STATISTICAL ANALYSIS OF LIFETIME DATA USING NEW MODIFIED WEIBULL DISTRIBUTIONS

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The University of Manchester

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Doctor of Philosophy
Statistical Analysis of Lifetime Data Using New Modified Weibull Distributions
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The Weibull distribution is a popular and widely used distribution in reliability and in lifetime data analysis. Since 1958, the Weibull distribution has been modified by many researchers to allow for non-monotonic hazard functions. Many modifications of the Weibull distribution have achieved the above purpose. On the other hand, the number of parameters has increased, the forms of the survival and hazard functions have become more complicated and the estimation problems have risen.

This thesis provides an extensive review of some discrete and continuous versions of the modifications of the Weibull distribution, which could serve as an important reference and encourage further modifications of the Weibull distribution. Four different modifications of the Weibull distribution are proposed to address some of the above problems using different techniques. First model, with five parameters, is constructed by considering a two-component serial system with one component following a Weibull distribution and another following a modified Weibull distribution. A new method has been proposed to reduce the number of parameters of the new modified Weibull distribution from five to three parameters to simplify the distribution and address the estimation problems. The reduced version has the same desirable properties of the original distribution in spite of having two less parameters. It can be an alternative distribution for all modifications of the Weibull distribution with bathtub shaped hazard rate functions. To deal with unimodal shaped hazard rates, the third model with four parameters, named as the exponentiated reduced modified Weibull distribution is introduced. This model is flexible, has a nice physical interpretation and has the ability to capture monotonically increasing, unimodal and bathtub shaped hazard rates. It is a generalization of the reduced modified Weibull distribution. The proposed distribution gives the best fit comparing to other modifications of the Weibull distribution including those having similar properties. A three-parameter discrete distribution is introduced based on the reduced distribution. It is one of only three discrete distributions allowing for bathtub shaped hazard rate functions. Four real data sets have applied to this distribution. The new distribution is shown to outperform at least three other models including the ones allowing for bathtub shaped hazard rate functions.

The new models show flexibility and can be used to model different kinds of real data sets better than other modified versions of Weibull distribution including those having the same number of parameters. The mathematical properties and statistical inferences of the new models are studied. Based on a simulation study the performances of the MLEs of each model are assessed with respect to sample size n.

We find no evidence that the generalized modified Weibull distribution can provide a better fit than the exponentiated Weibull distribution for data sets exhibiting the modified unimodal hazard function.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

Notation

Symbol	**Description**
$F(\cdot)$	The cumulative distribution function
$f(\cdot)$	The probability density function
$p(\cdot)$	The probability mass function
$S(\cdot)$	The survival function
$h(\cdot)$	The hazard rate function
$H(\cdot)$	The cumulative hazard function
$\hat{F}(\cdot)$	The empirical distribution function
$\mathrm{int}\{\cdot\}$	The integer part
$\Gamma(\cdot)$	Gamma function
$\gamma(\cdot, x)$	Incomplete gamma function
$\Gamma_{\delta}(\cdot, \cdot)$	Generalized gamma function
$\Gamma_{x,\delta}(\cdot, \cdot)$	Incomplete generalized gamma function
$B(\cdot, \cdot)$	Beta function
$I_x(\cdot, \cdot)$	Incomplete beta function
$Q(\cdot)$	The quintile function
$F_w(\cdot)$	The CDF of the traditional Weibull distribution
$S_W(\cdot)$	The survival function of the Weibull distribution
$h_W(\cdot)$	The hazard function of the Weibull distribution
$S_{MW}(\cdot)$	The survival function of the modified Weibull distribution
$h_{MW}(\cdot)$	The hazard function of the modified Weibull distribution
μ_r'	The $r-th$ moment of the new modified Weibull distribution
$M_X(\cdot)$	The moment generating function
$f_{r:n}(x)$	The pdf of the rth order statistic $X_{(r)}$
$\mu_k^{\prime(r:n)}$	The kth moment of the rth order statistic $X_{(r)}$

$L(\cdot)$	The likelihood function
$\mathcal{L}(\cdot)$	The log-likelihood function
$H\,(p)$	Renyi entropy
I^{-1}	The variance- covariance matrix
ω	The log-likelihood ratio statistic

