A Truncated Prediction Approach to Consensus Control of Lipschitz Nonlinear Multi-Agent Systems with Input Delay

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Abstract—This paper deals with the consensus control problem for Lipschitz nonlinear multi-agent systems with input delay. A prediction of the agent state over the delay period is approximated by the zero input solution of the agent dynamics. The structure of a linear state feedback control algorithm is assumed for each agent based on such approximated state prediction. By transforming the Laplacian matrix into the real Jordan form, sufficient conditions are established under which the multi-agent systems under the proposed control algorithms achieve global consensus. The feedback gain is then designed by solving these conditions with an iterative LMI procedure. A simulation study is given to validate the proposed control design.

Index Terms—Consensus control, truncated prediction, input delay, Lipschitz nonlinearity.

I. INTRODUCTION

Motivated by the extensive applications including flocking [1], synchronization [2] and formation control [3], distributed cooperative control of multi-agent systems has received considerable attention in recent years. In many applications, the subsystems or agents are required to achieve the same task or to reach an agreement upon certain quantities of interest, which is referred to as consensus or agreement. Consensus problems have attracted significant interests from the control theory community and have been intensively studied in the literature [4]–[10], to mention a few.

The network links between the agents are important for the consensus analysis and design. In graph theory, Laplacian matrix is often used to characterize the network connection in consensus control [4]. A theoretical framework for consensus problems of multi-agent systems with fixed and switching topologies was presented in [5]. The results were extended in [9], in which a unified approach for the consensus of multi-agent systems with general linear node dynamics was introduced. Many early results on consensus problems are limited to linear dynamics. Consensus for nonlinear systems is more involved than for their linear systems counterparts. The difficulty of consensus control for nonlinear systems owes to certain restrictions the nonlinearity imposes on using the information of the individual systems. Consensus control for second-order Lipschitz nonlinear multi-agent systems was addressed in [11]. The consensus problem of high-order multi-agent systems with nonlinear dynamics was studied in [12]–[14]. The works [15]–[16] address the consensus output regulation problem of nonlinear multi-agent systems.

Because of the time taken for transportation of materials, transmission of signals etc., delays are inevitable in physical systems. If not taken into consideration a priori, delays will degrade the performance of the closed-loop systems, and in the extreme situations, may even cause the loss of stability. The importance of addressing delays has been well recognized for a long time (see [17] and the references therein). With the internet and other communication tools used in the consensus control of multi-agent systems, time delay due to data transmission occurs more often. In particular, the consensus time delay occurs in the control input when the protocols depend on the relative state information transmitted over the network.

One basic idea for tackling input delay is to predict the evolution of state variable for the delay period and then use the predicted state for control [18], [19]. State prediction is based on the explicit solution of the state equation, which consists of the zero input and the zero state solutions. The zero state solution involves the integral of the past control input and causes difficulty in control implementation [20]. An alternative method based on the prediction is to ignore the troublesome zero state solution, and use the zero input solution as the prediction. The resulting prediction is referred to as the truncated prediction [21]–[23]. The existing studies [24]–[32] of consensus control with communication delays mainly focus on linear systems. Consensus problems for nonlinear multi-agent systems with time delay are expected to be more complicated just as stabilization of a nonlinear system with time delay is much more involved than its linear counterpart. Additional care is required to tackle the influence of the nonlinearity that appears in the agent dynamics.

In this paper, we consider the truncated prediction feedback for consensus control of Lipschitz nonlinear multi-agent systems with input delay. By using the real Jordan form, global consensus analysis is put in the framework of Lyapunov analysis. The proposed analysis ensures that the integral terms of the system state are carefully considered using Krasovskii functionals. Furthermore, the conditions can be solved as Linear Matrix Inequalities (LMIs) with a set of...
iterative scalar parameters, in a way similar to the iterative procedures developed in [33]. In contrast with [13]–[14], in which Artstein-Kwon-Pierson reduction method renders the closed-loop system delay free at the cost of an infinite dimensional control algorithm, and with [32], in which the agent dynamics are restricted to be linear, the proposed control design for Lipschitz nonlinear multi-agent systems with input delay are more involved. A simulation study is carried out to demonstrate the results obtained in the paper.

The remainder of this paper is organized as follows. Section II presents some notations and the problem formulation. A few preliminary results for the consensus analysis are given in Section III. Section IV presents the main results on the consensus control design. Simulation results are given in Section V. Section VI concludes the paper.

