CONSENSUS CONTROL FOR MULTI-AGENT SYSTEMS WITH INPUT DELAY

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By
Chunyan Wang
School of Electrical and Electronic Engineering

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Abstract

This thesis applies predictor-based methods for the distributed consensus control of multi-agent systems with input delay. “Multi-agent systems” is a term used to describe a group of agents which are connected together to achieve specified control tasks over a communication network. In many applications, the subsystems or agents are required to reach an agreement upon certain quantities of interest, which is referred to as “consensus control”. This input delay may represent delays in the network communication. The main contribution of this thesis is to provide feasible methods to deal with the consensus control for general multi-agent systems with input delay.

The consensus control for general linear multi-agent systems with parameter uncertainties and input delay is first investigated under directed network connection. Artstein reduction method is applied to deal with the input delay. By transforming the Laplacian matrix into the real Jordan form, delay-dependent conditions are derived to guarantee the robust consensus control for uncertain multi-agent systems with input delay. Then, the results are extended to a class of Lipschitz nonlinear multi-agent systems and the impacts of Lipschitz nonlinearity and input delay in consensus control are investigated. By using tools from control theory and graph theory, sufficient conditions based on the Lipschitz constant are identified for proposed protocols to tackle the nonlinear terms in the system dynamics.

Other than the time delay, external disturbances are inevitable in various practical systems including the multi-agent systems. The consensus disturbance rejection problems are investigated. For linear multi-agent systems with bounded external disturbances, Truncated Predictor Feedback (TPF) approach is applied to deal with the input delay and the H_∞ consensus analysis is put in the framework of Lyapunov analysis. Sufficient conditions are derived to guarantee the H_∞ consensus in time domain. Some disturbances in real engineering problems have inherent characteristics such as harmonics

and unknown constant load. For those kinds of disturbances in Lipschitz nonlinear multi-agent systems with input delay, Disturbance Observer-Based Control (DOBC) technique is applied to design the disturbance observers. A new predictor-based control scheme is constructed for each agent by utilizing the estimate of the disturbance and the prediction of the relative state information. Sufficient delay-dependent conditions are derived to guarantee consensus with disturbance rejection.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Publications

1). C. Wang, Z. Zuo, Z. Lin and Z. Ding. Consensus control of a class of Lipschitz nonlinear systems with input delay, IEEE Trans. Circuit and Systems I., vol. 62, no. 11, pp. 2730-2738, 2015.

2). C. Wang, Z. Zuo, Z. Lin and Z. Ding. A truncated prediction approach to consensus control of Lipschitz nonlinear multi-agent systems with input delay, IEEE Trans. Control Netw. Syst., 2016. Online Published.

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Abbreviations

MASs	**M**ulti-**A**gent **S**ystems
ODEs	**O**rdinary **D**ifferential **E**quations
UAV	**U**nmanned **A**erial **V**ehicles
AUV	**A**utonomous **U**nderwater **V**ehicles
WRS	**W**heeled **R**obots **S**ystem
AHS	**A**utomated **H**ighway **S**ystem
LMI	**L**inear **M**atrix **I**nequality
TPF	**T**runcated **P**rediction **F**eedback
DOBC	**D**isturbance **O**bserver-**B**ased **C**ontrol

Symbols

$\mathbb{R}$ set of real numbers

$\mathbb{C}$ set of complex numbers

$\mathbb{R}^n$ n-dimensional Euclidean space

$\mathbb{R}^{n\times m}$ set of $n\times m$ real matrices

$0, 0_{n\times m}$ $n\times m$ zero matrix

$\mathbf{1}, \mathbf{1}_n$ a column vector with all entries equal to one

I, I_n the identity matrix with appropriate dimension

$||x||$ the Euclidean norm of the vector x

A^T transpose of the matrix A

A^H conjugate transpose of the matrix A

A^{-1} inverse of the matrix A

$\mathrm{tr}(A)$ trace of the matrix A

$\mathrm{diag}(A_i)$ a block-diagonal matrix with $A_i, i = 1, \cdots, N$, on the diagonal

$||A||$ the induced 2-norm of the matrix A

$||A||_F$ $||A||_F = \sqrt{\mathrm{tr}(A^T A)}$, the Frobenius norm of the matrix A

$\lambda_\sigma(A)$ the maximum singular value of the matrix A

$\lambda_{\min}(A)$ the minimum eigenvalue of the matrix A

$\lambda_{\max}(A)$ the maximum eigenvalue of the matrix A

$\mathcal{L}_2\,[0,\infty)$ the space of square integrable functions over $[0,\infty)$

$\mathcal{L}_2^p\,[0,\infty)$ the space of p-dimensional square integrable functions over $[0,\infty)$

$||f||_2$ $\left(\int_0^\infty ||f(t)||^2\,\mathrm{d}t\right)^{1/2}$ for $f \in \mathcal{L}_2\,[0,\infty)$

$||\mathcal{G}||_\infty$ H_∞ norm of the operator $\mathcal{G}$

$\otimes$ the Kronecker product of matrices

$\in$ belong to

$A > B$ $A - B$ is positive definite

$A \geq B$ $A - B$ is positive semi-definite

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Chapter 1

Introduction

1.1 Background

In decades, the cooperative control of multi-agent systems has emerged as an attractive area of research. In the formulation of cooperative control, there are two types of methods in the literature: centralized method and distributed method. Compared to centralized method which uses a core cell to control and connect a group of agents, the distributed method, which uses the local information to design distributed controllers, brings a number of benefits. By using low-cost sensors and processors to replace the expensive core unit, distributed multi-agent systems can reduce the cost effectively; the motion of the agent only relay on the local relative information from the neighbours, which reduces the signal communication and computational workload [1]; furthermore, distributed multi-agent systems are more tolerant to bad environment, since failure of one agent does not seriously effect the performance of the whole system.

The developments of multi-agent systems also get well support from the improvement of sensor technology, communication technique, and modern control theory. As smaller, more accurate and reliable sensor and communication system come out, the strategies using cooperative group of agents to implement a certain task become possible and applicable [2].

As a result, the distributed control of multi-agent systems has drawn increased attention in recent years. Typical problems include consensus [3, 4], flocking [5], swarming [6], formation control [7], and synchronization [8]. There are numerous applications of multi-agent systems in the real world including wheeled robotics system [9], satellites [10, 11], autonomous underwater vehicles (AUV) [12, 13], spacecraft [14], unmanned aerial vehicles(UAV) [15, 16], automated highway systems [17], sensor network [8], surveillance [18], smart grid [19], and so on. Additionally, due to different aims, the cooperative control of multiple mobile robots can implement some specified tasks such as distributed manipulation [20], mapping of unknown or environments [21, 22], rural search and rescue [23], transportation of large objects [24, 25].

While the multi-agent systems research brings many benefits, it also introduces challenges for the researchers. There are three key components for the control of a group of agents: the agent dynamics, the network communication between the agents and the cooperative control laws required to achieve the desired group behaviours [26]. In many applications, the technical challenge is to design consensus algorithms for the agents to reach an agreement upon certain quantities of interest. In addition, time delay may exist during the process of information communication between the agents. If not taken into consideration a priori, delays will degrade the performance of the closed-loop systems and, in the extreme situations, may even cause the loss of stability. Last but not the least, in the real-world application, it is vital to handle a dynamic changing environment. The agents must have the ability that cope with various unknown events which may interfere with the process or disrupt the implementation.

1.2 Research Problems

This thesis studies the consensus problems though the tools from control theory and graph theory with emphasis on the input delay, intrinsic nonlinear agent dynamics, and external disturbances existing in the multi-agent systems. Specifically, we will focus on the following consensus problems.

- For general linear multi-agent systems with input delay and parametric uncertainties under directed network connection, the distributed protocols design for the closed-loop systems to reach robust consensus will be investigated.
- The consensus analysis for nonlinear systems with delay is more involved due to certain restrictions the nonlinearity imposes on using the information of the individual systems. The impacts of Lipschitz nonlinearity and input delay in consensus control will be investigated.
- The practical physical systems often suffer from external disturbances. For general multi-agent systems with input delay and bounded external disturbances, the H_∞ consensus analysis will be investigated under directed network connection.

- For Lipschitz nonlinear multi-agent systems with input delay and a class of unknown external disturbances, the consensus disturbance rejection problem based on DOBC approach will be investigated.

1.3 Overview of Related Work

The concept of multi-agent coordination is initially inspired by the observations and descriptions of collective behaviour in nature, such as fish schooling, bird flocking and insect swarming [27]. These behaviours may have advantages of seeking foods, migrating, or avoiding predators and obstacles, and therefore the study of such behaviours has drawn increased attention from researchers in various fields [28]. In 1987, three simple rules, separation (collision avoidance), alignment (velocity matching) and cohesion (flock centring), were proposed by Reynolds [29] to summarise the key characteristics of a group of biology agents. After that, a simple model was introduced by Vicsek [30] in 1995 to investigate the emergence of self-ordered motion in systems of particles with biologically motivated interaction. The flocking behaviours were later theoretically studied in [31–34].

In this section, the related work will be reviewed. First, we introduce the general overview of consensus control. Then, the delay effects existing in the multi-agent systems are reviewed. Finally, we will introduce the uncertainties and disturbances issues in control of multi-agent systems which may interfere the process or disrupt implementation.

1.3.1 Consensus Control

Consensus is a fundamental problem in cooperative control of multi-agent system, since many applications are based on the consensus algorithm design. In multi-agent systems, consensus problem means how to design the control strategy for a group of agents to reach a consensus (or agreement) as whole. The basic idea is that each agent updates its information state based on the information states of its local neighbours in such a way that the final information state of each agent converges to a common value [18]. One significant contribution in consensus control is due to the application of graph theory, particularly, the Laplacian matrix of the network connection, to the control design of the multi-agent systems for the network links between the agents [14,35,36]. A general framework of the consensus problem for networks of integrators was proposed in [4]. Since then, consensus problems have been intensively studied in different directions in the literature.

In terms of the agent dynamics, consensus problems under various systems dynamics have been massively investigated. The system dynamics has impactful influence on the final consensus state of the multi-agent systems. For example, the consensus state of multi-agent systems with single integrator dynamics often converges to a constant value, meanwhile, consensus for second-order dynamics might converge to a dynamic final value (i.e., a time function) [2]. Many early results on consensus problems are based on simple agent dynamics such as first or second-order integrators dynamics [3, 37–41]. However, in reality a large class of practical physical systems cannot be feedback linearised as first or second-order dynamical model. For instance, for a group of Unmanned Air Vehicles (UAV) [42], higher-order dynamic models may be needed. More complicated agent dynamics are described by high-order linear multi-agent systems in [43–48]. After that, the results were extended to nonlinear multi-agent systems [49–54]. Consensus for nonlinear systems is more involved than that for their linear systems counterparts. The difficulty of consensus control for nonlinear systems owes to certain restrictions the nonlinearity imposes on using the information of the individual systems. Consensus control for second-order Lipschitz nonlinear multi-agent systems was addressed in [55]. The consensus problems of high-order multi-agent systems with nonlinear dynamics were studied in [49–51, 54]. The works [52, 53] address the consensus output regulation problem of nonlinear multi-agent systems. A common assumption in the previous results is that the dynamics of the agents are identical and precisely known, which might not be practical in many circumstances. Due to the existence of the non-identical uncertainties, the consensus control of heterogeneous multi-agent systems was studied in [56–59].

The communication connections between the agents are also playing an important role in consensus problems. Most of the existing results are based on fixed communication topology, which indicates that the Laplacian matrix $\mathcal{L}$ is a constant matrix (see Chapter 2 for graph theory notations). It is pointed out in [26, 60] that the consensus is reachable if and only if zero is a simple eigenvalue of $\mathcal{L}$. If zero is not a simple eigenvalue of $\mathcal{L}$, the agents cannot reach consensus asymptotically as there exist at least two separate subgroups or at least two agents in the group who do not receive any information [61]. It is also known that zero is a simple eigenvalue of $\mathcal{L}$ if and only if the directed communication topology has a directed spanning tree or the undirected communication topology is connected [3, 4]. The results with directed graphs are more involved than those with undirected graphs. The main problem is that the Laplacian matrix associated with a directed graph is generally not positive semi-definite [28]. The decomposition method for the undirected systems cannot be applied to the directed one due to this unfavourable feature. For the consensus control with directed communication graphs, the balanced and/or strong connected conditions are needed in [62, 63], which are stronger than the directed spanning tree condition. In practice, the communication between the agents may not be fixed due to technological limitations of sensors or link failures. The consensus control of multi-agent systems with switching topologies has been investigated in [4,54,64]. In [55] and [65], consensus control with communication constraint and Markovian communication failure were studied.

In term of the number of leader, the above researches can also be roughly specified into three classes, that is, leaderless consensus (consensus without a leader) [43, 51] whose agreement value depends on the initial states of the agents, leader-follower consensus (or consensus tracking) [38, 54] which has a leader agent to determine the final consensus value, and containment control [40, 41] where has more than one leader in agent networks. Compared to leaderless consensus, consensus tracking, and containment control have the advantages to determine the final consensus value in advance [48].

1.3.2 Delay Effects in Consensus Control

Time delays widely exist in practical systems due to the time taken for transmission of signals, transport of materials, etc. The presence of time delays, if not considered in the controller design, may seriously degrade the performance of the controlled system and, in the extreme situations, may even cause the loss of stability. Therefore, the stabilization of time-delay systems has attracted much attention in both academic and industrial communities; see the surveys [66, 67], the monographs [68–70], and the references therein.

In the formulation of stabilization of time-delay systems, there are two types of feedback methods in the literature: standard (memoryless) feedback and predictive (memory) feedback. Memoryless controllers are useful for the systems with state delays [71–75]. However, it is known that system with input delay is more difficult to handle in control theory [67]. For predictive feedback, compensation is added in the controller design to offset the adverse effect of the time delay and the stabilization problems are reduced to similar problems for ordinary differential equations. A wide variety of predictor-based methods such as Smith predictor [76], modified

Smith predictor [77], finite spectrum assignment [?], and Artstein-Kwon-Pearson reduction method [78, 79] are effective and efficient when the delay is too large to be neglected and a standard (memoryless) feedback would fail. However a drawback of the predictor-based methods is that the controllers involve integral terms of the control input, resulting in difficulty for the control implementation [80]. A halfway solution between these two methods is to ignore the troublesome integral part, and use the prediction based on the exponential of the system matrix, which is known as the Truncated Prediction Feedback (TPF) approach. This idea started from low-gain control of the systems with input saturation [81], then it was developed for linear systems [82–84], and nonlinear systems [85, 86].

With the Internet and other communication tools used in the consensus control, time delay due to data transmission occurs more often [85]. In particular, the consensus time delay occurs in the control input when the protocols depend on the relative state information transmitted over the network. Consensus with input delay has been extensively studied in the literature (see [16, 57, 87–97] and the references therein). The early results of consensus with time delay in [4, 98–105] focus on analysing the stability of consensus algorithms with time delay for first or second-order integrators dynamics and finding the upper bounds on the time delays such that the consensus can still be achieved in the presence of time delay. Based on a Linear Matrix Inequality (LMI) method, consensus for directed networks of integrators with non-uniform time-varying delays was investigated in [105]. Necessary and sufficient condition was derived in [57] for a class of high-order multi-agent systems with communication delays. TPF approach was applied in [106] for linear high-order multi-agent systems where the open-loop dynamics of the agents is restricted to be not exponentially unstable. The controlled consensus problem of multi-agent systems with nonlinear agent dynamics and communication delay are more complicated and just a few results have been obtained [107, 108]. Furthermore, the challenge to improve the convergence speed of a consensus protocol with communication delay seems to be rarely studied in the literature [109].

1.3.3 Uncertainties and Disturbances in Consensus Control

A common assumption in the previous studies is that the dynamics of the agents are identical and precisely known, which might be restrictive in many circumstances. The practical physical systems often suffer from uncertainties which may be caused by mutations in system parameters, modelling errors or some ignored factors [110]. The robust consensus problem of multi-agent systems has formed into a hot topic in recent years. Han et al. investigated the robust consensus problem for multi-agent systems with continuous-time and discrete-time dynamics in [111] and [112], where the weighted adjacency matrix is a polynomial function of uncertain parameters. In particular, the H_∞ robust control problem was investigated in [113] for a group of autonomous agents governed by uncertain general linear node dynamics. However, most of the existing results on consensus control of uncertain multi-agent systems were often restricted to certain conditions, like single or double integrators [114, 115], or undirected network connections [116].

Other than the parameter uncertainties, the agents may also be subject to unknown external disturbances, which might degrade the system performance and even cause the network system to diverge or oscillate. The consensus problems of multi-agent systems with performance requirements have emerged as a challenge topic in recent years. The robust H_∞ consensus problems were investigated for multi-agent systems with first and second-order integrators dynamics in [115, 117]. The H_∞ consensus problems for general linear dynamics with undirected graphs were studied in [118, 119]. The results obtained in [118] were extended to directed graph in [63]. The H_∞ consensus problems for switching directed topologies were investigated in [120, 121]. The nonlinear H_∞ consensus problem was studied in [62] with directed graph. Global H_∞ pinning synchronization problem for a class of directed networks with aperiodic sampled-data communications was addressed in [122]. It is worth noting that the directed graphs in [62,63] are restricted to be balanced or strongly connected. The main problem is that the Laplacian matrix associated with a directed graph is generally not positive semi-definite [28]. The decomposition method developed in [118] for the undirected systems cannot be applied to the directed one due to this unfavourable feature. Until now, it is still an active research area to achieve H_∞ consensus control in general directed multi-agent systems. Besides the external disturbances, the unavoidable model and parameter uncertainties in the multi-agent systems, which may resulting from modelling errors and varying environmental parameters, were also considered in [115, 117].

H_∞ control has been proved to be effective for disturbance rejection of the multi-agent systems with external disturbances bounded by H_2 norms. However, disturbances in real engineering problems are often periodic and have inherent characteristics such as harmonics and unknown constant load [123]. For those kinds of disturbances, it is desirable by utilizing the disturbance information in the design of control input to cancel the disturbances directly. One common design method is to estimate the disturbance by using the measurements of states or outputs and then use the disturbance estimate to compensate the influence of the disturbance on the system, which is referred to as Disturbance Observer-Based Control (DOBC) [124]. Using DOBC, consensus of second-order multi-agent dynamical systems with exogenous disturbances was studied in [125, 126] for matched disturbances and in [127] for unmatched disturbances. Disturbance observer based tracking controllers for high-order integrator-type and general multi-agent systems were proposed in [128, 129], respectively. A systematic study on consensus disturbance rejection via disturbance observers could be found in [42]. Note that most existing results are limited to linear systems and there is a lack of study on consensus disturbance rejection for nonlinear multi-agent systems. The protocol design for nonlinear multi-agent systems with input delay is more involved due to the unpredictable disturbances. For example, the well-known model reduction method [79] cannot be applied for the consensus disturbance rejection problem.

1.4 Contributions and Organization

The objective of this thesis is to study consensus control problems for general multi-agent systems with input delay. The main contributions include the followings:

- For general linear multi-agent systems with input delay and parameter uncertainties, sufficient conditions for the robust consensus problem under directed communication topology are identified using Lyapunov tools in the time domain.

- For a class of Lipschitz nonlinear multi-agent systems with input delay, the impacts of Lipschitz nonlinearity and input delay in consensus control is investigated. Conditions based on the Lipschitz constant are identied for proposed consensus protocols to tackle Lipschitz nonlinear terms in the system dynamics.

- The H_∞ consensus control for general multi-agent systems with input delay and external disturbances bounded by H_2 norms is investigated. Sufficient delay-dependent conditions are derived for the multi-agent systems to guarantee the H_∞ consensus.

- For Lipschitz nonlinear multi-agent systems with input delay and unknown external disturbances, DOBC method is applied and sufficient conditions is derived to guarantee consensus with disturbance rejection.

Overall, this thesis consists of seven chapters. The organization of each chapter is described in detail at the beginning of that chapter. To understand the whole thesis structure, a brief introduction of these chapters is given as follows.

In Chapter 1, we first introduce the background of consensus control of multi-agent systems. Then, we review the related work, including the overview of consensus control, delay effects, uncertainties and external disturbances existing in consensus control. The main contributions of this thesis is also introduced.

In Chapter 2, some related preliminaries, including mathematical notations, matrix theory, stability theory, basic algebraic graph theory, and preliminary results used in this thesis, are introduced.

In Chapter 3, we systematically investigate the consensus control for general linear multi-agent systems with parametric uncertainties and input delay. Artstein model reduction method, one of the most well-know predictor feedback approaches, is used to deal with the input delay. Due to the existence of parametric uncertainties, the system can not be completely transformed into a delay-free one. Further endeavours are made to ensure that the extra integral term, which remains in the system dynamics after transformation, is properly considered. The significance of this research is to provide a feasible method to deal with the robust consensus control for uncertain multi-agent systems with input delay.

In Chapter 4, we systematically investigate the consensus control problem for Lipschitz nonlinear multi-agent systems with input delay. Artstein model reduction method and TPF approach are adopted to design the consensus protocols such that the delays can be compensated. Judicious analysis is carried out to tackle the influence of the nonlinear terms under the state transformation. By transforming the Laplacian matrix into the real Jordan form, global stability analysis is put in the framework of Lyapunov functions in real domain. For the control design, only the relative information obtained via the network connection is used, without local feedback control of the agents. Sufficient conditions are derived for the multi-agent systems to guarantee the global consensus in the time domain. The conditions can be solved as LMIs (linear matrix inequalities) with a set of iterative scalar parameters.

In Chapter 5, we consider the H_∞ consensus control for high-order multi-agent systems with general directed graph and input delay. The connection graph between the agents only needs a directed spanning tree, which is essential for consensus control, rather than the balanced or strongly connected conditions. The input delay caused by the communications between the agents is considered by using the TPF method.

The troublesome integral term involved in traditional predictor feedback approaches is ignored, and only the prediction based on the exponential of the systems matrix is used for control design. The proposed TPF controller is infinite and easy to implement.

In Chapter 6, we consider the consensus disturbance rejection problem for Lipschitz nonlinear multi-agent systems with input delay based on the DOBC approach. Different from the conventional predictor feedback approach, a non-ideal predictor based control scheme is constructed for each agent by using the estimate of the disturbance and the prediction of the relative state. Rigorous analysis within the framework of Lyapunov-Krasovskii functionals is carried out to guarantee that the extra integral terms of the system state associated with nonlinear functions are properly considered. Sufficient conditions are derived for the multi-agent systems to guarantee the consensus disturbance rejection.

In Chapter 7, we summarize the thesis and discuss the future research directions.

Chapter 2

Preliminaries

2.1 Matrix Theory

In this section, some mathematical notations and basic definitions that will be used in the remainder of this thesis are provided. The main references in this section are [28, 85, 130].

Definition 2.1.1. *The Kronecker product of matrix $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is defined as*

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix},$$

and they have following properties:

1. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$,
2. $(A \otimes B) + (A \otimes C) = A \otimes (B + C)$,
3. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$,
4. $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$,
5. $(A \otimes B)^T = A^T \otimes B^T$,
6. *If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ are both positive definite (positive semi-definite), so is $A \otimes B$,*

where the matrices are assumed to be compatible for multiplication.

Lemma 2.1.1 (Gershgorin's Disc Theorem [130]). *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, let*

$$R_i'(A) \equiv \sum_{j=1, j\neq i}^{n} \left|a_{ij}\right|, i = 1, 2, \cdots, n$$

denote the deleted absolute row sums of A, and consider the n Gersgorin discs

$$\left\{z \in \mathbb{C} : |z - a_{ii}| \leq R_i'(A)\right\}, i = 1, 2, \cdots, n.$$

The eigenvalues of A in the union of Gersgorin discs are given by

$$G(A) = \bigcup_{i=1}^{n} \left\{z \in \mathbb{C} : |z - a_{ii}| \leq R_i'(A)\right\}.$$

Furthermore, if the union of k of the n discs that comprise $G(A)$ forms a set $G_k(A)$ that is disjoint from the remaining $n-k$ discs, then $G_k(A)$ contains exactly k eigenvalues of A, counted according to their algebraic multiplicities.

Definition 2.1.2. *[130] A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is diagonally dominant if*

$$|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} \left|a_{ij}\right| = R_i'(A), \ \forall i = 1, 2, \cdots, n.$$

It is strictly diagonally dominant if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} \left|a_{ij}\right| = R_i'(A), \ \forall i = 1, 2, \cdots, n.$$

Definition 2.1.3 (M-matrix, Definition 6 in [28]). *A square matrix $A \in \mathbb{R}^{n\times n}$ is called a singular (nonsingular) M-matrix, if all its off-diagonal elements are non-positive and all eigenvalues of A have nonnegative (positive) real parts.*

Lemma 2.1.2 (Schur Complement Lemma). *For any constant symmetric matrix*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix},$$

the following statements are equivalent:

(1) $S < 0$,

(2) $S_{11} < 0, \ S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$,

(3) $S_{22} < 0, S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0$.

Lemma 2.1.3 (Youngs Inequality)**.** *For any given* $a, b \in \mathbb{R}^n$*, we have*

$$2a^T SQb \leq a^T SPS^T a + b^T Q^T P^{-1} Qb,$$

where $P > 0$*, S and Q have appropriate dimensions.*

Lemma 2.1.4 (Hölder's Inequality)**.** *For* $x \in \mathbb{R}^n$ *and* $y \in \mathbb{R}^n$*, if* $p > 1$ *and* $q > 1$ *are real numbers such that* $1/p + 1/q = 1$*, then*

$$\sum_{i=1}^{n} |x_i y_i| \leq \left\{ \sum_{i=1}^{n} |x_i|^p \right\}^{1/p} \left\{ \sum_{i=1}^{n} |y_i|^q \right\}^{1/q}.$$

Lemma 2.1.5 (Jensen's Inequality in [68])**.** *For a positive definite matrix P, and a function* $x : [a, b] \to \mathbb{R}^n$*, with* $a, b \in \mathbb{R}$ *and* $b > a$*, the following inequality holds:*

$$\left(\int_a^b x^T(\tau) \mathrm{d}\tau \right) P \left(\int_a^b x(\tau) \mathrm{d}\tau \right) \leq \ (b-a) \int_a^b x^T(\tau) P x(\tau) \mathrm{d}\tau. \tag{2.1}$$

Lemma 2.1.6 ([85])**.** *For a positive definite matrix P, the following identity holds*

$$e^{A^T t} P e^{At} - e^{\omega_1 t} P = -e^{\omega_1 t} \int_0^t e^{-\omega_1 \tau} e^{A^T \tau} R e^{A\tau} \mathrm{d}\tau, \tag{2.2}$$

where

$$\bar{R} = -A^T P - PA + \omega_1 P.$$

Furthermore, if $\bar{R}$ *is positive definite,* $\forall t > 0$*,*

$$e^{A^T t} P e^{At} < e^{\omega_1 t} P. \tag{2.3}$$

2.2 Stability Theory

In this section, some basic concepts of stability theorems based on Lyapunov functions are provided. The material in this section is from Ding [131].

Consider a nonlinear system

$$\dot{x} = f(x), \tag{2.4}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$ is the state of the system, and $f : \mathcal{D} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function, with $x = 0$ as an equilibrium point, that is $f(0) = 0$, and with $x = 0$ as an interior point of $\mathcal{D}$. $\mathcal{D}$ denotes a domain around the equilibrium $x = 0$.

Definition 2.2.1 (Lyapunov stability, Definition 4.1 in [131])**.** *For the system (2.4), the equilibrium point $x = 0$ is said to be Lyapunov stable if for any given positive real number R, there exists a positive real number r to ensure that$\|x(t)\| < R$ for all $t > 0$ if $\|x(0)\| < r$. Otherwise, the equilibrium point is unstable*

Definition 2.2.2 (Asymptotic stability, Definition 4.2 in [131])**.** *For the system (2.4), the equilibrium point $x = 0$ is asymptotically stable if it is stable (Lyapunov) and furthermore* $\lim_{t\to\infty} x(t) = 0$.

Definition 2.2.3 (Exponential stability, Definition 4.3 in [131])**.** *For the system (2.4), the equilibrium point $x = 0$ is exponential stable if there exist two positive real numbers α and λ such that the following inequality holds:*

$$\|x(t)\| < \alpha \|x(0)\| \exp^{-\lambda t},$$

for $t > 0$ in some neighbourhood $\mathcal{D} \subset \mathbb{R}^n$ containing the equilibrium point.

Definition 2.2.4 (Globally asymptotic stability, Definition 4.4 in [131])**.** *If the asymptotic stability defined in Definition 2.2.2 holds for any initial state in $\mathbb{R}^n$, the equilibrium point is said to be globally asymptotically stable.*

Definition 2.2.5 (Globally exponential stability, Definition 4.5 in [131])**.** *If the exponential stability defined in Definition 2.2.3 holds for any initial state in $\mathbb{R}^n$, the equilibrium point is said to be globally exponentially stable.*

Definition 2.2.6 (Positive definite function, Definition 4.6 in [131])**.** *A function $V(x) \in \mathcal{D} \subset \mathbb{R}^n$ is said to be locally positive definite if $V(x) > 0$ for $x \in \mathcal{D}$ except at $x = 0$ where $V(x) = 0$. If $\mathcal{D} = \mathbb{R}^n$, i.e., the above property holds for the entire state space, $V(x)$ is said to be globally positive definite.*

Definition 2.2.7 (Lyapunov function, Definition 4.7 in [131])**.** *If in $\mathcal{D} \in \mathbb{R}^n$ containing the equilibrium point $x = 0$, the function $V(x)$ is positive definite and has continuous partial derivatives, and if its time derivative along any state trajectory of system (2.4) is non-positive, i.e.,*

$$\dot{V}(x) \leq 0,$$

then $V(x)$ is a Lyapunov function.

Definition 2.2.8 (Radially unbounded function, Definition 4.8 in [131]). *A positive definite function $V(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be radially unbounded if $V(x) \longrightarrow \infty$ as $\|x\| \longrightarrow \infty$.*

Theorem 2.2.1 (Lyapunov theorem for global stability, Theorem 4.3 in [131]). *For the system (2.4) with $\mathcal{D} \in \mathbb{R}^n$, if there exists a function $V(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ with continuous first order derivatives such that*

- *$V(x)$ is positive definite*
- *$\dot{V}(x)$ is negative definite*
- *$V(x)$ is radially unbounded*

then the equilibrium point $x = 0$ is globally asymptotically stable.

2.3 Basic Algebraic Graph Theory

This section introduces some knowledge relates to graph theory, which is fundamental in consensus control. The material in this section is from [28, 51, 132].

The graph theory has been introduced by Leonard Euler in year 1736. Generally it is convenient to model the information exchanges among agents by directed or undirected graphs. A directed graph $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$, in which $\mathcal{V} \triangleq \{v_1, v_2, \cdots, v_N\}$ is the set of nodes, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edge with the ordered pair of nodes. A vertex represents an agent, and each edge represents a connection. A weighted graph associates a weight with every edge in the graph. Self loops in the form of (v_i, v_i) are excluded unless otherwise indicated. The edge (v_i, v_j) in the edge set $\mathcal{E}$ denotes that agent v_j can obtain information from agent v_i, but not necessarily vice versa. For an edge (v_i, v_j), node v_i is called the parent node, v_j is the child node, and v_i is a neighbour of v_j . The set of neighbours of node v_i is denoted as $\mathcal{N}_i$, whose cardinality is called the in-degree of node v_i.

A graph is defined as being balanced when it has the same number of ingoing and outgoing edges for all the nodes (edge (v_i, v_j) is said to be outgoing with respect to node v_i and incoming with respect to v_j). A graph with the property that $(v_i, v_j) \in \mathcal{E}$ implies $(v_j, v_i) \in \mathcal{E}$ for any $v_i, v_j \in \mathcal{V}$ is said to be undirected, where the edge (v_i, v_j) denotes that agents v_i and v_j can obtain information from each other. Clearly, an undirected graph is a special balanced graph.

A directed path from node v_{i_1} to node v_{i_l} is a sequence of ordered edges of the form $(v_{i_k}, v_{i_{k+1}}), k = 1, 2, \cdots, l-1$. An undirected path in an undirected graph is defined analogously. A cycle is a directed path that starts and ends at the same node. A directed graph is strongly connected if there is a directed path from every node to every other node. Note that for an undirected graph, strong connectedness is simply termed connectedness. A directed graph is complete if there is an edge from every node to every other node. A undirected tree is an undirected graph where all the nodes can be connected by the way of a single undirected path. A (rooted) directed tree is a directed graph in which every node has exactly one parent except for one node, called the root, which has no parent and has directed paths to all other nodes. A directed tree is defined as spanning when it connects all the nodes in the graph. It can be demonstrated that this implies that there is at least one root node connected with a simple path to all the other nodes. A graph is said to have or contain a directed spanning tree if a subset of the edges forms a directed spanning tree. This is equivalent to saying that the graph has at least one node with directed paths to all other nodes. For undirected graphs, the existence of a directed spanning tree is equivalent to being connected. However, in directed graphs, the existence of a directed spanning tree is a weaker condition than being strongly connected. A strongly connected graph contains at least one directed spanning tree.

Associated with the communication graph is its adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, where the element a_{ij} denotes the connection between the agent i and agent j. $a_{ij} = 1$ if $(j,i) \in \mathcal{E}$, otherwise is zero, and $a_{ii} = 0$ for all nodes with the assumption that there exists no self loop. In the directed graph $\mathcal{G}$, $(i, j) \in \mathcal{E}$ denotes that the jth agent can obtain the information from the ith agent, but not vice versa. A directed path on the graph $\mathcal{G}$ from node i_1 to node i_s is a sequence of ordered edges as $(i_1,i_2), \cdots, (i_{s-1},i_s)$. A directed graph that contains a spanning tree is that there exists a node called the root, and this root has a directed path to every other node of the graph. The Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ is defined by $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$ when $i \neq j$.

From the definition of the Laplacian matrix and also the above example, it is easy to see that $\mathcal{L}$ is diagonally dominant and has nonnegative diagonal entries. Since $\mathcal{L}$ has zero row sums, 0 is an eigenvalue of $\mathcal{L}$ with an associated eigenvector 1. According to Gershgorins disc theorem, all nonzero eigenvalues of $\mathcal{L}$ are located within a disk in the complex plane centred at $d_{\max}$ and having radius of $d_{\max}$, where $d_{\max}$ denotes the maximum in-degree of all nodes. According to the definition of M-matrix in the last subsection, we know that the Laplacian matrix $\mathcal{L}$ is a singular M-matrix.