Abbreviations

CDF	Cumulative hazard function
PDF	Probability density function
HF	Hazard function
PMF	Probability mass functions
SF	Survival function
MLE	Maximum likelihood estimate
DW(I)	Type I discrete Weibull distribution
DW(II)	Type II discrete Weibull distribution
DW(III)	Type III discrete Weibull distribution
DIW	Discrete Inverse Weibull distribution
DMW	Discrete modified Weibull distribution
DAddW	Discrete additive Weibull distribution
IW	Inverse Weibull distribution
KumIW	Kumaraswamy inverse Weibull distribution
BIW	Beta inverse Weibull distribution
LogW	Log-Weibull distribution
RefW	Reflected Weibull distribution
KiesMW	Kies's modified Weibull distribution
PhMW	Phani's modified Weibull distribution
EW	Exponentiated Weibull distribution
EP	Exponentiated Pareto distribution
GR	Generalized Rayleigh distribution
EGam	Exponentiated gamma distribution
EGum	Exponentiated Gumbel distribution
EF	Eponentiated Fréchet distribution

GLFR	Generalized linear failure rate distribution
GG	Generalized Gompertz distribution
EGLE	Exponentiated generalized linear exponential distribution
GE	Generalized exponential distribution
GW	Generalized Weibull distribution
AddW	Additive Weibull distribution
SZMW	Sarhan and Zaindin's modified Weibull distribution
ExW	Extended Weibull distribution
PL	Power Lindley distribution
MW	Modified Weibull distribution
GMW	Generalized modified Weibull distribution
BMW	Beta modified Weibull distribution
MCMC	Markov chain Monte Carlo
GPW	Generalized power Weibull
MWEx	Modified Weibull extension
EMWEx	Exponentiated modified Weibull extension
BGW	Beta generalized Weibull distribution
BW	Beta-Weibull distribution
BEW	Beta exponentiated Weibull
BGE	Beta generalized Exponential
OddW	Odd Weibull distribution
FlxWE	Flexible Weibull extension
KumW	Kumaraswamy Weibull distribution
KumE	Kumaraswamy exponential distribution
KumR	Kumaraswamy Rayleigh distribution
EKum	Exponentiated Kumaraswamy Weibull distribution
KumMW	Kumaraswamy modified Weibull distribution
NMW	New Modified Weibull Distribution
ERMW	Exponentiated reduced modified Weibull distribution
DRMW	Discrete modified Weibull distribution
LFR	The linear failure rate distribution
AIC	Akaike information criterion

BIC	Bayesian information criterion
CAIC	Consistent Akaike information criterion
K-S	Kolmogorov-Smirnov statistic
TTT	Total Time on Test
LRT	Likelihood ratio statistic
RNMW	Reduced New Modified Weibull Distribution
MLE	Maximum likelihood estimates
TBF	Time between failures
NDMW	A new discrete modified Weibull distribution
DGD	The discrete gamma distribution
DND	The discrete normal distribution
DRD	The discrete Rayleigh distribution
DBD	The discrete Burr distribution
DPD	The discrete Pareto distribution
SDD	A finite range simple discrete life distribution
DB3	The discrete Burr type III
DLD	The discrete Lindley distribution
RGMW	Reduced generalized modified Weibull distribution
REW	Reduced exponentiated Weibull distribution
RMW	Reduced Lai et al.'s modified Weibull distribution
RW	Reduced Weibull distribution
MSE	Mean squared error
CP	Coverage probability
CL	Coverage length
WPP	Weibull Probability Plot
IWPP	Inverse Weibull Probability Plot

Dedication

Exclusively dedicated to my brother, Fahad Almalki, who passed away in a car crash on the morning of Thursday the 29^{th} of September 2011 in Jeddah, Saudi Arabia .

"May Allah forgive and have mercy upon him, excuse him and pardon him, and make honorable his reception."

It was the saddest day of my life, and no words can describe my feeling when I lost my brother. He was not just my brother, but my dearest, most loyal, respectful, trustworthy, and honest friend. My head and my heart will never forget him; everything around me, and all inside of me, reminds me of him.

It is difficult, and may be impossible, to combine a time of such great happiness with memories such great sadness, but I would like to do so. I would like to apologise to the many people around me who I should also dedicate this work to. You deserve that and more, but I am so sorry, it must be dedicated to my brother. My father, my mother, my wife, my eldest daughter Juri and my youngest daughter Joud, my eldest son Abdulrahman and my youngest son Muhammad, my sisters, my brothers, my relatives and my friends; I thank you so much for your support, but this work must be dedicated to my brother.

