Theoretical Models of Sports Leagues and Other Contests

A thesis submitted to the University of Manchester for the degree of PhD in Economics in the Faculty of Humanities

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List of Contents

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Abstract

This thesis was submitted by Luke Devonald to the University of Manchester for the degree of PhD in Economics on 1st February 2017. The thesis consists of three separate chapters all of which investigate Theoretical Models of Sports Leagues and Other Contests.

Chapter One outlines a new approach for modelling sports leagues, which complements traditional analyses of clubs' off-field talent recruitments with a subsequent analysis of players' on-field efforts. Most notably, the approach reveals a new theoretical basis for the hypothesis that sports fans prefer outcome uncertainty.

Chapter Two provides a new theoretical model of the soft budget constraint phenomenon, in which governments provide bailouts for loss-making clubs in European soccer leagues. Most notably, the model indicates that governments provide an inefficiently high level of bailout funding to clubs. However, the model reveals that some positive level of bailout funding may be optimal.

Chapter Three analyses a generic contest model with the possibility of a draw; an outcome in which no contestant is the winner. Most notably, our analysis reveals that introducing the possibility of a draw reduces homogeneous contestants' efforts. However, with heterogeneous contestants, introducing the possibility of a draw may induce greater effort from the strongest contestant.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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A Game of Two Halves: Sports League Models with Off-field Owner Talent Recruitment and On-field Player Effort Supply.

Abstract:

This paper introduces a new two-stage approach for the theoretical modelling of sports leagues, in which the traditional analysis of clubs' off-field talent recruitment is complemented by a subsequent analysis of players' on-field effort exertions. This approach allows us to specify a team's performance as a product of their talent and their effort, with the exponent on effort labelled as the 'effort intensity' of the league. We are then able to describe league outcomes, such as competitive balance, as functions of performances (i.e., of talents and efforts) rather than talents alone. We present a benchmark two-stage model of a North American sports league, which delivers a number of novel insights. First, we find that i) players exert greater efforts in leagues with greater competitive balance. It follows that ii) fans' preference for competitive balance depends positively on a league's effort intensity and iii) equilibrium competitive balance depends positively on a North American sports league's effort intensity. These results have a number of significant implications for the sports economics literature. Notably, results ii) and iii) indicate that models which do not account for the importance of efforts may understate competitive balance. Results i) and ii) indicate a new theoretical basis for Rottenberg's (1956) Uncertainty of Outcome Hypothesis (UOH). Lastly, result iii) may help to explain a perceived competitive balance problem in Major League Baseball.

“Hard work beats talent when talent doesn’t work hard” – Kevin Durant

“Football is not just about quality, it is about effort” – Jose Mourinho

“Even a great player must always work” – Sir Alex Ferguson

1. Introduction

A sizable section of the sports economics literature considers theoretical models of sports leagues. The vast majority of existing models describe leagues in which clubs (or their owners) simultaneously recruit talent from a given labour market, in order to maximise some objective function (see El-Hodiri & Quirk (1971), Fort & Quirk (1995), Szymanski & Kesenne (2004), Lang et al (2011) and Dietl et al (2012) amongst many others[1]). Ensuing equilibrium talent recruitments are then responsible for determining outcomes such as clubs’ win percentages, revenues and profits, as well as the league’s competitive balance (i.e., the extent to which clubs have equal win percentages) and fans’ utility (see Vrooman (1995), Kesenne (2000), Falconieri et al (2004), Dietl & Lang (2008)[2] and Madden (2012) amongst many others). Of particular interest is the effect of various regulations/policy interventions, such as revenue sharing or salary caps, on these outcomes (see Fort & Quirk (1995), Kesenne (2000) and Szymanski & Kesenne (2004) amongst many others).

However, the quotes at the top of this page reveal a deficiency in this approach. Sports league outcomes are determined not only by players’ talents, but also by players’ efforts. A big club may recruit a team of highly talented players, but if these players stroll the playing field contributing little or no on-field effort, the club will struggle with a far lower win percentage than their talents would predict. Conversely, a small club may only be able to acquire players

[1] Fort & Quirk (1995), Szymanski (2003) and Kesenne (2014) provide surveys of this extensive literature.
[2] Note that Dietl & Lang (2008) differs ostensibly in that fan preferences depend on ‘match quality’; however, these match qualities depend solely on clubs’ talent levels. Others to follow this approach include Cyrenne (2001), Dietl et al (2008) and Dietl et al (2009).

with limited talent, but if these players compensate for their limitations with conscientious effort, fans will acquire greater utility than their talents would predict.

As such, this paper suggests a new general modelling approach, in which the traditional analysis of clubs' off field talent recruitments is complemented by a subsequent analysis of players' on-field effort exertions. This approach resembles 'a game of two halves'. The first 'off field half' represents a pre-season stage, in which clubs recruit players with a certain talent level. The second 'on-field half' represents the season itself, in which recruited players then exert efforts trying to win the league. This approach allows us to specify each team's sporting 'performance' level as a product of their talents and efforts, where we label the exponent on efforts as the 'effort intensity' of the league[3]. Crucially, we then assume that league outcomes are determined by these performances (i.e., by both talents and efforts)[4], rather than by talents alone.

We exemplify this new approach with a specific model of a two-club North American sports league. The first half of this model follows a standard analysis of clubs' talent recruitment in a North American setting (i.e., clubs face a perfectly inelastic talent supply and choose expenditures to maximise profits). The second half, meanwhile, follows a standard contest literature analysis of players' efforts (i.e., players value winning, but find exerting effort costly, so choose efforts to maximise expected payoffs).

This benchmark example delivers a number of novel insights. First, we find that i) players exert greater efforts in leagues with a greater degree of competitive balance. It follows that ii) the greater is the effort intensity of a sports league, the greater is fans' preference for

[3] Our conjecture in this paper is that less physical sports, such as chess or darts say, have a lower effort intensity than more physical sports, such as soccer or American football (see Sections 2 and 7 for further discussion of this conjecture).

[4] Note that Cyrenne (2009) also proposes that league outcomes depend on performance levels. However, in their model, performances depend solely on talent levels.

competitive balance. Finally, iii) the greater is the effort intensity of a North American sports league, the greater is its equilibrium level of competitive balance.

These results have a wide-range of implications for the sports economics literature. For instance, noting that traditional sports league models implicitly assume zero effort intensity (i.e., efforts play no role in determining performances and therefore league outcomes), results ii) and iii) potentially indicate that the existing literature under-estimates equilibrium competitive balance as well as fans' desired competitive balance.

Meanwhile, results i) and ii) reveal a new endogenous mechanism by which sports fans have a preference for competitive balance. We illustrate this by showing that even fans with no exogenous preference for competitive balance, may still have a preference for competitive balance, due to its positive effect on player efforts. This provides a new theoretical basis for Rottenberg's (1956) Uncertainty of Outcome Hypothesis (UOH) (i.e., the hypothesis that fans have some preference for competitive balance[5]). We consider this a timely contribution to the debate surrounding UOH. To the best of our knowledge, Coates et al (2014) is the only other contribution that provides a potential underlying theoretical basis for UOH.

Finally, if we interpret efforts as physical energy expenditures, result iii) indicates that North American sports leagues may feature systematically different levels of competitive balance according to their physical intensity. With Ainsworth et al (2000) reporting that baseball requires relatively low physical energy expenditures compared to other North American sports, our model predicts that ceteris paribus baseball leagues have lower equilibrium competitive balance than football leagues say. This may help to explain a long-standing competitive balance problem in Major League Baseball (Schmidt & Berri (2001), Sanderson & Seigfried (2003) and Maxcy & Mondello (2006)).

[5] We are aware that uncertainty of outcome and competitive balance are different concepts, but for the purposes of this paper, with our two-club setting, we may use the terms interchangeably.

From here, Section 2 outlines our general approach in more detail. Sections 3-5 then present our benchmark model exemplifying this approach. We solve this model by backward induction. Thus, Section 3 begins by solving for players' efforts in the second on-field half of our game, which gives result i). Section 4 introduces fan preferences and presents result ii). Section 5 completes our backward induction solution by solving for clubs' talent recruitments in the first off-field half of our game, which delivers result iii). Sections 6 and 7 then discuss the insights of our model in the context of UOH and Major League Baseball respectively. Finally, Section 8 concludes.

2. General Approach

As indicated in the introduction, this paper proposes a new approach for modelling sports leagues, in which the traditional analysis of clubs' off-field talent recruitments is complemented by a subsequent analysis of players' on-field effort exertions. This section outlines this general approach.

For this, we consider a sports league with n clubs, $L = \{1 \dots n\}$. We propose to analyse this league as a two-stage game. The game's first stage is a pre-season period, in which each club, $i \in L$, recruits a team with playing talent, $t_i \in \mathbb{R}_+$. Specification of the talent recruitment problem facing clubs in this stage should be informed by the existing sports league literature. For instance, when analysing a European sports league, it is standard to assume that clubs recruit talents to maximise their win percentage and face a perfectly elastic talent supply (see Kesenne (2007), Garcia-del-Barrio & Szymanski (2009) and Madden (2012) amongst many others). Conversely, when analysing a North American sports league, it is standard to assume that clubs recruit talents to maximise profits and face a perfectly inelastic talent supply (see Fort & Quirk (1995), Szymanski (2004) and Madden (2011) amongst many others).

Teams recruited in stage one then represent their clubs over the course of the coming season. This season is described by the game's second stage, in which teams compete to win the league by exerting on-field efforts, $e_i \in \mathbb{R}_+$. Specification of teams' effort choice problem in this stage should be informed by the contest literature, which provides a great number of models for analysing efforts in competitive scenarios (see Corchon (2007), Konrad (2007) and Corchon & Serena (2016) for surveys of this literature).

This two-stage approach allows us to define each team's performance level as a product of their talents and their efforts, $x_i = t_i e_i{}^{\varepsilon}$, where the parameter $\varepsilon \in [0,1]$ captures the relative importance of efforts for performance production[6]. We label this parameter the 'effort intensity' of the league. The greater is this effort intensity the more on-field efforts matter for performances. Crucially, we conjecture that this varies across different sports leagues. For instance, intuition suggests that on-field efforts are relatively less important in sports like chess, darts or snooker[7], compared with sports like American football, soccer or even baseball[8]. Thus, our assertion is that football leagues have a greater effort intensity, ε, than darts leagues say.

We are then able to specify outcomes, such as (for example) clubs' win percentages, p_i, revenues, R_i, and profits, π_i, as well as the league's competitive balance, CB, and fans' utility, u_i, as functions of teams' performances $x = (x_1 \dots x_n)$. We consider this to provide a more accurate portrayal of sports leagues than traditional models, in which outcomes are determined by talents alone (thereby ignoring the importance of on-field efforts). Note that in this context traditional models implicitly assume $\varepsilon = 0$ (i.e., note that $\varepsilon = 0 \Rightarrow x_i = t_i$).

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[6] Note that $\varepsilon \leq 1$ ensures that talents are always weakly more important than efforts for performance production.
[7] In these sports, cognitive efforts may be required to produce performances but physical efforts are generally not required to any great extent.
[8] In these sports, both cognitive and physical efforts are required in order to produce performances. In Section 7, we further argue that football requires greater physical efforts than baseball, indicating that football has the greater effort intensity.

The next three sections illustrate the general approach espoused here with an example model of a two-club (i.e., $n = 2$) North American sports league. We solve this model by backward induction so begin in the next section with our stage two analysis of players' on-field efforts.

3. Stage Two: On-Field Efforts

Consider a stage two sub-game, following stage one pre-season talent recruitments of $t = (t_1, t_2) \in \mathbb{R}_+^2$, where we label total talent recruitments T (i.e., $t_1 + t_2 = T$).

This sub-game represents the actual sporting season, in which recruited teams compete to win the league by exerting on-field efforts, $e_i \in \mathbb{R}_+$. Recall that, by exerting efforts, each team i produces a performance level, $x_i = t_i {e_i}^{\varepsilon}$, which is a product of their talent and their effort (where $\varepsilon \in [0,1]$). We assume that these performances determine win percentages (i.e., the probability that each team wins the league) via a standard Tullock (1980) contest success function, $p_i = \frac{x_i}{x_i + x_j}$.

We further assume that each team receives a common payoff of $v \in \mathbb{R}_+$ when they win the league and 0 when they lose. Note that this payoff may represent athletes' intrinsic desire to win the league and/or some financial reward from doing so[9]. Finally, we assume that teams have a constant marginal effort cost normalised to one. It follows that each team's stage 2 effort choice problem is given by;

$$\max_{e_i \in \mathbb{R}_+} v \frac{t_i {e_i}^{\varepsilon}}{t_i {e_i}^{\varepsilon} + t_j {e_j}^{\varepsilon}} - e_i$$

[9] In future we hope to relax the assumption that teams' payoff for winning is a) homogeneous and b) exogenous. Introducing heterogeneity in this payoff will allow us to study equilibria in which contestants exert heterogeneous efforts. Meanwhile, endogenising this payoff could allow us to study clubs' choices of player bonuses and/or managerial recruitments as a means to motivating teams in the on-field stage.

Note that this represents a standard specification in the contest literature (see Gradstein (1995)). Simultaneously solving this problem for both teams yields second stage sub-game equilibrium effort levels;

$$e_i{}^* = v\varepsilon\frac{t_i t_j}{T^2}$$

Quite naturally, these effort levels are increasing in teams' motivation to win the league (i.e., $\frac{\partial e_i{}^*}{\partial v} \geq 0$) and in the league's effort intensity (i.e., the importance of efforts for performances, $\frac{\partial e_i{}^*}{\partial \varepsilon} \geq 0$).

It follows that sub-game equilibrium performances are $x_i{}^* = t_i e_i{}^{*\varepsilon} = (v\varepsilon)^\varepsilon \frac{t_i{}^{1+\varepsilon} t_j{}^\varepsilon}{T^{2\varepsilon}}$, while, since contestants' efforts are homogeneous (i.e., $e_i{}^* = e_j{}^*$), sub-game equilibrium win percentages follow a standard Tullock contest success function in talents, $p_i{}^* = \frac{x_i{}^*}{x_i{}^* + x_j{}^*} = \frac{t_i}{T}$.

Now, we introduce a measure of competitive balance in our league (i.e., the extent to which teams have equal win percentages). For this, we follow[10] Szymanski (2003) and define competitive balance to be the product of teams' win percentages, $CB = p_i p_j$. Note that this measure of competitive balance is maximised (with a value $\frac{1}{4}$) when teams have equal win percentages (i.e., $p_i = p_j = \frac{1}{2}$) and minimised (with a value 0) whenever one team wins the league with certainty (i.e., $p_i = 1$ or $p_j = 1$). More generally, this measure increases as win percentages become more equally distributed.

Crucially, note that sub-game equilibrium competitive balance is given by $CB^* = p_i{}^* p_j{}^* = \frac{t_i t_j}{T^2}$. Thus, we may rewrite sub-game equilibrium efforts as $e_i{}^* = v\varepsilon CB^*$. It follows that;

[10] Others to follow this specification of competitive balance include Dietl & Lang (2008), Vrooman (2009), Dietl et al (2009) and Kesenne (2015).

Result 1: Players exert greater efforts in leagues with a greater degree of competitive balance (i.e., $\frac{\partial e_i{}^*}{\partial CB^*} \geq 0$).

Put simply, the more equally distributed teams' talents are the more on-field effort they exert. This represents a well-established result in the contest literature (see Gradstein (1995), Stein (2002) and Brown (2011) for example). However, to the best of our knowledge, this result has not yet been discussed in the context of a sports league model. The next section begins this discussion by identifying the implications of Result 1 for fan preferences.

4. Fan Preferences

Standard sports league models typically assume that fans' utility from live game attendance (i.e., from watching their team) depends on teams' talents. Our two-stage approach allows us to instead specify fans' utility as a function of performances. Our conjecture is that this provides a more accurate depiction of fan preferences (i.e., that sports fans are primarily concerned with the performance levels teams actually produce rather than the talents they intrinsically possess).

To illustrate let us assume that fans of team i have a Cobb-Douglas utility function in performances;

$$\bar{u}_i = x_i{}^\alpha x_j{}^\beta \quad : \quad \alpha > \beta, \alpha > 0 \; and \; \beta > -\alpha$$

We assume $\alpha > \beta$ so that fans prefer their own team's performance to the opposition team's performance. We impose $\alpha > 0$, but do not impose any sign restriction on β. Thus, fans may have a taste for opposition performances (i.e., $\beta > 0$) or a distaste for opposition performances (i.e., $\beta < 0$). Finally, we impose $\beta > -\alpha$ so that fans' distaste for opposition performance is not overwhelmingly strong[11].

The remainder of this section discusses fans' preference for competitive balance with this utility specification. First, we discuss the exogenous preference for competitive balance implied by this specification. Then, we note that Result 1 in the previous section implies that fans have a further endogenous preference for competitive balance, due to its positive effect on player efforts and therefore performances[12].

1. *Exogenous Preference for Competitive Balance*

We first derive a measure of fans' *exogenous preference for competitive balance* implied by our utility specification, $\bar{u}_i = {x_i}^{\alpha}{x_j}^{\beta}$.

To this end, let us suppose that fans of team i could hypothetically allocate a given total performance X across teams in order to maximise their own utility. We denote the performance level they would choose to allocate to their own team ${x_i}^X$ (i.e., ${x_i}^X = \operatorname{argmax}_{x_i \in [0,X]} {x_i}^{\alpha}(X - x_i)^{\beta}$). It is easy to see that if $\beta \leq 0$ then fans prefer their team to have the maximum possible performance level (i.e., ${x_i}^X = X$). However, if $\beta > 0$, fans prefer to allocate some performance to their rival team, ${x_i}^X = \frac{\alpha}{\alpha+\beta} X \in \left(\frac{X}{2}, X\right)$.

The competitive balance level associated with ${x_i}^X$ then provides our measure of fans' *exogenous preference for competitive balance*;

$$CB^X = {p_i}^X {p_j}^X = \frac{{x_i}^X {x_j}^X}{X^2} = \begin{cases} \dfrac{\alpha\beta}{(\alpha+\beta)^2} & if\ \beta > 0 \\ 0 & if\ \beta \leq 0 \end{cases}$$

[11] Note that $\beta > -\alpha$ ensures that if teams had equal performances and fans of team i were given the choice between marginally improving their own team's performance or marginally reducing the opposition's performance, they would prefer to marginally improve their own team's performance.

[12] To immediately see this endogenous preference for competitive balance, note that we can write fans' second stage sub-game equilibrium utility as ${\bar{u}_i}^* = {x_i}^{*\alpha}{x_j}^{*\beta} = {t_i}^{\alpha}{t_j}^{\beta}{e_i}^{*\varepsilon(\alpha+\beta)} = {t_i}^{\alpha}{t_j}^{\beta}(v\varepsilon CB^*)^{\varepsilon(\alpha+\beta)}$.

Thus, when $\beta \leq 0$, fans have no exogenous preference for competitive balance (they prefer their team to perform infinitely better than the opposition and win with certainty). However, when $\beta > 0$, fans have some exogenous preference for competitive balance (they prefer their rival to produce some performance and have some chance of winning). Note that $\beta \to \alpha \Rightarrow CB^X \to \frac{1}{4}$, so that fans exogenously prefer perfect balance (they prefer both teams to have equal performance and equal chance of winning).

2. Total Preference for Competitive Balance

Now, on top of this exogenous preference for competitive balance, our two-stage approach reveals that fans have an *endogenous preference for competitive balance*. To see this endogenous preference intuitively, recall from Result 1 in the previous section that teams' on-field efforts are increasing in the league level of competitive balance. It follows that, fans endogenously prefer competitive balance due to its positive effect on efforts (and therefore performances). Our measure of fans' exogenous preference for competitive balance abstracts from this endogenous preference by simply considering different allocations of performances across teams, without considering the pivotal role of players' stage two efforts in producing these performances[13].

Thus, to capture fans' combined endogenous and exogenous preference for competitive balance, we now derive a measure of fans' *total preference for competitive balance*. To this end, we suppose that fans of team i can now hypothetically allocate total talent T across teams[14], prior to the second stage sub-game described in the previous section (this allows us to capture the additional endogenous importance of competitive balance for on-field efforts

[13] In reality, allocating performances across teams is not feasible. A critical component of a team's performance is their on-field effort, which is necessarily an endogenous outcome of players' stage two effort choice problem. In this sense, efforts and therefore performances cannot readily be transferred from one team to another.
[14] While transferring performances across teams is not feasible, transferring talents is. Thus, this measure of competitive balance should be considered fans' actual preference for competitive balance.

and fan preferences, which is not captured by the exogenous measure). In this instance, fans choose a talent allocation to maximise their second stage sub-game equilibrium utility[15];

$$\bar{u}_i^* = x_i^{*\alpha} x_j^{*\beta} = \left(\frac{v\varepsilon}{T^2}\right)^{\varepsilon(\alpha+\beta)} t_i^{\hat{\alpha}} t_j^{\hat{\beta}} \quad : \quad \hat{\alpha} = \varepsilon(\alpha+\beta)+\alpha \text{ } and \text{ } \hat{\beta} = \varepsilon(\alpha+\beta)+\beta$$

We denote the talent level they would choose to allocate to their own team t_i^T (i.e., $t_i^T = \text{argmax}_{t_i \in [0,T]}\, t_i^{\hat{\alpha}}(T-t_i)^{\hat{\beta}}$). Now if $\hat{\beta} \leq 0$ fans prefer their team to have the maximum possible talent level (i.e., $t_i^T = T$). However, if $\hat{\beta} > 0$, fans prefer to allocate some talent to their rival team, $t_i^T = \frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}} T \in \left(\frac{T}{2}, T\right)$.

The competitive balance level associated with t_i^T then provides our measure of fans' *total preference for competitive balance*;

$$CB^T = p_i^T p_j^T = \frac{t_i^T t_j^T}{T^2} = \begin{cases} \dfrac{\hat{\alpha}\hat{\beta}}{\left(\hat{\alpha}+\hat{\beta}\right)^2} & if\ \hat{\beta} > 0 \\ 0 & if\ \hat{\beta} \leq 0 \end{cases}$$

Thus, when $\hat{\beta} \leq 0$, fans have no total preference for competitive balance. However, when $\hat{\beta} > 0$, fans have some total preference for competitive balance. Finally, note that $\beta \to \alpha \Rightarrow \hat{\beta} \to \hat{\alpha} \Rightarrow CB^T \to \frac{1}{4}$, so that fans totally prefer perfect balance.

Appendix 1 shows that fans' total preference for competitive balance exceeds their exogenous preference for competitive balance (i.e., $CB^T \geq CB^X$). The difference between the two measures is the *endogenous preference for competitive balance* identified by our two-stage approach (i.e., $CB^N = CB^T - CB^X$). Recall that this endogenous preference reflects the fact that fans prefer competitive balance due to its positive effect on player efforts and

[15] To derive $\bar{u}_i^*$ recall from the previous section that $x_i^* = (v\varepsilon)^{\varepsilon} \frac{t_i^{1+\varepsilon} t_j^{\varepsilon}}{T^{2\varepsilon}}$. Note that we implicitly here assume that fans can perfectly foresee teams' subsequent second-stage efforts when they hypothetically allocate talents.

thereby performances and utility (i.e., not because of any exogenous taste for competitive balance). When the importance of effort is removed from our model (i.e., $\varepsilon = 0$), fans no longer have this endogenous preference (i.e., $CB^N = 0 \Leftrightarrow CB^T = CB^X$).

It follows intuitively (and is shown in Appendix 2) that;

Result 2: Fans' total preference for competitive balance is increasing in the effort intensity of a sports league (i.e., $\frac{\partial CB^T}{\partial \varepsilon} \geq 0$).

Put simply, the more important efforts are for performances, the more concerned fans are with competitive balance, due to its catalyst effect on effort exertions. That fans' preferred level of competitive balance depends in this systematic way on the effort intensity of sports leagues is a significant new result for the sports league literature. We discuss the implications of this result for the Uncertainty of Outcome Hypothesis in Section 6. First, the next section shows that Result 2 has repercussions for clubs' equilibrium talent recruitments in North American sports leagues.

5. Stage One: Off-Field Talent Recruitment

In this section, we return to solve for clubs' stage one off-field talent recruitments $t = (t_1, t_2) \in \mathbb{R}^2_+$. This completes the backward induction solution of our example model.

For this pre-season stage, we consider a North American sports league[16]. Thus, in keeping with traditional sports league models, we assume a fixed supply of talent to the league, $T \in \mathbb{R}_+$, and that each club i seeks to maximise their profits, π_i.

[16] Note that the analyses of players' on-field efforts and fan preferences presented in the previous two sections do not assume a North American setting. It follows that Results 1 and 2 apply equally to European sports leagues.

We specify clubs' profits as the difference between their revenues, R_i, and their expenditures, z_i (i.e., $\pi_i = R_i - z_i$). We specify expenditures as simply the cost of playing talent recruitment, $z_i = wt_i$, where $w \in \mathbb{R}_+$ is the market clearing wage for talent. Further, we invoke a Falconieri et al (2004) micro-foundation for revenues from fans' utility[17] so that $R_i = \frac{N_i}{4}\bar{u}_i = \frac{N_i}{4}{x_i}^{\alpha}{x_j}^{\beta}$, where $N_i \in \mathbb{R}_+$ is the size of club i's supporter base. Crucially, we assume that clubs (and their fans) can perfectly foresee players' subsequent stage two on-field efforts so that, in stage one, they are aware of their second-stage sub-game equilibrium revenues;

$$R_i^{\ *} = \frac{N_i}{4}\bar{u}_i^{\ *} = \frac{N_i}{4}\left(\frac{v\varepsilon}{T^2}\right)^{\varepsilon(\alpha+\beta)} {t_i}^{\widehat{\alpha}}{t_j}^{\widehat{\beta}}$$

Finally, acknowledging the sensitivity of the market wage to clubs' stage one talent recruitments, we follow Madden (2011) and suppose that clubs' choice variable in this stage is their expenditure (rather than their talent recruitment directly). Noting[18] that $t_i = \frac{z_i}{z_i + z_j}T$, we can now write clubs' stage one problem as;

$$\max_{z_i \in \mathbb{R}_+} A_i \frac{{z_i}^{\widehat{\alpha}}{z_j}^{\widehat{\beta}}}{\left(z_i + z_j\right)^{\widehat{\alpha}+\widehat{\beta}}} - z_i \quad : \quad A_i = \frac{N_i}{4}T^{\alpha+\beta}(v\varepsilon)^{\varepsilon(\alpha+\beta)}$$

[17] This micro-foundation is as follows. Suppose a club has N_i fans. Assume that fans' utility from live match attendance is $\theta\bar{u}_i - q_i$, where q_i is the ticket price and $\theta \sim U[0,1]$. Supposing fans receive zero utility from non-attendance, their total demand for tickets is $N_i(1 - \frac{q_i}{\bar{u}_i})$. Profit-maximising clubs with no stadium capacity constraints or stadium costs will then set $q_i = \frac{\bar{u}_i}{2}$, so that $\frac{N_i}{2}$ fans attend and revenues are $R_i = \frac{N_i}{4}\bar{u}_i$. Note that implicit to this micro-foundation is some 'round-robin' format to our league, in which each club takes turns hosting home games (for which only home fans attend).

[18] To see this, note that if w is the market clearing wage, we have $z_i = wt_i$ and $z_i + z_j = wT$ so that $t_i = \frac{z_i}{w} = \frac{z_i}{z_i + z_j}T$.

If we now further assume[19] that $\hat{\alpha} < 1$ and $\alpha - \beta < 1$, Appendix 3 shows that the first order conditions characterising interior solutions to this problem are for clubs i and j respectively;

$$A_i \tau^{\hat{\alpha}-1}\left(\hat{\alpha} - \hat{\beta}\tau\right) = (1+\tau)^{\hat{\alpha}+\hat{\beta}}(z_i + z_j)$$

$$A_j \tau^{\hat{\beta}}\left(\hat{\alpha}\tau - \hat{\beta}\right) = (1+\tau)^{\hat{\alpha}+\hat{\beta}}(z_i + z_j)$$

where $\tau = \frac{z_i}{z_j} = \frac{t_i}{t_j}$ denotes our league's talent ratio. Combining these conditions, Appendix 4 then shows that we have a unique interior sub-game perfect Nash equilibrium, in which clubs recruit talent t_i^{**} (and players subsequently in stage two exert on-field effort $e_i^{**} = v\varepsilon \frac{t_i^{**} t_j^{**}}{T^2}$).

The sub-game perfect Nash equilibrium level of competitive balance is then $CB^{**} = \frac{t_i^{**} t_j^{**}}{T^2}$. Crucially, Appendix 5 finds that;

Result 3: Equilibrium competitive balance in North American sports leagues is increasing in the effort intensity of the league (i.e., $\frac{\partial CB^{**}}{\partial \varepsilon} \geq 0$).

This result follows quite intuitively from Result 2 in the previous section, which found that fans' preference for competitive balance is also increasing in the effort intensity of the league. To see the link, recall that, in our North American sports league, clubs are profit-maximisers and have revenues which are proportional to fans' utility (i.e., $R_i^* = \frac{N_i}{4}\bar{u}_i^*$). This gives clubs an obvious incentive to satiate fans. Thus, the greater is a league's effort intensity, the greater is fans' preference for competitive balance and therefore the greater is clubs' incentive to ensure a balanced league with their talent recruitments.

[19] These restrictions serve to ensure existence of a unique equilibrium. Note that together they are equivalent to $\alpha < \min\{\frac{1-\varepsilon\beta}{1+\varepsilon}, 1+\beta\}$.

Like Result 2, Result 3 is a potentially significant new result for the sports economics literature; reporting that leagues may vary systematically in their competitive balance according to the importance of efforts for performance production (i.e., the league's effort intensity). Noting that traditional sports league models do not account for the importance of efforts, Results 2 and 3 respectively suggest that existing models may under-state fans' preference for competitive balance as well as equilibrium competitive balance in North American sports leagues.

In the next two sections, we examine some further implications of Results 2 and 3, and our model more generally, for the sports economics literature.

6. Uncertainty of Outcome Hypothesis

The Uncertainty of Outcome Hypothesis (UOH) attributed to Rottenberg (1956) is a cornerstone of sports economics. It surmises that sports fans prefer sports events with some degree of outcome uncertainty (i.e., fans prefer to watch matches in which the eventual winner is unknown)[20]. In the context of our model, this is equivalent to fans having some preference for competitive balance (i.e., $CB^T > 0$)[21].

The hypothesis has garnered widespread support from sports economists and policy-makers alike; frequently informing theoretical specifications of fan preferences (see Dietl & Lang (2008), Dietl et al (2009) and Lang et al (2011) for instance) as well as practical decisions from governing bodies (for instance, legislators commonly justify collusive practices by sports clubs on UOH grounds (Szymanski (2003))). However, the empirical evidence for

[20] Note that this represents a minimal statement of UOH. Cairns et al (1986) discuss a variety of proposed definitions of UOH, all of which imply this statement of UOH.
[21] Note that in general outcome uncertainty and competitive balance are related but distinct concepts (Forrest & Simmons (2002)). However, in the context of our two-club model, there is an equivalence; fans prefer some outcome uncertainty (i.e., ${p_i}^T < 1$) if and only if they prefer some competitive balance (i.e., $CB^T > 0$). Note also that CB^T measures fans' preference for competitive balance here, rather than CB^X, since CB^X ignores the importance of competitive balance for player efforts and therefore underestimates fans' true preference for competitive balance.

UOH remains ambiguous (see Forrest et al (2005) and Coates et al (2014)), with attendance data consistently failing to identify any fan preference for outcome uncertainty (see Forrest & Simmons (2002), Coates & Humphreys (2010), Beckman et al (2011) amongst many others).

Given this lack of empirical evidence for UOH, Coates et al (2014) attempt to instead provide a theoretical basis for UOH. To this end, they model sports fans with reference dependent preferences (i.e., with Koszegi & Rabin's (2006) gain-loss utility). They successfully show that such fans may have an endogenous preference for outcome uncertainty, thereby providing a potential theoretical basis for UOH. However, this theoretical basis is predicated on an assumption that fans are not loss-averse, whereas Tversky & Kahneman (1991) establishes loss-aversion as the behavioural norm in models of reference dependent preferences.

Crucially, our two-stage model provides an alternative theoretical basis for UOH, by showing (in Section 4) that fans have an endogenous preference for competitive balance (which is equivalent to outcome uncertainty in our model). The intuition for this is as follows. Fans in our model acquire utility from teams' performances, which depend critically on teams' on-field efforts. Result 1 in Section 3 shows that teams exert more effort in leagues with greater competitive balance. Thus, fans in our model endogenously prefer competitive balance as it helps to extract the greatest efforts and therefore performances from teams (again, this equates to an endogenous preference for outcome uncertainty).

This constitutes significant new theoretical evidence in favour of UOH. Indeed, we can now show that UOH can hold (i.e., $CB^T > 0$) even when fans have no exogenous preference for outcome uncertainty/competitive balance (i.e., $CB^X = 0$). To see this, recall from Section 4 that $CB^X = 0 \Leftrightarrow \beta \leq 0$, while $CB^T > 0 \Leftrightarrow \hat{\beta} > 0 \Leftrightarrow \beta > -\frac{\varepsilon}{1+\varepsilon}\alpha$. Thus, whenever $\beta \in$

$\left(-\frac{\varepsilon}{1+\varepsilon}\alpha, 0\right]$, UOH holds entirely as a result of fans' endogenous preference for competitive balance.

Interestingly, recall that Result 2 in Section 4 states that fans' preference for competitive balance is increasing in the effort intensity of a sports league. It follows that UOH is more likely to hold in sports leagues with a greater effort intensity. To see this clearly note that $CB^T > 0 \Leftrightarrow \hat{\beta} > 0 \Leftrightarrow \varepsilon > -\frac{\beta}{\alpha+\beta}$. Thus, our model indicates that the relevance of UOH may vary across sports according to their effort intensity. Future research/policy-making with regards to UOH may be improved by accounting for such variation.

To this end, the next section presents a tentative discussion of effort intensities in North American sports leagues. We argue that football leagues have a greater effort intensity than baseball leagues say. In the context of this section's discussion, this implies that UOH is less likely to hold in baseball leagues as compared to football leagues.

7. <u>Effort Intensity of North American Sports Leagues</u>

Our two-stage approach introduces a new effort intensity parameter, ε, to the standard sports league model. The equilibrium results for our example model presented in Sections 3-5 reveal the significance of this parameter. For instance, Result 2 finds that fans' preferred level of competitive balance is increasing in ε, while Result 3 finds that equilibrium competitive balance in North American sports leagues is also increasing in ε. In order to interpret these results, this section tentatively discusses the effort intensity in the major North American sports leagues.

Recall that the effort intensity of a league measures the importance of efforts, e_i, for performances (i.e., $x_i = t_i {e_i}^{\varepsilon}$). For what follows, it serves to interpret efforts more specifically as *physical* efforts[22]. Thus, Section 3's stage two sub-game equilibrium effort, ${e_i}^* = v\varepsilon CB^*$, represents our estimate of a team's on-field physical effort exertions. Note that these physical effort exertions are proportional to the league's effort intensity (i.e., ${e_i}^* \propto \varepsilon$). It follows that all else (i.e., v and CB^*) being equal, we can rank sports leagues according to their effort intensity by comparing the physical efforts involved.

For this, we refer to Ainsworth et al (2000), which describes the physical efforts involved in a variety of different sports (as measured by the Metabolic Equivalent of Task (MET), which gives the relative energy expenditure involved in a task (or sport in our case) compared with the resting rate of energy expenditure[23]). For North American sports leagues their findings are as follows;

North American Sport (Major League)	MET Estimate of Physical Effort Involved
Baseball (MLB)	5.0
Basketball (NBA)	8.0
Football (NFL)	9.0
Hockey (NHL)	8.0

Thus, football involves greater physical effort than basketball, which in turn involves greater physical effort than baseball (i.e., ${e_i}^{*NFL} > {e_i}^{*NBA} > {e_i}^{*MLB}$). In the context of our model, this indicates a greater effort intensity for football than basketball and baseball (i.e., $\varepsilon^{NFL} > \varepsilon^{NBA} > \varepsilon^{MLB}$). Put simply, performances depend more on efforts in football than in basketball, than in baseball.

[22] Note that this interpretation of e_i effectively abstracts from the co-importance of cognitive effort for performance in sports. In a richer model, it may be informative to allow players to choose separately a cognitive effort as well as a physical effort. Note also though that this interpretation corresponds neatly with the Oxford Dictionaries' definition of sport as *"An activity involving physical exertion and skill"*.

[23] An MET of k indicates that a sport requires k times the physical effort that a seated rest requires.

Result 2 therefore indicates that football fans prefer more competitive balance than baseball fans say. Intuitively, this follows from the assertion that physical efforts are relatively more important in football than in baseball. Thus, football fans care more about efforts than baseball fans and thereby have a greater endogenous preference for competitive balance.

Meanwhile, Result 3 indicates that, all else being equal[24], football leagues have greater equilibrium competitive balance than baseball leagues. To see why, recall that profit-maximising clubs have an incentive to satiate fans' preferences, thus, when making their talent recruitments, football clubs have a greater incentive to maintain competitive balance than baseball clubs do.

In this sense, our two-stage approach may help to explain a perceived historical competitive balance problem in baseball leagues (Schmidt & Berri (2001), Sanderson & Seigfried (2003) and Maxcy & Mondello (2006)). This competitive balance problem was initially conjectured by journalists and industry insiders at the turn of the millennium (see Associated Press (1999) and Levin et al (2000)). This conjecture has subsequently been re-enforced by empirical observations in the sports economics literature[25], which show that, as our model would predict, MLB has historically suffered from a low level of competitive balance compared with the NFL (see Quirk & Fort (1992), Schmidt & Berri (2003), Fort (2006) and Trandel & Maxcy (2011)).

A number of authors offer potential explanations for the relative lack of competitive balance in baseball leagues. For instance, Maxcy & Mondello (2006) argue that competitive balance

[24]Note that this is very much a ceteris paribus statement. In reality, institutional differences between the NFL and MLB will also have a major effect in determining relative competitive balance outcomes.
[25] Note that there is also some evidence that the competitive balance problem in baseball is not as severe as the associated conjecture suggests (see Depken (1999) and Schmidt & Berri (2001) and note that Fort (2006) finds that MLB has a greater level of competitive balance than the NBA).

is greater in the NFL than MLB because of a greater importance of television revenues[26]. Meanwhile, Sanderson & Siegfried (2003) propose that differences in competitive balance across the leagues owe to institutional differences in revenue sharing agreements. Similarly, Maxcy & Modello (2006) discuss the impact of players' free agency rights. Our model now provides a further contribution to this discussion, indicating that the NFL may have a greater level of competitive balance than MLB due to its greater physical intensity.