Lemma 2.3.1 ([3, 28]). *The Laplacian matrix $\mathcal{L}$ of a directed graph $\mathcal{G}$ has at least one zero eigenvalue with a corresponding right eigenvector $\mathbf{1} = [1, 1, \ldots, 1]^T$ and all nonzero eigenvalues have positive real parts. Furthermore, zero is a simple eigenvalue of $\mathcal{L}$ if and only if $\mathcal{G}$ has a directed spanning tree. In addition, there exists a nonnegative left eigenvector r of $\mathcal{L}$ associated with the zero eigenvalue, satisfying $r^T\mathcal{L} = 0$ and $r^T 1 = 1$. Moreover, r is unique if $\mathcal{G}$ has a directed spanning tree.*

Lemma 2.3.2 ([51]). *For a Laplacian matrix $\mathcal{L}$ that zero is a simple eigenvalue, there exists a similarity transformation T, with its first column being $T_1 = \mathbf{1}$, such that*

$$T^{-1}\mathcal{L}T = J, \tag{2.5}$$

with J being a block diagonal matrix in the real Jordan form

$$J = \begin{bmatrix} 0 & & & & & & \\ & J_1 & & & & & \\ & & \ddots & & & & \\ & & & J_p & & & \\ & & & & J_{p+1} & & \\ & & & & & \ddots & \\ & & & & & & J_q \end{bmatrix}, \tag{2.6}$$

where $J_k \in \mathbb{R}^{n_k}$, $k = 1, 2, \ldots, p$, are the Jordan blocks for real eigenvalues $\lambda_k > 0$ with the multiplicity n_k in the form

$$J_k = \begin{bmatrix} \lambda_k & 1 & & & \\ & \lambda_k & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_k & 1 \\ & & & & \lambda_k \end{bmatrix},$$

and $J_k \in \mathbb{R}^{2n_k}$, $k = p+1, p+2, \ldots, q$, are the Jordan blocks for conjugate eigenvalues

$\alpha_k \pm j\beta_k$, $\alpha_k > 0$ *and* $\beta_k > 0$, *with the multiplicity* n_k *in the form*

$$J_k = \begin{bmatrix} \nu(\alpha_k,\beta_k) & I_2 & & & \\ & \nu(\alpha_k,\beta_k) & I_2 & & \\ & & \ddots & \ddots & \\ & & & \nu(\alpha_k,\beta_k) & I_2 \\ & & & & \nu(\alpha_k,\beta_k) \end{bmatrix},$$

with I_2 *being the identity matrix in* $\mathbb{R}^{2\times 2}$ *and*

$$\nu(\alpha_k,\beta_k) = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

2.4 Basic Idea of Predictor Feedback Design

2.4.1 Predictor Feedback Design

In this section, we will recall the basic idea of predictor-based feedback design. Consider a linear input-delayed system

$$\dot{x}(t) = Ax(t) + Bu(t-h), \tag{2.7}$$

where $x \in \mathbb{R}^n$ denotes the state, $u \in \mathbb{R}^m$ denotes the control input, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices, $h \in \mathbb{R}_+$ is input delay, which is known and constant.

If delay is absent, we only need to find a stabilizing gain vector K such that the matrix $A+BK$ is Hurwitz. Accordingly, for input-delayed system (2.7), if we can have a control that achieves

$$u(t-h) = Kx(t), \tag{2.8}$$

the control problem of the input-delayed system (2.7) is solved with a stabilizing gain vector K.

Control (2.8) can be alternatively written as

$$u(t) = Kx(t+h). \tag{2.9}$$

It is unrealistic since the controller requires future values of the state x at time $t+h$

which cannot be obtained with direct measurement. However, with the variation-of-constants formula, the vector $x(t+h)$ can be calculated as follows

$$\begin{aligned} x(t+h) &= e^{A(t+h)}x(0) + \int_0^{t+h} \mathrm{e}^{A(t+h-\tau)}Bu(\tau-h)\mathrm{d}\tau \\ &= e^{Ah}x(t) + \int_{t-h}^{t} \mathrm{e}^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau, \qquad \forall t \geq 0. \end{aligned} \tag{2.10}$$

Therefore we can express the controller as

$$u(t) = K\left[e^{Ah}x(t) + \int_{t-h}^{t} \mathrm{e}^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau\right], \qquad \forall t \geq 0,$$

which is implementable, but it is infinite-dimensional, since it contains the distributed delay term involving past controls [70], $\int_{t-h}^{t} \mathrm{e}^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau$. The closed-loop system is fully delay-compensated,

$$\dot{x}(t) = (A+BK)x(t), \qquad t \geq h. \tag{2.11}$$

During the interval $t \in [0,h]$, the system state is governed by

$$x(t) = e^{A(t)}x(0) + \int_0^t \mathrm{e}^{A(t-\tau)}Bu(\tau-h)\mathrm{d}\tau, \forall t \in [0,h]. \tag{2.12}$$

2.4.2 Model Reduction Method

Based on the basic idea of predictor feedback design, the model reduction method for linear system with input delay was first developed by Kwon and Pierson in [78]. The results were then extended to time-varying system with distributed delays by Artstein in [79]. The outline of the method is as follows.

Consider a system

$$\dot{x}(t) = Ax(t) + Bu(t-h), \tag{2.13}$$

where $x \in \mathbb{R}^n$ denotes the state, $u \in \mathbb{R}^m$ denotes the control input, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices, h is input delay.

Let

$$z(t) = x(t) + \int_t^{t+h} \mathrm{e}^{A(t-\tau)}Bu(\tau-h)\mathrm{d}\tau, \tag{2.14}$$

Differentiating $z(t)$ against time yields

$$\begin{aligned}\dot{z}(t) &= Ax(t) + Bu(t-h) + \mathrm{e}^{-Ah}Bu(t) - Bu(t-h) + A\int_t^{t+h} \mathrm{e}^{A(t-\tau)}Bu(\tau-h)\mathrm{d}\tau \\ &= Az(t) + Du(t), \end{aligned} \tag{2.15}$$

where $D = \mathrm{e}^{-Ah}B$. The general form of the Leibniz integral rule has been used for this derivation.

The controllability of (A,B) and $(A, e^{-Ah}B)$ are equivalent as proved in [133]. We consider a controller

$$u(t) = Kz(t). \tag{2.16}$$

From (2.14) and (2.16), we have

$$\|x(t)\| \leq \|z(t)\| + h\left(\max_{-h\leq\theta\leq 0} \|e^{A\theta}\|\right)\|B\|\|K\|\|z_t(\theta)\|,$$

where $z_t(\theta) := z(t+\theta)$, $-h \leq \theta \leq 0$. Thus, $x(t) \to 0$ as $z(t) \to 0$. In other words, if the controller (2.16) stabilizes the transformed system (2.15), then the original system (2.13) is also stable with the same controller [78].

Remark 2.4.1. *With any given bounded initial condition $u(\theta)$, $\theta \in [-h, 0]$, a stable feedback controller* (2.16) *implies that $u(t)$ in* (2.14) *is locally integrable, which allows for the model reduction as* (2.15).

Remark 2.4.2. *By introducing a state transformation, the input-delayed linear system is transformed to a delay-free one which is finite dimensional, where the reduction in dimension is achieved.*

Chapter 3

Robust Consensus for Linear MASs

In this chapter, we systematically investigate the consensus control for general linear multi-agent systems with parameter uncertainties and communication delay. This kind of network communication delay can be formulated as the input delay when the inputs only depend on the relative state information transmitted via the network. The main contributions of this chapter are summarized as follows: (1) In this chapter, the robust consensus problem for general linear multi-agent systems is considered. Compared with [114, 115], the model under consideration is more general. (2) Compared to the previous works [111, 113, 116], the requirement for the communication graph in this chapter is more general. The connection graph between the agents only needs a directed spanning tree, which is essential for consensus control, rather than the undirected, balanced or strongly connected conditions. (3) Model reduction method is used to deal with the input delay and further endeavours are made to ensure that the extra integral term, which remains in the system dynamics after transformation due to the parameter uncertainties, is properly considered.

The rest of this chapter is organized as follows. Some notations and the problem formulation are given in Section 3.1. Section 3.2 presents the main results for the consensus analysis. Based on the analysis result in Section 3.2, Section 3.3 presents a consensus controller design method. Simulation results are included in Section 3.4. The content of this chapter is summarized in Section 3.5.

3.1 Problem Formulation

In this chapter, we consider the control design for a group of N uncertain subsystems with input delay, of which the subsystems are described by

$$\dot{x}_i(t) = [A + \Delta A(t)]x_i(t) + [B + \Delta B(t)]u_i(t-h), \tag{3.1}$$

where for subsystems $i = 1, 2, \ldots, N$, $x_i \in \mathbb{R}^n$ is the state vector, $u_i \in \mathbb{R}^m$ is the control input vector, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices with (A, B) being controllable, $h > 0$ is the input delay, the initial conditions $x_i(0)$ and $u_i(s)$, $s \in [-h, 0]$ are given. $\Delta A(t)$ and $\Delta B(t)$ are time-varying uncertain matrices, and are assumed to be of the form

$$\Delta A(t) = E\Sigma(t)F_1 \quad \text{and} \quad \Delta B(t) = E\Sigma(t)F_2, \tag{3.2}$$

where E, F_1 and F_2 are real constant matrices with appropriate dimensions, and $\Sigma(t)$ is an unknown real time-varying matrix that satisfies $\Sigma^T(t)\Sigma(t) \le I$.

Remark 3.1.1. *It is worth noticing that the agents in the network are nominally identical and the model uncertainty matrices satisfy the same form as (3.2). Different from the existing works that focus on the identical agents in the network, the terms ΔA and ΔB in (3.1) allow the agents to have different dynamics and the uncertainty is characterised by the time-varying matrix $\Sigma(t)$, which implies that each subsystem in the group can be non-identical. For the consensus design, only the bound of $\Sigma(t)$ (i.e., the worst case) is needed.*

Associated with the communication graph is its adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, where the element a_{ij} denotes the connection between the agent i and agent j. $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$, otherwise is zero, and $a_{ii} = 0$ for all nodes with the assumption that there exists no self loop. A directed graph that contains a spanning tree is that there exists a node called the root, and this root has a directed path to every other node of the graph. The Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ is defined by $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$ when $i \neq j$.

Assumption 3.1.1. *The eigenvalues of the Laplacian matrix $\mathcal{L}$ are distinct.*

Remark 3.1.2. *In terms of the network connections, this condition implies that the network has a spanning tree to connect any two subsystems in the system. As mentioned in Lemma 2.3.1, for consensus design, we only need that the eigenvalue at 0 is simple.*

Assumption 3.1.1 is a bit stronger than necessary, and it is adopted for the convenience of presentation of the proposed design.

The aim of the current chapter is to design a control strategy, using the relative state information to ensure that all the uncertain subsystems converge to an identical trajectory.

3.2 Consensus Analysis

By the model reduction method, we consider the following linear transformation for each agent

$$z_i(t) = x_i(t) + \int_t^{t+h} \mathrm{e}^{A(t-\tau)} B u_i(\tau - h) \mathrm{d}\tau.$$

The original multi-agent systems are transformed to

$$\begin{aligned} \dot{z}_i(t) =& (A + \Delta A) z_i(t) + D u_i(t) \\ &+ \Delta B u_i(t-h) - \Delta A \int_t^{t+h} \mathrm{e}^{A(t-\tau)} B u_i(\tau - h) \mathrm{d}\tau, \end{aligned} \tag{3.3}$$

where $D = e^{-Ah}B$. As seen in (3.3), (3.1) is not completely reduced to a delay-free system due to the parameter uncertainties.

We propose a control design using the relative state information. The control input takes the structure

$$\begin{aligned} u_i(t) =& -K \sum_{j=1}^{N} a_{ij} [z_i(t) - z_j(t)] \\ =& -K \sum_{j=1}^{N} l_{ij} z_j(t), \end{aligned} \tag{3.4}$$

where $K \in \mathbb{R}^{m \times n}$ is a constant control gain matrix to be designed later.

Remark 3.2.1. *It is worth noting from* (3.4) *that the proposed control only uses the relative state information of the subsystems via network connections.*

Remark 3.2.2. *Note that the information on each control input $u_i(t)$ on the time interval $[t-h, t]$ can be stored and used for control. In practical implementations, the discretization of an integral or some numerical quadrature method [134] can be used to approximate the integral term in the control input $u_i(t)$.*

Let $z(t) = [z_1^T, z_2^T, \dots, z_N^T]^T$, and the closed-loop system is then written as

$$\begin{aligned}\dot{z}(t) = & [I_N \otimes (A + \Delta A) - \mathcal{L} \otimes DK] z(t) \\ & - (\mathcal{L} \otimes \Delta BK) z(t-h) - (I_N \otimes \Delta A)\sigma(t),\end{aligned} \tag{3.5}$$

where $\sigma = [\sigma_1^T, \sigma_2^T, \dots, \sigma_N^T]^T$ with the elements defined by

$$\sigma_i = -\int_t^{t+h} \mathrm{e}^{A(t-\tau)} BK \sum_{j=1}^{N} l_{ij} z_j(\tau - h) \mathrm{d}\tau. \tag{3.6}$$

Let us define $\bar{r}^T = [\bar{r}_1, \bar{r}_2, \dots, \bar{r}_N] \in \mathbb{R}^N$ as the left eigenvector of $\mathcal{L}$ corresponding to the eigenvalue at 0, that is, $\bar{r}^T \mathcal{L} = 0$, and furthermore, we set $\bar{r}^T \mathbf{1} = 1$. It can be shown from Assumption 3.1.1 and Lemma 2.3.2 that there exists a non-singular matrix $\overline{T}$ with the first column $\overline{T}_{(1)} = \mathbf{1}$ and the first row of $\overline{T}^{-1}$, $\overline{T}^{-1}_{(1)} = \bar{r}^T$, such that

$$\overline{T}^{-1} \mathcal{L} \overline{T} = \bar{J}, \tag{3.7}$$

with $\bar{J}$ being a block diagonal matrix in the real Jordan form

$$\bar{J} = \begin{bmatrix} 0 & & & & & & \\ & \lambda_2 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_{n_\lambda} & & & \\ & & & & \nu_1 & & \\ & & & & & \ddots & \\ & & & & & & \nu_{n_\nu} \end{bmatrix},$$

where $\lambda_i \in \mathbb{R}$ for $i = 2, 3, \dots, n_\lambda$ and

$$\nu_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \in \mathbb{R}^{2\times 2},$$

for $i = n_\lambda + 1, n_\lambda + 2, \dots, N$. In the above expression of $\bar{J}$, λ_i, α_i and β_i are positive real numbers with λ_i being real eigenvalues and $\alpha_i \pm \beta_i$ conjugate eigenvalues of $\mathcal{L}$, respectively. Clearly we have $n_\lambda + 2n_\nu = N$. Note that the non-zero eigenvalues of $\mathcal{L}$ are positive or with positive real parts.

Based on the vector $\bar{r}$, we introduce a state transformation

$$\xi_i = z_i - \sum_{j=1}^{N} \bar{r}_j z_j, \tag{3.8}$$

for $i = 1, 2, \ldots, N$. With $\xi = [\xi_1^T, \xi_2^T, \ldots, \xi_N^T]^T$, we have

$$\xi = z - [(\mathbf{1}\bar{r}^T) \otimes I_n]z = (M \otimes I_n)z,$$

where $M \triangleq I_N - \mathbf{1}\bar{r}^T$. Since $\bar{r}^T\mathbf{1} = 1$, it can be shown that $M\mathbf{1} = 0$. Therefore the consensus of system (3.1) is achieved when $\xi = 0$, as $\xi = 0$ implies $z_1 = z_2 = \ldots = z_N$. The dynamics of ξ can then be obtained as

$$\begin{aligned} \dot{\xi}(t) &= (M \otimes I_n)\dot{z}(t) \\ &= [I_N \otimes (A + \Delta A) - \mathcal{L} \otimes DK]\,\xi(t) \\ &\quad - (\mathcal{L} \otimes \Delta BK)\xi(t-h) - (M \otimes I_n)(I_N \otimes \Delta A)\sigma. \end{aligned} \tag{3.9}$$

To explore the structure of $\mathcal{L}$, let us introduce another state transformation

$$\eta = (\overline{T}^{-1} \otimes I_n)\xi. \tag{3.10}$$

Then we have

$$\begin{aligned} \dot{\eta}(t) = [I_N \otimes (A + \Delta A) - \bar{J} \otimes DK]\eta(t) \\ - (\bar{J} \otimes \Delta BK)\eta(t-h) - \Psi(z), \end{aligned} \tag{3.11}$$

where $\Psi(z) = (\overline{T}^{-1} \otimes I_n)(M \otimes I_n)(I_N \otimes \Delta A)\sigma$. For the convenience, let $\eta = [\eta_1^T, \eta_2^T, \ldots, \eta_N^T]^T$ and $\Psi(z) = [\psi_1^T(z), \psi_2^T(z), \ldots, \psi_N^T(z)]^T$ with $\eta_i \in \mathbb{R}^n$ and $\psi_i : \mathbb{R}^{nN} \to \mathbb{R}^n$ for $i = 1, 2, \ldots, N$.

Hence, if we can design a control gain matrix K to ensure that η converges to zero asymptotically, the consensus control is achieved. With the control law shown in (3.4), the control gain matrix K is chosen as

$$K = D^T P, \tag{3.12}$$

where P is a positive definite matrix. In the remaining part of this section, we will use Lyapunov-function-based analysis to identify a condition for P to ensure that consensus is achieved by using the control algorithm (3.4) with the control gain K in (3.12).

Based on the state transformations (3.8) and (3.10), we have

$$\begin{aligned}\eta_1 &= (\bar{r}^T \otimes I_n)\xi \\ &= [(\bar{r}^T M) \otimes I_n]z \\ &\equiv 0.\end{aligned}$$

As discussed earlier, the consensus control can be guaranteed by showing that η converges to zero, which is sufficed by showing that η_i converge to zero for $i = 2, 3, \ldots, N$, since we have shown that $\eta_1 \equiv 0$.

In view of (3.11), for $i \in \{2, 3, \ldots, n_\lambda\}$, the dynamics of the subsystem state variables are given by

$$\dot{\eta}_i(t) = (A + \Delta A - \lambda_i DK)\eta_i(t) - \lambda_i \Delta BK\eta_i(t-h) - \psi_i.$$

For $k \in \{1, 2, \ldots, n_\nu\}$, we consider the dynamics of the subsystem state variables in pairs. For convenience of presentation, define

$$\begin{aligned}i_1(k) &= 1 + n_\lambda + 2k - 1, \\ i_2(k) &= 1 + n_\lambda + 2k.\end{aligned}$$

The dynamics of η_{i_1} and η_{i_2} are expressed by

$$\begin{aligned}\dot{\eta}_{i_1}(t) = &(A + \Delta A - \alpha_k DK)\eta_{i_1}(t) - \beta_k DK\eta_{i_2}(t) - \alpha_k \Delta BK\eta_{i_1}(t-h) \\ &- \beta_k \Delta BK\eta_{i_2}(t-h) - \psi_{i_1}, \\ \dot{\eta}_{i_2}(t) = &(A + \Delta A - \alpha_k DK)\eta_{i_2}(t) + \beta_k DK\eta_{i_1}(t) - \alpha_k \Delta BK\eta_{i_2}(t-h) \\ &+ \beta_k \Delta BK\eta_{i_1}(t-h) - \psi_{i_2}.\end{aligned}$$

Let

$$V_1 = \sum_{i=2}^{N} \eta_i^T(t) P \eta_i(t) \tag{3.13}$$

for $i = 2, 3, \ldots, N$. Then the derivative of V_1 can be directly calculated as

$$
\begin{aligned}
\dot{V}_1 = & \sum_{i=2}^{n_\lambda} \eta_i^T(t)\left(A^T P + PA - 2\lambda_i PDD^T P\right)\eta_i(t) \\
& + \sum_{i=n_\lambda+1}^{N} \eta_i^T(t)\left(A^T P + PA - 2\alpha_i PDD^T P\right)\eta_i(t) \\
& - 2\sum_{i=2}^{n_\lambda} \lambda_i \eta_i^T(t) P\Delta BD^T P\eta_i(t-h) - 2\sum_{i=n_\lambda+1}^{N} \alpha_i \eta_i^T(t) P\Delta BD^T P\eta_i(t-h) \\
& - 2\sum_{k=1}^{n_\nu/2} \beta_k \eta_{i_1(k)}^T(t) P\Delta BD^T P\eta_{i_2(k)}(t-h) + 2\sum_{k=1}^{n_\nu/2} \beta_k \eta_{i_2(k)}^T(t) P\Delta BD^T P\eta_{i_1(k)}(t-h) \\
& - 2\sum_{i=2}^{N} \left[\eta_i^T(t) P\Delta A\eta_i(t) + \eta_i^T(t) P\psi_i\right] \\
\leq & \sum_{i=2}^{N} \Bigg[\eta_i^T(t)\left(A^T P + PA - 2\underline{\alpha} PDD^T P\right)\eta_i(t) + \frac{1}{\mu}\eta_i^T(t) PEE^T P\eta_i(t) \\
& + \mu\eta_i^T(t) F_1^T F_1 \eta_i(t) + \frac{\bar{\alpha} + \bar{\beta}}{\varepsilon}\eta_i^T(t) PEE^T P\eta_i(t) \\
& - 2\eta_i^T P\Psi_i + \varepsilon(\bar{\alpha} + \bar{\beta})\eta_i^T(t-h) PDF_2^T F_2 D^T P\eta_i(t-h)\Bigg],
\end{aligned} \tag{3.14}
$$

where $\underline{\alpha} \triangleq \min\{\lambda_2, \ldots, \lambda_{n_\lambda}, \alpha_1, \ldots, \alpha_{n_\nu}\}$, $\bar{\alpha} \triangleq \max\{\lambda_2, \ldots, \lambda_{n_\lambda}, \alpha_1, \cdots, \alpha_{n_\nu}\}$, $\bar{\beta} \triangleq \max\{\beta_1, \ldots, \beta_{n_\nu}\}$. The inequality $\pm 2a^T b \leq \kappa a^T a + b^T b/\kappa$ has been used to deal with the uncertain terms $\Delta A(t) = E\Sigma(t)F_1$ and $\Delta B(t) = E\Sigma(t)F_2$.

The extra integral term Ψ in the transformed systems dynamic model (3.11) is expressed as a function of the state z. For the stability analysis, first we need to establish a bound of the integral function $-2\eta_i^T P\psi_i$ in terms of the transformed state η. From the state transformations (3.8) and (3.10), we have

$$
\Psi(z) = \left(\overline{T}^{-1} \otimes I_n\right)(M \otimes I_n)(I_N \otimes \Delta A)\sigma.
$$

Let

$$
\begin{aligned}
\Phi &= [\phi_1, \cdots, \phi_N]^T = (M \otimes I_n)\bar{\sigma}, \\
\bar{\sigma} &= [\bar{\sigma}_1, \cdots, \bar{\sigma}_N]^T = (I_N \otimes \Delta A)\sigma.
\end{aligned}
$$

Recalling $M = I_N - \mathbf{1}\bar{r}^T$, we can get

$$\begin{aligned}\phi_k &= \bar{\sigma}_k - \sum_{j=1}^{N} \bar{r}_j \bar{\sigma}_j \\ &= \Delta A \left(\sigma_k - \sum_{j=1}^{N} \bar{r}_j \sigma_j \right), \\ \psi_i &= (\tau_i \otimes I_n)\Phi \\ &= \sum_{k=1}^{N} \tau_{ik} \phi_k \\ &= \sum_{k=1}^{N} \tau_{ik} \Delta A \left(\sigma_k - \sum_{j=1}^{N} \bar{r}_j \sigma_j \right) \\ &= \Delta A \sum_{k=1}^{N} \tau_{ik} \sigma_k - \Delta A \sum_{k=1}^{N} \tau_{ik} \sum_{j=1}^{N} \bar{r}_j \sigma_j,\end{aligned}$$

where τ_i is the ith row of $\overline{T}^{-1}$. It then follows that

$$\begin{aligned}-2\eta_i^T P \psi_i =& 2\eta_i^T P \Delta A \sum_{k=1}^{N} \tau_{ik} \sum_{j=1}^{N} \bar{r}_j \sigma_j - 2\eta_i^T P \Delta A \sum_{k=1}^{N} \tau_{ik} \sigma_k \\ \le& \frac{2}{\rho} \eta_i^T P E E^T P \eta_i + \rho \left(\sum_{k=1}^{N} \tau_{ik} \sigma_k \right)^T F_1^T F_1 \left(\sum_{k=1}^{N} \tau_{ik} \sigma_k \right) \\ &+ \rho \left(\sum_{k=1}^{N} \tau_{ik} \sum_{j=1}^{N} r_j \sigma_j \right)^T F_1^T F_1 \left(\sum_{k=1}^{N} \tau_{ik} \sum_{j=1}^{N} r_j \sigma_j \right) \\ \le& \frac{2}{\rho} \eta_i^T P E E^T P \eta_i + \rho \left(\|\tau_i\| 2^2 + \|\tau_i\|^2 \|r\|^2 \right) \sum_{k=1}^{N} \|F_1 \sigma_k\|^2. \end{aligned} \tag{3.15}$$

From (3.4) and (3.6), we have

$$\begin{aligned}
\sigma_k =& -\int_t^{t+h} \mathrm{e}^{A(t-\tau)} BK \sum_{j=1}^N l_{kj} z_j(\tau-h)\mathrm{d}\tau \\
=& \int_t^{t+h} \mathrm{e}^{A(t-\tau)} BK \sum_{j=1}^N a_{kj} \left(z_j(\tau-h) - z_k(\tau-h)\right) \mathrm{d}\tau \\
=& \int_t^{t+h} \mathrm{e}^{A(t-\tau)} BK \sum_{j=1}^N a_{kj} \left[(t_j - t_k) \otimes I_n\right] \eta(\tau-h)\mathrm{d}\tau \\
=& \int_t^{t+h} \mathrm{e}^{A(t-\tau)} BK \sum_{j=1}^N a_{kj} \sum_{l=1}^N (t_{jl} - t_{kl})\eta_l(\tau-h)\mathrm{d}\tau \\
=& \sum_{j=1}^N a_{kj} \sum_{l=1}^N (t_{jl} - t_{kl})\delta_l, \qquad (3.16)
\end{aligned}$$

where t_i is the ith row of $\overline{T}$ and

$$\delta_l = \int_t^{t+h} \mathrm{e}^{A(t-\tau)} BK\eta_l(\tau-h)\mathrm{d}\tau. \qquad (3.17)$$

It then follows that

$$\begin{aligned}
\sum_{k=1}^N \|F_1\sigma_k\|^2 =& \sum_{k=1}^N \| \sum_{j=1}^N a_{kj} \sum_{l=1}^N (t_{jl} - t_{kl}) F_1\delta_l\|^2 \\
\leq& \sum_{k=1}^N \| \sum_{j=1}^N a_{kj} \sum_{l=1}^N t_{jl} F_1\delta_l\|^2 + \sum_{k=1}^N \| \sum_{j=1}^N a_{kj} \sum_{l=1}^N t_{kl} F_1\delta_l\|^2 \\
\leq& 2\|\mathcal{A}\|_F^2 \|\overline{T}\|_F^2 \sum_{l=1}^N \|F_1\delta_l\|^2. \qquad (3.18)
\end{aligned}$$

We next deal with $\|F_1\delta_l\|_2^2$ and have

$$\begin{aligned}
\|F_1\delta_l\|_2^2 =& \delta_l^T F_1^T F_1 \delta_l \\
=& \left(\int_t^{t+h} F_1\mathrm{e}^{A(t-\tau)} BK\eta_l(\tau-h)\mathrm{d}\tau\right)^T \left(\int_t^{t+h} F_1\mathrm{e}^{A(t-\tau)} BK\eta_l(\tau-h)\mathrm{d}\tau\right) \\
\leq& h\int_t^{t+h} \eta_l^T(\tau-h) K^T D^T e^{A^T(t-\tau+h)} F_1^T F_1 e^{A(t-\tau+h)} DK\eta_l(\tau-h)\mathrm{d}\tau \\
=& h\int_0^h \eta_l^T(t-\tau) PDD^T e^{A^T\tau} F_1^T F_1 e^{A\tau} DD^T P\eta_l(t-\tau)\mathrm{d}\tau. \qquad (3.19)
\end{aligned}$$

With (3.18) and (3.19), the summation of $-2\eta_i^T P\psi_i$ can be obtained as

$$\begin{aligned}
&-\sum_{i=2}^{N} 2\eta_i^T P\psi_i \\
&\leq \frac{2}{\rho}\sum_{i=2}^{N}\eta_i^T(t)PEE^TP\eta_i(t) + 2\rho\sum_{i=2}^{N}\left(\|\tau_i\|^2 + \|\tau_i\|^2\|r\|^2\right)\|\mathcal{A}\|_F^2\|\overline{T}\|_F^2\sum_{l=2}^{N}\|F_1\delta_l\|^2 \\
&\leq \frac{2}{\rho}\sum_{i=2}^{N}\eta_i^T(t)PEE^TP\eta_i(t) + \rho\gamma_0^2\sum_{i=2}^{N} h\int_0^h \eta_l^T(t-\tau)K^TD^Te^{A^T\tau}F_1^TF_1e^{A\tau}DK\eta_l(t-\tau)\mathrm{d}\tau
\end{aligned} \tag{3.20}$$

with

$$\begin{aligned}
\gamma_0^2 &= 2\sum_{i=2}^{N}\left(\|\tau_i\|^2 + \|\tau_i\|^2\|\bar{r}\|^2\right)\|\mathcal{A}\|_F^2\|\overline{T}\|_F^2 \\
&\leq 2\left\|T^{-1}\right\|_F^2\left(1+N\|\bar{r}\|^2\right)\|\mathcal{A}\|_F^2\|\overline{T}\|_F^2,
\end{aligned}$$

where we have used

$$\sum_{i=1}^{N}\|\tau_i\|^2 = \left\|\overline{T}^{-1}\right\|_F^2.$$

Hence, together with (3.14) and (3.20), we get

$$\begin{aligned}
\dot{V}_1 \leq& \sum_{i=2}^{N}\eta_i^T(t)\left(A^TP + PA - 2\underline{\alpha}PDD^TP + \left(\frac{1}{\mu} + \frac{\bar{\alpha}+\bar{\beta}}{\varepsilon} + \frac{2}{\rho}\right)PEE^TP + \mu F_1^TF_1\right)\eta_i(t) \\
&+ \sum_{i=2}^{N}\varepsilon(\bar{\alpha}+\bar{\beta})\eta_i^T(t-h)PDF_2^TF_2D^TP\eta_i(t-h) \\
&+ \rho\gamma_0^2\sum_{i=2}^{N} h\int_0^h \eta_i^T(t-\tau)PDD^Te^{A^T\tau}F_1^TF_1e^{A\tau}DD^TP\eta_i(t-\tau)\mathrm{d}\tau.
\end{aligned} \tag{3.21}$$

For the delayed term shown in (3.21), we consider the following Krasovskii functional

$$V_2 = (\bar{\alpha}+\bar{\beta})\sum_{i=2}^{N}\int_{t-h}^{t}\eta_i^T(\tau)R\eta_i(\tau)\mathrm{d}\tau, \tag{3.22}$$

where

$$R - \varepsilon PDF_2^TF_2D^TP > 0. \tag{3.23}$$

A direct evaluation gives that

$$\dot{V}_2 = (\bar{\alpha}+\bar{\beta})\sum_{i=2}^{N}\eta_i^T(t)R\eta_i(t) - (\bar{\alpha}+\bar{\beta})\sum_{i=2}^{N}\eta_i^T(t-h)R\eta_i(t-h). \tag{3.24}$$

For the integral term shown in (3.22), we consider the following Krasovskii functional

$$V_3 = \rho h\gamma_0^2\sum_{i=2}^{N}\int_0^h\int_{t-s}^{t}\eta_i^T(\tau)PDD^T\mathrm{e}^{A^Ts}F_1^TF_1\mathrm{e}^{As}DD^TP\eta_i(\tau)\mathrm{d}\tau\mathrm{d}s.$$

A direct evaluation gives that

$$\begin{aligned}\dot{V}_3 = {} & \rho h\gamma_0^2\sum_{i=2}^{N}\int_0^h\eta_i^T(t)PDD^T\mathrm{e}^{A^Ts}F_1^TF_1\mathrm{e}^{As}DD^TP\eta_i(t)\mathrm{d}s\\ & -\rho h\gamma_0^2\sum_{i=2}^{N}\int_0^h\eta_i^T(t-s)PDD^T\mathrm{e}^{A^Ts}F_1^TF_1\mathrm{e}^{As}DD^TP\eta_i(t-s)\mathrm{d}s\\ \le {} & \rho\gamma_0^2\sum_{i=2}^{N}\eta_i^T(t)PDD^T\overline{W}^{-1}DD^TP\eta_i(t)\mathrm{d}s\\ & -\rho h\gamma_0^2\sum_{i=2}^{N}\int_0^h\eta_i^T(t-s)PDD^T\mathrm{e}^{A^Ts}F_1^TF_1\mathrm{e}^{As}DD^TP\eta_i(t-s)\mathrm{d}s,\end{aligned} \tag{3.25}$$

where

$$\overline{W}^{-1} \ge h\int_0^h\mathrm{e}^{A^Ts}F_1^TF_1\mathrm{e}^{As}\mathrm{d}s. \tag{3.26}$$

Let

$$V = V_1 + V_2 + V_3.$$

From (3.21), (3.24) and (3.25), we obtain that

$$\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 \le \sum_{i=2}^{N}\eta_i^T(t)H\eta_i(t), \tag{3.27}$$

where

$$\begin{aligned}H \triangleq {} & A^TP + PA - 2\underline{\alpha}PDD^TP + \mu F_1^TF_1 + (\bar{\alpha}+\bar{\beta})R\\ & + \left(\frac{1}{\mu} + \frac{\bar{\alpha}+\bar{\beta}}{\varepsilon} + \frac{2}{\rho}\right)PEE^TP + \rho\gamma_0^2PDD^T\overline{W}^{-1}DD^TP.\end{aligned} \tag{3.28}$$

3.3 Controller Design

In this section, we consider the problem of controller design for the multi-agent systems (3.1). Based on the consensus analysis in last section, a controller design method is derived in the following theorem.