II. PROBLEM STATEMENT

Consider a group of $N$ agents, each represented by a nonlinear agent subject to input delay and Lipschitz nonlinearity,

$$
\dot{x}_i(t) = A x_i(t) + B u_i(t - h) + \phi(x_i(t)),
$$

where for agent $i$, $i = 1, 2, \ldots, N$, $x_i = [x_{i,1}, x_{i,2}, \ldots, x_{i,n}]^T \in \mathbb{R}^n$ is the state vector, $u_i \in \mathbb{R}^{n \times n}$ is the control input vector, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are constant matrices with $(A, B)$ being controllable, $h > 0$ is input delay, and $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\phi(0) = 0$, is a Lipschitz nonlinear function with a Lipschitz constant $\gamma$, i.e., for any two constant vectors $a, b \in \mathbb{R}^n$,

$$
\|\phi(a) - \phi(b)\| \leq \gamma \|a - b\|.
$$

The communications among the agents are described by a directed graph $G(V, E)$, where $V$ is a set of vertices and $E$ is a set of edges. A vertex represents an agent, and each edge represents a connection. Associated with the graph is its adjacency matrix $Q$, where element $q_{ij}$ denotes the connection between two agents. More specifically, if a connection exists from agent $j$ to agent $i$, $q_{ij} = 1$; otherwise $q_{ij} = 0$. The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ is defined by

$$
l_{ij} = \begin{cases}
\sum_{k=1, k \neq i}^{N} q_{ik}, & i = j,
-q_{ij}, & i \neq j.
\end{cases}
$$

For a directed graph, the Laplacian matrix $L$ has the following properties.

Lemma 1 ([35]): The Laplacian matrix $L$ of a directed graph $G$ has at least one zero eigenvalue with 1 as a corresponding right eigenvector and all nonzero eigenvalues have positive real parts. Furthermore, zero is a simple eigenvalue of $L$ if and only if $G$ has a directed spanning tree. In addition, there exists a nonnegative left eigenvector $r$ of $L$ associated with the zero eigenvalue, satisfying $r^T L = 0$ and $r^T 1 = 1$. Moreover, $r$ is unique if $G$ has a directed spanning tree.

The objective of this paper is to design a control algorithm for each agent such that the multi-agent systems (1) achieve global consensus. That is, under these control algorithms, the following hold for all initial conditions,

$$
\lim_{t \rightarrow \infty} (x_i(t) - x_j(t)) = 0, \ \forall i \neq j.
$$

Assumption 1: The eigenvalues of the Laplacian matrix are distinct.

Remark 1: This assumption ensures that the eigenvalue zero is algebraically simple and the directed graph contains a spanning tree. In a directed graph, the existence of a directed spanning tree is a weaker condition than being strongly connected. A strongly connected graph contains at least one directed spanning tree [35]. A stronger condition imposed by Assumption 1 is for the convenience of presentation. The results shown in this paper can be extended to the cases where the Laplacian matrix has multiple eigenvalues other than at zero, by following the procedures shown in [13].

III. PRELIMINARY RESULTS

In this section, a few preliminary results are presented. We first recall the truncated predictor feedback (TPF) method [21]–[23]. Consider an input-delayed system

$$
\dot{x}(t) = Ax(t) + Bu(t - h).
$$

From the system dynamics, we have

$$
x(t) = e^{Ah} x(t - h) + \int_{t-h}^{t} e^{A(t-\tau)} Bu(\tau - h) d\tau.
$$

The first term, $e^{Ah} x(t - h)$, is a truncated predictor of the state $x(t)$ based on $x(t - h)$. We take the control input as

$$
u(t) = Ke^{Ah} x(t),
$$

where $K \in \mathbb{R}^{n \times m}$ is a control gain matrix. The resultant closed-loop dynamics are given by

$$
\dot{x}(t) = (A + BK)x(t) - BKd(t),
$$

where

$$
d(t) = \int_{t-h}^{t} e^{A(t-\tau)} Bu(\tau - h) d\tau.
$$

In the TPF method, the troublesome integral term is ignored, and only the prediction based on the exponential of the systems matrix is used for control design.

We also need the following lemmas.

Lemma 2 ([36]): For a positive definite matrix $P$, and a function $x : [a, b] \rightarrow \mathbb{R}^n$, with $a, b \in \mathbb{R}$ and $b > a$, the following inequality holds:

$$
\left(\int_{a}^{b} x^T(x) dx \right) P \left(\int_{a}^{b} x(x) dx \right) \leq (b - a) \int_{a}^{b} x^T(x) P x(x) dx.
$$

Lemma 3 ([37],[38]): For a positive definite matrix $P$, the following identity holds

$$
e^{AT} P e^{At} - e^{\omega t} P = -e^{\omega t} \int_{0}^{t} e^{-\omega \tau} e^{AT} R e^{A\tau} d\tau,
$$

where $\omega \geq 0$ is a scalar and

$$
R = -AT P - PA + \omega P.
$$

Furthermore, if $R$ is positive definite, $\forall t > 0$,

$$
e^{AT} P e^{At} < e^{\omega t} P.
$$
IV. Consensus Control

For the multi-agent systems (1), we have

\[ x_i(t) = e^{Ah}x_i(t-h) + \int_{t-h}^t e^{A(t-\tau)} (Bu_i(\tau-h) + \phi(x_i(\tau))) d\tau. \]