8. Conclusion

This paper introduces (in Section 2) a new two-stage approach for the theoretical modelling of sports leagues, in which the traditional analysis of clubs' off-field talent recruitment is complemented by a subsequent analysis of players' on-field effort exertions. This approach allows us to specify a team's performance as a product of their talent and their effort, with the exponent on effort labelled as the effort intensity of the league. We are then able to describe league outcomes, such as competitive balance, as functions of performances (i.e., of talents and efforts) rather than talents alone.

To illustrate this approach (in Sections 3-5), we present an example of a two-stage North American sports league model. With this benchmark model, we find that i) players exert greater efforts in leagues with greater competitive balance. It follows that ii) fans have a greater preference for competitive balance in leagues with a greater effort intensity. Subsequently, iii) equilibrium competitive balance is greater in North American sports leagues with a greater effort intensity.

These findings deliver a variety of insights for the sports economics literature. Most notably (in Section 6), we show that i) and ii) indicate a new theoretical basis for the long-standing

[26] Forrest & Simmons (2002) show that TV audiences prefer a greater level of competitive balance than partisan crowds. Thus, Maxcy & Mondello (2006) argue that this gives football clubs a greater incentive to maintain competitive balance.

Uncertainty of Outcome Hypothesis. Further (in Section 7), we show that iii) indicates that all else being equal football leagues should have a greater level of competitive balance than baseball leagues. This may in part help to explain a historical competitive balance problem in Major League Baseball.

Crucially, these insights are delivered by perhaps the simplest possible specification of a two-stage model. Our hope and belief is that alternative specifications can deliver further insight.

To this end, from here[27] we a) plan to pursue alternative specifications of clubs' stage-one off-field talent recruitment problem. Most obviously, we wish to consider the European case in which clubs are win-maximisers (rather than profit-maximisers) and face a perfectly elastic talent supply (rather than perfectly inelastic). Beyond this, it remains to study the effect of various regulations, such as revenue sharing agreements and salary caps etc in the context of our two-stage model. Finally, we may ultimately wish to expand clubs' stage one choice set to include choices of bonus payments or managerial recruitments as a means to influencing players' stage two efforts.

We then b) plan to pursue alternative specifications of clubs' stage-two on-field effort choice problem. Here, we first wish to investigate a model in which players have a heterogeneous prize for winning $v_i \neq v_j$. This will allow for heterogeneous equilibrium efforts. From there, introducing endogenous prizes may also be of interest (i.e., allowing clubs to provide bonuses or motivational management that increase v_i, or potentially more simply allowing v_i to depend on attendances). Finally, we may also wish to investigate alternative contest literature specifications (i.e., we may introduce the possibility of draws (Yildizparlak (2013)), maximum effort caps (Gavious et al (2002)) or behavioural considerations (Baharad & Nitzan (2008)) etc).

[27] A major priority from here should also be to analyse a two-stage model with $n > 2$ clubs.

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10. Appendix 1

In this appendix we show that $CB^T \geq CB^X$. For this, it suffices[28] to show that ${p_i}^X \geq {p_i}^T$, where we have;

$$
{p_i}^X = \frac{{x_i}^X}{X} = \begin{cases} \dfrac{\alpha}{\alpha+\beta} & if\ \beta > 0 \\ 1 & if\ \beta \leq 0 \end{cases} \quad and \quad {p_i}^T = \frac{{t_i}^T}{T} = \begin{cases} \dfrac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}} & if\ \hat{\beta} > 0 \\ 1 & if\ \hat{\beta} \leq 0 \end{cases}
$$

First suppose $\beta \leq 0$, it follows that ${p_i}^X = 1$, which implies ${p_i}^X \geq {p_i}^T$ (since ${p_i}^T$ is a win percentage and therefore less than or equal to 1).

Second suppose $\beta > 0$, it follows that $\hat{\beta} = \varepsilon(\alpha+\beta)+\beta > 0$ also. Therefore, ${p_i}^X \geq {p_i}^T \Leftrightarrow \frac{\alpha}{\alpha+\beta} \geq \frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}$. Recalling that $\hat{\alpha} = \varepsilon(\alpha+\beta)+\alpha$, we can write $\frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}} = \frac{\hat{\varepsilon}+\alpha}{2\hat{\varepsilon}+\alpha+\beta}$ where $\hat{\varepsilon} = \varepsilon(\alpha+\beta) \geq 0$. Then note that $\frac{\partial\left(\frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}\right)}{\partial\hat{\varepsilon}} = \frac{\beta-\alpha}{(2\hat{\varepsilon}+\alpha+\beta)^2} < 0$. It follows that $\frac{\alpha}{\alpha+\beta} \geq \frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}$, which in turn implies that ${p_i}^X \geq {p_i}^T$.

[28] To see this, note that ${p_i}^T$ is necessarily greater than $\frac{1}{2}$ (i.e., fans never prefer the opposition to have a greater win percentage). Thus, ${p_i}^X \geq {p_i}^T \Rightarrow CB^T \geq CB^X$.

11. Appendix 2

In this appendix we show that $\frac{\partial CB^T}{\partial \varepsilon} \geq 0$. For this, it suffices[29] to show that $\frac{\partial p_i{}^T}{\partial \varepsilon} \leq 0$. Recall that;

$$p_i{}^T = \frac{t_i{}^T}{T} = \begin{cases} \dfrac{\hat{\alpha}}{\hat{\alpha} + \hat{\beta}} & if\ \hat{\beta} > 0 \\ 1 & if\ \hat{\beta} \leq 0 \end{cases}$$

First suppose $\hat{\beta} \leq 0$, it follows that $p_i{}^T = 1$. Thus, since $p_i{}^T$ is a win percentage always defined on the unit interval, we have $\frac{\partial p_i{}^T}{\partial \varepsilon} \leq 0$.

Second suppose $\hat{\beta} > 0$, so that $p_i{}^T = \frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}$. Recalling that $\hat{\alpha} = \varepsilon(\alpha + \beta) + \alpha$ and $\hat{\beta} = \varepsilon(\alpha + \beta) + \beta$, we can write $p_i{}^T = \frac{\hat{\varepsilon}+\alpha}{2\hat{\varepsilon}+\alpha+\beta}$ where[30] $\hat{\varepsilon} = \varepsilon(\alpha + \beta) \geq 0$. Then note that $\frac{\partial p_i{}^T}{\partial \hat{\varepsilon}} = \frac{\beta-\alpha}{(2\hat{\varepsilon}+\alpha+\beta)^2} < 0$. It follows that $\frac{\partial p_i{}^T}{\partial \varepsilon} = \frac{\partial p_i{}^T}{\partial \hat{\varepsilon}} \frac{\partial \hat{\varepsilon}}{\partial \varepsilon} = \frac{(\beta-\alpha)(\alpha+\beta)}{(2\hat{\varepsilon}+\alpha+\beta)^2} < 0$.

[29] To see this, note that $p_i{}^T$ is necessarily greater than $\frac{1}{2}$ (i.e., fans never prefer the opposition to have a greater win percentage). Thus, $\frac{\partial p_i{}^T}{\partial \varepsilon} \leq 0 \Rightarrow \frac{\partial CB^T}{\partial \varepsilon} \geq 0$.

[30] Recall that $\beta > -\alpha$ so that $\alpha + \beta > 0$.

12. Appendix 3

In this appendix we derive the first order conditions characterising interior solutions to clubs' first stage problem;

$$\max_{z_i \in \mathbb{R}_+} \pi_i{}^* = A_i \frac{z_i{}^{\hat{\alpha}} z_j{}^{\hat{\beta}}}{\left(z_i + z_j\right)^{\hat{\alpha}+\hat{\beta}}} - z_i \quad : \quad A_i = \frac{N_i}{4} T^{\alpha+\beta} (v\varepsilon)^{\varepsilon(\alpha+\beta)}$$

To this end, note that $\frac{\partial \pi_i{}^*}{\partial z_i} = A_i \frac{z_i{}^{\hat{\alpha}-1} z_j{}^{\hat{\beta}}}{\left(z_i+z_j\right)^{\hat{\alpha}+\hat{\beta}+1}} \left(\hat{\alpha} z_j - \hat{\beta} z_i\right) - 1$. It follows that in any interior solution to clubs' maximisation problem we must have $\frac{\partial \pi_i{}^*}{\partial z_i} = 0$ or equivalently;

$$A_i \frac{z_i{}^{\hat{\alpha}-1} z_j{}^{\hat{\beta}}}{\left(z_i + z_j\right)^{\hat{\alpha}+\hat{\beta}+1}} \left(\hat{\alpha} z_j - \hat{\beta} z_i\right) = 1$$

To show that this first order condition does indeed deliver a maximum, it remains to show that $\frac{\partial^2 \pi_i{}^*}{\partial z_i{}^2} < 0$ when evaluated at $\frac{\partial \pi_i{}^*}{\partial z_i} = 0$. For this, check that;

$$\frac{\partial^2 \pi_i{}^*}{\partial z_i{}^2} = \hat{\alpha} A_i \frac{z_i{}^{\hat{\alpha}-2} z_j{}^{\hat{\beta}}}{\left(z_i + z_j\right)^{\hat{\alpha}+\hat{\beta}+1}} \left(\hat{\alpha} z_j - \hat{\beta} z_i\right) - \hat{\alpha} A_i \frac{z_i{}^{\hat{\alpha}-2} z_j{}^{\hat{\beta}+1}}{\left(z_i + z_j\right)^{\hat{\alpha}+\hat{\beta}+1}}$$

$$- (1 + \hat{\alpha} + \hat{\beta}) A_i \frac{z_i{}^{\hat{\alpha}-1} z_j{}^{\hat{\beta}}}{\left(z_i + z_j\right)^{\hat{\alpha}+\hat{\beta}+2}} \left(\hat{\alpha} z_j - \hat{\beta} z_i\right)$$

Thus, when $\frac{\partial \pi_i{}^*}{\partial z_i} = 0 \Leftrightarrow A_i \frac{z_i{}^{\hat{\alpha}-1} z_j{}^{\hat{\beta}}}{\left(z_i+z_j\right)^{\hat{\alpha}+\hat{\beta}+1}} \left(\hat{\alpha} z_j - \hat{\beta} z_i\right) = 1$, we have;

$$\frac{\partial^2 \pi_i{}^*}{\partial z_i{}^2} = \frac{\hat{\alpha}}{z_i} - \frac{\hat{\alpha} z_j}{z_i\left(\hat{\alpha} z_j - \hat{\beta} z_i\right)} - \frac{1 + \hat{\alpha} + \hat{\beta}}{z_i + z_j}$$

It serves to rewrite this second order derivative as

$$z_i \frac{\partial^2 \pi_i{}^*}{\partial z_i{}^2} = \hat{\alpha}\left[\frac{\hat{\alpha} z_j - \hat{\beta} z_i - z_j}{\hat{\alpha} z_j - \hat{\beta} z_i} - \frac{z_i}{z_i + z_j}\right] - (1 + \hat{\beta})\frac{z_i}{z_i + z_j}$$

Now, noting that in any interior solution we must have $z_i, z_j > 0$ and that[31] $1 + \hat{\beta} > 0$, we have $\frac{\partial^2 \pi_i{}^*}{\partial z_i{}^2} < 0 \Leftrightarrow \frac{\hat{\alpha} z_j - \hat{\beta} z_i - z_j}{\hat{\alpha} z_j - \hat{\beta} z_i} < \frac{z_i}{z_i + z_j}$. Further noting that $\frac{\partial \pi_i{}^*}{\partial z_i} = 0 \Rightarrow \hat{\alpha} z_j - \hat{\beta} z_i > 0$, it follows that $\frac{\partial^2 \pi_i{}^*}{\partial z_i{}^2} < 0$ holds whenever i) $\hat{\alpha} z_j - \hat{\beta} z_i - z_j \leq 0$. If on the other hand, we have ii) $\hat{\alpha} z_j - \hat{\beta} z_i - z_j > 0$ then we note that[32] $\frac{\partial^2 \pi_i{}^*}{\partial z_i{}^2} < 0$ is equivalent to $z_i + z_j > \hat{\alpha} z_j - \hat{\beta} z_i$. Rearranging, this is in turn equivalent to $z_i > \frac{\hat{\alpha} - 1}{1 + \hat{\beta}} z_j$, which necessarily holds since $\hat{\alpha} < 1$.

Thus, $\frac{\partial \pi_i{}^*}{\partial z_i} = 0 \Leftrightarrow A_i \frac{z_i{}^{\hat{\alpha}-1} z_j{}^{\hat{\beta}}}{(z_i + z_j)^{\hat{\alpha}+\hat{\beta}+1}}\left(\hat{\alpha} z_j - \hat{\beta} z_i\right) = 1$ characterises the interior solution to clubs' maximisation problem. For convenience we now rewrite club i's first order condition in terms of $\tau = \frac{z_i}{z_j}$;

$$A_i \tau^{\hat{\alpha}-1}\left(\hat{\alpha} - \hat{\beta}\tau\right) = (1 + \tau)^{\hat{\alpha}+\hat{\beta}}(z_i + z_j)$$

and similarly for club j, $\frac{\partial \pi_j{}^*}{\partial z_j} = 0 \Leftrightarrow A_j \frac{z_j{}^{\hat{\alpha}-1} z_i{}^{\hat{\beta}}}{(z_i + z_j)^{\hat{\alpha}+\hat{\beta}+1}}\left(\hat{\alpha} z_i - \hat{\beta} z_j\right) = 1$ is equivalent to;

$$A_j \tau^{\hat{\beta}}\left(\hat{\alpha}\tau - \hat{\beta}\right) = (1 + \tau)^{\hat{\alpha}+\hat{\beta}}(z_i + z_j)$$

[31] To see this, recall that $\hat{\beta} = \varepsilon(\alpha + \beta) + \beta$ and that $\beta > -\alpha$ so that $\varepsilon(\alpha + \beta) \geq 0$. It follows that $1 + \hat{\beta} > 0$ is implied by $1 + \beta > 0$. Finally, since $\alpha < 1$, we must have $\beta > -1 \Leftrightarrow 1 + \beta > 0$.

[32] To see this, note that $\frac{\partial^2 \pi_i{}^*}{\partial z_i{}^2} < 0 \Leftrightarrow \frac{h_1 - z_j}{h_1} < \frac{h_2 - z_j}{h_2}$ where $h_1 = \hat{\alpha} z_j - \hat{\beta} z_i, h_2 = z_i + z_j$. Since we have $h_1 - z_j > 0$, this holds if and only if $h_2 > h_1$.

13. Appendix 4

In this appendix we show that clubs' first order conditions characterising interior solutions;

$$A_i \tau^{\widehat{\alpha}-1}\left(\hat{\alpha}-\hat{\beta}\tau\right) = (1+\tau)^{\widehat{\alpha}+\widehat{\beta}}(z_i+z_j)$$

$$A_j \tau^{\widehat{\beta}}\left(\hat{\alpha}\tau-\hat{\beta}\right) = (1+\tau)^{\widehat{\alpha}+\widehat{\beta}}(z_i+z_j)$$

together identify a unique interior sub-game perfect Nash equilibrium talent recruitment $t_i^{**} \in \mathbb{R}_+$. For this, it suffices to show that there exists a unique $\tau^{**} \in \mathbb{R}_+$ that solves this system of equations (i.e., note that $\tau^{**} = \frac{t_i^{**}}{t_j^{**}.} = \frac{t_i^{**}}{T-t_i^{**}}$ uniquely identifies $t_i^{**} = \frac{\tau^{**}}{1+\tau^{**}}T$). We begin by proving existence of τ^{**} and then proceed to prove uniqueness. In what follows, for ease of exposition and without loss of generality, we assume $N_i > N_j$ (this ensures that $\tau^{**} = \frac{t_i^{**}}{t_j^{**}.}$ will be greater than one).

1. Existence

By subtracting j's first order condition from i's[33], we see that $\tau^{**} \in \mathbb{R}_+$ satisfies both clubs' first order conditions if and only if $\varphi(\tau^{**}) = 0$ where;

$$\varphi(\tau) = N_i \tau^{\widehat{\alpha}-1}\left(\hat{\alpha}-\hat{\beta}\tau\right) - N_j \tau^{\widehat{\beta}}\left(\hat{\alpha}\tau-\hat{\beta}\right)$$

Now to show that such a τ^{**} necessarily exists, first note that $\varphi(1) = \left(N_i - N_j\right)(\alpha-\beta) > 0$. Second, note that $\lim_{\tau\to\infty}\varphi(\tau) \to -\infty$. It follows that, by continuity of $\varphi(\tau)$, there must exist some $\tau^{**} > 1$ with $\varphi(\tau^{**}) = 0$.

[33] And noting that $A_i = \frac{N_i}{4}T^{\alpha+\beta}(\nu\varepsilon)^{\varepsilon(\alpha+\beta)}$ so that the common term $\frac{1}{4}T^{\alpha+\beta}(\nu\varepsilon)^{\varepsilon(\alpha+\beta)}$ drops out.

2. Uniqueness

For uniqueness, we begin by showing that $\varphi(\tau) > 0$ for any $\tau \in [0,1)$. For this, first recall that i) $N_i > N_j$. Then note[34] that $\alpha - \beta < 1$ implies that ii) $\tau^{\hat{\alpha}-1} > \tau^{\hat{\beta}}$. Finally, note[35] that $\beta > -\alpha$, implies that iii) $\hat{\alpha} - \hat{\beta}\tau > \hat{\alpha}\tau - \hat{\beta}$. Together i-iii) imply that $\varphi(\tau) = N_i\tau^{\hat{\alpha}-1}(\hat{\alpha} - \hat{\beta}\tau) - N_j\tau^{\hat{\beta}}(\hat{\alpha}\tau - \hat{\beta}) > 0$.

Having shown in our proof of existence that $\varphi(1) > 0$, we can complete our proof for uniqueness by showing that $\varphi''(\tau) < 0$ for any $\tau > 1$. For this, check that $\varphi''(\tau) < 0 \Leftrightarrow \tau > \frac{\hat{\beta}}{\hat{\alpha}}\frac{\beta-\alpha}{2+\beta-\alpha}$. First, note[36] that $\frac{\beta-\alpha}{2+\beta-\alpha} \in (-1,0)$. Second, note[37] that $\frac{\hat{\beta}}{\hat{\alpha}} > -1$. It follows that $\frac{\hat{\beta}}{\hat{\alpha}}\frac{\beta-\alpha}{2+\beta-\alpha} < 1$. Thus, $\varphi''(\tau) < 0$ whenever $\tau > 1$.

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[34] To see this, check that $\tau \in [0,1)$ implies that $\tau^{\hat{\alpha}-1} > \tau^{\hat{\beta}}$ if and only if $\hat{\alpha} - 1 < \hat{\beta}$. Then recall that $\hat{\alpha} = \varepsilon(\alpha+\beta) + \alpha$ and $\hat{\beta} = \varepsilon(\alpha+\beta) + \beta$ so that $\hat{\alpha} - \hat{\beta} = \alpha - \beta$. Thus, we have $\tau^{\hat{\alpha}-1} > \tau^{\hat{\beta}} \Leftrightarrow \alpha - \beta < 1$.

[35] To see this, check that $\hat{\alpha} - \hat{\beta}\tau > \hat{\alpha}\tau - \hat{\beta} \Leftrightarrow \hat{\alpha} + \hat{\beta} > (\hat{\alpha} + \hat{\beta})\tau$. Thus, since $\tau \in [0,1)$, this holds if and only if $\hat{\alpha} + \hat{\beta} > 0$. Recalling that $\hat{\alpha} = \varepsilon(\alpha+\beta) + \alpha$ and $\hat{\beta} = \varepsilon(\alpha+\beta) + \beta$, this is equivalent to $(2\varepsilon+1)(\alpha+\beta) > 0$, which necessarily holds since $\beta > -\alpha$.

[36] To see this, recall that $\beta < \alpha$ and $\alpha - \beta < 1$. It follows that $\beta - \alpha \in (-1,0)$. Meanwhile, it also follows that $2 + \beta - \alpha > 1$. Thus, $\frac{\beta-\alpha}{2+\beta-\alpha} \in (-1,0)$.

[37] To see this, recall that $\hat{\alpha} = \varepsilon(\alpha+\beta) + \alpha$ and $\hat{\beta} = \varepsilon(\alpha+\beta) + \beta$. Thus, $\frac{\hat{\beta}}{\hat{\alpha}} > -1 \Leftrightarrow (2\varepsilon+1)(\alpha+\beta) > 0$, which necessarily holds since $\beta > -\alpha$.

14. Appendix 5

In this appendix we show that $\frac{\partial CB^{**}}{\partial \varepsilon} \geq 0$. For this, it suffices[38] to show that $\frac{\partial \tau^{**}}{\partial \varepsilon} \leq 0$. For this begin by recalling that τ^{**} is identified by

$$N_i {\tau^{**}}^{\hat{\alpha}-1}\left(\hat{\alpha} - \hat{\beta}\tau^{**}\right) - N_j {\tau^{**}}^{\hat{\beta}}\left(\hat{\alpha}\tau^{**} - \hat{\beta}\right) = 0$$

Taking logs, we have that $\ln\left(\frac{N_i}{N_j}\right) + \ln\left(\hat{\alpha} - \hat{\beta}\tau^{**}\right) = (1 + \beta - \alpha)\ln(\tau^{**}) + \ln\left(\hat{\alpha}\tau^{**} - \hat{\beta}\right)$.

Totally differentiating[39] with respect to ε yields;

$$\frac{\partial \tau^{**}}{\partial \varepsilon} = \frac{\tau^{**}\left(1 - {\tau^{**}}^2\right)(\alpha + \beta)\left(\hat{\alpha} - \hat{\beta}\right)}{(1 + \beta - \alpha)\left(\hat{\alpha} - \hat{\beta}\tau^{**}\right)\left(\hat{\alpha}\tau^{**} - \hat{\beta}\right) + \tau^{**}\left(\hat{\alpha}^2 - \hat{\beta}^2\right)}$$

Then, we have that $\frac{\partial \tau^{**}}{\partial \varepsilon} < 0$ is implied[40] by $\hat{\alpha} - \hat{\beta}\tau^{**} > 0$. To see that this holds note that $\hat{\alpha} - \hat{\beta}\tau^{**} \leq 0 \Rightarrow \varphi(\tau^{**}) < 0$, which yields a contradiction.

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[38] To see this, note from the previous appendix that τ^{**} is necessarily greater than 1 (i.e., the big club i always recruits a larger talent than their rival club j) so that ${p_i}^{**} > \frac{1}{2}$. Thus, $\frac{\partial \tau^{**}}{\partial \varepsilon} \leq 0 \Rightarrow \frac{\partial {p_i}^{**}}{\partial \varepsilon} \leq 0 \Rightarrow \frac{\partial CB^{**}}{\partial \varepsilon} \geq 0$.

[39] For this, recall that $\hat{\alpha} = \varepsilon(\alpha + \beta) + \alpha$ and $\hat{\beta} = \varepsilon(\alpha + \beta) + \beta$ so that $\frac{\partial \hat{\alpha}}{\partial \varepsilon} = \frac{\partial \hat{\beta}}{\partial \varepsilon} = \alpha + \beta$.

[40] To see this, note from the previous appendix that $\tau^{**} > 1$ so that $\left(1 - {\tau^{**}}^2\right) < 1$. Further note that $\hat{\alpha} - \hat{\beta} = \alpha - \beta > 0$ since $\alpha > \beta$. It follows that $\hat{\alpha}\tau^{**} - \hat{\beta} > 0$ and $\hat{\alpha}^2 - \hat{\beta}^2 > 0$ also. Finally, note that $1 + \beta - \alpha > 0$ is implied by $\alpha - \beta < 1$, while $\alpha + \beta > 0$ is implied by $\beta > -\alpha$.

A Theoretical Model of the Soft Budget Constraint Phenomenon in European Soccer Leagues.

Abstract:

Recent contributions to the sports economics literature observe that the soft budget constraint (SBC) phenomenon is prevalent amongst professional European soccer clubs. However, to the best of our knowledge, Franck & Lang (2014) is the only contribution that provides a theoretical model of SBCs in a sports league setting. This paper provides a new model, in which local governments provide bailouts for loss-making clubs. Crucially, unlike in Franck & Lang (2014), bailouts occur ex-post and, as such, governments suffer from a dynamic commitment problem. We derive a measure of the equilibrium budget softness, provided by governments, and a measure of the optimal budget softness, which governments would provide if they could make bailout decisions ex-ante. We find that i) the equilibrium budget softness is greater than optimal, but that ii) some budget softness can be optimal and that iii) the optimal budget softness converges to the equilibrium budget softness in a special case. These findings challenge the conventional wisdom that SBCs in sports leagues are a negative phenomenon.

1. Introduction

The soft budget constraint (SBC) phenomenon, as originally described by Kornai (1980), is the economic scenario in which a persistently loss-making organisation receives bailouts from a supporting organisation. Recent contributions to the sports economics literature observe that SBCs are prevalent amongst professional European soccer clubs (Andreff (2007 and 2011), Storm and Nielsen (2012) and Franck (2014)). Motivated by these contributions, this paper provides a new theoretical model of SBCs in European soccer leagues. To the best of our knowledge, the model is only the second to apply the SBC concept to a sports league setting, after that of Franck and Lang (2014). As such, we believe it represents a timely contribution to both the SBC and sports economics literatures.

We briefly discuss these literatures in the next section, before presenting our model, in this paper's third and fourth sections. The model consists of two stages. The first stage provides a relatively standard analysis of European soccer clubs' talent recruitment, in which clubs' objective is win maximisation and there is a perfectly elastic supply of talent. However, the second stage is unique in providing a subsequent analysis of local government bailouts of loss-making clubs. Crucially, since governments make bailout decisions *ex-post* in our model, after clubs recruit talent, they suffer from a dynamic commitment problem.

We solve for the sub-game perfect equilibrium of our model in section 5 and provide an expression for the equilibrium bailout-revenue ratio of clubs. This represents a convenient measure of the equilibrium budget softness. In section 6, we then derive an expression for the optimal bailout-revenue ratio, which local governments would choose if they had no dynamic commitment problem (i.e., if they could credibly commit ex-ante). This provides a measure of the optimal budget softness.

We examine these measures and present our conclusions in section 7. We find that, i) clubs' equilibrium budget softness is greater than the optimal budget softness, but that ii) some budget softness can be optimal and iii) as fans' taste for talent goes to its upper bound, the optimal budget softness converges to the equilibrium budget softness. These results challenge the established discourse in the sports economics literature, which views SBCs in sports leagues as a largely negative phenomenon.

2. Related Literature

The SBC literature, originated by Kornai (1980), concerns the economic phenomenon in which a persistently loss-making organisation receives bailout assistance from a supporting organisation. Much of the literature presents theoretical models of SBCs, in order to examine the efficiency and welfare implications of the phenomenon (Kornai et al (2003)). While a selection of these models attempt to provide general analyses (see Schaffer (1989), Dewatripont and Maskin (1995) and Stiglitz (1996)), many are tailored to analyse specific environments in which SBCs are prevalent. For instance, Crivelli & Staal (2013) provide a model, which analyses SBCs within federal governments, while the model of Mitchell (2000) concerns SBCs in financial markets.

The sports economics literature has recently identified European soccer leagues as a further environment in which SBCs are prevalent (Andreff (2007 and 2011), Storm and Nielsen (2012) and Franck (2014)), with loss-making clubs frequently receiving bailout assistance from private benefactors and public authorities (Garcia and Rodriguez (2003), Buraimo et al (2006) and Hamil and Walters (2010)). For instance, Franck (2014) reports that, in 2011 alone, private benefactor bailouts of Europe's elite clubs amounted to over 1 billion Euros. Meanwhile, Van Rompuy (2012) reports that, as of 2012, public authority bailouts of Spanish clubs also exceeded 1 billion Euros.

To the best of our knowledge, Franck and Lang (2014) is the only existing contribution that provides a theoretical model of SBCs within a sports league environment. The model of Franck and Lang (2014) features two stages. In the first stage, a public authority (or private benefactor) commits to bailout a club, in the event of insolvency, with a certain probability. In the second stage, the club engages in an investment strategy. The model reveals that greater first stage bailout probability commitments generate greater second stage investments.

Our model, which we present in the next section, differs from that of Franck and Lang (2014) in a number of key respects. Most significantly, our model reverses the two-stage dynamics of Franck and Lang (2014), so that public authorities make bailout decisions after clubs engage in investment strategies, rather than before. In this way, governments in our model have a dynamic commitment problem. This is motivated by both the sports economics and SBC literatures. In the sports literature, Van Rompuy (2012) and Franck (2014) argue that public authorities cannot make credible ex-ante commitments, since clubs are aware they have an incentive to provide bailouts ex-post. In the SBC literature, many argue that this type of dynamic commitment problem is a fundamental characteristic of the SBC phenomenon (Schaffer (1989), Dewatripont and Maskin (1995), Schaffer (1998) and Kornai et al (2003)).

Further, as compared to Franck & Lang (2014), our model features an endogenous measure of fan welfare. This seems important as fan welfare is generally considered the primary motivation for public authority bailouts (Storm and Nielsen (2012)). This also allows our model to feature endogenous bailout costs and endogenous collateral damage costs for non-bailouts (both of which are specified exogenously in Franck & Lang (2014)).

One ostensible limitation of our model, as compared to that of Franck & Lang (2014), is the lack of any uncertainty. Franck (2014) considers uncertainty to be an inherent feature of European soccer clubs' investment returns. In the model of Franck and Lang (2014), this uncertainty is necessary for the SBC phenomenon to arise. Crucially though, Schaffer (1998) argues that uncertainty is not fundamental to the SBC phenomenon. Our model re-enforces this by showing that European soccer clubs can have SBCs even in the absence of any uncertainty.

3. Game Setup

3.1 Overview

In this sub-section, we describe an overview of our model. The model is an extensive form game representation of a two-club league, which is unique in acknowledging that clubs, $C \equiv \{C_1, C_2\}$, may receive bailouts from local governments, $G \equiv \{G_1, G_2\}$. We assume that local governments are distinct (i.e., G_1 and G_2 are not the same government). This assumption applies equally to national leagues with clubs from various sub-national regions (i.e., Spain's La Liga, in which case local governments are regional governments) and European leagues with clubs from various European nations (i.e., the European Champions League, in which case local governments are national governments).

In the game's first period, each of the league's clubs, $C_i \in C$, simultaneously chooses a positive talent recruitment level, $t_i > 0$. The subsequent talent recruitment vector, $t = (t_1, t_2)$, is associated with a club expenditure, $E_i(t) > 0$, and a club revenue, $R_i(t) > 0$. For expenditures, we assume a perfectly elastic supply of talent at the wage rate w, so that $E_i(t) = wt_i$. For revenues, we assume that clubs in our league play each other twice in a round-robin home and away format, with revenues accrued from home game ticket sales to local fans. We provide a Falconieri et al (2004) micro-foundation for these revenues in the next section. Clubs then have a first period deficit/surplus given by; $D_i(t) = E_i(t) - R_i(t)$ (observe that $D_i(t) > 0$ indicates a deficit while $D_i(t) < 0$ indicates a surplus).

In the game's second period, each of the local governments, $G_i \in G$, whose club has a first period deficit, $D_i(t) > 0$, must simultaneously decide whether they are willing to provide a bailout, $B_i(t) = 1$, or not, $B_i(t) = 0$. Note that, by analysing the case in which governments make bailout decisions ex-post, we implicitly assume that governments have no credible recourse to any ex-ante commitment devices and, therefore, have a dynamic commitment problem.

In the event that either government is unwilling to provide a bailout for a loss-making club, $B_1(t) = 0$ or $B_2(t) = 0$, the league fails, fixtures are unfulfilled, and both clubs are forced to refund their local fans for ticket purchases. In this case, we assume that any other bailout commitments are rescinded, and both clubs suffer insolvency due to unpaid player wages[41].

In any other case, both clubs remain solvent, the league proceeds and local fans, of which there are $F_i > 0$, having paid $R_i(t)$ for tickets, receive a total attendance utility, $V_i(t)$, which we again micro-found in the next section. Local governments that committed to a bailout, $B_i(t) = 1$, make a payment, $D_i(t)$, to their local club, which ensures their solvency. Crucially, we assume that this bailout is financed by a uniform lump-sum per capita tax, $\tau_i(t) = \frac{D_i(t)}{T_i}$, on the local population, $T_i > F_i$.

The next sub-sections define our game's extensive form more precisely. First, we define our game's players, sequencing and information, in the next sub-section, and then define our game's strategy set, S, and payoff-function, $\pi: S \to \mathbb{R}^4$, in the following two sub-sections.

[41] Note the contagion here. If one club suffers bankruptcy, the entire league fails, since no fixtures can be fulfilled. This ensures that the league's other club also suffers bankruptcy (even if they may have otherwise been profitable).

3.2 Players, Sequencing and Information

Our game features four players, two clubs, $C \equiv \{C_1, C_2\}$, and two local governments $G \equiv \{G_1, G_2\}$. The game's player set is, therefore, $\{C, G\}$.

The game's sequencing is as follows. In the game's first period, each club, $C_i \in C$, simultaneously makes a positive talent recruitment choice, $t_i > 0$, generating a talent recruitment vector, $t = (t_1, t_2)$. In the game's second period, governments whose local clubs have a first period deficit, $G_i \in G: D_i(t) > 0$, then simultaneously decide whether or not they are willing to provide their club with a bailout, $B_i(t) \in \{0,1\}$.

Finally, we assume that the game's information is complete. That is, clubs and governments' strategies sets and payoff functions are common knowledge amongst all players.

3.3 Strategy Set

Our game's strategy set, $S \equiv S^C \times S^G$, is composed of a strategy set for clubs, $S^C \equiv S_1^C \times S_2^C$, and a strategy set for governments, $S^G \equiv S_1^G \times S_2^G$.

Since each club, $C_i \in C$, acts only in the first period of our game, in which they choose a talent recruitment, $t_i \in \mathbb{R}_{++}$, their strategy set is simply; $S_i^C \equiv \mathbb{R}_{++}$.

Each local government, $G_i \in G$, meanwhile, acts in the second period of our game, if and only if their club has a first period deficit, $D_i(t) > 0$. In which case, they must make a binary bailout decision, $B_i(t) \in \{0,1\}$. Denoting the set of first period talent recruitments that generate a deficit for club i, $Z_i \equiv \{t \in S^C: D_i(t) \ > 0\}$, any government strategy, $B_i \in S_i^G$, must specify an action, $B_i(t) \in \{0,1\}$, for any $t \in Z_i$. The local government strategy set is therefore the set of all possible functions, B_i, with domain Z_i and range $\{0,1\}$. That is, $S_i^G \equiv \mathcal{F}$, where; $B_i \in \mathcal{F} \Leftrightarrow \ B_i: Z_i \rightarrow \{0,1\}$.

3.4 Payoff Functions

To complete the definition of our game, it remains to define its payoff function, $\pi: S \rightarrow \mathbb{R}^4$. For this, we begin by defining the club payoff function, $\pi_i^C: S \rightarrow \mathbb{R}$, before defining the government payoff function, $\pi_i^G: S \rightarrow \mathbb{R}$. The payoff function is then, $\pi = \pi_1^C \times \pi_2^C \times \pi_1^G \times \pi_2^G$.

3.4.1 Club Payoff Function

For the club payoff function, $\pi_i^C: S \rightarrow \mathbb{R}$, we follow the standard specification of European soccer league models and assume that clubs pursue an objective of win-maximisation, where their win percentage, $p_i(t) \in (0,1)$, is strictly increasing in their talent recruitment (i.e., $\frac{\partial p_i(t)}{\partial t_i} > 0$).

We denote the set of strategies that result in league failure and club bankruptcies $\hat{Z} \equiv \{s \in S: D_i(t) > 0 \cap B_i(t) = 0 \; for\ some\ i \in \{1,2\}\}$. For any case in which the league does not fail, $s \notin \hat{Z}$, we assume that clubs receive a payoff equal to their win percentage, $\pi_i^C(s) = p_i(t)$. For any case in which the league does fail, $s \in \hat{Z}$, both clubs suffer bankruptcy. We assume that this bankruptcy has an associated cost, $r(t_i) > 0$, which is increasing in the extent of the club's liabilities, $r'(t_i) > 0$. The club payoff function is then defined as;

Definition 1: Club Payoff Function – Each Club's payoff function is;

$$\pi_i^C(s) = \begin{cases} -r(t_i) & if\ s \in \hat{Z} \\ p_i(t) & if\ s \notin \hat{Z} \end{cases}$$

Where $r(t_i)$ is a bankruptcy cost, incurred whenever the league fails, $s \in \hat{Z}$, which is increasing in clubs' talent recruitment, $r'(t_i) > 0$. While, $p_i(t)$ is club i's win percentage whenever the league does not fail, $s \notin \hat{Z}$ ■

3.4.2 Local Government Payoff Function

For the local government payoff function, $\pi_i^G: S \to \mathbb{R}$, we suppose that the government is additively concerned with the welfare of its citizens, of which there are T_i. Within this population, there are two disjoint types; non-fans, N_i, and fans, F_i, of the local club. The proportion of fans out of the total population is then $f_i = \frac{F_i}{T_i}$, we interpret this as a measure of the 'popular support' of club i in their local area. Non-fans have total welfare, $U_i^N: S \to \mathbb{R}$, while fans have total welfare, $U_i^F: S \to \mathbb{R}$. The government's additive social welfare function is then;

$$\pi_i^G(s) = U_i^N(s) + \omega_i U_i^F(s)$$

where $\omega_i > 0$ is the government's relative weighting of fans' welfare compared to non-fans' welfare.

