```
(*The basic function, derived from Voinova et al. Phys. Scr. 1999, 59, 391-396
   β is the complex response function (see below)
   μfilm is the film's elastic shear modulus
   ηfilm is the film's dynamic (shear) viscosity
   ρfilm is the film's density
   hfilm is the film's thickness
   ηbulk is the fluid's dynamic viscosity
   ρbulk is the fluid's density
   ⅈ is the square root of -1
   Harm is the harmonic number

   QCM-D Frequency reponse: Δf =
 Im[β/(2π ρq hq)] where ρq is the density of the
    quartz resonator and hq is its thickness
    QCM-D Dissipation response: Δd = -Re[β/(π Harm f1 ρq hq)] where
   f1 is the fundamental resonant frequency of the quartz resonator
*)
$Assumptions = μfilm ∈ Reals && ηfilm ∈ Reals && ρfilm ∈ Reals && hfilm ∈ Reals &&
  ηbulk ∈ Reals && ρbulk ∈ Reals && f1 ∈ Reals && hq ∈ Reals && ρq ∈ Reals &&
  Harm ∈ Reals && μfilm > 0 && ηfilm > 0 && ρfilm > 0 && hfilm > 0 && ηbulk > 0 &&
  ρbulk > 0 && f1 > 0 && hq > 0 && ρq > 0 && Harm > 0 && Harm ∈ Integers
β[μfilm_, ηfilm_, ρfilm_, hfilm_, ηbulk_, ρbulk_] = FullSimplify[
  Simplify[Refine[κ1 ξ1 (1 - A Exp[2 ξ1 hfilm]) / (1 + A Exp[2 ξ1 hfilm]) //.
      {κ1 → ηfilm - ⅈ μfilm / (2 π Harm f1), κf → ηbulk,
       ξ1 → Sqrt[-ρfilm (2 π Harm f1)^2 / (μfilm + ⅈ ηfilm (2 π Harm f1))],
       ξf → Sqrt[ⅈ ρbulk (2 π Harm f1) / ηbulk], A → (κ1 ξ1 + κf ξf) / (κ1 ξ1 - κf ξf)}]]]
```

$$\mu\text{film} \in \text{Reals} \;\&\&\; \eta\text{film} \in \text{Reals} \;\&\&\; \rho\text{film} \in \text{Reals} \;\&\&\; \text{hfilm} \in \text{Reals} \;\&\&\; \eta\text{bulk} \in \text{Reals} \;\&\&\; \rho\text{bulk} \in \text{Reals} \;\&\&\; \text{f1} \in \text{Reals} \;\&\&\; \text{hq} \in \text{Reals} \;\&\&\; \rho\text{q} \in \text{Reals} \;\&\&\; \text{Harm} \in \text{Reals} \;\&\&\; \mu\text{film} > 0 \;\&\&\; \eta\text{film} > 0 \;\&\&\; \rho\text{film} > 0 \;\&\&\; \text{hfilm} > 0 \;\&\&\; \eta\text{bulk} > 0 \;\&\&\; \rho\text{bulk} > 0 \;\&\&\; \text{f1} > 0 \;\&\&\; \text{hq} > 0 \;\&\&\; \rho\text{q} > 0 \;\&\&\; \text{Harm} > 0 \;\&\&\; \text{Harm} \in \text{Integers}$$

$$-\left(\left(\left(2\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film} - \text{ⅈ}\,\mu\text{film}\right)\sqrt{\rho\text{film}}\right.\right.$$

$$\left(\left(1+\text{ⅈ}\right)\sqrt{\pi}\,\sqrt{\frac{\text{f1}\,\text{Harm}\,\eta\text{bulk}\,\rho\text{bulk}}{\rho\text{film}}}\;\text{Cos}\left[\frac{2\,\text{f1}\,\text{Harm}\,\text{hfilm}\,\pi\sqrt{\rho\text{film}}}{\sqrt{2\,\text{ⅈ}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]+\right.$$

$$\left.\left.\text{ⅈ}\,\sqrt{2\,\text{ⅈ}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}\;\text{Sin}\left[\frac{2\,\text{f1}\,\text{Harm}\,\text{hfilm}\,\pi\sqrt{\rho\text{film}}}{\sqrt{2\,\text{ⅈ}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]\right)\right)\Big/$$

$$\left(\left(2\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film} - \text{ⅈ}\,\mu\text{film}\right)\text{Cos}\left[\frac{2\,\text{f1}\,\text{Harm}\,\text{hfilm}\,\pi\sqrt{\rho\text{film}}}{\sqrt{2\,\text{ⅈ}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]+\right.$$

$$\left(1+\text{ⅈ}\right)\sqrt{\pi}\,\sqrt{\frac{\text{f1}\,\text{Harm}\,\eta\text{bulk}\left(2\,\text{ⅈ}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}\right)\rho\text{bulk}}{\rho\text{film}}}$$