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Chapter 1

Introduction and Background

1.1 Introduction

After an extensive review of modifications to the Weibull distribution, this thesis contributes four new modifications to the Weibull distribution that can be used to analyse different kinds of lifetime data sets with different shapes of hazard rate. This chapter is a prefatory chapter to the thesis. Section 1.2 provides some historical and background information of lifetime distributions, the hazard function (HF) and their applications. The motivations of this thesis are presented in Section 1.3 and the organisation of the thesis is described in Section 1.4. Large parts of this thesis have been published in or submitted to journals for publication and some have been presented in conferences, these are presented in Section 1.5.

1.2 Historical and Background Information

Statistical lifetime distributions (sometimes called ageing distributions) are widely used in many different fields for modelling data sets. They have been applied to many areas: reliability engineering such as machine life cycles, medical sciences such as the survival times of patients after surgery or duration to recurrence of a kind of cancer after surgical removal, computer sciences such as the failure rates of a software system, insurance such as the durations without claims of customers policies, marketing such as the lifetimes of customers, and social sciences such as duration of marriage till divorce or duration that a graduate remains unemployed, among others. For more details about the above lifetime applications we refer to Lai and Xie (2006), Pham (2006), Ohishi *et al* (2009), Lai (2012) and Bemmaor and Glady (2012).

If X is a lifetime random variable with $f(x)$ being its probability density function (PDF), $F(x)$ being its cumulative distribution function (CDF), and $S(x)$ being its survival function (SF), then the hazard function (HF) $h(x)$ is defined as the ratio of $f(x)$ to $S(x)$

$$h(x) = \frac{f(x)}{S(x)}, \tag{1.1}$$

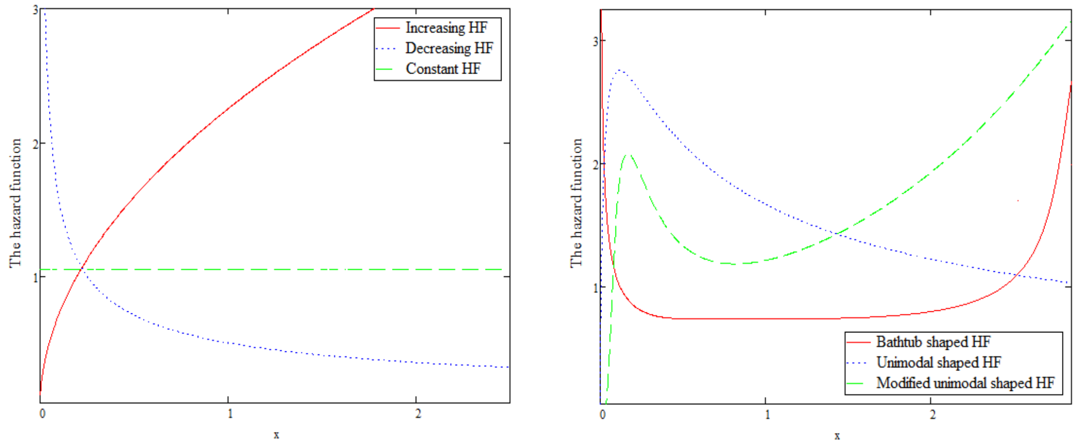
and $h(x)\triangle x$ represents the approximate probability of failure in the interval $[x, x+\triangle x)$, cf. Lawless (1982).

The hazard rate function plays a fundamental role in lifetime modelling. A lifetime distribution is said to have an increasing hazard rate if its HF $h(x)$ is monotonically increasing over time and to have a decreasing hazard rate if $h(x)$ is monotonically decreasing. If $h(x)$ initially decreases, followed by an approximately constant period (called useful life period), then followed by an increasing period, the distribution is said to have a bathtub shape. The distribution is said to have a unimodal hazard rate if its hazard rate function has a unique mode, also called an upside-down bathtub shape. The different shapes of the hazard rate function can be investigated using the first derivative of the hazard function as follows. The shape of the HF can be:

- Monotonically increasing (non-decreasing) shape if the values of the first derivative of $h(x)$ respect to x is positive for all values of x.
- Monotonically decreasing (non-increasing) shape if the values of the first derivative of $h(x)$ respect to x is negative for all values of x.
- Constant if the values of the first derivative of $h(x)$ respect to x equals 0.
- Bathtub shaped if the values of the first derivative of $h(x)$ respect to x are negative for $x \in (0, x_0)$ and positive for $x > x_0$ and the value x_0 is a unique and positive solution of $h'(x_0) = 0$.
- Unimodal (upside-down bathtub) shaped if $h'(x) > 0$ for $x \in (0, x_0)$ and $h'(x) < 0$ for $x > x_0$ and the value x_0 is a unique and positive solution of $h'(x_0) = 0$

- Unimodal followed by increasing (modified unimodal (bathtub) shaped) if the HF first has a unimodal and then followed by increasing.