We assume that total non-fan welfare, $U_i^N(s)$, is simply their total income, Y_i^N, minus total uniform tax liabilities of $\frac{N_i D_i(t)}{T_i}$, if the government bails out their local club ($\Leftrightarrow s \notin \hat{Z} \cap t \in Z_i$);

$$U_i^N(s) = \begin{cases} Y_i^N - (1 - f_i)D_i(t) & s \notin \hat{Z} \cap t \in Z_i \\ Y_i^N & otherwise \end{cases}$$

We assume that total fan welfare, $U_i^F(s)$, is their total income, Y_i^F, plus an additive total utility from match attendance, $V_i(t)$, minus total ticket expenditures, $R_i(t)$, so long as the league does not fail ($\Leftrightarrow s \notin \hat{Z}$), minus total uniform tax liabilities, $\frac{F_i D_i(t)}{T_i}$, if the government bails out their club ($\Leftrightarrow s \notin \hat{Z} \cap t \in Z_i$);

$$U_i^F(s) = \begin{cases} Y_i^F + V_i(t) - R_i(t) - f_i D_i(t) & s \notin \hat{Z} \cap t \in Z_i \\ Y_i^F + V_i(t) - R_i(t) & s \notin \hat{Z} \cap t \notin Z_i \\ Y_i^F & otherwise \end{cases}$$

With this, we can define the local government payoff function;

Definition 2: Local Government Payoff Function – Each local government's payoff function is;

$$\pi_i^G(s) = \begin{cases} Y_i^N - (1 - f_i)D_i(t) + \omega_i[Y_i^F + V_i(t) - R_i(t) - f_i D_i(t)] & s \notin \hat{Z} \cap t \in Z_i \\ Y_i^N + \omega_i[Y_i^F + V_i(t) - R_i(t)] & s \notin \hat{Z} \cap t \notin Z_i \\ Y_i^N + \omega_i[Y_i^F] & s \in \hat{Z} \end{cases}$$

Where, if the league fails, $s \in \hat{Z}$, local governments receive a benchmark payoff, $Y_i^N + \omega_i[Y_i^F]$. Otherwise, if the league proceeds and the local government does not have to bailout their club, $s \notin \hat{Z} \cap t \notin Z_i$, they receive their benchmark payoff, plus an extra payoff reflecting their valuation of fans' consumer surplus, $\omega_i[V_i(t) - R_i(t)]$. Finally, if the league proceeds thanks to a local government bailout, $s \notin \hat{Z} \cap t \in Z_i$, they receive their benchmark payoff, plus a payoff from fans' consumer surplus, minus their perceived bailout cost, $(1 - f_i)D_i(t) + \omega_i f_i D_i(t)$ ■

4. Falconieri et al (2004) Micro-Foundation

Before we present our game's equilibrium, it remains to specify a functional form for clubs' revenues, $R_i(t)$, and fans' total attendance utility, $V_i(t)$. For this, we follow Falconieri et al's (2004) widely adopted micro-foundation.

This micro-foundation assumes that fans vary uniformly in the utility they receive from attendance; with the least passionate fan receiving zero utility and the most passionate receiving $\bar{v}_i(t)$. Further, it is assumed that for each club's home game i) only home fans, F_i, can attend, ii) there are no stadium capacity constraints iii) there are no marginal costs to ticket sales and iv) clubs must set a single fixed ticket price, z_i.

It can be shown that win-maximising clubs set ticket prices at the profit-maximising level, $z_i = \frac{\bar{v}_i(t)}{2}$. At this ticket price, half of the club's fan base, $\frac{F_i}{2}$, (i.e., those for which attendance utility is greater than $\frac{\bar{v}_i(t)}{2}$) purchase a ticket for their club's home match, which generates revenues, $R_i(t) = \frac{F_i \bar{v}_i(t)}{4}$ (i.e., attendance, $\frac{F_i}{2}$, multiplied by the ticket price, $\frac{\bar{v}_i(t)}{2}$) . Fans' total utility is then given by $V_i(t) = \frac{3F_i \bar{v}_i(t)}{8}$ (i.e., the average utility of attending fans, $\frac{3\bar{v}_i(t)}{4}$, multiplied by the number of attending fans, $\frac{F_i}{2}$).

Crucially, we assume a Cobb-Douglas form for fans' utility;

$$\bar{v}_i(t) = {t_i}^{\alpha} {t_j}^{\beta} \qquad : \ 1 > \alpha + \beta \geq \alpha > \beta \geq 0$$

Here, α is a measure of fans' taste for their own team's talent and β is a measure of fans' taste for opposition talent. We refer to $\alpha + \beta$ simply as 'fans' taste for talent'. $1 > \alpha + \beta$ ensures that clubs' talent recruitment problem has a positive finite solution in the next section. Thus, we say that fans' taste for talent is at its maximum when $\alpha + \beta \to 1$. $\alpha > \beta$ ensures that fans prefer their own team's talent more than the opposition's talent. $\beta \geq 0$ ensures that fans have a non-negative preference for opposition talent, as justified by the empirical findings of Buraimo & Simmons (2008).

With this, we can define each club's micro-founded revenues, fans' total utility and club deficits as follows;

Definition 3: Each club $C_i \in C$ has;

i) *Revenue Function* $R_i(t) = \frac{F_i {t_i}^{\alpha} {t_j}^{\beta}}{4}$,

ii) *Total Fan Utility Function* $V_i(t) = \frac{3F_i {t_i}^{\alpha} {t_j}^{\beta}}{8}$,

iii) *Deficit Function* $D_i(t) = wt_i - \frac{F_i {t_i}^{\alpha} {t_j}^{\beta}}{4}$ ■

This allows us to present the Nash equilibrium of our model in the next section.

5. <u>Equilibrium Budget Softness</u>

In this section, we can now derive a measure of the equilibrium budget softness in our model. For this, it is convenient to first rewrite the government payoff function (presented in Definition 2) as follows;

$$\pi_i^G(s) = \begin{cases} Y_i^N + \omega_i \left[Y_i^F + \dfrac{\overline{D}_i(t) - D_i(t)}{2\mu(\omega_i, f_i)} \right] & s \notin \hat{Z} \cap t \in Z_i \\ Y_i^N + \omega_i \left[Y_i^F + \dfrac{\overline{D}_i(t)}{2\mu(\omega_i, f_i)} \right] & s \notin \hat{Z} \cap t \notin Z_i \\ Y_i^N + \omega_i [Y_i^F] & s \in \hat{Z} \end{cases}$$

where $\mu(\omega_i, f_i) = \frac{\omega_i}{2(f_i \omega_i + (1 - f_i))}$ and $\overline{D}_i(t) = 2\mu(\omega_i, f_i)[V_i(t) - R_i(t)] = \mu(\omega_i, f_i) \frac{F_i {t_i}^{\alpha} {t_j}^{\beta}}{4}$.

In order to interpret the significance of $\overline{D}_i(t)$, note that, when $D_i(t) = \overline{D}_i(t)$, the local government is indifferent between bailing out, $s \notin \hat{Z} \cap t \in Z_i$, and allowing the league to fail, $s \in \hat{Z}$. Whereas, when $D_i(t) < \overline{D}_i(t)$, it prefers to bailout and when $D_i(t) > \overline{D}_i(t)$, it prefers to allow the league to fail. As such, $\overline{D}_i(t)$ represents a deficit threshold, beyond which the local government will not provide a bailout, but below which they will. The significance of $\mu(\omega_i, f_i)$ will become clear shortly when we present Proposition 2.

First, we solve for our game's equilibria by backward induction in Appendix 1. Proposition 1 reports that;

Proposition 1: *Any sub-game perfect Nash equilibrium of our game, $(t^*, B^*) \in S$, in which neither local government plays a weakly dominated action in any period 2 sub-game*[42] *is such that;*

i) *Both clubs have a first period deficit equal to their deficit threshold (i.e., $D_i(t^*) = \overline{D}_i(t^*)$),*

ii) *Both local governments provide a second period bailout on the equilibrium path (i.e., ${B_i}^*(t^*) = 1$).*

iii) *Clubs have a payoff, $\pi_i^C(t^*, B^*) = p_i(t^*)$, and governments have a payoff $\pi_i^G(t^*, B^*) = Y_i^N + \omega_i Y_i^F$,*

iv) *Each club's talent recruitment is given by ${t_i}^* = {A_i}^{\gamma(1-\theta\beta)}{A_j}^{\gamma\theta\beta}$ where $A_i = \left(\frac{1+\mu(\omega_i, f_i)}{4w}\right)F_i$, $\gamma = \frac{1}{1-\beta-\alpha}$ and $\theta = \frac{1}{1+\beta-\alpha}$.*

<u>Proof:</u> *See Appendix 1* ■

Crucially, we find that i) in equilibrium clubs are systematically loss-making, $t^* \in Z_i$ such that deficits are at the threshold level, $D_i(t^*) = \overline{D}_i(t^*)$, beyond which governments do not bailout. This corresponds with observed loss-making behaviours of European soccer clubs in particular (Garcia & Rodriguez (2003), Szymanski & Zimbalist (2006) and Franck (2014)).

Further, we find that ii) in equilibrium local governments are willing to provide bailouts for clubs' losses, ${B_i}^*(t^*) = 1$. As such, clubs perennially remain solvent and league fixtures are

[42] This restriction is useful in delivering a unique period 1 equilibrium talent recruitment. Without proof, we note that the unique period 1 equilibrium talent recruitment delivered by this restriction is also the only equilibrium talent recruitment possible if we were to model local governments' bailout decisions sequentially rather than simultaneously.

always fulfilled, $(t^*, B^*) \notin \hat{Z}$. This again corresponds with observed bailout behaviour of many real-world authorities, who frequently provide favourable tax and/or municipal stadium rental rates and have a notable reluctance to liquidate clubs to retrieve unpaid tax debts (Hamil & Walters (2010), Van Rompuy (2012) and Franck (2014)). This also rationalises real-world observations of miraculously high survival rates among loss-making clubs (Kuper & Szymanski (2010) and Storm & Nielsen (2012)).

Finally, we find that iii) in equilibrium clubs extract all possible rents from the league, leaving governments with exactly their league failure payoff, $Y_i^N + \omega_i Y_i^F$. That governments extract zero rents is a result of their dynamic commitment problem. They are unable to make credible ex-ante commitments not to bailout while they have an ex-post incentive to do so (i.e., whenever $D_i(t^*) \leq \overline{D}_i(t^*)$). Knowing governments' ex-post incentive to bailout, win-maximising clubs are able to extract the entirety of available rents by recruiting talent up to the deficit threshold, $D_i(t^*) = \overline{D}_i(t^*)$ (at which the government is exactly indifferent between bailing out and allowing the league to fail).

Now, a convenient measure of equilibrium budget softness is clubs' equilibrium deficit-revenue ratio, $DR_i{}^* = \frac{D_i(t^*)}{R_i(t^*)}$. This budget softness measure potentially ranges from 0, if a club had no deficit, to ∞, if a club had an infinitely large deficit relative to their revenues. In between, a value of 1 would indicate that the club has a deficit equal to 100% of their revenues.

Proposition 2: *Equilibrium budget softness, for any club, $C_i \in C$, as measured by their equilibrium deficit-revenue ratio, is given by $DR_i{}^* = \frac{D_i(t^*)}{R_i(t^*)} = \mu(\omega_i, f_i) > 0$ which;*

i) is increasing in the government's weight on fan welfare ($\frac{\partial DR_i{}^*}{\partial \omega_i} > 0$), *ii)* *is decreasing in the popular support for the local club whenever the government is more concerned with fan welfare* ($\omega_i > 1 \Leftrightarrow \frac{\partial DR_i^*}{\partial f_i} < 0$)*, but is increasing in the popular support for the local club whenever the government is more concerned with non-fan welfare* ($\omega_i < 1 \Leftrightarrow \frac{\partial DR_i^*}{\partial f_i} > 0$)*,*

iii) *goes to* 0 *in the limit as governments are exclusively concerned with non-fan welfare* ($\lim_{\omega_i \to 0} DR_i^* = 0$)*, equals* $\frac{1}{2}$ *with a utilitarian government* ($DR_i^*|_{\omega_i=1} = \frac{1}{2}$) *and goes to* $\frac{1}{2f_i}$ *in the limit as governments are exclusively concerned with fan welfare* ($\lim_{\omega_i \to \infty} DR_i^* = \frac{1}{2f_i}$)*.*

Proof: *See Appendix 2*■

Thus, we can now interpret $\mu(\omega_i, f_i)$ as clubs' equilibrium budget softness. What's more, with this proposition, we report that i) the more concerned local governments are with fans' welfare, the softer are clubs' budget constraints, $\frac{\partial DR_i^*}{\partial \omega_i} > 0$. This is an intuitive result, which simply follows from the fact that governments have a greater incentive to bailout loss-making clubs when they place a higher value on fans' utility.

Further, we find that ii) if the government cares more (less) about fans' welfare than non-fans' welfare (i.e., $\omega_i \underset{(<)}{>} 1$), budget softness is decreasing (increasing) in the popular support for club i (i.e., $\frac{\partial DR_i^*}{\partial f_i} \underset{(>)}{<} 0$. This somewhat paradoxical result follows from the fact that, if the government cares disproportionately about fan welfare, then the perceived cost of financing bailouts is increasing in the proportion of fans in the total population (i.e., note that the perceived cost of financing a bailout of D_i via universal taxes is $(1 - f_i)D_i + \omega_i f_i D_i = [1 + (\omega_i - 1)f_i]D_i$). Governments are then less willing to soften clubs' budget constraints when the popular support for the local club is high. On the other hand, if the government cares disproportionately about non-fan welfare, then the perceived cost of financing bailouts is decreasing in the proportion of fans in the total population. Governments are then more willing to soften clubs' budget constraints when the popular support for the local club is high

Finally, we find that iii) there is no budget softness when governments are unconcerned with fans' welfare. This follows naturally from the fact that governments have no incentive to bailout clubs when $\omega_i \rightarrow 0$. Meanwhile a utilitarian government generates a deficit to revenue ratio of 50%. Note that the utilitarian budget softness is independent of the popular support, f_i. Finally, a government solely concerned with fans' welfare generates a deficit to revenue ratio of $\frac{50}{f_i}$%. That increases in f_i reduce this budget softness follows from ii).

6. Optimal Budget Softness For Local Governments

In this section, we present a measure of the optimal budget softness for local governments.

In our model, local governments make bailout decisions ex-post; after clubs make their talent recruitments. This implies a dynamic commitment problem, in which clubs, aware of the government's ex-post incentive to provide bailouts, are able to extract all available rents. In this way, the equilibrium budget softness is sub-optimal for local governments (Schaffer (1989), Dewatripont & Maskin (1995) and Kornai et al (2003)).

To identify the optimal budget softness for local governments, we abstract from their dynamic commitment problem, by solving for the bailout-revenue ratio they would choose if they could credibly commit ex-ante (i.e., before clubs make their talent recruitments). This ex-ante optimal bailout-revenue ratio provides our measure of the optimal budget softness for local governments. To the extent that local governments are representative, we may consider the local government optimal budget softness to be optimal for the local population also.

For the following proposition, it serves to define $\varphi = \gamma(\alpha(1-\theta\beta)+\theta\beta^2)$, where $\gamma = \frac{1}{1-\beta-\alpha}$ and $\theta = \frac{1}{1+\beta-\alpha}$.

Proposition 3: *Local governments' ex-ante optimal bailout-revenue ratio, that would occur if they could credibly commit before period 1, is given by* $DR_i{}^C = \begin{cases} \frac{\varphi DR_i{}^{*}-1}{1+\varphi} & if\ \varphi DR_i{}^{*} > 1 \\ 0 & if\ \varphi DR_i{}^{*} \leq 1 \end{cases}$, *which is increasing in fans' taste for talent and the government weight on fan welfare (i.e.,* $\frac{\partial DR_i{}^C}{\partial \alpha}, \frac{\partial DR_i{}^C}{\partial \beta}, \frac{\partial DR_i{}^C}{\partial \omega_i} \geq 0$*).*

<u>Proof:</u> *See Appendix 3*■

Firstly, with this proposition, we find that there is a case in which a completely hardened budget constraint is optimal for local governments (i.e., when $\varphi DR_i{}^{*} \leq 1$). In this case, governments' dynamic commitment problem forces them to provide bailout funding ex-post (i.e., $DR_i{}^{*} > 0$), whereas ex-ante they prefer not to fund their local club at all (i.e., $DR_i{}^C = 0$). It follows that policy-makers should seek to eliminate SBCs. This result is broadly consistent with the current discourse within the sports economics literature, which describes SBCs as a negative phenomenon.

However, with this proposition, we also find that there is a case in which some degree of budget softness is optimal (i.e., when $\varphi DR_i{}^{*} > 1$). This is a significant result, which is quite against the prevailing discourse on SBCs in the sports league literature. It suggests that some government bailout support for sports clubs may indeed be optimal. In this case, there is an implied under-investment problem associated with clubs' unsupported talent recruitment, which government bailouts help to alleviate. This under-investment problem may be due, in part, to the fact that sports entertainment represents a quasi-public non-rivalrous good (i.e., when fans purchase match tickets they do not detract from the enjoyment of fellow fans[43]).

Lemma 1 qualifies this result by reporting that the optimal budget softness is always lesser than the equilibrium budget softness;

Lemma 1: *Local governments' optimal budget softness is strictly less than the equilibrium softness ($DR_i{}^C < DR_i{}^*$).*

<u>Proof:</u> *See Appendix 4*■

Thus, we find that governments' dynamic commitment problem ensures that clubs' budgets are excessively soft. That is, governments provide an inefficiently high level of bailout funding to their local clubs[44].

Finally, Proposition 3 also reports that the optimal budget softness is increasing in i) fans' taste for their own team's talent (i.e., $\frac{\partial DR_i{}^C}{\partial \alpha} \geq 0$), ii) fans' taste for their rival team's talent (i.e., $\frac{\partial DR_i{}^C}{\partial \beta} \geq 0$) and iii) the government's weight on fan welfare (i.e., $\frac{\partial DR_i{}^C}{\partial \omega_i} \geq 0$, note that each of these inequalities are strict when $DR_i{}^C > 0$). Intuitively, the more utility fans acquire from clubs' talent recruitment[45] (i.e., the greater are α and β) the more bailout funding governments optimally provide. Similarly, the more a local government cares about fans' utility (i.e., the greater is ω_i) the more bailout funding is optimal.

[43] Whereas in other industries recruiting a talented workforce increases the quantity of a firm's output, in sports leagues recruiting playing talent increases the quality of clubs' fixed output.
[44] Note that this result holds purely from the perspective of the local government, which is only concerned with the welfare of its own citizens. A supra-national or supra-regional government, which is concerned with the welfare of citizens in both regions 1 and 2 may optimally prefer a budget softness that is even greater than the equilibrium level (since $\beta \geq 0$ fans in region j receive a positive external benefit from the bailout funding of local government i).
[45] To see why the optimal softness is increasing in β, note that when government i softens their club's budget constraint, this encourages greater talent recruitment from club j as well as club i. Thus, the greater is fans' taste for opposition talent (i.e., β) the greater are the welfare benefits of budget softness.

Crucially, Lemma 2 notes that as fans' taste for talent increases to its maximum possible level, the optimal budget softness increases to the equilibrium budget softness;

Lemma 2: *Local governments' optimal budget softness approaches the equilibrium softness, in the limit as fans' taste for talent reaches its maximum (*$\lim_{\alpha+\beta\to 1} DR_i{}^C = DR_i{}^*$*).*

<u>*Proof:*</u> *See Appendix 4*■

Thus, as a limiting case, the equilibrium budget softness becomes optimal as the welfare fans acquire reaches its greatest intensity (i.e., as $\alpha + \beta \to 1$). In this special case, we find that governments' dynamic commitment problem is effectively nullified; the deviation between the equilibrium budget softness and the optimal budget softness goes to zero, (i.e., $\lim\limits_{\alpha+\beta\to 1} DR_i{}^* - DR_i{}^C = 0$). It follows that there is almost no excess budget softness, in this case, and policy-makers should not seek to harden clubs' budget constraints.

7. Conclusion

This paper provides a new model of SBCs in a European soccer league setting. In this model, local governments soften budget constraints by providing bailouts for loss-making clubs ex-post. In this way, governments are subject to a dynamic commitment problem. We derive a measure of the equilibrium budget softness, provided by governments in our model, and a measure of the optimal budget softness, which governments would provide if they could make bailout decisions ex-ante (i.e., if they were not subject to any dynamic commitment problem).

We report three significant results. First, we find that;

Result 1: Completely hardened budget constraints are not always optimal.

In other words, even when governments are not subject to any dynamic commitment problem, they may still optimally choose to soften clubs' budget constraints. This result challenges the prevailing wisdom of the sports economics literature, which generally considers SBCs in sports leagues as a negative phenomenon. However, second, we find that;

Result 2: In equilibrium, clubs have a sub-optimally high level of budget softness.

In other words, where governments suffer from a dynamic commitment problem, they are forced to provide an excessively high level of budget softness to clubs. This result does establish a potential need for policy-makers to restrict the prevalence of SBCs in sports leagues. Finally, though, we also find that;

Result 3: There exists a special case, as fans' taste for talent reaches its maximum, in which clubs' equilibrium budget softness becomes very nearly optimal.

In this special case, the need for policy intervention to limit SBCs in sports leagues is apparently nullified.

These results are particularly relevant to the recent implementation of Financial Fair Play (FFP); a set of rules designed to reduce SBCs in European football leagues (Franck (2014)). Result 1 reports that it may not be optimal for FFP to completely harden clubs' budget constraints (and that governments may attempt to resist such efforts). However, Result 2 indicates that FFP's hardening of budget constraints may be beneficial for governments suffering from a dynamic commitment problem (and for citizens of such governments). Finally, though, Result 3 reports that the prevailing softness of clubs' budget constraints, before FFP, may have already been close to optimal (in which case, any significant hardening of budget constraints is likely to be sub-optimal).

From this benchmark, a number of extensions to our model may be informative for both the sports economics and SBC literatures. First, in line with the model of Franck & Lang (2014), it may be of interest to study the effect of introducing uncertainty to our model. Though our initial conjecture is that qualitative results would be largely unchanged, there are surely some inter-relationships between uncertainty and the SBC phenomenon in sports leagues, not currently captured by this model. Second, it would be desirable to investigate the optimal budget softness from the perspective of some federal government that is concerned with the welfare of citizens in both regions. Our conjecture is that such a government would prefer a greater level of budget softness than local governments, as they would account for the positive externality effects of clubs' talent recruitments on rival fans (i.e., recall that $\beta \geq 0$). Finally, from a political economy perspective, it would certainly be of interest to introduce an election stage, before clubs' talent recruitments, in which fans and non-fans vote for their preferred governments. Our conjecture is that narrowly selfish non-fans would seek to elect a government that is only concerned with their welfare (i.e., $\omega_i = 0$). However, narrowly selfish fans may seek to elect a government that is also concerned with non-fan welfare (i.e., ω_i finite), in order to limit the dynamic commitment problem and thereby achieve the level of budget softness which is optimal for them.

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9. Appendix 1

In this appendix, we solve for our game's sub-game perfect equilibria, $(t^*, B^*) \in S$, by backward induction. For this, we begin by solving for the possible period 2 sub-game equilibria.

9.1 Period 2 Sub-Game Equilibria

Any possible period 2 sub-game, follows a given period 1 talent recruitment, t, and is such that there exists some government $G_i \in G$ whose local club has period 1 losses of $D_i(t) > 0$. This government then has a binary decision between a bailout action, $B_i(t) = 1$ and a non-bailout action $B_i(t) = 0$. The payoff matrix for G_i, in terms of this period 2 decision, is as follows;

	$B_i(t) = 1$	$B_i(t) = 0$
$B_j(t) = 1 \; or \; t \notin Z_j$	$Y_i^N + \omega_i \left[Y_i^F + \frac{\overline{D}_i(t) - D_i(t)}{2\mu(\omega_i, f_i)} \right]$	$Y_i^N + \omega_i [Y_i^F]$
$B_j(t) = 0$	$Y_i^N + \omega_i [Y_i^F]$	$Y_i^N + \omega_i [Y_i^F]$

Where, following from the specification of governments' payoff function at the beginning of Section 5, government i has payoff $Y_i^N + \omega_i [Y_i^F]$ whenever the league fails and payoff $Y_i^N + \omega_i \left[Y_i^F + \frac{\overline{D}_i(t) - D_i(t)}{2\mu(\omega_i, f_i)} \right]$ whenever the league proceeds (this payoff reflects the government's valuation of fans' consumer surplus and the cost of raising bailout funds).

Here, it serves to eliminate the possibility that a government plays a weakly dominated action in any period 2 sub-game equilibrium.

First, from the payoff matrix, note that, if club i's deficit is below their deficit threshold, $\overline{D}_i(t) > D_i(t)$, then the no bailout action, $B_i(t) = 0$, is weakly dominated by the bailout action, $B_i(t) = 1$. On the other hand, note that if club i's deficit is over their deficit threshold, $\overline{D}_i(t) < D_i(t)$, then the bailout action, $B_i(t) = 1$, is weakly dominated by the no bailout action, $B_i(t) = 0$.

Thus, eliminating weakly dominated actions, we have that, in any sub-game in which a club's deficit is below threshold, their local government must always bailout in equilibrium (i.e., ${B_i}^*(t) = 1$ if $\overline{D}_i(t) > D_i(t)$). Further, we have that, in any sub-game in which a club's deficit is above threshold, their local government can never bailout in equilibrium (i.e., ${B_i}^*(t) = 0$ if $\overline{D}_i(t) < D_i(t)$).

Finally, we note that if club i's deficit is equal to their deficit threshold, $\overline{D}_i(t) = D_i(t)$, then the government is indifferent between bailing out and not bailing out regardless of any bailout decision by government j (their payoff at every element of the payoff matrix is $Y_i^N + \omega_i[Y_i^F]$). In this case, we have that, in any sub-game in which a club's deficit is equal to threshold, their local government can both bailout or not bailout in equilibrium (i.e., ${B_i}^*(t) \in \{0,1\}$ if $\overline{D}_i(t) = D_i(t)$).

Thus, to summarise, for any possible sub-game following a generic talent recruitment t, we have that if Ai) $D_i(t) \leq 0$ government i makes no bailout action in equilibrium, (we notate this case ${B_i}^*(t) = \emptyset$ from here). Meanwhile, if Bi) $\overline{D}_i(t) > D_i(t) > 0$ government i always bails out in equilibrium (${B_i}^*(t) = 1$), since in this case not bailing out is a weakly dominated action. Further, if Ci) $\overline{D}_i(t) = D_i(t)$ government i may either bailout or not bailout in equilibrium (${B_i}^*(t) \in \{0,1\}$), since independent of government j they are always indifferent between the bailout and no bailout action. Finally, if Di) $\overline{D}_i(t) < D_i(t)$ government i never bails out in equilibrium ($B_i^*(t) = 0$), since in this case bailing out is a weakly dominated action.

By this reasoning, it follows that the following table describes the complete set of sub-game equilibria, in which no government plays a weakly dominated action, given any possible period 1 talent recruitment t,

	$D_i(t) \leq 0$ (Ai)	$\bar{D}_i(t) > D_i(t) > 0$ (Bi)	$\bar{D}_i(t) = D_i(t)$ (Ci)	$\bar{D}_i(t) < D_i(t)$ (Di)
$D_j(t) \leq 0$ (Aj)	$B_i^*(t) = \emptyset$ $B_j^*(t) = \emptyset$	$B_i^*(t) = 1$ $B_j^*(t) = \emptyset$	$B_i^*(t) \in \{0,1\}$ $B_j^*(t) = \emptyset$	$B_i^*(t) = 0$ $B_j^*(t) = \emptyset$
$\bar{D}_j(t) > D_j(t) > 0$ (Bj)	$B_i^*(t) = \emptyset$ $B_j^*(t) = 1$	$B_i^*(t) = 1$ $B_j^*(t) = 1$	$B_i^*(t) \in \{0,1\}$ $B_j^*(t) = 1$	$B_i^*(t) = 0$ $B_j^*(t) = 1$
$\bar{D}_j(t) = D_j(t)$ (Cj)	$B_i^*(t) = \emptyset$ $B_j^*(t) \in \{0,1\}$	$B_i^*(t) = 1$ $B_j^*(t) \in \{0,1\}$	$B_i^*(t) \in \{0,1\}$ $B_j^*(t) \in \{0,1\}$	$B_i^*(t) = 0$ $B_j^*(t) \in \{0,1\}$
$\bar{D}_j(t) < D_j(t)$ (Dj)	$B_i^*(t) = \emptyset$ $B_j^*(t) = 0$	$B_i^*(t) = 1$ $B_j^*(t) = 0$	$B_i^*(t) \in \{0,1\}$ $B_j^*(t) = 0$	$B_i^*(t) = 0$ $B_j^*(t) = 0$

9.2 Sub-Game Perfect Nash Equilibria

We now solve for our game's sub-game perfect Nash equilibria. For this, recall from Definition 1 in Section 3 that club payoffs are;

$$\pi_i^C(s) = \begin{cases} -r(t_i) & if\ s \in \hat{Z} \\ p_i(t) & if\ s \notin \hat{Z} \end{cases}$$

Where clubs suffer a bankruptcy cost, $r(t_i)$, which is increasing in their liabilities, $r'(t_i) > 0$, if the league fails, $s \in \hat{Z}$. While, as win-maximisers, they receive a payoff equal to their win percentage, $p_i(t)$, which is increasing in their talent, $\frac{\partial p_i}{\partial t_i} > 0$, if the league proceeds, $s \notin \hat{Z}$.

First, note there can be no equilibrium in which the league fails (i.e. $s \in \hat{Z}$). Otherwise, clubs' equilibrium payoff is their bankruptcy cost; $\pi_i^C(t^*, B^*) = -r(t_i{}^*)$. However, since $S_i^C \equiv \mathbb{R}_{++}$, clubs can reduce their talent recruitment marginally to some $t_i{}' = t_i{}^* - \varepsilon$. This reduction may or may not prevent league failure. In any case, since in reducing their liabilities, the club has reduced bankruptcy costs, $r(t_i{}^*) > r(t_i{}')$, the club achieves a greater payoff with this deviation , $\pi_i^C\left(t_i{}'t_j^*, B^*\right) \geq -r(t_i{}') > r(t_i{}^*) = \pi_i^C(t^*, B^*)$. It follows that clubs could profitably deviate from any equilibrium featuring league failure.

Thus, we have that, in equilibrium, clubs' deficits must be less than or equal to their deficit threshold (i.e., $\overline{D}_i(t^*) \geq D_i(t^*)$), since, otherwise governments' strategies ensures the league would fail in period 2. Note also that clubs' equilibrium payoffs must be their win percentage, $\pi_i^C(t^*, B^*) = p_i(t^*)$.

Next, we show that we cannot have an equilibrium in which some club i generates a first period deficit which is strictly lesser than their deficit threshold (i.e., $\overline{D}_i(t^*) > D_i(t^*)$). To see this, note that otherwise, by continuity of $\overline{D}_i$ and D_i, club i could increase their talent recruitment marginally to some $t_i{}' = t_i{}^* + \varepsilon$, while ensuring their deficit remains below their deficit threshold; $\overline{D}_i\left(t_i{}', t_j^*\right) > D_i(t_i{}', t_j^*)$). After this deviation, since we have $\overline{D}_j(t^*) \geq D_j(t^*)$ and $\left.\frac{\partial[\overline{D}_j(t) - D_j(t)]}{\partial t_i}\right|_{t_i > 0, t_j > 0} > 0$ (see Lemma 3.3 at the end of this appendix), club j's deficit must be strictly less than their threshold, $D_j\left(t_i', t_j^*\right) < \overline{D}_j\left(t_i', t_j^*\right)$. As such, the league does not fail in period 2 and club i has a higher payoff since they have increased their win percentage; $\pi_i^C\left(t_i{}'t_j^*, B^*\right) = p_i\left(t_i', t_j^*\right) > p_i(t^*) = \pi_i^C(t^*, B^*)$. It follows that clubs could profitably deviate from any equilibrium in which $\overline{D}_i(t^*) > D_i(t^*)$.

Thus, in any possible equilibrium it must be that both clubs' first period deficit is equal to their deficit threshold (i.e., $\overline{D}_i(t^*) = D_i(t^*)$ for any club i). Lemma 3.1 at the end of this appendix shows that this implies that $t_i{}^*$ is uniquely defined as asserted in part iv) of Proposition 1. Further, since the league cannot fail in any possible equilibrium, both governments must bailout their local club in period 2, (i.e., $B_i{}^*(t^*) = 1$ for any club i).

To confirm that we do have equilibria in this case, note that clubs cannot profitably reduce their talent recruitment to any $t_i' < t_i^*$, since doing so necessarily reduces their win percentage; $\pi_i^C\left(t_i{}'t_j^*, B^*\right) \leq p_i\left(t_i', t_j^*\right) < p_i(t^*) = \pi_i^C(t^*, B^*)$. Further, clubs cannot profitably increase their talent recruitment to any $t_i' > t_i^*$. Since $\left.\frac{\partial[\overline{D}_i(t) - D_i(t)]}{\partial t_i}\right|_{t_i \geq t_i{}^*, t_j = t_j{}^*} < 0$ (see Lemma 3.2 at the end of this appendix), their subsequent deficit would exceed their deficit threshold, $D_i\left(t_i', t_j^*\right) > \overline{D}_i\left(t_i', t_j^*\right)$, which causes the league to fail in period 2; $\pi_i^C\left(t_i', t_j^*, B^*\right) = -r(t_i') < p_i(t^*) = \pi_i^C(t^*, B^*)$.

Thus, $(t^*, B^*) \in S$ is a sub-game perfect equilibrium of our game, in which no local government plays a weakly dominated action in any period 2 sub-game, if and only if i) $\overline{D}_i(t^*) = D_i(t^*)$ for any club i and ii) $B_i{}^*(t^*) = 1$ for any club i, while off the equilibrium path (for sub-game perfection) B^* also satisfies;

$$B_i{}^*(t) \in \begin{cases} \{1\} & if\ \overline{D}_i(t) > D_i(t) \\ \{0,1\} & if\ \overline{D}_i(t) = D_i(t) \\ \{0\} & if\ \overline{D}_i(t) < D_i(t) \end{cases}$$

This completes our proof of Proposition 1, subject to the following proof of Lemma 3, which is divided into three parts;

Lemma 3.1: $\overline{D}_i(t) - D_i(t) = 0$ *for any* $i \in \{1,2\}$ *if and only if* $t = t^*$

For this, recall that, $\overline{D}_i(t) = 2\mu(\omega_i, f_i)[V_i(t) - R_i(t)]$ and that $D_i(t) = wt_i - R_i(t)$, where, by the Falconieri et al (2004) micro-foundation, $V_i(t) = \frac{3F_i t_i^{\alpha} t_j^{\beta}}{8}$ and $R_i(t) = \frac{F_i t_i^{\alpha} t_j^{\beta}}{4}$. It follows that $\overline{D}_i(t) - D_i(t) = 0 \Leftrightarrow t_i = A_i t_i^{\alpha} t_j^{\beta}$, where $A_i = \left(\frac{1+\mu(\omega_i, f_i)}{4w}\right) F_i$. So that, if $\overline{D}_i(t) - D_i(t) = 0$ for any $i \in \{1,2\}$ then we must have $t_j = \left(\frac{A_j}{A_i}\right)^{\theta} t_i$, where $\theta = \frac{1}{1+\beta-\alpha}$.

Substituting $t_j = \left(\frac{A_j}{A_i}\right)^{\theta} t_i$ into $t_i = A_i t_i^{\alpha} t_j^{\beta}$, we find that $t_i = t_i^* = A_i^{\gamma(1-\theta\beta)} A_j^{\gamma\theta\beta}$, where $\gamma = \frac{1}{1-\beta-\alpha}$.

Lemma 3.2: $\left.\frac{\partial[\overline{D}_i(t) - D_i(t)]}{\partial t_i}\right|_{t_i \geq t_i^*, t_j = t_j^*} < 0$

First, note that $\overline{D}_i(t) = \mu_i R_i(t)$. So that $\overline{D}_i(t) - D_i(t) = (1+\mu_i)R_i(t) - wt_i$. Given that $R_i(t) = \frac{F_i t_i^{\alpha} t_j^{\beta}}{4}$, our first derivative with respect to t_i is given by; $\frac{\partial[\overline{D}_i(t) - D_i(t)]}{\partial t_i} = \frac{1}{t_i}[\alpha(1+\mu_i)R_i(t) - wt_i]$. Meanwhile, our second derivative with respect to t_i is given by; $\frac{\partial^2[\overline{D}_i(t) - D_i(t)]}{\partial t_i^2} = \frac{1}{t_i^2}\alpha(\alpha-1)(1+\mu_i)R_i(t) < 0$. Since this second derivative is strictly negative, we must have $\left.\frac{\partial[\overline{D}_i(t) - D_i(t)]}{\partial t_i}\right|_{t_i \geq t_i^*, t_j = t_j^*} \leq \left.\frac{\partial[\overline{D}_i(t) - D_i(t)]}{\partial t_i}\right|_{t_i = t_i^*, t_j = t_j^*}$.

Thus, to show that $\left.\frac{\partial[\overline{D}_i(t) - D_i(t)]}{\partial t_i}\right|_{t_i \geq t_i^*, t_j = t_j^*} < 0$, it is sufficient to show that $\left.\frac{\partial[\overline{D}_i(t) - D_i(t)]}{\partial t_i}\right|_{t_i = t_i^*, t_j = t_j^*} = \frac{1}{t_i^*}[\alpha(1+\mu_i)R_i(t^*) - wt_i^*] < 0$. For this, recall from part i) of this lemma that $\overline{D}_i(t^*) - D_i(t^*) = 0$. From the first line of this proof, we have $\overline{D}_i(t^*) - D_i(t^*) = (1+\mu_i)R_i(t^*) - wt_i^*$. This implies that $wt_i^* = (1+\mu_i)R_i(t^*)$. Therefore, $\left.\frac{\partial[\overline{D}_i(t) - D_i(t)]}{\partial t_i}\right|_{t_i = t_i^*, t_j = t_j^*} = \frac{1}{t_i^*}(\alpha-1)(1+\mu_i)R_i(t^*) < 0$.

Lemma 3.3: $\left.\frac{\partial[\bar{D}_i(t)-D_i(t)]}{\partial t_j}\right|_{t_i>0,t_j>0} \geq 0$

First, note that $\bar{D}_i(t) = \mu_i R_i(t)$. So that $\bar{D}_i(t) - D_i(t) = (1+\mu_i)R_i(t) - wt_i$. Given that $R_i(t) = \frac{F_i {t_i}^{\alpha} {t_j}^{\beta}}{4}$, our first derivative with respect to t_j is given by; $\frac{\partial[\bar{D}_i(t)-D_i(t)]}{\partial t_j} = \frac{\beta(1+\mu_i)R_i(t)}{t_j} \geq 0$.