$$\left.\left.\text{Sin}\left[\frac{2\,\text{f1}\,\text{Harm}\,\text{hfilm}\,\pi\sqrt{\rho\text{film}}}{\sqrt{2\,\text{ⅈ}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]\right)\right)$$

```
(*Derivative of the beta function with respect to film density ∂β/∂ρfilm.
   This needs to be done in stages for Mathematica
 to be able to process it in a sensible length of time
*)
$Assumptions = True
FullSimplfy[
 Simplify[Refine[D[β[μfilm, ηfilm, ρfilm, hfilm, ηbulk, ρbulk], ρfilm],
   Assumptions → μfilm ∈ Reals && ηfilm ∈ Reals && ρfilm ∈ Reals && hfilm ∈ Reals &&
     ηbulk ∈ Reals && ρbulk ∈ Reals && f1 ∈ Reals && hq ∈ Reals && ρq ∈ Reals &&
     Harm ∈ Reals && μfilm > 0 && ηfilm > 0 && ρfilm > 0 && hfilm > 0 && ηbulk > 0 &&
     ρbulk > 0 && f1 > 0 && hq > 0 && ρq > 0 && Harm > 0 && Harm ∈ Integers]]
 (*The assumptions here are the same as those included
 below using the $Assumptions tag*)
```

```
True
```

$$\texttt{FullSimplfy}\Big[\left(\left(2\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mathrm{i}\,\mu\texttt{film}\right)\left(-\left(-1+e^{\frac{4\,\texttt{f1}\,\texttt{Harm}\,\texttt{hfilm}\,\pi\sqrt{\rho\texttt{film}}}{\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}}}\right)\right.\right.$$

$$\mu\texttt{film}\left(-2\,\mathrm{i}\left(-1+e^{\frac{4\,\texttt{f1}\,\texttt{Harm}\,\texttt{hfilm}\,\pi\sqrt{\rho\texttt{film}}}{\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}}}\right)\sqrt{2\,\pi}\;\eta\texttt{bulk}\sqrt{\frac{\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\rho\texttt{bulk}}{\eta\texttt{bulk}}}+\right.$$

$$\left.\left(1+e^{\frac{4\,\texttt{f1}\,\texttt{Harm}\,\texttt{hfilm}\,\pi\sqrt{\rho\texttt{film}}}{\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}}}\right)\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}\;\sqrt{\rho\texttt{film}}\right)\sqrt{\rho\texttt{film}}-$$

$$16\,\mathrm{i}\,e^{\frac{4\,\texttt{f1}\,\texttt{Harm}\,\texttt{hfilm}\,\pi\sqrt{\rho\texttt{film}}}{\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}}}\,\texttt{f1}^2\,\texttt{Harm}^2\,\texttt{hfilm}\,\pi^2\sqrt{\rho\texttt{film}}\;\left(-\eta\texttt{bulk}\,\rho\texttt{bulk}+\eta\texttt{film}\,\rho\texttt{film}\right)+$$

$$2\,\texttt{f1}\,\texttt{Harm}\,\pi\left(-\left(-1+e^{\frac{4\,\texttt{f1}\,\texttt{Harm}\,\texttt{hfilm}\,\pi\sqrt{\rho\texttt{film}}}{\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}}}\right)\eta\texttt{bulk}\right.$$

$$\left(\mathrm{i}\left(1+e^{\frac{4\,\texttt{f1}\,\texttt{Harm}\,\texttt{hfilm}\,\pi\sqrt{\rho\texttt{film}}}{\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}}}\right)\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}\;\rho\texttt{bulk}+\right.$$

$$\left.2\left(-1+e^{\frac{4\,\texttt{f1}\,\texttt{Harm}\,\texttt{hfilm}\,\pi\sqrt{\rho\texttt{film}}}{\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}}}\right)\sqrt{2\,\pi}\;\eta\texttt{film}\sqrt{\frac{\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\rho\texttt{bulk}}{\eta\texttt{bulk}}}\;\sqrt{\rho\texttt{film}}\right)-$$

$$\mathrm{i}\left(-1+e^{\frac{8\,\texttt{f1}\,\texttt{Harm}\,\texttt{hfilm}\,\pi\sqrt{\rho\texttt{film}}}{\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}}}\right)\eta\texttt{film}\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}\;\rho\texttt{film}-$$

$$\left.\left.\left.4\,e^{\frac{4\,\texttt{f1}\,\texttt{Harm}\,\texttt{hfilm}\,\pi\sqrt{\rho\texttt{film}}}{\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}}}\,\texttt{hfilm}\,\mu\texttt{film}\,\rho\texttt{film}^{3/2}\right)\right)\right)\Big/$$

$$\left(2\left(\left(\left(-1+e^{\frac{4\,\texttt{f1}\,\texttt{Harm}\,\texttt{hfilm}\,\pi\sqrt{\rho\texttt{film}}}{\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}}}\right)\sqrt{2\,\pi}\;\eta\texttt{bulk}\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}\right.\right.\right.$$

$$\left.\left.\sqrt{\frac{\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\rho\texttt{bulk}}{\eta\texttt{bulk}}}+\left(1+e^{\frac{4\,\texttt{f1}\,\texttt{Harm}\,\texttt{hfilm}\,\pi\sqrt{\rho\texttt{film}}}{\sqrt{-2\,\mathrm{i}\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mu\texttt{film}}}}\right)\right.\right.$$

$$\left.\left.\left(2\,\texttt{f1}\,\texttt{Harm}\,\pi\,\eta\texttt{film}-\mathrm{i}\,\mu\texttt{film}\right)\sqrt{\rho\texttt{film}}\right)^2\sqrt{\rho\texttt{film}}\right)\Big]$$