Theorem 3.3.1. *For the input-delayed uncertain multi-agent systems (3.1) with the associated Laplacian matrix that satisfies Assumption 3.1.1, if there exist matrices* $X = P^{-1} > 0, Y > 0$ *and scalars* $\mu > 0, \varepsilon > 0, \rho > 0$*, such that*

$$\begin{bmatrix} Y & DF_2^T \\ F_2D^T & \frac{1}{\varepsilon}I \end{bmatrix} > 0, \tag{3.29}$$

$$\begin{bmatrix} \bar{H} & XF_1^T & DD^T \\ F_1X & -\frac{1}{\mu}I & 0 \\ DD^T & 0 & -\frac{1}{\rho\gamma_0^2}\overline{W} \end{bmatrix} < 0, \tag{3.30}$$

where

$$\begin{aligned}
\bar{H} =& XA^T + AX - 2\underline{\alpha}Y^TD^T + \left(\frac{1}{\mu} + \frac{\bar{\alpha}+\bar{\beta}}{\varepsilon} + \frac{2}{\rho}\right)EE^T + (\bar{\alpha}+\bar{\beta})Y, \\
\bar{\alpha} =& \max\{\lambda_2, \ldots, \lambda_{n_\lambda}, \alpha_1, \ldots, \alpha_{n_\nu}\}, \\
\bar{\beta} =& \max\{\beta_1, \ldots, \beta_{n_\nu}\}, \\
\underline{\alpha} =& \min\{\lambda_2, \ldots, \lambda_{n_\lambda}, \alpha_1, \ldots, \alpha_{n_\nu}\}, \\
\gamma_0^2 =& 2\left\|\overline{T}^{-1}\right\|_F^2 \left(1 + N\|r\|_2^2\right) \|\mathcal{A}\|_F^2 \left\|\overline{T}\right\|_F^2,
\end{aligned}$$

and $\overline{W}$ *is a positive-definite matrix satisfying*

$$\overline{W}^{-1} \geq h\int_0^h \mathrm{e}^{A^Ts}F_1^TF_1e^{As}\mathrm{d}s, \tag{3.31}$$

then the consensus control problem of system (3.1) can be solved by the control design (3.4) with the control gain $K = D^TX^{-1}$*.*

Proof From the analysis above, we know that the control (3.4) stabilizes η if the conditions (3.23), (3.26) and $H < 0$ in (3.28) are satisfied. Indeed, it is easy to see the conditions (3.23) and (3.26) are equivalent to the conditions specified in (3.29) and

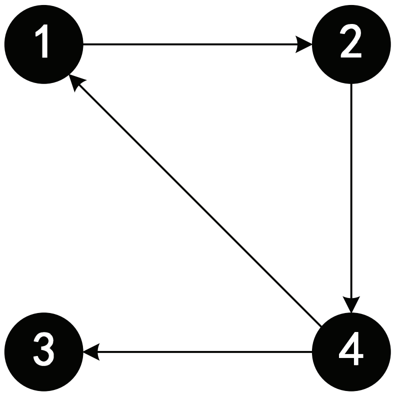

Figure 3.1: Network connection topology

(3.30) with $Y = P^{-1}RP^{-1}$. From (3.28), it can be obtained that $H < 0$ is equivalent to

$$P^{-1}A^T + AP^{-1} - 2\underline{\alpha}DD^T + \left(\frac{1}{\mu} + \frac{\bar{\alpha}+\bar{\beta}}{\varepsilon} + \frac{2}{\rho}\right)EE^T$$
$$+\mu P^{-1}F_1^T F_1 P^{-1} + (\bar{\alpha}+\bar{\beta})P^{-1}RP^{-1} + \rho\gamma_0^2 DD^T \overline{W}^{-1} DD^T < 0, \qquad (3.32)$$

which is further equivalent to (3.31) with $X = P^{-1}$. Hence, we conclude that η converges to zero asymptotically. This completes the proof.

Remark 3.3.1. *The conditions shown in (3.29) to (3.31) can be checked by standard LMI routines for a set of fixed values R and $\overline{W}^{-1}$. The iterative methods developed in [135] for single linear system may also be applied here.*

3.4 Numerical Examples

In this section, the scenario under consideration is a connection of 4 subsystems in networks as shown in Fig.3.1. The dynamics of each subsystem are described by (3.1) with

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Sigma(t) = \begin{bmatrix} \sin(t) & 0 \\ 0 & \sin(2t) \end{bmatrix},$$
$$E = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \ F_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \ F_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

The Laplacian matrix associated with the graph in Fig.3.1 is

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of $\mathcal{L}$ are $[0, 1, (3+\sqrt{3}\mathrm{j})/2, (3-\sqrt{3}\mathrm{j})/2]$, and the Assumption 3.1.1 is satisfied. Then we obtain

$$\bar{J} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{3}{2} \end{bmatrix},$$

with the matrices

$$\overline{T} = \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 0 & -1 & 0 \\ 1 & -2 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix},$$

and

$$\overline{T}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \frac{1}{3} & -\frac{2}{3} & 0 & \frac{1}{3} \\ \frac{\sqrt{3}}{3} & 0 & 0 & -\frac{\sqrt{3}}{3} \end{bmatrix}.$$

Thus, we have $\bar{r}^T = [1/3, 1/3, 0, 1/3]^T$, $\underline{\alpha} = 1$, $\bar{\alpha} = 1.5$ and $\bar{\beta} = \sqrt{3}/2$.

Remark 3.4.1. *For the directed communication graph, as seen in this example, the Laplacian matrix $\mathcal{L}$ is asymmetric, which is more involved than the undirected case in [50]. Furthermore, the third element of the left eigenvector $\bar{r}$ associated with the zero eigenvalue is 0. This suggests that the Lyapunov function used in [62] cannot be used for the consensus analysis here.*

The input delay of the system is $h = 0.03$s. The positive definite matrix X can be computed with $\mu = 1$, $\varepsilon = 1$ and $\rho = 1$, as

$$X = \begin{bmatrix} 276.6367 & -0.0436 \\ -0.0436 & 0.6394 \end{bmatrix},$$

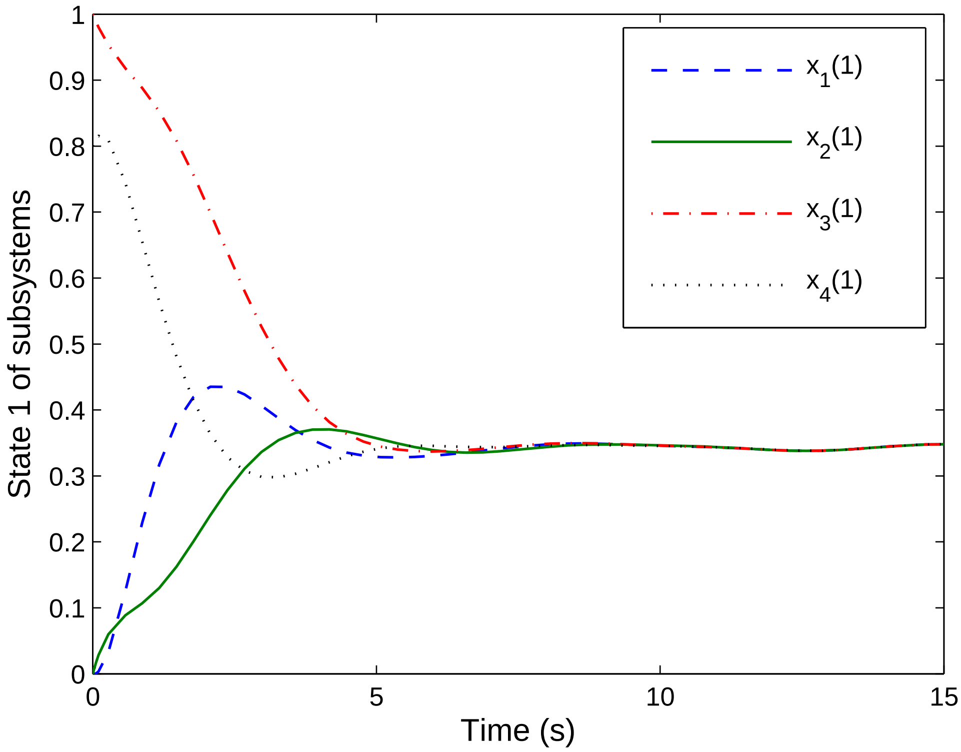

Figure 3.2: The state 1 of subsystems with $h = 0.03$.

and thus the control gain is obtained as

$$K = \begin{bmatrix} 0.0001 & 1.5640 \end{bmatrix}.$$

Simulation study has been carried out with the results shown in Figs. 3.2 and 3.3 for the states of each subsystems. Clearly the conditions specified in Theorem 3.3.1 are sufficient for the control gain to achieve consensus control. Without any re-tuning the control gain, the consensus control is still achieved for the multi-agent system with much larger uncertainty and delay, as shown in Figs. 3.4 and 3.5, which implies the proposed protocol has preferred robustness and the conditions might be conservative for a given input delay.

3.5 Summary

In this chapter, with aid of the reduction method, we have solved the consensus problem of the multi-agent systems with parameter uncertainties and input delay. Further

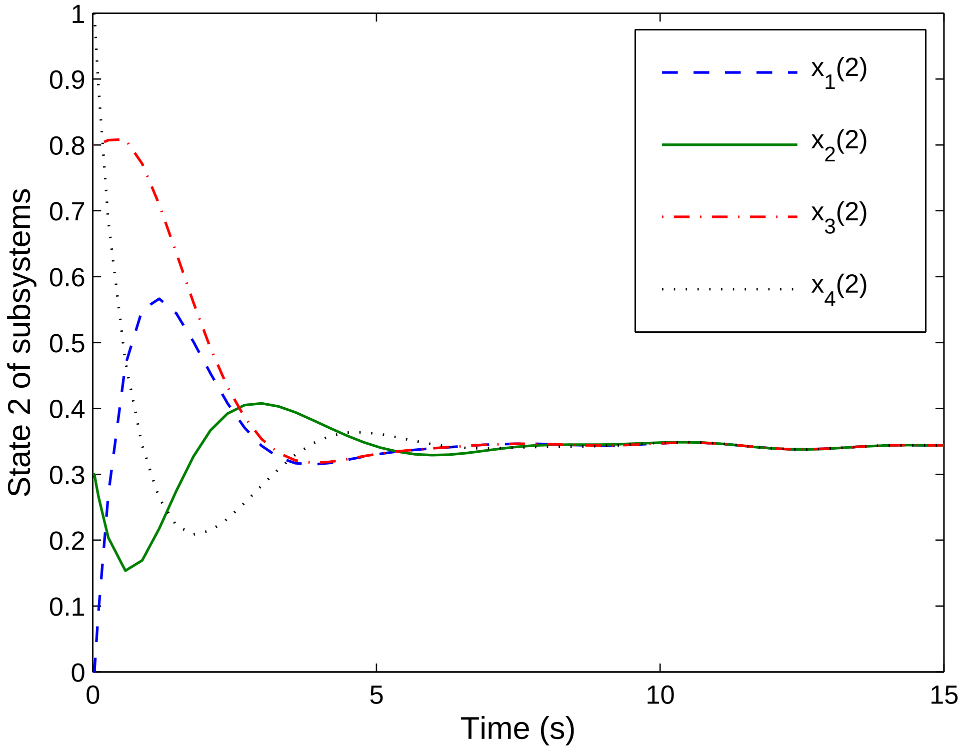

Figure 3.3: The state 2 of subsystems with $h = 0.03$.

analysis has been developed to tackle the influence of the extra integral term under the transformations. Two sufficient conditions are derived for the closed-loop system to achieve global consensus using Lyapunov-Krasovskii method in the time domain. The significance of this research is to provide a feasible method to deal with the robust consensus control for uncertain multi-agent systems with input delay.

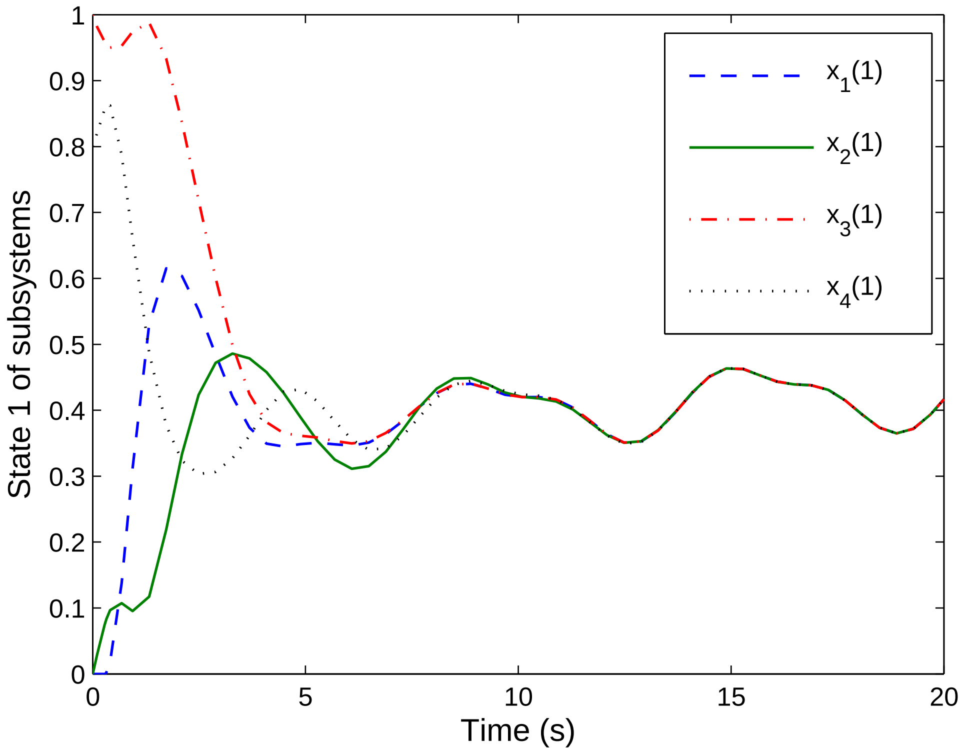

Figure 3.4: The state 1 of subsystems with $E = \text{diag}\{2,2\}$ and $h = 0.3$.

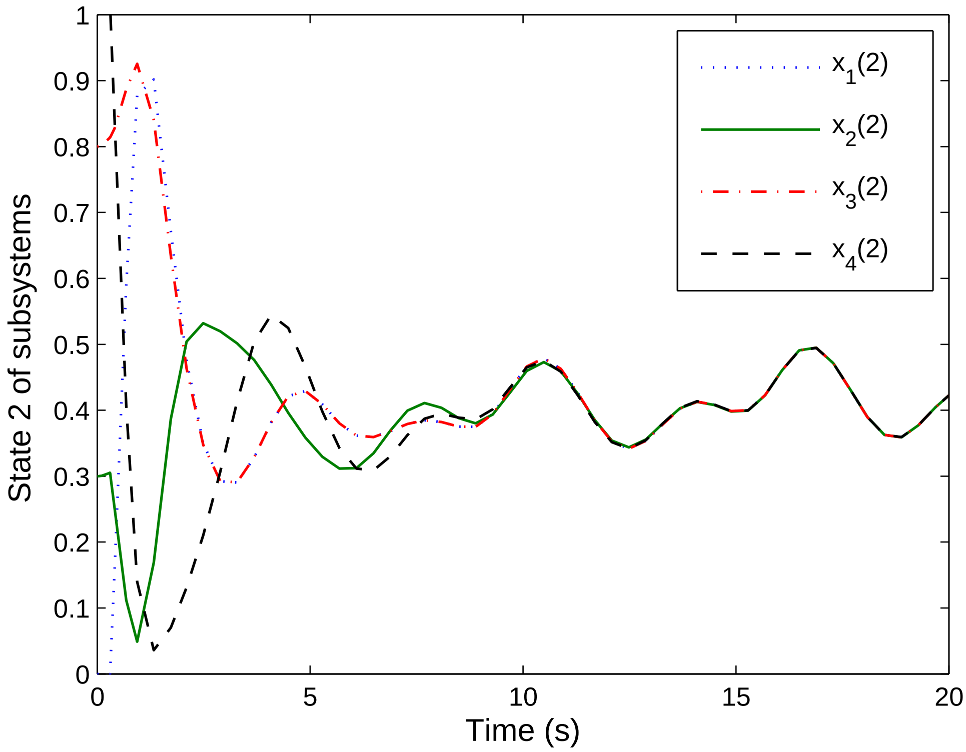

Figure 3.5: The state 2 of subsystems with $E = \text{diag}\{2,2\}$ and $h = 0.3$.

Chapter 4

Consensus Control for Nonlinear MASs

In this chapter, we systematically investigate the consensus control problem for multi-agent systems with nonlinearity and input delay. To deal with the input delay, model reduction method is employed by a state transformation in the presence of nonlinearity in the agent dynamics. Compensation based on the relative input information is added in the controller design to offset any constant input delay. Due to the limitations of sensors or link failures, sometimes, the relative input information is unobtainable. For such cases, TPF approach is adopted to deal with the input delay and a finite-dimensional TPF controller is constructed for each agent. To tackle the influence of the nonlinear terms under the state transformations, further rigorous analysis is carried out to ensure that the extra integral terms of the system state associated with nonlinear functions are properly considered by means of Krasovskii functionals. By transforming the Laplacian matrix into the real Jordan form, global stability analysis is put in the framework of Lyapunov functions in real domain. Conditions based on the Lipschitz constant are identified for proposed consensus protocols with/without relative input information to tackle Lipschitz nonlinear terms in the system dynamics. Simulations are carried out to demonstrate the results obtained in the chapter.

The main contributions of this chapter are summarized as follows: (1) Many of the results on consensus control are based on linear system dynamics [43–48]. In this chapter, the consensus problem for Lipschitz nonlinear multi-agent systems is considered. (2) Compared to the previous works [49, 50, 52, 54], where the communication graphs are undirected or strong connected, the requirement for the communication graph in this chapter is more general. (3) Input delay is taken into consideration and further rigorous analysis is carried out to ensure that the extra integral terms of the system state associated with nonlinear functions are properly considered by means of Krasovskii functionals.

The rest of this chapter is organized as follows. The problem formulation is given in Section 4.1. Section 4.2 presents the main results on consensus control with model reduction method. Based on the TPF approach, Section 4.3 presents a finite-dimensional consensus controller design method without the relative input information. Simulation results are included in Section 4.4. Section 4.5 summarises this chapter.

4.1 Problem Formulation

In this section, we consider control design for a group of N agents, each represented by a nonlinear subsystem that is subject to input delay and Lipschitz nonlinearity,

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t-h) + \phi(x_i), \tag{4.1}$$

where for agent i, $i = 1,2,\ldots,N$, $x_i \in \mathbb{R}^n$ is the state vector, $u_i \in \mathbb{R}^m$ is the control input vector, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices with (A,B) being controllable, $h > 0$ is input delay, and the initial conditions $x_i(\theta)$, $\theta \in [-h,0]$, are given and bounded, and $\phi : \mathbb{R}^n \to \mathbb{R}^n$, $\phi(0) = 0$, is a Lipschitz nonlinear function with a Lipschitz constant γ, i.e., for any two constant vectors a, $b \in \mathbb{R}^n$,

$$\|\phi(a) - \phi(b)\| \leq \gamma\|a-b\|.$$

Associated with the communication graph is its adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, where the element a_{ij} denotes the connection between the agent i and agent j. $a_{ij} = 1$ if $(j,i) \in \mathcal{E}$, otherwise is zero, and $a_{ii} = 0$ for all nodes with the assumption that there exists no self loop. A directed graph that contains a spanning tree is that there exists a node called the root, and this root has a directed path to every other node of the graph. The Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ is defined by $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$ when $i \neq j$.

Assumption 4.1.1. *Zero is a simple eigenvalue of the Laplacian matrix $\mathcal{L}$.*

Remark 4.1.1. *This assumption implies that is a directed spanning tree in the network. For undirected graph or strongly connected and balanced graph, the condition, $\bar{x}^T\mathcal{L}\bar{x} \geq 0, \ \forall\, \bar{x} \in \mathbb{R}^N$, holds. However, for the general direction graph in Assumption*

(4.1.1), the Laplacian matrix $\mathcal{L}$ is asymmetric and $\bar{x}^T \mathcal{L} \bar{x}$ can be sign-indefinite [28]. The decomposition method developed in [118] for the undirected multi-agent systems cannot be applied here due to this unfavourable feature.

The consensus control problem considered in this chapter is to design a control strategy, using the relative state information, to ensure that all agents asymptotically converge to an identical trajectory.

4.2 Consensus Control of Nonlinear MASs with Input Delay: Model Reduction Method

4.2.1 Stability Analysis for Single Nonlinear System

In this section, we first consider the Artstein model reduction method for a single nonlinear system. Consider an input-delayed system

$$\dot{x}(t) = Ax(t) + Bu(t-h) + \phi(x(t)), \tag{4.2}$$

with $\phi : \mathbb{R}^n \to \mathbb{R}^n$, $\phi(0) = 0$, being a Lipschitz nonlinear function, and the initial conditions $x(\theta)$, $\theta \in [-h, 0]$, being bounded. Let

$$z(t) = x(t) + \int_t^{t+h} \mathrm{e}^{A(t-\tau)} Bu(\tau - h)\mathrm{d}\tau, \tag{4.3}$$

Differentiating $z(t)$ against time yields

$$\begin{aligned}\dot{z}(t) &= Ax(t) + \phi(x(t)) + \mathrm{e}^{-Ah}Bu(t) + A\int_t^{t+h} \mathrm{e}^{A(t-\tau)} Bu(\tau - h)\mathrm{d}\tau \\ &= Az(t) + Du(t) + \phi(x(t)),\end{aligned} \tag{4.4}$$

where $D = \mathrm{e}^{-Ah}B$.

We consider a controller

$$u(t) = Kz(t). \tag{4.5}$$

From (4.3) and (4.5), we have

$$\|x(t)\| \le \|z(t)\| + h\left(\max_{-h\le\theta\le 0} \|e^{A\theta}\|\right) \|B\|\|K\|\|z_t(\theta)\|,$$

where $z_t(\theta) := z(t+\theta)$, $-h \le \theta \le 0$. Thus, $x(t) \to 0$ as $z(t) \to 0$. In other words, if the controller (4.5) stabilizes the transformed system (4.4), then the original system (4.2) is also stable with the same controller [78].

4.2.2 Consensus Analysis

For the multi-agent systems (4.1), we use (4.3) to transform the agent dynamics to

$$\dot{z}_i(t) = Az_i(t) + Du_i(t) + \phi(x_i(t)), \tag{4.6}$$

where $D = \mathrm{e}^{-Ah}B$.

We propose a control design using the relative state information. The control input takes the structure,

$$\begin{aligned} u_i(t) &= -K\sum_{j=1}^{N} l_{ij}z_j(t) \\ &= -K\sum_{j=1}^{N} a_{ij}\left(z_i(t) - z_j(t)\right), \end{aligned} \tag{4.7}$$

where $K \in \mathbb{R}^{m\times n}$ is a constant control gain matrix to be designed later, a_{ij} and l_{ij} being the elements of the graph adjacency matrix $\mathcal{A}$ and the Laplacian matrix $\mathcal{L}$, respectively.

Remark 4.2.1. *From (4.3) and (4.7), we have*

$$u_i(t) = -K\sum_{j=1}^{N} l_{ij}\left(x_j(t) + \int_t^{t+h} \mathrm{e}^{A(t-\tau)}Bu_j(\tau-h)\mathrm{d}\tau\right).$$

It is observed that certain compensation is added in the controller design based on the model reduction method. With the state transformation (4.3), the original input-delayed multi-agent systems (4.2) are reduced to delay-free systems (4.6). In this way, conventional finite-dimensional techniques can be used for the consensus analysis and controller design.

The closed-loop system is then described by

$$\dot{z}(t) = (I_N \otimes A - \mathcal{L} \otimes DK)z(t) + \Phi(x), \tag{4.8}$$

where

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_N(t) \end{bmatrix}, \quad \Phi(x) = \begin{bmatrix} \phi(x_1) \\ \phi(x_2) \\ \vdots \\ \phi(x_N) \end{bmatrix}.$$

With lemmas (2.3.1) and (2.3.2), we can define $r^T \in \mathbb{R}^N$ as the left eigenvector of $\mathcal{L}$ corresponding to the eigenvalue at 0, that is, $r^T\mathcal{L} = 0$. Furthermore, let r be scaled such that $r^T\mathbf{1} = 1$ and let the first row of T^{-1} be $(T^{-1})_1 = r^T$, where T is the transformation matrix defined in Lemma (2.3.2).

Based on the vector r, we introduce a state transformation

$$\overline{\xi}_i(t) = z_i(t) - \sum_{j=1}^{N} r_j z_j(t), \tag{4.9}$$

for $i = 1, 2\ldots, N$. Let

$$\overline{\xi} = [\overline{\xi}_1^T, \overline{\xi}_2^T, \cdots, \overline{\xi}_N^T]^T.$$

We have

$$\overline{\xi} = z - ((\mathbf{1}r^T) \otimes I_n)z = (M \otimes I_n)z,$$

where $M = I_N - \mathbf{1}r^T$. Since $r^T\mathbf{1} = 1$, it can be shown that $M\mathbf{1} = 0$. Therefore the consensus of system (4.6) is achieved when $\lim_{t\to\infty}\overline{\overline{\xi}}(t) = 0$, as $\overline{\xi} = 0$ implies $z_1 = z_2 = \cdots = z_N$, due to the fact that the null space of M is span($\mathbf{1}$). The dynamics of $\overline{\xi}$ can then be obtained as

$$\begin{aligned}
\dot{\overline{\xi}} &= \dot{z} - (\mathbf{1}r^T \otimes I_N)\,\dot{z} \\
&= (I_N \otimes A - \mathcal{L} \otimes DK)z \\
&\quad - \mathbf{1}r^T \otimes I_N[I_N \otimes A - \mathcal{L} \otimes DK]z + (M \otimes I_n)\Phi(x) \\
&= (I_N \otimes A - \mathcal{L} \otimes DK)\overline{\xi} + (M \otimes I_n)\Phi(x).
\end{aligned} \tag{4.10}$$

To explore the structure of $\mathcal{L}$, let us introduce another state transformation

$$\overline{\eta} = (T^{-1} \otimes I_n)\overline{\xi}. \tag{4.11}$$

Then with Lemma (2.3.2), we have

$$\dot{\overline{\eta}} = (I_N \otimes A - J \otimes DK)\overline{\eta} + \Psi(x), \tag{4.12}$$

where $\Psi(x) = (T^{-1}M \otimes I_n)\Phi(x)$, and

$$\overline{\eta} = \begin{bmatrix} \overline{\eta}_1 \\ \overline{\eta}_2 \\ \vdots \\ \overline{\eta}_N \end{bmatrix}, \qquad \Psi(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_N(x) \end{bmatrix},$$

with $\overline{\eta}_i \in \mathbb{R}^n$ and $\psi_i : \mathbb{R}^{n\times N} \to \mathbb{R}^n$ for $i = 1,2,\ldots,N$.

Then from (4.9) and (4.11), we have:

$$\begin{aligned} \overline{\eta}_1 &= \left(r^T \otimes I_n\right)\overline{\xi} \\ &= \left((r^T M) \otimes I_n\right) z \\ &\equiv 0. \end{aligned}$$

The nonlinear term $\Psi(x)$ in the transformed system dynamic model (4.12) is expressed as a function of the state x. For the stability analysis, first we need to establish a bound of this nonlinear function in terms of the transformed state $\overline{\eta}$. The following lemma gives a bound of $\Psi(x)$.

Lemma 4.2.1. *For the nonlinear term $\Psi(x)$ in the transformed system dynamics (4.12), a bound can be established in terms of the state $\overline{\eta}$ as*

$$\|\Psi\|^2 \leq \gamma_0^2(\|\overline{\eta}\|^2 + 4\lambda_\sigma^2(\mathcal{A})\,\|\delta\|^2), \tag{4.13}$$

with

$$\begin{aligned} \gamma_0 &= 2\sqrt{2N}\gamma\|r\|\,\|T\|_F\,\lambda_\sigma(T^{-1}), \\ \delta &= -\int_t^{t+h} \mathrm{e}^{A(t-\tau)}BK\overline{\eta}(\tau-h)\mathrm{d}\tau. \end{aligned}$$

Proof. Based on the state transformations (4.9) and (4.11), we have

$$\begin{aligned} \Psi(x) &= \left(T^{-1} \otimes I_n\right)(M \otimes I_n)\,\Phi(x) \\ &= \left(T^{-1} \otimes I_n\right)\mu, \end{aligned}$$

where $\mu = (M \otimes I_n)\Phi(x)$. Then, we have

$$\|\Psi(x)\| \leq \lambda_\sigma(T^{-1})\,\|\mu\|, \tag{4.14}$$

where $\mu = [\mu_1, \mu_2, \ldots, \mu_N]^T$.

Recalling that $M = I_N - \mathbf{1}r^T$, we have

$$\begin{aligned} \mu_i &= \phi(x_i) - \sum_{k=1}^{N} r_k \phi(x_k) \\ &= \sum_{k=1}^{N} r_k \left(\phi(x_i) - \phi(x_k)\right). \end{aligned}$$

It then follows that

$$\|\mu_i\| \leq \gamma \sum_{k=1}^{N} |r_k|\,\|x_i - x_k\|. \tag{4.15}$$

From the state transformation (4.3), we have

$$\begin{aligned} x_i - x_k &= (z_i - \sigma_i) - (z_k - \sigma_k) \\ &= (z_i - z_k) - (\sigma_i - \sigma_k), \end{aligned}$$

where

$$\sigma_i = \int_t^{t+h} \mathrm{e}^{A(t-\tau)} B u_i(\tau - h)\mathrm{d}\tau.$$

Then, we have

$$\|\mu_i\| \leq \gamma \sum_{k=1}^{N} |r_k| \left(\|z_i - z_k\| + \|\sigma_i - \sigma_k\|\right). \tag{4.16}$$

From $\overline{\eta} = (T^{-1} \otimes I_n)\overline{\xi}$, we obtain $\overline{\xi} = (T \otimes I_n)\overline{\eta}$, and from the state transformations (4.9), we have

$$\begin{aligned} z_i - z_k &= \overline{\xi}_i - \overline{\xi}_k \\ &= ((t_i - t_k) \otimes I_n)\,\overline{\eta} \\ &= \sum_{j=1}^{N} (t_{ij} - t_{kj})\overline{\eta}_j, \end{aligned}$$

where t_k denotes the kth row of T. Then, we obtain

$$\|z_i - z_k\| \leq (\|t_i\| + \|t_k\|)\,\|\overline{\eta}\|. \tag{4.17}$$

Here we used the inequality

$$\begin{aligned}\sum_{i=1}^{N}(a_i b_i) &\leq \|a\|\,\|b\| \\ &= \sqrt{\sum_{i=1}^{N} a_i^2 \sum_{i=1}^{N} b_i^2}.\end{aligned}$$

We next deal with the derived terms σ_i and σ_k. We have

$$\begin{aligned}\sum_{k=1}^{N} |r_k|\,\|\sigma_i - \sigma_k\| &\leq \sum_{k=1}^{N} |r_k|\,\|\sigma_i\| + \sum_{k=1}^{N} |r_k|\,\|\sigma_k\| \\ &\leq \|r\|\sqrt{N}\,\|\sigma_i\| + \|r\|\,\|\sigma\|, \end{aligned} \tag{4.18}$$

where $\sigma = [\sigma_1^T, \sigma_2^T, \cdots, \sigma_N^T]^T$, and we used the inequality

$$\sum_{i=1}^{N} |a_i| \leq \sqrt{N}\,\|a\|.$$

Then, from (4.16), (4.17) and (4.18), we can obtain that

$$\begin{aligned}\|\mu_i\| &\leq \gamma \sum_{k=1}^{N} |r_k| \left(\|t_i\| + \|t_k\|\right) \|\overline{\eta}\| + \gamma\sqrt{N}\,\|r\|\,\|\sigma_i\| + \gamma\|r\|\,\|\sigma\| \\ &\leq \gamma\Big(\|r\|\sqrt{N}\,\|t_i\| + \|r\|\,\|T\|_F\Big)\|\overline{\eta}\| + \gamma\sqrt{N}\,\|r\|\,\|\sigma_i\| + \gamma\|r\|\,\|\sigma\| \\ &= \gamma\|r\|\Big(\big(\sqrt{N}\,\|t_i\| + \|T\|_F\big)\|\overline{\eta}\| + \sqrt{N}\,\|\sigma_i\| + \|\sigma\|\Big). \end{aligned} \tag{4.19}$$

It then follows that

$$\begin{aligned}\|\mu\|^2 &= \sum_{i=1}^{N} \left(\|\mu_i\|\right)^2 \\ &\leq \left(\sum_{i=1}^{N} \|\mu_i\|\right)^2 \\ &\leq 4\gamma^2\,\|r\|^2 \sum_{i=1}^{N}\Big(N\,\|t_i\|^2 + \|T\|_F^2\Big)\|\overline{\eta}\|^2 + 4\gamma^2\,\|r\|^2 \sum_{i=1}^{N}\Big(N\,\|\sigma_i\|^2 + \|\sigma\|^2\Big) \\ &= 4\gamma^2\,\|r\|^2 N\Big(2\,\|T\|_F^2\,\|\overline{\eta}\|^2 + 2\,\|\sigma\|^2\Big) \\ &= 8\gamma^2\,\|r\|^2 N\Big(\|T\|_F^2\,\|\overline{\eta}\|^2 + \|\sigma\|^2\Big), \end{aligned} \tag{4.20}$$

where we have used

$$\sum_{k=1}^{N} \|t_k\|^2 = \|T\|_F^2,$$

and the inequality

$$(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2).$$

Next we need to deal with $\|\sigma\|^2$. From (4.7), we can get

$$\begin{aligned}
\sigma_i &= \int_t^{t+h} \mathrm{e}^{A(t-\tau)} B u_i(\tau-h)\mathrm{d}\tau \\
&= -\int_t^{t+h} \mathrm{e}^{A(t-\tau)} BK \sum_{j=1}^{N} l_{ij} z_j(\tau-h)\mathrm{d}\tau.
\end{aligned}$$

From the relationship between $\mathcal{A}$ and $\mathcal{L}$, we have

$$\begin{aligned}
\sum_{j=1}^{N} l_{ij} z_j &= \sum_{j=1}^{N} a_{ij}(z_i - z_j) \\
&= \sum_{j=1}^{N} a_{ij}\left((t_i - t_j)\otimes I_n\right)\overline{\eta} \\
&= \sum_{j=1}^{N} a_{ij} \sum_{l=1}^{N} (t_{il} - t_{jl})\overline{\eta}_l.
\end{aligned} \tag{4.21}$$

Here we define δ_l

$$\delta_l = -\int_t^{t+h} \mathrm{e}^{A(t-\tau)} BK\overline{\eta}_l(\tau-h)\mathrm{d}\tau. \tag{4.22}$$

Then we can obtain that

$$\sigma_i = \sum_{j=1}^{N} a_{ij} \sum_{l=1}^{N} (t_{il} - t_{jl})\delta_l.$$

It then follows that

$$\|\sigma_i\| \leq \sum_{j=1}^{N} a_{ij}\left(\|t_i\| + \left\|t_j\right\|\right)\|\delta\|. \tag{4.23}$$

where $\delta = [\delta_1^T, \delta_2^T, \cdots, \delta_N^T]^T$. With (4.23), the sum of the $\|\sigma_i\|$ can be obtained

$$\begin{aligned}\sum_{i=1}^{N} \|\sigma_i\| &\leq \|\delta\| \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\|t_i\| + \|t_j\|) \\ &= \|\delta\| \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\|t_i\| + \|\delta\| \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\|t_j\| \\ &\leq \lambda_\sigma(\mathcal{A})\|T\|_F\|\delta\| + \lambda_\sigma(\mathcal{A}^T)\|T\|_F\|\delta\| \\ &= \left(\lambda_\sigma(\mathcal{A}) + \lambda_\sigma(\mathcal{A}^T)\right)\|T\|_F\|\delta\| \\ &= 2\lambda_\sigma(\mathcal{A})\|T\|_F\|\delta\|, \end{aligned} \tag{4.24}$$

with $\lambda_\sigma(\mathcal{A}) = \lambda_\sigma(\mathcal{A}^T)$. In (4.24), we have used the following inequalities

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\|t_i\| = \begin{bmatrix} a_{11} & \cdots & a_{N1} \\ \vdots & \ddots & \vdots \\ a_{1N} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} \|t_1\| \\ \vdots \\ \|t_N\| \end{bmatrix} \leq \lambda_\sigma\left(\mathcal{A}^T\right)\|T\|_F,$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\|t_j\| = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} \|t_1\| \\ \vdots \\ \|t_N\| \end{bmatrix} \leq \lambda_\sigma(\mathcal{A})\|T\|_F.$$

Therefore we have

$$\begin{aligned}\|\sigma\|^2 &= \sum_{i=1}^{N}(\|\sigma_i\|)^2 \\ &\leq \left(\sum_{i=1}^{N}\|\sigma_i\|\right)^2 \\ &\leq 4\lambda_\sigma^2(\mathcal{A})\|T\|_F^2\|\delta\|^2. \end{aligned} \tag{4.25}$$

Hence, together with (4.20) and (4.25), we get

$$\|\mu\|^2 \leq 8\gamma^2\|r\|^2 N\|T\|_F^2\left(\|\overline{\eta}\|^2 + 4\lambda_\sigma^2(\mathcal{A})\|\delta\|^2\right). \tag{4.26}$$

Finally, we obtain the bound for Ψ as

$$\begin{aligned}\|\Psi\|^2 &\le \lambda_\sigma^2(T^{-1})\,\|\mu\|^2 \\ &\le \gamma_0^2\left(\|\overline{\eta}\|^2 + 4\lambda_\sigma^2(\mathcal{A})\,\|\delta\|^2\right),\end{aligned} \tag{4.27}$$

with

$$\begin{aligned}\gamma_0 &= 2\sqrt{2N}\gamma\|r\|\,\|T\|_F\,\lambda_\sigma(T^{-1}),\\ \delta &= -\int_t^{t+h} \mathrm{e}^{A(t-\tau)}BK\overline{\eta}(\tau-h)\mathrm{d}\tau.\end{aligned}$$

This completes the proof. □

With the control law shown in (4.7), the control gain matrix K is chosen as

$$K = D^T P, \tag{4.28}$$

where P is a positive definite matrix. In the remaining part of this section, we will use Lyapunov-function-based analysis to identify a condition for P to ensure that consensus is achieved by using the control algorithm (4.7) with the control gain K in (4.28).