10. Appendix 2

In this appendix, we provide a proof for Proposition 2, presented in Section 5, which concerns, our measure of equilibrium budget softness, the equilibrium deficit-revenue ratio $DR_i^* = \frac{D_i(t^*)}{R_i(t^*)}$. We start by showing that $DR_i^* = \mu(\omega_i, f_i)$ before proving the three parts of our proposition in turn.

First, recall that, from the previous Appendix, $D_i(t^*) = \bar{D}_i(t^*)$ so that $DR_i^* = \frac{\bar{D}_i(t^*)}{R_i(t^*)}$. Then, from our Falconieri et al (2004) micro-foundation, we have that $\bar{D}_i(t) = 2\mu(\omega_i, f_i)[V_i(t) - R_i(t)] = \mu(\omega_i, f_i)R_i(t)$, so that $DR_i^* = \mu(\omega_i, f_i)$.

Recalling that $\mu(\omega_i, f_i) = \frac{\omega_i}{2(f_i\omega_i + (1-f_i))} > 0$, our proposition is then that this equilibrium deficit-revenue ratio has the following three properties;

i) *is increasing in the government's weight on fan welfare ($\frac{\partial DR_i^*}{\partial \omega_i} > 0$),*

This follows simply from the observation that $\frac{\partial DR_i^*}{\partial \omega_i} = \frac{\partial \mu}{\partial \omega_i} = \frac{(1-f_i)}{2(f_i\omega_i + (1-f_i))^2} > 0$.

ii) *is decreasing in the popular support for the local club whenever the government is more concerned with fan welfare ($\omega_i > 1 \Leftrightarrow \frac{\partial DR_i^*}{\partial f_i} < 0$), but is decreasing in the popular support for the local club whenever the government is more concerned with non-fan welfare ($\omega_i < 1 \Leftrightarrow \frac{\partial DR_i^*}{\partial f_i} > 0$),*

This follows from the observation that $\frac{\partial DR_i^*}{\partial f_i} = \frac{\partial \mu}{\partial f_i} = \frac{\omega_i(1-\omega_i)}{2(f_i\omega_i + (1-f_i))^2}$, so that $\omega_i < 1 \Leftrightarrow \frac{\partial DR_i^*}{\partial f_i} > 0$ and $\omega_i > 1 \Leftrightarrow \frac{\partial DR_i^*}{\partial f_i} < 0$.

iii) *goes to* 0 *as governments are exclusively concerned with non-fan welfare* ($\lim_{\omega_i \to 0} DR_i^* = 0$), $\frac{1}{2}$ *with a utilitarian government* ($DR_i^*|_{\omega_i=1} = \frac{1}{2}$) *and* $\frac{1}{2f_i}$ *as governments are exclusively concerned with fan welfare* ($\lim_{\omega_i \to \infty} DR_i^* = \frac{1}{2f_i}$).

This follows simply from the observation that $\lim_{\omega_i \to 0} DR_i^* = \lim_{\omega_i \to 0} \mu(\omega_i, f_i) = 0$, while $DR_i^*|_{\omega_i=1} = \mu(1, f_i) = \frac{1}{2}$ and, finally, by l'hopital's rule,

$\lim_{\omega_i \to \infty} DR_i^* = \lim_{\omega_i \to \infty} \mu(\omega_i, f_i) = \frac{1}{2f_i}$.

11. Appendix 3

In this appendix, we provide a proof for Proposition 3, presented in Section 6, which concerns local governments' ex-ante optimal bailout-revenue ratio, that would occur if they could credibly commit before period 1, ${DR_i}^C$. We start by showing that

$${DR_i}^C = \begin{cases} \frac{\varphi {DR_i}^* - 1}{1+\varphi} & if\ \varphi {DR_i}^* > 1 \\ 0 & if\ \varphi {DR_i}^* \leq 1 \end{cases}.$$

For this, suppose that the dynamics of our model are reversed, so that local governments can now pre-commit to a certain bailout-revenue ratio, DR_i, before clubs recruit talent. Solving by backward induction, we can see that win-maximising clubs will recruit talent to satisfy the following break-even condition; $(1 + DR_i)R_i(t) = wt_i$. Combining this condition with that of opposition club j, we can identify clubs' talent recruitment strategy; ${t_i}^C = {\hat{A}_i}^{\gamma(1-\theta\beta)} {\hat{A}_j}^{\gamma\theta\beta}$ where $\hat{A}_i = \left(\frac{1+DR_i}{4w}\right) F_i$.

Given clubs' subsequent talent recruitment strategy vector $t^c = ({t_1}^C, {t_2}^C)$, we now allow local governments to pre-commit to a given bailout-revenue ratio, in order to maximise their payoff, π_i^G. We can see that the local government's ex-ante bailout-revenue ratio choice problem is then; $\max_{DR_i \geq 0} Y_i^N + \omega_i \left[Y_i^F + \frac{\overline{D}_i(t^c) - D_i(t^c)}{2\mu_i(\omega_i, f_i)}\right]$. Recalling that Y_i^N, Y_i^F, ω_i and f_i are exogenous parameters and noting, from our Falconieri et al (2004) micro-foundation, that $\overline{D}_i(t^c) = \mu(\omega_i, f_i) \frac{F_i {t_i}^{C\alpha} {t_j}^{C\beta}}{4}$ and $D_i(t^c) = w{t_i}^C - \frac{F_i {t_i}^{C\alpha} {t_j}^{C\beta}}{4}$, the local government's problem can be rewritten;

$$\max_{DR_i \geq 0} (1 + \mu(\omega_i, f_i)) \frac{F_i {t_i}^{C\alpha} {t_j}^{C\beta}}{4} - w{t_i}^C$$

Differentiating, and noting that $R_i(t^c) = \frac{F_i {t_i^C}^{\alpha} {t_j^C}^{\beta}}{4}$, yields the first order condition; $(1 + DR_i^{\,c})R_i(t^c)\varphi\left(\frac{1+\mu(\omega_i,f_i)}{1+DR_i^{\,c}}\right) = wt_i^{\,C}\,\gamma(1-\theta\beta)$. Then, recalling that we must have $(1 + DR_i^{\,c})R_i(t^c) = wt_i^{\,C}$ and noting that $\gamma(1-\theta\beta) = 1+\varphi$ and $DR_i^{\,*} = \mu(\omega_i, f_i)$, we find that, as asserted, $DR_i^{\,C} = \begin{cases} \frac{\varphi DR_i^{\,*}-1}{1+\varphi} & if\ \varphi DR_i^{\,*} > 1 \\ 0 & if\ \varphi DR_i^{\,*} \le 1 \end{cases}$.

Next, we prove that $\frac{\partial DR_i^{\,C}}{\partial \alpha}, \frac{\partial DR_i^{\,C}}{\partial \beta}, \frac{\partial DR_i^{\,C}}{\partial \omega_i} \ge 0$.

For this, first suppose $\varphi DR_i^{\,*} \le 1$ so that $DR_i^{\,C} = 0$. Recall that $DR_i^{\,*} = \mu(\omega_i, f_i) = \frac{\omega_i}{2(f_i\omega_i+(1-f_i))}$, which from Appendix 2 is increasing in ω_i (i.e., $\frac{\partial DR_i^{\,*}}{\partial \omega_i} > 0$) and is independent of α and β. Further, note from Lemma 4.2 at the end of this appendix that $\varphi = \gamma(\alpha(1-\theta\beta)+\theta\beta^2)$ is increasing in α and β (i.e., $\frac{\partial \varphi}{\partial \alpha} > 0$ and $\frac{\partial \varphi}{\partial \beta} > 0$) and is independent of ω_i. Thus, marginal increases in α, β or ω_i strictly increase $\varphi DR_i^{\,*}$ (i.e., $\frac{\partial \varphi DR_i^{\,*}}{\partial \alpha}, \frac{\partial \varphi DR_i^{\,*}}{\partial \beta}, \frac{\partial \varphi DR_i^{\,*}}{\partial \omega_i} > 0$). After, this we either have i) $\varphi DR_i^{\,*} \le 1 \Rightarrow DR_i^{\,C} = 0$, in which case $\frac{\partial DR_i^{\,C}}{\partial \alpha}, \frac{\partial DR_i^{\,C}}{\partial \beta}, \frac{\partial DR_i^{\,C}}{\partial \omega_i} = 0$ or ii) $\varphi DR_i^{\,*} > 1 \Rightarrow DR_i^{\,C} > 0$, in which case $\frac{\partial DR_i^{\,C}}{\partial \alpha}, \frac{\partial DR_i^{\,C}}{\partial \beta}, \frac{\partial DR_i^{\,C}}{\partial \omega_i} > 0$. It follows that $\frac{\partial DR_i^{\,C}}{\partial \alpha}, \frac{\partial DR_i^{\,C}}{\partial \beta}, \frac{\partial DR_i^{\,C}}{\partial \omega_i} \ge 0$ when $\varphi DR_i^{\,*} \le 1$.

Second, suppose $\varphi DR_i^{\,*} > 1$ so that $DR_i^{\,C} = \frac{\varphi DR_i^{\,*}-1}{1+\varphi}$. Note that $\frac{\partial DR_i^{\,C}}{\partial DR_i^{\,*}} = \frac{\varphi}{1+\varphi} > 0$. It follows that, since $\frac{\partial DR_i^{\,*}}{\partial \omega_i} > 0$, we have $\frac{\partial DR_i^{\,C}}{\partial \omega_i} = \frac{\partial DR_i^{\,C}}{\partial DR_i^{\,*}}\frac{\partial DR_i^{\,*}}{\partial \omega_i} > 0$. Next, note that $\frac{\partial DR_i^{\,C}}{\partial \varphi} = \frac{1+DR_i^{\,*}}{(1+\varphi)^2} > 0$. It follows that, since $\frac{\partial \varphi}{\partial \alpha} > 0$ and $\frac{\partial \varphi}{\partial \beta} > 0$, we have $\frac{\partial DR_i^{\,C}}{\partial \alpha} = \frac{\partial DR_i^{\,C}}{\partial \varphi}\frac{\partial \varphi}{\partial \alpha} > 0$ and $\frac{\partial DR_i^{\,C}}{\partial \beta} = \frac{\partial DR_i^{\,C}}{\partial \varphi}\frac{\partial \varphi}{\partial \beta} > 0$. Thus, $\frac{\partial DR_i^{\,C}}{\partial \alpha}, \frac{\partial DR_i^{\,C}}{\partial \beta}, \frac{\partial DR_i^{\,C}}{\partial \omega_i} > 0$ when $\varphi DR_i^{\,*} > 1$.

To see that both $\varphi DR_i{}^* \leq 1$ and $\varphi DR_i{}^* > 1$ are possible, note that $DR_i{}^* > 0$, while the following lemma notes that φ can take any positive real value;

Lemma 4.1: $\varphi \in (0, \infty)$

First we show that $\lim_{\alpha+\beta\to 0} \varphi = 0$. For this, note that, since $\alpha > \beta > 0$, if $\alpha + \beta \to 0$ then we necessarily have $\alpha \to 0$ and $\beta \to 0$. Further, note that, after rearrangement, $\varphi = \gamma\theta(\alpha - \alpha^2 + \beta^2)$. Thus, since $\lim_{\alpha+\beta\to 0} \gamma = \lim_{\alpha+\beta\to 0} \theta = 1$, we must have $\lim_{\alpha+\beta\to 0} \varphi = \lim_{\alpha+\beta\to 0} \alpha - \alpha^2 + \beta^2 = 0$.

Second we show that $\lim_{\alpha+\beta\to 1} \varphi \to \infty$. For this, again note that $\varphi = \gamma\theta(\alpha - \alpha^2 + \beta^2)$. Then note that $\lim_{\alpha+\beta\to 1} \gamma \to \infty$. Thus, to show that $\lim_{\alpha+\beta\to 1} \varphi \to \infty$, it remains to show that $\lim_{\alpha+\beta\to 1} \theta(\alpha - \alpha^2 + \beta^2) = \lim_{\alpha+\beta\to 1} \frac{\alpha-\alpha^2+\beta^2}{1+\beta-\alpha}$ is positive and finite. For this, note that $\lim_{\alpha+\beta\to 1} \beta = 1 - \alpha$. Therefore, $\lim_{\alpha+\beta\to 1} \frac{\alpha-\alpha^2+\beta^2}{1+\beta-\alpha} = \frac{\alpha-\alpha^2+(1-\alpha)^2}{1+(1-\alpha)-\alpha} = \frac{1}{2}$.

Now, given that $1 > \alpha + \beta \geq \alpha > \beta \geq 0$, while the following lemma shows that $\frac{\partial\varphi}{\partial\alpha} > 0$ and $\frac{\partial\varphi}{\partial\beta} > 0$, we must have $\varphi \in (0, \infty)$.

Lemma 4.2: $\frac{\partial\varphi}{\partial\alpha} > 0$ *and* $\frac{\partial\varphi}{\partial\beta} > 0$

First we show that $\frac{\partial\varphi}{\partial\alpha} > 0$. For this note that, after rearrangement, we have $\varphi = \frac{\alpha-\alpha^2+\beta^2}{1-2\alpha+\alpha^2-\beta^2}$. By the quotient rule, we have that $\frac{\partial\varphi}{\partial\alpha} = \frac{(1-2\alpha)(1-2\alpha+\alpha^2-\beta^2)-(2\alpha-2)(\alpha-\alpha^2+\beta^2)}{(1-2\alpha+\alpha^2-\beta^2)^2}$. Cancelling common terms and rearranging, we find that $\frac{\partial\varphi}{\partial\alpha} > 0 \Leftrightarrow (1-\alpha)^2 + \beta^2 > 0$, which holds since both square terms are necessarily positive.

Second we show that $\frac{\partial\varphi}{\partial\beta} > 0$. For this again note that $\varphi = \frac{\alpha-\alpha^2+\beta^2}{1-2\alpha+\alpha^2-\beta^2}$. Using the quotient rule again, we have that $\frac{\partial\varphi}{\partial\beta} = \frac{2\beta\left(1-2\alpha+\alpha^2-\beta^2\right)-(-2\beta)\left(\alpha-\alpha^2+\beta^2\right)}{(1-2\alpha+\alpha^2-\beta^2)^2}$. Cancelling common terms and rearranging once more yields $\frac{\partial\varphi}{\partial\beta} > 0 \Leftrightarrow 2\beta(1-\alpha) > 0$, which holds since $\beta > 0$ and $\alpha < 1$.

12. Appendix 4

In this appendix, we provide a proof of Lemmas 1 and 2, presented in Section 6, which concern local governments' optimal budget softness, $DR_i{}^C = \begin{cases} \frac{\varphi DR_i{}^* - 1}{1+\varphi} & if\ \varphi DR_i{}^* > 1 \\ 0 & if\ \varphi DR_i{}^* \leq 1 \end{cases}$.

First, Lemma 1 asserts that the optimal budget softness is strictly less than the equilibrium softness (i.e., $DR_i{}^C < DR_i{}^*$). To see this, note that $DR_i{}^* > 0$, so that $DR_i{}^C < DR_i{}^* \Leftarrow \frac{\varphi DR_i{}^* - 1}{1+\varphi} < DR_i{}^*$. Rearranging, we find that $\frac{\varphi DR_i{}^* - 1}{1+\varphi} < DR_i{}^* \Leftrightarrow 1 + DR_i{}^* > 0$, which necessarily holds.

Second, Lemma 2 asserts that the optimal budget softness approaches the equilibrium softness, in the limit as fans' taste for talent reaches its maximum (i.e., $\lim_{\alpha+\beta\to 1} DR_i{}^C = DR_i{}^*$). To see this, recall that we refer to $\alpha + \beta$ as fans' taste for talent and that $\alpha + \beta < 1$. Thus, as fans' taste for talent reaches its maximum we have $\alpha + \beta \to 1$. Recall that $DR_i{}^* > 0$ is independent of $\alpha + \beta$ and that, from Lemma 4.1 in the previous appendix, $\lim\limits_{\alpha+\beta\to 1} \varphi \to \infty$. It follows that $\lim\limits_{\alpha+\beta\to 1} \varphi DR_i{}^* \to \infty$, so that $DR_i{}^C = \frac{\varphi DR_i{}^* - 1}{1+\varphi}$. By l'hopital's rule, we have $\lim_{\alpha+\beta\to 1} \frac{\varphi DR_i{}^* - 1}{1+\varphi} = DR_i{}^*$. Thus, $\lim_{\alpha+\beta\to 1} DR_i{}^C = DR_i{}^*$.

Contest Success Function with the Possibility of a Draw: An Equilibrium Analysis

Abstract:

The contest literature concerns the economic scenario in which multiple contestants compete for a prize. Central to any contest model is the specification of a contest success function (CSF), which assigns contestants' win probabilities according to their efforts. The most commonly employed CSF is the Tullock CSF. A notable limitation of the Tullock CSF is that it always produces a winner, whereas many real-world contests may end without a winner (e.g., sports contests can end in a tie, military conflicts can end in stalemate, patent races can end with no successful applications). Blavatskyy (2010) describes this outcome as a 'draw' and axiomatises a new CSF, which generalises the Tullock CSF to allow for the possibility of draws. This current paper provides the first equilibrium analysis of a contest with Blavatskyy's CSF. We present a wide range of findings, which are particularly pertinent to the literatures on patent races and labour market tournaments amongst others. Most notably, we find that introducing the possibility of a draw necessarily reduces homogeneous contestants' efforts. However, with heterogeneous contestants, introducing the possibility of a draw may increase the effort of the strongest contestant. What's more, introducing a prize for a draw may increase the effort of the weakest contestant.

1. Introduction

An active segment of the economics literature concerns the analysis of contest models. Contest models describe the scenario in which multiple contestants (or agents) compete against one another, by exerting costly efforts, to win some rivalrous prize[46]. Each contest model has a contest success function (CSF) [47], which assigns contestants' win probabilities according to their efforts. Different contest models employ different CSFs. Owing to its versatility, tractability and axiomatisation by Skaperdas (1996) and Clark & Riis (1998), the Tullock (1980) or logit CSF is perhaps the most commonly employed.

However, the Tullock CSF has a notable limitation. Like many of its alternatives, the Tullock CSF always produces a winner[48]. This is incompatible with many real-life contests that may end with no winner. For instance, patent races may end with no researcher able to make a desired technological breakthrough. Labour market tournaments may end with no worker earning a promotion/bonus. Political lobbying contests may end with no interest group winning governmental support and matching contests, in which agents compete to become the partner of a principal, may end with the principal electing to remain single or with some incumbent partner.

Blavatksyy (2010) describes the outcome in which no contestant wins as a 'draw' and axiomatises a new CSF that generalises the Tullock CSF to allow for the possibility of draws. This current paper provides the first equilibrium analysis of a contest model with Blavatksyy's CSF. We present a wide-range of comparative static results, which are pertinent to a variety of real-world contests. Most notably, we find that introducing the possibility of a

[46] Consult Corchon (2007), Konrad (2007) and Corchon & Serena (2016) for broad surveys of the contest literature.
[47] In some cases, this CSF may only be implicitly defined.
[48] This follows from the fact that, with the Tullock CSF, the sum over all contestants' winning probabilities is equal to one.

draw necessarily reduces homogeneous contestants' efforts. However, when contestants have heterogeneous abilities, introducing the possibility of a draw may lead to a greater effort from the strongest contestant. In this instance, introducing a prize for a draw can also induce a greater effort from the weakest contestant.

From here, Section 2 summarises the existing literature on contests with draws. Section 3 discusses Blavatskyy's CSF. Section 4 identifies a number of real-world contests consistent with this CSF. Section 5 presents our model. Section 6 analyses the equilibrium of our model with homogeneous contestants. Section 7 analyses the equilibrium with heterogeneous contestants. Section 8 provides a tentative discussion of the implications of these analyses. Section 9 considers some extensions. Finally, Section 10 concludes.

2. Related Literature

An emerging segment of the contest literature considers contests that may end without producing a winner; an outcome Blavatksyy (2010) describes as a draw[49].

Nalebuff & Stiglitz (1983) is perhaps the first contribution in this regard. They study a two-agent labour market tournament that ends in a draw if neither agent is able to 'win by a gap'. They find that introducing the possibility of a draw can improve homogeneous risk averse agents' welfare. Key to this result is Nalebuff & Stiglitz's assumption that agents share the tournament prize equally in the event of a draw. This assumption ensures that, in any symmetric equilibrium, introducing the possibility of a draw reduces agents' risk exposure, while maintaining their expected prize value.

[49] Draws have also been considered outside of the contest literature. For example, in the industrial organisation literature, Loury (1979) considers a dynamic patent race model in which it is possible that, at any given point in time, no firm successfully makes a technological breakthrough. Meanwhile, in the statistics literature, Rao & Kupper (1967) and Davidson (1970) also consider the possibility of draws in Bradley & Terry (1952) paired comparison experiments. Finally, in the closely related literature on all pay auctions, Gelder et al (2015) introduces the possibility of draws if no bidder wins by a sufficient gap (see also Che & Gale (1998), Cohen & Sala (2007) and Szech (2015)).

Imhof & Krakel (2011) extends the model of Nalebuff & Stiglitz. They find that, with homogeneous risk neutral agents, introducing the possibility of a draw reduces employers' profits. However, with heterogeneous agents, introducing the possibility of a draw can improve employers' profits.

The models of Nalebuff & Stiglitz/Imhof & Krakel relate specifically to a labour market tournament. Jia (2012) and Yildizparlak (2013) study the first generic contest models with the possibility of a draw.

Jia provides a stochastic micro-foundation for a new CSF with the possibility of a draw. In the two-contestant case, this CSF has draw probability $p_D(e) = \frac{(b^2-1)t_1e_1t_2e_2}{(t_1e_1+bt_2e_2)(t_2e_2+bt_1e_1)}$, where we interpret e_i as contestant i's effort, t_i as contestant i's ability and $b > 1$ as the exogenous likelihood of a draw. Jia studies a contest with this CSF and finds that introducing the possibility of a draw ensures contestants with heterogeneous abilities exert different effort levels. This solves a 'homogeneity paradox' associated with standard contests without draws, in which heterogeneous contestants exert homogeneous efforts. Note that Jia's analysis is predicated on the assumption that contestants receive zero payoff in the event of a draw (i.e., contestants are indifferent between drawing and losing).

Yildizparlak also suggests a new CSF with the possibility of a draw. In the n-contestant case, this CSF has draw probability $p_D(e) = \frac{[\sum f(e_i)]^b - [\sum f(e_i)^b]}{[\sum f(e_i)]^b}$, where we again interpret e_i as contestant i's effort, $b > 1$ as the exogenous draw likelihood and note that $f(.)$ is a strictly increasing and concave function. Yilidizparlak analyses a contest with this CSF and finds that, with three or more homogeneous contestants, introducing the possibility of a draw necessarily reduces contestants' efforts. However, with just two homogeneous contestants, introducing the possibility of a draw may increase efforts. Unlike Jia, Yildizparlak allows contestants to receive a positive payoff in the event of a draw, but finds that homogeneous contestants' efforts are independent of this payoff.

The models of Jia and Yildizparlak (as well as Nalebuff & Stiglitz and Imhof & Krakel), share one important feature; the probability of a draw is always maximised when homogenous contestants exert equal efforts, regardless of the magnitude of these efforts. This feature is broadly consistent with many sports contests, such as soccer, hockey and chess, in which opponents with identical abilities and efforts have a high probability of cancelling each other out and producing a 'stalemate' (Peeters & Szymanski (2012), Yildizparlak (2013) and Corchon & Serena (2016)). This feature is also appealing for military conflicts, in which a stalemate is likely whenever combatants possess equal military strength.

Blavatskyy (2010) axiomatises a new CSF with the possibility of a draw[50], which behaves very differently to those of Jia and Yildizparlak. In the n-contestant case, Blavatskyy's CSF has draw probability $p_D(e) = \frac{b}{b+\sum t_i {e_i}^{\varepsilon}}$. Thus, we see that the probability of a draw is strictly decreasing in contestants' efforts. It follows that, unlike the CSFs of Jia or Yildizparlak, Blavatskyy's CSF assigns a very low draw probability when homogeneous contestants have equal but very large efforts. This makes Blavatskyy's CSF inappropriate for modelling soccer matches, chess games or military conflicts (Peeters & Szymanski (2012), Yildizparlak (2013) and Corchon & Serena (2016)).

However, Blavatskyy's CSF is uniquely appropriate in many other instances, as we argue in Section 4 of this current paper. Crucially, the existing literature does not yet provide any analysis of a contest model with Blavatskyy's CSF. Since this current paper addresses this gap in the literature, it serves to examine Blavatskyy's CSF in more detail in the next section.

[50] The relevance of Blavatskyy's CSF is also underlined by Jia (2012), which provides a separate stochastic micro-foundation to complement Blavatskyy's axiomatic foundation.

3. Blavatskyy's CSF

Blavatskyy (2010) axiomatises the first closed form CSF with the possibility of a draw[51]. Blavatskyy's axiomatisation is particularly appealing as it generalises the axiomatisation, provided by Skaperdas (1996) and Clark & Riis (1998), for the popular Tullock CSF[52] (in which, contestant i has win probability $p_i(e) = \frac{t_i e_i{}^{\varepsilon}}{\sum t_j e_j{}^{\varepsilon}}$).

For any given vector of contestants' efforts, $e = (e_1 \ldots e_n) \in \mathbb{R}_+^n$, Blavatskyy's CSF assigns a win probability, $p_i(e) \in [0,1)$, to each contestant $i \in \{1 \ldots n\}$ and a 'draw probability', $p_D(e) \in (0,1]$, to the outcome in which no contestant wins. Blavatskyy's CSF has the following functional form[53];

$$p_i(e) = \frac{x_i(e_i)}{b + \sum_{j=1}^{n} x_j(e_j)}, p_D(e) = \frac{b}{b + \sum_{j=1}^{n} x_j(e_j)} \quad : \quad x_i(e_i) = t_i e_i{}^{\varepsilon}$$

where $t_i \in \mathbb{R}_{++}$, $\varepsilon \in \mathbb{R}_{++}$ and $b \in \mathbb{R}_{++}$.

In this paper, we interpret $x_i(e_i) \in \mathbb{R}_+$ as a measure of the performance level of contestant i. Further, we interpret $t_i \in \mathbb{R}_{++}$ as a measure of contestant i's ability. It follows that performances are a product of a contestant's ability and effort, $e_i \in \mathbb{R}_+$, with the parameter $\varepsilon \in \mathbb{R}_{++}$ determining the relative importance of efforts (i.e., $x_i(e_i) = t_i e_i{}^{\varepsilon}$)[54]. It also follows that a contestant's win probability is strictly increasing in their own performance (i.e., $\frac{\partial p_i(e)}{\partial x_i} > 0$) and strictly decreasing in their opposition's performance (i.e., $\frac{\partial p_i(e)}{\partial x_j} < 0$).

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[51] Note that Blavatskyy also axiomatises a more general CSF, without a closed form, in which draws may occur between different subsets of contestants.

[52] In this way, Blavatskyy's CSF generalises the Tullock CSF. Note that the CSFs of Jia (2012) and Yildizparlak (2013) also generalise the Tullock CSF.

[53] Note that in Blavatskyy (2010) the b parameter that we present here is effectively normalised to one (i.e., $p_i = \frac{t_i e_i{}^{\varepsilon}}{b + \sum_{j=1}^{n} t_j e_j{}^{\varepsilon}} = \frac{\frac{t_i}{b} e_i{}^{\varepsilon}}{1 + \sum_{j=1}^{n} \frac{t_j}{b} e_j{}^{\varepsilon}}$ and $p_D = \frac{b}{b + \sum_{j=1}^{n} t_j e_j{}^{\varepsilon}} = \frac{1}{1 + \sum_{j=1}^{n} \frac{t_j}{b} e_j{}^{\varepsilon}}$).

[54] Note that this functional form also appears in Chapter 1 of this thesis.

Crucially, the probability that no contestant wins and the contest ends in a draw is strictly decreasing in contestants' performances (i.e., $\frac{\partial p_D(e)}{\partial x_i} < 0$).

We interpret $b \in \mathbb{R}_{++}$ as a measure of the exogenous 'draw likelihood'. To see why, note that the probability that the contest ends in a draw is increasing in b (i.e., $\frac{\partial p_D(e)}{\partial b} > 0$). This follows from the fact that as b increases the contest becomes harder for contestants to win (i.e., $\frac{\partial p_i(e)}{\partial b} < 0$). Thus, we also interpret b as a measure of the exogenous 'difficulty' of a contest. As the draw likelihood/difficulty of a contest goes to zero, Blavatskyy's CSF collapses to a standard Tullock CSF[55] (i.e., $\lim_{b \to 0} \; p_i(e) = \frac{x_i(e_i)}{\sum_{j=1}^{n} x_j(e_j)}$ and $p_D(e) = 0$).

Blavatksyy's CSF has several notable features, which set it apart from the CSFs proposed by Jia (2012) and Yildizparlak (2013). Firstly, with Blavatskyy's CSF, contestants have no chance of winning a contest when they exert zero effort[56] (i.e., $e_i = 0 \Rightarrow p_i(e) = 0$). Thus, if all contestants exert zero effort, no contestant can win and the contest ends in a draw with certainty (i.e., $p_D(0) = 1$). Crucially, as contestants increase their efforts, it becomes strictly less likely that no contestant wins (i.e., $\frac{\partial p_D(e)}{\partial e_i} < 0$). Finally, in the limit, as contestants' efforts go to infinity, the outcome in which no contestant wins becomes almost impossible (i.e., $\lim_{\sum e_j \to \infty} p_D(e) = 0$).

To reiterate from the previous section, these features make Blavatskyy's CSF inappropriate for analysing stalemates in soccer, chess, military conflicts etc. In these examples, contestants can often improve their chances of securing a draw by increasing their efforts (i.e., $\frac{\partial p_D(e)}{\partial e_i} > 0$), while, the probability of a draw remains high when homogeneous contestants have large

[55] Note also that with Blavatskyy's CSF the probability that contestant i wins given that the contest does not end in a draw is determined by a standard Tullock CSF (i.e., $p_{i|\bar{D}} = \frac{p_i}{\sum p_j} = \frac{x_i}{\sum x_j}$).

[56] Note that this removes a discontinuity problem associated with the standard Tullock CSF.

but equal effort levels (i.e., $\lim_{\sum e_j \to \infty} p_D(e) \neq 0$). The next section, though, identifies a number of real-world contests that are uniquely consistent with these features of Blavatskyy's CSF.

4. Motivating Examples

The purpose of this section is to identify a selection of real-world contests, which are broadly consistent with the main properties of Blavatskyy's CSF. Namely, this section identifies real-world contests in which a draw (i.e., the outcome in which no contestant wins) i) occurs with certainty when contestants exert zero effort (i.e., $p_D(0) = 1$), ii) occurs less frequently as efforts increase (i.e., $\frac{\partial p_D(e)}{\partial e_i} < 0$) and iii) becomes extremely unlikely as efforts go to infinity (i.e., $\lim_{\sum e_j \to \infty} p_D(e) = 0$). Note that no other CSF in the existing literature shares these properties. In this sense, Blavatskyy's CSF is currently uniquely positioned to analyse the contests described below. This provides the motivation for the modelling that follows in the next sections.

1. *Patent Races*

Patent races represent perhaps the most natural application of Blavatskyy's CSF. In a patent race firms compete, by making R&D investments, to develop some new technology. The first firm to develop the technology wins and is awarded a patent. However, if no firm is able to develop the technology then there is no winner and the race ends in a draw. As with Blavatskyy's CSF, this is a default outcome, which occurs with certainty if firms make no R&D investments and becomes less likely as investments increase. Ultimately, as investments go to infinity the probability of a draw goes to zero.

2. Labour Market Tournaments

Nalebuff & Stiglitz (1983) and Imhof & Krakel (2011) study a two-agent labour market tournament that ends in a draw if no worker outperforms their rival by a sufficient gap. In their model, draws occur when the employer considers both workers inseparably worthy of a promotion/bonus. However, it is also possible for labour market tournaments to end in a draw because no worker is considered worthy of reward. As is consistent with Blavatskyy's CSF, this is likely to occur with certainty when workers exert no effort, less frequently as efforts increase and almost never as efforts go to infinity.

3. Lobbying/Procurement Contests

In political lobbying contests, special interest groups pressure governments for funding and/or policy interventions. Such contests can end without a winner if the government resists all competing groups. Again, this occurs with certainty if lobbying groups exert no pressure and becomes less likely (and ultimately impossible) as pressure is increased. Similarly, in procurement contests, private companies compete for public contracts. A draw occurs if the government decides against awarding the contract to any company. The probability of this occurring again decreases from 1 to 0 as efforts increase.

4. Matching Contests

In matching contests, agents compete to become the partner of a principal. Often the principal reserves the right to remain single or with some incumbent partner, in which case no agent wins and the contest ends in a draw. We again expect this to occur with certainty when agents exert zero effort, less frequently as efforts increase and almost never when efforts become infinite.

5. Sports Races

In sports races, athletes compete to be the first to complete a specified course. The winner is the athlete with the quickest time. However, it is also possible for a race to end without a winner if no athlete successfully completes the course. As in Blavatskyy's CSF, this is actually the default outcome, which occurs with certainty when athletes exert zero effort, but becomes less likely as efforts increase. The 'Barkley marathons', a 100-mile ultra-marathon held annually in the mountains of Tennessee, is an example of a race that frequently ends without a winner[57].

6. Marketing Contests/Political Elections

Blavatskyy's CSF may also be useful for describing marketing contests in which firms compete for market share. In this case, we interpret the draw probability as the proportion of the market not penetrated by any firm. Intuitively, this proportion decreases from 1 to 0 as firms increase their marketing expenditures. Similarly, Blavatksyy's CSF may be used to analyse political elections, with the draw probability describing the proportion of the electorate abstaining or failing to 'turn out'.

7. Contests with Artificial Contestants

Finally, note that Blavatskyy's CSF is equivalent to a Tullock CSF with some additional exogenous contestant(s) with performance b - a draw being the outcome in which an exogenous contestant wins. In this way, Blavatskyy's CSF can be used to analyse contests with artificial contestants. For instance, before the advent of super-computers, a number of

[57] Although most conventional sports races have a relatively low draw likelihood (since it is usually almost inconceivable that no athlete finishes), organisers could readily increase the draw likelihood of any race by imposing a maximum time limit (as is the case with the Barkley marathons, which must be completed in 60 hours). For instance, the Olympic Committee could impose a 10-second time limit on the 100-metre sprint. Result 3 in Section 8 suggests that this may lead to faster world record times.

chess tournaments featured both human and computer players. Meanwhile, many multi-player video games feature Non-Player Characters (NPCs) competing against human players.

5. Model

Motivated by the examples of the previous section, this section describes our model of a generic contest with Blavatksyy's CSF. This model generalises the standard Tullock contest model to allow for the possibility that no contestant wins and the contest ends in a draw.

Our model assumes that;

i) contestants' winning probabilities and the draw probability are consistent with Blavatskyy's axiomatisation (i.e., $p_i(e) = \frac{x_i(e_i)}{b+\sum_{j=1}^{n} x_j(e_j)}$ and $p_D(e) = \frac{b}{b+\sum_{j=1}^{n} x_j(e_j)}$),

ii) contestants' abilities, $t_i \in \mathbb{R}_+$, and efforts, $e_i \in \mathbb{R}_+$, are equally important for their performance production (i.e., $x_i(e_i) = t_i e_i \Leftrightarrow \varepsilon = 1$),

iii) contestants receive a prize $v \in \mathbb{R}_{++}$ if they win the contest, $av \in \mathbb{R}_+$ if they draw and 0 if they lose[58],

iv) contestants have a constant marginal effort cost normalised to one,

v) contestants are risk neutral expected payoff, $\pi_i(e) = vp_i(e) + avp_D(e) - e_i$, maximisers,

It follows that our model consists of n contestants, with each contestant $i \in \{1 \dots n\}$ choosing their effort exertion to solve;

$$\max_{e_i \in \mathbb{R}_+} \pi_i(e) = v\frac{x_i(e_i)}{b + \sum_{j=1}^{n} x_j(e_j)} + av\frac{b}{b + \sum_{j=1}^{n} x_j(e_j)} - e_i \quad : \quad x_i(e_i) = t_i e_i$$

[58] Note that this implies each contestant values winning, losing and drawing equally. In future models, it may be of interest to examine the effect of heterogeneous valuations of the draw prize in particular.

This model represents a generalised Tullock contest with two additional parameters; a 'draw likelihood' parameter, b, and a 'draw prize' parameter, a.

The draw likelihood parameter describes the exogenous likelihood that no contestant wins and can take any positive value (i.e., $b \in \mathbb{R}_{++}$). This parameter introduces the possibility that no contestant wins to the standard Tullock contest (i.e., note that our model collapses to a standard Tullock model as $b \to 0$). In the context of our motivating examples, b may represent the difficulty of developing a technological innovation in a patent race, the likelihood that no worker is awarded a bonus/promotion in a labour market tournament, the likelihood that no company is awarded a contract in a procurement contest or the likelihood that a principal remains single in a matching contest.

The draw prize parameter describes the relative value of a draw compared to a win or a loss. For the remainder of this paper, we assume that $a \in \left[0, \frac{1}{n}\right)$. This ensures that contestants weakly prefer a draw to a loss (i.e., $a \geq 0$) and that the total payoff for a draw is less than for a win[59] (i.e., $a < \frac{1}{n}$). In the context of our motivating examples, a may represent the value to a researcher of not being beaten by a rival in a patent race, a consolation bonus offered when no worker wins the full bonus/promotion in a labour market tournament, a compensation payment made when no company is awarded a contract in a procurement contest or some satisfaction acquired by agents when a principal remains single in a matching contest.

In the next two sections, we identify the Nash equilibrium of our model by simultaneously solving this effort choice problem for each contestant. In so doing, we are able to provide a comprehensive set of comparative statics for the draw likelihood and draw prize parameters.

[59] This assumption ensures that, where contestants' payoffs are derived from prizes offered by a contest organiser, the total prize offered for a draw does not exceed the prize for a win.

6. Nash Equilibrium with Homogeneous Contestants

First, let us consider the case in which contestants have homogeneous abilities (i.e., $t_1 = \cdots = t_n = t$). It follows that each contestant $i \in \{1 \dots n\}$ solves;

$$\max_{e_i \in \mathbb{R}_+} \pi_i(e) = v \frac{ab + te_i}{b + t\sum_{j=1}^{n} e_j} - e_i$$

Simultaneously solving this problem for all contestants, we find that our contest has a unique Nash equilibrium, $e^* = ({e_1}^* \dots {e_n}^*)$;

Proposition 1: Nash Equilibrium with Homogeneous Contestants

With homogeneous contestants, there exists a unique Nash equilibrium, e^*, in which all contestants exert the same effort level, ${e_1}^* = \cdots = {e_n}^*$.