(*Below is included for comparison and can be deleted later*)

$$\left(-i\,f1\,Harm\sqrt{2\pi}\,\sqrt{-2\,i\,f1\,Harm\,\pi\,\eta film-\mu film}\,\sqrt{\rho film}\,\rho bulk\right.$$
$$\left.Cosh\left[\frac{2\,f1\,hfilm\,Harm\,\pi\sqrt{\rho film}}{\sqrt{-2\,i\,f1\,Harm\,\pi\,\eta film-\mu film}}\right]+\left(-2\,f1\,Harm\,\pi\,\eta film+i\,\mu film\right)\right.$$
$$\left.\rho film\sqrt{\frac{i\,f1\,Harm\,\rho bulk}{\eta bulk}}\,Sinh\left[\frac{2\,f1\,hfilm\,Harm\,\pi\sqrt{\rho film}}{\sqrt{-2\,i\,f1\,Harm\,\pi\,\eta film-\mu film}}\right]\right)\Big/$$
$$\left(\sqrt{-2\,i\,f1\,Harm\,\pi\,\eta film-\mu film}\,\sqrt{\rho film}\,\sqrt{\frac{i\,f1\,Harm\,\rho bulk}{\eta bulk}}\right.$$
$$\left.Cosh\left[\frac{2\,f1\,hfilm\,Harm\,\pi\sqrt{\rho film}}{\sqrt{-2\,i\,f1\,Harm\,\pi\,\eta film-\mu film}}\right]+\right.$$
$$\left.f1\,Harm\sqrt{2\pi}\,\rho bulk\,Sinh\left[\frac{2\,f1\,hfilm\,Harm\,\pi\sqrt{\rho film}}{\sqrt{-2\,i\,f1\,Harm\,\pi\,\eta film-\mu film}}\right]\right)$$

```
(*Assumptions are that all material parameters are positive
 real numbers and that the harmonic number is an integer*)
$Assumptions = μfilm ∈ Reals && ηfilm ∈ Reals && ρfilm ∈ Reals && hfilm ∈ Reals &&
   ηbulk ∈ Reals && ρbulk ∈ Reals && f1 ∈ Reals && hq ∈ Reals && ρq ∈ Reals &&
   Harm ∈ Reals && μfilm > 0 && ηfilm > 0 && ρfilm > 0 && hfilm > 0 && ηbulk > 0 &&
   ρbulk > 0 && f1 > 0 && hq > 0 && ρq > 0 && Harm > 0 && Harm ∈ Integers
```

```
μfilm ∈ Reals && ηfilm ∈ Reals && ρfilm ∈ Reals && hfilm ∈ Reals &&
 ηbulk ∈ Reals && ρbulk ∈ Reals && f1 ∈ Reals && hq ∈ Reals && ρq ∈ Reals &&
 Harm ∈ Reals && μfilm > 0 && ηfilm > 0 && ρfilm > 0 && hfilm > 0 && ηbulk > 0 &&
 ρbulk > 0 && f1 > 0 && hq > 0 && ρq > 0 && Harm > 0 && Harm ∈ Integers
```

```
(*Derivative of the beta function
 with respect to film shear modulus ∂β/∂μfilm*)
FullSimplify[D[β[μfilm, ηfilm, ρfilm, hfilm, ηbulk, ρbulk], μfilm]]
```