We propose a control design based on the truncated prediction method. The control input takes the structure

\[ u_i(t) = Ke^{Ah} \sum_{j=1}^N q_{ij} (x_i(t) - x_j(t)) \]

\[ = Ke^{Ah} \sum_{j=1}^N l_{ij} x_j(t), \]  \hspace{1cm} (5)

where \( K \in \mathbb{R}^{m \times n} \) is a constant control gain matrix to be designed later. Under control algorithm (5), the multi-agent systems (1) can be written as

\[ \dot{x}_i = Ax_i + BK \sum_{j=1}^N l_{ij} x_j + \phi(x_i) \]

\[ -BK \sum_{j=1}^N l_{ij} \int_{t-h}^t e^{A(t-\tau)} (Bu_j(\tau-h) + \phi(x_j)) d\tau. \]

The closed-loop system is then described by

\[ \dot{x} = (I_N \otimes A + L \otimes BK) x \]

\[ + (L \otimes BK) (d_1 + d_2) + \Phi(x), \]  \hspace{1cm} (6)

where

\[ d_1 = -\int_{t-h}^t e^{A(t-\tau)} Bu(\tau-h) d\tau, \]

\[ d_2 = -\int_{t-h}^t e^{A(t-\tau)} \Phi(x) d\tau, \]

with

\[ x(t) = [x_1^T(t), x_2^T(t), \ldots, x_N^T(t)]^T, \]

\[ u(t) = [u_1^T(t), u_2^T(t), \ldots, u_N^T(t)]^T, \]

\[ \Phi(x) = [\phi^T(x_1), \phi^T(x_2), \ldots, \phi^T(x_N)]^T, \]

and \( \otimes \) denotes the Kronecker product of matrices.

Define \( r^T \in \mathbb{R}^{1 \times N} \) as the left eigenvector of the Laplacian matrix \( L \) associated with the eigenvalue zero, satisfying \( r^T L = 0 \). Furthermore, let \( r \) be scaled such that \( r^T 1 = 1 \). By Assumption 1, we know that a non-singular matrix \( T \), with its first column \( T(1) = 1 \) and the first row of \( T^{-1} \), \( T^{-1}(1) = r^T \), can be constructed such that

\[ T^{-1} LT = J, \]  \hspace{1cm} (7)

where \( J \) being a block diagonal matrix in the real Jordan form with the structure

\[ J = \begin{bmatrix}
0 & & \\
\lambda_1 & \ddots & \\
& \ddots & \ddots \\
& & \lambda_{n_\lambda} & \\
& & & \nu_1 \\
& & & & \ddots \\
& & & & & \nu_{n_\nu}
\end{bmatrix}, \]

where \( \lambda_i \in \mathbb{R} \) for \( i = 1, 2, \ldots, n_\lambda \), and

\[ \nu_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \]

for \( i = 1, 2, \ldots, n_\nu \). In the above expression of \( J \), \( \lambda_i \), \( \alpha_i \) and \( \beta_i \) are positive real numbers with \( \lambda_i \) denoting real eigenvalues of \( L \), and \( \alpha_i \pm i\beta_i \) represent conjugate complex eigenvalues of \( L \). Clearly we have \( 1 + n_\lambda + 2n_\nu = N \). Moreover, all the non-zero eigenvalues of \( L \) are positive or with positive real parts.

Based on the vector \( r \), we introduce a state transformation

\[ \xi_i = x_i - \sum_{j=1}^N r_j x_j, \]

\[ \text{for } i = 1, 2, \ldots, N. \]  \hspace{1cm} (8)

Let \( \xi = [\xi_1^T, \xi_2^T, \ldots, \xi_N^T]^T \). Then we have

\[ \dot{\xi} = \begin{bmatrix}
(1^T \otimes I_N) x \\
(M \otimes I_n) x
\end{bmatrix}, \]

where \( M = I_N - 1r^T \). Since \( r^T 1 = 1 \), it can be shown that \( M1 = 0 \). Therefore the consensus of system (6) is achieved when \( \lim_{t \to \infty} \xi(t) = 0 \), as \( \xi = 0 \) implies that \( x_1 = x_2 = \cdots = x_N \), due to the fact that the null space of \( M \) is span\{1\}. The dynamics of \( \xi \) can then be derived as

\[ \dot{\xi} = (I_N \otimes A + L \otimes BK) x - (1r^T \otimes I_N) [I_N \otimes A + L \otimes BK] x \\
+ (M \otimes I_n) (L \otimes BK) (d_1 + d_2) + (M \otimes I_n) \Phi(x) \\
+ (L \otimes BK) (d_1 + d_2), \]  \hspace{1cm} (9)

where we have used \( r^T L = 0 \).