If $b < vt(1-a)$, this effort level is positive, ${e_i}^* > 0$, and defined by $f({e_i}^*) = 0$ where;

$$f(e_i) = vt[b(1-a) + t(n-1)e_i] - (b + tne_i)^2$$

Otherwise, if $b \geq vt(1-a)$, this effort level is zero, ${e_i}^* = 0$.

Proof: *See Section 12.1 of Appendix 1* ■

Thus, with homogeneous contestants, we have a unique equilibrium in which contestants exert positive effort while the draw likelihood is sufficiently small, $b < vt(1-a)$, but exert zero effort once the draw likelihood becomes too large, $b \geq vt(1-a)$. To see why, recall that as b increases the contest becomes harder to win (i.e., recall that $\frac{\partial p_i(e)}{\partial b} < 0$). Thus, when $b \geq vt(1-a)$, the contest becomes prohibitively difficult and it is no longer profitable for contestants to exert effort trying to win.

The remainder of this section provides a set of equilibrium comparative statics for the draw prize, a, and the draw likelihood, b. For these analyses, we assume that $b < vt(1-a)$ so that equilibrium efforts are always positive.

6.1 Comparative Statics for the Draw Prize

We report the following effects of the draw prize on the Nash equilibrium with homogeneous contestants;

Proposition 1.1a: Comparative Statics for the Draw Prize with Homogeneous Contestants

#	***Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Prize*** $(\frac{\partial z}{\partial a})$
i	*Effort* (e_i^*)	$-$
ii	*Performance* $(x_i(e_i^*))$	$-$
iii	*Win Probability* $(p_i(e^*))$	$-$
iv	*Draw Probability* $(p_D(e^*))$	$+$
v	*Expected Payoff* $(\pi_i(e^*))$	$+$

Proof: *See Section 12.2 of Appendix 1* ■

Thus, we find that i) increasing the draw prize decreases contestants' equilibrium efforts (i.e., $\frac{\partial e_i^*}{\partial a} < 0$). In this way, contest organisers seeking to elicit maximum efforts from homogeneous contestants should not offer a draw prize. This result differs from that of Yildizparlak (2013), which reports that equilibrium efforts are independent of the draw prize (i.e., $\frac{\partial e_i^*}{\partial a} = 0$). This difference is attributable to the contrasting nature of Blavatskyy's and Yildizparlak's CSFs. With Blavatskyy's CSF, the probability of a draw is always decreasing in contestants' efforts. Thus, increasing the prize for a draw reduces contestants' incentive to exert effort. In contrast, with Yildizparlak's CSF, the probability of a draw is maximised whenever contestants have equal efforts. It happens that Yildizparlak's equilibrium is symmetric, so that contestants always have the maximum chance of attaining a draw. It follows that increasing the prize for a draw does not cause contestants to change their efforts.

Given that performances are proportional to efforts (i.e., recall that $x_i({e_i}^*) = t{e_i}^*$), we also have ii) increasing the draw prize decreases contestants' equilibrium performances (i.e., $\frac{\partial x_i({e_i}^*)}{\partial a} < 0$). What's more, we find that increasing the draw prize iii) decreases contestants' equilibrium win probabilities (i.e., $\frac{\partial p_i(e^*)}{\partial a} < 0$) and iv) increases the equilibrium draw probability (i.e., $\frac{\partial p_D(e^*)}{\partial a} > 0$). In this way, offering a prize for a draw is a self-fulfilling mechanism that increases the frequency of a draw. Finally, we find that v) increasing the draw prize increases contestants' expected payoffs (i.e., $\frac{\partial \pi_i(e^*)}{\partial a} > 0$).

6.2 Comparative Statics for the Draw Likelihood

We report the following effects of the draw likelihood on the Nash equilibrium with homogeneous contestants;

Proposition 1.1b: Comparative Statics for the Draw Likelihood with Homogeneous Contestants

#	***Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Likelihood ($\frac{\partial z}{\partial b}$)***
i	*Effort ($e_i{}^*$)*	$-$
ii	*Performance ($x_i(e_i{}^*)$)*	$-$
iii	*Win Probability ($p_i(e^*)$)*	$-$
iv	*Draw Probability ($p_D(e^*)$)*	$+$
v	*Expected Payoff ($\pi_i(e^*)$)*	$-$ *when* $b < \bar{b}_\pi$ $+$ *when* $b > \bar{b}_\pi$

where $\bar{b}_\pi = \frac{vt[2-(n+1)a][2-n(n+1)a]}{4(1-na)} \leq vt(1-a)$

Proof: *See Section 12.3 of Appendix 1* ■

Thus, we find that i) increasing the draw likelihood decreases contestants' equilibrium efforts (i.e., $\frac{\partial e_i{}^*}{\partial b} < 0$). Intuitively, this result says that making a contest harder to win only disincentivises efforts; contest organisers cannot increase the difficulty of a contest in order to induce greater efforts. It follows that the standard Tullock contest, without any possibility of a draw, elicits the greatest efforts from contestants. This result again differs with that of Yildizparlak (2013), which reports that introducing a small draw possibility to a Tullock contest can increase contestants' effort (when $n = 2$). This difference should again be attributed to the contrasting nature of Yildizparlak's CSF.

Again, it follows simply that ii) increasing the draw likelihood decreases contestants' equilibrium performances (i.e., $\frac{\partial x_i(e_i{}^*)}{\partial b} < 0$). We also find that increasing the exogenous draw likelihood iii) reduces contestants' win probabilities (i.e., $\frac{\partial p_i(e^*)}{\partial b} < 0$), and iv) increases the equilibrium draw probability (i.e., $\frac{\partial p_D(e^*)}{\partial b} > 0$). Finally, we find that v) up to a certain threshold, increasing the draw likelihood decreases contestants' equilibrium expected payoffs (i.e., $\frac{\partial \pi_i(e^*)}{\partial b} < 0$ when $b < \bar{b}_\pi$), but above this threshold, increasing the draw likelihood increases expected payoffs (i.e., $\frac{\partial \pi_i(e^*)}{\partial b} > 0$ when $b > \bar{b}_\pi$).

The following lemma notes that the size of the draw prize determines the optimal draw likelihood for contestants;

Lemma 1[60]: $a \in [0, \frac{1}{n^2}) \Rightarrow argmax_{b \in (0, vt(1-a))} \pi_i(e^*) \to 0$ *and*

$a \in \left(\frac{1}{n^2}, \frac{1}{n}\right) \Rightarrow argmax_{b \in (0, vt(1-a))} \pi_i(e^*) \to vt(1-a)$.

Proof: *See part v) in Section 12.3 of Appendix 1* ■

Thus, if the associated prize is high enough (i.e., $a > \frac{1}{n^2}$), contestants prefer a high draw likelihood. However, with a low draw prize (i.e., $a < \frac{1}{n^2}$), contestants prefer no draw likelihood.

7. Nash Equilibrium with Heterogeneous Contestants

Having analysed the Nash equilibrium with homogeneous contestants, we now present the Nash equilibrium with heterogeneous contestants. We consider the special case with just 2 contestants; a strong contestant, s, and a weak contestant, w, where the strong contestant is more able than the weak contestant (i.e., $t_s > t_w$). We denote contestants' total ability T (i.e., $T = t_s + t_w$). Each contestant $i \in \{s, w\}$ chooses their effort to solve;

$$\max_{e_i \in \mathbb{R}_+} \pi_i(e) = v \frac{ab + t_i e_i}{b + t_s e_s + t_w e_w} - e_i$$

[60] Note that the proof of this lemma also shows $a = 0 \Rightarrow \frac{\partial \pi_i(e^*)}{\partial b} < 0$ and $a \in \left(\frac{2}{n(n+1)}, \frac{1}{n}\right) \Rightarrow \frac{\partial \pi_i(e^*)}{\partial b} > 0$.

Solving this problem simultaneously for both contestants, yields a unique Nash equilibrium, $e^* = (e_s{}^*, e_w{}^*)$;

Proposition 2: Nash Equilibrium with Heterogeneous Contestants

With heterogeneous contestants, there exists a unique Nash equilibrium, e^*, in which either i) $e_s{}^* > 0$ and $e_w{}^* > 0$, ii) $e_s{}^* > 0$ and $e_w{}^* = 0$ or iii) $e_s{}^* = e_w{}^* = 0$.

If $b < \frac{vt_s(1-a)}{\left[(1-a)\frac{t_s}{t_w}+a\right]^2} = \bar{b}_w$, we have that i) each contestant's equilibrium effort is positive, $e_s{}^* > 0$ and $e_w{}^* > 0$, and is implicitly defined by $f_s(e_s{}^*) = 0$ and $f_w(e_w{}^*) = 0$, where;

$$f_s(e_s) = vt_w(b(1-a) + t_s e_s) - \left[b(1-a)\frac{t_w}{t_s} + ba + Te_s\right]^2$$

$$f_w(e_w) = vt_s(b(1-a) + t_w e_w) - \left[b(1-a)\frac{t_s}{t_w} + ba + Te_w\right]^2$$

If $\bar{b}_w \leq b < vt_s(1-a) = \bar{b}_s$, we have that ii) the strong contestant's equilibrium effort is positive, $e_s{}^* = \frac{[vt_s b(1-a)]^{\frac{1}{2}} - b}{t_s} > 0$, while the weak contestant's equilibrium effort is zero, $e_w{}^* = 0$.

If $b \geq \bar{b}_s$, we have that iii) both contestants' equilibrium effort is zero, $e_s{}^* = e_w{}^* = 0$.

Proof: *See Section 13.1 of Appendix 2*■

Thus, we have a unique equilibrium in which i) both contestants exert positive effort when $b < \bar{b}_w$, ii) only the strong contestant exerts positive effort when $\bar{b}_w \leq b < \bar{b}_s$, and iii) both contestants exerts zero effort when $b \geq \bar{b}_s$. Again, this follows from the fact that as b increases the contest becomes harder to win (i.e., recall that $\frac{\partial p_i(e)}{\partial b} < 0$). Thus, the contest eventually becomes prohibitively difficult to win, first for the weak contestant (when $b \geq \bar{b}_w$) and then for the strong contestant (when $b \geq \bar{b}_s$).

The following lemma notes that, whenever they provide effort, the strong contestant provides a greater effort than the weak contestant;

Lemma 2: $b < \bar{b}_s \Rightarrow {e_s}^* > {e_w}^*$.

Proof: *See Section 13.1 of Appendix 2*■

This reaffirms Jia's (2012) finding that introducing the possibility of a draw to a Tullock contest removes the 'homogeneity paradox', in which heterogeneous contestants provide homogeneous efforts.

The remainder of this section provides a set of equilibrium comparative statics for the draw prize, a, and the draw likelihood, b. For these analyses, we assume that $b < \bar{b}_w$ so that both contestants' equilibrium efforts are positive.

7.1 Comparative Statics for the Draw Prize

In the heterogeneous case, we report the effects of the draw prize on 1) the strong contestant and 2) the weak contestant. We then report 3) the total effects of the draw prize. We begin with 1) the strong contestant;

Proposition 2.1a: Comparative Statics for the Draw Prize with Heterogeneous Contestants (Strong Contestant Outcomes)

#	***Strong Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Prize ($\frac{\partial z}{\partial \alpha}$)***
i	*Effort ($e_s{}^*$)*	$-$
ii	*Performance ($x_s(e_s{}^*)$)*	$-$
iii	*Win Probability ($p_s(e^*)$)*	$-$
iv	*Draw Probability ($p_D(e^*)$)*	$+$
v	*Expected Payoff ($\pi_s(e^*)$)*	$+$

Proof: *See Section 13.2 of Appendix 2*■

Thus, in the heterogeneous case, the effects of the draw prize on the strong contestant are qualitatively identical to those described in the homogeneous case. Specifically, the strong contestant's i) effort, ii) performance and iii) win probability are all decreasing in the draw prize, while the strong contestant's iv) draw probability and v) expected payoff are increasing in the draw prize.

Next, we describe the effects of the draw prize on 2) the weak contestant;

Proposition 2.2a: Comparative Statics for the Draw Prize with Heterogeneous Contestants (Weak Contestant Outcomes)

#	***Weak Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Prize*** $(\frac{\partial z}{\partial a})$
i	*Effort* $(e_w{}^*)$	$+$ *when* $a < \bar{a}_{e_w}$ $-$ *when* $a > \bar{a}_{e_w}$
ii	*Performance* $(x_w(e_w{}^*))$	$+$ *when* $a < \bar{a}_{e_w}$ $-$ *when* $a > \bar{a}_{e_w}$
iii	*Win Probability* $(p_w(e^*))$	$+$ *when* $a < \bar{a}_{p_w}$ $-$ *when* $a > \bar{a}_{p_w}$
iv	*Draw Probability* $(p_D(e^*))$	$+$
v	*Expected Payoff* $(\pi_w(e^*))$	$+$

where $\bar{a}_{e_w} = \frac{1}{2} + \frac{vt_st_w}{4b(t_s-t_w)}\left[1 - \frac{T}{2(t_s-t_w)}\right] < \bar{a}_{p_w} = \frac{1}{2} + \left[\frac{vt_st_w}{4b(t_s-t_w)}\right]^{\frac{1}{2}} - \frac{T}{2(t_s-t_w)}$

Proof: *See Section 13.3 of Appendix 2*■

Thus, in a significant departure from the homogeneous case, increasing the draw prize may increase the weak contestant's i) effort, ii) performance and iii) win probability (i.e., $\frac{\partial e_w{}^*}{\partial a} > 0, \frac{\partial x_w(e_w{}^*)}{\partial a} > 0$ and $\frac{\partial p_w(e^*)}{\partial a} > 0$ when $a < \bar{a}_{e_w}$). Crucially then, contest organisers may offer a draw prize as a mechanism for inducing more effort from a weaker contestant. Indeed, the following lemma reports that this is the case whenever the weak contestant is sufficiently weak;

Lemma 3: $t_w < \frac{t_s}{3} \Rightarrow \frac{\partial e_w{}^*}{\partial a} > 0, \frac{\partial x_w(e_w{}^*)}{\partial a} > 0$ *and* $\frac{\partial p_w(e^*)}{\partial a} > 0.$

Proof: *See part iii) of Section 13.3 in Appendix 2*■

Finally, we describe 3) the total effects of the draw prize;

Proposition 2.3a: Comparative Statics for the Draw Prize with Heterogeneous Contestants (Total Contestant Outcomes)

#	*Total Contestants' Equilibrium Outcome (z)*	*Effect of the Draw Prize* ($\frac{\partial z}{\partial a}$)
i	*Effort* ($e_s{}^* + e_w{}^*$)	$+$ *when* $a < \bar{a}_{e_s+e_w}$ $-$ *when* $a > \bar{a}_{e_s+e_w}$
ii	*Performance* ($x_s(e_s{}^*) + x_w(e_w{}^*)$)	$-$
iii	*Win Probability* ($p_s(e^*) + p_w(e^*)$)	$-$
iv	*Draw Probability* ($p_D(e^*)$)	$+$
v	*Expected Payoff* ($\pi_s(e^*) + \pi_w(e^*)$)	$+$

where $\bar{a}_{e_s+e_w} = \frac{1}{2} + \frac{v t_s t_w T}{4b(t_s - t_w)^2}\left[1 - \frac{T^2}{2(t_s - t_w)^2}\right]$

Proof: *See Section 13.4 of Appendix 2*■

Thus, in the heterogeneous case, we find that i) increasing the draw prize can be a mechanism for increasing total efforts (i.e., $\frac{\partial e_s{}^* + e_w{}^*}{\partial a} > 0$ when $a < \bar{a}_{e_s+e_w}$). Again, the following lemma reports that this is the case if the weak contestant is sufficiently weak;

Lemma 4: $t_w < \frac{\sqrt{2}-1}{\sqrt{2}+1} t_s \Rightarrow \frac{\partial e_s{}^* + e_w{}^*}{\partial a} > 0.$

Proof: *See part i) of Section 13.4 in Appendix 2*■

However, all other effects are unchanged from the homogeneous case. In particular, ii) increasing the draw prize necessarily reduces total performances.

7.2 Comparative Statics for the Draw Likelihood

We now report the effects of the draw likelihood on 1) the strong contestant and 2) the weak contestant. We then report 3) the total effects. We begin with 1) the strong contestant;

Proposition 2.1b: Comparative Statics for the Draw Likelihood with Heterogeneous Contestants (Strong Contestant Outcomes)

#	***Strong Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Likelihood ($\frac{\partial z}{\partial b}$)***
i	*Effort ($e_s{}^*$)*	$+$ *when* $b < \bar{b}_{e_s}$ $-$ *when* $b > \bar{b}_{e_s}$
ii	*Performance ($x_s(e_s{}^*)$)*	$+$ *when* $b < \bar{b}_{e_s}$ $-$ *when* $b > \bar{b}_{e_s}$
iii	*Win Probability ($p_s(e^*)$)*	$-$
iv	*Draw Probability ($p_D(e^*)$)*	$+$
v	*Expected Payoff ($\pi_s(e^*)$)*	$-$ *when* $b < \bar{b}_{\pi_s}$ $+$ *when* $b > \bar{b}_{\pi_s}$[61]

where $\bar{b}_{e_s} = \frac{vt_st_w(1-a)[(1-3a)t_s-(1-a)t_w]}{4(1-2a)[(1-a)t_w+at_s]^2}, \bar{b}_{\pi_s} = \frac{vt_st_w(2-3a)[(2-5a)t_s-at_w]}{4(1-2a)[(1-a)t_w+at_s]^2}$

Proof: *See Section 13.5 of Appendix 2*■

Crucially, we find that increasing the draw likelihood may increase the strong contestant's i) efforts and ii) performances (i.e., $\frac{\partial e_s{}^*}{\partial b} > 0$ and $\frac{\partial x_s(e_s{}^*)}{\partial b} > 0$ when $b < \bar{b}_{e_s}$). Thus, with heterogeneous contestants, introducing a draw likelihood (i.e., making the contest more difficult to win) may be a mechanism for inducing greater efforts/performances from the

[61] Note that part v) of Section 13.5 in Appendix 2 shows that $a > \frac{2}{5} \Rightarrow \frac{\partial \pi_s(e^*)}{\partial b} > 0$.

most able contestant. The following lemma notes that this can only be the case when the draw prize is less than $\frac{1}{3}$;

Lemma 5: $a > \frac{1}{3} \Rightarrow \frac{\partial e_s{}^*}{\partial b} < 0$.

Proof: *See part i) of Section 13.5 in Appendix 2*■

Next, we report the effects of the draw likelihood on 2) the weak contestant;

Proposition 2.2b: Comparative Statics for the Draw Likelihood with Heterogeneous Contestants (Weak Contestant Outcomes)

#	***Weak Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Likelihood ($\frac{\partial z}{\partial b}$)***
i	*Effort ($e_w{}^*$)*	$-$
ii	*Performance ($x_w(e_w{}^*)$)*	$-$
iii	*Win Probability ($p_w(e^*)$)*	$-$
iv	*Draw Probability ($p_D(e^*)$)*	$+$
v	*Expected Payoff ($\pi_w(e^*)$)*	$-$ *when* $b < \bar{b}_{\pi_w}$ $+$ *when* $b > \bar{b}_{\pi_w}$

where $\bar{b}_{\pi_w} = \frac{vt_s t_w(2-3a)[(2-5a)t_w - at_s]}{4(1-2a)[(1-a)t_s + at_w]^2} < \bar{b}_w$

Proof: *See Section 13.6 of Appendix 2*■

Thus, we find that the effects of the draw likelihood on the weak contestant are qualitatively identical to the homogeneous case. In particular, increasing the draw likelihood always decreases the weak contestant's i) efforts, ii) performances and iii) win probability (i.e, $\frac{\partial e_w{}^*}{\partial b} < 0$, $\frac{\partial x_w(e_w{}^*)}{\partial b} < 0$ and $\frac{\partial p_w(e^*)}{\partial b} < 0$). Meanwhile, the following lemma shows that the weak contestant benefits from a greater draw likelihood whenever the associated prize is greater than a third;

Lemma 6: $a > \frac{1}{3} \Rightarrow \frac{\partial \pi_w(e^*))}{\partial b} > 0.$

Proof: *See part v) of Section 13.6 in Appendix 2*■

Finally, we describe the 3) the total effects of the draw likelihood;

Proposition 2.3b: Comparative Statics for the Draw Likelihood with Heterogeneous Contestants (Total Contestant Outcomes)

#	***Total Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Likelihood ($\frac{\partial z}{\partial b}$)***
i	*Effort ($e_s{}^* + e_w{}^*$)*	$-$
ii	*Performance ($x_s(e_s{}^*) + x_w(e_w{}^*)$)*	$-$
iii	*Win Probability ($p_s(e^*) + p_w(e^*)$)*	$-$
iv	*Draw Probability ($p_D(e^*)$)*	$+$
v	*Expected Payoff ($\pi_s(e^*) + \pi_w(e^*)$)*	$-$ *when* $b < \bar{b}_{\pi_w}$ $+$ *when* $b > \bar{b}_{\pi_s}$

Proof: *See Section 13.7 of Appendix 2*■

Again, the total results are qualitatively identical to the homogeneous case. Most notably, increasing the draw likelihood reduces total i) efforts and ii) performances. Thus, although we find that a contest organiser may increase the draw likelihood to induce a greater effort/performance from the strong contestant, doing so necessarily reduces total efforts/performances.

8. Discussion

This section provides a tentative discussion of the results of the previous two sections. First, Table 1 summarises our results for the homogeneous case.

Table 1: Comparative Statics for the Homogeneous Case

#	***Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Prize*** $(\frac{\partial z}{\partial a})$	***Effect of the Draw Likelihood*** $(\frac{\partial z}{\partial b})$
i	*Effort* $(e_i{}^*)$	$-$	$-$
ii	*Performance* $(x_i(e_i{}^*))$	$-$	$-$
iii	*Win Probability* $(p_i(e^*))$	$-$	$-$
iv	*Draw Probability* $(p_D(e^*))$	$+$	$+$
v	*Expected Payoff* $(\pi_i(e^*))$	$+$	$-$ *then* $+$

Most notably, we find that;

Result 1: Homogeneous contestants' efforts/performances are decreasing in the draw likelihood.

Thus, with homogeneous contestants, contest organisers looking to maximise efforts/performances should aim to minimise the likelihood that no contestant wins. For instance, in labour market tournaments, employers should guarantee the selection of one worker for promotions/bonuses; threatening to withhold a bonus if no worker performs satisfactorily only makes under-performance more likely. Similarly, in matching contests, principals should guarantee the selection of a new partner; a principal that retains the option of an incumbent partner (or threatens to remain single) reduces the attention they receive from agents.

We also find that;

Result 2: Homogeneous contestants' efforts/performances are decreasing in the draw prize.

Thus, with homogenous contestants, contest organisers looking to maximise efforts/performances should not offer a consolation prize if no contestant wins. For instance, in procurement contests, authorities should not compensate firms when no contract is awarded. Similarly, in patent races, policy makers should not seek to compensate researchers for fruitless races.

Next, Table 2 summarises our results for the heterogeneous case.

Table 2: Comparative Statics for the Heterogeneous Case

#	*Contestants' Equilibrium Outcome (z)*		*Effect of the Draw Prize* $(\frac{\partial z}{\partial a})$	*Effect of the Draw Likelihood* $(\frac{\partial z}{\partial b})$
i	*Effort*	*Strong* $(e_i{}^*)$	$-$	$+$ *then* $-$
		Weak $(e_j{}^*)$	$+$ *then* $-$	$-$
		Total $(e_i{}^* + e_j{}^*)$	$+$ *then* $-$	$-$
ii	*Performance*	*Strong* $(x_i(e_i{}^*))$	$-$	$+$ *then* $-$
		Weak $(x_j(e_j{}^*))$	$+$ *then* $-$	$-$
		Total $(x_i(e_i{}^*) + x_j(e_j{}^*))$	$-$	$-$
iii	*Win Probability*	*Strong* $(p_i(e^*))$	$-$	$-$
		Weak $(p_j(e^*))$	$+$ *then* $-$	$-$
		Total $(p_i(e^*) + p_j(e^*))$	$-$	$-$
iv	*Draw Probability*	Strong/Weak/Total$(p_D(e^*))$	$+$	$+$
v	*Expected Payoff*	*Strong* $(\pi_i(e^*))$	$+$	$-$ *then* $+$
		Weak $(\pi_j(e^*))$	$+$	$-$ *then* $+$
		Total $(\pi_i(e^*) + \pi_j(e^*))$	$+$	$-$ *then* $+$

Crucially, qualitative differences from the homogeneous case are indicated in red. Most notably, we find that;

Result 3: With heterogeneous contestants, the effort/performance of the strongest contestant can be increasing in the draw likelihood.

In this case, the more likely it is that no contestant wins (i.e., the more difficult a contest is), the more effort the strongest contestant exerts. For instance, in patent races, the more difficult a technology is to develop, the more the most efficient firm invests on R&D. Meanwhile, in matching contests, a principal with an established incumbent partner (or an established will to remain single) receives more attention from the most desirable agent than one without. Finally, onerous sports races, like the Barkley marathons, with a high chance that no athlete wins produce greater performances from the most able athletes.

On the other hand, we also find that;

Result 4: With heterogeneous contestants, the effort/performance of the weakest contestant can be increasing in the draw prize.

In this case, contest organisers can induce a greater effort from the weakest contestant by offering a consolation prize when no contestant wins. For instance, somewhat paradoxically, employers can improve the outputs of their least able worker by offering a bonus when no worker performs well enough to earn a promotion (or a larger bonus). Similarly, in sports races, the performance of the weakest athlete can be improved by the offer of a prize if no athlete wins.

Next, we discuss some extended comparative statics for our model.

9. Extensions

This section acknowledges that contest organisers may also be concerned with i) the equity of a (heterogeneous) contest and ii) the profitability of a contest. We discuss each in turn.

9.1 Equity

In many cases, contest organisers are concerned with the equity or fairness of a contest. This concern may be driven by a desire to make the contest unpredictable/entertaining (for instance, in sports, Rottenberg (1956) suggests that uncertainty of outcome is important for revenues). Alternatively, organisers may be concerned that excessive inequity might generate dissatisfaction among contestants (for instance, in labour market tournaments, employers may be concerned that workers' morale is damaged by wide disparities in efforts or payoffs). Finally, organisers may simply have a benevolent desire to improve the relative welfare of disadvantaged contestants (for instance, in matching contests the principal may not wish to allow a high level of inequity to persist among agents).

Motivated by this we report the following effects of the draw prize and the draw likelihood on the difference in heterogeneous contestant outcomes;

Proposition 2.4: Comparative Statics for the Difference in Heterogeneous Contestant Outcomes

#	***Difference in Contestants' Equilibrium Outcomes (z)***	***Effect of the Draw Prize*** $(\frac{\partial z}{\partial a})$	***Effect of the Draw Likelihood*** $(\frac{\partial z}{\partial b})$
i	*Effort* $({e_s}^* - {e_w}^*)$	$-$	$+$
ii	*Performance* $(x_s({e_s}^*) - x_w({e_w}^*))$	$-$	$+$
iii	*Win Probability* $(p_s(e^*) - p_w(e^*))$	$-$	$+$
iv	*Draw Probability* $(p_D(e^*) - p_D(e^*)))$	*N/A*	*N/A*
v	*Expected Payoff* $(\pi_s(e^*) - \pi_w(e^*))$	$+$ *when* $a < \bar{a}_{\pi_s - \pi_w}$ $-$ *when* $a > \bar{a}_{\pi_s - \pi_w}$	$-$

where $\bar{a}_{\pi_s-\pi_w} = \frac{1}{2} - \frac{3vt_st_w}{8bT}$

Proof: *See Sections 13.8 and 13.9 of Appendix 2*■

Thus, we find that increasing the draw prize reduces the inequity in contestants' i) efforts, ii) performances and iii) win probabilities. However, increasing the draw likelihood increases the inequity in these outcomes. Interestingly though, increasing the draw likelihood reduces the inequity between contestants' v) expected payoff outcomes, whereas, increasing the draw prize can increase the inequity in payoffs[62].

9.2 Profitability

Contest organisers may also be concerned with the profitability of a contest. In this sub-section, we consider an employer's profits from a labour market tournament that ends in a draw if no worker performs satisfactorily. In this example, the employer pays a bonus v to a

[62] Note though that part v) of Section 13.8 in Appendix 2 shows that $t_w < \frac{3}{13}t_s \Rightarrow \frac{\partial \pi_s(e^*) - \pi_w(e^*)}{\partial a} < 0$.

winning worker and a consolation bonus of av to each worker in the event that no worker wins the full bonus. Thus, the employer has expected expenditure $E(e) = v \sum p_i(e) + navp_D(e)$. If we further interpret $x_i(e_i)$ as worker i's output then the employer has expected profit $Q(e) = \sum x_i(e_i) - E(e)$.

In the homogeneous case, we report the following effects of the draw prize and the draw likelihood on expenditures and profits;

Proposition 1.2: Comparative Statics on Employers' Expenditures/Profits with Homogeneous Contestants

#	***Employer's Equilibrium Outcome (z)***	***Effect of the Draw Prize ($\frac{\partial z}{\partial a}$)***	***Effect of the Draw Likelihood ($\frac{\partial z}{\partial b}$)***
i	*Expected Expenditure ($E(e^*)$)*	$+$	$-$
ii	*Expected Profit ($Q(e^*)$)*	$-$	$+$ *when* $b < \bar{b}_Q{}^{Hom}$ $-$ *when* $b > \bar{b}_Q{}^{Hom}$

where $\bar{b}_Q{}^{Hom} = \frac{v[t(1-a)+(1-na)][n-an(t+n)-t(n-2)]}{4t(1-na)}$*)*

Proof: *See Sections 12.4 and 12.5 of Appendix 1* ■

Crucially, we find that contest organisers can increase the profitability of a contest by introducing the possibility of a draw (i.e., $\frac{\partial Q(e^*)}{\partial b} > 0$, note that this requires both $a < \frac{n-(n-2)t}{n(t+n)}$ and $t < \frac{n}{(n-2)}$). This follows from the fact that, although introducing the possibility of a draw reduces workers' outputs, it also reduces the employer's expected expenditures (i.e., $\frac{\partial E(e^*)}{\partial b} < 0$). Note that this result contrasts with Imhof & Krakel (2011), which finds that, with homogeneous workers, employers' profits fall when the possibility of a draw is introduced. We also note that these results are preserved in the heterogeneous case;

Proposition 2.5: Comparative Statics on Employers' Expenditures/Profits with Heterogeneous Contestants

#	***Employer's Equilibrium Outcome (z)***	***Effect of the Draw Prize*** $(\frac{\partial z}{\partial a})$	***Effect of the Draw Likelihood*** $(\frac{\partial z}{\partial b})$
i	*Expected Expenditure* $(E(e^*))$	$+$	$-$
ii	*Expected Profit* $(Q(e^*))$	$-$	$+$ *when* $b < \bar{b}_Q{}^{Het}$ $-$ *when* $b > \bar{b}_Q{}^{Het}$

where $\bar{b}_Q{}^{Het} = \frac{v[(1-2a)T+2t_s t_w(1-a)][(1-2a)T-2at_s t_w]}{4T(1-2a)t_s t_w}$)

Proof: *See Sections 13.10 and 13.11 of Appendix 2*■

10. Conclusion

Motivated by the observation that many real-world contests may end without producing a winner (i.e., in a draw), this paper provides the first equilibrium analysis of a contest with Blavatskyy's (2010) CSF. We present a wide-range of comparative statics results. Most notably, we find that, with homogeneous contestants, introducing the possibility of a draw reduces contestants' effort exertions. However, with heterogeneous contestants, introducing the possibility of a draw may be useful as a mechanism for extracting more effort from the strongest contestant. In this instance, contest organisers may also wish to offer a prize for a draw in order to improve the effort of the weakest contestant. In the case of a labour market tournament, we further find that, with homogeneous or heterogeneous contestants, it can be profitable for employers to introduce the possibility of a draw.

Beyond this analysis, there remain many open avenues for future research with Blavatskyy's CSF. First, in this paper we have solved for the equilibrium of a contest whose CSF has a unit exponent on efforts (i.e., $\varepsilon = 1$). It remains to study the more general Blavatskyy case in which $\varepsilon \in \mathbb{R}_{++}$. Similarly, we have restricted the possible values of the draw prize. It remains to study a contest with a negative draw prize, or a total draw prize (i.e., na) that is greater than the payoff for a win. Further, studying a heterogeneous contest with three or more contestants may generate results not captured in the two contestant case studied here. Other potential extensions include introducing behavioural considerations (i.e., considering contestants with reference dependent preferences or risk averse attitudes), generalising to non-linear effort cost functions or studying different organiser objective functions (i.e., considering the relationship between the optimal draw likelihood/prize and the objectives of contest organisers).

More generally, the literature on contests with draws is surely one which is worthy of further development. Upon inspection, real-world contests with the possibility of draw outcomes are perhaps even more common than contests without the possibility of draws. In this paper, as in other contributions to this literature, we discuss in very general terms the 'effect of the draw likelihood' or the 'effect of the draw prize' on outcomes such as efforts or payoffs. However, this abstracts from the fact that real-world contests feature many different types of draws. For instance, a draw in soccer, which occurs because neither team is able to decisively out-perform their rival, is different from a draw in a patent race, which occurs because no researcher is able to overcome the exogenous natural barriers to discovering a new technology. Different CSFs in the literature are clearly better suited to analysing the occurrence of different draw types. For instance, Blavatskyy's CSF is better suited to describing draws in patent races, whereas other CSFs, such as Yildizparlak's (2013), are better suited to describing draws in soccer. This explains why different CSFs deliver markedly different equilibrium results (for instance, with Blavatskyy's CSF, we find that efforts are decreasing in the draw prize, but, with Yildizparlak's CSF, efforts are independent of the draw prize). Thus, perhaps the greatest immediate challenge for the literature is to attempt to distinguish between the different real-world contest outcomes that are currently defined under the umbrella term 'draw'.

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12. Appendix 1: Homogeneous Equilibrium

This appendix contains proofs for the Nash equilibrium with homogeneous contestants.

12.1 Proposition 1

This sub-section provides a proof of;

Proposition 1: Nash Equilibrium with Homogeneous Contestants

With homogeneous contestants, there exists a unique Nash equilibrium, in which all contestants exert the same effort level, ${e_1}^* = \cdots = {e_n}^*$.

If $b < vt(1-a)$, *contestants' equilibrium effort is positive,* ${e_i}^* > 0$, *and defined by* $f({e_i}^*) = 0$ *where;*

$$f(e_i) = vt[b(1-a) + t(n-1)e_i] - (b + tne_i)^2$$

Otherwise, if $b \geq vt(1-a)$, *contestants' equilibrium effort is zero,* ${e_i}^* = 0$.

First note that $\pi_i(e) = v\frac{ab+te_i}{b+t\sum_{j=1}^{n} e_j} - e_i$ is strictly concave in e_i. To see this, note that our first order derivative is;

$$\frac{\partial \pi_i(e)}{\partial e_i} = vt\frac{b(1-a) + t\sum_{j\neq i} e_j}{\left(b + t\sum_{j=1}^{n} e_j\right)^2} - 1$$

so that our second order derivative is strictly negative;

$$\frac{\partial^2 \pi_i(e)}{\partial {e_i}^2} = -2vt^2\frac{b(1-a) + t\sum_{j\neq i} e_j}{\left(b + t\sum_{j=1}^{n} e_j\right)^3} < 0$$

Then note that we have three different potential equilibrium types;

TYPE 1: NON-PARTICIPATION EQUILIBRIUM ($\forall i, e_i^* = 0$)

By concavity of $\pi_i(e)$, there exists a non-participation equilibrium, e^*, in which $e_i^* = 0$ for any $i \in [1 \dots n]$, if and only if $\frac{\partial \pi_i(0)}{\partial e_i} \leq 0 \Leftrightarrow b \geq vt(1-a)$.

TYPE 2: PARTIAL NON-PARTICIPATION EQUILIBRIUM ($\exists i, k: e_i^* = 0, e_k^* > 0$)

By concavity of $\pi_i(e)$, there exists a partial non-participation equilibrium, e^*, in which $e_i^* = 0$ and $e_k^* > 0$ for some $i, k \in [1 \dots n]$ if and only if i) $\frac{\partial \pi_i(e^*)}{\partial e_i} \leq 0 \Leftrightarrow vt \frac{b(1-a)+t\sum_{j \neq i} e_j^*}{\left(b+t\sum_{j=1}^{n} e_j^*\right)^2} \leq 1$ and ii) $\frac{\partial \pi_k(e^*)}{\partial e_k} = 0 \Leftrightarrow vt \frac{b(1-a)+t\sum_{j \neq k} e_j^*}{\left(b+t\sum_{j=1}^{n} e_j^*\right)^2} = 1$. However, note that, if ii) holds, then i) is equivalent to $e_k^* \leq e_i^*$, which is a contradiction. Thus, we exclude the possibility of a partial non-participation equilibrium.

TYPE 3: PARTICIPATION EQUILIBRIUM (($\forall i, e_i^* > 0$)

By concavity of $\pi_i(e)$, there exists a participation equilibrium, e^*, in which $e_i^* > 0$ for any $i \in [1 \dots n]$, if and only if $\frac{\partial \pi_i(e^*)}{\partial e_i} = 0 \Leftrightarrow vt \frac{b(1-a)+t\sum_{j \neq i} e_j^*}{\left(b+t\sum_{j=1}^{n} e_j^*\right)^2} = 1$ for any i. It follows that any participation equilibrium must be symmetric, $e_1^* = \cdots = e_n^*$, with $f(e_i^*) = 0$ where;

$$f(e_i) = vt[b(1-a) + t(n-1)e_i] - (b + tne_i)^2$$

Note that $f(e_i)$ is a strictly concave and continuous function in e_i. Our first order derivative is;

$$f'(e_i) = vt^2(n-1) - 2tn(b + tne_i)$$

So that our second order derivative is strictly negative;

$$f''(e_i) = -2t^2n^2 < 0$$

Now, let us first suppose that $b \geq vt(1-a)$. With this, we have $f(0) \leq 0$. Further, since $f'(0) = t(vt(n-1) - 2nb)$ and the minimum value of b is $vt(1-a)$, we have $f'(0) \leq vt^2((2a-1)n-1) < 0$. Since $f(e_i)$ is concave, it follows that $f(e_i^*) = 0 \Rightarrow e_i^* \leq 0$, which is a contradiction. Thus, we exclude the possibility of a participation equilibrium when $b \geq vt(1-a)$.