The stability analysis will be carried out in terms of $\overline{\eta}$. As discussed earlier, the consensus control can be guaranteed by showing that $\overline{\eta}$ converges to zero, which is sufficed by showing that $\overline{\eta}_i$ converges to zero for $i = 2, 3, \ldots, N$, since we have shown that $\overline{\eta}_1 = 0$.

From the structure of the Laplacian matrix shown in Lemma (2.3.2), we can see that

$$N_k = 1 + \sum_{j=1}^{k} n_j,$$

for $k = 1, 2, \ldots, q$. Note that $N_q = N$.

The agent state variables $\overline{\eta}_i$ from $i = 2$ to N_p are the state variables which are associated with the Jordan blocks of real eigenvalues, and $\overline{\eta}_i$ for $i = N_p + 1$ to N are with Jordan blocks of complex eigenvalues.

For the state variables associated with the Jordan blocks J_k of real eigenvalues, i.e., for $k \le p$, we have the dynamics given by

$$\dot{\overline{\eta}}_i = (A - \lambda_k DD^T P)\overline{\eta}_i - DD^T P\overline{\eta}_{i+1} + \psi_i(x),$$

for $i = N_{k-1}+1, N_{k-1}+2, \cdots, N_k - 1$, and

$$\dot{\overline{\eta}}_i = (A - \lambda_k DD^T P)\overline{\eta}_i + \psi_i(x),$$

for $i = N_k$.

For the state variables associated with the Jordan blocks J_k, i.e., for $k > p$, corresponding to complex eigenvalues, we consider the dynamics of the state variables in pairs. For notational convenience, let

$$\begin{aligned} i_1(j) &= N_{k-1} + 2j - 1, \\ i_2(j) &= N_{k-1} + 2j, \end{aligned}$$

for $j = 1, 2, \ldots, n_k/2$. The dynamics of $\overline{\eta}_{i_1}$ and $\overline{\eta}_{i_2}$ for $j = 1, 2, \ldots, n_k/2 - 1$ are expressed by

$$\begin{aligned} \dot{\overline{\eta}}_{i_1} &= (A - \alpha_k DD^T P)\overline{\eta}_{i_1} - \beta_k DD^T P\overline{\eta}_{i_2} - DD^T P\overline{\eta}_{i_1+2} + \psi_{i_1}, \\ \dot{\overline{\eta}}_{i_2} &= (A - \alpha_k DD^T P)\overline{\eta}_{i_2} + \beta_k DD^T P\overline{\eta}_{i_1} - DD^T P\overline{\eta}_{i_2+2} + \psi_{i_2}. \end{aligned}$$

For $j = n_k/2$, we have

$$\begin{aligned} \dot{\overline{\eta}}_{i_1} &= (A - \alpha_k DD^T P)\overline{\eta}_{i_1} - \beta_k DD^T P\overline{\eta}_{i_2} + \psi_{i_1}, \\ \dot{\overline{\eta}}_{i_2} &= (A - \alpha_k DD^T P)\overline{\eta}_{i_2} + \beta_k DD^T P\overline{\eta}_{i_1} + \psi_{i_2}. \end{aligned}$$

Let

$$V_i = \overline{\eta}_i^T P \overline{\eta}_i, \tag{4.29}$$

for $i = 2, 3 \ldots, N$. Let

$$V_0 = \sum_{i=2}^{N} \overline{\eta}_i^T P \overline{\eta}_i. \tag{4.30}$$

For the convenience of presentation, we borrow the following results for V_0 from [51].

Lemma 4.2.2. *For a network-connected dynamic system (4.1) with the transformed state $\overline{\eta}$, $\dot{V}_0$ has following bounds specified in one of the following two cases:*
1) If the eigenvalues of the Laplacian matrix $\mathcal{L}$ are distinct, i.e., $n_k = 1$ for $k =$

$1,2,\ldots,q$, $\dot{V}_0$ satisfies

$$\dot{V}_0 \le \sum_{i=2}^{N} \overline{\eta}_i^T \left(A^T P + PA - 2\alpha PDD^T P + \kappa PP\right) \overline{\eta}_i + \frac{1}{\kappa}\|\Psi\|^2, \tag{4.31}$$

with κ being any positive real number and

$$\alpha = \min\{\lambda_1, \lambda_2, \ldots, \lambda_p, \alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_q\}.$$

2) If the Laplacian matrix $\mathcal{L}$ has multiple eigenvalues, i.e., $n_k > 1$ for any $k \in \{1,2,\cdots,q\}$, $\dot{V}_0$ satisfies

$$\dot{V}_0 \le \sum_{i=2}^{N} \overline{\eta}_i^T \left(A^T P + PA - 2(\alpha-1) PDD^T P + \kappa PP\right) \overline{\eta}_i + \frac{1}{\kappa}\|\Psi\|^2, \tag{4.32}$$

with κ being any positive real number.

Using Lemmas (4.2.1) and (4.2.2), we easily obtain

$$\dot{V}_0 \le \sum_{i=2}^{N} \overline{\eta}_i^T \left(A^T P + PA - 2\alpha PDD^T P + \kappa PP + \frac{\gamma_0^2}{\kappa} I_n\right) \overline{\eta}_i + \frac{4\gamma_0^2}{\kappa}\lambda_\sigma^2(\mathcal{A})\widetilde{\Delta}, \tag{4.33}$$

for Case 1) with $\widetilde{\Delta} = \delta^T\delta$, and

$$\dot{V}_0 \le \sum_{i=2}^{N} \overline{\eta}_i^T \left(A^T P + PA - 2(\alpha-1) PDD^T P + \kappa PP + \frac{\gamma_0^2}{\kappa} I_n\right) \overline{\eta}_i + \frac{4\gamma_0^2}{\kappa}\lambda_\sigma^2(\mathcal{A})\widetilde{\Delta}, \tag{4.34}$$

for Case 2). Here we have used $\|\overline{\eta}\|^2 = \sum_{i=2}^{N} \|\overline{\eta}_i\|^2$.

The remaining analysis is to explore the bound of $\widetilde{\Delta}$. With δ_l in (4.22) and Lemma (2.1.5), we have

$$\begin{aligned}
\widetilde{\Delta}_i &= \int_t^{t+h} \overline{\eta}_i^T(\tau-h) K^T B^T e^{A^T(t-\tau)} d\tau \int_t^{t+h} e^{A(t-\tau)} BK\overline{\eta}_i(\tau-h) d\tau \\
&\le h \int_t^{t+h} \overline{\eta}_i^T(\tau-h) PDD^T e^{A^T h} e^{A^T(t-\tau)} e^{A(t-\tau)} e^{Ah} DD^T P \overline{\eta}_i(\tau-h) d\tau.
\end{aligned}$$

In view of Lemma (2.1.6) with $P = I_n$, provided that

$$\bar{R} = -A^T - A + \omega_1 I_n > 0, \tag{4.35}$$

we have

$$\mathrm{e}^{A^T t}\mathrm{e}^{At} < \mathrm{e}^{\omega_1 t} I_n,$$

and

$$\begin{aligned}
\widetilde{\Delta}_i &\leq h \int_t^{t+h} \mathrm{e}^{\omega_1(t-\tau)} \overline{\eta}_i^T(\tau-h) PDD^T e^{A^T h}\mathrm{e}^{Ah} DD^T P \overline{\eta}_i(\tau-h)\mathrm{d}\tau \\
&\leq h\mathrm{e}^{\omega_1 h} \int_t^{t+h} \mathrm{e}^{\omega_1(t-\tau)} \overline{\eta}_i^T(\tau-h) PDD^T DD^T P \overline{\eta}_i(\tau-h)\mathrm{d}\tau \\
&\leq \rho^2 h\mathrm{e}^{\omega_1 h} \int_t^{t+h} \mathrm{e}^{\omega_1(t-\tau)} \overline{\eta}_i^T(\tau-h) \overline{\eta}_i(\tau-h)\mathrm{d}\tau \\
&\leq \rho^2 h\mathrm{e}^{2\omega_1 h} \int_t^{t+h} \overline{\eta}_i^T(\tau-h) \overline{\eta}_i(\tau-h)\mathrm{d}\tau,
\end{aligned}$$

where ρ is a positive real number satisfying

$$\rho^2 I_n \geq PDD^T DD^T P. \tag{4.36}$$

Then the summation of $\widetilde{\Delta}_i$ can be obtained as

$$\begin{aligned}
\widetilde{\Delta} &= \sum_{i=2}^N \widetilde{\Delta}_i \\
&\leq \sum_{i=2}^N \rho^2 h\mathrm{e}^{2\omega_1 h} \int_t^{t+h} \overline{\eta}_i^T(\tau-h) \overline{\eta}_i(\tau-h)\mathrm{d}\tau.
\end{aligned} \tag{4.37}$$

For the integral term $\widetilde{\Delta}$ shown in (4.37), we consider the following Krasovskii functional

$$\widetilde{W}_i = \int_t^{t+h} \mathrm{e}^{\tau-t} \overline{\eta}_i^T(\tau-h)\overline{\eta}_i(\tau-h)\mathrm{d}\tau + \int_t^{t+h} \overline{\eta}_i^T(\tau-2h)\overline{\eta}_i(\tau-2h)\mathrm{d}\tau.$$

A direct evaluation gives that

$$\begin{aligned}
\dot{\widetilde{W}}_i &= -\int_t^{t+h} \mathrm{e}^{\tau-t} \overline{\eta}_i^T(\tau-h)\overline{\eta}_i(\tau-h)\mathrm{d}\tau \\
&\quad -\overline{\eta}_i(t-2h)^T\overline{\eta}_i(t-2h) + \mathrm{e}^h \overline{\eta}_i^T(t)\overline{\eta}_i(t) \\
&\leq -\int_t^{t+h} \overline{\eta}_i^T(\tau-h)\overline{\eta}_i(\tau-h)\mathrm{d}\tau + \mathrm{e}^h \overline{\eta}_i^T(t)\overline{\eta}_i(t).
\end{aligned}$$

With $\widetilde{W}_0 = \sum_{i=2}^{N} \widetilde{W}_i$, we have

$$\begin{aligned}\dot{\widetilde{W}}_0 &= \sum_{i=2}^{N} \dot{\widetilde{W}}_i \\ &\leq -\sum_{i=2}^{N} \int_{t}^{t+h} \overline{\eta}_i^T(\tau - h)\overline{\eta}_i(\tau - h)\mathrm{d}\tau + \sum_{i=2}^{N} \mathrm{e}^h \overline{\eta}_i^T(t)\overline{\eta}_i(t). \end{aligned} \tag{4.38}$$

Let

$$V = V_0 + \rho^2 h \mathrm{e}^{2\omega_1 h} \frac{4\gamma_0^2}{\kappa} \lambda_\sigma^2(\mathcal{A})\widetilde{W}_0. \tag{4.39}$$

From (4.33), (4.34), (4.37) and (4.38), we obtain that

$$\dot{V} \leq \overline{\eta}^T(t)\,(I_N \otimes H_1)\,\overline{\eta}(t), \tag{4.40}$$

where

$$H_1 := A^T P + PA - 2\alpha PDD^T P + \kappa PP + \frac{\gamma_0^2}{\kappa}\left(1 + \lambda_\sigma^2(\mathcal{A})\rho^2 h \mathrm{e}^{(2\omega_1+1)h}\right) I_n, \tag{4.41}$$

for Case 1), and

$$H_1 := A^T P + PA - 2(\alpha - 1)PDD^T P + \kappa PP + \frac{\gamma_0^2}{\kappa}\left(1 + \lambda_\sigma^2(\mathcal{A})\rho^2 h \mathrm{e}^{(2\omega_1+1)h}\right) I_n, \tag{4.42}$$

for Case 2).

4.2.3 Consensus Controller Design

The above expressions can be used for consensus analysis of network-connected systems with Lipschitz nonlinearity and input delay. The following theorem summarizes the results.

Theorem 4.2.1. *For an input-delayed multi-agent system (4.1) with the associated Laplacian matrix that satisfies Assumption (4.1.1), the consensus control problem can be solved by the control algorithm (4.7) with the control gain $K = D^T P$ specified in one of the following two cases:*

1) If the eigenvalues of the Laplacian matrix $\mathcal{L}$ are distinct, the consensus is *achieved if the following conditions are satisfied for* $W = P^{-1}$ *and* $\rho > 0$, $\omega_1 \geq 0$,

$$(A - \frac{1}{2}\omega_1 I_n)^T + (A - \frac{1}{2}\omega_1 I_n) < 0, \tag{4.43}$$

$$\rho W \geq DD^T, \tag{4.44}$$

$$\begin{bmatrix} WA^T + AW - 2\alpha DD^T + \kappa I_n & W \\ W & \dfrac{-\kappa I_n}{\gamma_0^2(1 + 4h_0\rho^2)} \end{bmatrix} < 0, \tag{4.45}$$

where κ *is any positive real number and* $h_0 = \lambda_\sigma^2(\mathcal{A}) h e^{(2\omega_1+1)h}$.

2) If the Laplacian matrix $\mathcal{L}$ *has multiple eigenvalues, the consensus is achieved if the conditions (4.43), (4.44) and the following condition are satisfied for* $W = P^{-1}$ *and* $\rho > 0$, $\omega_1 \geq 0$,

$$\begin{bmatrix} WA^T + AW - 2(\alpha - 1)DD^T + \kappa I_n & W \\ W & \dfrac{-\kappa I_n}{\gamma_0^2(1 + 4h_0\rho^2)} \end{bmatrix} < 0, \tag{4.46}$$

where κ *is any positive real number and* $h_0 = \lambda_\sigma^2(\mathcal{A}) h e^{(2\omega_1+1)h}$.

Proof. When the eigenvalues are distinct, from the analysis in this section, we know that the feedback law (4.7) will stabilize $\overline{\eta}$ if the conditions (4.35), (4.36) and $H_1 < 0$ in (4.41) are satisfied. Indeed, it is easy to see the conditions (4.35) and (4.36) are equivalent to the conditions specified in (4.43) and (4.44). From (4.41), it can be obtained that $H_1 < 0$ is equivalent to

$$P^{-1}A^T + AP^{-1} - 2\alpha DD^T + \kappa I_n + \frac{\gamma_0^2}{\kappa}(1 + 4h_0\rho^2)P^{-1}P^{-1} < 0, \tag{4.47}$$

which is further equivalent to (4.45). Hence we conclude that $\overline{\eta}$ converges to zero asymptotically.

When the Laplacian matrix has multiple eigenvalues, the feedback law (4.7) will stabilize $\overline{\eta}$ if the conditions (4.35), (4.35) and $H_1 < 0$ in (4.42) are satisfied. Following the similar procedure as Case 1), we can show that, under the conditions (4.43), (4.44) and (4.46), $\overline{\eta}$ converges to zero asymptotically. The proof is completed. □

Remark 4.2.2. *The conditions shown in (4.43) to (4.46) can be checked by standard LMI routines for a set of fixed values* ρ *and* ω_1. *The iterative methods developed in [135] for single linear system may also be applied here.*

4.3 Consensus Control of Nonlinear MASs with Input Delay: TPF Method

4.3.1 Finite-Dimensional Consensus Controller Design

The consensus controller (4.7) designed in last section

$$u_i(t) = -K\sum_{j=1}^{N} l_{ij}z_j(t),$$

alternatively can be written as

$$u_i(t) = -K\sum_{j=1}^{N} l_{ij}\left(x_j(t) + \int_t^{t+h} \mathrm{e}^{A(t-\tau)}Bu_j(\tau-h)\mathrm{d}\tau\right).$$

It is obvious that the controller for each agent requires the relative input signals among the agents. It may be unreachable sometimes due to the sensor restriction. To overcome this problem, in this section, we will propose another alternative dynamic consensus protocol based on the TPF method. The control input takes the structure

$$\begin{aligned} u_i(t) &= Ke^{Ah}\sum_{j=1}^{N} a_{ij}\left(x_i(t) - x_j(t)\right) \\ &= Ke^{Ah}\sum_{j=1}^{N} l_{ij}x_j(t), \end{aligned} \tag{4.48}$$

where $K \in \mathbb{R}^{m\times n}$ is a constant control gain matrix to be designed later, a_{ij} and l_{ij} being the elements of the graph adjacency matrix $\mathcal{A}$ and the Laplacian matrix $\mathcal{L}$, respectively.

Remark 4.3.1. *It is worth noting from* (4.48) *that the proposed control only uses the relative state information of the agents via network connections. The controller for each agent is finite-dimensional and easy to implement since the integral of the relative input information is not needed.*

4.3.2 Overview of TPF Approach

Consider a linear input-delayed system

$$\dot{x}(t) = Ax(t) + Bu(t-h), \tag{4.49}$$

where $x \in \mathbb{R}^n$ denotes the state, $u \in \mathbb{R}^m$ denotes the control input, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices, $h > 0$ is input delay.

As introduced in Chapter 2, the main idea of the predictor feedback is to design the controller

$$u(t-h) = Kx(t), \ \ \forall t \geq h. \tag{4.50}$$

The resultant closed-loop system is then given by

$$\dot{x}(t) = (A+BK)x(t), \tag{4.51}$$

where the control gain matrix K is chosen so that $A+BK$ is Hurwitz. The input (4.50) can be rewritten as

$$\begin{aligned} u(t) =& Kx(t+h) \\ =& Ku_1(t) + Ku_2(t), \end{aligned} \tag{4.52}$$

where

$$\begin{aligned} u_1(t) =& Ke^{Ah}x(t), \\ u_2(t) =& K\int_{t-h}^{t} \mathrm{e}^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau. \end{aligned}$$

As point out in [82], no matter how large the value of the input delay is, the infinite dimensional predictor term $u_2(t)$ in (4.52) is dominated by the finite dimensional predictor term $u_1(t)$ and thus might be safely neglected in $u(t)$ under certain conditions. As a result, the predictor feedback law (4.52) can be truncated as

$$u(t) = Ku_1(t) = Ke^{Ah}x(t), \tag{4.53}$$

which is refer to as the TPF control method. The resultant closed-loop system is then given by

$$\dot{x}(t) = (A+BK)x(t) + \widetilde{d}(t), \tag{4.54}$$

where

$$\widetilde{d}(t) = -BK\int_{t-h}^{t} \mathrm{e}^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau.$$

Now the control problem is to find a proper gain matrix K to stabilize the resultant closed-loop system (4.54).

4.3.3 Consensus Analysis

For the multi-agent systems (4.1), we have

$$x_i(t) = e^{Ah}x_i(t-h) + \int_{t-h}^{t} e^{A(t-\tau)}\left(Bu_i(\tau-h) + \phi\left(x_i(\tau)\right)\right)\mathrm{d}\tau.$$

Under control algorithm (4.48), the multi-agent systems (4.1) can be written as

$$\begin{aligned}\dot{x}_i = {} & Ax_i + BK\sum_{j=1}^{N} l_{ij}x_j + \phi(x_i)\\ & - BK\sum_{j=1}^{N} l_{ij}\int_{t-h}^{t} e^{A(t-\tau)}\left(Bu_j(\tau-h) + \phi(x_j)\right)\mathrm{d}\tau.\end{aligned}$$

The closed-loop system is then described by

$$\dot{x} = (I_N \otimes A + \mathcal{L} \otimes BK)\,x + (\mathcal{L} \otimes BK)\,(d_1 + d_2) + \Phi(x), \tag{4.55}$$

where

$$\begin{aligned} d_1 &= -\int_{t-h}^{t} e^{A(t-\tau)}Bu(\tau-h)\mathrm{d}\tau,\\ d_2 &= -\int_{t-h}^{t} e^{A(t-\tau)}\Phi(x)\mathrm{d}\tau,\end{aligned}$$

with

$$\begin{aligned} x(t) &= \left[x_1^T(t), x_2^T(t), \cdots, x_N^T(t)\right]^T,\\ u(t) &= \left[u_1^T(t), u_2^T(t), \cdots, u_N^T(t)\right]^T,\\ \Phi(x) &= \left[\phi^T(x_1), \phi^T(x_2), \cdots, \phi^T(x_N)\right]^T.\end{aligned}$$

Based on the vector r in Lemma (2.1.6), we introduce a state transformation

$$\xi_i = x_i - \sum_{j=1}^{N} r_j x_j, \tag{4.56}$$

for $i = 1, 2, \ldots, N$. Let $\xi = \left[\xi_1^T, \xi_2^T, \cdots, \xi_N^T\right]^T$. Then we have

$$\begin{aligned}\xi &= x - \left(\left(\mathbf{1}r^T\right) \otimes I_n\right) x \\ &= (M \otimes I_n)\, x,\end{aligned}$$

where $M = I_N - \mathbf{1}r^T$. Since $r^T\mathbf{1} = 1$, it can be shown that $M\mathbf{1} = 0$. Therefore the consensus of system (4.55) is achieved when $\lim_{t\to\infty}\xi(t) = 0$, as $\xi = 0$ implies that $x_1 = x_2 = \cdots = x_N$, due to the fact that the null space of M is span$\{\mathbf{1}\}$. The dynamics of ξ can then be derived as

$$\begin{aligned}\dot{\xi} &= (I_N \otimes A + \mathcal{L} \otimes BK)\, x - \left(\mathbf{1}r^T \otimes I_n\right)[I_N \otimes A + \mathcal{L} \otimes BK]\, x \\ &\quad + (M \otimes I_n)(\mathcal{L} \otimes BK)(d_1 + d_2) + (M \otimes I_n)\,\Phi(x) \\ &= (I_N \otimes A + \mathcal{L} \otimes BK)\,\xi + (M \otimes I_n)\,\Phi(x) + (\mathcal{L} \otimes BK)(d_1 + d_2), \qquad (4.57)\end{aligned}$$

where we have used $r^T\mathcal{L} = 0$.

To explore the structure of $\mathcal{L}$, we propose another state transformation

$$\eta = \left(T^{-1} \otimes I_n\right)\xi, \qquad (4.58)$$

with $\eta = \left[\eta_1^T, \eta_2^T, \cdots, \eta_N^T\right]^T$. Then, based on Lemma (2.1.6) we have

$$\dot{\eta} = (I_N \otimes A + J \otimes BK)\,\eta + \Pi(x) + \Delta(x) + \Psi(x), \qquad (4.59)$$

where

$$\begin{aligned}\Pi(x) &= \left(T^{-1}\mathcal{L} \otimes BK\right) d_1, \\ \Delta(x) &= \left(T^{-1}\mathcal{L} \otimes BK\right) d_2, \\ \Psi(x) &= \left(T^{-1}M \otimes I_n\right)\Phi(x).\end{aligned}$$

For the notational convenience, let

$$\Pi = \begin{bmatrix}\pi_1 \\ \pi_2 \\ \vdots \\ \pi_N\end{bmatrix}, \quad \Delta = \begin{bmatrix}\delta_1 \\ \delta_2 \\ \vdots \\ \delta_N\end{bmatrix}, \quad \Psi = \begin{bmatrix}\psi_1 \\ \psi_2 \\ \vdots \\ \psi_N\end{bmatrix},$$

with $\pi_i, \delta_i, \psi_i, : \mathbb{R}^{n\times N} \to \mathbb{R}^n$ for $i = 1, 2, \ldots, N$.

From state transformations (4.56) and (4.58), we have:

$$\eta_1 = \left(r^T \otimes I_n\right)\xi = \left((r^T M) \otimes I_n\right)x \equiv 0.$$

With the control law shown in (4.48), the control gain matrix K is chosen as

$$K = -B^T P, \tag{4.60}$$

where P is a positive definite matrix.

The consensus analysis will be carried out in terms of η. By (4.58), the consensus is achieved if η converges to zero, or equivalently if η_i converges to zero for $i = 2,3,\ldots,N$, since it has been shown that $\eta_1 = 0$. Let

$$V_i = \eta_i^T P \eta_i, \tag{4.61}$$

for $i = 2,3,\cdots,N$. By employing the similar Lyapunov functions developed in last section, we have the following results.

Lemma 4.3.1. *For a multi-agent systems (4.1) with the transformed state η, $\dot{V}_0$ has following bounds specified in one of the following two cases:*
1) If the eigenvalues of the Laplacian matrix $\mathcal{L}$ are distinct, i.e., $n_k = 1$ for $k = 1,2,\ldots,q$, $\dot{V}_0$ satisfies

$$\begin{aligned}\dot{V}_0 \le \eta^T &\left[I_N \otimes \left(A^T P + PA - 2\alpha PBB^T P + \sum_{\iota=1}^{3} \kappa_\iota PP\right)\right]\eta \\ &+ \frac{1}{\kappa_1}\|\Pi\|^2 + \frac{1}{\kappa_2}\|\Delta\|^2 + \frac{1}{\kappa_3}\|\Psi\|^2, \end{aligned} \tag{4.62}$$

with $\kappa_1, \kappa_2, \kappa_3$ being any positive real numbers and α is the real part of the smallest non-zero eigenvalue of the Laplacian matrix $\mathcal{L}$

$$\alpha = \min\{\lambda_1, \lambda_2, \ldots, \lambda_p, \alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_q\}.$$

2) If the Laplacian matrix $\mathcal{L}$ has multiple eigenvalues, i.e., $n_k > 1$ for any $k \in \{1,2,\cdots,q\}$,

$\dot{V}_0$ *satisfies*

$$\dot{V}_0 \leq \eta^T \left[I_N \otimes \left(A^T P + PA - 2(\alpha - 1) PBB^T P + \sum_{\iota=1}^{3} \kappa_\iota PP \right) \right] \eta$$
$$+ \frac{1}{\kappa_1} \|\Pi\|^2 + \frac{1}{\kappa_2} \|\Delta\|^2 + \frac{1}{\kappa_3} \|\Psi\|^2, \tag{4.63}$$

with κ *being any positive real number.*

The following lemmas give the bounds of $\|\Pi\|^2$, $\|\Delta\|^2$ and $\|\Psi\|^2$.

Lemma 4.3.2. *For the integral term* $\|\Pi\|^2$ *shown in the transformed system dynamics (4.59), the bounds can be established as*

$$\|\Pi\|^2 \leq \gamma_0 \int_{t-h}^{t} \eta^T (\tau - h) \eta (\tau - h) \mathrm{d}\tau, \tag{4.64}$$

with

$$\gamma_0 = 4h\rho^4 e^{2\omega_1 h} \lambda_\sigma^2 \left(T^{-1}\right) \|\mathcal{L}\|_F^2 \|\mathcal{A}\|_F^2 \|T\|_F^2,$$

where $\mathcal{A}$ *is the adjacency matrix,* $\mathcal{L}$ *is the Laplacian matrix,* T *is the non-singular matrix,* ρ *and* ω_1 *are positive numbers such that*

$$\rho^2 I \geq PBB^T BB^T P, \tag{4.65}$$
$$\omega_1 I > A^T + A. \tag{4.66}$$

Proof. By the definition of $\Pi(x)$ in (4.59), we have

$$\|\Pi\| = \left\| \left(T^{-1} \otimes I_n\right) (\mathcal{L} \otimes BK) d_1 \right\|$$
$$\leq \lambda_\sigma \left(T^{-1}\right) \|\mu\|, \tag{4.67}$$

where $\mu = (\mathcal{L} \otimes BK) d_1$. For the notational convenience, let $\mu = \left[\mu_1^T, \mu_2^T, \ldots, \mu_N^T\right]^T$. Then from (4.48) and (4.55), we have

$$\mu_i = -BK \sum_{j=1}^{N} l_{ij} \int_{t-h}^{t} e^{A(t-\tau)} Bu_j(\tau - h) \mathrm{d}\tau$$
$$= BB^T P \sum_{j=1}^{N} l_{ij} \int_{t-h}^{t} e^{A(t-\tau)} BB^T P e^{Ah} \sum_{k=1}^{N} a_{jk} \left(x_k(\tau - h) - x_j(\tau - h)\right) \mathrm{d}\tau. \tag{4.68}$$

From $\eta = \left(T^{-1} \otimes I_n\right) \xi$, we obtain $\xi = (T \otimes I_n) \eta$, and from the state transformations (4.56) and (4.58), we have

$$
\begin{aligned}
x_k(t) - x_j(t) &= \xi_k(t) - \xi_j(t) \\
&= \left(\left(T_k - T_j\right) \otimes I_n\right) \eta(t) \\
&= \sum_{l=1}^{N} \left(T_{kl} - T_{jl}\right) \eta_l(t),
\end{aligned}
\tag{4.69}
$$

where T_k denotes the kth row of T. We define

$$
\sigma_l = BB^T P \int_{t-h}^{t} \mathrm{e}^{A(t-\tau)} BB^T P e^{Ah} \eta_l(\tau - h) \mathrm{d}\tau.
$$

Then, from (4.68) and (4.69), we can obtain that

$$
\mu_i = \sum_{j=1}^{N} l_{ij} \sum_{k=1}^{N} a_{jk} \sum_{l=1}^{N} \left(T_{kl} - T_{jl}\right) \sigma_l.
$$

For the notational convenience, let $\sigma = \left[\sigma_1^T, \sigma_2^T, \ldots, \sigma_N^T\right]^T$. It then follows that

$$
\begin{aligned}
\|\mu_i\| &\leq \sum_{j=1}^{N} \left|l_{ij}\right| \sum_{k=1}^{N} \left|a_{jk}\right| \|T_k\| \|\sigma\| + \sum_{k=1}^{N} \sum_{j=1}^{N} \left|l_{ij}\right| \left|a_{jk}\right| \left\|T_j\right\| \|\sigma\| \\
&\leq \sum_{j=1}^{N} \left|l_{ij}\right| \left\|a_j\right\| \|T\|_F \|\sigma\| + \sum_{k=1}^{N} \sum_{j=1}^{N} \left|l_{ij}\right| \|a_k\| \|T\|_F \|\sigma\| \\
&\leq \|l_i\| \|\mathcal{A}\|_F \|T\|_F \|\sigma\| + \|l_i\| \|\mathcal{A}\|_F \|T\|_F \|\sigma\| \\
&= 2 \|l_i\| \|\mathcal{A}\|_F \|T\|_F \|\sigma\|,
\end{aligned}
$$

where l_i and a_i denote the ith row of $\mathcal{L}$ and $\mathcal{A}$, respectively. Therefore, we have

$$
\begin{aligned}
\|\mu\|^2 &= \sum_{i=1}^{N} \|\mu_i\|^2 \\
&\leq 4 \sum_{i=1}^{N} \|l_i\|^2 \|\mathcal{A}\|_F^2 \|T\|_F^2 \|\sigma\|^2 \\
&= 4 \|L\|_F^2 \|\mathcal{A}\|_F^2 \|T\|_F^2 \|\sigma\|^2.
\end{aligned}
\tag{4.70}
$$

Next we need to deal with $\|\sigma\|^2$. With the Jensen's Inequality in Lemma (2.1.5) and the condition (4.83), we have

$$\begin{aligned}\|\sigma_i\|^2 \leq h \int_{t-h}^{t} &\eta_i^T(\tau-h)e^{A^Th}PBB^Te^{A^T(t-\tau)}PBB^T \\ &\times BB^TPe^{A(t-\tau)}BB^TPe^{Ah}\eta_i(\tau-h)\mathrm{d}\tau \\ &\leq h\rho^2\int_{t-h}^{t}\eta_i^T(\tau-h)e^{A^Th}PBB^Te^{A^T(t-\tau)}e^{A(t-\tau)}BB^TPe^{Ah}\eta_i(\tau-h)\mathrm{d}\tau.\end{aligned}$$

In view of Lemma (2.1.6), with the condition (4.84), we have

$$\begin{aligned}\|\sigma_i\|^2 &\leq h\rho^2\int_{t-h}^{t}e^{\omega_1(t-\tau)}\eta_i^T(\tau-h)e^{A^Th}PBB^TBB^TPe^{Ah}\eta_i(\tau-h)\mathrm{d}\tau \\ &\leq h\rho^4e^{\omega_1 h}\int_{t-h}^{t}\eta_i^T(\tau-h)e^{A^Th}e^{Ah}\eta_i(\tau-h)\mathrm{d}\tau \\ &\leq h\rho^4e^{2\omega_1 h}\int_{t-h}^{t}\eta_i^T(\tau-h)\eta_i(\tau-h)\mathrm{d}\tau.\end{aligned}$$

Then, $\|\sigma\|^2$ can be bounded as

$$\begin{aligned}\|\sigma\|^2 &= \sum_{i=1}^{N}\|\sigma_i\|^2 \\ &\leq h\rho^4e^{2\omega_1 h}\int_{t-h}^{t}\eta^T(\tau-h)\eta(\tau-h)\mathrm{d}\tau.\end{aligned} \tag{4.71}$$

Therefore, from Equations (4.67), (4.70) and (4.71), we have

$$\|\Pi\|^2 \leq \gamma_0\int_{t-h}^{t}\eta^T(\tau-h)\eta(\tau-h)\mathrm{d}\tau.$$

This completes the proof. □

Lemma 4.3.3. *For the integral terms $\Delta(x)$ in the transformed system dynamics (4.59), a bound can be established as*

$$\|\Delta\|^2 \leq \gamma_1\int_{t-h}^{t}\eta^T(\tau)\eta(\tau)\mathrm{d}\tau, \tag{4.72}$$

where

$$\gamma_1 = 4\rho^2he^{\omega_1 h}\gamma^2\lambda_\sigma^2\left(T^{-1}\right)\lambda_\sigma^2(\mathcal{A})\|T\|_F^2,$$

with ρ and ω_1 being as defined in (4.65) and (4.66).

