathbb{E}\left[1_{\{\tau_1>T,\tau_2>t\}}X_T|\mathcal{G}_t\right] = 1_{\{\tau_1>t,\tau_2>t\}}\mathbb{E}\left[1_{\{\tau_1>T\}}X_T|\mathcal{G}_t\right]\\
=&1_{\{\tau_1>t,\tau_2>t\}}\mathbb{E}\left[X_T 1_{\{\tau_1>T,\tau_2>T\}} + X_T 1_{\{\tau_1>T,t<\tau_2\le T\}}|\mathcal{G}_t\right]\\
=&1_{\{\tau_1>t,\tau_2>t\}}\mathbb{E}\left[X_T 1_{\{\tau_1>T,\tau_2>T\}}|\mathcal{G}_t\right] + 1_{\{\tau_1>t,\tau_2>t\}}\mathbb{E}\left[X_T 1_{\{\tau_1>T,\tau_2\le T\}}|\mathcal{G}_t\right]\\
=&1_{\{\tau_1>t,\tau_2>t\}}\frac{\mathbb{E}\left[X_T 1_{\{\tau_1>T,\tau_2>T\}}|\mathcal{F}_t\right]}{\mathbb{Q}(\tau_1>t,\tau_2>t|\mathcal{F}_t)} + 1_{\{\tau_1>t,\tau_2>t\}}\frac{\mathbb{E}\left[X_T 1_{\{\tau_1>T,t<\tau_2\le T\}}|\mathcal{F}_t\right]}{\mathbb{Q}(\tau_1>t,\tau_2>t|\mathcal{F}_t)}\\
=&1_{\{\tau_1>t,\tau_2>t\}}\left(\frac{\mathbb{E}\left[X_T 1_{\{\tau_1>T,\tau_2>T\}}|\mathcal{F}_t\right]}{e^{-\int_0^t \lambda_1(u)+\lambda_2(u)du}} + \frac{\mathbb{E}\left[X_T 1_{\{\tau_1>T,t<\tau_2\le T\}}|\mathcal{F}_t\right]}{e^{-\int_0^t \lambda_1(u)+\lambda_2(u)du}}\right)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\Bigg(\mathbb{E}\left[X_T e^{-\int_t^T \lambda_1(u)+\lambda_2(u)du}\Big|\mathcal{F}_t\right]\\
+&\mathbb{E}\left[\int_t^T e^{-\int_t^{s^-}\lambda_2(u)du}\lambda_2(s^-)e^{-\int_t^{s^-}\lambda_1(u)du-\int_s^T\lambda_1(u)du}X_T ds^-\Big|\mathcal{F}_t\right]\Bigg)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\Bigg(\mathbb{E}\left[X_T e^{-\int_t^T \lambda_1(u)+\lambda_2(u)du}\Big|\mathcal{F}_t\right]\\
+&\mathbb{E}\left[\int_t^T e^{-\int_t^{s^-}\lambda_1(u)+\lambda_2(u)du}\lambda_2(s^-)e^{-\int_s^T\lambda_1(u)du}X_T ds^-\Big|\mathcal{F}_t\right]\Bigg)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\Bigg(\mathbb{E}\left[X_T e^{-\int_t^T \lambda_1(u)+\lambda_2(u)du}\Big|\mathcal{F}_t\right]\\
+&\mathbb{E}\left[\int_t^T e^{-\int_t^{s^-}\lambda_1(u)+\lambda_2(u)du}\lambda_2(s^-)\mathbb{E}\left[e^{-\int_{s-}^T\lambda_1(u)du}X_T\Big|\mathcal{G}_{s^-}^{-1}\right]ds^-\Big|\mathcal{F}_t\right]\Bigg)
\end{aligned}
$$

□

The second part of the last equality due to the property of conditional expectation given $\mathcal{F}_t \subseteq \mathcal{G}_s^{-1}$ for $s \geq t$. The implication of Corollary 3 is as follow. When we evaluate a payment X_T conditional on firm 1 survive to time T at state $\mathcal{A}$, where $\{\tau_1 > t, \tau_2 > t\}$ or $H_t^1 = 0, H_t^2 = 0$, the expected value of such a payment should be measured in two different situations. One is the situation that the firm 2 survival to time T, which means default contagion does not happen. This is the first part of the last equality in Corollary 3. The other situation is that the default contagion happened before T, or firm 2 defaults before T, which leads to the second part of the last equation in Corollary 3. When firm 2 defaults at time s, the economy changes from state $\mathcal{A}$ to $\mathcal{B}$ and the payment X_T is therefore evaluated as $\mathbb{E}[1_{\{\tau_1>T,\tau_2=s\}}X_T|\mathcal{G}_s^{-1}]$, which is implied by Corollary 2.