  Figure 1.1 shows the monotonic shapes of the HF (left side) and some non-monotonic shapes (right side)



Monotone HFs Non-monotonic HFs

Figure 1.1: Monotonic and non-monotonic hazard rate functions

The most popular lifetime distributions including the exponential, Weibull, gamma, Rayleigh, Pareto and Gompertz have monotonic hazard rate functions, cf. Lawless (1982). However, certain lifetime data (for example, human mortality, machine life cycles and data from some biological and medical studies) require non-monotonic shapes like the bathtub shape, the unimodal (upside-down bathtub) or modified unimodal shape.

Human mortality is shown to have a bathtub shaped HF. Initially, the hazard rate of the death (newborn babies) is very high level specially in the first six months after birth, that is caused by deformities, heart dysfunctions or other infant diseases. Then, the risk of level of death decreases rapidly until it reaches its lowest level and remains approximately constant. At some point, during the ages between 30 and 40 the death risk increases over time. The first period is called infant mortality, the second is called useful life period (normal life period) and the third period is a wear out period. This can be interpreted as the fate of the individual person during his life or it can be interpreted as the hazard rates of three groups, first group includes the infants (about 0-6 months) with decreasing hazard rate, second group with approximately constant hazard rate contains people below the ages between 30 and 40 and the third group with increasing hazard rate for the people above the ages between 30 and 40. The bathtub shape can also be seen in the failures times of some electrical products, where the early failures are caused by some manufactory defects, handling or types of storage. The useful life period, when the products are used, have relatively constant failure rate, and is the longest period among the three. At the end of the product life the wear out period appears with increasing failure rate over time.

On the other hand, when the main reasons of the failures of products are caused by fatigue and corrosion, the failure rates of those products will exhibit unimodal shapes, cf. Lai and Xie (2006). In some medical situations, for example breast cancer, the hazard rate has been shown to be unimodal or modified unimodal shape. The hazard rate for breast cancer recurrence after surgical removal has been observed to have unimodal shape, cf. Demicheli *et al* (2004). Initially, the hazard rate for breast cancer recurrence begins with a low level, then increases gradually after a finite period time after the surgical removal until reaching a peak before then decreasing. Another example of the unimodal shape is the hazard of infection with some new viruses, where it increases in the early viruses' ages from low level till it reaches a peak and then decreases.

The hazard rate of death of breast cancer patients represents a modified unimodal shape (unimodal shape followed by increasing). It has a modified unimodal shape with three phases, first increasing, then decreasing, then again increasing, which has nice and useful interpretations, cf. Zajicek (2011). It can be interpreted as a description of three groups of patients, first group is represented by the first phase that contains the weak patients who die within the first three years, so the hazard rate of this group is increasing, while the second phase represents the group with strong patients, their bodies have became familiar with the disease and they are getting better. The hazard rate of death of these patients is decreasing. In the third phase they become weaker and their ability to cope with the disease declines, then the hazard rate of death increases. Also, that can be interpreted in the case of the statuses of patient with the disease. At first the patient has been surprised and dose not recognize the appropriate ways to deal and cope with the disease, as a result the hazard rate will be increasing. After while, the patient becomes more knowledgeable and her ability to cope the disease increases, so the hazard rate of death decreases. Finally, as the age of the patient increases, the hazard rate increases again over the time, see Zajicek (2011).

The Weibull distribution is one of the most important, desirable and widely used lifetime distributions. It has been used in many different fields with many applications. The CDF of the Weibull distribution is simple and has a closed form which gives a simple expression of its survival and hazard functions. It is a flexible distribution that can be used to fit different kinds of of lifetime data sets in different fields. Moreover, it has a physical meaning and interpretations of its parameters.

The two-parameter Weibull distribution is specified by the cumulative distribution function CDF

$$F(x) = 1 - \exp\left(-\alpha x^{\theta}\right), \; x > 0, \tag{1.2}$$

where $\alpha > 0$ and $\theta > 0$ are the scale and shape parameters, respectively. The corresponding probability density function PDF is

$$f(x) = \alpha\theta x^{\theta-1} \exp\left(-\alpha x^{\theta}\right), \; x > 0. \tag{1.3}$$

The corresponding hazard function HF is

$$h(x) = \alpha\theta x^{\theta-1}, \; x > 0, \tag{1.4}$$

which can be increasing, decreasing or constant depending on $\theta > 1$, $\theta < 1$ or $\theta = 1$. Unfortunately, it dose not exhibit any kind of non-monotonic hazard rate shape.