Conversely, let us suppose that $b < vt(1-a)$. With this, we have $f(0) > 0$. We also have $\lim_{e_i \to \infty} f(e_i) = -\infty$. Thus, by continuity of $f(e_i)$, there exists some $e_i^* > 0$ for which $f(e_i^*) = 0$. Further, by concavity of $f(e_i)$, this $e_i^* > 0$ is unique (note that $f'(e_i^*)$ must be negative). Therefore, we have a unique participation equilibrium when $b < vt(1-a)$.

12.2 Proposition 1.1a

This sub-section provides a proof of;

Proposition 1.1a: Comparative Statics for the Draw Prize with Homogeneous Contestants

#	***Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Prize ($\frac{\partial z}{\partial a}$)***
i	*Effort (e_i^*)*	$-$
ii	*Performance ($x_i(e_i^*)$)*	$-$
iii	*Win Probability ($p_i(e^*)$)*	$-$
iv	*Draw Probability ($p_D(e^*)$)*	$+$
v	*Expected Payoff ($\pi_i(e^*)$)*	$+$

We prove each comparative static in turn;

i) EFFORTS ($\frac{\partial e_i{}^*}{\partial a} < 0$)

Recall that we have assumed $b < vt(1-a)$ so that contestants' equilibrium efforts are uniquely defined by $f(e_i{}^*) = 0$ where;

$$f(e_i) = vt[b(1-a) + t(n-1)e_i] - (b + tne_i)^2$$

Totally differentiating with respect to a and rearranging, we have;

$$\frac{\partial e_i{}^*}{\partial a} = \frac{vb}{vt(n-1) - 2(b + tne_i{}^*)n}$$

It follows that $\frac{\partial e_i{}^*}{\partial a} < 0$ if and only if;

$$e_i{}^* > \frac{v(n-1)}{2n^2} - \frac{b}{tn} = \bar{e}_1$$

Now, since $f(e_i)$ is concave with $f(e_i{}^*) = 0$ and $f'(e_i{}^*) < 0$, we have that $e_i{}^* > \bar{e}_1$ is implied by $f(\bar{e}_1) > 0$. To see this holds, check that;

$$f(\bar{e}_1) = \left[\frac{vt(n-1)}{2n}\right]^2 + \frac{vtb(1-na)}{n} > 0$$

ii) PERFORMANCES ($\frac{\partial x_i(e_i{}^*)}{\partial a} < 0$)

Recall that contestants' performances are proportional to efforts, $x_i(e_i{}^*) = te_i{}^*$. Thus, $\frac{\partial x_i(e_i{}^*)}{\partial a} = t\frac{\partial e_i{}^*}{\partial a} < 0$.

iii) WIN PROBABILITIES ($\frac{\partial p_i(e^*)}{\partial a} < 0$)

Given symmetry of contestants' efforts, we have $p_i(e^*) = \frac{1-p_D(e^*)}{n}$. Thus, $\frac{\partial p_i(e^*)}{\partial a} = -\frac{1}{n}\frac{\partial p_D(e^*)}{\partial a}$. Next, we prove $\frac{\partial p_D(e^*)}{\partial a} > 0$, so that $\frac{\partial p_i(e^*)}{\partial a} < 0$.

iv) DRAW PROBABILITY ($\frac{\partial p_D(e^*)}{\partial a} > 0$)

With symmetry, we have $p_D(e^*) = \frac{b}{b+tne_i{}^*}$. Differentiating yields

$\frac{\partial p_D(e^*)}{\partial a} = -\frac{bnt}{(b+tne_i{}^*)^2}\frac{\partial e_i{}^*}{\partial a} > 0.$

v) EXPECTED PAYOFFS ($\frac{\partial \pi_i(e^*)}{\partial a} > 0$)

With symmetry, we have $\pi_i(e) = v\frac{ab+te_i{}^*}{b+tne_i{}^*} - e_i{}^*$. Differentiating yields;

$$\frac{\partial \pi_i(e^*)}{\partial a} = \frac{v}{(b+tne_i{}^*)^2}\left\{\left(b + t\frac{\partial e_i{}^*}{\partial a}\right)(b+tne_i{}^*) - tn\frac{\partial e_i{}^*}{\partial a}(ab+te_i{}^*)\right\} - \frac{\partial e_i{}^*}{\partial a}$$

Recalling that $\frac{\partial e_i{}^*}{\partial a} < 0$, we have $\frac{\partial \pi_i(e^*)}{\partial a} > 0$ if $\left(b + t\frac{\partial e_i{}^*}{\partial a}\right)(b+tne_i{}^*) - tn\frac{\partial e_i{}^*}{\partial a}(ab + te_i{}^*) > 0$. This sufficient condition is equivalent to;

$$\frac{\partial e_i{}^*}{\partial a} > -\frac{b+tne_i{}^*}{t(1-na)}$$

Recalling that $\frac{\partial e_i{}^*}{\partial a} = \frac{vb}{vt(n-1)-2(b+tne_i{}^*)n} < 0$ and rearranging gives;

$$2(b+tne_i{}^*)^2 > vt[b(1-a) + (n-1)te_i{}^*]$$

Now, note that $f(e_i{}^*) = 0 \Rightarrow vt[b(1-a) + (n-1)te_i{}^*] = (b+tne_i{}^*)^2$. Thus, our sufficient condition for $\frac{\partial \pi_i(e^*)}{\partial a} > 0$ is $(b+tne_i{}^*)^2 > 0$.

12.3 <u>Proposition 1.1b</u>

This sub-section provides a proof of;

Proposition 1.1b: Comparative Statics for the Draw Likelihood with Homogeneous Contestants

#	***Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Likelihood ($\frac{\partial z}{\partial b}$)***
i	*Effort (e_i^*)*	$-$
ii	*Performance ($x_i(e_i^*)$)*	$-$
iii	*Win Probability ($p_i(e^*)$)*	$-$
iv	*Draw Probability ($p_D(e^*)$)*	$+$
v	*Expected Payoff ($\pi_i(e^*)$)*	$-$ *when* $b < \bar{b}_\pi$ $+$ *when* $b > \bar{b}_\pi$

where $\bar{b}_\pi = \frac{vt[2-(n+1)a][2-n(n+1)a]}{4(1-na)} \leq vt(1-a)$

We prove each comparative static in turn;

i) EFFORTS ($\frac{\partial e_i^*}{\partial b} < 0$)

Recall that we have assumed $b < vt(1-a)$ so that contestants' equilibrium efforts are uniquely defined by $f(e_i^*) = 0$ where;

$$f(e_i) = vt[b(1-a) + t(n-1)e_i] - (b + tne_i)^2$$

Totally differentiating with respect to b and rearranging, we have;

$$\frac{\partial e_i^*}{\partial b} = \frac{1}{t}\left[\frac{2(b + tne_i^*) - vt(1-a)}{vt(n-1) - 2(b + tne_i^*)n}\right]$$

Note from part i) of the proof for Proposition 1.1a that $vt(n-1) - 2(b + tne_i^*)n < 0$.

Thus, $\frac{\partial e_i^*}{\partial b} < 0 \Leftrightarrow 2(b + tne_i^*) > vt(1-a)$, which holds if and only if;

$$e_i{}^* > \frac{v(1-a)}{2n} - \frac{b}{tn} = \bar{e}_2$$

Now, since $f(e_i)$ is concave with $f(e_i{}^*) = 0$ and $f'(e_i{}^*) < 0$, we have that $e_i{}^* > \bar{e}_2$ is implied by $f(\bar{e}_2) > 0$. To see this holds, check that;

$$f(\bar{e}_2) = \frac{(vt)^2(1-a)(n-2+na)}{4n} + \frac{vtb(1-na)}{n} > 0$$

ii) PERFORMANCES ($\frac{\partial x_i(e_i{}^*)}{\partial b} < 0$))

Recall that contestants' performances are proportional to efforts, $x_i(e_i{}^*) = te_i{}^*$. Thus, $\frac{\partial x_i(e_i{}^*))}{\partial b} = t\frac{\partial e_i{}^*}{\partial b} < 0$.

iii) WIN PROBABILITIES ($\frac{\partial p_i(e^*)}{\partial b} < 0$)

Given symmetry of contestants' efforts, we have $p_i(e^*) = \frac{1-p_D(e^*)}{n}$. Thus, $\frac{\partial p_i(e^*)}{\partial b} = -\frac{1}{n}\frac{\partial p_D(e^*)}{\partial b}$. Next, we prove $\frac{\partial p_D(e^*)}{\partial b} > 0$, so that $\frac{\partial p_i(e^*)}{\partial b} < 0$.

iv) DRAW PROBABILITY ($\frac{\partial p_D(e^*)}{\partial b} > 0$)

With symmetry, we have $p_D(e^*) = \frac{b}{b+tne_i{}^*}$. Differentiating yields $\frac{\partial p_D(e^*)}{\partial b} = \frac{nt}{(b+tne_i{}^*)^2}\left[e_i{}^* - b\frac{\partial e_i{}^*}{\partial b}\right] > 0$ (since $\frac{\partial e_i{}^*}{\partial b} < 0$).

v) EXPECTED PAYOFFS ($\frac{\partial \pi_i(e^*)}{\partial b} \underset{(<)}{>} 0$ when $b \underset{(<)}{>} \bar{b}_\pi$)

With symmetry, we have $\pi_i(e) = v\frac{ab+te_i{}^*}{b+tne_i{}^*} - e_i{}^*$. Differentiating yields;

$$\frac{\partial \pi_i(e^*)}{\partial b} = \frac{v}{(b+tne_i{}^*)^2}\left\{\left(a + t\frac{\partial e_i{}^*}{\partial b}\right)(b+tne_i{}^*) - \left(1 + tn\frac{\partial e_i{}^*}{\partial b}\right)(ab+te_i{}^*)\right\} - \frac{\partial e_i{}^*}{\partial b}$$

Rearranging, we have $\frac{\partial \pi_i(e^*)}{\partial b} = \frac{vt(1-na)}{(b+tne_i^*)^2}\left\{b\frac{\partial e_i^*}{\partial b} - e_i^*\right\} - \frac{\partial e_i^*}{\partial b}$. Therefore, $\frac{\partial \pi_i(e^*)}{\partial b} \underset{(<)}{>} 0 \Leftrightarrow$ $[vtb(1-na) - (b+tne_i^*)^2]\frac{\partial e_i^*}{\partial b} \underset{(<)}{>} vt(1-na)e_i^*$. Recalling that $\frac{\partial e_i^*}{\partial b} = \frac{1}{t}\left[\frac{2(b+tne_i^*)-vt(1-a)}{vt(n-1)-2(b+tne_i^*)n}\right] < 0$, this condition is equivalent to;

$$\left[vtb(1-na) - z^{*2}\right][2z^* - vt(1-a)] \underset{(>)}{<} vt^2(1-na)e_i^*[vt(n-1) - 2z^*n]$$

where $z^* = b + tne_i^*$. Rearranging, we have;

$$vt[3-(2n+1)a]z^{*2} - 2z^{*3} \underset{(>)}{<} v^2t^2(1-na)[b(1-a) + (n-1)te_i^*]$$

Now, note that $f(e_i^*) = 0 \Rightarrow vt[b(1-a) + (n-1)te_i^*] = z^{*2}$. Thus, our if and only if condition for $\frac{\partial \pi_i(e^*)}{\partial b} \underset{(<)}{>} 0$ is $z^* \underset{(<)}{>} \frac{vt[2-(n+1)a]}{2}$ or equivalently;

$$e_i^* \underset{(<)}{>} \frac{v[2-(n+1)a]}{2n} - \frac{b}{tn} = \bar{e}_3$$

Next, check that $f'(\bar{e}_3) = vt^2(n+1)(na-1) < 0$. It follows that[63], since $f(e_i)$ is strictly concave with $f(e_i^*) = 0$ and $f'(e_i^*) < 0$, we have $e_i^* \underset{(<)}{>} \bar{e}_3$ if and only if $f(\bar{e}_3) \underset{(<)}{>} 0$. We have;

$$f(\bar{e}_3) = \left(\frac{vt}{2}\right)^2\left[\frac{2-(n+1)a}{n}\right][n(n+1)a - 2] + \frac{vtb(1-na)}{n}$$

Thus, we have $f(\bar{e}_3) \underset{(<)}{>} 0$ if and only if;

[63] To see the logic here, note that, since $f(.)$ is concave, $\bar{e}_3$ and e_i^* must be on the same downward sloping portion of the domain for $f(.)$ (i.e., $f'(e_i) < 0$ for any $e_i > \min\{\bar{e}_3, e_i^*\}$). Thus, we have $e_i^* \underset{(<)}{>} \bar{e}_3$ if and only if $f(\bar{e}_3) \underset{(<)}{>} f(e_i^*) = 0$.

$$b \underset{(<)}{>} \frac{vt[2-(n+1)a][2-n(n+1)a]}{4(1-na)} = \bar{b}_\pi$$

This completes the proof that $\frac{\partial \pi_i(e^*)}{\partial b} \underset{(<)}{>} 0$ when $b \underset{(<)}{>} \bar{b}_\pi$. Next, we prove the lemma associated with this comparative static;

Lemma 1: $a \in [0, \frac{1}{n^2}) \Rightarrow argmax_{b \in (0, vt(1-a))} \pi_i(e^*) \to 0$ *and*

$a \in \left(\frac{1}{n^2}, \frac{1}{n}\right) \Rightarrow argmax_{b \in (0, vt(1-a))} \pi_i(e^*) \to vt(1-a)$.

For this, note that $\frac{\partial \pi_i(e^*)}{\partial b} \underset{(<)}{>} 0$ when $b \underset{(<)}{>} \bar{b}_\pi$ implies that $argmax_{b \in (0, vt(1-a))} \pi_i(e^*) \vec{\in} \{0, vt(1-a)\}$. From the standard Tullock contest result, we have $\lim_{b \to 0} \pi_i(e^*) = \frac{v}{n^2}$. Meanwhile, since $\lim_{b \to vt(1-a)} {e_i}^* = 0$, we have $\lim_{b \to vt(1-a)} \pi_i(e^*) = av$. It follows that $a \in [0, \frac{1}{n^2}) \Rightarrow argmax_{b \in (0, vt(1-a))} \pi_i(e^*) \to 0$ and $a \in \left(\frac{1}{n^2}, \frac{1}{n}\right) \Rightarrow argmax_{b \in (0, vt(1-a))} \pi_i(e^*) \to vt(1-a)$.

Here, we also show that $a = 0 \Rightarrow \frac{\partial \pi_i(e^*)}{\partial b} < 0$ and $a \in \left(\frac{2}{n(n+1)}, \frac{1}{n}\right) \Rightarrow \frac{\partial \pi_i(e^*)}{\partial b} > 0$. To see this, first, note that $a = 0 \Rightarrow \bar{b}_\pi = vt$. Meanwhile, when $a = 0$, our initial participation constraint is $b < vt$. Thus $a = 0 \Rightarrow b < \bar{b}_\pi \Rightarrow \frac{\partial \pi_i(e^*))}{\partial b} < 0$. Second, note that $a \in \left(\frac{2}{n(n+1)}, \frac{1}{n}\right) \Rightarrow \bar{b}_\pi < 0 < b \Rightarrow \frac{\partial \pi_i(e^*))}{\partial b} > 0$.

12.4 Proposition 1.2a

This sub-section provides a proof of;

Proposition 1.2a: Comparative Statics on Employers' Expenditures/Profits with Homogeneous Contestants (Draw Prize)

#	***Employer's Equilibrium Outcome (z)***	***Effect of the Draw Prize ($\frac{\partial z}{\partial a}$)***
i	*Expected Expenditure ($E(e^*)$)*	$+$
ii	*Expected Profit ($Q(e^*)$)*	$-$

We prove each comparative static in turn;

i) EXPENDITURES ($\frac{\partial E(e^*)}{\partial a} > 0$)

To see this, note that, given our symmetric equilibrium, we have $E(e^*) = v[np_i(e^*) + nap_D(e^*)]$. Recall further that $np_i(e^*) + p_D(e^*) = 1$, so that $E(e^*) = v[1 - (1 - na)p_D(e^*)]$. It follows that $\frac{\partial E(e^*)}{\partial a} = v\left[np_D(e^*) - (1 - na)\frac{\partial p_D(e^*)}{\partial a}\right]$. Thus, we have $\frac{\partial E(e^*)}{\partial a} > 0$ if and only if;

$$\frac{\partial p_D(e^*)}{\partial a} < \frac{np_D(e^*)}{(1 - na)}$$

Recalling that $\frac{\partial p_D(e^*)}{\partial a} = -\frac{bnt}{(b+tne_i{}^*)^2}\frac{\partial e_i{}^*}{\partial a} = -\frac{p_D(e^*)nt}{b+tne_i{}^*}\frac{\partial e_i{}^*}{\partial a}$, this condition is equivalent to;

$$\frac{\partial e_i{}^*}{\partial a} > -\frac{b + tne_i{}^*}{t(1 - na)}$$

Next, recalling that $\frac{\partial e_i{}^*}{\partial a} = \frac{vb}{vt(n-1)-2(b+tne_i{}^*)n} < 0$ and rearranging gives;

$$vt[b(1 - a) + (n - 1)te_i{}^*] < 2(b + tne_i{}^*)^2$$

Now, note that $f(e_i{}^*) = 0 \Rightarrow vt[b(1-a) + (n-1)te_i{}^*] = (b + tne_i{}^*)^2$. Thus, our condition for $\frac{\partial E(e^*)}{\partial a} > 0$ becomes $(b + tne_i{}^*)^2 > 0$, which necessarily holds.

ii) PROFITS ($\frac{\partial Q(e^*)}{\partial a} < 0$)

To see this, recall that in our symmetric equilibrium $Q(e^*) = nx_i(e_i{}^*) - E(e^*)$. Thus, $\frac{\partial Q(e^*)}{\partial a} = n\frac{\partial x_i(e_i{}^*)}{\partial a} - \frac{\partial E(e^*)}{\partial a} < 0$.

12.5 <u>Proposition 1.2b</u>

This sub-section provides a proof of;

Proposition 1.2b: Comparative Statics on Employers' Expenditures/Profits with Homogeneous Contestants (Draw Likelihood)

#	***Employer's Equilibrium Outcome (z)***	***Effect of the Draw Likelihood ($\frac{\partial z}{\partial b}$)***
i	*Expected Expenditure ($E(e^*)$)*	$-$
ii	*Expected Profit ($Q(e^*)$)*	$+$ *when* $b < \bar{b}_Q{}^{Hom}$ $-$ *when* $b > \bar{b}_Q{}^{Hom}$

where $\bar{b}_Q{}^{Hom} = \frac{v[t(1-a)+(1-na)][n-an(t+n)-t(n-2)]}{4t(1-na)}$*)*

We prove each comparative static in turn;

i) EXPENDITURES ($\frac{\partial E(e^*)}{\partial b} < 0$)

Recall that in our symmetric equilibrium $E(e^*) = v[1-(1-na)p_D(e^*)]$. It follows that $\frac{\partial E(e^*)}{\partial b} = -v(1-na)\frac{\partial p_D(e^*)}{\partial b} < 0$.

ii) PROFITS ($\frac{\partial Q(e^*)}{\partial b}\underset{(<)}{>}0$ when $b\underset{(>)}{<}\bar{b}_Q{}^{Hom}$)

Recall that in our symmetric equilibrium $Q(e^*) = nx_i(e_i{}^*) - E(e^*)$. Thus, $\frac{\partial Q(e^*)}{\partial b} = nt\frac{\partial e_i{}^*}{\partial b} + v(1-na)\frac{\partial p_D(e^*)}{\partial b}$. Now, recall that $\frac{\partial p_D(e^*)}{\partial b} = \frac{nt}{z^{*2}}\left[e_i{}^* - b\frac{\partial e_i{}^*}{\partial b}\right]$, where $z^* = b + tne_i{}^*$. Thus, $\frac{\partial Q(e^*)}{\partial b}\underset{(<)}{>}0$ is equivalent to;

$$\left[z^{*2} - bv(1-na)\right]\frac{\partial e_i{}^*}{\partial b}\underset{(<)}{>} - v(1-na)e_i{}^*$$

Recalling that $\frac{\partial e_i{}^*}{\partial b} = \frac{1}{t}\left[\frac{2z^* - vt(1-a)}{vt(n-1) - 2z^*n}\right]$ with $vt(n-1) - 2z^*n < 0$ and rearranging yields;

$$\left[z^{*2} - bv(1-na)\right]\left[2z^* - vt(1-a)\right]\underset{(>)}{<}vt(1-na)e_i{}^*\left[2z^*n - vt(n-1)\right]$$

Which we can rewrite as a cubic function in z^*;

$$2z^{*3} - vt(1-a)z^{*2} - 2v(1-na)[b + tne_i{}^*]z^*$$
$$+ v^2t(1-na)[b(1-a) + t(n-1)e_i{}^*]\underset{(>)}{<}0$$

Again, noting that $b + tne_i{}^* = z^*$ and that $f(e_i{}^*) = 0 \Rightarrow vt[b(1-a) + (n-1)te_i{}^*] = z^{*2}$, this is equivalent to $z^*\underset{(>)}{<}\frac{v[t(1-a)+(1-na)]}{2}$ or alternatively;

$$e_i{}^*\underset{(>)}{<}\frac{v[t(1-a) + (1-na)]}{2tn} - \frac{b}{tn} = \bar{e}_4$$

Next, check that $f'(\bar{e}_4) = vt(n+t)(na-1) < 0$. It follows that, since $f(e_i)$ is strictly concave with $f(e_i{}^*) = 0$ and $f'(e_i{}^*) < 0$, we have $e_i{}^*\underset{(>)}{<}\bar{e}_4$ if and only if $f(\bar{e}_4)\underset{(>)}{<}0$. Now, note that we have;

$$f(\bar{e}_4) = \left(\frac{v}{2}\right)^2 \left[\frac{t(1-a)+(1-na)}{n}\right][an(t+n)+t(n-2)-n] + \frac{vtb(1-na)}{n}$$

Thus, we have $f(\bar{e}_4) \underset{(>)}{<} 0$ if and only if;

$$b \underset{(>)}{<} \frac{v[t(1-a)+(1-na)][n-an(t+n)-t(n-2)]}{4t(1-na)} = \bar{b}_Q{}^{Hom}$$

13. Appendix 2: Heterogeneous Equilibrium

This appendix contains proofs for the Nash equilibrium with heterogeneous contestants.

13.1 Proposition 2

This sub-section provides a proof of;

Proposition 2: Nash Equilibrium with Heterogeneous Contestants

With heterogeneous contestants, there exists a unique Nash equilibrium, e^*, in which either i) $e_s{}^* > 0$ and $e_w{}^* > 0$, ii) $e_s{}^* > 0$ and $e_w{}^* = 0$ or iii) $e_s{}^* = e_w{}^* = 0$.

If $b < \frac{vt_s(1-a)}{\left[(1-a)\frac{t_s}{t_w}+a\right]^2} = \bar{b}_w$, we have that i) each contestant's equilibrium effort is positive, $e_s{}^* > 0$ and $e_w{}^* > 0$, and is implicitly defined by $f_s(e_s{}^*) = 0$ and $f_w(e_w{}^*) = 0$, where;

$$f_s(e_s) = vt_w(b(1-a) + t_s e_s) - \left[b(1-a)\frac{t_w}{t_s} + ba + Te_s\right]^2$$

$$f_w(e_w) = vt_s(b(1-a) + t_w e_w) - \left[b(1-a)\frac{t_s}{t_w} + ba + Te_w\right]^2$$

If $\bar{b}_w \leq b < vt_s(1-a) = \bar{b}_s$, we have that ii) the strong contestant's equilibrium effort is positive, $e_s{}^* = \frac{[vt_s b(1-a)]^{\frac{1}{2}}-b}{t_s} > 0$, while the weak contestant's equilibrium effort is zero, $e_w{}^* = 0$.

If $b \geq \bar{b}_s$, we have that iii) both contestants' equilibrium effort is zero, $e_s{}^* = e_w{}^* = 0$.

First note that $\pi_i(e) = v\frac{ab+t_i e_i}{b+t_s e_s+t_w e_w} - e_i$ is strictly concave in e_i. To see this, check that our first order derivative is;

$$\frac{\partial \pi_i(e)}{\partial e_i} = vt_i \frac{b(1-a) + t_j e_j}{(b + t_s e_s + t_w e_w)^2} - 1$$

so that our second order derivative is strictly negative;

$$\frac{\partial^2 \pi_i(e)}{\partial {e_i}^2} = -2v{t_i}^2 \frac{b(1-a) + t_j e_j}{(b + t_s e_s + t_w e_w)^3} < 0$$

We then have four different potential equilibrium types;

TYPE 1: NON-PARTICIPATION EQUILIBRIUM (${e_s}^* = {e_w}^* = 0$)

By concavity of $\pi_i(e)$, there exists a non-participation equilibrium, e^*, in which ${e_s}^* = {e_w}^* = 0$, if and only if i) $\frac{\partial \pi_s(0,0)}{\partial e_s} \leq 0 \Leftrightarrow b \geq vt_s(1-a)$ and ii) $\frac{\partial \pi_w(0,0)}{\partial e_w} \leq 0 \Leftrightarrow b \geq vt_w(1-a)$. Since $t_s > t_w$, we have that i) ⇒ ii). Thus, we have a non-participation equilibrium if and only if $b \geq vt_s(1-a) = \bar{b}_s$.

TYPE 2: STRONG NON-PARTICIPATION EQUILIBRIUM (${e_s}^* = 0$, ${e_w}^* > 0$)

By concavity of $\pi_i(e)$, there exists a strong non-participation equilibrium, e^*, in which ${e_s}^* = 0$ and ${e_w}^* > 0$, if and only if i) $\frac{\partial \pi_s(0,{e_w}^*)}{\partial e_s} \leq 0 \Leftrightarrow vt_s \frac{b(1-a)+t_w {e_w}^*}{(b+t_w {e_w}^*)^2} \leq 1$ and ii) $\frac{\partial \pi_w(0,{e_w}^*)}{\partial e_w} = 0 \Leftrightarrow vt_w \frac{b(1-a)}{(b+t_w {e_w}^*)^2} = 1$. However, note that, if ii) holds, then i) is equivalent to $t_s[b(1-a) + t_w {e_w}^*] \leq t_w b(1-a)$. Thus, since ${e_w}^* > 0$ and $t_s > t_w$, we have a contradiction and can exclude the possibility of a strong non-participation equilibrium.

TYPE 3: WEAK NON-PARTICIPATION EQUILIBRIUM (${e_s}^* > 0$, ${e_w}^* = 0$)

By concavity of $\pi_i(e)$, there exists a weak non-participation equilibrium, e^*, in which ${e_s}^* > 0$ and ${e_w}^* = 0$, if and only if i) $\frac{\partial \pi_s({e_s}^*,0)}{\partial e_s} = 0 \Leftrightarrow vt_s \frac{b(1-a)}{(b+t_s {e_s}^*)^2} = 1$ and ii) $\frac{\partial \pi_w({e_s}^*,0)}{\partial e_w} \leq 0 \Leftrightarrow vt_w \frac{b(1-a)+t_s {e_s}^*}{(b+t_s {e_s}^*)^2} \leq 1$. Note that i) implies ${e_s}^* = \frac{[vt_s b(1-a)]^{\frac{1}{2}} - b}{t_s}$. Thus, since ${e_s}^* > 0$, we must have $[vt_s b(1-a)]^{\frac{1}{2}} > b \Leftrightarrow b < vt_s(1-a) = \bar{b}_s$. Further, note that, if i) holds, then $b + t_s e_s{}^* = [vt_s b(1-a)]^{\frac{1}{2}}$, so that, after rearrangement, ii) is equivalent to $b \geq \frac{vt_s(1-a)}{\left[(1-a)\frac{t_s}{t_w}+a\right]^2} = \bar{b}_w$. Thus, we have a weak non-participation equilibrium if and only if $\bar{b}_w \leq b < \bar{b}_s$. Note that this is necessarily a non-empty interval since $\left[(1-a)\frac{t_s}{t_w} + a\right]^2 > 1 \Rightarrow \bar{b}_s > \bar{b}_w$.

TYPE 4: PARTICIPATION EQUILIBRIUM ($e_s{}^* > 0$, $e_w{}^* > 0$)

By concavity of $\pi_i(e)$, there exists a participation equilibrium, e^*, in which $e_s{}^* > 0$ and $e_w{}^* > 0$, if and only if i) $\frac{\partial \pi_s(e_s{}^*, e_w{}^*)}{\partial e_s} = 0 \Leftrightarrow vt_s \frac{b(1-a)+t_w e_w{}^*}{(b+t_s e_s{}^*+t_w e_w{}^*)^2} = 1$ and ii) $\frac{\partial \pi_w(e_s{}^*, e_w{}^*)}{\partial e_w} = 0 \Leftrightarrow vt_w \frac{b(1-a)+t_s e_s{}^*}{(b+t_s e_s{}^*+t_w e_w{}^*)^2} = 1$. Combining i) and ii), we have $t_s[b(1-a) + t_w e_w{}^*] = t_w[b(1-a) + t_s e_s{}^*]$, which implies;

$$e_s{}^* = e_w{}^* + \frac{b(1-a)(t_s - t_w)}{t_w t_s} \Leftrightarrow e_w{}^* = e_s{}^* - \frac{b(1-a)(t_s - t_w)}{t_w t_s}$$

Substituting, i) and ii) hold respectively if and only if $f_s(e_s{}^*) = 0$ and $f_w(e_w{}^*) = 0$ where;

$$f_s(e_s) = vt_w(b(1-a) + t_s e_s) - \left[b(1-a)\frac{t_w}{t_s} + ba + Te_s\right]^2$$

$$f_w(e_w) = vt_s(b(1-a) + t_w e_w) - \left[b(1-a)\frac{t_s}{t_w} + ba + Te_w\right]^2$$

Note that these functions are both strictly concave and continuous. To see this check that our first order derivatives are;

$$f_s{}'(e_s) = vt_s t_w - 2T\left[b(1-a)\frac{t_w}{t_s} + ba + Te_s\right]$$

$$f_w{}'(e_w) = vt_st_w - 2T\left[b(1-a)\frac{t_s}{t_w} + ba + Te_w\right]$$

So that our second order derivatives are strictly negative;

$$f_s{}''(e_s) = f_w{}''(e_w) = -2T^2 < 0$$

Now, let us first suppose that $b \geq \frac{vt_s(1-a)}{\left[(1-a)\frac{t_s}{t_w}+a\right]^2} = \bar{b}_w$. It follows that $f_w(0) \leq 0$. Further, we show that $f_w{}'(0) = vt_st_w - 2T\left[b(1-a)\frac{t_w}{t_s} + ba + Te_s\right] < 0$. For this note that $f_w{}'(0)$ is strictly increasing in a, so that $f_w{}'(0) \leq vt_st_w - \frac{T^2b}{t_w}$ (since $a \leq \frac{1}{2}$). Further, given $b \geq \bar{b}_w$, we have $f_w{}'(0) \leq vt_st_w\left\{1 - \frac{T^2(1-a)}{[(1-a)t_s+at_w]^2}\right\}$. Therefore, $f_w{}'(0) < 0 \Leftrightarrow T^2(1-a) > [(1-a)t_s + at_w]^2$. This holds for any $a \in \left[0, \frac{1}{2}\right]$, since $T^2(1-a) \geq T^2(1-a)^2 \geq [(1-a)t_s + at_w]^2$ with at least one strict inequality for any $a \in \left[0, \frac{1}{2}\right]$. Since $f_w(e_w)$ is concave, it follows that $f_w(e_w{}^*) = 0 \Rightarrow e_w{}^* \leq 0$, which is a contradiction. Thus, we exclude the possibility of a participation equilibrium when $b \geq \bar{b}_w$.

Now, let us suppose that $b < \frac{vt_s(1-a)}{\left[(1-a)\frac{t_s}{t_w}+a\right]^2} = \bar{b}_w$. It follows that $f_w(0) > 0$ and we show that $f_s(0) = vt_wb(1-a) - \left[b(1-a)\frac{t_w}{t_s} + ba\right]^2 > 0$. For this, note that $f_s(0) > 0 \Leftrightarrow b < \frac{vt_w(1-a)}{\left[(1-a)\frac{t_w}{t_s}+a\right]^2}$. Simple algebraic manipulation then reveals that $\frac{vt_w(1-a)}{\left[(1-a)\frac{t_w}{t_s}+a\right]^2} > \frac{vt_s(1-a)}{\left[(1-a)\frac{t_s}{t_w}+a\right]^2} > b$. Thus, we have $f_s(0) > 0$ and $f_w(0) > 0$. Further, $\lim_{e_s\to\infty} f_s(e_s) = \lim_{e_w\to\infty} f_w(e_w) = -\infty$. Thus, by continuity, there exists some $e_s{}^* > 0$ and $e_w{}^* > 0$ with $f_s(e_s{}^*) = 0$ and $f_w(e_w{}^*) = 0$. Concavity of these functions guarantees uniqueness (note that $f_s{}'(e_s{}^*)$ and $f_w{}'(e_w{}^*)$ must be negative). Therefore, we have a unique participation equilibrium, when $b < \bar{b}_w$.

This completes our proof of Proposition 2, which shows that, with heterogeneous contestants, our model has a unique equilibrium, e^*, with ${e_w}^* > 0$ when $b < \bar{b}_w$ and ${e_s}^* > 0$ when $b < \bar{b}_s$ (${e_w}^* = 0$ and ${e_s}^* = 0$ otherwise respectively). Here, we also provide a proof for Lemma 2;

Lemma 2: $b < \bar{b}_s \Rightarrow {e_s}^* > {e_w}^*$.

First, suppose $\bar{b}_w \leq b < \bar{b}_s$. We have ${e_s}^* > 0$ and ${e_w}^* = 0$, so that ${e_s}^* > {e_w}^*$.

Second, suppose $b < \bar{b}_w$. We have ${e_s}^* > 0$ and ${e_w}^* > 0$. From the proof above, we must have ${e_s}^* = {e_w}^* + \frac{b(1-a)(t_s-t_w)}{t_w t_s}$. Thus, $t_s > t_w \Rightarrow {e_s}^* > {e_w}^*$.

13.2 Proposition 2.1a

This sub-section provides a proof of;

Proposition 2.1a: Comparative Statics for the Draw Prize with Heterogeneous Contestants (Strong Contestant Outcomes)

#	***Strong Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Prize ($\frac{\partial z}{\partial \alpha}$)***
i	*Effort (${e_s}^*$)*	$-$
ii	*Performance ($x_s({e_s}^*)$)*	$-$
iii	*Win Probability ($p_s(e^*)$)*	$-$
iv	*Draw Probability ($p_D(e^*)$)*	$+$
v	*Expected Payoff ($\pi_s(e^*)$)*	$+$

We prove each comparative static in turn;

i) EFFORTS ($\frac{\partial e_s{}^*}{\partial a} < 0$)

Given $b < \bar{b}_w$, the strong contestant's equilibrium effort, $e_s{}^*$, is uniquely defined by $f_s(e_s{}^*) = 0$ where;

$$f_s(e_s) = vt_w(b(1-a) + t_s e_s) - \left[b(1-a)\frac{t_w}{t_s} + ba + Te_s\right]^2$$

Totally differentiating with respect to a and rearranging, we have;

$$\frac{\partial e_s{}^*}{\partial a} = \frac{b}{t_s}\left[\frac{vt_s t_w + 2z_s{}^*(t_s - t_w)}{vt_s t_w - 2z_s{}^* T}\right]$$

where $z_s{}^* = b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^* > 0$. It follows that $\frac{\partial e_s{}^*}{\partial a} < 0$ if and only if $z_s{}^* > \frac{vt_s t_w}{2T}$ or equivalently;

$$e_s{}^* > \frac{vt_s t_w}{2T^2} - \frac{b(1-a)}{T}\frac{t_w}{t_s} - \frac{ba}{T} = \bar{e}_{1s}$$

Now, since $f_s(e_s)$ is concave with $f_s(e_s{}^*) = 0$ and $f_s{}'(e_s{}^*) < 0$, we have that $e_s{}^* > \bar{e}_{1s}$ is implied by $f_s(\bar{e}_{1s}) > 0$. To see this holds, check that;

$$f(\bar{e}_{1s}) = \left[\frac{vt_s t_w}{2T}\right]^2 + \frac{vt_s t_w b(1-2a)}{T} > 0$$

ii) PERFORMANCES ($\frac{\partial x_s(e_s{}^*)}{\partial a} < 0$)

Recall that contestants' performances are proportional to efforts, $x_s(e_s{}^*) = t_s e_s{}^*$. Thus, $\frac{\partial x_s(e_s{}^*)}{\partial a} = t_s\frac{\partial e_s{}^*}{\partial a} < 0$.

iii) WIN PROBABILITIES ($\frac{\partial p_s(e^*)}{\partial a} < 0$)

Recall that the strong contestant has equilibrium win probability $p_s(e^*) = \frac{t_s e_s{}^*}{b+t_s e_s{}^*+t_w e_w{}^*}$.