To explore the structure of \( L \), we propose another state transformation

\[ \eta = (T^{-1} \otimes I_n) \xi, \]

\[ \text{with } \eta = [\eta_1^T, \eta_2^T, \ldots, \eta_N^T]^T. \]  \hspace{1cm} (10)

Then we have

\[ \dot{\eta} = (I_N \otimes A + J \otimes BK) \eta + \Pi(x) + \Delta(x) + \Psi(x), \]

\[ \text{with } \Pi(x) = (T^{-1} L \otimes BK) d_1, \]

\[ \Delta(x) = (T^{-1} L \otimes BK) d_2, \]

\[ \Psi(x) = (T^{-1} M \otimes I_n) \Phi(x). \]
For the notational convenience, let

\[
\Pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_N \end{bmatrix}, \quad \Delta = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_N \end{bmatrix}, \quad \Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{bmatrix},
\]

with \( \pi_i, \delta_i, \psi_i : \mathbb{R}^{n \times N} \rightarrow \mathbb{R}^n \) for \( i = 1, 2, \ldots, N \).

From state transformations (8) and (10), we have:

\[
\eta_1 = (r^T \otimes I_n) \xi = ((r^T M) \otimes I_n) x \equiv 0.
\]

With the control law shown in (5), the control gain matrix \( K \) is chosen as

\[
K = -B^T P, \quad (12)
\]

where \( P \) is a positive definite matrix.

The following theorem presents sufficient conditions to ensure that the consensus problem is solved by using the control algorithm (5) with control gain \( K \) in (12).

**Theorem 1:** For the Lipschitz nonlinear multi-agent systems (1) with input delay, the consensus problem can be solved by the control algorithm (5) with \( K = -B^T P \) if there exists a positive definite matrix \( P \) and constants \( \omega_1 \geq 0, \rho, \kappa_1, \kappa_2, \kappa_3 > 0 \) such that

\[
\rho W \geq BB^T, \quad (13)
\]

\[
\begin{bmatrix} A - \frac{1}{2} \omega_1 I_n \\ W \end{bmatrix} = 0,
\]

where

\[
\begin{cases}
\alpha = \min \{ \lambda_1, \lambda_2, \ldots, \lambda_{n_\lambda}, \alpha_1, \alpha_2, \ldots, \alpha_{n_\alpha} \}, \\
\Gamma = \frac{\gamma_0}{\kappa_1} e^h + \frac{21}{\kappa_2} e^h + \frac{22}{\kappa_3}, \\
\gamma_0 = 4 h^3 \rho^4 e^{2 \omega_1 h} \lambda_2^2 (T^{-1}) \| L \|_F^2 \| Q \|_F^2 \| T \|_F^2, \\
\gamma_1 = 4 h^2 \rho^2 e^{\omega_1 h} \lambda_2^2 (T^{-1}) \lambda_2^2 (Q) \| T \|_F^2, \\
\gamma_2 = 4 n \gamma_1^2 \| r \|_F^2 \lambda_2^2 (T^{-1}) \| T \|_F^2,
\end{cases}
\]

are satisfied with \( W = P^{-1} \) and

\[
\begin{align*}
\alpha &= \min \{ \lambda_1, \lambda_2, \ldots, \lambda_{n_\lambda}, \alpha_1, \alpha_2, \ldots, \alpha_{n_\alpha} \}, \\
\Gamma &= \frac{\gamma_0}{\kappa_1} e^h + \frac{21}{\kappa_2} e^h + \frac{22}{\kappa_3}, \\
\gamma_0 &= 4 h^3 \rho^4 e^{2 \omega_1 h} \lambda_2^2 (T^{-1}) \| L \|_F^2 \| Q \|_F^2 \| T \|_F^2, \\
\gamma_1 &= 4 h^2 \rho^2 e^{\omega_1 h} \lambda_2^2 (T^{-1}) \lambda_2^2 (Q) \| T \|_F^2, \\
\gamma_2 &= 4 n \gamma_1^2 \| r \|_F^2 \lambda_2^2 (T^{-1}) \| T \|_F^2,
\end{align*}
\]

where \( Q \) is the adjacency matrix, \( L \) is the Laplacian matrix, \( T \) is the non-singular matrix defined in (7), \( \rho \) and \( \omega_1 \) are positive numbers such that

\[
\rho^2 I \geq PBB^T BB^T P, \quad (16)
\]

\[
\omega_1 I > A^T + A, \quad (17)
\]

and \( \lambda_{n_\alpha}(\cdot) \) and \( \| \cdot \|_F \) are the maximum singular value and the Frobenius norm of a matrix, respectively.