$$\left(\text{f1}\,\text{Harm}\left(4\,\text{i}\,\text{f1}^2\,\text{Harm}^2\,\text{hfilm}\,\pi\,\mu\text{film}\,\rho\text{film}^{7/2}+\right.\right.$$
$$8\,\text{f1}^3\,\text{Harm}^3\,\text{hfilm}\,\pi^2\,\rho\text{film}^{5/2}\left(\eta\text{bulk}\,\rho\text{bulk}-\eta\text{film}\,\rho\text{film}\right)+$$
$$\left(2+2\,\text{i}\right)\sqrt{\pi}\left(\mu\text{film}\sqrt{\text{f1}^3\,\text{Harm}^3\,\eta\text{bulk}\,\rho\text{bulk}\,\rho\text{film}^5}\;+\right.$$
$$\left.2\,\text{i}\,\pi\,\eta\text{film}\sqrt{\text{f1}^5\,\text{Harm}^5\,\eta\text{bulk}\,\rho\text{bulk}\,\rho\text{film}^5}\right)-$$
$$\left(2+2\,\text{i}\right)\sqrt{\pi}\left(\mu\text{film}\sqrt{\text{f1}^3\,\text{Harm}^3\,\eta\text{bulk}\,\rho\text{bulk}\,\rho\text{film}^5}\;+\right.$$
$$\left.2\,\text{i}\,\pi\,\eta\text{film}\sqrt{\text{f1}^5\,\text{Harm}^5\,\eta\text{bulk}\,\rho\text{bulk}\,\rho\text{film}^5}\right)$$
$$\text{Cos}\left[\frac{4\,\text{f1}\,\text{Harm}\,\text{hfilm}\,\pi\sqrt{\rho\text{film}}}{\sqrt{2\,\text{i}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]+\text{f1}\,\text{Harm}\sqrt{2\,\text{i}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}$$
$$\left.\left.\rho\text{film}^2\left(-\text{i}\,\mu\text{film}\,\rho\text{film}+2\,\text{f1}\,\text{Harm}\,\pi\left(\eta\text{bulk}\,\rho\text{bulk}+\eta\text{film}\,\rho\text{film}\right)\right)\right)\right)\Bigg/$$
$$\left.\text{Sin}\left[\frac{4\,\text{f1}\,\text{Harm}\,\text{hfilm}\,\pi}{\sqrt{\frac{2\,\text{i}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}{\rho\text{film}}}}\right]\right)\Bigg/$$
$$\left(4\left(2\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}-\text{i}\,\mu\text{film}\right)^2\sqrt{\rho\text{film}}\right.$$
$$\left(\text{i}\,\text{f1}\,\text{Harm}\,\rho\text{film}\,\text{Cos}\left[\frac{2\,\text{f1}\,\text{Harm}\,\text{hfilm}\,\pi\sqrt{\rho\text{film}}}{\sqrt{2\,\text{i}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]+\right.$$
$$\left(1+\text{i}\right)\sqrt{\pi}\sqrt{-\frac{\text{f1}^3\,\text{Harm}^3\,\eta\text{bulk}\,\rho\text{bulk}\,\rho\text{film}}{2\,\text{i}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}$$
$$\left.\left.\text{Sinh}\left[2\,\text{f1}\,\text{Harm}\,\text{hfilm}\,\pi\sqrt{-\frac{\rho\text{film}}{2\,\text{i}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]\right)^2\right)$$

$$\Bigg(\mathbf{f1\,Harm}\left(4\,i\,\mathbf{f1}^2\,\mathbf{hfilm\,Harm}^2\,\pi\,\mu\mathbf{film}\,\rho\mathbf{film}^{7/2}\right.$$
$$+8\,\mathbf{f1}^3\,\mathbf{hfilm\,Harm}^3\,\pi^2\,\rho\mathbf{film}^{5/2}\,\left(-\eta\mathbf{film}\,\rho\mathbf{film}+\eta\mathbf{bulk}\,\rho\mathbf{bulk}\right)$$
$$+\left(2+2\,i\right)\sqrt{\pi}\,\left(\mu\mathbf{film}\sqrt{\mathbf{f1}^3\,\mathbf{Harm}^3\,\eta\mathbf{bulk}\,\rho\mathbf{film}^5\,\rho\mathbf{bulk}}\,+\right.$$
$$\left.2\,i\,\pi\,\eta\mathbf{film}\sqrt{\mathbf{f1}^5\,\mathbf{Harm}^5\,\eta\mathbf{bulk}\,\rho\mathbf{film}^5\,\rho\mathbf{bulk}}\right)$$
$$-\left(2+2\,i\right)\sqrt{\pi}\,\left(\mu\mathbf{film}\sqrt{\mathbf{f1}^3\,\mathbf{Harm}^3\,\eta\mathbf{bulk}\,\rho\mathbf{film}^5\,\rho\mathbf{bulk}}\,+\right.$$
$$\left.2\,i\,\pi\,\eta\mathbf{film}\sqrt{\mathbf{f1}^5\,\mathbf{Harm}^5\,\eta\mathbf{bulk}\,\rho\mathbf{film}^5\,\rho\mathbf{bulk}}\right)$$
$$\mathbf{Cos}\left[\frac{4\,\mathbf{f1\,hfilm\,Harm}\,\pi\sqrt{\rho\mathbf{film}}}{\sqrt{2\,i\,\mathbf{f1\,Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}}}\right]+\mathbf{f1\,Harm}\sqrt{2\,i\,\mathbf{f1\,Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}}$$
$$\rho\mathbf{film}^2\,\left(-i\,\mu\mathbf{film}\,\rho\mathbf{film}+2\,\mathbf{f1\,Harm}\,\pi\,\left(\eta\mathbf{film}\,\rho\mathbf{film}+\eta\mathbf{bulk}\,\rho\mathbf{bulk}\right)\right)$$
$$\left.\left.\mathbf{Sin}\left[\frac{4\,\mathbf{f1\,hfilm\,Harm}\,\pi\sqrt{\rho\mathbf{film}}}{\sqrt{2\,i\,\mathbf{f1\,Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}}}\right]\right)\right)\Big/$$
$$\Bigg(4\,\left(2\,\mathbf{f1\,Harm}\,\pi\,\eta\mathbf{film}-i\,\mu\mathbf{film}\right)^2\sqrt{\rho\mathbf{film}}$$
$$\left(i\,\mathbf{f1\,Harm}\,\rho\mathbf{film}\,\mathbf{Cos}\left[\frac{2\,\mathbf{f1\,hfilm\,Harm}\,\pi\sqrt{\rho\mathbf{film}}}{\sqrt{2\,i\,\mathbf{f1\,Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}}}\right]+\right.$$
$$\left(1+i\right)\sqrt{\pi}\sqrt{-\frac{\mathbf{f1}^3\,\mathbf{Harm}^3\,\eta\mathbf{bulk}\,\rho\mathbf{film}\,\rho\mathbf{bulk}}{2\,i\,\mathbf{f1\,Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}}}$$
$$\left.\left.\mathbf{Sinh}\left[2\,\mathbf{f1\,hfilm\,Harm}\,\pi\sqrt{-\frac{\rho\mathbf{film}}{2\,i\,\mathbf{f1\,Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}}}\right]\right)^2\right)\Bigg)$$

```
(*Derivative of the beta function with respect to film viscosity ∂β/∂ηfilm*)
FullSimplify[D[β[μfilm, ηfilm, ρfilm, hfilm, ηbulk, ρbulk], ηfilm]]
```