For many years, using different techniques, many researchers have developed various modified forms of the Weibull distribution to achieve non-monotonic shapes. The two-parameter flexible Weibull extension of Bebbington *et al* (2007) has a hazard function that can be increasing, decreasing or bathtub shaped. Zhang and Xie (2011) studied the characteristics and application of the truncated Weibull distribution which has a bathtub shaped hazard function. A three-parameter model, called the exponentiated Weibull distribution, was introduced by Mudholkar and Srivastave (1993). Another three-parameter model is by Marshall and Olkin (1997) and called the extended Weibull distribution. Xie *et al* (2002) proposed a three-parameter modified Weibull extension with a bathtub shaped hazard function. The modified Weibull (MW) distribution of Lai *et al* (2003) multiplies the Weibull cumulative hazard function αx^{θ} by $e^{\lambda x}$, which was later generalized to exponentiated form by Carrasco *et al* (2008).

Recent studies of the modified Weibull include Jiang *et al* (2010), Soliman *et al* (2012) and Upadhyaya and Gupta (2010). Among the four-parameter distributions, the additive Weibull distribution (AddW) of Xie and Lai (1996) with cumulative distribution function CDF

$$F(x) = 1 - e^{-\alpha x^{\theta} - \beta x^{\gamma}},\ x \geq 0,$$

has a bathtub-shaped hazard function consisting of two Weibull hazards, one increasing ($\theta > 1$) and one decreasing ($0 < \gamma < 1$). The modified Weibull distribution of Sarhan and Zaindin (SZMW) (2009) can be derived from the additive Weibull distribution by setting $\theta = 1$. A four-parameter beta Weibull distribution was proposed by Famoye *et al* (2005). Five-parameter modified Weibull distributions include Phani's modified Weibull (1987) which generalizes the four-parameter Weibull distribution which was proposed by Kies (1958), the Kumaraswamy Weibull by Cordeiro *et al* (2010) and the beta modified Weibull (BMW) introduced by Silva *et al* (2010) and further studied by Nadarajah *et al* (2011). The latest examples include the beta generalized Weibull distribution by Singla, *et al* (2012), exponentiated generalized linear exponential distribution by Sarhan *et al* (2013), the generalized Gomprtz distribution by El-Gohary *et al* (2013) and the exponentiated modified weibull extension distribution by Sarhan and Apaloo (2013) which exhibits a bathtub-shaped pattern.

It has been pointed out by Nadrajah and Kotz (2005) that some of the modified Weibull distributions can be obtained as special cases of Gurvich *et al* (1997)'s form

$$F(x) = 1 - \exp\left(-H(x)\right),\ x \geq 0, \tag{1.5}$$

where $H(x)$ is a monotonically increasing function of x. This is true for any lifetime distribution F as $H(x) = -\log(1 - F(x))$ will be the cumulative hazard function.

Although some flexible distributions exist among these modifications with only two or three parameters, such as the flexible Weibull extension of Bebbington *et al* (2007), the MW distribution of Lai *et al* (2003) and the modified Weibull extension of Xie *et al* (2002), the effective and flexible modified Weibull distributions have four or five parameters. In contrast, although the distributions with four parameters or more that exhibit a bathtub shaped hazard rate are useful and flexible, they are also complex Nelson (1990) and cause estimation problems as a consequence of the number of parameters, especially when the sample size is not large.

If $G(x)$ is a CDF, then a distribution with CDF, $F(x) = G(x)^{\theta}$, is called an exponentiated distribution. Using this exponentiation, several lifetime distributions have been generalized. Kundu and Gupta (1999) generalized the standard exponential distribution to the generalized exponential (GE) distribution with CDF

$$F(x;\alpha,\theta) = \left(1 - e^{-\alpha x}\right)^{\theta}$$

for $x > 0$, where $\alpha > 0$ and $\theta > 0$. This distribution has been extensively studied by several authors, see, for example, Gupta and Kundu (2001a, 2001b), Raqab (2004), Gupta and Kundu (2007), and Kundu and Gupta (2008). Mudholkar and Srivastava (1993) generalized the Weibull distribution to the exponentiated Weibull (EW) distribution. The CDF of the EW distribution is

$$F(x;\alpha,\gamma,\theta) = \left(1 - e^{-\alpha x^{\gamma}}\right)^{\theta} \tag{1.6}$$

for $x > 0$, where $\alpha > 0$, $\gamma > 0$ and $\theta > 0$. The exponentiated Pareto (EP) distribution was proposed by Gupta *et al.* (1998), while the generalized Rayleigh (GR) distribution was studied by Surles and Padgett (2001, 2005) and Kundu and Raqab (2005). The CDF of the GR distribution is

$$F(x;\alpha,\theta) = \left(1 - e^{-\alpha x^{2}}\right)^{\theta}$$

for $x > 0$, where $\alpha > 0$ and $\theta > 0$.