**Proof:** The consensus analysis will be carried out in terms of \( \eta \). By (10), the consensus is achieved if \( \eta \) converges to zero, or equivalently if \( \eta \) converges to zero for \( i = 2, 3, \ldots, N \), since it has been shown that \( \eta_1 = 0 \). Let

\[
\text{subject to } \eta_{i+1} = (A - \lambda_i B B^T P) \eta_i + \pi_i + \delta_i + \psi_i,
\]

for \( i = 1, 2, \ldots, N \). For \( i = 2, 3, \ldots, n_\lambda + 1 \), we have

\[
\eta_i = (A - \lambda_i B B^T P) \eta_i + \pi_i + \delta_i + \psi_i,
\]

and hence

\[
\dot{V}_i = \eta_i^T (A^T P + PA - 2 \lambda_i PBB^T P) \eta_i + 2 \eta_i^T P \pi_i + 2 \eta_i^T P \delta_i + 2 \eta_i^T P \psi_i \\
\leq \eta_i^T \left( A^T P + PA - 2 \lambda_i PBB^T P + \sum_{i=1}^{3} \kappa_i P P \right) \eta_i \\
+ \frac{1}{\kappa_1} \| \pi_i \|^2 + \frac{1}{\kappa_2} \| \delta_i \|^2 + \frac{1}{\kappa_3} \| \psi_i \|^2,
\]

(19)

where \( \kappa_1, \kappa_2 \) and \( \kappa_3 \) are any positive numbers. In the above derivation, we used the well-known inequality

\[
2 a^T b \leq \kappa a^T a + \frac{1}{\kappa} b^T b, \quad \forall a, b \in \mathbb{R}^n.
\]

For \( i = n_\lambda + 2, n_\lambda + 3, \ldots, N \), we consider the evolution of the states in pairs corresponding to each pair of conjugate eigenvalues.

For a \( k \in \{1, 2, \ldots, n_\lambda\} \), let

\[
i_1 = 1 + n_\lambda + 2k - 1, \\
i_2 = 1 + n_\lambda + 2k.
\]

The dynamics of \( \eta_{i_1} \) and \( \eta_{i_2} \) are expressed as

\[
\dot{\eta}_{i_1} = (A - \alpha_k B B^T P) \eta_{i_1} - \beta_k B B^T P \eta_{i_2} + \pi_{i_1} + \delta_{i_1} + \psi_{i_1}, \\
\dot{\eta}_{i_2} = (A - \alpha_k B B^T P) \eta_{i_2} + \beta_k B B^T P \eta_{i_1} + \pi_{i_2} + \delta_{i_2} + \psi_{i_2}.
\]

Let

\[
V_i = \eta_i^T P \eta_{i_1} + \eta_i^T P \eta_{i_2}.
\]

Using the dynamics shown above, we can compute in a similar way as in the real eigenvalue case that

\[
\dot{V}_i = \eta_{i_1}^T \left( A^T P + PA - 2 \alpha_k PBB^T P \right) \eta_{i_1} + \eta_{i_2}^T \left( A^T P + PA - 2 \alpha_k PBB^T P + \sum_{i=1}^{3} \kappa_i P P \right) \eta_{i_2} \\
+ 2 \eta_{i_1}^T P \pi_{i_1} + 2 \eta_{i_1}^T P \delta_{i_1} + 2 \eta_{i_2}^T P \psi_{i_1} \\
+ 2 \eta_{i_1}^T P \pi_{i_2} + 2 \eta_{i_1}^T P \delta_{i_2} + 2 \eta_{i_2}^T P \psi_{i_2}
\]

\[
\leq \eta_{i_1}^T \left( A^T P + PA - 2 \alpha_k PBB^T P + \sum_{i=1}^{3} \kappa_i P P \right) \eta_{i_1} \\
+ \frac{1}{\kappa_1} \| \pi_{i_1} \|^2 + \frac{1}{\kappa_2} \| \delta_{i_1} \|^2 + \frac{1}{\kappa_3} \| \psi_{i_1} \|^2 \\
+ \eta_{i_2}^T \left( A^T P + PA - 2 \alpha_k PBB^T P + \sum_{i=1}^{3} \kappa_i P P \right) \eta_{i_2} \\
+ \frac{1}{\kappa_1} \| \pi_{i_2} \|^2 + \frac{1}{\kappa_2} \| \delta_{i_2} \|^2 + \frac{1}{\kappa_3} \| \psi_{i_2} \|^2,
\]

(20)

where \( \kappa_1, \kappa_2 \), and \( \kappa_3 \) are any positive numbers. Let

\[
V_0 = \sum_{i=2}^{N} V_i,
\]

In view of (19) and (20), we have

\[
V_0 = \sum_{i=2}^{N} V_i + \sum_{i=n_\lambda+2}^{N} V_i \\
\leq \eta^T \left[ I_N \otimes \left( A^T P + PA - 2 \alpha_k PBB^T P + \sum_{i=1}^{3} \kappa_i P P \right) \right] \eta \\
+ \frac{1}{\kappa_1} \| \Pi \|^2 + \frac{1}{\kappa_2} \| \Delta \|^2 + \frac{1}{\kappa_3} \| \Psi \|^2.
\]

(21)
From (25), (26), and (27), we obtain that
if the conditions (16) and (17) are satisfied.

Lemma 5: For the nonlinear term \( \Psi(x) \) in the transformed system dynamics (11), a bound can be established as

\[
\|\Psi\|^2 \leq \gamma_2 \|\eta\|^2.
\]

Proof: See the Appendix.