Lemma B.0.3. *Let Z_t is an $\mathbb{F}$-predictable process such that the random variable*

$Z_{\tau_1} 1_{\{\tau_1 \le T\}}$ *is* $\mathbb{Q}$*-integrable. Suppose at time* $t \le T$ *with* $\{\tau_1 > t, \tau_2 \le t\}$*, we have*

$$\begin{aligned}
&1_{\{\tau_1>t,\tau_2\le t\}}\mathbb{E}[Z_{\tau_1}1_{\{\tau_1\le T\}}|\mathcal{G}_t] \\
=&1_{\{\tau_1>t,\tau_2\le t\}}\frac{\mathbb{E}\left[Z_{\tau_1}1_{\{t<\tau_1\le T\}}|\mathcal{G}_t^{-1}\right]}{\mathbb{Q}(\tau_1>t|\mathcal{G}_t^{-1})} \\
=&1_{\{\tau_1>t,\tau_2\le t\}}\frac{\mathbb{E}\left[\int_t^T e^{-\int_0^{\tau_2^-}\lambda_1(u)du-\int_{\tau_2}^s\lambda_1(u)du}\lambda_1(s)Z_sds|\mathcal{G}_t^{-1}\right]}{e^{-\int_0^{\tau_2^-}\lambda_1(u)du-\int_{\tau_2}^t\lambda_1(u)du}} \\
=&1_{\{\tau_1>t,\tau_2\le t\}}\mathbb{E}\left[\int_t^T e^{-\int_t^s\lambda_1(u)du}\lambda_1(s)Z_sds\middle|\mathcal{G}_t^{-1}\right]
\end{aligned} \tag{B.6}$$

The third equality of Lemma B.0.3 is similar to Lemma 2.2.2 but consider the discontinuous of sample path $\lambda_1(t)$ at $0 < \tau_2 \le t$.

Lemma B.0.4. *Let* Z_t *is an* $\mathbb{F}$*-predictable process such that the random variable* $Z_{\tau_1} 1_{\{\tau_1 \le T\}}$ *is* $\mathbb{Q}$*-integrable. Suppose at time* $t \le T$ *with* $\{\tau_1 > t, \tau_2 > t\}$*, we have*

$$\begin{aligned}
\mathbb{E}[1_{\{\tau_1>t,\tau_2>t\}}Z_{\tau_1}1_{\{\tau_1\le T\}}|\mathcal{G}_t] = 1_{\{\tau_1>t,\tau_2>t\}}\Bigg(\mathbb{E}\left[\int_t^T e^{-\int_t^s\lambda_1(u)+\lambda_2(u)du}\lambda_1(s)Z_sds\middle|\mathcal{F}_t\right] \\
+\mathbb{E}\left[\int_t^T e^{-\int_t^{s^-}\lambda_1(u)+\lambda_2(u)du}\lambda_2(s^-)\mathbb{E}\left[\int_{s^-}^T e^{-\int_s^r\lambda_1(u)du}Z_rdr\middle|\mathcal{G}_{s^-}^{-1}\right]ds^-\middle|\mathcal{F}_t\right]\Bigg)
\end{aligned} \tag{B.7}$$

The proof of Lemma B.0.4 is as follow.