Differentiating with respect to a and rearranging yields;

$$\frac{\partial p_s(e^*)}{\partial a} = \frac{1}{z^{*2}}\left[t_s \frac{\partial e_s{}^*}{\partial a}(b + t_w e_w{}^*) - t_s t_w \frac{\partial e_w{}^*}{\partial a} e_s{}^*\right]$$

where $z^* = b + t_s e_s{}^* + t_w e_w{}^* > 0$. Now, recall from our proof of Proposition 2 that $e_w{}^* = e_s{}^* - \frac{b(1-a)(t_s-t_w)}{t_w t_s}$ and, therefore, $\frac{\partial e_w{}^*}{\partial a} = \frac{\partial e_s{}^*}{\partial a} + \frac{b(t_s-t_w)}{t_w t_s}$. Substituting and rearranging gives;

$$\frac{\partial p_s(e^*)}{\partial a} = \frac{1}{z^{*2}}\left\{t_s \frac{\partial e_s{}^*}{\partial a}\left[ab + b(1-a)\frac{t_w}{t_s}\right] - b(t_s - t_w)e_s{}^*\right\} < 0$$

Where negativity of this differential follows from $\frac{\partial e_s{}^*}{\partial a} < 0$ and $t_s > t_w$.

iv) DRAW PROBABILITY ($\frac{\partial p_D(e^*)}{\partial a} > 0$)

We show that $\frac{\partial p_D(e^*)}{\partial a} > 0$ in part iv) of the proof for Proposition 2.3a.

v) EXPECTED PAYOFFS ($\frac{\partial \pi_s(e^*)}{\partial a} > 0$)

Recall that the strong contestant has expected equilibrium payoff $\pi_s(e^*) = v\frac{ab+t_s e_s{}^*}{b+t_s e_s{}^*+t_w e_w{}^*} - e_s{}^*$. Check that, since $e_w{}^* = e_s{}^* - \frac{b(1-a)(t_s-t_w)}{t_w t_s}$, we have $z^* = b + t_s e_s{}^* + t_w e_w{}^* = b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^* = z_s{}^*$. Thus;

$$\pi_s(e^*) = v\frac{ab + t_s e_s{}^*}{b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^*} - e_s{}^*$$

Differentiating with respect to a and rearranging yields;

$$\frac{\partial \pi_s(e^*)}{\partial a} = \frac{vbt_w}{{z_s^*}^2 t_s}\left[t_s \frac{\partial e_s^*}{\partial a}(1-2a) + b + 2t_s e_s^*\right] - \frac{\partial e_s^*}{\partial a}$$

Since $\frac{\partial e_s^*}{\partial a} < 0$, $\frac{\partial \pi_s(e^*)}{\partial a} > 0$ is implied by $t_s \frac{\partial e_s^*}{\partial a}(1-2a) + b + 2t_s e_s^* > 0$, which is equivalent to;

$$\frac{\partial e_s^*}{\partial a} > -\frac{b + 2t_s e_s^*}{t_s(1-2a)}$$

Recalling that $\frac{\partial e_s^*}{\partial a} = \frac{b}{t_s}\left[\frac{vt_s t_w + 2z_s^*(t_s - t_w)}{vt_s t_w - 2z_s^* T}\right] < 0$, this is equivalent to;

$$(b + 2t_s e_s^*)[2z_s^* T - vt_s t_w] > b(1-2a)[vt_s t_w + 2z_s^*(t_s - t_w)]$$

Rearranging gives;

$$2z_s^*\left[b(1-a)\frac{t_w}{t_s} + ba + Te_s^*\right] > vt_w(b(1-a) + t_s e_s^*)$$

Now, recall that $z_s^* = \left[b(1-a)\frac{t_w}{t_s} + ba + Te_s^*\right]$ and that $f_s(e_s^*) = 0 \Rightarrow vt_w(b(1-a) + t_s e_s^*) = {z_s^*}^2$. Our sufficient condition for $\frac{\partial \pi_s(e^*)}{\partial a} > 0$ then becomes ${z_s^*}^2 > 0$.

13.3 Proposition 2.2a

This sub-section provides a proof of;

Proposition 2.2a: Comparative Statics for the Draw Prize with Heterogeneous Contestants (Weak Contestant Outcomes)

#	***Weak Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Prize ($\frac{\partial z}{\partial a}$)***
i	*Effort* ($e_w{}^*$)	$+$ *when* $a < \bar{a}_{e_w}$ $-$ *when* $a > \bar{a}_{e_w}$
ii	*Performance* ($x_w(e_w{}^*)$)	$+$ *when* $a < \bar{a}_{e_w}$ $-$ *when* $a > \bar{a}_{e_w}$
iii	*Win Probability* ($p_w(e^*)$)	$+$ *when* $a < \bar{a}_{p_w}$ $-$ *when* $a > \bar{a}_{p_w}$
iv	*Draw Probability* ($p_D(e^*)$)	$+$
v	*Expected Payoff* ($\pi_w(e^*)$)	$+$

where $\bar{a}_{e_w} = \frac{1}{2} + \frac{vt_st_w}{4b(t_s-t_w)}\left[1 - \frac{T}{2(t_s-t_w)}\right] < \bar{a}_{p_w} = \frac{1}{2} + \left[\frac{vt_st_w}{4b(t_s-t_w)}\right]^{\frac{1}{2}} - \frac{T}{2(t_s-t_w)}$

We prove each comparative static in turn;

i) EFFORTS ($\frac{\partial e_w{}^*}{\partial a} \underset{(<)}{>} 0$ when $a \underset{(>)}{<} \bar{a}_{e_w}$)

Given $b < \bar{b}_w$, the weak contestant's equilibrium effort, $e_w{}^*$, is uniquely defined by $f_w(e_w{}^*) = 0$ where;

$$f_w(e_w) = vt_s(b(1-a) + t_we_w) - \left[b(1-a)\frac{t_s}{t_w} + ba + Te_w\right]^2$$

Totally differentiating with respect to a and rearranging, we have;

$$\frac{\partial e_w{}^*}{\partial a} = \frac{b}{t_w}\left[\frac{vt_st_w - 2z_w{}^*(t_s-t_w)}{vt_st_w - 2z_w{}^*T}\right]$$

Where $z_w{}^* = b(1-a)\frac{t_s}{t_w} + ba + Te_w{}^*$. Now note that, since $e_w{}^* = e_s{}^* - \frac{b(1-a)(t_s-t_w)}{t_w t_s}$,

$z_w{}^* = b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^* = z_s{}^*$. Thus, we write;

$$\frac{\partial e_w{}^*}{\partial a} = \frac{b}{t_w}\left[\frac{vt_st_w - 2z_w{}^*(t_s-t_w)}{vt_st_w - 2z_s{}^*T}\right]$$

Further, recall from part i) of the proof for proposition 2.1a that $vt_st_w - 2z_s{}^*T < 0$. It follows that $\frac{\partial e_w{}^*}{\partial a} \underset{(<)}{>} 0 \Leftrightarrow z_w{}^* \underset{(<)}{>} \frac{vt_st_w}{2(t_s-t_w)}$, which is equivalent to;

$$e_w{}^* \underset{(<)}{>} \frac{vt_st_w}{2(t_s-t_w)T} - \frac{b(1-a)}{T}\frac{t_s}{t_w} - \frac{ba}{T} = \bar{e}_{1w}$$

Next, check that $f_w'(\bar{e}_{1w}) = -\frac{2vt_st_w{}^2}{t_s-t_w} < 0$. It follows that, since $f_w(e_w)$ is concave with $f_w(e_w{}^*) = 0$ and $f_w{}'(e_w{}^*) < 0$ we have $e_w{}^* \underset{(<)}{>} \bar{e}_{1w}$ if and only if $f_w(\bar{e}_{1w}) \underset{(<)}{>} 0$. Since;

$$f_w(\bar{e}_{1w}) = \frac{(vt_st_w)^2}{2(t_s-t_w)}\left[\frac{1}{T} - \frac{1}{2(t_s-t_w)}\right] + \frac{vt_st_wb(1-2a)}{T}$$

we have $f_w(\bar{e}_{1w}) \underset{(<)}{>} 0$ if and only if;

$$a \underset{(>)}{<} \frac{1}{2} + \frac{vt_st_w}{4b(t_s-t_w)}\left[1 - \frac{T}{2(t_s-t_w)}\right] = \bar{a}_{e_w}$$

ii) PERFORMANCES ($\frac{\partial x_w(e_w{}^*)}{\partial a} \underset{(<)}{>} 0$ when $a \underset{(>)}{<} \bar{a}_{e_w}$)

Recall that contestants' performances are proportional to efforts, $x_w(e_w{}^*) = t_we_w{}^*$. Thus, $\frac{\partial x_w(e_w{}^*)}{\partial a} = t_w\frac{\partial e_w{}^*}{\partial a}$. From the previous part of this proof we have $\frac{\partial e_w{}^*}{\partial a} \underset{(<)}{>} 0$ when $a \underset{(>)}{<} \bar{a}_{e_w}$.

iii) WIN PROBABILITIES ($\frac{\partial p_w(e^*)}{\partial a} \underset{(<)}{>} 0$ when $a \underset{(>)}{<} \bar{a}_{p_w}$)

Recall that the weak contestant has equilibrium win probability $p_w(e^*) = \frac{t_w e_w{}^*}{b + t_s e_s{}^* + t_w e_w{}^*}$. Note that, since $e_s{}^* = e_w{}^* + \frac{b(1-a)(t_s - t_w)}{t_w t_s}$, we have $b + t_s e_s{}^* + t_w e_w{}^* = b(1-a)\frac{t_s}{t_w} + ba + Te_w{}^* = z_w{}^*$. Thus, $p_w(e^*) = \frac{t_w e_w{}^*}{z_w{}^*}$. Differentiating with respect to a and rearranging yields;

$$\frac{\partial p_w(e^*)}{\partial a} = \frac{b}{z_w{}^{*2}}\left\{\frac{\partial e_w{}^*}{\partial a}[(1-a)t_s + at_w] + (t_s - t_w)e_w{}^*\right\}$$

It follows that $\frac{\partial p_w(e^*)}{\partial a} \underset{(<)}{>} 0$ if and only if;

$$\frac{\partial e_w{}^*}{\partial a} \underset{(<)}{>} -\frac{(t_s - t_w)e_w{}^*}{(1-a)t_s + at_w}$$

Recall that $\frac{\partial e_w{}^*}{\partial a} = \frac{b}{t_w}\left[\frac{vt_s t_w - 2z_w{}^*(t_s - t_w)}{vt_s t_w - 2z_w{}^* T}\right]$ where $vt_s t_w - 2z_w{}^* T < 0$. Thus, $\frac{\partial p_w(e^*)}{\partial a} \underset{(<)}{>} 0$ if and only if;

$$b[vt_s t_w - 2z_w{}^*(t_s - t_w)][(1-a)t_s + at_w] \underset{(>)}{<} t_w(t_s - t_w)e_w{}^*[2z_w{}^* T - vt_s t_w]$$

Rearranging, this is equivalent to;

$$vt_s[b(1-a) + t_w e_w{}^*] + \frac{vbt_s t_w}{t_s - t_w} + \underset{(>)}{<} 2z_w{}^*\left[b(1-a)\frac{t_s}{t_w} + ba + Te_w{}^*\right]$$

Now, on the right hand side note that $b(1-a)\frac{t_s}{t_w} + ba + Te_w{}^* = z_w{}^*$. Further, on the left hand side note that $f_w(e_w{}^*) = 0 \Rightarrow vt_s(b(1-a) + t_w e_w{}^*) = z_w{}^{*2}$. Thus, we have $\frac{\partial p_w(e^*)}{\partial a} \underset{(<)}{>} 0$ if and only if $z_w{}^* \underset{(<)}{>} \left[\frac{vbt_s t_w}{t_s - t_w}\right]^{\frac{1}{2}}$ or equivalently;

$$e_w{}^* \underset{(<)}{>} \frac{1}{T}\left[\frac{vbt_st_w}{t_s - t_w}\right]^{\frac{1}{2}} - \frac{b(1-a)}{T}\frac{t_s}{t_w} - \frac{ba}{T} = \bar{e}_{2w}$$

Next, check that;

$$f_w(\bar{e}_{2w}) = \frac{vt_st_wb}{T(t_s - t_w)}\left\{\left[\frac{vt_st_w(t_s - t_w)}{b}\right]^{\frac{1}{2}} - T\right\} + \frac{vt_st_wb(1-2a)}{T}$$

and note that, since $f_w(e_w{}^*) = 0$, we have[64] $\bar{e}_{2w} = e_w{}^* \Leftrightarrow f_w(\bar{e}_{2w}) = 0$, which is equivalent to;

$$a = \frac{1}{2} + \left[\frac{vt_st_w}{4b(t_s - t_w)}\right]^{\frac{1}{2}} - \frac{T}{2(t_s - t_w)} = \bar{a}_{p_w}$$

Further recall that $\frac{\partial e_w{}^*}{\partial a} < 0$ and note that $\frac{\partial \bar{e}_{2w}}{\partial a} > 0$. It follows that $e_w{}^* \underset{(<)}{>} \bar{e}_{2w}$ is equivalent to $a \underset{(>)}{<} \bar{a}_{p_w}$.

This completes our proof that $\frac{\partial p_w(e^*)}{\partial a} \underset{(<)}{>} 0$ when $a \underset{(>)}{<} \bar{a}_{p_w}$. We can now also prove Lemma 3, which relates to this comparative static;

Lemma 3: $t_w < \frac{t_s}{3} \Rightarrow \frac{\partial e_w{}^*}{\partial a} > 0, \frac{\partial x_w(e_w{}^*)}{\partial a} > 0$ *and* $\frac{\partial p_w(e^*)}{\partial a} > 0.$

To see this, first recall from part i) of this proof that $\frac{\partial e_w{}^*}{\partial a} > 0$ if and only if;

$$f_w(\bar{e}_{1w}) = \frac{(vt_st_w)^2}{2(t_s - t_w)}\left[\frac{1}{T} - \frac{1}{2(t_s - t_w)}\right] + \frac{vt_st_wb(1-2a)}{T} > 0$$

[64] Note that $\bar{e}_{2w} = e_w{}^* \Leftrightarrow f_w(\bar{e}_{2w}) = 0$ represents a slight simplification for parsimony. In fact, $f_w(.)$ has two roots. Thus, $\bar{e}_{2w} = e_w{}^*$ is actually equivalent to $f_w(\bar{e}_{2w}) = 0$ with $\bar{e}_{2w}$ being the largest of the two possible roots. For this reason, we take the positive root of $\left[\frac{vt_st_w}{4b(t_s - t_w)}\right]^{\frac{1}{2}}$ when evaluating $\bar{a}_{p_w}$ (since $\frac{\partial \bar{e}_{2w}}{\partial a} > 0$).

A sufficient condition for this is $\frac{1}{T} > \frac{1}{2(t_s - t_w)}$ or equivalently $t_w < \frac{t_s}{3}$. Second recall from part ii) of this proof that $\frac{\partial x_w(e_w^*)}{\partial a}$ is proportional to $\frac{\partial e_w^*}{\partial a}$. Finally, recall that $\frac{\partial p_w(e^*)}{\partial a} = \frac{b}{z_w^{*2}}\left\{\frac{\partial e_w^*}{\partial a}[(1-a)t_s + at_w] + (t_s - t_w)e_w^*\right\}$, which is necessarily positive whenever $\frac{\partial e_w^*}{\partial a} > 0$.

iv) DRAW PROBABILITY ($\frac{\partial p_D(e^*)}{\partial a} > 0$)

We show that $\frac{\partial p_D(e^*)}{\partial a} > 0$ in part iv) of the proof for Proposition 2.3a.

v) EXPECTED PAYOFFS ($\frac{\partial \pi_w(e^*)}{\partial a} > 0$)

Recall that the weak contestant has expected equilibrium payoff $\pi_w(e^*) = v\frac{ab + t_w e_w^*}{b + t_s e_s^* + t_w e_w^*} - e_w^*$. Further recall that, since $e_s^* = e_w^* + \frac{b(1-a)(t_s - t_w)}{t_w t_s}$, we have $b + t_s e_s^* + t_w e_w^* = b(1-a)\frac{t_s}{t_w} + ba + Te_w^* = z_w^*$. Thus;

$$\pi_w(e^*) = v\frac{ab + t_w e_w^*}{b(1-a)\frac{t_s}{t_w} + ba + Te_w^*} - e_w^*$$

Differentiating with respect to a and rearranging yields;

$$\frac{\partial \pi_w(e^*)}{\partial a} = \frac{vbt_s}{z_w^{*2} t_w}\left[t_w \frac{\partial e_w^*}{\partial a}(1-2a) + b + 2t_w e_w^*\right] - \frac{\partial e_w^*}{\partial a}$$

Thus, $\frac{\partial \pi_w(e^*)}{\partial a} > 0$ is equivalent to;

$$t_w \frac{\partial e_w^*}{\partial a}\left[vbt_s(1-2a) - z_w^{*2}\right] > -vbt_s[b + 2t_w e_w^*]$$

Recalling that $\frac{\partial e_w^*}{\partial a} = \frac{b}{t_w}\left[\frac{vt_s t_w - 2z_w^*(t_s - t_w)}{vt_s t_w - 2z_w^* T}\right]$ where $vt_s t_w - 2z_w^* T < 0$, we have;

$$[vt_st_w - 2z_w{}^*(t_s - t_w)]\left[vbt_s(1-2a) - z_w{}^{*2}\right] < vt_s[b + 2t_we_w{}^*][2z_w{}^*T - vt_st_w]$$

Rearranging yields;

$$2(t_s - t_w)z_w{}^{*3} - vt_st_wz_w{}^{*2} - 4vt_st_w\left[b(1-a)\frac{t_s}{t_w} + ba + Te_w{}^*\right]$$
$$< -2v^2t_s{}^2t_w(b(1-a) + t_we_w{}^*)$$

Recalling that $\left[b(1-a)\frac{t_s}{t_w} + ba + Te_w{}^*\right] = z_w{}^*$ and that $f_w(e_w{}^*) = 0 \Rightarrow vt_s(b(1-a) + t_we_w{}^*) = z_w{}^{*2}$, we have $\frac{\partial \pi_w(e^*)}{\partial a} > 0$ if and only if $z_w{}^* < \frac{3vt_st_w}{2(t_s - t_w)}$ or equivalently;

$$e_w{}^* < \frac{3vt_st_w}{2T(t_s - t_w)} - \frac{b(1-a)}{T}\frac{t_s}{t_w} - \frac{ba}{T} = \bar{e}_{3w}$$

Next, check that $f_w'(\bar{e}_{3w}) = -\frac{2vt_st_w(t_s + 2t_w)}{t_s - t_w} < 0$. It follows that, since $f_w(e_w)$ is concave with $f_w(e_w{}^*) = 0$ and $f_w{}'(e_w{}^*) < 0$ we have $e_w{}^* < \bar{e}_{3w}$ if and only if $f_w(\bar{e}_{3w}) < 0$. Since;

$$f_w(\bar{e}_{3w}) = \frac{vt_st_wb(1-2a)}{T} - \frac{3(t_s + 5t_w)}{4T}\left[\frac{vt_st_w}{(t_s - t_w)}\right]^2$$

we have $f_w(\bar{e}_{3w}) < 0$ if and only if;

$$b < \frac{3vt_st_w(t_s + 5t_w)}{4(1-2a)(t_s - t_w)^2}$$

Recalling that our participation constraint ensures $b < \frac{vt_s(1-a)}{\left[(1-a)\frac{t_s}{t_w} + a\right]^2}$, this condition must hold if;

$$\frac{3vt_st_w(t_s + 5t_w)}{(t_s - t_w)^2} > \frac{4vt_st_w{}^2(1-a)(1-2a)}{[(1-a)t_s + at_w]^2}$$

To see that this is necessarily true, note that (since $t_s + 5t_w > 6t_w$ and $\frac{(1-a)(1-2a)}{[(1-a)t_s+at_w]^2} < \frac{1}{\left[\frac{t_s+t_w}{2}\right]^2}$ respectively);

$$\frac{3vt_st_w(t_s+5t_w)}{(t_s-t_w)^2} > \frac{18vt_st_w{}^2}{(t_s-t_w)^2} > \frac{16vt_st_w{}^2}{(t_s+t_w)^2} > \frac{4vt_st_w{}^2(1-a)(1-2a)}{[(1-a)t_s+at_w]^2}$$

13.4 Proposition 2.3a

This sub-section provides a proof of;

Proposition 2.3a: Comparative Statics for the Draw Prize with Heterogeneous Contestants (Total Contestant Outcomes)

#	***Total Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Prize ($\frac{\partial z}{\partial a}$)***
i	*Effort ($e_s{}^* + e_w{}^*$)*	$+$ *when* $a < \bar{a}_{e_s+e_w}$ $-$ *when* $a > \bar{a}_{e_s+e_w}$
ii	*Performance ($x_s(e_s{}^*) + x_w(e_w{}^*)$)*	$-$
iii	*Win Probability ($p_s(e^*) + p_w(e^*)$)*	$-$
iv	*Draw Probability ($p_D(e^*)$)*	$+$
v	*Expected Payoff ($\pi_s(e^*) + \pi_w(e^*)$)*	$+$

where $\bar{a}_{e_s+e_w} = \frac{1}{2} + \frac{vt_st_wT}{4b(t_s-t_w)^2}\left[1 - \frac{T^2}{2(t_s-t_w)^2}\right]$

We prove each comparative static in turn;

i) EFFORTS ($\frac{\partial e_s{}^*+e_w{}^*}{\partial a} \underset{(<)}{>} 0$ when $a \underset{(>)}{<} \bar{a}_{e_s+e_w}$)

Recall that $e_w{}^* = e_s{}^* - \frac{b(1-a)(t_s-t_w)}{t_wt_s}$. It follows that $e_s{}^* + e_w{}^* = 2e_s{}^* - \frac{b(1-a)(t_s-t_w)}{t_wt_s}$ and $\frac{\partial e_s{}^*+e_w{}^*}{\partial a} = 2\frac{\partial e_s{}^*}{\partial a} + \frac{b(t_s-t_w)}{t_wt_s}$. Thus, we have $\frac{\partial e_s{}^*+e_w{}^*}{\partial a} \underset{(<)}{>} 0$ if and only if;

$$\frac{\partial e_s{}^*}{\partial a} \underset{(<)}{>} -\frac{b(t_s - t_w)}{2t_w t_s}$$

Recalling that $\frac{\partial e_s{}^*}{\partial a} = \frac{b}{t_s}\left[\frac{vt_st_w + 2z_s{}^*(t_s - t_w)}{vt_st_w - 2z_s{}^*T}\right] < 0$, this is equivalent to;

$$(t_s - t_w)[2z_s{}^*T - vt_st_w] \underset{(<)}{>} 2t_w[vt_st_w + 2z_s{}^*(t_s - t_w)]$$

Rearranging gives $z_s{}^* \underset{(<)}{>} \frac{vt_st_wT}{2(t_s - t_w)^2}$ or equivalently;

$$e_s{}^* \underset{(<)}{>} \frac{vt_st_w}{2(t_s - t_w)^2} - \frac{b(1-a)}{T}\frac{t_w}{t_s} - \frac{ba}{T} = \bar{e}_{2s}$$

Next, check that $f_s'(\bar{e}_{2s}) = vt_st_w\left[1 - \left(\frac{T}{t_s - t_w}\right)^2\right] < 0$. It follows that, since $f_s(.)$ is concave with $f_s(e_s{}^*) = 0$ and $f_s{}'(e_s{}^*) < 0$ we have $e_s{}^* \underset{(<)}{>} \bar{e}_{2s}$ if and only if $f_s(\bar{e}_{2s}) \underset{(<)}{>} 0$. Since;

$$f_s(\bar{e}_{2s}) = \frac{(vt_st_w)^2}{2(t_s - t_w)^2}\left[1 - \frac{T^2}{2(t_s - t_w)^2}\right] + \frac{vt_st_wb(1-2a)}{T}$$

we have $f_s(\bar{e}_{2s}) \underset{(<)}{>} 0$ if and only if;

$$a \underset{(>)}{<} \frac{1}{2} + \frac{vt_st_wT}{4b(t_s - t_w)^2}\left[1 - \frac{T^2}{2(t_s - t_w)^2}\right] = \bar{a}_{e_s + e_w}$$

This completes our proof that $\frac{\partial e_s{}^* + e_w{}^*}{\partial a} \underset{(<)}{>} 0$ when $a \underset{(>)}{<} \bar{a}_{e_s + e_w}$. We now also prove Lemma 4, which relates to this comparative static;

Lemma 4: $t_w < \frac{\sqrt{2}-1}{\sqrt{2}+1} t_s \Rightarrow \frac{\partial e_s{}^* + e_w{}^*}{\partial a} > 0.$

Recall that $\frac{\partial e_s{}^* + e_w{}^*}{\partial a} > 0$ if and only if $f_s(\bar{e}_{2s}) = \frac{(vt_st_w)^2}{2(t_s - t_w)^2}\left[1 - \frac{T^2}{2(t_s - t_w)^2}\right] + \frac{vt_st_wb(1-2a)}{T} > 0$. A sufficient condition for this is $1 - \frac{T^2}{2(t_s - t_w)^2} > 0 \Leftrightarrow t_w < \frac{\sqrt{2}-1}{\sqrt{2}+1} t_s$.

ii) PERFORMANCES ($\frac{\partial x_s(e_s{}^*)+x_w(e_w{}^*)}{\partial a} < 0$)

Again, recall that $e_w{}^* = e_s{}^* - \frac{b(1-a)(t_s-t_w)}{t_w t_s}$. It follows that $x_s(e_s{}^*) + x_w(e_w{}^*) = t_s e_s{}^* + t_w e_w{}^* = T e_s{}^* - \frac{b(1-a)(t_s-t_w)}{t_s}$, so that $\frac{\partial x_s(e_s{}^*)+x_w(e_w{}^*)}{\partial a} = T\frac{\partial e_s{}^*}{\partial a} + \frac{b(t_s-t_w)}{t_s}$. Thus, we have $\frac{\partial x_s(e_s{}^*)+x_w(e_w{}^*)}{\partial a} < 0$ if and only if;

$$\frac{\partial e_s{}^*}{\partial a} < -\frac{b(t_s-t_w)}{T t_s}$$

Recalling that $\frac{\partial e_s{}^*}{\partial a} = \frac{b}{t_s}\left[\frac{v t_s t_w + 2 z_s{}^*(t_s-t_w)}{v t_s t_w - 2 z_s{}^* T}\right] < 0$, this is equivalent to;

$$T[v t_s t_w + 2 z_s{}^*(t_s - t_w)] > (t_s - t_w)[2 z_s{}^* T - v t_s t_w]$$

Rearranging gives $2 v t_s{}^2 t_w > 0$.

iii) WIN PROBABILITIES ($\frac{\partial p_s(e^*)+p_w(e^*)}{\partial a} < 0$)

Recall that $p_s(e^*) + p_w(e^*) = 1 - p_D(e^*)$. Thus, $\frac{\partial p_s(e^*)+p_w(e^*)}{\partial a} < 0 \Leftrightarrow \frac{\partial p_D(e^*)}{\partial a} > 0$, which we prove next.

iv) DRAW PROBABILITY ($\frac{\partial p_D(e^*)}{\partial a} > 0$)

Recall that $p_D(e^*) = \frac{b}{b+x_s(e_s{}^*)+x_w(e_w{}^*)}$. Thus, $\frac{\partial p_D(e^*)}{\partial a} > 0$ is implied by $\frac{\partial x_s(e_s{}^*)+x_w(e_w{}^*)}{\partial a}$.

v) EXPECTED PAYOFFS ($\frac{\partial \pi_s(e^*)+\pi_w(e^*)}{\partial a} > 0$)

This follows from our earlier proofs that $\frac{\partial \pi_s(e^*)}{\partial a} > 0$ and $\frac{\partial \pi_w(e^*)}{\partial a} > 0$ (i.e., $\frac{\partial \pi_s(e^*)+\pi_w(e^*)}{\partial a} = \frac{\partial \pi_s(e^*)}{\partial a} + \frac{\partial \pi_w(e^*)}{\partial a}$).

13.5 <u>Proposition 2.1b</u>

This sub-section provides a proof of;

Proposition 2.1b: Comparative Statics for the Draw Likelihood with Heterogeneous Contestants (Strong Contestant Outcomes)

#	***Strong Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Likelihood*** $(\frac{\partial z}{\partial b})$
i	*Effort* $(e_s{}^*)$	$+$ *when* $b < \bar{b}_{e_s}$ $-$ *when* $b > \bar{b}_{e_s}$
ii	*Performance* $(x_s(e_s{}^*))$	$+$ *when* $b < \bar{b}_{e_s}$ $-$ *when* $b > \bar{b}_{e_s}$
iii	*Win Probability* $(p_s(e^*))$	$-$
iv	*Draw Probability* $(p_D(e^*))$	$+$
v	*Expected Payoff* $(\pi_s(e^*))$	$-$ *when* $b < \bar{b}_{\pi_s}$ $+$ *when* $b > \bar{b}_{\pi_s}$[65]

where $\bar{b}_{e_s} = \frac{vt_st_w}{2(1-2a)(\frac{a}{1-a}t_s+t_w)}\left[\frac{T}{2\left(\frac{a}{1-a}t_s+t_w\right)} - 1\right]$, $\bar{b}_{\pi_s} = \frac{vt_st_w(2-3a)}{2(1-2a)[(1-a)t_w+at_s]}\left\{\frac{T(2-3a)}{2[(1-a)t_w+at_s]} - 1\right\}$

We prove each comparative static in turn;

i) EFFORTS $\left(\frac{\partial e_s{}^*}{\partial b} \underset{(<)}{>} 0 \text{ when } b \underset{(>)}{<} \bar{b}_{e_s}\right)$

Given $b < \bar{b}_w$, the strong contestant's equilibrium effort, $e_s{}^*$, is uniquely defined by $f_s(e_s{}^*) = 0$ where;

$$f_s(e_s) = vt_w(b(1-a) + t_se_s) - \left[b(1-a)\frac{t_w}{t_s} + ba + Te_s\right]^2$$

[65] Note that part v) of Section 13.5 in Appendix 2 shows that $a > \frac{2}{5} \Rightarrow \frac{\partial \pi_s(e^*)}{\partial b} > 0$.

Totally differentiating with respect to b and rearranging, we have;

$$\frac{\partial e_s{}^*}{\partial b} = \frac{1}{t_s}\left[\frac{2z_s{}^*[(1-a)t_w + at_s] - vt_st_w(1-a)}{vt_st_w - 2z_s{}^*T}\right]$$

Recall from part i) of the proof for proposition 2.1a that $vt_st_w - 2z_s{}^*T < 0$. It follows that $\frac{\partial e_s{}^*}{\partial b} \underset{(<)}{>} 0 \Leftrightarrow z_s{}^* \underset{(>)}{<} \frac{vt_st_w(1-a)}{2[(1-a)t_w+at_s]}$, which is equivalent to;

$$e_s{}^* \underset{(>)}{<} \frac{vt_st_w(1-a)}{2T[(1-a)t_w + at_s]} - \frac{b(1-a)}{T}\frac{t_w}{t_s} - \frac{ba}{T} = \bar{e}_{3s}$$

Next, check that $f_s'(\bar{e}_{3s}) = \frac{vt_s{}^2t_w(2a-1)}{(1-a)t_w+at_s} < 0$. It follows that, since $f_s(.)$ is concave with $f_s(e_s{}^*) = 0$ and $f_s{}'(e_s{}^*) < 0$ we have $e_s{}^* \underset{(>)}{<} \bar{e}_{3s}$ if and only if $f_s(\bar{e}_{3s}) \underset{(>)}{<} 0$. Since;

$$f_s(\bar{e}_{3s}) = \frac{(1-a)[(1-a)t_w - (1-3a)t_s]}{4T}\left[\frac{vt_st_w}{(1-a)t_w + at_s}\right]^2 + \frac{vt_st_wb(1-2a)}{T}$$

we have $f_s(\bar{e}_{3s}) \underset{(>)}{<} 0$ if and only if;

$$b \underset{(>)}{<} \frac{vt_st_w(1-a)[(1-3a)t_s - (1-a)t_w]}{4(1-2a)[(1-a)t_w + at_s]^2} = \bar{b}_{e_s}$$

This completes our proof that $\frac{\partial e_s{}^*}{\partial b} \underset{(<)}{>} 0$ when $b \underset{(>)}{<} \bar{b}_{e_s}$. We can now also prove Lemma 5, which relates to this comparative static;

Lemma 5: $a > \frac{1}{3} \Rightarrow \frac{\partial e_s{}^*}{\partial b} < 0$.

To see this, note that $\frac{\partial e_s{}^*}{\partial b} < 0$ if and only if $b > \bar{b}_{e_s}$. Further note that $a > \frac{1}{3} \Rightarrow (1-3a)t_s < 0 \Rightarrow \bar{b}_{e_s} < 0 < b$.

ii) PERFORMANCES ($\frac{\partial x_s(e_s{}^*)}{\partial b} \underset{(<)}{>} 0$ when $b \underset{(>)}{<} \bar{b}_{e_s}$)

Recall that contestants' performances are proportional to efforts, $x_s(e_s{}^*) = t_s e_s{}^*$. Thus, $\frac{\partial x_s(e_s{}^*)}{\partial b} = t_s \frac{\partial e_s{}^*}{\partial b}$. From the previous part of this proof we have $\frac{\partial e_s{}^*}{\partial b} \underset{(<)}{>} 0$ when $b \underset{(>)}{<} \bar{b}_{e_s}$.

iii) WIN PROBABILITIES ($\frac{\partial p_s(e^*)}{\partial b} < 0$)

Recall that the strong contestant has equilibrium win probability $p_s(e^*) = \frac{t_s e_s{}^*}{b + t_s e_s{}^* + t_w e_w{}^*}$. Note that, since $e_w{}^* = e_s{}^* - \frac{b(1-a)(t_s - t_w)}{t_w t_s}$, we have $b + t_s e_s{}^* + t_w e_w{}^* = b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^* = z_s{}^*$. Thus, $p_s(e^*) = \frac{t_s e_s{}^*}{z_s{}^*}$. Differentiating with respect to b and rearranging yields;

$$\frac{\partial p_s(e^*)}{\partial b} = \frac{(1-a)t_w + at_s}{z_s{}^{*2}} \left\{ \frac{\partial e_s{}^*}{\partial b} b - e_s{}^* \right\}$$

It follows that $\frac{\partial p_s(e^*)}{\partial b} < 0 \Leftrightarrow \frac{\partial e_s{}^*}{\partial b} < \frac{e_s{}^*}{b}$. Recalling that $\frac{\partial e_s{}^*}{\partial b} = \frac{1}{t_s}\left[\frac{2z_s{}^*[(1-a)t_w + at_s] - vt_s t_w(1-a)}{vt_s t_w - 2z_s{}^* T}\right]$ and $vt_s t_w - 2z_s{}^* T < 0$, this is equivalent to;

$$b[2z_s{}^*[(1-a)t_w + at_s] - vt_s t_w(1-a)] > t_s e_s{}^*[vt_s t_w - 2z_s{}^* T]$$

Which after rearrangement yields;

$$2z_s{}^* \left[b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^*\right] > vt_w(b(1-a) + t_s e_s{}^*)$$

Recalling that $b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^* = z_s{}^*$ and $f_s(e_s{}^*) = 0 \Rightarrow vt_w(b(1-a) + t_s e_s{}^*) = z_s{}^{*2}$, this equivalent to $z_s{}^{*2} > 0$.

iv) DRAW PROBABILITY ($\frac{\partial p_D(e^*)}{\partial b} > 0$)

We show that $\frac{\partial p_D(e^*)}{\partial b} > 0$ in part iv) of the proof for Proposition 2.3b.

v) EXPECTED PAYOFFS ($\frac{\partial \pi_s(e^*)}{\partial b} \underset{(<)}{>} 0$ when $b \underset{(<)}{>} \bar{b}_{\pi_s}$)

Recall that the strong contestant has expected equilibrium payoff $\pi_s(e^*) = v\frac{ab+t_s e_s{}^*}{b+t_s e_s{}^*+t_w e_w{}^*} - e_s{}^*$. Note that, since $e_w{}^* = e_s{}^* - \frac{b(1-a)(t_s-t_w)}{t_w t_s}$, we have $b + t_s e_s{}^* + t_w e_w{}^* = b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^* = z_s{}^*$. Thus;

$$\pi_s(e^*) = v\frac{ab + t_s e_s{}^*}{b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^*} - e_s{}^*$$

Differentiating with respect to b and rearranging yields;

$$\frac{\partial \pi_s(e^*)}{\partial b} = \frac{v(1-2a)t_w}{z_s{}^{*2}}\left[\frac{\partial e_s{}^*}{\partial b}b - e_s{}^*\right] - \frac{\partial e_s{}^*}{\partial b}$$

It follows that $\frac{\partial \pi_s(e^*)}{\partial b} \underset{(<)}{>} 0$ if and only if;

$$v(1-2a)t_w e_s{}^* \underset{(>)}{<} \left[vb(1-2a)t_w - z_s{}^{*2}\right]\frac{\partial e_s{}^*}{\partial b}$$

Recalling that $\frac{\partial e_s{}^*}{\partial b} = \frac{1}{t_s}\left[\frac{2z_s{}^*[(1-a)t_w + at_s] - vt_s t_w(1-a)}{vt_s t_w - 2z_s{}^* T}\right]$ and $vt_s t_w - 2z_s{}^* T < 0$, we can rewrite this condition as a cubic in $z_s{}^*$;

$$\begin{aligned} &2[(1-a)t_w + at_s]z_s{}^{*3} - vt_s t_w(1-a)z_s{}^{*2} \\ &- 2vt_s t_w(1-2a)\left[b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^*\right]z_s{}^* \\ &\qquad + v^2 t_s t_w{}^2(1-2a)[b(1-a) + t_s e_s{}^*] \underset{(<)}{>} 0 \end{aligned}$$

Noting that $\left[b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^*\right] = z_s{}^*$ and $f_s(e_s{}^*) = 0 \Rightarrow vt_w[b(1-a) + t_s e_s{}^*] = z_s{}^{*2}$, this is equivalent to $z_s{}^* \underset{(<)}{>} \frac{vt_s t_w(2-3a)}{2[(1-a)t_w + at_s]}$ or;

$$e_s{}^* \underset{(<)}{>} \frac{vt_s t_w(2-3a)}{2T[(1-a)t_w + at_s]} - \frac{b(1-a)}{T}\frac{t_w}{t_s} - \frac{ba}{T} = \bar{e}_{4s}$$

Next, check that $f_s'(\bar{e}_{4s}) = \frac{vt_s t_w(2a-1)(t_w+2t_s)}{(1-a)t_w+at_s} < 0$. It follows that, since $f_s(.)$ is concave with $f_s(e_s{}^*) = 0$ and $f_s{}'(e_s{}^*) < 0$ we have $e_s{}^* \underset{(<)}{>} \bar{e}_{4s}$ if and only if $f_s(\bar{e}_{4s}) \underset{(<)}{>} 0$. Since;

$$f_s(\bar{e}_{4s}) = \frac{(2-3a)[at_w - (2-5a)t_s]}{4T}\left[\frac{vt_s t_w}{(1-a)t_w + at_s}\right]^2 + \frac{vt_s t_w b(1-2a)}{T}$$

we have $f_s(\bar{e}_{4s}) \underset{(<)}{>} 0$ if and only if;

$$b \underset{(<)}{>} \frac{vt_s t_w(2-3a)[(2-5a)t_s - at_w]}{4(1-2a)[(1-a)t_w + at_s]^2} = \bar{b}_{\pi_s}$$

This completes our proof that $\frac{\partial \pi_s(e^*)}{\partial b} \underset{(<)}{>} 0$ when $b \underset{(<)}{>} \bar{b}_{\pi_s}$. Here, we also quickly note that $a > \frac{2}{5} \Rightarrow \bar{b}_{\pi_s} < 0 < b \Rightarrow \frac{\partial \pi_s(e^*)}{\partial b} > 0$.