For the second integral term shown in (25), we consider the following Krasovskii functional

\[
W_1 = e^h \int_{t-h}^t e^{\tau-h} \eta^T(\tau-h) \eta(\tau-h) d\tau + e^h \int_{t-h}^t \eta^T(\tau) \eta(\tau) d\tau.
\]

A direct evaluation gives that

\[
W_1 = \int_{t-h}^t e^{\tau-t} \eta^T(\tau-h) \eta(\tau-h) d\tau + e^h \int_{t-h}^t \eta^T(\tau) \eta(\tau) d\tau.
\]

For the second integral term shown in (25), we consider the following Krasovskii functional

\[
W_2 = e^h \int_{t-h}^t e^{\tau-t} \eta^T(\tau) \eta(\tau) d\tau.
\]

A direct evaluation gives that

\[
W_2 = -e^h \int_{t-h}^t e^{\tau-t} \eta^T(\tau) \eta(\tau) d\tau + e^h \eta^T(t) \eta(t)
\]

\[
- \eta^T(t-h) \eta(t-h)
\]

\[
- \int_{t-h}^t \eta^T(\tau-h) \eta(\tau-h) d\tau + e^h \eta^T(t) \eta(t).
\]

Let

\[
V = V_0 + \frac{\gamma_0}{\kappa_1} W_1 + \frac{\gamma_1}{\kappa_2} W_2.
\]

From (25), (26), and (27), we obtain that

\[
\dot{V} \leq \eta^T(t) (I_N \otimes H) \eta(t),
\]

where

\[
H := A^T P + PA - 2\alpha PBB^T P + \sum_{i=1}^3 \kappa_i PP
\]

\[
+ \left( \frac{\gamma_0}{\kappa_1} \eta + \frac{\gamma_1}{\kappa_2} \eta + \frac{\gamma_2}{\kappa_3} \right) I_n.
\]

From the analysis in this section, we know that the feedback law (5) will stabilize \( \eta \) if the conditions (16), (17) and \( H < 0 \) in (29) are satisfied. Indeed, it is easy to see the conditions (16) and (17) are equivalent to the conditions specified in (13) and (14). From (30), it can be obtained that \( H < 0 \) is equivalent to

\[
W A^T + AW - 2\alpha B B^T P + (\kappa_1 + \kappa_2 + \kappa_3) I_n
\]

\[
+ \left( \frac{\gamma_0}{\kappa_1} \eta + \frac{\gamma_1}{\kappa_2} \eta + \frac{\gamma_2}{\kappa_3} \right) WW < 0,
\]

which is further equivalent to (15). It implies that \( \eta \) converges to zero asymptotically. Hence, the consensus control is achieved.

It is observed that (15) is more likely to be satisfied if the values of \( \rho, \omega_1, \kappa_1, \kappa_2, \kappa_3 \) are small. Therefore, the algorithm for finding a feasible solution of the conditions shown in (13) to (15) can be designed by following the iterative methods developed in [33] for an individual linear system. In particular, we suggest the following step by step algorithm.

1) Set \( \omega_1 = \lambda_{\text{max}}(A + A^T) \) if \( \lambda_{\text{max}}(A + A^T) > 0 \); otherwise set \( \omega_1 = 0 \).

2) Fix the value of \( \rho, \omega_1, \kappa_1, \kappa_2, \kappa_3 \) to some constants \( \hat{\omega}_1 > \omega_1 \) and \( \hat{\rho}, \hat{\kappa}_1, \hat{\kappa}_2, \hat{\kappa}_3 > 0 \); make an initial guess for the values of \( \hat{\rho}, \hat{\omega}_1, \hat{\kappa}_1, \hat{\kappa}_2, \hat{\kappa}_3 \).

3) Solve the LMI equation (15) for \( W \) with the fixed values; if a feasible value of \( W \) cannot be found, return to Step 2) and reset the values of \( \hat{\rho}, \hat{\omega}_1, \hat{\kappa}_1, \hat{\kappa}_2, \hat{\kappa}_3 \).

4) Solve the LMI equation (13) for \( \rho \) with the feasible value of \( W \) obtained in Step 3) and make sure that the value of \( \rho \) is minimized.

5) If the condition \( \hat{\rho} \geq \rho \) is satisfied, then \( (\hat{\rho}, \hat{\omega}_1, \hat{\kappa}_1, \hat{\kappa}_2, \hat{\kappa}_3, W) \) is a feasible solution for Theorem 1; otherwise, set \( \hat{\rho} = \rho \) and return to Step 3).

Remark 2: Given the input delay \( h \) and the Lipschitz constant \( \gamma \), it is concluded that the existence of a feasible solution is related to the matrices \( (A, B) \) and the Laplacian matrix \( L \). Additionally, since the values of \( h \) and \( \gamma \) are fixed and they are not the decision variables of the LMIs, a feasible solution may not exist if the values of \( h \) and \( \gamma \) are too large. Therefore, a trigger should be added in the algorithm to stop the iteration procedure if the values of \( \hat{\rho}, \hat{\omega}_1, \hat{\kappa}_1, \hat{\kappa}_2, \hat{\kappa}_3 \) are out of the preset range.