Proof.

$$
\begin{aligned}
&\mathbb{E}[1_{\{\tau_1>t,\tau_2>t\}}Z_{\tau_1}1_{\{\tau_1\le T\}}|\mathcal{G}_t] = 1_{\{\tau_1>t,\tau_2>t\}}\mathbb{E}[Z_{\tau_1}1_{\{\tau_1\le T\}}|\mathcal{G}_t]\\
=&1_{\{\tau_1>t,\tau_2>t\}}\mathbb{E}\left[Z_{\tau_1}1_{\{\tau_1<\tau_2,\tau_1\le T\}} + Z_{\tau_1}1_{\{t<\tau_2<\tau_1\le T\}}|\mathcal{G}_t\right]\\
=&1_{\{\tau_1>t,\tau_2>t\}}\left(\mathbb{E}\left[Z_{\tau_1}1_{\{\tau_1<\tau_2,\tau_1\le T\}}|\mathcal{G}_t\right] + \mathbb{E}\left[Z_{\tau_1}1_{\{\tau_2<\tau_1\le T\}}|\mathcal{G}_t\right]\right)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\left(\frac{\mathbb{E}\left[Z_{\tau_1}1_{\{t<\tau_1<\tau_2,\tau_1\le T\}}|\mathcal{F}_t\right]}{\mathbb{Q}(\tau_1>t,\tau_2>t|\mathcal{F}_t)} + \frac{\mathbb{E}\left[Z_{\tau_1}1_{\{t<\tau_2<\tau_1\le T\}}|\mathcal{F}_t\right]}{\mathbb{Q}(\tau_1>t,\tau_2>t|\mathcal{F}_t)}\right)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\left(\frac{\mathbb{E}\left[\int_t^T e^{-\int_0^s \lambda_1(u)+\lambda_2(u)du}\lambda_1(s)Z_s ds\Big|\mathcal{F}_t\right]}{e^{-\int_0^t \lambda_1(u)+\lambda_2(u)du}} + \frac{\mathbb{E}\left[\int_t^T Z_{\tau_1}1_{\{s^-<\tau_1\le T\}}dF^2_{s^-}\Big|\mathcal{F}_t\right]}{\mathbb{Q}(\tau_1>t,\tau_2>t|\mathcal{F}_t)}\right)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\Bigg(\frac{\mathbb{E}\left[\int_t^T e^{-\int_0^s \lambda_1(u)+\lambda_2(u)du}\lambda_1(s)Z_s ds\Big|\mathcal{F}_t\right]}{e^{-\int_0^t \lambda_1(u)+\lambda_2(u)du}}\\
&+\frac{\mathbb{E}\left[\int_t^T e^{-\int_0^{s^-} \lambda_2(u)du}Z_{\tau_1}1_{\{s^-<\tau_1\le T\}}ds^-\Big|\mathcal{F}_t\right]}{\mathbb{Q}(\tau_1>t,\tau_2>t|\mathcal{F}_t)}\Bigg)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\Bigg(\mathbb{E}\left[\int_t^T e^{-\int_t^s \lambda_1(u)+\lambda_2(u)du}\lambda_1(s)Z_s ds\Bigg|\mathcal{F}_t\right]\\
&+\frac{\mathbb{E}\left[\int_t^T e^{-\int_0^{s^-} \lambda_2(u)du}\int_{s^-}^T e^{-\int_0^{s^-} \lambda_1(u)du-\int_s^r \lambda_1(u)du}\lambda_1(r)Z_r dr ds^-\Big|\mathcal{F}_t\right]}{\mathbb{Q}(\tau_1>t,\tau_2>t|\mathcal{F}_t)}\Bigg)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\Bigg(\mathbb{E}\left[\int_t^T e^{-\int_t^s \lambda_1(u)+\lambda_2(u)du}\lambda_1(s)Z_s ds\Bigg|\mathcal{F}_t\right]\\
&+\frac{\mathbb{E}\left[\int_t^T e^{-\int_0^{s^-} \lambda_1(u)+\lambda_2(u)du}\int_{s^-}^T e^{-\int_s^r \lambda_1(u)du}\lambda_1(r)Z_r dr ds^-\Big|\mathcal{F}_t\right]}{\mathbb{Q}(\tau_1>t,\tau_2>t|\mathcal{F}_t)}\Bigg)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\Bigg(\mathbb{E}\left[\int_t^T e^{-\int_t^s \lambda_1(u)+\lambda_2(u)du}\lambda_1(s)Z_s ds\Bigg|\mathcal{F}_t\right]\\
&+\mathbb{E}\left[\int_t^T e^{-\int_t^{s^-} \lambda_1(u)+\lambda_2(u)du}\int_{s^-}^T e^{-\int_s^r \lambda_1(u)du}\lambda_1(r)Z_r dr ds^-\Bigg|\mathcal{F}_t\right]\Bigg)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\Bigg(\mathbb{E}\left[\int_t^T e^{-\int_t^s \lambda_1(u)+\lambda_2(u)du}\lambda_1(s)Z_s ds\Bigg|\mathcal{F}_t\right]\\
&+\mathbb{E}\left[\int_t^T e^{-\int_t^{s^-} \lambda_1(u)+\lambda_2(u)du}\mathbb{E}\left[\int_{s^-}^T e^{-\int_s^r \lambda_1(u)du}\lambda_1(r)Z_r dr\Bigg|\mathcal{G}^{-1}_{s^-}\right]ds^-\Bigg|\mathcal{F}_t\right]\Bigg)
\end{aligned}
$$