Proof. In a way similar to Lemma (4.3.2), we have

$$\begin{aligned}\|\Delta(x)\| &= \left\|\left(T^{-1}\otimes I_n\right)(\mathcal{L}\otimes BK)\,d_2\right\| \\ &\leq \lambda_\sigma\left(T^{-1}\right)\left\|\bar{\delta}\right\|, \end{aligned} \tag{4.73}$$

where $\bar{\delta} = (\mathcal{L}\otimes BK)d_2$. Let $\bar{\delta} = \left[\bar{\delta}_1^T, \bar{\delta}_2^T, \ldots, \bar{\delta}_N^T\right]^T$. Then, from (4.48) and (4.55), we have

$$\bar{\delta}_i = \sum_{j=1}^{N} a_{ij} BB^T P \int_{t-h}^{t} e^{A(t-\tau)}\left[\phi(x_i)-\phi(x_j)\right]\mathrm{d}\tau.$$

It follows that

$$\begin{aligned}\left\|\bar{\delta}_i\right\|^2 = \sum_{j=1}^{N} a_{ij}^2 \int_{t-h}^{t} &\left[\phi(x_i)-\phi(x_j)\right]^T e^{A^T(t-\tau)}\mathrm{d}\tau \\ &\times PBB^TBB^TP\int_{t-h}^{t} e^{A(t-\tau)}\left[\phi(x_i)-\phi(x_j)\right]\mathrm{d}\tau.\end{aligned}$$

With Jensen's Inequality in Lemma (2.1.5) and the condition (4.65), we have

$$\begin{aligned}\left\|\bar{\delta}_i\right\|^2 &\leq h\sum_{j=1}^{N} a_{ij}^2 \int_{t-h}^{t}\left[\phi(x_i)-\phi(x_j)\right]^T e^{A^T(t-\tau)}PBB^T \\ &\quad \times BB^TPe^{A(t-\tau)}\left[\phi(x_i)-\phi(x_j)\right]\mathrm{d}\tau \\ &\leq \rho^2 h\sum_{j=1}^{N} a_{ij}^2 \int_{t-h}^{t}\left[\phi(x_i)-\phi(x_j)\right]^T e^{A^T(t-\tau)}e^{A(t-\tau)}\left[\phi(x_i)-\phi(x_j)\right]\mathrm{d}\tau.\end{aligned}$$

In view of Lemma (2.1.6), with the condition (4.66), we have

$$\begin{aligned}\left\|\bar{\delta}_i\right\|^2 &\leq \rho^2 h\sum_{j=1}^{N} a_{ij}^2 \int_{t-h}^{t} e^{\omega_1(t-h)}\left\|\phi(x_i)-\phi(x_j)\right\|^2\mathrm{d}\tau \\ &\leq \rho^2 h e^{\omega_1 h}\gamma^2\sum_{j=1}^{N} a_{ij}^2\int_{t-h}^{t}\left\|x_i(\tau)-x_j(\tau)\right\|^2\mathrm{d}\tau.\end{aligned}$$

From the state transformations, we have

$$\begin{aligned}x_i(t)-x_j(t) &= \xi_i(t)-\xi_j(t)\\ &= \left(\left(t_i-t_j\right)\otimes I_n\right)\eta(t)\\ &= \sum_{l=1}^{N}\left(t_{il}-t_{jl}\right)\eta_l(t).\end{aligned}$$

Let us define $\overline{\sigma}_l = \int_{t-h}^{t}\eta_l(\tau)\mathrm{d}\tau$ for $l = 1,2, \cdots, N$. Then,

$$\begin{aligned}\left\|\bar{\delta}_i\right\|^2 &\le \rho^2 he^{\omega_1 h}\gamma^2\sum_{j=1}^{N}a_{ij}^2\sum_{l=1}^{N}\left|t_{il}-t_{jl}\right|^2\|\overline{\sigma}_l\|^2\\ &\le 2\rho^2 he^{\omega_1 h}\gamma^2\sum_{j=1}^{N}a_{ij}^2\sum_{l=1}^{N}\left(|t_{il}|^2+\left|t_{jl}\right|^2\right)\|\overline{\sigma}_l\|^2\\ &\le 2\rho^2 he^{\omega_1 h}\gamma^2\sum_{j=1}^{N}a_{ij}^2\left(\|t_i\|^2+\left\|t_j\right\|^2\right)\|\overline{\sigma}\|^2,\end{aligned}$$

where $\overline{\sigma} = \left[\overline{\sigma}_1^T, \overline{\sigma}_2^T, \ldots, \overline{\sigma}_N^T\right]^T$. Consequently,

$$\begin{aligned}\left\|\bar{\delta}\right\|^2 &= \sum_{i=1}^{N}\left\|\bar{\delta}_i\right\|^2\\ &\le 2\rho^2 he^{\omega_1 h}\gamma^2\|\overline{\sigma}\|^2\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}^2\left(\|t_i\|^2+\left\|t_j\right\|^2\right)\\ &\le 2\rho^2 he^{\omega_1 h}\gamma^2\lambda_\sigma^2(\mathcal{A})\|T\|_F^2\|\overline{\sigma}\|^2+2\rho^2 he^{\omega_1 h}\gamma^2\lambda_\sigma^2(\mathcal{A})\|T\|_F^2\|\overline{\sigma}\|^2\\ &= 4\rho^2 he^{\omega_1 h}\gamma^2\lambda_\sigma^2(\mathcal{A})\|T\|_F^2\|\overline{\sigma}\|^2.\end{aligned}\tag{4.74}$$

Putting (4.73) and (4.74) together, we have

$$\|\Delta\|^2 \le \gamma_1\int_{t-h}^{t}\eta^T(\tau)\eta(\tau)\mathrm{d}\tau.$$

with

$$\gamma_1 = 4\rho^2 hNe^{\omega_1 h}\gamma^2\lambda_\sigma^2(T^{-1})\lambda_\sigma^2(\mathcal{A})\|T\|_F^2.$$

□

Lemma 4.3.4. *For the nonlinear term $\Psi(x)$ in the transformed system dynamics (4.59),* *a bound can be established as*

$$\|\Psi\|^2 \leq \gamma_2 \|\eta\|^2, \tag{4.75}$$

where

$$\gamma_2 = 4N\gamma^2 \|r\|^2 \lambda_\sigma^2 \left(T^{-1}\right) \|T\|_F^2.$$

Proof. By the definition of $\Psi(x)$ in (4.59), we have

$$\begin{aligned}\|\Psi\| &= \left\|\left(T^{-1} \otimes I_n\right)(M \otimes I_n)\Phi(x)\right\| \\ &\leq \lambda_\sigma\left(T^{-1}\right)\|\bar{z}\|,\end{aligned}$$

where $\bar{z} = (M \otimes I_n)\Phi(x)$. For the notational convenience, let $\bar{z} = \left[\bar{z}_1^T, \bar{z}_2^T, \ldots, \bar{z}_N^T\right]^T$. Then from (4.56), we have

$$\begin{aligned}\bar{z}_i &= \phi(x_i) - \sum_{k=1}^{N} r_k \phi(x_k) \\ &= \sum_{k=1}^{N} r_k(\phi(x_i) - \phi(x_k)).\end{aligned}$$

It then follows that

$$\begin{aligned}\|\bar{z}_i\| &\leq \sum_{k=1}^{N} |r_k| \|(\phi(x_i) - \phi(x_k))\| \\ &\leq \gamma \sum_{k=1}^{N} |r_k| \|x_i - x_k\|.\end{aligned}$$

In light of (4.69), we have

$$\begin{aligned}\|\bar{z}_i\| &\leq \gamma \sum_{k=1}^{N} |r_k| \left(\|T_i\| + \|T_k\|\right)\|\eta\| \\ &\leq \gamma\|\eta\| \left(\sum_{k=1}^{N} |r_k| \|T_i\| + \|r\| \|T\|_F\right).\end{aligned}$$

Therefore we have

$$\begin{aligned}
\|\bar{z}\|^2 &= \sum_{i=1}^{N} \|\bar{z}_i\|^2 \\
&\leq 2\gamma^2 \|\eta\|^2 \sum_{i=1}^{N} \left(\|T_i\|^2 \left(\sum_{k=1}^{N} |r_k| \right)^2 + \|r\|^2 \|T\|_F^2 \right) \\
&\leq 2\gamma^2 \|\eta\|^2 \sum_{i=1}^{N} \left(\|T_i\|^2 N \|r\|^2 + \|r\|^2 \|T\|_F^2 \right) \\
&= 4N\gamma^2 \|r\|^2 \|T\|_F^2 \|\eta\|^2,
\end{aligned}$$

and

$$\|\Psi\|^2 \leq \gamma_2 \|\eta\|^2.$$

This completes the proof. □

Using (4.63), (4.64), (4.72) and (4.75), we can obtain

$$\begin{aligned}
\dot{V}_0 \leq{}& \eta^T \left[I_N \otimes \left(A^T P + PA - 2\hat{\alpha} PBB^T P + \sum_{\iota=1}^{3} \kappa_\iota PP + \frac{\gamma_2}{\kappa_3} I_n \right) \right] \eta \\
&+ \frac{\gamma_0}{\kappa_1} \int_{t-h}^{t} \eta^T(\tau - h)\eta(\tau - h) \mathrm{d}\tau + \frac{\gamma_1}{\kappa_2} \int_{t-h}^{t} \eta^T(\tau)\eta(\tau) \mathrm{d}\tau, \qquad (4.76)
\end{aligned}$$

where $\hat{\alpha} = \alpha$ for Case 1 and $\hat{\alpha} = \alpha - 1$ for Case 2 in Lamma (4.3.1).

For the first integral term shown in (4.76), we consider the following Krasovskii functional

$$W_3 = e^h \int_{t-h}^{t} e^{\tau - t} \eta^T(\tau - h)\eta(\tau - h) \mathrm{d}\tau + e^h \int_{t-h}^{t} \eta^T(\tau)\eta(\tau) \mathrm{d}\tau.$$

A direct evaluation gives that

$$\begin{aligned}
\dot{W}_3 &= -e^h \int_{t-h}^{t} e^{\tau - t} \eta^T(\tau - h)\eta(\tau - h) \mathrm{d}\tau - \eta^T(t - 2h)\eta(t - 2h) + e^h \eta^T(t)\eta(t) \\
&\leq -\int_{t-h}^{t} \eta^T(\tau - h)\eta(\tau - h) \mathrm{d}\tau + e^h \eta(t)^T \eta(t). \qquad (4.77)
\end{aligned}$$

For the second integral term shown in (4.76), we consider the following Krasovskii functional

$$W_4 = e^h \int_{t-h}^{t} e^{\tau - t} \eta^T(\tau)\eta(\tau)\mathrm{d}\tau.$$

A direct evaluation gives that

$$\begin{aligned}\dot{W}_4 =& -e^h \int_{t-h}^{t} e^{\tau-t}\eta^T(\tau)\eta(\tau)\mathrm{d}\tau + e^h\eta^T(t)\eta(t) - \eta^T(t-h)\eta(t-h) \\ \leq& -\int_{t-h}^{t}\eta^T(\tau)\eta(\tau)\mathrm{d}\tau + e^h\eta^T(t)\eta(t).\end{aligned} \tag{4.78}$$

Let

$$V = V_0 + \frac{\gamma_0}{\kappa_1}W_3 + \frac{\gamma_1}{\kappa_2}W_4. \tag{4.79}$$

From (4.76), (4.77), and (4.78), we obtain that

$$\dot{V} \leq \eta^T(t)\left(I_N \otimes H_3\right)\eta(t), \tag{4.80}$$

where

$$H_3 := A^T P + PA - 2\alpha PBB^T P + \sum_{\iota=1}^{3} \kappa_\iota PP + \left(\frac{\gamma_0}{\kappa_1}e^h + \frac{\gamma_1}{\kappa_2}e^h + \frac{\gamma_2}{\kappa_3}\right) I_n. \tag{4.81}$$

for Case 1 and

$$H_3 := A^T P + PA - 2(\alpha - 1)PBB^T P + \sum_{\iota=1}^{3} \kappa_\iota PP + \left(\frac{\gamma_0}{\kappa_1}e^h + \frac{\gamma_1}{\kappa_2}e^h + \frac{\gamma_2}{\kappa_3}\right) I_n. \tag{4.82}$$

for Case 2.

Basd on the analysis above, the following theorem presents sufficient conditions to ensure that the consensus problem is solved by using the control algorithm (4.48).

Theorem 4.3.1. *For the Lipschitz nonlinear multi-agent systems (1) with input delay,*

1) If the eigenvalues of the Laplacian matrix $\mathcal{L}$ are distinct, the consensus control problem can be solved by the control algorithm (4.48) with $K = -B^T P$ if there exists a *positive definite matrix P and constants $\omega_1 \geq 0$, $\rho, \kappa_1, \kappa_2, \kappa_3 > 0$ such that*

$$\rho W \geq BB^T, \tag{4.83}$$

$$\left(A - \frac{1}{2}\omega_1 I_n\right)^T + \left(A - \frac{1}{2}\omega_1 I_n\right) < 0, \tag{4.84}$$

$$\begin{bmatrix} WA^T + AW - 2\alpha BB^T + (\kappa_1 + \kappa_2 + \kappa_3) I_n & W \\ W & -\dfrac{I_n}{\Gamma} \end{bmatrix} < 0, \tag{4.85}$$

are satisfied with $W = P^{-1}$ and

$$\Gamma = \frac{\gamma_0}{\kappa_1} e^h + \frac{\gamma_1}{\kappa_2} e^h + \frac{\gamma_2}{\kappa_3}.$$

2) If the Laplacian matrix $\mathcal{L}$ has multiple eigenvalues, the consensus control problem can be solved by the control algorithm (4.48) with $K = -B^T P$ if there exists a positive definite matrix P and constants $\omega_1 \geq 0$, $\rho, \kappa_1, \kappa_2, \kappa_3 > 0$ such that the conditions (4.83), (4.84) and

$$\begin{bmatrix} WA^T + AW - 2(\alpha - 1) BB^T + (\kappa_1 + \kappa_2 + \kappa_3) I_n & W \\ W & -\dfrac{I_n}{\Gamma} \end{bmatrix} < 0, \tag{4.86}$$

are satisfied.

Proof. From the analysis in this section, we know that the feedback law (4.48) will stabilize η if the conditions (4.65), (4.66) and $H_3 < 0$ in (4.80) are satisfied. Indeed, it is easy to see the conditions (4.65) and (4.66) are equivalent to the conditions specified in (4.83) and (4.84). From (4.81), it can be obtained that $H_3 < 0$ is equivalent to

$$WA^T + AW - 2\hat{\alpha} BB^T + (\kappa_1 + \kappa_2 + \kappa_3) I_n + \left(\frac{\gamma_0}{\kappa_1} e^h + \frac{\gamma_1}{\kappa_2} e^h + \frac{\gamma_2}{\kappa_3}\right) WW < 0,$$

which is further equivalent to (4.85). It implies that η converges to zero asymptotically. Hence, the consensus control is achieved. □

It is observed that (4.85) is more likely to be satisfied if the values of $\rho, \omega_1, \kappa_1, \kappa_2, \kappa_3$ are small. Therefore, the algorithm for finding a feasible solution of the conditions shown in (4.83) to (4.85) can be designed by following the iterative methods developed in [135] for an individual linear system. In particular, we suggest the following step by step algorithm.

1) Set $\omega_1 = \lambda_{\max}(A+A^T)$ if $\lambda_{\max}(A+A^T) > 0$; otherwise set $\omega_1 = 0$.

2) Fix the value of $\rho, \omega_1, \kappa_1, \kappa_2, \kappa_3$ to some constants $\tilde{\omega}_1 > \omega_1$ and $\tilde{\rho}, \tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\kappa}_3 > 0$; make an initial guess for the values of $\tilde{\rho}, \tilde{\omega}_1, \tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\kappa}_3$.

3) Solve the LMI equation (4.85) for W with the fixed values; if a feasible value of W cannot be found, return to Step 2) and reset the values of $\tilde{\rho}, \tilde{\omega}_1, \tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\kappa}_3$.

4) Solve the LMI equation (4.83) for ρ with the feasible value of W obtained in Step 3) and make sure that the value of ρ is minimized.

5) If the condition $\tilde{\rho} \geq \rho$ is satisfied, then $(\tilde{\rho}, \tilde{\omega}_1, \tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\kappa}_3, W)$ is a feasible solution for Theorem (4.3.1); otherwise, set $\tilde{\rho} = \rho$ and return to Step 3).

Remark 4.3.2. *Given the input delay h and the Lipschitz constant γ, it is concluded that the existence of a feasible solution is related to the matrices (A, B) and the Laplacian matrix $\mathcal{L}$. Additionally, since the values of h and γ are fixed and they are not the decision variables of the LMIs, a feasible solution may not exist if the values of h and γ are too large. Therefore, a trigger should be added in the algorithm to stop the iteration procedure if the values of $\tilde{\rho}, \tilde{\omega}_1, \tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\kappa}_3$ are out of the preset range.*

4.4 Simulations

4.4.1 A Circuit Example

In this section, we will illustrate in some details the proposed consensus control design through a circuit example. The system under consideration is a connection of four agents (i.e. $N = 4$) as shown in Figure (4.1), each of which is described by a second-order dynamic model as

$$\begin{cases} \dot{p}_i(t) = v_i(t), \\ \dot{v}_i(t) = f(v_i) + u_i(t-h), \end{cases} \tag{4.87}$$

where $p_i = [p_{ix}, p_{iy}, p_{iz}]^T \in \mathbb{R}^3$ denotes the position vector of agent i, $v_i = [v_{ix}, v_{iy}, v_{iz}]^T \in \mathbb{R}^3$ the velocity vector, $f(v_i) \in \mathbb{R}^3$ the intrinsic dynamics of agent i, governed by the chaotic Chua circuit [136]

$$f(v_i) = \begin{bmatrix} -0.59v_{ix} + v_{iy} - 0.17(|v_{ix}+1| - |v_{ix}-1|) \\ v_{ix} - v_{iy} + v_{iz} \\ -v_{iy} - 5v_{iz} \end{bmatrix}.$$

Let $x_i = [p_i^T, v_i^T]^T \in \mathbb{R}^6$. The dynamic equation (4.87) of each agent can be re-arranged as the state space model (4.1) with

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -0.59 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -5 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $\phi(x_i) = [0,0,0,-0.17(|v_{ix}+1|-|v_{ix}-1|),0,0]^T$. The adjacency matrix is given

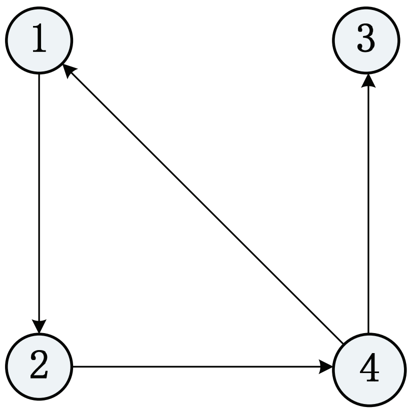

Figure 4.1: Communication topology.

by

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and the resultant Laplacian matrix is obtained as

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of $\mathcal{L}$ are $\{0, 1, 3/2 \pm j\sqrt{3}/2\}$, and therefore Assumption (4.1.1) is satisfied. Furthermore, the eigenvalues are distinct. We obtain that

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{3}{2} \end{bmatrix},$$

with the matrices

$$T = \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 0 & -1 & 0 \\ 1 & -2 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix},$$

and $r^T = [1/3, 1/3, 0, 1/3]^T$.

The nonlinear function $\phi(x_i)$ in each agent dynamics is globally Lipschitz with a Lipschitz constant $\gamma = 0.34$, which gives $\gamma_0 = 3.7391$ by (4.27). Based on the Laplacian matrix $\mathcal{L}$, we have $\alpha = 1$. In simulation, the input delay is set as $h = 0.03$s. A positive definite matrix P can be obtained with $\kappa = 0.01$, $\omega_1 = 1.5$ and $\rho = 2$, as

$$P = \begin{bmatrix} 5.03 & -0.53 & 0.18 & 2.58 & 0.29 & 0.08 \\ -0.53 & 5.37 & 0.43 & 0.28 & 2.39 & 0.47 \\ 0.18 & 0.43 & 7.75 & -0.08 & -0.38 & 1.58 \\ 2.58 & 0.28 & -0.08 & 2.65 & 0.93 & 0.17 \\ 0.29 & 2.39 & -0.38 & 0.93 & 2.17 & 0.25 \\ 0.08 & 0.47 & 1.58 & 0.17 & 0.25 & 0.92 \end{bmatrix},$$

to satisfy the conditions of Theorem 4.1. Consequently, the control gain is obtained as

$$K = \begin{bmatrix} -2.19 & -0.12 & -0.01 & -2.46 & -0.74 & -0.15 \\ -0.13 & -2.10 & 0.30 & -0.75 & -2.08 & -0.32 \\ -0.09 & -0.43 & -1.64 & -0.18 & -0.18 & -1.27 \end{bmatrix}.$$

Simulation study has been carried out with the results shown in Figure (4.3) for the positions state disagreement of each agent. Clearly the conditions specified in Theorem 4.1 are sufficient for the control gain to achieve consensus control for the multi-agent systems. The same control gain has also been used for different values of input delay. The results shown in Figure (4.4) indicate that the conditions could be conservative in the control gain design for a given input delay and Lipschitz nonlinear function.

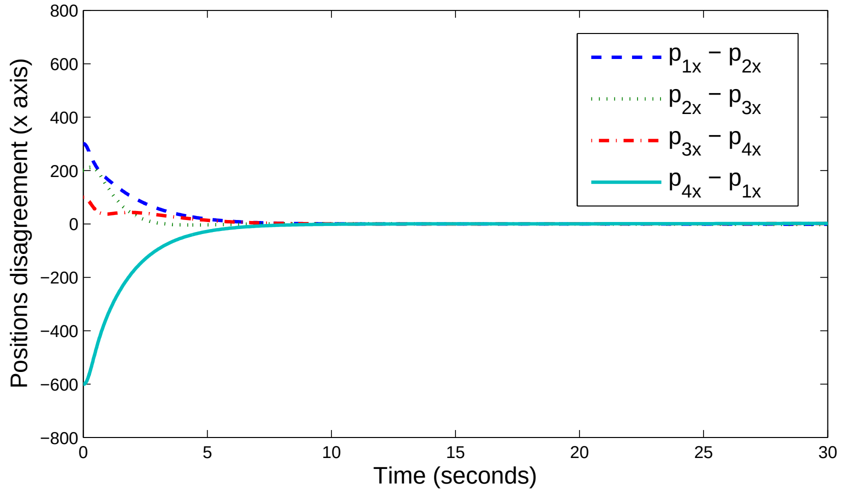

Figure 4.2: Schematic representation of coordinate frames.

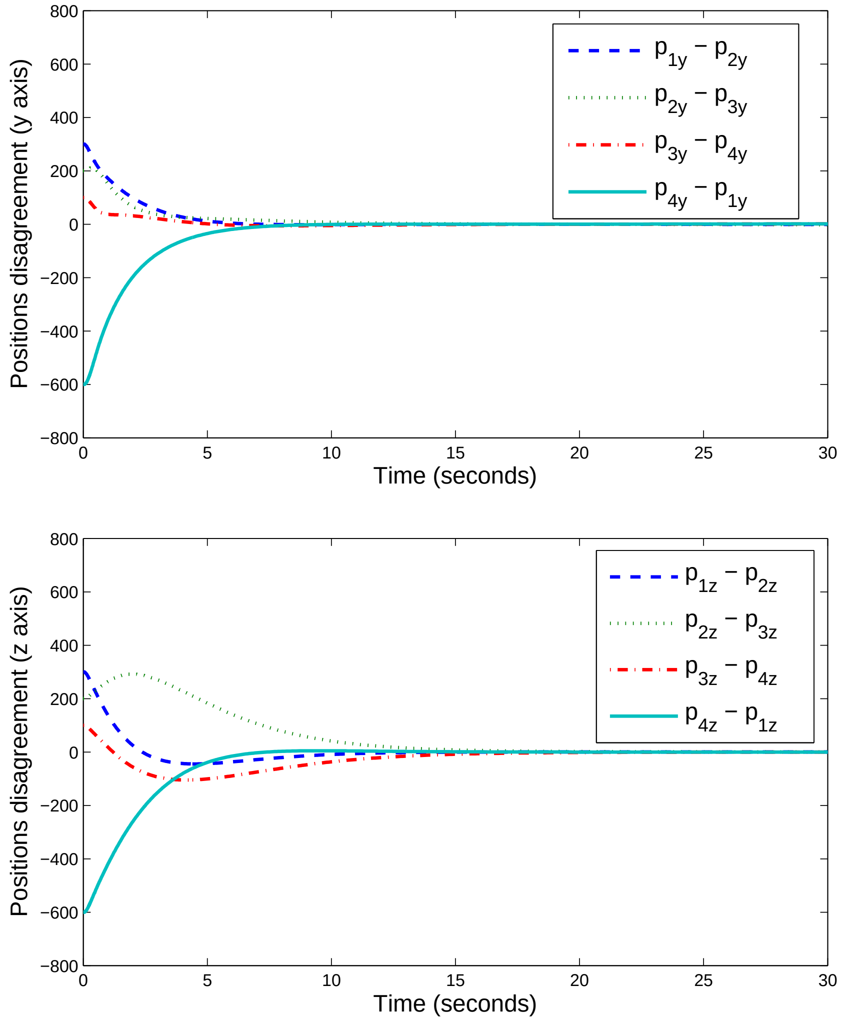

Figure 4.3: The positions disagreement of 4 agents: $h = 0.03$s.

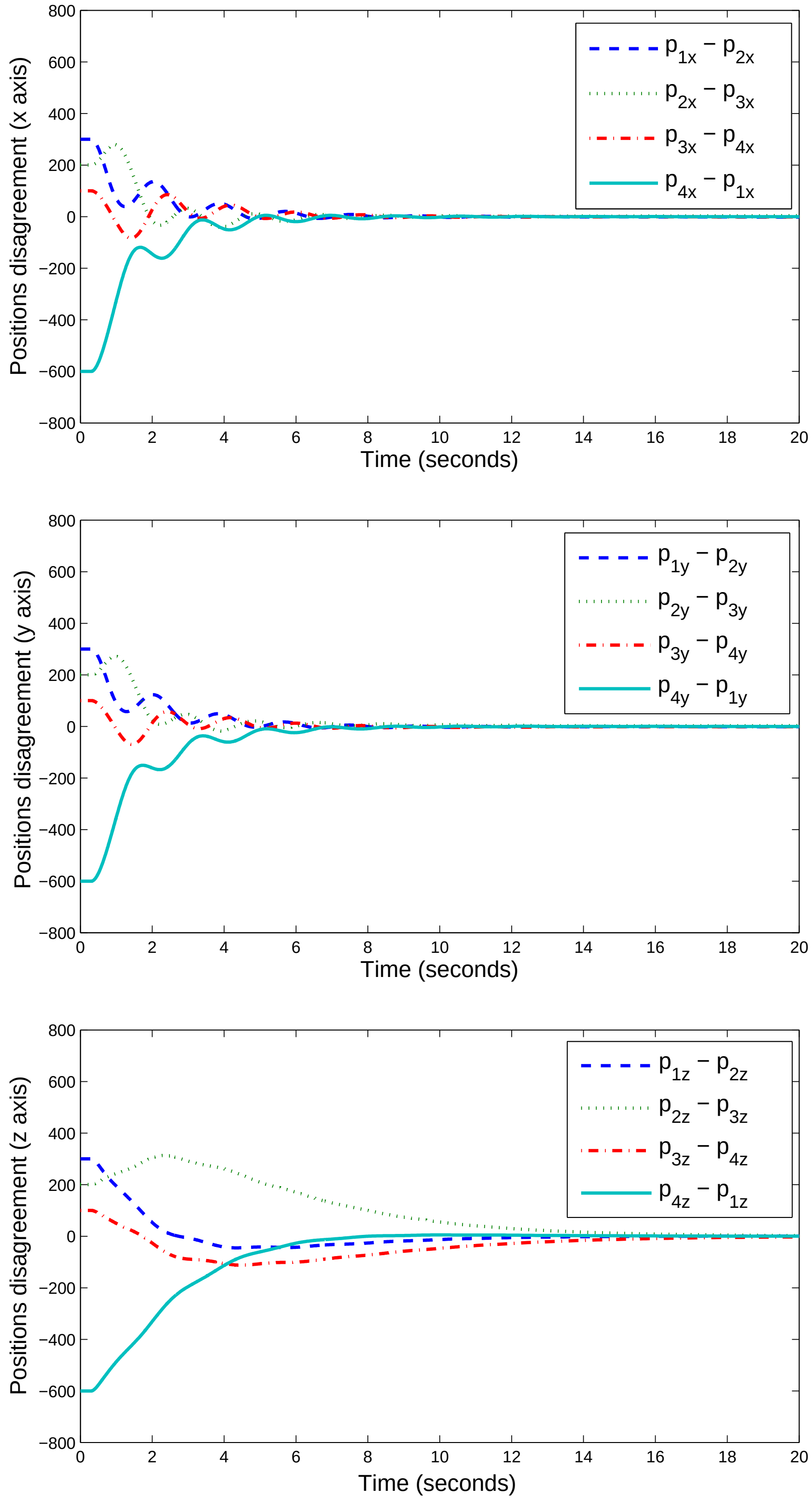

Figure 4.4: The positions disagreement of 4 agents: $h = 0.3$s.

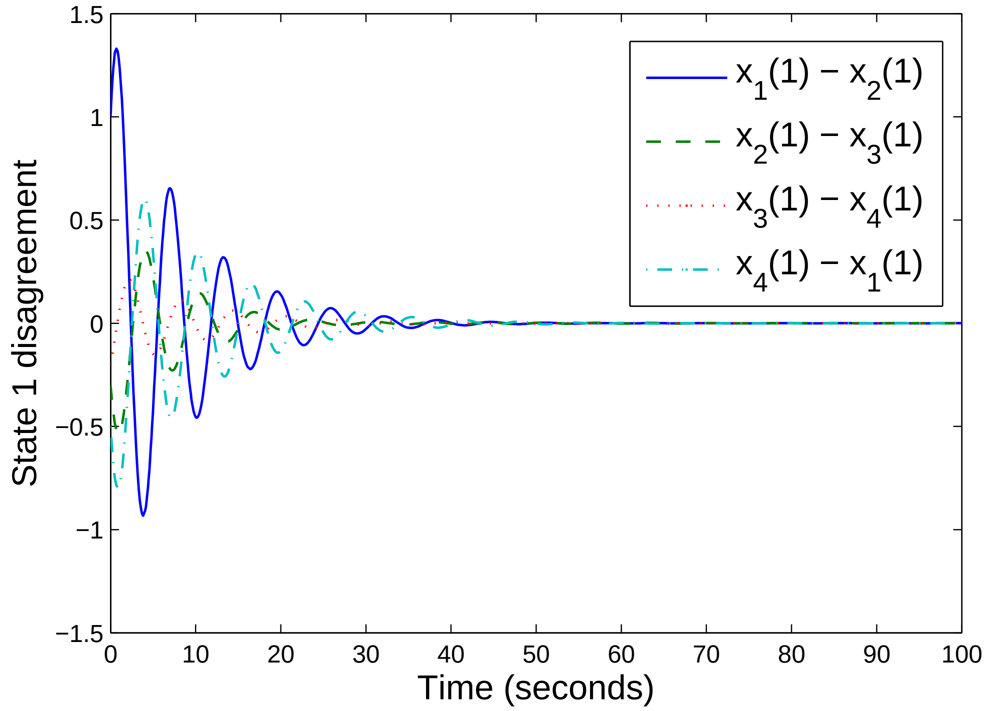

Figure 4.5: The state 1 disagreement of agents with $h = 0.1$ and $g = 0.03$.

Indeed, extensive simulation shows that the same control gain can possibly achieve consensus control for the system with a much larger delay and Lipschitz constant.

4.4.2 A Numerical Example

In this section, a simulation study is carried out to demonstrate the effectiveness of the proposed TPF controller design. Consider a connection of four agents as shown in Figure 4.1. The dynamics of the ith agent is described by a second-order model as

$$\dot{x}_i(t) = \begin{bmatrix} -0.09 & 1 \\ -1 & -0.09 \end{bmatrix} x_i(t) + g \begin{bmatrix} \sin(x_{i1}(t)) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t-0.1).$$

The linear part of the system represents a decayed oscillator. The time delay of the system is 0.1 seconds, and the Lipschitz constant $\gamma = g$. The eigenvalues of $\mathcal{L}$ are $\{\ 0,\ 1,\ 3/2 \pm j\sqrt{3}/2\ \}$ and $r^T = [\ \frac{1}{3},\ \frac{1}{3},\ 0,\ \frac{1}{3}\ ]$.

In this case, we choose $\gamma = g = 0.03$, and the initial conditions for the agents as $x_1(\theta) = [1,\ 1]^T$, $x_2(\theta) = [0,\ 0]^T$, $x_3(\theta) = [0.3,\ 0.5]^T$, $x_4(\theta) = [0.5,\ 0.3]^T$, $u(\theta) = [0,\ 0,\ 0,\ 0]^T$, for $\theta \in [-h, 0]$. With the values of $\omega_1 = 0, \rho = 0.3, \kappa_1 = \kappa_2 = 0.05$, and

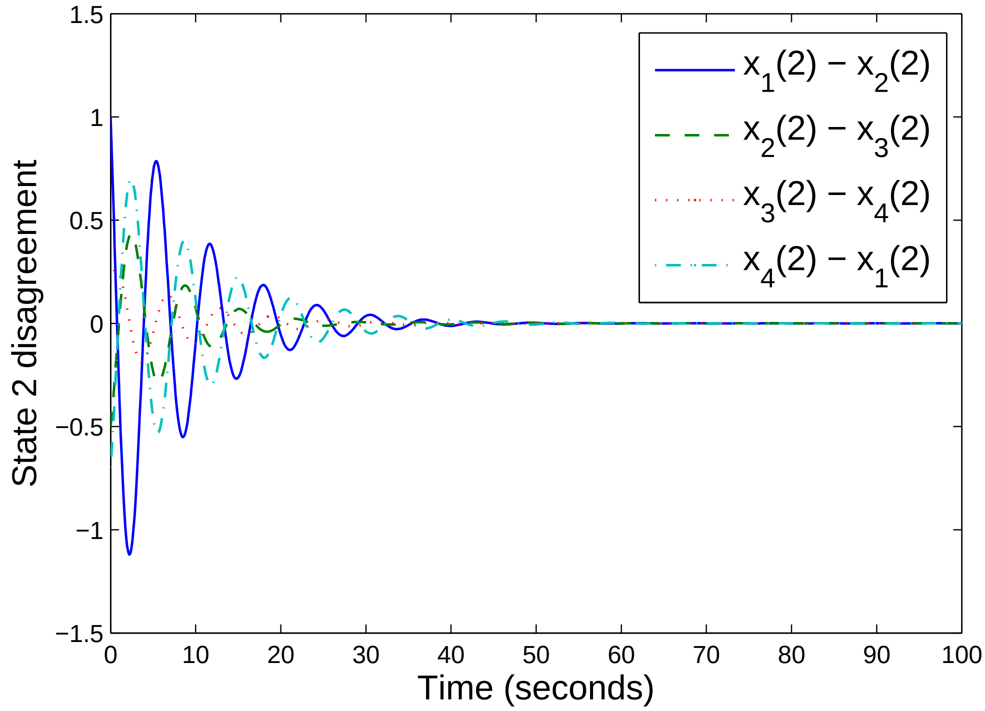

Figure 4.6: The state 2 disagreement of agents with $h = 0.1$ and $g = 0.03$.