Using the same technique, Nadarajah and Kotz (2006b) introduced the exponentiated gamma (EGam) distribution, the exponentiated Gumbel (EGum) distribution and the exponentiated Fréchet (EF) distribution. The generalized modified Weibull (GMW) distribution of Carrasco *et al.* (2008) is an exponentiated distribution. Sarhan and Kundu (2009) introduced a three-parameter distribution called the generalized linear failure rate (GLFR) distribution with CDF

$$F(x) = \left(1 - e^{-\alpha x - \beta x^{2}}\right)^{\theta}$$

for $x > 0$, where $\alpha > 0$, $\beta > 0$ and $\theta > 0$.

Recently, different versions of exponentiated distributions have been introduced by several authors: Sarhan and Apaloo (2013), El-Gohary *et al.* (2013) and Sarhan *et al.* (2013) introduced the exponentiated modified Weibull extension (EMWEx) distribution, the generalized Gompertz (GG) distribution and the exponentiated generalized linear exponential (EGLE) distribution, respectively.

The unimodal or modified unimodal hazard rate function has many applications in reliability and survival analysis as described earlier. The HFs of the GMW and EW distributions can exhibit unimodal shapes. There are other distributions exhibiting unimodal HFs: the inverse Weibull (IW) distribution, Kumaraswamy inverse Weibull (KumIW) distribution due to Shahbaz *et al.* (2012), the beta inverse Weibull (BIW) distribution due to Hanook *et al.* (2013), the generalized Weibull (GW) distribution due to Mudholkar and Kollia (1994), the extended Weibull distribution due to Marshall and Olkin (1997), and the beta Weibull distribution due to Famoye *et al.* (2005) among others.

In reliability and lifetime analysis, when the failures (lifetimes) are truly discrete, continuous models may not be appropriate. Discrete models will be consistent with such data. Example of discrete data include: the number of rounds fired by a weapon till the first failure; the number of deaths at a given place over a given period; the number of cycles prior to the first failure when devices work in cycles; the number of periods successfully completed without failure when devices are observed just once a week, a month or a year. Sometimes continuous data are grouped as discrete because the amount of data is huge or the individual observations are either unknown or their ranges are more important than the values. During the last few decades, a large number of continuous lifetime distributions has been proposed and some of them have been extensively studied and modified. On the other hand, although the number of discrete distributions has increased slightly during the last a few years, more studies are needed in this area.

It is well known that the geometric distribution is the discrete analogue of the exponential distribution, while the negative binomial distribution is the discrete analogue of the gamma distribution. Nakagawa and Osaki (1975), Stein and Dattero (1984) and Padgett and Spurrier (1985) proposed three different discrete versions of the Weibull distribution which were further studied by Khan *et al* (1989) and Kulasekera (1994). A two-parameter discrete gamma distribution (DGD) was introduced by Yang (1994). Chakraborty and Chakravarty (2012) recently discussed its parameter estimation using different methods. Roy (2003) and Roy(2004) proposed the discrete normal distribution (DND) and the discrete Rayleigh distribution (DRD). The discrete Burr distribution (DBD) and the discrete Pareto distribution (DPD) were proposed by

Krishna and Singh (2009). Lai and Wang (1995) introduced a finite range simple discrete life distribution (SDD). Jazi *et al.* (2010) discretized the inverse Weibull (DIW) distribution while Gomez-Deniz *et al.* (2011) introduced a new discrete distribution with actuarial applications. A discrete analogue of the continuous modified Weibull distribution of Lai *et al* (2003) (DMW) was introduced by Noughabi *et al.* (2011). Recently, Bebbington *et al.* (2012), Al-Huniti and Al-Dayian (2012) and Bakouch *et al.* (2014) introduced the discrete additive Weibull (DAddW), discrete Burr type III (DB3) and discrete Lindley (DLD) distributions, respectively.