□

The next results will be used for valuation dividend payments prior to default time.

Lemma B.0.5. *Assume that A_t is a bounded, $\mathbb{F}$-predictable process of finite variation. Then for every $t \le s \le T$, the continuous dividend payment conditional on firm 1* *survive when* $\{\tau_1 > t, \tau_2 \le t\}$ *is*

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau_1>t,\tau_2\le t\}}\int_t^T 1_{\{\tau_1>s\}}dA_s\middle|\mathcal{G}_t\right]\\
=&1_{\{\tau_1>t,\tau_2\le t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T 1_{\{\tau_1>s\}}dA_s\middle|\mathcal{G}_t\right]\\
=&1_{\{\tau_1>t,\tau_2\le t\}}\frac{\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau_1>t\}}\int_t^T 1_{\{\tau_1>s\}}dA_s\middle|\mathcal{G}_t^{-1}\right]}{\mathbb{Q}(\tau_1>t|\mathcal{G}_t^{-1})}\\
=&1_{\{\tau_1>t,\tau_2\le t\}}\frac{\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_0^{\tau_2^-}\lambda_1(u)du}e^{-\int_{\tau_2}^s\lambda_1(u)du}dA_s\middle|\mathcal{G}_t^{-1}\right]}{e^{-\int_0^{\tau_2^-}\lambda_1(u)du}e^{-\int_{\tau_2}^t\lambda_1(u)du}}\\
=&1_{\{\tau_1>t,\tau_2\le t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s\lambda_1(u)du}dA_s\middle|\mathcal{G}_t^{-1}\right]
\end{aligned}\tag{B.8}$$

Lemma B.0.6. *Assume that A_t is a bounded, $\mathbb{F}$-predictable process of finite variation. Then for every $t \le r \le T$, the continuous dividend payment conditional on firm* 1 *survive when* $\{\tau_1 > t, \tau_2 > t\}$ *is*

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau_1>t,\tau_2>t\}}\int_t^T 1_{\{\tau_1>r\}}dA_r\middle|\mathcal{G}_t\right]=1_{\{\tau_1>t,\tau_2>t\}}\Bigg(\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^r\lambda_1(u)+\lambda_1(u)du}dA_r\middle|\mathcal{F}_t\right]\\
&+\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^{s^-}\lambda_1(u)+\lambda_2(u)du}\lambda_2(s^-)\mathbb{E}\left[\int_{s^-}^T e^{-\int_{s^-}^r\lambda_1(u)du}dA_r\middle|\mathcal{G}_{s^-}^{-1}\right]ds^-\middle|\mathcal{F}_t\right]\Bigg)
\end{aligned}\tag{B.9}$$

The proof of (B.0.6) is as follow

Proof.