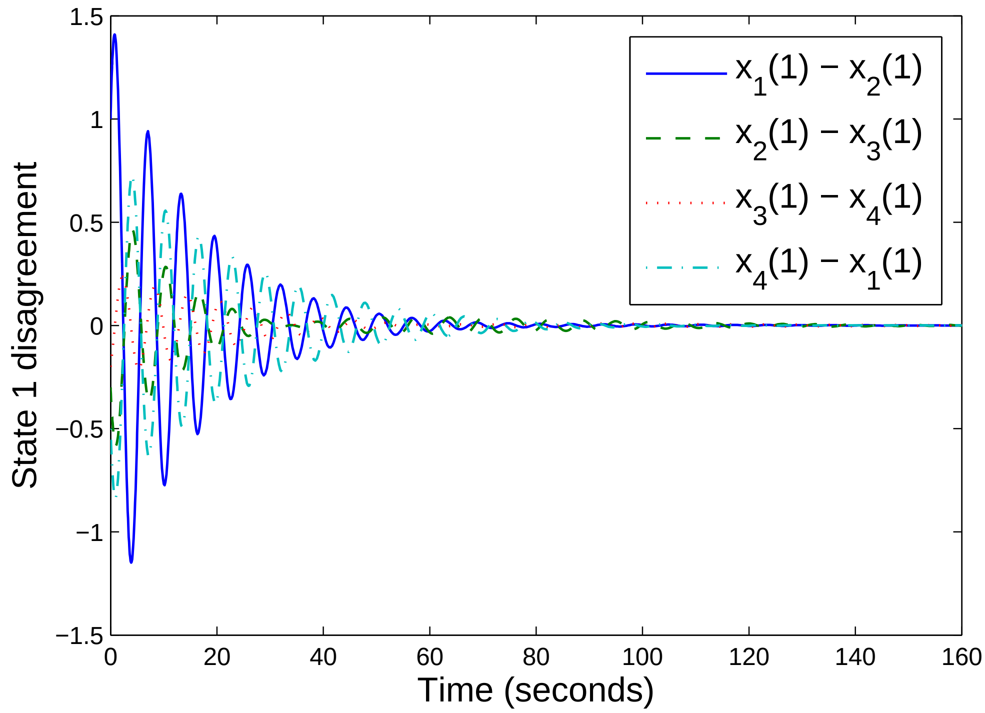

Figure 4.7: The state 1 disagreement of agents with $h = 0.5$ and $g = 0.15$.

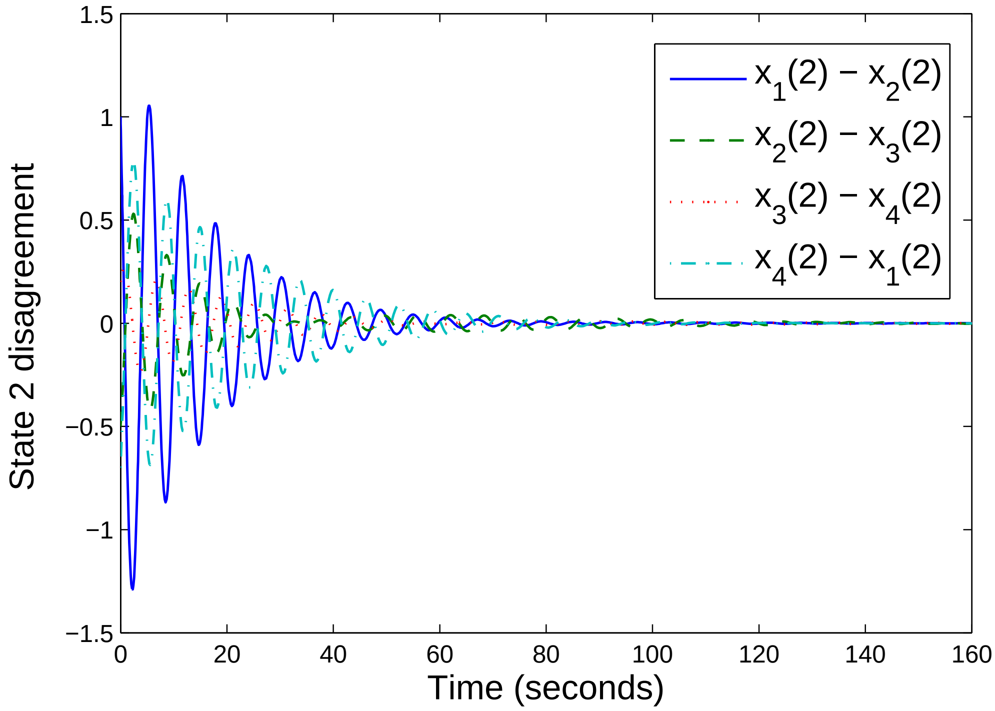

Figure 4.8: The state 2 disagreement of agents with $h = 0.5$ and $g = 0.15$.

$\kappa_3 = 0.5$, a feasible solution of the feedback gain K is found to be

$$K = \left[\begin{array}{cc} -0.1480 & -0.6359 \end{array}\right].$$

Figures 4.4 and 4.5 show the simulation results for the state disagreement of each agent. Clearly the conditions specified in Theorem 4.3 are sufficient for the control gain to achieve consensus control. Without re-tuning the control gain, the consensus control is still achieved for the multi-agent systems with a larger delay of 0.5 seconds and a bigger Lipschitz constant of $g = 0.15$, as shown in Figures 4.6 and 4.7.

4.5 Summary

This chapter has investigated the impacts of nonlinearity and input delay in consensus control. This input delay may represent some delays in the network communication. Sufficient conditions are derived for the multi-agent systems to guarantee the global consensus using Lyapunov-Krasovskii method in the time domain. The significance of this research is to provide feasible methods to deal with consensus control of a class of Lipschitz nonlinear multi-agent systems with input delay which includes some common circuits such as Chua circuits.

Chapter 5

Disturbance Rejection with H_∞ Consensus Control

In this chapter, we consider the H_∞ consensus control for general multi-agent systems with directed graph and input delay. To deal with input delay, a truncated prediction of the agent state over the delay period is approximated by the finite dimensional term of the classical state predictor. The truncated predictor feedback method is used for the consensus protocol design. By exploring certain features of the Laplacian matrix, the H_∞ consensus analysis is put in the framework of Lyapunov analysis. The integral terms that remain in the transformed systems are carefully analyzed by using Krasovskii functional. Sufficient conditions are derived for the multi-agent systems to guarantee the H_∞ consensus in the time domain. The feedback gain is then designed by solving these conditions with an iterative LMI procedure. A simulation study is carried out to validate the proposed control design. The contributions of this chapter are two folds. Firstly, upon exploring the certain features of the Laplacian matrix in the real Jordan form, the H_∞ consensus analysis is put in the framework of Lyapunov analysis for multi-agent systems connected by a general directed graph. Compared to the previous works, the requirement for the communication graph in this chapter is more general. The connection graph between the agents only needs a directed spanning tree, which is essential for consensus control, rather than the balanced or strongly connected conditions. Secondly, we consider the H_∞ consensus control of multi-agent systems in the presence of input delay. This input delay may represent some delays in the network communication. By using the TPF method, the troublesome integral term is ignored, and only the prediction based on the exponential of the systems matrix is used for control design. Furthermore, rigorous analysis is carried out to ensure that the extra integral terms under the transformations, including the ones for external disturbances and input delay, are properly considered using Krasovskii functionals. Sufficient conditions are derived for the multi-agent systems to guarantee the H_∞ consensus in the time domain. The conditions can be solved as LMIs (linear matrix inequalities) with a set of iterative scalar parameters.

The remainder of this chapter is organized as follows. Section 5.1 presents some notations and the problem formulation. Section 5.2 presents the main results on the H_∞ consensus control design. Simulation results are given in Section 5.3. Section 5.4 summarises this chapter.

5.1 Problem Formulation

Consider a group of N agents, each represented by a linear dynamic subject to input delay and external disturbance,

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t-h) + D_1\omega_i(t), \tag{5.1}$$

where for agent i, $i = 1,2,\ldots,N$, $x_i \in \mathbb{R}^n$ is the state vector, $u_i \in \mathbb{R}^{m\times n}$ is the control input vector, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $D_1 \in \mathbb{R}^{n\times m}$ are constant matrices with (A,B) being controllable, $h > 0$ is input delay, and $\omega_i \in \mathcal{L}_2^m\,[0,\infty)$ is the external disturbance.

The communications among the agents are described by a directed graph $\mathcal{G}\,(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ is a set of vertices and $\mathcal{E}$ is a set of edges. A vertex represents an agent, and each edge represents a connection. Associated with the graph is its adjacency matrix $\mathcal{A}$, where element a_{ij} denotes the connection between two agents. More specifically, if a connection exists from agent j to agent i, a_{ij} = 1; otherwise a_{ij} = 0. The Laplacian matrix $\mathcal{L}$ is defined by $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$ when $i \neq j$. For a directed graph, the Laplacian matrix $\mathcal{L}$ has the following properties.

Lemma 5.1.1 ([3,28]). *The Laplacian matrix $\mathcal{L}$ of a directed graph $\mathcal{G}$ has at least one zero eigenvalue with* $\mathbf{1}$ *as a corresponding right eigenvector and all nonzero eigenvalues have positive real parts. Furthermore, zero is a simple eigenvalue of $\mathcal{L}$ if and only if $\mathcal{G}$ has a directed spanning tree. In addition, there exists a nonnegative left eigenvector r of $\mathcal{L}$ associated with the zero eigenvalue, satisfying $r^T\mathcal{L} = 0$ and $r^T\mathbf{1} = 1$. Moreover, r is unique if $\mathcal{G}$ has a directed spanning tree.*

The objective of this chapter is to design a control algorithm for each agent such that the multi-agent systems (5.1) achieve consensus and meanwhile maintain a desirable disturbance rejection performance. In view of this, we introduce a state transformation

$$\xi_i = x_i - \sum_{j=1}^{N} r_j x_j, \tag{5.2}$$

where $i = 1, 2, \ldots, N$, $\xi_i \in \mathbb{R}^n$, r_j denotes the jth element of the nonnegative left eigenvector r of $\mathcal{L}$ associated with the zero eigenvalue, satisfying $r^T \mathcal{L} = 0$ and $r^T 1 = 1$. Based on the new variable ξ_i, we define the performance variable as $e_i(t) = C\xi_i(t)$, where $e_i \in \mathbb{R}^m$, $C \in \mathbb{R}^{m\times n}$ is a constant matrix. Let $e(t) = \left[e_1^T, e_2^T, \cdots, e_N^T\right]^T$, $\omega = \left[\omega_1^T, \omega_2^T, \cdots, \omega_N^T\right]^T$.

Remark 5.1.1. *The consensus value is related to the weight average of the initial conditions of the states of the nodes. It depends on the graph topology through the left eigenvector $r = [r_1, r_2, \cdots, r_N]^T$ of the zero eigenvalue of Laplacian matrix $\mathcal{L}$ [132]. To this end, ξ_i could be formulated as the consensus state error for each agent.*

The H_∞ consensus control problem can be defined as below.

Definition 5.1.1. *Given a positive scalar $\bar{\gamma}$, the H_∞ consensus is achieved if the two requirements listed below are satisfied:*

1. The multi-agent systems (5.1) *with $\omega_i \equiv 0$ can reach consensus. That is, under these control algorithms, the following hold for all initial conditions,*

$$\lim_{t\to\infty} \left(x_i(t) - x_j(t)\right) = 0, \ \ \forall i \neq j.$$

2. Under the zero-initial condition, the performance variable $e(t)$ satisfies

$$J_1 = \int_0^\infty \left[e^T(t)e(t) - \bar{\gamma}^2 \omega^T(t)\omega(t)\right] \mathrm{d}\tau < 0. \tag{5.3}$$

Assumption 5.1.1. *Zero is a simple eigenvalue of the Laplacian matrix $\mathcal{L}$ associated with the network connection in this chapter.*

Remark 5.1.2. *This assumption implies that the directed graph contains a spanning tree, which is essential for consensus control. If zero is not a simple eigenvalue of $\mathcal{L}$, the agents cannot reach consensus asymptotically as there exist at least two separate subgroups or at least two agents in the group who do not receive any information [3].*

Remark 5.1.3. *The left eigenvector r of the Laplacian matrix $\mathcal{L}$ with the zero eigenvalue is crucial for the consensus design with directed graph. Feasible methods are* *given in [3] to calculate this vector. In addition, the elements of r in this chapter could be zero. This suggests that the methods developed before may not be suitable for the H_∞ consensus analysis here.*

5.2 Main Results

For the multi-agent system (5.1), we have

$$x_i(t) = e^{Ah}x_i(t-h) + \int_{t-h}^{t} e^{A(t-\tau)}\left(Bu_i(\tau-h) + D_1\omega_i\right)\mathrm{d}\tau.$$

By employing the TPF approach, the control input can be constructed as following

$$\begin{aligned} u_i(t) &= Ke^{Ah}\sum_{j=1}^{N} a_{ij}\left(x_i(t) - x_j(t)\right) \\ &= Ke^{Ah}\sum_{j=1}^{N} l_{ij}x_j(t), \end{aligned} \tag{5.4}$$

where $K \in \mathbb{R}^{m\times n}$ is a constant control gain to be designed later, a_{ij} and l_{ij} being the elements of the graph adjacency matrix $\mathcal{A}$ and the Laplacian matrix $\mathcal{L}$, respectively. Under control algorithm (5.4), the multi-agent system (5.1) can be written as

$$\begin{aligned} \dot{x}_i(t) = {} & Ax_i(t) + BK\sum_{j=1}^{N} l_{ij}x_j(t) + D_1\omega_i \\ & - BK\sum_{j=1}^{N} l_{ij}\int_{t-h}^{t} e^{A(t-\tau)}\left(Bu_j(\tau-h) + D_1\omega_j\right)\mathrm{d}\tau. \end{aligned}$$

Let $x = \left[x_1^T, x_2^T, \cdots, x_N^T\right]^T$, $u = \left[u_1^T, u_2^T, \cdots, u_N^T\right]^T$. The closed-loop system is then described by

$$\dot{x}(t) = (I_N \otimes A + \mathcal{L}\otimes BK)x(t) + (\mathcal{L}\otimes BK)(d_3 + d_4) + (I_N \otimes D_1)\omega, \tag{5.5}$$

where

$$\begin{aligned} d_3 &= -\int_{t-h}^{t}\left(I_N \otimes e^{A(t-\tau)}B\right)u(\tau-h)\mathrm{d}\tau, \\ d_4 &= -\int_{t-h}^{t}\left(I_N \otimes e^{A(t-\tau)}D_1\right)\omega(t)\mathrm{d}\tau. \end{aligned}$$

From the state transformation (5.2), we have

$$\begin{aligned}\xi &= x - \left(\left(\mathbf{1}r^T\right) \otimes I_n\right) x \\ &= (M \otimes I_n)\, x,\end{aligned} \tag{5.6}$$

where $\xi = \left[\xi_1^T, \xi_2^T, \cdots, \xi_N^T\right]^T$, $M = I_N - \mathbf{1}r^T$. Since $r^T\mathbf{1} = 1$, it can be shown that $M\mathbf{1} = 0$. Therefore the consensus of system (5.1) is achieved when $\lim_{t\to\infty}\xi(t) = 0$, as $\xi = 0$ implies that $x_1 = x_2 = \cdots = x_N$, due to the fact that the null space of M is span$\{\mathbf{1}\}$. The dynamics of ξ can then be derived as

$$\dot{\xi} = (I_N \otimes A + \mathcal{L} \otimes BK)\,\xi + (\mathcal{L} \otimes BK)\,(d_3 + d_4) + (M \otimes D_1)\,\omega, \tag{5.7}$$

where we have used $r^T\mathcal{L} = 0$.

To explore the structure of $\mathcal{L}$, we propose another state transformation

$$\eta = \left(T^{-1} \otimes I_n\right) \xi, \tag{5.8}$$

with $\eta = \left[\eta_1^T, \eta_2^T, \cdots, \eta_N^T\right]^T$. Then we have

$$\dot{\eta} = (I_N \otimes A + J \otimes BK)\,\eta + \Delta_1(x) + \Delta_2(t) + \Omega(t), \tag{5.9}$$

where J is the Jordan form of the Laplacian matrix $\mathcal{L}$ defined in Lemma (2.1.6), and

$$\begin{aligned}\Delta_1 &= \left(T^{-1}\mathcal{L} \otimes BK\right) d_3, \\ \Delta_2 &= \left(T^{-1}\mathcal{L} \otimes BK\right) d_4, \\ \Omega &= \left(T^{-1}M \otimes D_1\right) \omega(t).\end{aligned}$$

From state transformations (5.6) and (5.8), we have:

$$\begin{aligned}\eta_1 &= \left(r^T \otimes I_n\right) \xi \\ &= \left((r^T M) \otimes I_n\right) x \\ &\equiv 0.\end{aligned}$$

With the control law shown in (5.4), the control gain matrix K is chosen as

$$K = -B^T P, \tag{5.10}$$

where P is a positive definite matrix to be designed. In the remainder of the paper, Lyapunov-function-based analysis will be carried out to identify a condition for P to ensure that the consensus problem is solved by using the control algorithm (5.10) with control gain K in (5.4).

The consensus analysis will be carried out in terms of η. Let

$$V_i = \eta_i^T P \eta_i, \tag{5.11}$$

for $i = 2, 3, \cdots, N$. Then, let

$$V_0 = \sum_{i=2}^{N} V_i.$$

For the convenience of presentation, we recall from [51] the following results on V_0.

Lemma 5.2.1. *For multi-agent systems (5.1) with the transformed state* η, $\dot{V}_0$ *has following bounds specified in one of the following two cases:*
1) If the eigenvalues of the Laplacian matrix $\mathcal{L}$ *are distinct,* $\dot{V}_0$ *satisfies*

$$\begin{aligned}\dot{V}_0 \le & \sum_{i=2}^{N} \eta_i^T \left(A^T P + PA - 2\alpha PBB^T P + (\kappa_1 + \kappa_2) PP\right) \eta_i \\ & + \frac{1}{\kappa_1} \|\Delta_1\|^2 + \frac{1}{\kappa_2} \|\Delta_2\|^2 + 2\eta^T (I_N \otimes P)\Omega,\end{aligned} \tag{5.12}$$

where Δ_1 *and* Δ_2 *are defined in (5.9),* κ_1 *and* κ_2 *are any positive real numbers, and* α *is the real part of the smallest non-zero eigenvalue of the Laplacian matrix* $\mathcal{L}$

$$\alpha = \min\{\lambda_1, \lambda_2, \ldots, \lambda_p, \alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_q\}.$$

2) If the Laplacian matrix $\mathcal{L}$ *has multiple eigenvalues,* $\dot{V}_0$ *satisfies*

$$\begin{aligned}\dot{V}_0 \le & \sum_{i=2}^{N} \eta_i^T \left(A^T P + PA - 2(\alpha - 1) PBB^T P + (\kappa_1 + \kappa_2) PP\right) \eta_i \\ & + \frac{1}{\kappa_1} \|\Delta_1\|^2 + \frac{1}{\kappa_2} \|\Delta_2\|^2 + 2\eta^T (I_N \otimes P)\Omega.\end{aligned} \tag{5.13}$$

The following lemmas give the bounds of $\|\Delta_1\|^2$ and $\|\Delta_2\|^2$.

Lemma 5.2.2. *For the term* $\Delta_1(x)$ *shown in the transformed system dynamics (5.9), a* *bound can be established as*

$$\|\Delta_1\|^2 \leq \rho_1 \int_{t-h}^{t} \eta^T(\tau-h)\,\eta(\tau-h)\,\mathrm{d}\tau, \tag{5.14}$$

where

$$\rho_1 = 4ha_1^4 e^{2\lambda h}\lambda_\sigma^2\left(T^{-1}\right)\|\mathcal{L}\|_F^2\,\|\mathcal{A}\|_F^2\,\|T\|_F^2,$$

with a_1 and λ being positive numbers such that

$$a_1^2 I \geq PBB^T BB^T P, \tag{5.15}$$

$$\lambda I > A^T + A. \tag{5.16}$$

Proof. By the definition of $\Delta_1(x)$ in (5.9), we have

$$\begin{aligned} \|\Delta_1\| &= \left\|\left(T^{-1}\otimes I_n\right)(\mathcal{L}\otimes BK)d_3\right\| \\ &\leq \lambda_\sigma\left(T^{-1}\right)\|\mu\|, \end{aligned} \tag{5.17}$$

where $\mu = (\mathcal{L}\otimes BK)d_3$.

Let $\mu = \left[\mu_1^T, \mu_2^T, \ldots, \mu_N^T\right]^T$. Then from (5.5) and (5.10), we have

$$\mu_i = BB^T P \sum_{j=1}^{N} l_{ij} \int_{t-h}^{t} e^{A(t-\tau)} BB^T P e^{Ah} \sum_{k=1}^{N} a_{jk}\left(x_k(\tau-h) - x_j(\tau-h)\right)\mathrm{d}\tau. \tag{5.18}$$

From $\eta = \left(T^{-1}\otimes I_n\right)\xi$, we obtain $\xi = (T\otimes I_n)\eta$. And from the state transformation (5.6), we have

$$\begin{aligned} x_k(t) - x_j(t) &= \xi_k(t) - \xi_j(t) \\ &= \left(\left(T_k - T_j\right)\otimes I_n\right)\eta(t) \\ &= \sum_{l=1}^{N}\left(T_{kl} - T_{jl}\right)\eta_l(t), \end{aligned} \tag{5.19}$$

where T_k denotes the kth row of T.

We define

$$\sigma_l = BB^T P \int_{t-h}^{t} \mathrm{e}^{A(t-\tau)} BB^T P e^{Ah}\eta_l(\tau-h)\mathrm{d}\tau. \tag{5.20}$$

Then, from (5.18) and (5.19), we can obtain that

$$\mu_i = \sum_{j=1}^{N} l_{ij} \sum_{k=1}^{N} a_{jk} \sum_{l=1}^{N} \left(T_{kl} - T_{jl}\right) \sigma_l .$$

For the notational convenience, let $\sigma = \left[\sigma_1^T, \sigma_2^T, \ldots, \sigma_N^T\right]^T$. It then follows that

$$\begin{aligned} \|\mu_i\| &\leq \sum_{j=1}^{N} \left|l_{ij}\right| \sum_{k=1}^{N} \left|a_{jk}\right| \|T_k\| \|\sigma\| + \sum_{k=1}^{N} \sum_{j=1}^{N} \left|l_{ij}\right| \left|a_{jk}\right| \left\|T_j\right\| \|\sigma\| \\ &\leq \sum_{j=1}^{N} \left|l_{ij}\right| \left\|a_j\right\| \|T\|_F \|\sigma\| + \sum_{k=1}^{N} \sum_{j=1}^{N} \left|l_{ij}\right| \|a_k\| \|T\|_F \|\sigma\| \\ &\leq 2 \|l_i\| \|\mathcal{A}\|_F \|T\|_F \|\sigma\| , \end{aligned} \tag{5.21}$$

where l_i denotes the ith row of $\mathcal{L}$. Therefore we have

$$\begin{aligned} \|\mu\|^2 = &\ \sum_{i=1}^{N} \|\mu_i\|^2 \\ \leq &\ 4 \|\mathcal{L}\|_F^2 \|\mathcal{A}\|_F^2 \|T\|_F^2 \|\sigma\|^2 , \end{aligned} \tag{5.22}$$

where we have used $\sum_{i=1}^{N} \|l_i\|^2 = (\|\mathcal{L}\|_F)^2$.

Next we need to deal with $\|\sigma\|^2$. By Lemma (2.1.5), we have

$$\begin{aligned} \|\sigma_i\|^2 \leq h \int_{t-h}^{t} \eta_i^T(\tau-h) e^{A^T h} PBB^T e^{A^T(t-\tau)} PBB^T BB^T P e^{A(t-\tau)} BB^T P e^{Ah} \eta_i(\tau-h) \mathrm{d}\tau \\ \leq h a_1^2 \int_{t-h}^{t} \eta_i^T(\tau-h) e^{A^T h} PBB^T e^{A^T(t-\tau)} e^{A(t-\tau)} BB^T P e^{Ah} \eta_i(\tau-h) \mathrm{d}\tau, \end{aligned}$$

where a_1 is a positive real number such that

$$a_1^2 I \geq PBB^T BB^T P.$$

In view of Lemma (2.1.6) with $P = I$, provided that

$$R = -A^T - A + \lambda I > 0,$$

we have

$$e^{A^T t} e^{At} < e^{\lambda t} I,$$

and

$$\begin{aligned}\|\sigma_i\|^2 &\leq ha_1^2 \int_{t-h}^{t} e^{\lambda(t-\tau)} \eta_i^T(\tau-h) e^{A^T h} PBB^T BB^T P e^{Ah} \eta_i(\tau-h) \mathrm{d}\tau \\ &\leq ha_1^4 e^{2\lambda h} \int_{t-h}^{t} \eta_i^T(\tau-h)\eta_i(\tau-h)\mathrm{d}\tau.\end{aligned}$$

Then, $\|\sigma\|^2$ can be bounded as

$$\begin{aligned}\|\sigma\|^2 &= \sum_{i=1}^{N} \|\sigma_i\|^2 \\ &\leq ha_1^4 e^{2\lambda h} \int_{t-h}^{t} \eta^T(\tau-h)\eta(\tau-h)\mathrm{d}\tau. \end{aligned} \tag{5.23}$$

Hence, together with (5.17), (5.22) and (5.23), we get

$$\|\Delta_1\|^2 \leq \rho_1 \int_{t-h}^{t} \eta^T(\tau-h)\,\eta(\tau-h)\,\mathrm{d}\tau.$$

This completes the proof. □

Lemma 5.2.3. *For the term $\Delta_2(t)$ in the transformed system dynamics (5.9), a bound can be established as*

$$\|\Delta_2\|^2 \leq \rho_2 \int_{t-h}^{t} \omega^T(\tau)\omega(\tau)\mathrm{d}\tau, \tag{5.24}$$

where

$$\rho_2 = ha_1^2 a_2 e^{\lambda h} \lambda_\sigma^2 \left(T^{-1}\right) \|\mathcal{L}\|_F^2,$$

with a_1 and λ being as defined in (5.15) and (5.16), and a_2 is a positive real number such that

$$a_2 I \geq D_1^T D_1. \tag{5.25}$$

Proof. In a way similar to Lemma 5.2.2, we have

$$\begin{aligned}\|\Delta_2\| &= \left\| \left(T^{-1} \otimes I_n\right)(\mathcal{L} \otimes BK)\, d_4 \right\| \\ &\leq \lambda_\sigma\left(T^{-1}\right) \left\|\bar{\zeta}\right\|, \end{aligned} \tag{5.26}$$

where $\bar{\zeta} = (\mathcal{L} \otimes BK)d_4$. Let $\bar{\zeta} = \left[\bar{\zeta}_1^T, \bar{\zeta}_2^T, \ldots, \bar{\zeta}_N^T\right]^T$. Then from (5.5) and (5.10), we have

$$\bar{\zeta}_i = \sum_{j=1}^{N} l_{ij} BB^T P \int_{t-h}^{t} e^{A(t-\tau)} D_1 \omega_j \mathrm{d}\tau.$$

It follows that

$$\|\bar{\zeta}_i\|^2 = \sum_{j=1}^{N} l_{ij}^2 \int_{t-h}^{t} \omega_j^T D_1^T e^{A^T(t-\tau)} \mathrm{d}\tau PBB^T BB^T P \int_{t-h}^{t} e^{A(t-\tau)} D_1 \omega_j \mathrm{d}\tau.$$

With Lemma (2.1.5) and the condition (5.15), we have

$$\|\bar{\zeta}_i\|^2 \leq h a_1^2 \sum_{j=1}^{N} l_{ij}^2 \int_{t-h}^{t} \omega_j^T D_1^T e^{A^T(t-\tau)} e^{A(t-\tau)} D_1 \omega_j \mathrm{d}\tau.$$

In view of Lemma (2.1.6), with the conditions (5.15) and (5.16), we have

$$\|\bar{\zeta}_i\|^2 \leq h a_1^2 a_2 e^{\lambda h} \int_{t-h}^{t} \sum_{j=1}^{N} l_{ij}^2 \|\omega_j\|^2 \mathrm{d}\tau.$$

Consequently,

$$\begin{aligned} \|\bar{\zeta}\|^2 &\leq h a_1^2 a_2 e^{\lambda h} \int_{t-h}^{t} \sum_{i=1}^{N} \sum_{j=1}^{N} l_{ij}^2 \|\omega_j\|^2 \mathrm{d}\tau \\ &\leq h a_1^2 a_2 e^{\lambda h} \|\mathcal{L}\|_F^2 \int_{t-h}^{t} \omega^T(\tau)\omega(\tau) \mathrm{d}\tau. \end{aligned} \tag{5.27}$$

Putting (5.26) and (5.27) together, we have

$$\|\Delta_2\|^2 \leq \rho_2 \int_{t-h}^{t} \omega^T(\tau)\omega(\tau) \mathrm{d}\tau.$$

This completes the proof. □

For the first integral term shown in (5.14), we consider the following Krasovskii functional

$$W_5 = e^h \int_{t-h}^{t} \eta^T(\tau)\eta(\tau) \mathrm{d}\tau + e^h \int_{t-h}^{t} e^{\tau - t} \eta^T(\tau - h)\eta(\tau - h) \mathrm{d}\tau.$$

A direct evaluation gives that

$$\begin{aligned}\dot{W}_5 =& -e^h \int_{t-h}^{t} e^{\tau-t}\eta^T(\tau-h)\eta(\tau-h)\mathrm{d}\tau - \eta^T(t-2h)\eta(t-2h) + e^h\eta^T(t)\eta(t)\\ \leq& -\int_{t-h}^{t} \eta^T(\tau-h)\eta(\tau-h)\mathrm{d}\tau + e^h\eta(t)^T\eta(t). \end{aligned} \tag{5.28}$$

For the second integral term shown in (5.24), we consider the following Krasovskii functional

$$W_6 = e^h \int_{t-h}^{t} e^{\tau-t}\omega^T(\tau)\omega(\tau)\mathrm{d}\tau.$$

A direct evaluation gives that

$$\begin{aligned}\dot{W}_6 =& -e^h \int_{t-h}^{t} e^{\tau-t}\omega^T(\tau)\omega(\tau)\mathrm{d}\tau + e^h\omega^T(\tau)\omega(\tau) - \omega^T(t-h)\omega(t-h)\\ \leq& -\int_{t-h}^{t} \omega^T(\tau)\omega(\tau)\mathrm{d}\tau + e^h\omega^T(\tau)\omega(\tau). \end{aligned} \tag{5.29}$$

Let

$$V = V_0 + \frac{\rho_1}{\kappa_1}W_5 + \frac{\rho_2}{\kappa_2}W_6. \tag{5.30}$$

From (5.12), (5.13), (5.28) and (5.29), we obtain that

$$\dot{V} \leq \eta^T(t)\left[I_N \otimes \left(H_5 + \frac{\rho_1}{\kappa_1}e^h I_n\right)\right]\eta(t) + \frac{\rho_2}{\kappa_2}e^h\omega^T(t)\omega(t) + 2\eta^T(I_N\otimes P)\Omega, \tag{5.31}$$

where

$$H_5 := A^TP + PA - 2\alpha PBB^TP + (\kappa_1+\kappa_2)PP,$$

for Case 1), and

$$H_5 := A^TP + PA - 2(\alpha-1)PBB^TP + (\kappa_1+\kappa_2)PP,$$

for Case 2).

The above expressions can be used for the H_∞ consensus analysis. The following theorem summarizes the results.

Theorem 5.2.1. *For an input-delayed multi-agent system (5.1) with the associated*

Laplacian matrix that satisfies Assumption 5.1.1, the H_∞ consensus control problem can be solved by the control algorithm (5.4) with the control gain $K = -B^T P$ specified in one of the following two cases:

1) If the eigenvalues of the Laplacian matrix $\mathcal{L}$ are distinct, the consensus is achieved if the following conditions are satisfied for $W = P^{-1}$ and $a_1 > 0$, $\lambda \geq 0$,

$$(A - \frac{1}{2}\lambda I_n)^T + (A - \frac{1}{2}\lambda I_n) < 0, \tag{5.32}$$

$$a_1 W \geq BB^T, \tag{5.33}$$

$$\begin{bmatrix} \Gamma_1 & W & D_1 \\ W & -\left(\frac{\rho_1 e^h}{\kappa_1} + a_3\right)^{-1} I_n & 0 \\ D_1^T & 0 & -\left(\bar{\gamma}^2 - \frac{\rho_2}{\kappa_2} e^h\right) a_4^{-1} \end{bmatrix} < 0, \tag{5.34}$$

where

$$\begin{aligned} &\kappa_1 > 0, \\ &\kappa_2 > \frac{\rho_2}{\bar{\gamma}^2} e^h, \\ &a_3 \geq \lambda_{\max}(T^T T \otimes C^T C), \\ &a_4 \geq \lambda_{\max}(T^{-1} M M^T \left(T^{-1}\right)^T), \\ &\Gamma_1 = WA^T + AW - 2\alpha BB^T + (\kappa_1 + \kappa_2) I_n. \end{aligned}$$

2) If the Laplacian matrix $\mathcal{L}$ has multiple eigenvalues, the consensus is achieved if the conditions (5.32), (5.33) and the following condition are satisfied for $W = P^{-1}$, $a_1 > 0$, $\lambda \geq 0$,

$$\begin{bmatrix} \Gamma_2 & W & D_1 \\ W & -\left(\frac{\rho_1 e^h}{\kappa_1} + a_3\right)^{-1} I_n & 0 \\ D_1^T & 0 & -\left(\gamma^2 - \frac{\rho_2}{\kappa_2} e^h\right) a_4^{-1} \end{bmatrix} < 0, \tag{5.35}$$

where $\Gamma_2 = WA^T + AW - 2(\alpha - 1)BB^T + (\kappa_1 + \kappa_2) I_n$.