$$
-\left(\left(\mathrm{f1}^3\,\mathrm{Harm}^3\,\pi\sqrt{-\frac{\rho\mathrm{film}}{2\,i\,\mathrm{f1}\,\mathrm{Harm}\,\pi\,\eta\mathrm{film}+\mu\mathrm{film}}}\right.\right.
$$

$$
\left(\left(2+2\,i\right)\sqrt{\pi}\,\left(\left(-1+i\right)\,\mathrm{f1}\,\mathrm{Harm}\,\mathrm{hfilm}\,\sqrt{\pi}\;\mu\mathrm{film}\,\rho\mathrm{bulk}\,\rho\mathrm{film}\right.\right.
$$

$$
\sqrt{-\frac{\rho\mathrm{film}}{2\,i\,\mathrm{f1}\,\mathrm{Harm}\,\pi\,\eta\mathrm{film}+\mu\mathrm{film}}}+i\,\mu\mathrm{film}\sqrt{-\frac{\mathrm{f1}\,\mathrm{Harm}\,\eta\mathrm{bulk}\,\rho\mathrm{bulk}^3\,\rho\mathrm{film}}{2\,i\,\mathrm{f1}\,\mathrm{Harm}\,\pi\,\eta\mathrm{film}+\mu\mathrm{film}}}-
$$

$$
2\,\pi\,\eta\mathrm{film}\sqrt{-\frac{\mathrm{f1}^3\,\mathrm{Harm}^3\,\eta\mathrm{bulk}\,\rho\mathrm{bulk}^3\,\rho\mathrm{film}}{2\,i\,\mathrm{f1}\,\mathrm{Harm}\,\pi\,\eta\mathrm{film}+\mu\mathrm{film}}}+
$$

$$
\left(2+2\,i\right)\mathrm{f1}^2\,\mathrm{Harm}^2\,\mathrm{hfilm}\,\pi^{3/2}\,\rho\mathrm{bulk}\sqrt{-\frac{\rho\mathrm{film}}{2\,i\,\mathrm{f1}\,\mathrm{Harm}\,\pi\,\eta\mathrm{film}+\mu\mathrm{film}}}
$$

$$
\left(\eta\mathrm{bulk}\,\rho\mathrm{bulk}-\eta\mathrm{film}\,\rho\mathrm{film}\right)+\left(-i\,\mu\mathrm{film}\sqrt{-\frac{\mathrm{f1}\,\mathrm{Harm}\,\eta\mathrm{bulk}\,\rho\mathrm{bulk}^3\,\rho\mathrm{film}}{2\,i\,\mathrm{f1}\,\mathrm{Harm}\,\pi\,\eta\mathrm{film}+\mu\mathrm{film}}}+\right.
$$

$$
\left.\left.2\,\pi\,\eta\mathrm{film}\sqrt{-\frac{\mathrm{f1}^3\,\mathrm{Harm}^3\,\eta\mathrm{bulk}\,\rho\mathrm{bulk}^3\,\rho\mathrm{film}}{2\,i\,\mathrm{f1}\,\mathrm{Harm}\,\pi\,\eta\mathrm{film}+\mu\mathrm{film}}}\right)\right.
$$

$$
\left.\left.\mathrm{Cos}\left[\frac{4\,\mathrm{f1}\,\mathrm{Harm}\,\mathrm{hfilm}\,\pi\sqrt{\rho\mathrm{film}}}{\sqrt{2\,i\,\mathrm{f1}\,\mathrm{Harm}\,\pi\,\eta\mathrm{film}+\mu\mathrm{film}}}\right]\right)+\right.
$$