1.3 Research Motivations

The main purpose of the modification and extension forms of the Weibull distribution is to describe and fit the data sets with non-monotonic hazard rate, such as the bathtub, unimodal and modified unimodal hazard rate. Many modifications of the Weibull distribution have achieved the above purpose. On the other hand, unfortunately, the number of parameters has increased, the forms of the survival and hazard functions have been complicated and estimation problems have risen. Moreover, some of the modifications do not have a closed form for their CDFs. As we have seen, the bathtub and the modified unimodal shapes have three phases: initially decreasing phase, relatively constant phase and then an increasing phase for the bathtub shape and the phases of the modified unimodal shape are initially increasing, then decreasing, then increasing again. The main weakness of some modified Weibull distributions is that they are unable to fit the last phase of the bathtub and the modified unimodal shapes, which are essential parts, as well as the first and middle phases.

Extensive reviews of some of these modifications have been presented, for example, see Rajarshi and Rajarshi (1988) and Murthy *et al* (2003). Pham and Lai (2007) and Lai *et al* (2011) introduced a brief review about the Weibull models. Most of the modifications of the Weibull distribution (both continuous and discrete) have been introduced in the last five years or so. In contract, the most recent intensive review was introduced before more than ten years. Also, most of them focused on the continuous distribution more than the discrete ones. So, it is timely that a review is written of the known modifications of the Weibull distribution. Chapter 2 provides an extensive review of the continuous and discrete modifications of the Weibull distributions. This Chapter could serve as an important reference and encourage further modifications of the Weibull distribution (and could be a useful reference complementing the book of Murthy *et al* (2003)).

To address some of the previous problems that appeared with some modifications of the Weibull, a new modification of the Weibull is provided in Chapter 3. The new modification is based on the Weibull distribution and one of the most interesting of its modifications. The new model is applied to two of the well known, popular and widely used lifetime data sets. Usually, most researchers compared their new modifications with its sub-models that have fewer parameters including the Weibull distribution. As an expected result, the new modification provides a better fit than its sub-models. They did not consider, or ignored, some other modifications that have the same number of parameters, have some similar properties or those which have the same purpose and are flexible enough to provide a good fit for this kind of data sets. The author believes, that is unfair to compare a distribution with its sub-models that have fewer parameters, it is just like when a football team with 11 players plays against its incomplete reserve team with 9 players or less. The proposed distribution in Chapter 3 will compare with its sub-models, other modifications of the Weibull distribution and with one of the best modifications of the Weibull that has the same number of parameters.

The majority of researches tend to increase the number of parameters to provide a better fit without considering the complexity or the problems of estimation. Rarely, researches tried to reduce the number of the parameters while maintaining the flexibility and the ability to fit data so well. The main purpose of Chapter 4 is to simplify the proposed model, that was introduced in the previous chapter, by reducing the number of parameters to address these problems while maintaining mach of the same flexibility and ability to fit data so well.

To cater to the need to model unimodal and modified unimodal shaped hazard rate lifetime data and to address the inability of some modifications of the Weibull to fit the third phase of the modified unimodal well, Chapter 5 introduces a four-parameter generalization distribution that can accommodate monotonically increasing, unimodal, modified unimodal and bathtub shaped hazard rate functions. The new distribution will be compared to other distributions including those having four parameters.

Many continuous Weibull distributions with bathtub shaped hazard rate functions have been introduced and studied. However, only a few discrete distributions have bathtub shaped hazard rate functions are introduced. The ones we are aware of are the discrete modified Weibull and the discrete additive Weibull distributions. This is the main motivation for Chapter 6 to introduce a new discrete distribution allowing for bathtub shaped hazard rate functions.

In recent years, many modifications of the Weibull distribution have been proposed. The author believes that there are some modified Weibull distributions with a small number of parameters which have not received the attention they deserve. Also, there are some modified Weibull distributions with a large number of parameters which need to be revalued with respect to what they really contribute. Some of these modifications have a large number of parameters and so their real benefits over simpler modifications are questionable. Chapter 7 shows that by considering the generalized modified Weibull distribution of Carrasco *et al.* (2008), a widely cited paper, and the exponentiated Weibull (EW) distribution of Mudholkar *et al.* (1995, 1996).

1.4 Organization of the thesis

This thesis contains eight chapters and proposes four new models, related to the Weibull distribution and its modifications, to improve the fitting of different kinds of lifetime data sets and to avoid some problems of some other modifications. This thesis is organized as the following:

- An intensive review of discrete and continuous versions of the modifications of the Weibull distribution is given in Chapter 2. This review includes the latest modifications that have been proposed in the last five years. The first part reviews continuous modifications of the Weibull distribution, including their probability density and hazard functions. The second part reviews discrete analogues of the Weibull distribution and their modifications, the probability mass and the hazard functions of those distributions are presented and their shapes are presented. Estimation methods, graphical tools for data analysis and goodness of fit are discussed.