Proof. From (5.8), we obtain that

$$\begin{aligned} e(t) &= (I_N \otimes C)\xi \\ &= (I_N \otimes C)(T \otimes I_n)\eta(t). \end{aligned}$$

It follows that

$$\begin{aligned} e^T(t)e(t) &= \eta^T(t)(T^T T \otimes C^T C)\eta(t) \\ &\le a_3 \eta^T(t)\eta(t). \end{aligned} \tag{5.36}$$

Under the zero-initial condition, $x(0) = 0$. It is clear that $V(0) = 0$. Next, for any non-zero ω, we have

$$\begin{aligned} J_1 &= \int_0^\infty \left[e^T(t)e(t) - \bar{\gamma}^2 \omega^T(t)\omega(t) + \dot{V} \right] \mathrm{d}\tau - V(\infty) + V(0) \\ &\le \int_0^\infty \eta^T \left[I_N \otimes \left(H_5 + \frac{\rho_1}{\kappa_1} e^h I_n + a_3 I_n \right) \right] \eta \mathrm{d}\tau \\ &\quad + \int_0^\infty \left(\frac{\rho_2}{\kappa_2} e^h - \bar{\gamma}^2 \right) \omega^T(t)\omega(t) + 2\eta^T (I_N \otimes P)\Omega \mathrm{d}\tau \\ &= \int_0^\infty \begin{bmatrix} \eta \\ \omega \end{bmatrix}^T \Theta \begin{bmatrix} \eta \\ \omega \end{bmatrix} \mathrm{d}\tau, \end{aligned} \tag{5.37}$$

where

$$\Theta = \begin{bmatrix} I_N \otimes \left(H_5 + \frac{\rho_1}{\kappa_1} e^h I_n + a_3 I_n \right) & T^{-1} M \otimes P D_1 \\ \left(T^{-1} M\right)^T \otimes D_1^T P & \left(\frac{\rho_2}{\kappa_2} e^h - \bar{\gamma}^2 \right) I \end{bmatrix}.$$

Thus, $J_1 < 0$ if $\Theta < 0$. By Schur complement lemma, we know that $\Theta < 0$ if the following inequality hold

$$H_5 + \frac{\rho_1}{\kappa_1} e^h I_n + a_3 I_n + a_4 \left(\bar{\gamma}^2 - \frac{\rho_2}{\kappa_2} e^h \right)^{-1} P D_1 D_1^T P < 0. \tag{5.38}$$

From $W = P^{-1}$, and condition (5.31), it is obtained that (5.38) is equivalent to

$$\begin{aligned} & W A^T + A W - 2\alpha B B^T + (\kappa_1 + \kappa_2) I_n \\ &\quad + \left(\frac{\rho_1}{\kappa_1} e^h + a_3 \right) W W + a_4 \left(\bar{\gamma}^2 - \frac{\rho_2}{\kappa_2} e^h \right)^{-1} D_1 D_1^T < 0, \end{aligned} \tag{5.39}$$

for Case 1), and

$$WA^T + AW - 2(\alpha - 1)BB^T + (\kappa_1 + \kappa_2)I_n \\ + \left(\frac{\rho_1}{\kappa_1}e^h + a_3\right)WW + a_4\left(\bar{\gamma}^2 - \frac{\rho_2}{\kappa_2}e^h\right)^{-1} D_1 D_1^T < 0, \tag{5.40}$$

for Case 2).

By Schur complement lemma, we know that the conditions (5.39) and (5.40) are equivalent to the conditions specified in (5.34) and (5.35). Considering conditions (5.32)–(5.35), we can obtain that $J_1 < 0$. Therefore the H_∞ consensus problem is solved. □

The consensus analysis for multi-agent systems with directed graph can clearly be applied to the systems with undirected graphs. Indeed, it can be treated as a special situation of the Case 1. A corollary is given for this special case..

Corollary 5.2.1. *For an input-delayed multi-agent system (5.1) with undirected graph, the H_∞ consensus control problem can be solved by the control algorithm (5.4) with the control gain $K = -B^T P$ where P is a positive definite matrix satisfying conditions (5.32)–(5.34).*

5.3 Numerical Examples

In this section, A simulation study is carried out to demonstrate the effectiveness of the proposed control design. Consider a connection of four agents as shown in Figure 1. The dynamics of each agent is described by (5.1), with

$$A = \begin{bmatrix} 0 & -0.1 \\ 0.1 & 0.1 \end{bmatrix}, B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, C = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}^T, D_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

The external disturbances $\omega = [2w, w, -2w, 1.5w]^T$, where $w(t)$ is a ten-period square wave starting at $t = 0$ with the width 5 and height 1. The input delay of the system is 0.1 seconds. The Laplacian matrix is given by

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

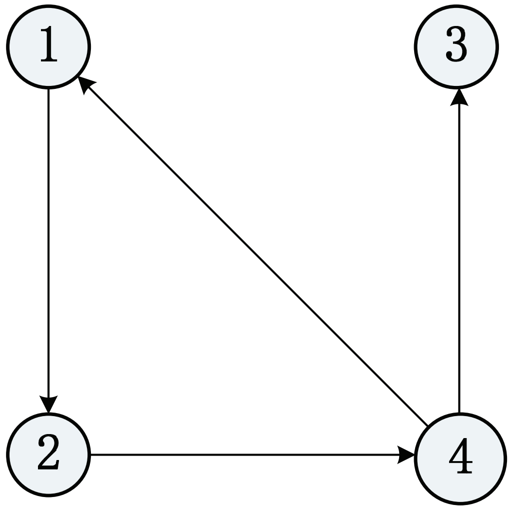

Figure 5.1: Communication topology.

The eigenvalues of $\mathcal{L}$ are $\left\{ 0,\ 1,\ 3/2 \pm j\sqrt{3}/2 \right\}$. Therefore, Assumption 5.1 is satisfied. We obtain that $\alpha = 1$ and $r^T = \left[\frac{1}{3},\ \frac{1}{3},\ 0,\ \frac{1}{3} \right]$. In this case, we choose the H_∞ performance index $\bar{\gamma} = 1$, and the initial states of agents are chosen randomly within $[-5,5]$, $u(\theta) = [0,\ 0,\ 0,\ 0]^T$, for $\theta \in [-h,0]$. With the values of $\lambda = 0.2, a_1 = 0.3$, and $\kappa_1 = \kappa_2 = 0.1$, a feasible solution of the feedback gain K is found to be

$$K = \begin{bmatrix} -1.5349 & -0.1504 \\ -0.1504 & -1.5541 \end{bmatrix}.$$

Figures 5.2 and 5.3 show the simulation results for the state of each agent under the case $\omega = 0$. Clearly the conditions specified in Theorem 5.4 are sufficient for the control gain to achieve consensus control. Figures 5.4 shows the trajectories of the performance variables $e_i(t), i = 1, \cdots, 4$ under the zero-initial condition. In addition, with the same control gain, the consensus control is still achieved for the multi-agent system with a much larger delay $h = 0.5$, as shown in Figures 6.5 and 6.6, which implies the conditions could be conservative in the control gain design for a given input delay.

5.4 Summary

In this chapter, we have addressed the H_∞ consensus problem for linear multi-agent systems with input delay and general directed graph. This input delay may represent

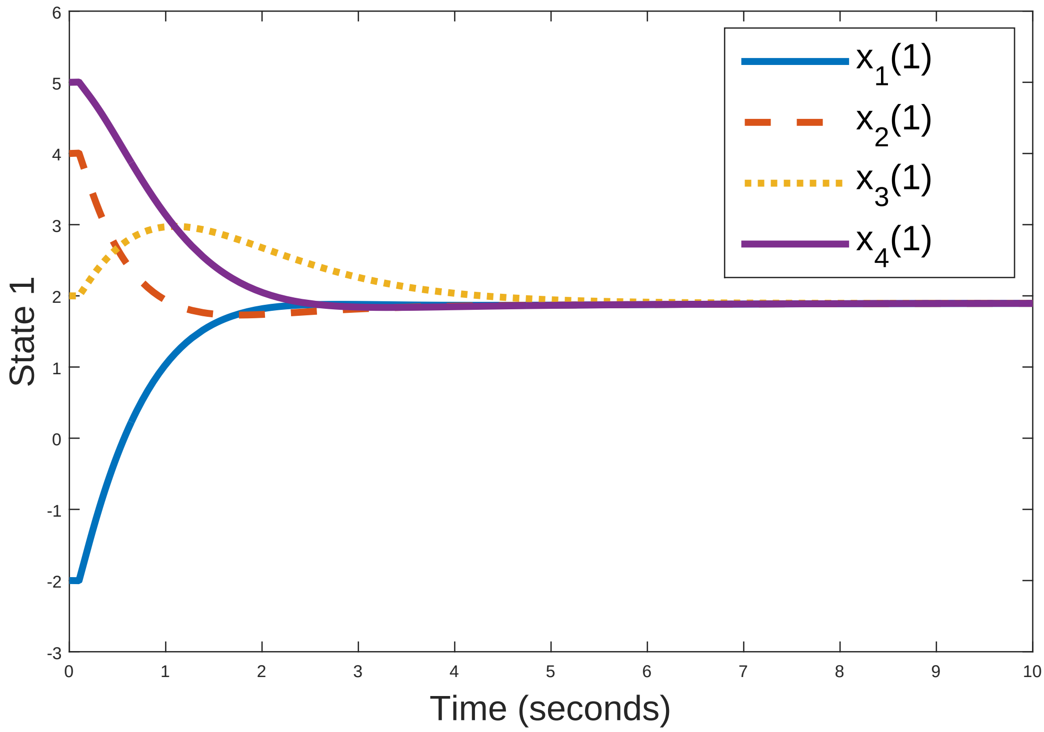

Figure 5.2: The state 1 of agents with $h = 0.1$.

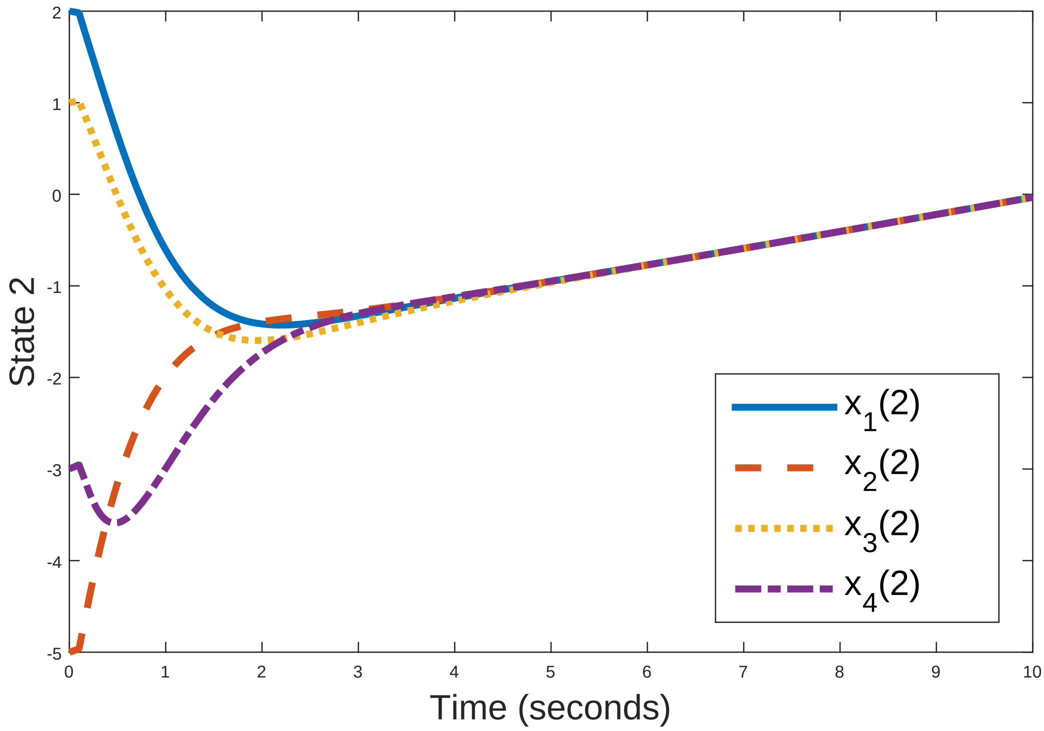

Figure 5.3: The state 2 of agents with $h = 0.1$.

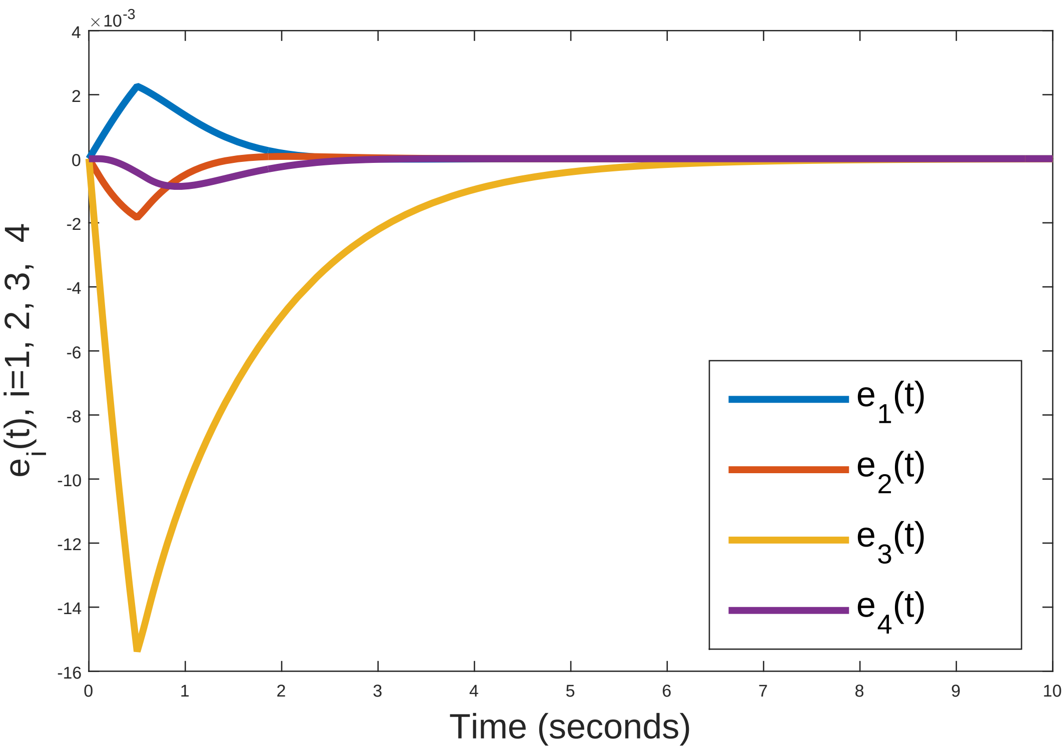

Figure 5.4: The trajectories of performance variables.

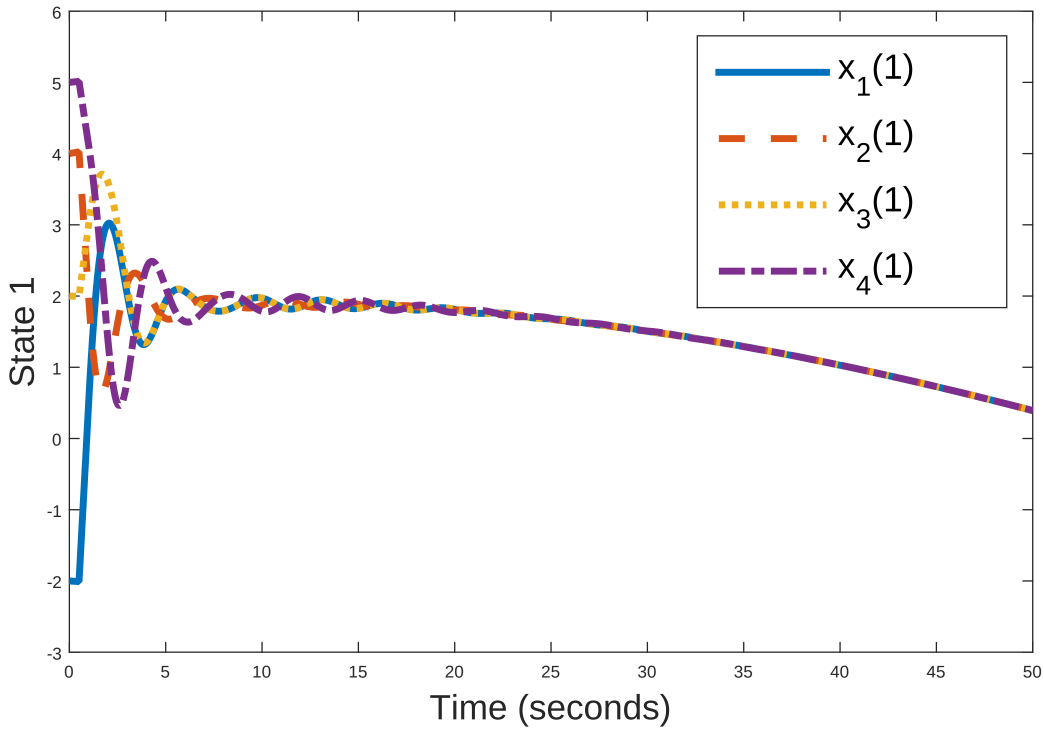

Figure 5.5: The state 1 of agents with $h = 0.5$.

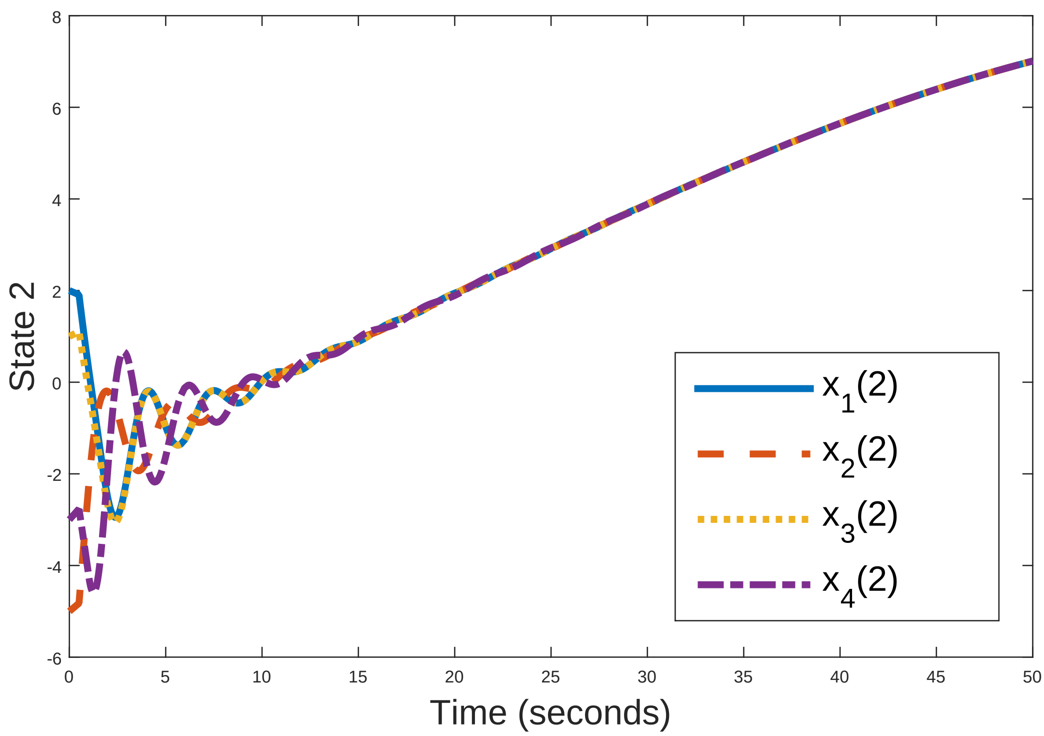

Figure 5.6: The state 2 of agents with $h = 0.5$.

some delays in the network communication. The truncated prediction feedback method is employed to deal with the input delay, and the integral terms that remain in the transformed systems are carefully analyzed by using Krasovskii functionals. By using the real Jordan form of the Laplacian matrix, sufficient conditions for the H_∞ consensus are identified through Lyapunov analysis. The conditions can be solved by employing LMIs with a set of iterative parameters. The requirement for the communication graph in this paper is much relaxed than the conditions specified in previous results, as the method presented in this section only requires the connection graph to have a spanning tree. Future work will focus on H_∞ consensus protocol design for multi-agent systems with time-varying input delay and Lipschitz nonlinearities.

Chapter 6

Disturbance Rejection with *DOBC* Approach

In this chapter, a new predictor-based consensus disturbance rejection method is proposed for high-order multi-agent systems with Lipschitz nonlinearity and input delay. First, a distributed disturbance observer for consensus control is developed for each agent to estimate the disturbance under the delay constraint. Based on the conventional predictor feedback approach, a non-ideal predictor based control scheme is constructed for each agent by utilizing the estimate of the disturbance and the prediction of the relative state information. Then, rigorous analysis is carried out to ensure that the extra terms associated with disturbances and nonlinear functions are properly considered. Sufficient conditions for the consensus of the multi-agent systems with disturbance rejection are derived based on the analysis in the framework of Lyapunov-Krasovskii functionals. A simulation example is included to demonstrate the performance of the proposed control scheme.

Compared with previous works, there are three main contributions of this chapter:

1) The consensus disturbance rejection problem is considered for general multi-agent systems with communication topology containing a directed spanning tree, which include the single, double, and high-order integrator-type systems [125–128] and undirected communication graphs [118, 119] as its special cases.

2) Input delay in considered for the consensus disturbance rejection. By revisiting the well-known predictor feedback approach, a non-ideal predictor based control scheme is constructed for each agent by adopting the estimate of the disturbance observer and the prediction of the relative state information.

3) Unlike [129], [42], where the agents are restricted to be linear, in this chapter, we

consider the Lipschitz nonlinearity in the system dynamics. Further rigorous analysis is carried out to guarantee that the extra integral terms of the system state associated with nonlinear functions are properly considered by using the tools of Krasovskii functionals.

The remainder of this chapter is organized as follows. In Section 6.1, the problem formulation is introduced. Section 6.2 presents the main results on the consensus disturbance rejection design. Simulation results are given in Section 6.3. Section 6.4 concludes the paper.

6.1 Problem Formulation

In this chapter, we consider leader-follower consensus control with a group of N agents. Assume that the dynamics of followers, labelled as $2,3,\ldots,N$, are described by

$$\dot{x}_i(t) = Ax_i(t) + \phi(x_i(t)) + Bu_i(t-h) + D_2\omega_i(t), \tag{6.1}$$

and the leader agent is indexed by 1, whose dynamics are represented by

$$\dot{x}_1(t) = Ax_1(t) + \phi(x_1(t)), \tag{6.2}$$

where $x_i \in \mathbb{R}^n$ denotes the state, $u_i \in \mathbb{R}^m$ denotes the control input, $x_1 \in \mathbb{R}^n$ is the leader's state, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $D_2 \in \mathbb{R}^{n\times s}$ are constant matrices with (A,B) being controllable, $h > 0$ is the constant and known input delay, $\omega_i \in \mathbb{R}^s$ is a disturbance that is generated by a linear exogenous system

$$\dot{\omega}_i(t) = S\omega_i(t), \tag{6.3}$$

with $S \in \mathbb{R}^{s\times s}$ being a known constant matrix, and the nonlinear function $\phi : \mathbb{R}^n \to \mathbb{R}^n$, $\phi(0) = 0$, is assumed to satisfy the Lipschitz condition as

$$\|\phi(\alpha) - \phi(\beta)\| \leq \gamma\|\alpha - \beta\|, \forall \alpha, \beta \in \mathbb{R}^n$$

where $\gamma > 0$ is the Lipschitz constant.

Let $\xi_i = x_i - x_1, i = 2,3,\cdots,N$ as the tracking error. Then, based on the system dynamics (6.1) and (6.2), the error dynamics of the ith agent can be obtained as

$$\dot{\xi}_i(t) = A\xi_i(t) + \psi_i + Bu_i(t-h) + D_2\omega_i(t), \tag{6.4}$$

where $\psi_i = \phi(x_i(t)) - \phi(x_1(t))$.

With the agent 1 as the leader, the control objective of this chapter is to design a control algorithm for each agent to follow the state of the leader x_1 under the disturbances. That is, under these control algorithms, the following hold for all initial conditions,

$$\lim_{t\to\infty} [x_i(t) - x_1(t)] = \lim_{t\to\infty} \xi_i(t) = 0, \ \ \forall i = 2,3,\cdots,N.$$

We make two assumptions about the dynamics of the agents and the connections between the agents.

Assumption 6.1.1. *The disturbance is matched. i.e., there exist a matrix $F \in \mathbb{R}^{m\times s}$ such that $D_2 = BF$.*

Assumption 6.1.2. *The communication topology $\mathcal{G}$ contains a directed spanning tree with the leader as the root.*

Remark 6.1.1. *The matching condition in Assumption (6.1.1) guarantees that the disturbance act via the same channel as that of the control input. This assumption could be relaxed in some circumstances because unmatched disturbances under uncertain conditions may be converted to the matched ones based on output regulation theory [42]. Furthermore, the disturbance condition given in* (6.3) *is commonly used for disturbance rejection and output regulation. Many kinds of disturbances in engineering can be described by this model. For instance, unknown constant disturbances or harmonics with unknown amplitudes and phases, belong to the allowed class of disturbances.*

Remark 6.1.2. *For the directed communication graphs in the previous sections, the final consensus value, which depends on the initial values, the network connection, and the agent dynamics, might be unknown a priori [28]. In some cases, it might be desirable for the agents states to converge onto a reference trajectory, which is known as consensus tracking (leader-follower consensus) problem. Compared to leaderless consensus, consensus tracking has the advantage to determine the final consensus value in advance.*

Because the leader has no neighbours, the Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ has the following structure

$$\mathcal{L} = \begin{bmatrix} 0 & 0_{1\times(N-1)} \\ \mathcal{L}_2 & \mathcal{L}_1 \end{bmatrix},$$

where $\mathcal{L}_1 \in \mathbb{R}^{(N-1)\times(N-1)}$ and $\mathcal{L}_2 \in \mathbb{R}^{(N-1)}$. From Definition 2.1.3, it is obvious that $\mathcal{L}_1$ is a nonsingular M-matrix. We also have the following result for $\mathcal{L}_1$:

Lemma 6.1.1 ([42]). *For the nonsingular M-matrix $\mathcal{L}_1$, there exists a positive diagonal matrix G such that*

$$G\mathcal{L}_1 + \mathcal{L}_1^T G \geq r_0 I, \tag{6.5}$$

for some positive constant r_0. It is also shown that G can be constructed by letting $G = \mathrm{diag}\{q_2, q_3, \cdots, q_N\} = (\mathrm{diag}(\pi))^{-1}$, where $\pi = [\pi_2, \pi_3, \cdots, \pi_N]^T = \left(\mathcal{L}_1^T\right)^{-1}[1, 1, \cdots, 1]^T$.

6.2 Consensus Controller and Disturbance Observer Design

The disturbance rejection design consists of disturbance estimation and rejection. The estimation is based on the relative state information obtained through the communication network. It is assumed that the ith agent collects the relative state information of its neighbouring agents as

$$\zeta_i(t) = \sum_{j=1}^{N} a_{ij}\left(x_i(t) - x_j(t)\right), \forall i = 2, 3, \cdots, N.$$

From the relationship between $\mathcal{A}$ and $\mathcal{L}$, it is easy to see that $\zeta_i(t) = \sum_{j=2}^{N} l_{ij}\xi_j(t)$. The disturbance estimation and rejection proposed in this chapter will be designed based on relative state information $\zeta_i(t)$.

The control input for disturbance rejection is designed as follows:

$$u_i(t) = cK\chi_i(t) - Fe^{Sh}\hat{\omega}_i(t), \tag{6.6}$$

where $\chi_i(t)$ and $\hat{\omega}_i(t)$ are generated by

$$\chi_i(t) = e^{Ah}\zeta_i(t) + \sum_{j=2}^{N} l_{ij} \int_{t-h}^{t} e^{A(t-\tau)} cBK\chi_j(t)\mathrm{d}\tau, \tag{6.7}$$

$$\hat{\omega}_i(t) = \eta_i(t) + L\zeta_i(t), \tag{6.8}$$

$$\begin{aligned}\dot{\eta}_i(t) =& S\eta_i(t) + (SL - LA)\zeta_i(t) - LBF\sum_{j=2}^{N} l_{ij}\left(\eta_j(t) - e^{Sh}\eta_j(t-h)\right) \\ & - LBF\sum_{j=2}^{N} l_{ij}\left(L\zeta_j(t) - e^{Sh}L\zeta_j(t-h)\right) - cLBK\sum_{j=2}^{N} l_{ij}\chi_j(t-h),\end{aligned} \tag{6.9}$$

where $c \geq 2q_{\max}/r_0$ is a positive real constant with $q_{\max} = \max\{q_2, q_3, \cdots, q_N\}$, K and L are constant gain matrices to be designed later.

Remark 6.2.1. *The integral term of $\chi_i(t)$ is added in the controller design to offset the adverse effect of the time delay. If the nonlinear and disturbance terms in* (6.1) *are absent, $\chi_i(t)$ is an ideal predictor of the relative state information of the ith agent. Due to the presence of disturbance, it is a non-ideal prediction of the relative state information. Furthermore,* (6.8)-(6.9) *are referred to as a distributed predictor-based consensus disturbance observer, which is only dependent on the relative state information, and independent of the information of the local state.*

Let $\tilde{z}_i(t) = \omega_i(t) - \hat{\omega}_i(t)$. A direct evaluation gives that

$$\begin{aligned}\dot{\tilde{z}}_i(t) &= S\omega_i(t) - \dot{\eta}_i(t) - L\sum_{j=2}^{N} l_{ij}\dot{\xi}_j(t) \\ &= S\tilde{z}_i(t) - L\sum_{j=2}^{N} l_{ij}\psi_j - LBF\sum_{j=2}^{N} l_{ij}\tilde{z}_j(t),\end{aligned} \tag{6.10}$$

which can be written in the compact form as

$$\dot{\tilde{z}}(t) = (I_{N-1}\otimes S)\tilde{z}(t) - (\mathcal{L}_1 \otimes LBF)\tilde{z}(t) - (\mathcal{L}_1 \otimes L)\Psi, \tag{6.11}$$

where $\Psi = \left[\psi_2^T, \psi_3^T, \cdots, \psi_N^T\right]^T$.

With the control input (6.6), the closed-loop dynamics of each agent in (6.4) can be written as

$$\begin{aligned}\dot{\xi}_i(t) =& A\xi_i(t) + \psi_i + BFe^{Sh}z_i(t-h) + cBKe^{Ah}\sum_{j=2}^{N} l_{ij}\xi_j(t-h) \\ &+ cBK\sum_{j=2}^{N} l_{ij}\int_{t-h}^{t} e^{A(t-\tau)}cBK\chi_j(\tau-h)\mathrm{d}\tau,\end{aligned} \tag{6.12}$$

where we have used $\omega_i(t) = e^{Sh}\omega_i(t-h)$ and $D_2 = BF$.

From the error dynamics (6.4), we have

$$\xi_i(t) = e^{Ah}\xi_i(t-h) + \int_{t-h}^{t} e^{A(t-\tau)}\left(\psi_i + Bu_i(\tau-h) + D_2\omega_i(\tau)\right)\mathrm{d}\tau. \tag{6.13}$$

Invoking (6.13) into (6.12), we obtain

$$\begin{aligned}\dot{\xi}_i(t) =& A\xi_i(t) + cBK\sum_{j=2}^{N} l_{ij}\xi_j(t) + \psi_i + BFe^{Sh}\tilde{z}_i(t-h) \\ &- cBK\sum_{j=2}^{N} l_{ij}\int_{t-h}^{t} e^{A(t-\tau)}\Big(\psi_j + BFe^{Sh}\tilde{z}_j(\tau-h)\Big)\mathrm{d}\tau.\end{aligned} \tag{6.14}$$

Let $\xi = \left[\xi_2^T, \xi_3^T, \cdots, \xi_N^T\right]^T$, $\tilde{z} = \left[\tilde{z}_2^T, \tilde{z}_3^T, \cdots, \tilde{z}_N^T\right]^T$. The error dynamics of $\xi(t)$ can be written in the compact form as

$$\dot{\xi}(t) = \left(I\otimes A + c\mathcal{L}_1\otimes BK\right)\xi(t) + \Psi + \Big(I\otimes BFe^{Sh}\Big)\tilde{z}(t-h) + \overline{\Delta}_1 + \overline{\Delta}_2, \tag{6.15}$$

where

$$\begin{aligned}\overline{\Delta}_1 &= -\left(c\mathcal{L}_1\otimes BK\right)\int_{t-h}^{t}\Big(I\otimes e^{A(t-\tau)}\Big)\Psi\mathrm{d}\tau, \\ \overline{\Delta}_2 &= -\left(c\mathcal{L}_1\otimes BK\right)\int_{t-h}^{t}\Big(I\otimes e^{A(t-\tau)}BFe^{Sh}\Big)\tilde{z}(\tau-h)\mathrm{d}\tau.\end{aligned}$$

For the convenience, let $\overline{\Delta}_1 = \left[\delta_2^T, \delta_3^T, \cdots, \delta_N^T\right]^T$ and $\overline{\Delta}_2 = \left[\bar{\delta}_2^T, \bar{\delta}_3^T, \cdots, \bar{\delta}_N^T\right]^T$.

Next, we will design the control gain K and the observer gain L. With the control law shown in (6.6), K and L are chosen as

$$K = -B^T P, \tag{6.16}$$

$$L = cQ^{-1}D_2^T, \tag{6.17}$$

where $P>0, Q>0$ are constant matrices to be designed.

In order to obtain the main results, the bounds on $\|\overline{\Delta}_1\|^2$ and $\|\overline{\Delta}_2\|^2$ are given in the following lemma.