13.6 Proposition 2.2b

This sub-section provides a proof of;

Proposition 2.2b: Comparative Statics for the Draw Likelihood with Heterogeneous Contestants (Weak Contestant Outcomes)

#	***Weak Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Likelihood ($\frac{\partial z}{\partial b}$)***
i	*Effort ($e_w{}^*$)*	$-$
ii	*Performance ($x_w(e_w{}^*)$)*	$-$
iii	*Win Probability ($p_w(e^*)$)*	$-$
iv	*Draw Probability ($p_D(e^*)$)*	$+$
v	*Expected Payoff ($\pi_w(e^*)$)*	$-$ *when* $b < \bar{b}_{\pi_w}$ $+$ *when* $b > \bar{b}_{\pi_w}$

where $\bar{b}_{\pi_w} = \frac{vt_st_w(2-3a)}{2(1-2a)[(1-a)t_s+at_w]}\left\{\frac{T(2-3a)}{2[(1-a)t_s+at_w]} - 1\right\} < \bar{b}_w$

We prove each comparative static in turn;

i) EFFORTS ($\frac{\partial e_w{}^*}{\partial b} < 0$)

Given $b < \bar{b}_w$, the weak contestant's equilibrium effort, $e_w{}^*$, is uniquely defined by $f_w(e_w{}^*) = 0$ where;

$$f_w(e_w) = vt_s(b(1-a) + t_we_w) - \left[b(1-a)\frac{t_s}{t_w} + ba + Te_w\right]^2$$

Totally differentiating with respect to b and rearranging, we have;

$$\frac{\partial e_w{}^*}{\partial b} = \frac{1}{t_w}\left[\frac{2z_w{}^*[(1-a)t_s + at_w] - vt_st_w(1-a)}{vt_st_w - 2z_w{}^*T}\right]$$

Recall from part i) of the proof for Proposition 2.2a that $vt_st_w - 2z_w{}^*T < 0$. It follows that $\frac{\partial e_w{}^*}{\partial b} < 0 \Leftrightarrow z_w{}^* > \frac{vt_st_w(1-a)}{2[(1-a)t_s+at_w]}$, which is equivalent to;

$$e_w{}^* > \frac{vt_s t_w(1-a)}{2T[(1-a)t_s + at_w]} - \frac{b(1-a)}{T}\frac{t_s}{t_w} - \frac{ba}{T} = \bar{e}_{4w}$$

Now, since $f_w(e_w)$ is concave with $f_w(e_w{}^*) = 0$ and $f_w{}'(e_w{}^*) < 0$, we have that $e_w{}^* > \bar{e}_{4w}$ is implied by $f_w(\bar{e}_{4w}) > 0$. To see this holds, check that;

$$f_w(\bar{e}_{4w}) = \frac{(1-a)[(1-a)t_s - (1-3a)t_w]}{4T}\left[\frac{vt_s t_w}{(1-a)t_s + at_w}\right]^2 + \frac{vt_s t_w b(1-2a)}{T}$$

with $t_s > t_w \Rightarrow (1-a)t_s - (1-3a)t_w > 0 \Rightarrow f_w(\bar{e}_{4w}) > 0$.

ii) PERFORMANCES ($\frac{\partial x_w(e_w{}^*)}{\partial b} < 0$)

Recall that contestants' performances are proportional to efforts, $x_w(e_w{}^*) = t_w e_w{}^*$. Thus, $\frac{\partial x_w(e_w{}^*)}{\partial b} = t_w \frac{\partial e_w{}^*}{\partial b} < 0$.

iii) WIN PROBABILITIES ($\frac{\partial p_w(e^*)}{\partial b} < 0$)

Recall that the weak contestant has equilibrium win probability $p_w(e^*) = \frac{t_w e_w{}^*}{b + t_s e_s{}^* + t_w e_w{}^*}$. Note that, since $e_s{}^* = e_w{}^* + \frac{b(1-a)(t_s - t_w)}{t_w t_s}$, we have $b + t_s e_s{}^* + t_w e_w{}^* = b(1-a)\frac{t_s}{t_w} + ba + Te_w{}^* = z_w{}^*$. Thus, $p_w(e^*) = \frac{t_w e_w{}^*}{z_w{}^*}$. Differentiating with respect to b and rearranging yields;

$$\frac{\partial p_w(e^*)}{\partial b} = \frac{(1-a)t_s + at_w}{z_w{}^{*2}}\left\{\frac{\partial e_w{}^*}{\partial b} b - e_w{}^*\right\}$$

It follows that $\frac{\partial e_w{}^*}{\partial b} < 0 \Rightarrow \frac{\partial p_w(e^*)}{\partial b} < 0$.

iv) DRAW PROBABILITY ($\frac{\partial p_D(e^*)}{\partial b} > 0$)

We show that $\frac{\partial p_D(e^*)}{\partial b} > 0$ in part iv) of the proof for Proposition 2.3b.

v) EXPECTED PAYOFFS ($\frac{\partial \pi_w(e^*)}{\partial b} \underset{(<)}{>} 0$ when $b \underset{(<)}{>} \bar{b}_{\pi_w}$)

This proof is identical to the corresponding proof for the strong contestant with reverse notation. Thus, following the proof presented for part v) of Proposition 2.1b, we have $\frac{\partial \pi_w(e^*)}{\partial b} \underset{(<)}{>} 0$ if and only if;

$$b \underset{(<)}{>} \frac{v t_s t_w (2-3a)[(2-5a)t_w - a t_s]}{4(1-2a)[(1-a)t_s + a t_w]^2} = \bar{b}_{\pi_w}$$

Here, we also prove Lemma 6, which relates to this comparative static;

Lemma 6: $a > \frac{1}{3} \Rightarrow \frac{\partial \pi_w(e^*))}{\partial b} > 0.$

To see this note that $t_s > t_w \Rightarrow (2-5a)t_w - a t_s < (2-6a)t_w$. It follows that $a > \frac{1}{3} \Rightarrow (2-5a)t_w - a t_s < 0 \Rightarrow \bar{b}_{\pi_w} < 0 < b \Rightarrow \frac{\partial \pi_w(e^*))}{\partial b} > 0.$

13.7 Proposition 2.3b

This sub-section provides a proof of;

Proposition 2.3b: Comparative Statics for the Draw Likelihood with Heterogeneous Contestants (Total Contestant Outcomes)

#	***Total Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Likelihood*** ($\frac{\partial z}{\partial b}$)
i	*Effort* ($e_s^* + e_w^*$)	$-$
ii	*Performance* ($x_s(e_s^*) + x_w(e_w^*)$)	$-$
iii	*Win Probability* ($p_s(e^*) + p_w(e^*)$)	$-$
iv	*Draw Probability* ($p_D(e^*)$)	$+$
v	*Expected Payoff* ($\pi_s(e^*) + \pi_w(e^*)$)	$-$ *when* $b < \bar{b}_{\pi_w}$ $+$ *when* $b > \bar{b}_{\pi_s}$

We prove each comparative static in turn;

i) EFFORTS ($\frac{\partial e_s^* + e_w^*}{\partial b} < 0$)

First, note that, since $\frac{\partial e_w^*}{\partial b} < 0$, we have $\frac{\partial e_s^* + e_w^*}{\partial b} < 0$ whenever $\frac{\partial e_s^*}{\partial b} < 0$. Second, note that if $\frac{\partial e_s^*}{\partial b} \geq 0$, we have $\frac{\partial x_s(e_s^*) + x_w(e_w^*)}{\partial b} = t_s \frac{\partial e_s^*}{\partial b} + t_w \frac{\partial e_w^*}{\partial b} < 0 \Rightarrow t_w \frac{\partial e_s^*}{\partial b} + t_w \frac{\partial e_w^*}{\partial b} < 0 \Leftrightarrow \frac{\partial e_s^* + e_w^*}{\partial b} < 0$. Thus, we prove that $\frac{\partial e_s^* + e_w^*}{\partial b} < 0$ by showing that $\frac{\partial x_s(e_s^*) + x_w(e_w^*)}{\partial b} < 0$ in the next part of this proof.

ii) PERFORMANCES ($\frac{\partial x_s(e_s^*) + x_w(e_w^*)}{\partial b} < 0$)

For this, note that since $e_w^* = e_s^* - \frac{b(1-a)(t_s - t_w)}{t_w t_s}$, we have $x_s(e_s^*) + x_w(e_w^*) = t_s e_s^* + t_w e_w^* = T e_s^* - \frac{b(1-a)(t_s - t_w)}{t_s}$. It follows that $\frac{\partial x_s(e_s^*) + x_w(e_w^*)}{\partial b} = T \frac{\partial e_s^*}{\partial b} - \frac{(1-a)(t_s - t_w)}{t_s}$. Thus, we have $\frac{\partial x_s(e_s^*) + x_w(e_w^*)}{\partial b} < 0$ if and only if;

$$\frac{\partial e_s{}^*}{\partial b} < \frac{(1-a)(t_s - t_w)}{T t_s}$$

Recalling that $\frac{\partial e_s{}^*}{\partial b} = \frac{1}{t_s}\left[\frac{2z_s{}^*[(1-a)t_w + at_s] - vt_s t_w(1-a)}{vt_s t_w - 2z_s{}^* T}\right]$ with $vt_s t_w - 2z_s{}^* T < 0$, this is equivalent to;

$$2z_s{}^*[(1-a)t_w + at_s]T - vt_s t_w(1-a)T > (1-a)(t_s - t_w)[vt_s t_w - 2z_s{}^* T]$$

Rearranging yields $z_s{}^* > \frac{vt_s t_w(1-a)}{T}$ or equivalently;

$$e_s{}^* > \frac{vt_s t_w(1-a)}{T^2} - \frac{b(1-a)}{T}\frac{t_w}{t_s} - \frac{ba}{T} = \bar{e}_{5s}$$

Now, since $f_s(e_s)$ is concave with $f_s(e_s{}^*) = 0$ and $f_s{}'(e_s{}^*) < 0$, we have that $e_s{}^* > \bar{e}_{5s}$ is implied by $f_s(\bar{e}_{5s}) > 0$. To see this holds, check that;

$$f_s(\bar{e}_{5s}) = a(1-a)\left[\frac{vt_s t_w}{T}\right]^2 + \frac{vt_s t_w b(1-2a)}{T} > 0$$

iii) WIN PROBABILITIES ($\frac{\partial p_s(e^*) + p_w(e^*)}{\partial b} < 0$)

This follows immediately from the fact that $\frac{\partial p_s(e^*)}{\partial b} < 0$ and $\frac{\partial p_w(e^*)}{\partial b} < 0$.

iv) DRAW PROBABILITY ($\frac{\partial p_D(e^*)}{\partial b} > 0$)

Recall that $p_D(e^*) = 1 - p_s(e^*) - p_w(e^*)$. It follows that $\frac{\partial p_D(e^*)}{\partial b} > 0 \Leftrightarrow \frac{\partial p_s(e^*) + p_w(e^*)}{\partial b} < 0$.

v) EXPECTED PAYOFFS ($\frac{\partial \pi_s(e^*) + \pi_w(e^*)}{\partial b} < 0$ when $b < \bar{b}_{\pi_w}$ and $\frac{\partial \pi_s(e^*) + \pi_w(e^*)}{\partial b} > 0$ when $b > \bar{b}_{\pi_s}$)

For this, we begin by showing that $\bar{b}_{\pi_s} = \frac{vt_st_w(2-3a)[(2-5a)t_s-at_w]}{4(1-2a)[(1-a)t_w+at_s]^2}$ is greater than or equal to $\bar{b}_{\pi_w} = \frac{vt_st_w(2-3a)[(2-5a)t_w-at_s]}{4(1-2a)[(1-a)t_s+at_w]^2}$. Equivalently;

$$\frac{(2-5a)t_s - at_w}{[(1-a)t_w + at_s]^2} \geq \frac{(2-5a)t_w - at_s}{[(1-a)t_s + at_w]^2}$$

Note that, since $a \leq \frac{1}{2},\ t_s > t_w \Rightarrow [(1-a)t_s + at_w]^2 \geq [(1-a)t_w + at_s]^2$. Thus, $\bar{b}_{\pi_s} \geq \bar{b}_{\pi_w}$ is implied by;

$$(2-5a)t_s - at_w \geq (2-5a)t_w - at_s$$

which holds since $(1-2a)t_s \geq (1-2a)t_w$.

Thus, $b < \bar{b}_{\pi_w} \Rightarrow b < \bar{b}_{\pi_s}$, so that we have $\frac{\partial \pi_s(e^*)}{\partial b} < 0$ and $\frac{\partial \pi_w(e^*)}{\partial b} < 0$. It follows that $\frac{\partial \pi_s(e^*)+\pi_w(e^*)}{\partial b} < 0$.

On the other hand, $b > \bar{b}_{\pi_s} \Rightarrow b > \bar{b}_{\pi_w}$, so that we have $\frac{\partial \pi_s(e^*)}{\partial b} > 0$ and $\frac{\partial \pi_w(e^*)}{\partial b} > 0$. It follows that $\frac{\partial \pi_s(e^*)+\pi_w(e^*)}{\partial b} > 0$.

13.8 Proposition 2.4a

This sub-section provides a proof of;

Proposition 2.4a: Comparative Statics for the Draw Prize with Heterogeneous Contestants (Difference in Contestant Outcomes)

#	***Difference in Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Prize ($\frac{\partial z}{\partial a}$)***
i	*Effort* ($e_s{}^* - e_w{}^*$)	$-$
ii	*Performance* ($x_s(e_s{}^*) - x_w(e_w{}^*)$)	$-$
iii	*Win Probability* ($p_s(e^*) - p_w(e^*)$)	$-$
iv	*Draw Probability* ($p_D(e^*) - p_D(e^*)$))	*N/A*
v	*Expected Payoff* ($\pi_s(e^*) - \pi_w(e^*)$)	$+$ *when* $a < \bar{a}_{\pi_s - \pi_w}$ $-$ *when* $a > \bar{a}_{\pi_s - \pi_w}$

where $\bar{a}_{\pi_s - \pi_w} = \frac{1}{2} - \frac{3vt_s t_w}{8bT}$

We prove each comparative static in turn;

i) EFFORTS ($\frac{\partial e_s{}^* - e_w{}^*}{\partial a} < 0$)

For this, note that $e_s{}^* - e_w{}^* = \frac{b(1-a)(t_s - t_w)}{t_w t_s}$. It follows that $\frac{\partial e_s{}^* - e_w{}^*}{\partial a} = -\frac{b(t_s - t_w)}{t_w t_s} < 0$.

ii) PERFORMANCES ($\frac{\partial x_s(e_s{}^*) - x_w(e_w{}^*)}{\partial a} < 0$)

For this, note that $x_s(e_s{}^*) - x_w(e_w{}^*) = t_s e_s{}^* - t_w e_w{}^* = t_w(e_s{}^* - e_w{}^*) + (t_s - t_w)e_s{}^*$. It follows that $\frac{\partial x_s(e_s{}^*) - x_w(e_w{}^*)}{\partial a} = t_w \frac{\partial e_s{}^* - e_w{}^*}{\partial a} + (t_s - t_w)\frac{\partial e_s{}^*}{\partial a} < 0$.

iii) WIN PROBABILITIES ($\frac{\partial p_s(e^*) - p_w(e^*)}{\partial a} < 0$)

Recall that $p_s(e^*) - p_w(e^*) = \frac{t_s e_s{}^* - t_w e_w{}^*}{b + t_s e_s{}^* + t_w e_w{}^*}$. Recall also that $e_w{}^* = e_s{}^* - \frac{b(1-a)(t_s - t_w)}{t_w t_s}$. It follows that;

$$p_s(e^*) - p_w(e^*) = \frac{t_s - t_w}{t_s}\frac{b(1-a) + t_s {e_s}^*}{{z_s}^*}$$

where $b + t_s {e_s}^* + t_w {e_w}^* = b(1-a)\frac{t_w}{t_s} + ba + T{e_s}^* = {z_s}^*$. Further, recall that $f_s({e_s}^*) = 0 \Rightarrow vt_w(b(1-a) + t_s {e_s}^*) = {z_s}^{*2}$. It follows that $p_s(e^*) - p_w(e^*) = \frac{t_s - t_w}{vt_s t_w}{z_s}^*$. Thus,

$$\frac{\partial p_s(e^*) - p_w(e^*)}{\partial a} = \frac{t_s - t_w}{vt_s t_w}\frac{\partial {z_s}^*}{\partial a}$$

Finally, $\frac{\partial {z_s}^*}{\partial a} < 0$ follows from the fact that $\frac{\partial {z_s}^*}{\partial a} = \frac{\partial x_s({e_s}^*) + x_w({e_w}^*)}{\partial a} < 0$.

v) EXPECTED PAYOFFS ($\frac{\partial \pi_s(e^*) - \pi_w(e^*)}{\partial a} \underset{(<)}{>} 0$ when $a \underset{(>)}{<} \bar{a}_{\pi_s - \pi_w}$)

Note that $\pi_s(e^*) - \pi_w(e^*) = [p_s(e^*) - p_w(e^*)] - [{e_s}^* - {e_w}^*]$. Recalling that $p_s(e^*) - p_w(e^*) = \frac{t_s - t_w}{vt_s t_w}{z_s}^*$ and ${e_s}^* - {e_w}^* = \frac{b(1-a)(t_s - t_w)}{t_w t_s}$ we have;

$$\pi_s(e^*) - \pi_w(e^*) = \frac{t_s - t_w}{vt_s t_w}[{z_s}^* - b(1-a)]$$

Since, ${z_s}^* = b(1-a)\frac{t_w}{t_s} + ba + T{e_s}^*$, we have;

$$\frac{\partial \pi_s(e^*) - \pi_w(e^*)}{\partial a} = \frac{t_s - t_w}{vt_s t_w}\left[\frac{b(2t_s - t_w)}{t_s} + T\frac{\partial {e_s}^*}{\partial a}\right]$$

Thus, we have $\frac{\partial \pi_s(e^*) - \pi_w(e^*)}{\partial a} \underset{(<)}{>} 0 \Leftrightarrow \frac{\partial {e_s}^*}{\partial a} \underset{(<)}{>} -\frac{b(2t_s - t_w)}{t_s T}$. Recalling that $\frac{\partial {e_s}^*}{\partial a} = \frac{b}{t_s}\left[\frac{vt_s t_w + 2{z_s}^*(t_s - t_w)}{vt_s t_w - 2{z_s}^* T}\right]$ with $vt_s t_w - 2{z_s}^* T < 0$, this is equivalent to;

$$T[vt_s t_w + 2{z_s}^*(t_s - t_w)] \underset{(>)}{<} (2t_s - t_w)[2{z_s}^* T - vt_s t_w]$$

Rearranging gives ${z_s}^* \underset{(<)}{>} \frac{3vt_s t_w}{2T}$ or equivalently;

$$e_s{}^* \underset{(<)}{>} \frac{3vt_st_w}{2T^2} - \frac{b(1-a)}{T}\frac{t_w}{t_s} - \frac{ba}{T} = \bar{e}_{6s}$$

Next, check that $f_s'(\bar{e}_{6s}) = -2vt_st_w < 0$. It follows that, since $f_s(.)$ is concave with $f_s(e_s{}^*) = 0$ and $f_s{}'(e_s{}^*) < 0$ we have $e_s{}^* \underset{(<)}{>} \bar{e}_{6s}$ if and only if $f_s(\bar{e}_{6s}) \underset{(<)}{>} 0$. Since;

$$f_s(\bar{e}_{6s}) = \frac{vt_st_wb(1-2a)}{T} - 3\left(\frac{vt_st_w}{2T}\right)^2$$

we have $f_s(\bar{e}_{6s}) \underset{(<)}{>} 0$ if and only if $a \underset{(>)}{<} \frac{1}{2} - \frac{3vt_st_w}{8bT} = \bar{a}_{\pi_s - \pi_w}$.

This completes our proof that $\frac{\partial \pi_s(e^*) - \pi_w(e^*)}{\partial a} \underset{(<)}{>} 0$ when $a \underset{(>)}{<} \bar{a}_{\pi_s - \pi_w}$. Here, we also show that $t_w < \frac{3}{13}t_s \Rightarrow \frac{\partial \pi_s(e^*) - \pi_w(e^*)}{\partial a} < 0$. For this, note that $\bar{a}_{\pi_s - \pi_w} < 0 \Rightarrow a > \bar{a}_{\pi_s - \pi_w} \Leftrightarrow \frac{\partial \pi_s(e^*) - \pi_w(e^*)}{\partial a} < 0$. Further, note that $\bar{a}_{\pi_s - \pi_w} < 0 \Leftrightarrow b < \frac{3vt_st_w}{4T}$. Now, since our participation constraint is $b < \frac{vt_s(1-a)}{\left[(1-a)\frac{t_s}{t_w} + a\right]^2}$, this is implied by $\frac{3vt_st_w}{4T} > \frac{vt_s(1-a)}{\left[(1-a)\frac{t_s}{t_w} + a\right]^2}$. Note that this is in turn implied by $\frac{3vt_st_w}{4T} > \frac{4vt_s}{\left[\frac{t_s}{t_w} + 1\right]^2} \Leftrightarrow t_w < \frac{3}{13}t_s$.

13.9 Proposition 2.4b

This sub-section provides a proof of;

Proposition 2.4b: Comparative Statics for the Draw Likelihood with Heterogeneous Contestants (Difference in Contestant Outcomes)

#	***Difference in Contestants' Equilibrium Outcome (z)***	***Effect of the Draw Likelihood ($\frac{\partial z}{\partial b}$)***
i	*Effort ($e_s{}^* - e_w{}^*$)*	$+$
ii	*Performance ($x_s(e_s{}^*) - x_w(e_w{}^*)$)*	$+$
iii	*Win Probability ($p_s(e^*) - p_w(e^*)$)*	$+$
iv	*Draw Probability ($p_D(e^*) - p_D(e^*)$))*	*N/A*
v	*Expected Payoff ($\pi_s(e^*) - \pi_w(e^*)$)*	$-$

We prove each comparative static in turn;

i) EFFORTS ($\frac{\partial e_s{}^* - e_w{}^*}{\partial b} > 0$)

For this, note that $e_s{}^* - e_w{}^* = \frac{b(1-a)(t_s - t_w)}{t_w t_s}$. It follows that $\frac{\partial e_s{}^* - e_w{}^*}{\partial b} = \frac{(1-a)(t_s - t_w)}{t_w t_s} > 0$.

ii) PERFORMANCES ($\frac{\partial x_s(e_s{}^*) - x_w(e_w{}^*)}{\partial b} > 0$)

For this, note that $\frac{\partial x_s(e_s{}^*) - x_w(e_w{}^*)}{\partial b} > 0 \Leftrightarrow t_s \frac{\partial e_s{}^*}{\partial b} > t_w \frac{\partial e_w{}^*}{\partial b}$. Then recall that;

$$\frac{\partial e_s{}^*}{\partial b} = \frac{1}{t_s}\left[\frac{2z_s{}^*[(1-a)t_w + at_s] - vt_s t_w(1-a)}{vt_s t_w - 2z_s{}^* T}\right]$$

$$\frac{\partial e_w{}^*}{\partial b} = \frac{1}{t_w}\left[\frac{2z_w{}^*[(1-a)t_s + at_w] - vt_s t_w(1-a)}{vt_s t_w - 2z_w{}^* T}\right]$$

Recall also that $z_s{}^* = z_w{}^*$ and $vt_s t_w - 2z_s{}^* T < 0$. It follows that $t_s \frac{\partial e_s{}^*}{\partial b} > t_w \frac{\partial e_w{}^*}{\partial b} \Leftrightarrow$ $(1-a)t_w + at_s < (1-a)t_s + at_w \Leftrightarrow t_s > t_w$.

iii) WIN PROBABILITIES ($\frac{\partial p_s(e^*)-p_w(e^*)}{\partial b} > 0$)

For this, note that $p_s(e^*) - p_w(e^*) = \frac{x_s(e_s{}^*)-x_w(e_w{}^*)}{b+t_s e_s{}^*+t_w e_w{}^*}$. Recall that, since $e_s{}^* = e_w{}^* + \frac{b(1-a)(t_s-t_w)}{t_w t_s}$, we have $b + t_s e_s{}^* + t_w e_w{}^* = b(1-a)\frac{t_s}{t_w} + ba + Te_w{}^* = z_w{}^*$. Thus, we have $p_s(e^*) - p_w(e^*) = \frac{x_s(e_s{}^*)-x_w(e_w{}^*)}{z_w{}^*}$. Now, given $\frac{\partial x_s(e_s{}^*)-x_w(e_w{}^*)}{\partial b} > 0$, we have that $\frac{\partial z_w{}^*}{\partial b} < 0 \Rightarrow \frac{\partial p_s(e^*)-p_w(e^*)}{\partial b} > 0$. Finally then, note that $\frac{\partial z_w{}^*}{\partial b} = T\frac{\partial e_w{}^*}{\partial b} - a\left(\frac{t_s-t_w}{t_w}\right) < 0$.

v) EXPECTED PAYOFFS ($\frac{\partial \pi_s(e^*)-\pi_w(e^*)}{\partial b} < 0$)

For this, recall that $\frac{\partial \pi_s(e^*)}{\partial b} = \frac{v(1-2a)t_w}{z_s{}^{*2}}\left[\frac{\partial e_s{}^*}{\partial b}b - e_s{}^*\right] - \frac{\partial e_s{}^*}{\partial b}$ and $\frac{\partial \pi_w(e^*)}{\partial b} = \frac{v(1-2a)t_s}{z_w{}^{*2}}\left[\frac{\partial e_w{}^*}{\partial b}b - e_w{}^*\right] - \frac{\partial e_w{}^*}{\partial b}$. Further, note that, since $e_w{}^* = e_s{}^* - \frac{b(1-a)(t_s-t_w)}{t_w t_s}$, we have $z_w{}^* = z_s{}^*$ and $\frac{\partial e_w{}^*}{\partial b}b - e_w{}^* = \frac{\partial e_s{}^*}{\partial b}b - e_s{}^*$. It follows that;

$$\frac{\partial \pi_s(e^*) - \pi_w(e^*)}{\partial b} = \frac{v(1-2a)(t_w-t_s)}{z_s{}^{*2}}\left[\frac{\partial e_s{}^*}{\partial b}b - e_s{}^*\right] - \left(\frac{\partial e_s{}^*}{\partial b} - \frac{\partial e_w{}^*}{\partial b}\right)$$

Noting further that $\frac{\partial e_s{}^*}{\partial b} - \frac{\partial e_w{}^*}{\partial b} = \frac{(1-a)(t_s-t_w)}{t_w t_s}$, we have $\frac{\partial \pi_s(e^*)-\pi_w(e^*)}{\partial b} < 0$ if and only if;

$$\frac{\partial e_s{}^*}{\partial b} > \frac{v(1-2a)t_w t_s e_s{}^* - (1-a)z_s{}^{*2}}{vb(1-2a)t_w t_s}$$

Recalling that $\frac{\partial e_s{}^*}{\partial b} = \frac{1}{t_s}\left[\frac{2z_s{}^*[(1-a)t_w+at_s]-vt_st_w(1-a)}{vt_st_w-2z_s{}^*T}\right]$ and $vt_st_w - 2z_s{}^*T < 0$, we can rewrite this condition as a cubic in $z_s{}^*$;

$$2T(1-a)z_s{}^{*3} - vt_st_w(1-a)z_s{}^{*2} - 2vt_st_w(1-2a)\left[b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^*\right]z_s{}^*$$
$$+ v^2t_st_w{}^2(1-2a)[b(1-a) + t_se_s{}^*] > 0$$

Noting that $\left[b(1-a)\frac{t_w}{t_s}+ba+Te_s{}^*\right]=z_s{}^*$ and $f_s(e_s{}^*)=0\Rightarrow vt_w[b(1-a)+t_se_s{}^*]=z_s{}^{*2}$, this is equivalent to $z_s{}^*>\frac{vt_st_w(2-3a)}{2T(1-a)}$ or;

$$e_s{}^*>\frac{vt_st_w(2-3a)}{2T^2(1-a)}-\frac{b(1-a)}{T}\frac{t_w}{t_s}-\frac{ba}{T}=\bar{e}_{7s}$$

Next, check that $f_s'(\bar{e}_{7s})=\frac{vt_st_w(2a-1)}{1-a}<0$. It follows that, since $f_s(.)$ is concave with $f_s(e_s{}^*)=0$ and $f_s{}'(e_s{}^*)<0$ we have $e_s{}^*>\bar{e}_{7s}$ if and only if $f_s(\bar{e}_{7s})>0$. To see that this hold note that;

$$f_s(\bar{e}_{7s})=a(2-3a)\left[\frac{vt_st_w}{2T(1-a)}\right]^2+\frac{vt_st_wb(1-2a)}{T}>0$$

13.10 Proposition 2.5a

This sub-section provides a proof of;

Proposition 2.5a: Comparative Statics on Employers' Expenditures/Profits with Heterogeneous Contestants (Draw Prize)

#	***Employer's Equilibrium Outcome (z)***	***Effect of the Draw Prize ($\frac{\partial z}{\partial a}$)***
i	*Expected Expenditure ($E(e^*)$)*	$+$
ii	*Expected Profit ($Q(e^*)$)*	$-$

We prove each comparative static in turn;

i) EXPENDITURES ($\frac{\partial E(e^*)}{\partial a} > 0$)

To see this, note that $E(e^*) = v[p_s(e^*) + p_w(e^*) + 2ap_D(e^*)] = v[1 - (1-2a)p_D(e^*)]$. It follows that $\frac{\partial E(e^*)}{\partial a} = v\left[2p_D(e^*) - (1-2a)\frac{\partial p_D(e^*)}{\partial a}\right]$. Recalling that $p_D(e^*) = \frac{b}{b + x_s(e_s{}^*) + x_w(e_w{}^*)}$, we have $\frac{\partial p_D(e^*)}{\partial a} = -\frac{p_D(e^*)}{b + x_s(e_s{}^*) + x_w(e_w{}^*)}\left[\frac{\partial x_s(e_s{}^*) + x_w(e_w{}^*)}{\partial a}\right]$. Further recalling that $\frac{\partial x_s(e_s{}^*) + x_w(e_w{}^*)}{\partial a} = T\frac{\partial e_s{}^*}{\partial a} + \frac{b(t_s - t_w)}{t_s}$ and that $b + x_s(e_s{}^*) + x_w(e_w{}^*) = z_s{}^*$, we have $\frac{\partial p_D(e^*)}{\partial a} = -\frac{p_D(e^*)}{z_s{}^*}\left[T\frac{\partial e_s{}^*}{\partial a} + \frac{b(t_s - t_w)}{t_s}\right]$, so that;

$$\frac{\partial E(e^*)}{\partial a} = \frac{vp_D(e^*)}{z_s{}^*}\left\{2z_s{}^* + (1-2a)\left[T\frac{\partial e_s{}^*}{\partial a} + \frac{b(t_s - t_w)}{t_s}\right]\right\}$$

It follows that $\frac{\partial E(e^*)}{\partial a} > 0$ if and only if;

$$\frac{\partial e_s{}^*}{\partial a} > -\frac{1}{T}\left[\frac{2z_s{}^*}{(1-2a)} + \frac{b(t_s - t_w)}{t_s}\right]$$

Next, recall that $\frac{\partial e_s{}^*}{\partial a} = \frac{b}{t_s}\left[\frac{vt_st_w + 2z_s{}^*(t_s - t_w)}{vt_st_w - 2z_s{}^*T}\right]$ with $vt_st_w - 2z_s{}^*T < 0$. This is then equivalent to;

$$b(1-2a)T[vt_st_w + 2z_s{}^*(t_s - t_w)] < [2z_s{}^*t_s + b(1-2a)(t_s - t_w)][2z_s{}^*T - vt_st_w]$$

Recalling that $z_s{}^* = \left[b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^*\right]$, we have $2z_s{}^*t_s + b(1-2a)(t_s - t_w) = T(b + 2t_se_s{}^*)$. Thus, after rearrangement, our condition becomes;

$$vt_w(b(1-a) + t_se_s{}^*) < 2z_s{}^*\left[b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^*\right]$$

Finally, recalling that $f_s(e_s{}^*) = 0 \Rightarrow vt_w(b(1-a) + t_s e_s{}^*) = z_s{}^{*2}$ and again substituting for $z_s{}^* = \left[b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^*\right]$, we have $\frac{\partial E(e^*)}{\partial a} > 0$ if and only if $z_s{}^{*2} > 0$, which necessarily holds.

ii) PROFITS ($\frac{\partial Q(e^*)}{\partial a} < 0$)

To see this, recall that $Q(e^*) = x_s(e_s{}^*) + x_w(e_w{}^*) - E(e^*)$. Thus,

$\frac{\partial Q(e^*)}{\partial a} = \frac{\partial x_s(e_s{}^*)+x_w(e_w{}^*)}{\partial a} - \frac{\partial E(e^*)}{\partial a} < 0.$

13.11 Proposition 2.5b

This sub-section provides a proof of;

Proposition 2.5b: Comparative Statics on Employers' Expenditures/Profits with Heterogeneous Contestants (Draw Likelihood)

#	***Employer's Equilibrium Outcome (z)***	***Effect of the Draw Likelihood*** ($\frac{\partial z}{\partial b}$)
i	*Expected Expenditure* ($E(e^*)$)	$-$
ii	*Expected Profit* ($Q(e^*)$)	$+$ *when* $b < \bar{b}_Q{}^{Het}$ $-$ *when* $b > \bar{b}_Q{}^{Het}$

where $\bar{b}_Q{}^{Het} = \frac{v[(1-2a)T+2t_s t_w(1-a)][(1-2a)T-2at_s t_w]}{4T(1-2a)t_s t_w}$*)*

We prove each comparative static in turn;

i) EXPENDITURES ($\frac{\partial E(e^*)}{\partial b} < 0$)

Recall that $E(e^*) = v[1-(1-2a)p_D(e^*)]$. It follows that $\frac{\partial E(e^*)}{\partial b} = -v(1-2a)\frac{\partial p_D(e^*)}{\partial b} < 0$.

ii) PROFITS ($\frac{\partial Q(e^*)}{\partial b} \overset{>}{(<)} 0$ when $b \overset{<}{(>)} \bar{b}_Q{}^{Het}$)

Recall that $Q(e^*) = t_s e_s{}^* + t_w e_w{}^* - v[1-(1-2a)p_D(e^*)]$. Further, recall that $e_w{}^* = e_s{}^* - \frac{b(1-a)(t_s-t_w)}{t_w t_s}$ and note that $\frac{\partial p_D(e^*)}{\partial b} = \frac{T}{z_s{}^{*2}}\left(e_s{}^* - b\frac{\partial e_s{}^*}{\partial b}\right)$. It follows that;

$$\frac{\partial Q(e^*)}{\partial b} = T\frac{\partial e_s{}^*}{\partial b} - \frac{(1-a)(t_s-t_w)}{t_s} + \frac{vT(1-2a)}{z_s{}^{*2}}\left(e_s{}^* - b\frac{\partial e_s{}^*}{\partial b}\right)$$

Thus, we have $\frac{\partial Q(e^*)}{\partial b} \overset{>}{(<)} 0$ if and only if;

$$Tt_s\frac{\partial e_s{}^*}{\partial b}\left[z_s{}^{*2} - vb(1-2a)\right] \overset{>}{(<)} (1-a)(t_s-t_w)z_s{}^{*2} - vT(1-2a)t_s e_s{}^*$$

Next, recalling that $\frac{\partial e_s{}^*}{\partial b} = \frac{1}{t_s}\left[\frac{2z_s{}^*[(1-a)t_w + at_s] - vt_st_w(1-a)}{vt_st_w - 2z_s{}^*T}\right]$, this is equivalent to;

$$T[2z_s{}^*[(1-a)t_w + at_s] - vt_st_w(1-a)]\left[z_s{}^{*2} - vb(1-2a)\right] \overset{<}{(>)} \left[(1-a)(t_s-t_w)z_s{}^{*2} - vT(1-2a)t_s e_s{}^*\right][vt_st_w - 2z_s{}^*T]$$

Which rearranges into the following cubic in $z_s{}^*$;

$$2t_s z_s{}^{*3} - \frac{2vt_s{}^2t_w(1-a)}{T}z_s{}^{*2} - 2v(1-2a)t_s\left[b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^*\right]z_s{}^* + v^2t_st_w(1-2a)(b(1-a) + t_se_s{}^*) \overset{<}{(>)} 0$$

Recalling that $b(1-a)\frac{t_w}{t_s} + ba + Te_s{}^* = z_s{}^*$ and $f_s(e_s{}^*) = 0 \Rightarrow vt_w(b(1-a) + t_se_s{}^*) = z_s{}^{*2}$, this equivalent to $z_s{}^* \overset{<}{(>)} v\left[\frac{t_st_w(1-a)}{T} + \frac{1-2a}{2}\right]$ or;

$$e_s{}^* \overset{<}{(>)} \frac{v}{T}\left[\frac{t_st_w(1-a)}{T} + \frac{1-2a}{2}\right] - \frac{b(1-a)}{T}\frac{t_w}{t_s} - \frac{ba}{T} = \bar{e}_{8s}$$

Next, check that $f_s'(\bar{e}_{8s}) = -v(1-2a)(T + t_s t_w) < 0$. It follows that, since $f_s(.)$ is concave with $f_s(e_s^*) = 0$ and $f_s'(e_s^*) < 0$ we have $e_s^* \underset{(>)}{<} \bar{e}_{8s}$ if and only if $f_s(\bar{e}_{8s}) \underset{(>)}{<} 0$.

Since;

$$f_s(\bar{e}_{8s}) = v^2 \left[\frac{t_s t_w (1-a)}{T} + \frac{1-2a}{2}\right]\left[\frac{t_s t_w a}{T} - \frac{1-2a}{2}\right] + \frac{v t_s t_w b(1-2a)}{T}$$

we have $f_s(\bar{e}_{8s}) \underset{(>)}{<} 0$ if and only if;

$$b \underset{(>)}{<} \frac{v[(1-2a)T + 2t_s t_w(1-a)][(1-2a)T - 2t_s t_w a]}{4T(1-2a)t_s t_w} = \bar{b}_Q{}^{Het}$$