V. SIMULATION

In this section, a simulation study is carried out to demonstrate the effectiveness of the proposed control design. Consider a connection of four agents as shown in Figure 1. The dynamics of the \( i \)th agent is described by a second-order model...
The eigenvalues of $L_0$ are errors of all the agents. Clearly the conditions specified the feedback gain $K$ as

$$K = \begin{bmatrix} -0.0021 & -0.0658 \end{bmatrix}.$$  

Figures 2 and 3 show the simulation results for the consensus errors of all the agents. Clearly the conditions specified

as

$$\dot{x}_1(t) = \begin{bmatrix} -0.09 & 1 \\ -1 & -0.09 \end{bmatrix} x_1(t) + g \begin{bmatrix} \sin(x_{11}(t)) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t - 0.1).$$  

The linear part of the system represents a decayed oscillator. The time delay of the system is 0.1 seconds, and the Lipschitz constant $\gamma = g$. The Laplacian matrix is given by

$$L = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$  

The eigenvalues of $L$ are $\{0, 1, 3/2 \pm j\sqrt{3}/2\}$. Therefore, Assumption 1 is satisfied. We obtain that

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 \sqrt{3} \\ 0 & 0 & -\sqrt{3} \sqrt{3} \end{bmatrix},$$  

with $\alpha = 1$ and $r^T = \begin{bmatrix} \frac{1}{3}, \frac{1}{7}, 0, \frac{1}{4}\end{bmatrix}$.

In this case, we choose $\gamma = g = 0.03$, and the initial conditions for the agents as $x_1(\theta) = [1, 1]^T$, $x_2(\theta) = [0, 0]^T$, $x_3(\theta) = [0.3, 0.5]^T$, $x_4(\theta) = [0.5, 0.3]^T$, $u(\theta) = [0, 0, 0, 0]^T$, for $\theta \in [-h, 0]$. With the values of $\omega_3 = 0$, $\rho = 0.05$, $\kappa_1 = \kappa_2 = 0.01$, and $\kappa_3 = 0.1$, a feasible solution of the feedback gain $K$ is found to be

$$K = \begin{bmatrix} -0.0021 & -0.0658 \end{bmatrix}.$$  

This paper has investigated the impacts of nonlinearity and input delay in consensus control. This input delay may represent some delays in the network communication. We propose a control design based on the truncated prediction for a class of Lipschitz nonlinear multi-agent systems with input delay. A complete consensus analysis is presented in

in Theorem 1 are sufficient for the control gain to achieve consensus control. Without re-tuning the control gain, the consensus control is still achieved for the multi-agent systems with a larger delay of 0.5 seconds and a bigger Lipschitz constant of $g = 0.15$, as shown in Figures 4 and 5, which imply the conditions might be conservative in the control gain design for a given input delay and Lipschitz condition.

VI. CONCLUSION

This paper has investigated the impacts of nonlinearity and input delay in consensus control. This input delay may represent some delays in the network communication. We propose a control design based on the truncated prediction for a class of Lipschitz nonlinear multi-agent systems with input delay. A complete consensus analysis is presented in

Fig. 4. The consensus errors of state 1 with $h = 0.5$ and $g = 0.15$.  

Fig. 5. The consensus errors of state 2 with $h = 0.5$ and $g = 0.15$.  

Fig. 3. The consensus errors of state 2 with $h = 0.1$ and $g = 0.03$.  

Fig. 2. The consensus errors of state 1 with $h = 0.1$ and $g = 0.03$.  

Fig. 1. Communication topology.  

Fig. 1. Communication topology.
a systematic framework of Lyapunov-Krasovskii functionals. Sufficient conditions are derived for the multi-agent systems to guarantee the global consensus in the time domain. The conditions can be solved by employing LMIs with a set of iterative parameters. This result extends the recent work on the truncation predictor to nonlinear multi-agent systems. Simulation results indicate that there is certain conservatism in the presented conditions. Further analysis to tighten the conditions is a topic of future research.

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APPENDIX

Proof of Lemma 4: By the definition of $\Pi(x)$ in (11), we have

$$\|\Pi\| = \|(T^{-1} \otimes I_n)(L \otimes BK)d_1\| \leq \lambda_\sigma (T^{-1}) \|\mu\|,$$  
(31)

where $\mu = (L \otimes BK)d_1$. For the notational convenience, let $\mu = [\mu^T_1, \mu^T_2, \ldots, \mu^T_N]^T$. Then from (6) and (12), we have

$$\mu_i = -BK \sum_{j=1}^{N} l_{ij} \int_{t-h}^{t} e^{A(t-\tau)} Bu_j(\tau - h) d\tau$$

$$= BB^T P \sum_{j=1}^{N} l_{ij} \int_{t-h}^{t} e^{A(t-\tau)} BBT Pe^{Ah}$$

$$\times \sum_{k=1}^{N} q_{jk} (x_j(\tau - h) - x_j(\tau - h)) d\tau.$$  
(32)