Lemma 6.2.1. *For the terms $\overline{\Delta}_1$ and $\overline{\Delta}_2$ in the error dynamics (6.15), bounds can be established as*

$$\|\overline{\Delta}_1\|^2 \leq \rho_1 \int_{t-h}^{t} \xi^T(\tau)\xi(\tau)\mathrm{d}\tau, \tag{6.18}$$

$$\|\overline{\Delta}_2\|^2 \leq \rho_2 \int_{t-h}^{t} \tilde{z}^T(\tau-h)\tilde{z}(\tau-h)\mathrm{d}\tau, \tag{6.19}$$

where

$$\begin{aligned}\rho_1 &= (N-1)\,c^2 h\rho^2 e^{\alpha_2 h}\gamma^2\,\|\mathcal{L}_1\|_F^2\,,\\ \rho_2 &= (N-1)h\alpha_1 c^2\rho^2 e^{(\alpha_0+\alpha_2)h}\|\mathcal{L}_1\|_F^2,\end{aligned}$$

with $\rho, \alpha_0, \alpha_1, \alpha_2$ being positive numbers such that

$$\rho^2 I \geq PBB^TBB^TP, \tag{6.20}$$

$$\alpha_0 > \lambda_{\max}(S+S^T), \tag{6.21}$$

$$\alpha_1 \geq \lambda_{\max}(F^TB^TBF), \tag{6.22}$$

$$\alpha_2 > \lambda_{\max}(A+A^T). \tag{6.23}$$

Proof. From the definition of $\overline{\Delta}_1$ in (6.15), we have $\left\|\overline{\Delta}_1\right\|^2 = \sum_{i=2}^{N}\|\delta_i\|^2$. With (6.16), we can get

$$\delta_i = cBB^TP\sum_{j=2}^{N} l_{ij}\int_{t-h}^{t} e^{A(t-\tau)}\psi_j \mathrm{d}\tau,$$

and

$$\|\delta_i\|^2 = c^2\int_{t-h}^{t}\left(\sum_{j=2}^{N} l_{ij}\psi_j^T\right)e^{A^T(t-\tau)}\mathrm{d}\tau PBB^TBB^TP\int_{t-h}^{t} e^{A(t-\tau)}\left(\sum_{j=2}^{N} l_{ij}\psi_j\right)\mathrm{d}\tau.$$

Based on Lemma (2.1.5) and the condition (6.20), one obtains

$$\|\delta_i\|^2 \leq c^2 h\rho^2\int_{t-h}^{t}\sum_{j=2}^{N} l_{ij}\psi_j^T e^{\left(A^T+A\right)(t-\tau)}\sum_{j=2}^{N} l_{ij}\psi_j \mathrm{d}\tau.$$

In light of Lemma (2.1.6) and the condition (6.23), one gets that

$$
\begin{aligned}
\|\delta_i\|^2 \leq\ & (N-1)\, c^2 h \rho^2 e^{\alpha_2 h} \sum_{j=2}^{N} l_{ij}^2 \int_{t-h}^{t} \left\|\phi(x_j) - \phi(x_1)\right\|^2 \mathrm{d}\tau \\
\leq\ & (N-1)\, c^2 h \rho^2 e^{\alpha_2 h} \gamma^2 \sum_{j=2}^{N} l_{ij}^2 \int_{t-h}^{t} \xi_j^T(\tau)\xi_j(\tau)\mathrm{d}\tau \\
\leq & (N-1)\, c^2 h \rho^2 e^{\alpha_2 h} \gamma^2 \|l_i\|^2 \int_{t-h}^{t} \xi^T(\tau)\xi(\tau)\mathrm{d}\tau.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\left\|\overline{\Delta}_1\right\|^2 \leq & (N-1)\, c^2 h \rho^2 e^{\alpha_2 h} \gamma^2 \sum_{i=2}^{N} \|l_i\|^2 \int_{t-h}^{t} \xi^T(\tau)\xi(\tau)\mathrm{d}\tau \\
\leq & (N-1)\, c^2 h \rho^2 e^{\alpha_2 h} \gamma^2 \|\mathcal{L}_1\|_F^2 \int_{t-h}^{t} \xi^T(\tau)\xi(\tau)\mathrm{d}\tau.
\end{aligned}
$$

In a similar way, we have

$$
\bar{\delta}_i = cBB^T P \sum_{j=2}^{N} l_{ij} \int_{t-h}^{t} e^{A(t-\tau)} BF e^{Sh} \tilde{z}_j(\tau - h)\mathrm{d}\tau.
$$

It follows that

$$
\begin{aligned}
\left\|\bar{\delta}_i\right\|^2 \leq\ & c^2 \rho^2 h \int_{t-h}^{t} \left(\sum_{j=2}^{N} l_{ij} \tilde{z}_j^T(\tau - h) \right) e^{S^T h} F^T B^T e^{A^T(t-\tau)} \\
& \times e^{A(t-\tau)} BF e^{Sh} \left(\sum_{j=2}^{N} l_{ij} \tilde{z}_j(\tau - h) \right) \mathrm{d}\tau \\
\leq\ & h\alpha_1 c^2 \rho^2 e^{(\alpha_0+\alpha_2)h} \int_{t-h}^{t} \sum_{j=2}^{N} l_{ij} \tilde{z}_j^T(\tau - h) \sum_{j=2}^{N} l_{ij} \tilde{z}_j(\tau - h)\mathrm{d}\tau \\
\leq\ & (N-1) h\alpha_1 c^2 \rho^2 e^{(\alpha_0+\alpha_2)h} \sum_{j=2}^{N} l_{ij}^2 \int_{t-h}^{t} \tilde{z}_j^T(\tau - h)\tilde{z}_j(\tau - h)\mathrm{d}\tau.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|\overline{\Delta}_2\|^2 \leq\ & (N-1) h\alpha_1 c^2 \rho^2 e^{(\alpha_0+\alpha_2)h} \sum_{i=2}^{N} \|l_i\|^2 \int_{t-h}^{t} \tilde{z}^T(\tau - h)\tilde{z}(\tau - h)\mathrm{d}\tau \\
\leq\ & (N-1) h\alpha_1 c^2 \rho^2 e^{(\alpha_0+\alpha_2)h} \|\mathcal{L}_1\|_F^2 \int_{t-h}^{t} \tilde{z}^T(\tau - h)\tilde{z}(\tau - h)\mathrm{d}\tau.
\end{aligned}
$$

This completes the proof. □

6.3 Consensus analysis

The following theorem presents sufficient conditions to ensure that the consensus disturbance rejection problem is solved by using the control algorithm (6.6) with the control gain K and the observer gain L in (6.16)-(6.17).

Theorem 6.3.1. *For multi-agent systems (6.1)–(6.2) with Assumptions 6.1 and 6.2, the consensus disturbance rejection problem can be solved by the control algorithm (6.6) with (6.16)–(6.17) if there exist positive definite matrices P, Q and constants $\omega_1 \geq 0$, $\rho, \kappa_i > 0, i = 1, 2, \cdots, 5$, such that*

$$\rho W - BB^T \geq 0, \tag{6.24}$$

$$\begin{bmatrix} AW + WA^T - 2BB^T + (\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4) I & W \\ W & -\varepsilon_1^{-1} \end{bmatrix} < 0, \tag{6.25}$$

$$QS + S^T Q - 2D_2^T D_2 + \varepsilon_2 I < 0, \tag{6.26}$$

are satisfied with $W = P^{-1}$ and

$$\varepsilon_1 = \left(\kappa_1^{-1} + c\kappa_5^{-1}\sigma_{\max}^2(\mathcal{L}_1)\right)\gamma^2 + \rho_1 \pi_{\min}^{-1} \pi_{\max} \kappa_3^{-1} e^h,$$
$$\varepsilon_2 = \pi_{\max}\pi_{\min}^{-1}\left(\alpha_1 \kappa_2^{-1} e^{(\alpha_0+1)h} + c\kappa_5 \lambda_{\max}(D_2^T D_2) + e^h \kappa_4^{-1} \rho_2\right),$$

where $\pi_{\min} = \min\{\pi_2, \pi_3, \cdots, \pi_N\}$, $\pi_{\max} = \max\{\pi_2, \pi_3, \cdots, \pi_N\}$.

Proof. To start the consensus analysis, we try a Lyapunov function candidate

$$V_0 = \xi^T (G \otimes P) \xi + \tilde{z}^T (G \otimes Q) \tilde{z} + \sigma_0 e^h \int_{t-h}^{t} e^{\tau - t} \tilde{z}^T(\tau) \tilde{z}(\tau) \mathrm{d}\tau, \tag{6.27}$$

where σ_0 is a positive value to be chosen later.

The derivative of V_0 along the trajectory of (6.11) and (6.15) can be obtained as

$$
\begin{aligned}
\dot{V}_0 = {} & \xi^T \left(G \otimes \left(PA + A^T P\right) - c\left(G\mathcal{L}_1 + \mathcal{L}_1^T G\right) \otimes PBB^T P\right)\xi \\
& + 2\sum_{i=2}^{N} \frac{1}{\pi_i}\xi_i^T P\left(BFe^{Sh}\tilde{z}_i(t-h) + \psi_i + \delta_i + \overline{\delta}_i\right) \\
& + \tilde{z}^T(t)\left(G \otimes \left(QS + S^T Q\right) - c(G\mathcal{L}_1 + \mathcal{L}_1^T G) \otimes D_2^T D_2\right)\tilde{z}(t) \\
& - 2c\tilde{z}^T(t)(G\mathcal{L}_1 \otimes D_2^T)\Psi - \sigma_0 e^h \int_{t-h}^{t} e^{\tau - t}\tilde{z}^T(\tau)\tilde{z}(\tau)\mathrm{d}\tau \\
& + \sigma_0 e^h \tilde{z}^T(t)\tilde{z}(t) - \sigma_0 \tilde{z}^T(t-h)\tilde{z}(t-h) \\
\leq {} & \xi^T \left(G \otimes \left(PA + A^T P + (\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)PP\right) - cr_0 I \otimes PBB^T P\right)\xi \\
& + \frac{\gamma^2}{\kappa_1}\sum_{i=2}^{N}\frac{1}{\pi_i}\xi_i^T\xi_i + \frac{\alpha_1}{\kappa_2}e^{\alpha_0 h}\sum_{i=2}^{N}\frac{1}{\pi_i}\tilde{z}_i^T(t-h)\tilde{z}_i(t-h) + \frac{\left\|\overline{\Delta}_1\right\|^2}{\kappa_3\pi_{\min}} + \frac{\left\|\overline{\Delta}_2\right\|^2}{\kappa_4\pi_{\min}} \\
& + \tilde{z}^T(t)\left(G \otimes \left(QS + S^T Q + \frac{\pi_{\max}}{\pi_{\min}}c\kappa_5\lambda_{\max}(D_2^T D_2)I\right) - cr_0 I \otimes D_2^T D_2\right)\tilde{z}(t) \\
& - \sigma_0\tilde{z}^T(t-h)\tilde{z}(t-h) + \sigma_0 e^h \tilde{z}^T(t)\tilde{z}(t) + \frac{c\gamma^2}{\kappa_5}\sigma_{\max}^2(\mathcal{L}_1)\sum_{i=2}^{N}\frac{1}{\pi_i}\xi_i^T\xi_i \\
\leq {} & \xi^T \left(G \otimes \left(PA + A^T P - 2PBB^T P + (\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)PP + \sigma_1 I\right)\right)\xi \\
& + \tilde{z}^T(t)\left(G \otimes \left(QS + S^T Q - 2D_2^T D_2 + \sigma_{11} I\right)\right)\tilde{z}(t) + \frac{\left\|\overline{\Delta}_1\right\|^2}{\kappa_3\pi_{\min}} + \frac{\left\|\overline{\Delta}_2\right\|^2}{\kappa_4\pi_{\min}} \\
& + \left(\alpha_1\kappa_2^{-1}\pi_{\min}^{-1}e^{\alpha_0 h} - \sigma_0\right)\tilde{z}^T(t-h)\tilde{z}(t-h),
\end{aligned}
\tag{6.28}
$$

where

$$
\begin{aligned}
\sigma_1 &= \left(\kappa_1^{-1} + c\kappa_5^{-1}\sigma_{\max}^2(\mathcal{L}_1)\right)\gamma^2, \\
\sigma_{11} &= \sigma_0\pi_{\max}e^h + c\kappa_5\pi_{\max}\pi_{\min}^{-1}\lambda_{\max}(D_2^T D_2),
\end{aligned}
$$

and Lemmas 6.1.1, 2.1.5 and 2.1.6 are used in above derivation.

By choosing $\sigma_0 = \alpha_1\kappa_2^{-1}\pi_{\min}^{-1}e^{\alpha_0 h}$, the derivative of V_0 could be written as

$$
\begin{aligned}
\dot{V}_0 \leq {} & \xi^T \left(G \otimes \left(PA + A^T P - 2PBB^T P + (\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)PP + \sigma_1 I\right)\right)\xi \\
& + \tilde{z}^T\left(G \otimes \left(QS + S^T Q - 2D_2^T D_2 + \sigma_2 I\right)\right)\tilde{z} + \frac{\rho_1}{\kappa_3\pi_{\min}}\int_{t-h}^{t}\xi^T(\tau)\xi(\tau)\mathrm{d}\tau \\
& + \frac{\rho_2}{\kappa_4\pi_{\min}}\int_{t-h}^{t}\tilde{z}^T(\tau - h)\tilde{z}(\tau - h)\mathrm{d}\tau,
\end{aligned}
\tag{6.29}
$$

where $\sigma_2 = \pi_{\max}\pi_{\min}^{-1}\left(\alpha_1\kappa_2^{-1}e^{(\alpha_0+1)h} + c\kappa_5\lambda_{\max}(D_2^T D_2)\right)$.

To deal with the first integral term shown in (6.29), we consider the following Krasovskii functional

$$V_1 = e^h \int_{t-h}^{t} e^{\tau-t}\xi^T(\tau)\xi(\tau)\mathrm{d}\tau,$$

With the direct calculations as

$$\begin{aligned}\dot{V}_1 =& -e^h \int_{t-h}^{t} e^{\tau-t}\xi^T(\tau)\xi(\tau)\mathrm{d}\tau + e^h\xi^T(t)\xi(t) - \xi^T(t-h)\xi(t-h)\\ \leq& -\int_{t-h}^{t}\xi^T(\tau)\xi(\tau)\mathrm{d}\tau + e^h\xi^T(t)\xi(t).\end{aligned} \tag{6.30}$$

Similarly, the second integral term in (6.29) is coped with as

$$V_2 = e^h\int_{t-h}^{t}\tilde{z}^T(\tau)\tilde{z}(\tau)\mathrm{d}\tau + e^h\int_{t-h}^{t}e^{\tau-t}\tilde{z}^T(\tau-h)\tilde{z}(\tau-h)\mathrm{d}\tau.$$

With the derivative as

$$\begin{aligned}\dot{V}_2 =& -e^h\int_{t-h}^{t}e^{\tau-t}\tilde{z}^T(\tau-h)\tilde{z}(\tau-h)\mathrm{d}\tau + e^h\tilde{z}^T(t)\tilde{z}(t) - \tilde{z}^T(t-2h)\tilde{z}(t-2h)\\ \leq& -\int_{t-h}^{t}\tilde{z}^T(\tau-h)\tilde{z}(\tau-h)\mathrm{d}\tau + e^h\tilde{z}^T(t)\tilde{z}(t).\end{aligned} \tag{6.31}$$

Let

$$V = V_0 + \rho_1\pi_{\min}^{-1}\kappa_3^{-1}V_1 + \rho_2\pi_{\min}^{-1}\kappa_4^{-1}V_2. \tag{6.32}$$

A direct evaluation gives that

$$\dot{V} \leq \xi^T(t)\left(G\otimes P_1\right)\xi(t) + \tilde{z}^T(t)\left(G\otimes Q_1\right)\tilde{z}(t), \tag{6.33}$$

where

$$P_1 = PA + A^T P - 2PBB^T P + (\kappa_1+\kappa_2+\kappa_3+\kappa_4)PP + \varepsilon_1 I, \tag{6.34}$$

$$Q_1 = QS + S^T Q - 2D_2^T D_2 + \varepsilon_2 I. \tag{6.35}$$

The condition in (6.24) is equivalent to the condition specified in (6.20). With (6.34) and (6.35), it can be shown by Schur Complement that conditions (6.25) and (6.26) are respectively equivalent to $P_1 < 0$ and $Q_1 < 0$, which further implies from

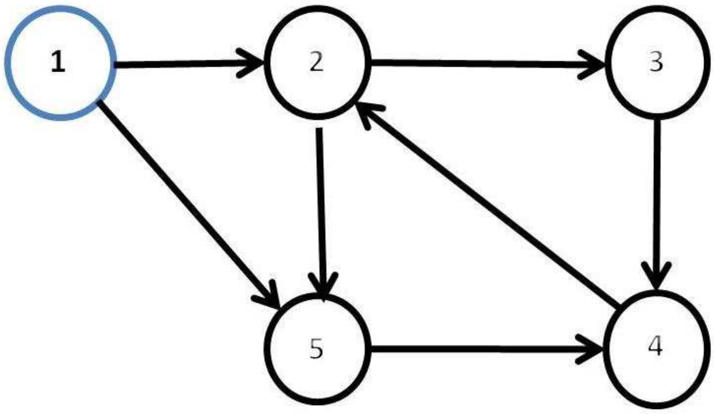

Figure 6.1: Communication topology.

(6.33) that $\dot{V}(t) < 0$. Thus, the error dynamics systems (6.4) are globally asymptotically stable at the origin, which implies that the consensus disturbance rejection of the multi-agent systems (6.1)–(6.2) is achieved. This completes the proof. □

6.4 Simulation

In this section, we will demonstrate the consensus disturbance rejection method under the leader-follower setup of five subsystems whose connection graph is specified by the adjacent matrix

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the first row all are zeros, as the agent indexed by 1 is taken as the leader. The communication graph is represented by Figure 6.1, and only the followers indexed by 2 and 5 have access to the leader's information. From Figure 6.1, it is easy to see that the communication topology contains a directed spanning tree. The dynamics of

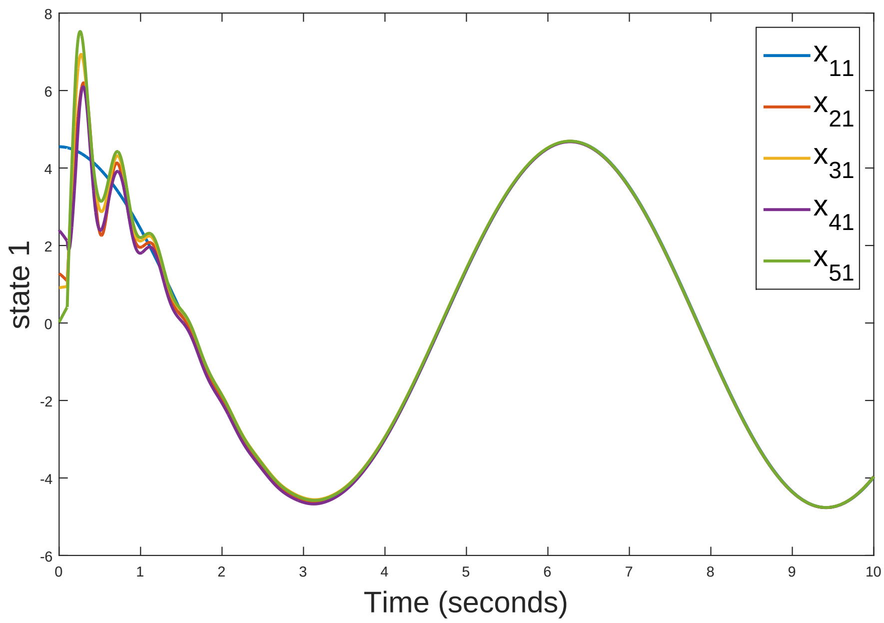

Figure 6.2: The trajectories of state 1 with $h = 0.1$ and $g = 0.05$.

the ith agent is described by (6.1), with

$$\dot{x}_i(t) = \begin{bmatrix} x_{i1}(t) \\ x_{i2}(t) \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \quad \phi(x_i) = g \begin{bmatrix} \sin(x_{i1}(t)) \\ \sin(x_{i2}(t)) \end{bmatrix},$$

which may present a practical dynamical model of unmanned aerial vehicle (UAV) [137]. In this work, it is supposed that external disturbance and time delay are exist in the control channel. The input disturbance $w_i(t)$ is generated by (6.3) with

$$S = \begin{bmatrix} 0 & -0.1 \\ 0.1 & 0 \end{bmatrix}, \; F = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix},$$

which represents an external periodic disturbance with known frequency but without any information of its magnitude and phase. The time delay of each agent is $0.1s$, and the Lipschitz constant $\gamma = g$. It can be checked that Assumptions (6.1.1) and (6.1.2) are satisfied.

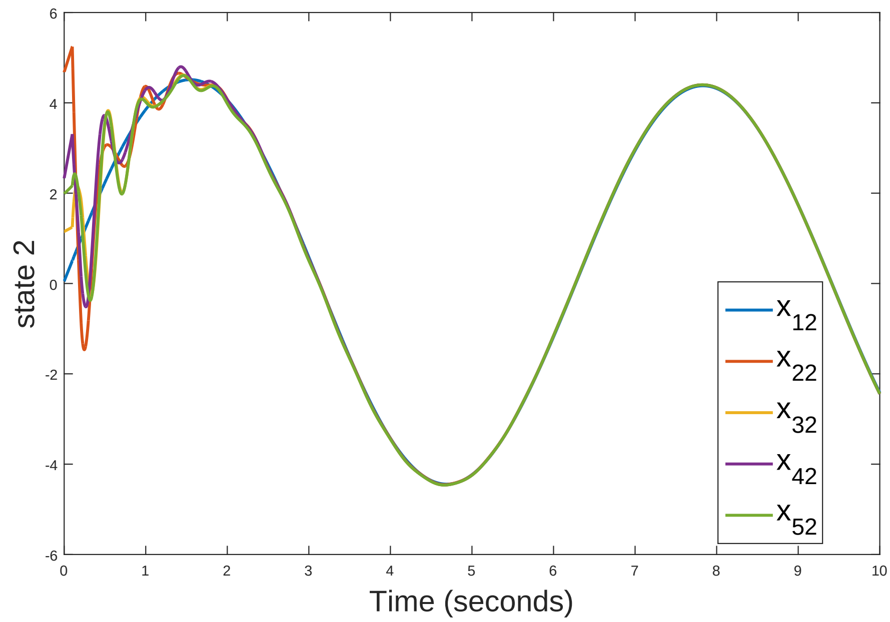

Figure 6.3: The trajectories of state 2 with $h = 0.1$ and $g = 0.05$.

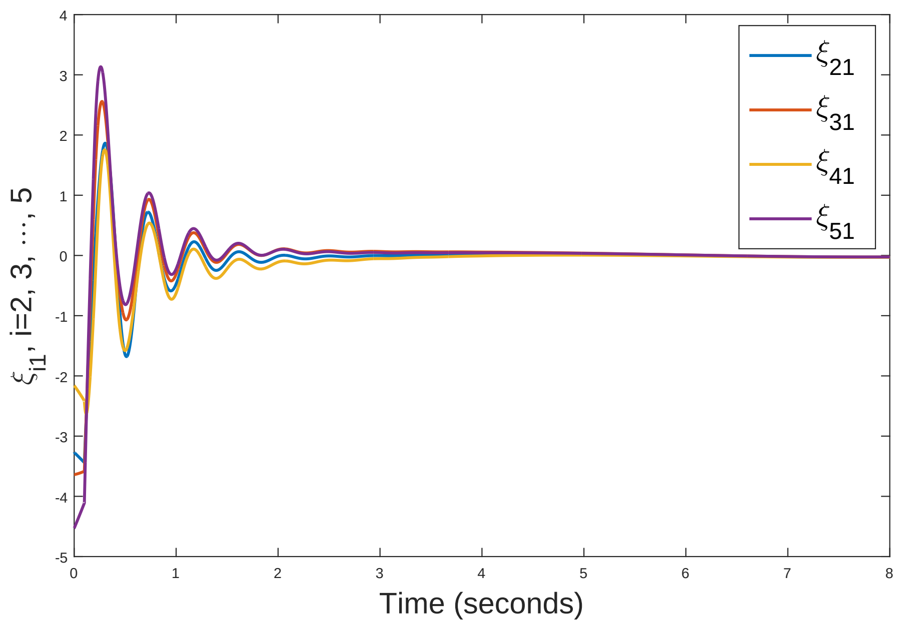

Figure 6.4: The evolutions of tracking error $x_{i1} - x_{11}$.

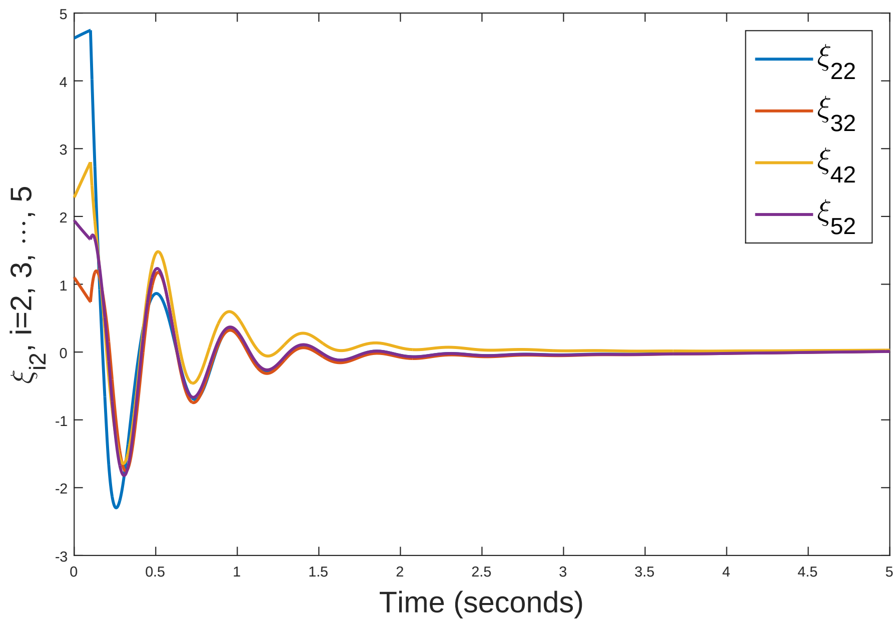

Figure 6.5: The evolutions of tracking error $x_{i2} - x_{12}$.

The Laplacian matrix $\mathcal{L}_1$ associated with $\mathcal{A}$ is that

$$\mathcal{L}_1 = \begin{bmatrix} 2 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 2 \end{bmatrix}.$$

Following the result shown in Lemma 2, we obtained that $G = \mathrm{diag}\{0.3846 \;\; 0.3571 \;\; 0.5556 \;\; 0.7143\}$ and $r_0 = 0.2573$. With $p_{\max} = 0.7143$ and $2p_{\max}/r_0 = 5.5523$, we set $c = 6$ in the control algorithm (6.6).

The initial states of agents are chosen randomly within $[\,0,\ 5\,]$, and $u(\theta) = [\,0,\ 0,\ 0,\ 0\,]^T$, $\forall \theta \in [-h, 0]$. The observer gain L is chosen as $L = \mu D_2^T$ with $\mu = 15$. With the values of $\rho = 0.2, g = 0.05$, a feasible solution of the feedback gain K is found to be

$$K = \begin{bmatrix} -0.1924 & -0.1233 \\ -0.0962 & -0.2466 \end{bmatrix}.$$

Simulation study has been carried out with different disturbances for agents. Figures 6.2 and 6.3 show the simulation results for the trajectories of the state. The tracking errors between the four followers and the leader are shown in Figures 6.4 and 6.5. The

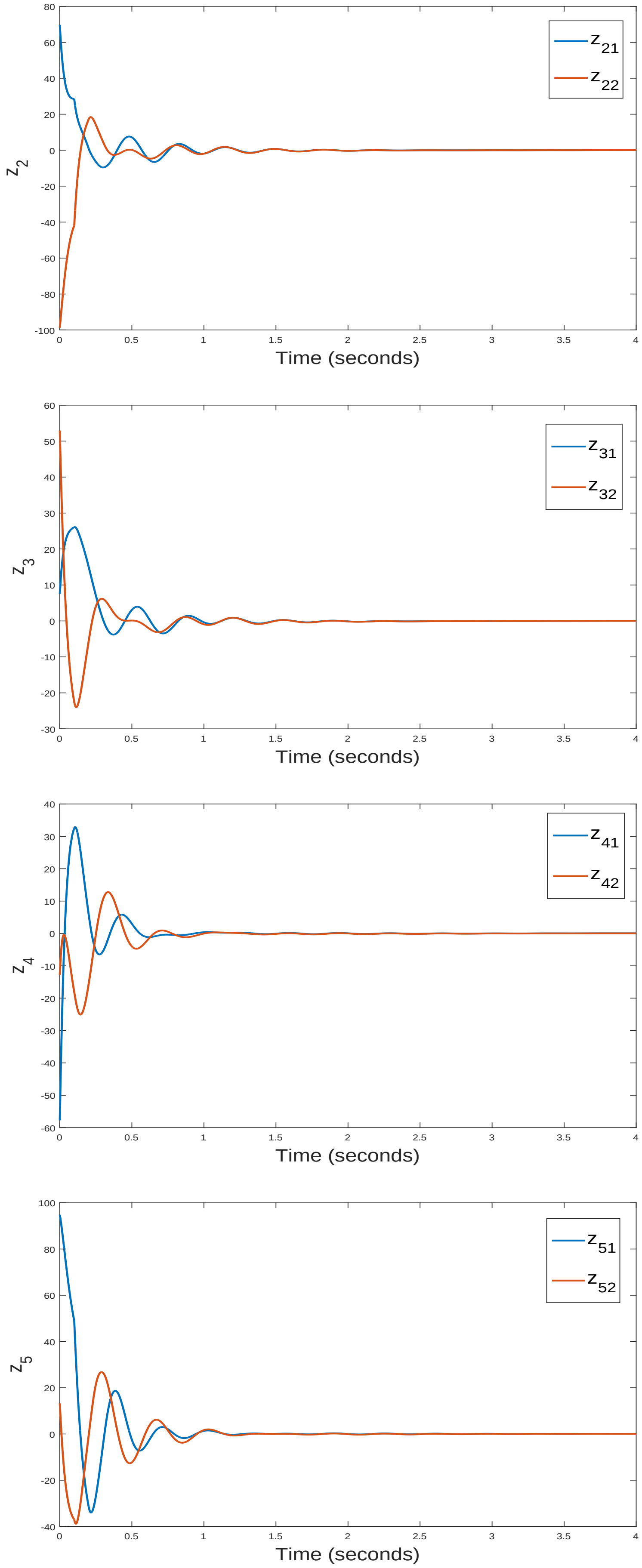

Figure 6.6: The tracking errors of the disturbance observers.

disturbance observer errors are shown in Figures 6.6. From the results shown in the figures, it can be seen that clearly all the five agents converge to the same set of states although they are under different disturbances; therefore, the conditions specified in Theorem 6.2 are sufficient to achieve the consensus disturbance rejection.

Remark 6.4.1. *As only the Lipschitz constant γ is used for the disturbance observer design and the exact information of the nonlinear functions is not required, this leads to conservatism in the presented conditions. With the same control gain, the consensus disturbance rejection is still achieved for the multi-agent systems with a larger Lipschitz constant.*

6.5 Summary

In this chapter, we have addressed the consensus disturbance rejection problem for Lipschitz nonlinear multi-agent systems with input delay under the directed communication graph. The input delay may represent some delays in the network communication or in the actuators. The conditions for designing disturbance observers for consensus control in presence of input delay are identified. In light of the well-known predictor-based feedback approach, a non-ideal predictor-based control scheme is constructed for each subsystem by using the estimate of the disturbance observer and the partial prediction of the relative state information. By exploring certain features of the Laplacian matrix, global consensus analysis is put in the framework of Lyapunov analysis. The proposed analysis ensures that the integral terms of the system state are carefully considered by using Krasovskii functionals. Sufficient conditions are derived for the input-delayed nonlinear systems to guarantee consensus disturbance rejection in the time domain. The gain in the controller can be obtained through an iterative LMI procedure. Finally, an example has demonstrated the effectiveness of the theoretical results.

Chapter 7

Conclusion and Future Work

This chapter concludes the thesis and discusses the future research directions related to the work done in this thesis.

7.1 Conclusion

This thesis considers the consensus problems of multi-agent systems with general agent dynamics and input delay under directed network connection between the agents. More specifically, the following research problems have been investigated rigorously.

1. The robust consensus problem of general linear multi-agent systems with input delay and parametric uncertainties has been considered. To deal with the input delay, Artstein model reduction method has been employed by a state transformation. The input-dependent integral term that remains in the transformed system, due to the model uncertainties, has been judiciously analysed. By carefully exploring certain features of the Laplacian matrix, sufficient conditions for the global consensus under directed communication topology have been identified using Lyapunov-Krasovskii functionals in the time domain. The proposed control only relies on relative state information of the agents via network connections. The effectiveness and robustness of the proposed control design has been demonstrated through a numerical simulation example.

2. The impacts of Lipschitz nonlinearity and input delay in consensus control have been investigated. Based on the predictor-based feedback approaches, distributed protocols have been proposed for a class of Lipschitz nonlinear multi-agent systems with input delay. Further rigorous analysis has been carried out to ensure

that the extra integral terms of the system state associated with nonlinear functions are properly considered by means of Krasovskii functionals. By transforming the Laplacian matrix into the real Jordan form, global stability analysis has been put in the framework of Lyapunov functions in real domain. Conditions based on the Lipschitz constant are identified for proposed consensus protocols to tackle Lipschitz nonlinear terms in the system dynamics under delay constraint.

3. The consensus disturbance rejection for multi-agent systems with input delay and external disturbances has been considered. First, the H_∞ consensus control for high-order multi-agent systems with input delay and external disturbances bounded by H_2 norms has been investigated. Sufficient conditions have been derived for the multi-agent systems to guarantee the H_∞ consensus in the time domain. Then, the multi-agent systems with unknown external disturbances has been studied. DOBC approach has been applied to design the disturbance observer and consensus protocols. A non-ideal predictor-based control scheme is constructed for each subsystem by utilizing the estimate of the disturbance and the prediction of the relative state information. Sufficient conditions have been derived to guarantee consensus with disturbance rejection in the time domain. Simulations have been employed to demonstrate the validity of the theoretical results.

7.2 Future Work

In this section, several potential extensions to our research are listed as follows.

1. The input delay existing in the multi-agent systems leads to extra integral terms in the transformed systems, and the analysis of the integral terms makes the derived conditions more conservative. Simulation results in this thesis also indicate that there is certain conservatism in the presented conditions. With the same control gain, the consensus is still achieved for the multi-agent system with a larger input delay. Further analysis to relax the conditions is a topic of future research.

2. In Chapters 4 and 6, consensus control for a class of Lipschitz nonlinear multi-agent systems has been studied. The impacts of Lipschitz nonlinearity and input delay in consensus control have been investigated. However, in reality, many nonlinear systems may not be satisfied with the Lipschitz condition. Hence,

consensus control for more general nonlinear multi-agent systems with input delay is a topic of future research from a practical point of view. The obstacle to solving consensus control problem for general nonlinear systems stems mainly from certain restrictions the nonlinearity imposes on using the information of the individual systems.

3. In recent years, distributed optimization based on multi-agent systems has been widely investigated due to its importance from a practical point of view. Most of existing results are based on simple agent dynamics such as first or second-order integrators dynamics. It is worthwhile investigating the distributed optimization based on general multi-agent systems. Furthermore, it is also meaningful to consider the imperfect communications, such as pocket drop and link failure in communication channels, for the consensus protocol design of multi-agent systems.

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