$$
\left.\rho\mathrm{bulk}\,\left(\mu\mathrm{film}\,\rho\mathrm{film}+2\,i\,\mathrm{f1}\,\mathrm{Harm}\,\pi\,\left(\eta\mathrm{bulk}\,\rho\mathrm{bulk}+\eta\mathrm{film}\,\rho\mathrm{film}\right)\right)\right.
$$

$$
\left.\left.\mathrm{Sinh}\left[4\,\mathrm{f1}\,\mathrm{Harm}\,\mathrm{hfilm}\,\pi\sqrt{-\frac{\rho\mathrm{film}}{2\,i\,\mathrm{f1}\,\mathrm{Harm}\,\pi\,\eta\mathrm{film}+\mu\mathrm{film}}}\right]\right)\right)\Big/
$$

$$
\left(2\,\rho\mathrm{bulk}\left(\frac{i\,\mathrm{f1}\,\mathrm{Harm}\,\rho\mathrm{film}\,\mathrm{Cos}\left[\frac{2\,\mathrm{f1}\,\mathrm{Harm}\,\mathrm{hfilm}\,\pi\sqrt{\rho\mathrm{film}}}{\sqrt{2\,i\,\mathrm{f1}\,\mathrm{Harm}\,\pi\,\eta\mathrm{film}+\mu\mathrm{film}}}\right]}{\sqrt{-\frac{\rho\mathrm{film}}{2\,i\,\mathrm{f1}\,\mathrm{Harm}\,\pi\,\eta\mathrm{film}+\mu\mathrm{film}}}}+\right.\right.
$$

$$
\left.\left(1+i\right)\sqrt{\pi}\,\sqrt{\mathrm{f1}^3\,\mathrm{Harm}^3\,\eta\mathrm{bulk}\,\rho\mathrm{bulk}}\right.
$$

$$
\left.\left.\left.\mathrm{Sinh}\left[2\,\mathrm{f1}\,\mathrm{Harm}\,\mathrm{hfilm}\,\pi\sqrt{-\frac{\rho\mathrm{film}}{2\,i\,\mathrm{f1}\,\mathrm{Harm}\,\pi\,\eta\mathrm{film}+\mu\mathrm{film}}}\right]\right)^2\right)\right)
$$

$$\left(\mathbf{f1}^2\,\mathbf{Harm}^2\,\pi\,\rho\mathbf{film}^2\right.$$

$$\left(-4\,\mathbf{f1}^2\,\mathbf{hfilm}\,\mathbf{Harm}^2\,\pi\,\mu\mathbf{film}\,\sqrt{2\,i\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}}\;\rho\mathbf{film}^{3/2}\,\rho\mathbf{bulk}+\right.$$

$$8\,i\,\mathbf{f1}^3\,\mathbf{hfilm}\,\mathbf{Harm}^3\,\pi^2\,\rho\mathbf{bulk}\left(-\eta\mathbf{film}\,\sqrt{2\,i\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}}\;\rho\mathbf{film}^{3/2}+\right.$$

$$\left.\eta\mathbf{bulk}\,\sqrt{\left(2\,i\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}\right)\rho\mathbf{film}}\;\rho\mathbf{bulk}\right)-\left(2-2\,i\right)\sqrt{\pi}\;\left(\mu\mathbf{film}\right.$$

$$\sqrt{\left(\mathbf{f1}^3\,\mathbf{Harm}^3\,\eta\mathbf{bulk}\left(2\,i\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}\right)\rho\mathbf{film}\,\rho\mathbf{bulk}^3\right)}+2\,i\,\pi\,\eta\mathbf{film}$$

$$\left.\sqrt{\left(\mathbf{f1}^5\,\mathbf{Harm}^5\,\eta\mathbf{bulk}\left(2\,i\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}\right)\rho\mathbf{film}\,\rho\mathbf{bulk}^3\right)}\right)+\left(2+2\,i\right)$$

$$\sqrt{\pi}\;\left(-i\,\mu\mathbf{film}\,\sqrt{\left(\mathbf{f1}^3\,\mathbf{Harm}^3\,\eta\mathbf{bulk}\left(2\,i\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}\right)\rho\mathbf{film}\,\rho\mathbf{bulk}^3\right)}+\right.$$

$$\left.2\,\pi\,\eta\mathbf{film}\,\sqrt{\left(\mathbf{f1}^5\,\mathbf{Harm}^5\,\eta\mathbf{bulk}\left(2\,i\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}\right)\rho\mathbf{film}\,\rho\mathbf{bulk}^3\right)}\right)$$

$$\mathbf{Cos}\left[\frac{4\,\mathbf{f1}\,\mathbf{hfilm}\,\mathbf{Harm}\,\pi\,\sqrt{\rho\mathbf{film}}}{\sqrt{2\,i\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}}}\right]-\mathbf{f1}\,\mathbf{Harm}\left(2\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}-i\,\mu\mathbf{film}\right)$$