From $\eta = (T^{-1} \otimes I_n) \xi$, we obtain $\xi = (T \otimes I_n) \eta$, and from the state transformation (8), we have

$$x_k(t) - x_j(t) = \xi_k(t) - \xi_j(t)$$

$$= ((T_k - T_j) \otimes I_n) \eta(t)$$

$$= \sum_{l=1}^{N} (T_{kl} - T_{jl}) \eta_l(t),$$  
(33)

where $T_k$ denotes the $k$th row of $T$. We define

$$\sigma_l = BB^T P \int_{t-h}^{t} e^{A(t-\tau)} BBT Pe^{Ah} \eta_2(\tau - h) d\tau.$$  
(34)

Next we need to deal with $\|\sigma\|^2$. With Lemma 2 and the condition (16), we have

$$\|\sigma\|^2 \leq h \int_{t-h}^{t} \eta_2^T(\tau - h)e^{A\tau} PBB^T e^{A^T(\tau - t)} PBB^T \times BB^T Pe^{A(\tau - t)} BB^T Pe^{Ah} \eta_2(\tau - h) d\tau$$

$$\leq h \lambda_2 h \int_{t-h}^{t} \eta_2^T(\tau - h)e^{A\tau} PBB^T e^{A^T(\tau - t)} BB^T Pe^{Ah} \eta_2(\tau - h) d\tau,$$

In view of Lemma 3, with the condition (17), we have

$$\|\sigma\|^2 \leq h \lambda_2 h \int_{t-h}^{t} e^{A_1(t-\tau)} \eta_1^T(\tau - h)e^{A_2^T(\tau - t)} PBB^T e^{A_1(\tau - t)} BB^T Pe^{A_2} \eta_1(\tau - h) d\tau$$

$$\leq h \lambda_2 h \int_{t-h}^{t} e^{A_1(t-\tau)} \eta_1^T(\tau - h)e^{A_2^T(\tau - t)} BB^T Pe^{A_2} \eta_1(\tau - h) d\tau.$$

Then, $\|\sigma\|^2$ can be bounded as

$$\|\sigma\|^2 = \sum_{i=1}^{N} \|\sigma_i\|^2$$

$$\leq h \lambda_2 h \int_{t-h}^{t} \eta_2^T(\tau - h)e^{A\tau} e^{Ah} \eta_2(\tau - h) d\tau.$$  
(35)

Therefore, from Equations (31), (34) and (35), we have

$$\|\Pi\|^2 \leq \gamma_0 \int_{t-h}^{t} \eta_2^T(\tau - h)e^{A\tau} e^{Ah} \eta_2(\tau - h) d\tau.$$  

This completes the proof. The proof of $\|\Delta\|$ is similar to that of $\|\Pi\|$ and hence omitted.

Proof of Lemma 5: By the definition of $\Psi(x)$ in (11), we have

$$\|\Psi\| = \|(T^{-1} \otimes I_n)(M \otimes I_n) \Phi(x)\| \leq \lambda_\sigma (T^{-1}) \|z\|,$$

where $z = (M \otimes I_n) \Phi(x)$. For the notational convenience, let $z = [z_1^T, z_2^T, \ldots, z_N^T]$. Then from (8), we have

$$z_i = \phi(x_i) - \sum_{k=1}^{N} r_k \phi(x_k) = \sum_{k=1}^{N} r_k (\phi(x_i) - \phi(x_k)).$$

It then follows that

$$\|z_i\| \leq \sum_{k=1}^{N} r_k \|\phi(x_i) - \phi(x_k)\| \leq \gamma \sum_{k=1}^{N} r_k \|x_i - x_k\|.$$
In light of (33), we have
\[
\|z_i\| \leq \gamma \sum_{k=1}^{N} |r_k| \left( \|T_i\| + \|T_k\| \right) \|\eta\| \leq \gamma \|\eta\| \left( \sum_{k=1}^{N} |r_k| \|T_i\| + \|\eta\| \|T\|_F \right).
\]
Therefore we have
\[
\|z\|^2 = \sum_{i=1}^{N} \|z_i\|^2 \leq 2\gamma^2 \|\eta\|^2 \sum_{i=1}^{N} \left( \|T_i\|^2 \left( \sum_{k=1}^{N} |r_k| \right)^2 + |r|^2 \|T\|_F^2 \right) \leq 2\gamma^2 \|\eta\|^2 \sum_{i=1}^{N} \left( \|T_i\|^2 N^2 |r|^2 + |r|^2 \|T\|_F^2 \right) = 4N^2 \gamma^2 \|r\|^2 \|T\|_F^2 \|\eta\|^2,
\]
and
\[
\|\Psi\|^2 \leq \gamma_2 \|\eta\|^2.
\]
This completes the proof. \(\blacksquare\)

REFERENCES

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