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau_1>t,\tau_2>t\}}\int_t^T 1_{\{\tau_1>r\}}dA_r\middle|\mathcal{G}_t\right] = 1_{\{\tau_1>t,\tau_2>t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T 1_{\{\tau_1>r\}}dA_r\middle|\mathcal{G}_t\right]\\
=&1_{\{\tau_1>t,\tau_2>t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T 1_{\{\tau_1>r,\tau_2>r\}}+1_{\{\tau_1>r,\tau_2\le r\}}dA_r\middle|\mathcal{G}_t\right]\\
=&1_{\{\tau_1>t,\tau_2>t\}}\left(\mathbb{E}^{\mathbb{Q}}\left[\int_t^T 1_{\{\tau_1>r,\tau_2>r\}}dA_r\middle|\mathcal{G}_t\right]+\mathbb{E}^{\mathbb{Q}}\left[\int_t^T 1_{\{\tau_1>r,\tau_2\le r\}}dA_r\middle|\mathcal{G}_t\right]\right)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\left(\frac{\mathbb{E}^{\mathbb{Q}}\left[\int_t^T 1_{\{\tau_1>r,\tau_2>r\}}dA_r\middle|\mathcal{F}_t\right]}{\mathbb{Q}(\tau_1>t,\tau_2>t|\mathcal{F}_t)}+\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau_2\le T\}}\int_{\tau_2}^T 1_{\{\tau_1>r\}}dA_r\middle|\mathcal{G}_t\right]\right)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\left(\frac{\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_0^r\lambda_1(u)+\lambda_2(u)du}dA_r\middle|\mathcal{F}_t\right]}{e^{-\int_0^t\lambda_1(u)+\lambda_2(u)du}}+\frac{\mathbb{E}^{\mathbb{Q}}\left[1_{\{t<\tau_2\le T\}}\int_{\tau_2}^T 1_{\{\tau_1>r\}}dA_r\middle|\mathcal{F}_t\right]}{\mathbb{Q}(\tau_1>t,\tau_2>t|\mathcal{F}_t)}\right)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\left(\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^r\lambda_1(u)+\lambda_2(u)du}dA_r\middle|\mathcal{F}_t\right]\right.\\
&\left.+\frac{\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_0^{s^-}\lambda_2(u)du}\lambda_2(s^-)\int_{s^-}^T e^{-\int_0^{s^-}\lambda_1(u)du-\int_s^T\lambda_1(u)du}dA_r ds^-\middle|\mathcal{F}_t\right]}{e^{-\int_0^t\lambda_1(u)+\lambda_1(u)du}}\right)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\left(\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^r\lambda_1(u)+\lambda_2(u)du}dA_r\middle|\mathcal{F}_t\right]\right.\\
&\left.+\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^{s^-}\lambda_1(u)+\lambda_2(u)du}\lambda_2(s^-)\int_{s^-}^T e^{-\int_s^T\lambda_1(u)du}dA_r ds^-\middle|\mathcal{F}_t\right]\right)\\
=&1_{\{\tau_1>t,\tau_2>t\}}\left(\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s\lambda_1(u)+\lambda_2(u)du}dA_s\middle|\mathcal{F}_t\right]\right.\\
&\left.+\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^{s^-}\lambda_1(u)+\lambda_2(u)du}\lambda_2(s^-)\mathbb{E}\left[\int_{s^-}^T e^{-\int_{s^-}^T\lambda_1(u)du}dA_r\middle|\mathcal{G}_{s^-}^{-1}\right]ds^-\middle|\mathcal{F}_t\right]\right)
\end{aligned}
$$

□

Corollary 3, Lemmas B.0.4 and B.0.6 are the three building blocks of any credit claims, whose combination can represent any credit claims referencing to firm 1 with specifications of X_T, Z_{τ_1} and A_t. Without consider simultaneous default, these three lemmas suggest that pricing a credit claim at time t with our default contagion model, under the event $\{\tau_1 > t, \tau_2 > t\}$, should be separated into two situations, which are firm 2 defaults earlier and later than firm 1. In the latter case, any payment at time $t < s \le T$ should also conditional on firm 2 survive. In the former case, the economy changes from state $\mathcal{A}$ to state $\mathcal{B}$ at firm 2's default time and the value of the remaining contract, from time s to maturity T, should be evaluated as Corollary 2, Lemmas B.0.3 and B.0.5, which evaluate credit claims with $\{\tau_1 > t, \tau_2 \leq t\}$. In this case, the valuation of credit claims with default contagion model is identical to the reduced-form model discussed in Section 2.2 since only one firm left.

Finally, suppose a credit claim referencing to firm 1 in state $\mathcal{A}$ is denoted as $u^{\mathcal{A}}$, Lemma 4.3.1 follows after combining Corollary 3, Lemmas B.0.4, B.0.6 and equation (B.1).