$$\rho\mathbf{bulk}\left(-i\,\mu\mathbf{film}\,\rho\mathbf{film}+2\,\mathbf{f1}\,\mathbf{Harm}\,\pi\left(\eta\mathbf{film}\,\rho\mathbf{film}+\eta\mathbf{bulk}\,\rho\mathbf{bulk}\right)\right)$$

$$\left.\left.\mathbf{Sin}\left[\frac{4\,\mathbf{f1}\,\mathbf{hfilm}\,\mathbf{Harm}\,\pi}{\sqrt{\frac{2\,i\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}}{\rho\mathbf{film}}}}\right]\right)\right)\Big/$$

$$\left(2\left(2\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}-i\,\mu\mathbf{film}\right)^2\sqrt{\left(2\,i\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}\right)\rho\mathbf{film}}\right.$$

$$\rho\mathbf{bulk}$$

$$\left(i\,\mathbf{f1}\,\mathbf{Harm}\,\rho\mathbf{film}\,\mathbf{Cos}\left[\frac{2\,\mathbf{f1}\,\mathbf{hfilm}\,\mathbf{Harm}\,\pi\,\sqrt{\rho\mathbf{film}}}{\sqrt{2\,i\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}}}\right]+\right.$$

$$\left(1+i\right)\sqrt{\pi}\;\sqrt{-\frac{\mathbf{f1}^3\,\mathbf{Harm}^3\,\eta\mathbf{bulk}\,\rho\mathbf{film}\,\rho\mathbf{bulk}}{2\,i\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}}}$$

$$\left.\left.\mathbf{Sinh}\left[2\,\mathbf{f1}\,\mathbf{hfilm}\,\mathbf{Harm}\,\pi\,\sqrt{-\frac{\rho\mathbf{film}}{2\,i\,\mathbf{f1}\,\mathbf{Harm}\,\pi\,\eta\mathbf{film}+\mu\mathbf{film}}}\right]\right)^2\right)$$

(*Derivative of the beta function with respect to film thickness $\partial\beta/\partial$hfilm*)
FullSimplify[D[β[μfilm, ηfilm, ρfilm, hfilm, ηbulk, ρbulk], hfilm]]

$$\left(2\,\text{f1}^3\,\text{Harm}^3\,\pi\,\rho\text{film}\left(-\mathrm{i}\,\mu\text{film}\,\rho\text{film}+2\,\text{f1}\,\text{Harm}\,\pi\left(-\eta\text{bulk}\,\rho\text{bulk}+\eta\text{film}\,\rho\text{film}\right)\right)\right)\Big/$$

$$\left(\frac{\mathrm{i}\,\text{f1}\,\text{Harm}\,\rho\text{film}\,\text{Cos}\left[\frac{2\,\text{f1}\,\text{Harm}\,\text{hfilm}\,\pi\sqrt{\rho\text{film}}}{\sqrt{2\,\mathrm{i}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]}{\sqrt{-\frac{\rho\text{film}}{2\,\mathrm{i}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}}+\left(1+\mathrm{i}\right)\sqrt{\pi}\,\sqrt{\text{f1}^3\,\text{Harm}^3\,\eta\text{bulk}\,\rho\text{bulk}}\right.$$

$$\left.\text{Sinh}\left[2\,\text{f1}\,\text{Harm}\,\text{hfilm}\,\pi\sqrt{-\frac{\rho\text{film}}{2\,\mathrm{i}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]\right)^2$$

$$\left(\mathbf{2\,f1^3\,Harm^3\,\pi\,\rho film\left(-i\,\mu film\,\rho film+2\,f1\,Harm\,\pi\left(\eta film\,\rho film-\eta bulk\,\rho bulk\right)\right)}\right)\Big/$$

$$\left(\frac{\mathbf{i\,f1\,Harm\,\rho film\,Cos\left[\frac{2\,f1\,hfilm\,Harm\,\pi\sqrt{\rho film}}{\sqrt{2\,i\,f1\,Harm\,\pi\,\eta film+\mu film}}\right]}}{\sqrt{-\frac{\rho film}{2\,i\,f1\,Harm\,\pi\,\eta film+\mu film}}}+\mathbf{\left(1+i\right)\sqrt{\pi}\,\sqrt{f1^3\,Harm^3\,\eta bulk\,\rho bulk}}\right.$$

$$\left.\mathbf{Sinh\left[2\,f1\,hfilm\,Harm\,\pi\sqrt{-\frac{\rho film}{2\,i\,f1\,Harm\,\pi\,\eta film+\mu film}}\right]}\right)^2$$

(*Derivative of the beta function with respect to fluid viscosity $\partial\beta/\partial\eta$bulk*)
FullSimplify[D[β[μfilm, ηfilm, ρfilm, hfilm, ηbulk, ρbulk], ηbulk]]

$$-\left(\left(\left(\frac{1}{2}+\frac{\mathrm{i}}{2}\right)\text{f1}\,\text{Harm}\sqrt{\pi}\,\left(2\,\mathrm{i}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}\right)\rho\text{bulk}\sqrt{\text{f1}\,\text{Harm}\,\eta\text{bulk}\,\rho\text{bulk}}\right.\right.$$

$$\left.\rho\text{film}\right)\Big/\left(\sqrt{\left(\text{f1}\,\text{Harm}\,\eta\text{bulk}\left(2\,\mathrm{i}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}\right)\rho\text{bulk}\,\rho\text{film}\right)}\right.$$

$$\text{Cos}\left[\frac{2\,\text{f1}\,\text{Harm}\,\text{hfilm}\,\pi\sqrt{\rho\text{film}}}{\sqrt{2\,\mathrm{i}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]-$$

$$\left.\left.\left(1-\mathrm{i}\right)\text{f1}\,\text{Harm}\sqrt{\pi}\,\eta\text{bulk}\,\rho\text{bulk}\,\text{Sin}\left[\frac{2\,\text{f1}\,\text{Harm}\,\text{hfilm}\,\pi\sqrt{\rho\text{film}}}{\sqrt{2\,\mathrm{i}\,\text{f1}\,\text{Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]\right)^2\right)$$

$$-\left(\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{\pi}\,\sqrt{\frac{\text{f1 Harm}\,\rho\text{bulk}}{\eta\text{bulk}}}\right)\Big/\left(\text{Cos}\left[\frac{2\,\text{f1 hfilm Harm}\,\pi\,\sqrt{\rho\text{film}}}{\sqrt{2\,i\,\text{f1 Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]-\left((1-i)\,\text{f1 Harm}\,\sqrt{\pi}\,\eta\text{bulk}\,\rho\text{bulk}\,\text{Sin}\left[\frac{2\,\text{f1 hfilm Harm}\,\pi\,\sqrt{\rho\text{film}}}{\sqrt{2\,i\,\text{f1 Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]\right)\Big/\left(\sqrt{\left(\text{f1 Harm}\,\eta\text{bulk}\,\left(2\,i\,\text{f1 Harm}\,\pi\,\eta\text{film}+\mu\text{film}\right)\,\rho\text{film}\,\rho\text{bulk}\right)}\right)^2\right)\right)$$

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(*Derivative of the beta function with respect to fluid density ∂β/∂ρbulk*)
FullSimplify[D[β[μfilm, ηfilm, ρfilm, hfilm, ηbulk, ρbulk], ρbulk]]
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$$-\left(\left(i\,\text{f1 Harm}\,\sqrt{\frac{\pi}{2}}\,\rho\text{film}^2\right)\Big/\left(\sqrt{\frac{i\,\text{f1 Harm}\,\rho\text{bulk}}{\eta\text{bulk}}}\left(\rho\text{film}\,\text{Cos}\left[\frac{2\,\text{f1 Harm}\,\text{hfilm}\,\pi}{\sqrt{\frac{2\,i\,\text{f1 Harm}\,\pi\,\eta\text{film}+\mu\text{film}}{\rho\text{film}}}}\right]+\left(\sqrt{2\,\pi}\,\sqrt{\text{f1 Harm}\,\eta\text{bulk}\,\rho\text{bulk}\,\rho\text{film}}\,\text{Sinh}\left[\frac{(1+i)\,\text{f1 Harm}\,\text{hfilm}\,\pi\,\rho\text{film}}{\sqrt{\text{f1 Harm}\,\pi\,\eta\text{film}\,\rho\text{film}-\frac{i\,\mu\text{film}\,\rho\text{film}}{2}}}\right]\Big/\left(\sqrt{2\,\text{f1 Harm}\,\pi\,\eta\text{film}-i\,\mu\text{film}}\right)\right)^2\right)\right)\right)$$

$$-\left(\left(i\,\text{f1 Harm}\,\sqrt{\frac{\pi}{2}}\right)\Big/\left(\sqrt{\frac{i\,\text{f1 Harm}\,\rho\text{bulk}}{\eta\text{bulk}}}\left(\text{Cos}\left[\frac{2\,\text{f1 hfilm Harm}\,\pi\,\sqrt{\rho\text{film}}}{\sqrt{2\,i\,\text{f1 Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]-\left((1-i)\,\text{f1 Harm}\,\sqrt{\pi}\,\eta\text{bulk}\,\rho\text{bulk}\,\text{Sin}\left[\frac{2\,\text{f1 hfilm Harm}\,\pi\,\sqrt{\rho\text{film}}}{\sqrt{2\,i\,\text{f1 Harm}\,\pi\,\eta\text{film}+\mu\text{film}}}\right]\right)\Big/\left(\sqrt{\left(\text{f1 Harm}\,\eta\text{bulk}\,\left(2\,i\,\text{f1 Harm}\,\pi\,\eta\text{film}+\mu\text{film}\right)\,\rho\text{film}\,\rho\text{bulk}\right)}\right)^2\right)\right)\right)$$

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(*Christopher Blanford August 2015*)
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