- Chapter 3 introduces a new modification of the Weibull distribution, to be known as new modified Weibull distribution (NMW). The sub-models of this new distribution are presented and the shapes of its probability density and hazard rate functions are investigated. Two different methods to simulate a random sample from this distribution are presented. Some of the mathematical properties of the NMW are studied including moments, moment generating function and order statistics. The estimation of its unknown parameters by maximum likelihood is discussed. The new modified Weibull distribution is applied to several real data sets and compared with some other modified Weibull distributions.

- Chapter 4 proposes a new simple method to reduce the number of parameters of the new modified Weibull of Chapter 3 from five to three parameters. The resulting distribution is to called reduced new modified Weibull distribution (RNMW) has been introduced. The shape of the hazard rate function is derived analytically. Some mathematical properties of the reduced model have been studied and maximum likelihood estimation is discussed. Four applications of complete and censored real data sets are used to compare the reduced version with the NMW.

- Chapter 5 introduces a new model, referred to as exponentiated reduced modified Weibull (ERMW), to fit the data with unimodal modified unimodal hazard rate. First, the definition, interpretation and special cases of the proposed model are presented. The shapes of the probability density and hazard rate function are shown. We derive expressions for its mathematical properties and maximum likelihood estimation. Using a simulation study the performance of the maximum likelihood estimates (MLEs) with respect to biases and mean squared errors is studied. Two applications with different hazard rate shapes are provided to show how the proposed distribution work in practice using real data sets.

- Chapter 6 proposes a new discrete model with three parameters to be called the new discrete modified Weibull distribution and noted as (DRMW). The new distribution has a bathtub or increasing hazard rate function. The definition of the new discrete distribution is given and a series expansion for its survival function is derived. The shapes of its probability mass function are shown and the shape of the hazard rate function are derived analytically. Some mathematical properties of the DRMW are studied including moments and order statistics. The point and interval estimators of the unknown parameters of the DRMW distribution are derived using the maximum likelihood method. Then, assessment of the performance of the MLEs with respect to sample size n are presented using a simulation study. The proposed distribution is applied to four applications with a bathtub and increasing hazard rate and compared with other discrete modified Weibull distributions.

- Chapter 7 uses two data sets with modified unimodal (unimodal followed by increasing) hazard function for comparing the exponentiated Weibull and generalized modified Weibull distributions. In a related issues, this chapter points out some incorrect results with regard to the generalized Weibull distribution of Carrasco *et al.* (2008), in the applications to the Serum-reversal and Radiotherapy data. The correct results are provided.

- Chapter 8 concludes the thesis and presents possible future work.

1.5 Publications and conferences

1. The intensive review that has been introduced in Chapter 2 was published in *Reliability Engineering & System Safety* in April 2014. The article entitled "*Modifications of the Weibull Distribution: A review*".

2. The main contribution in Chapter 3 was published in *Reliability Engineering & System Safety* in March 2013. The title of this article is "*A New Modified Weibull Distribution*".

3. The main contribution that is presented in Chapter 4 has been accepted to be published in *Communications in Statistics-Theory and Methods*. The final version of the paper has been forwarded to the publisher for copy editing and typesetting. The title of this paper is "*A Reduced New Modified Weibull Distribution*".

Also, this work was presented as a talk in the 8th World Congress in Probability and Statistics that was held in Istanbul, Turkey 2012.

4. The main results of Chapter 5 has been submitted to *Communications in Statistics-Theory and Methods* and it is under consideration. The article is entitled "*Exponentiated Reduced Modified Weibull Distribution*".

5. The new proposed discrete model in Chapter 6 has been published in *IEEE TRANSACTIONS ON RELIABILITY*. It is titled as "*A New Discrete Modified Weibull Distribution*".

6. Chapter 7 has been submitted to *IEEE TRANSACTIONS ON RELIABILITY* and now it is under consideration. It is titled as "*Comparing The Exponentiated and Generalized Modified Weibull Distributions*".

Chapter 2

Review of Modifications of the Weibull Distribution

2.1 Introduction

The Weibull distribution was introduced by the Swedish physicist Waloddi Weibull (Weibull, 1951). This work is one of the most widely cited papers. It has been used in many different areas such as material science, reliability engineering, physics, chemistry, medicine, psychology, pharmacy economics, quality control, maintenance and replacement, inventory control, biology, forestry, geology, geography, astronomy and other fields. A huge number of applications in the above areas have been listed in Rinne (2008), Section 7.1, Tables 7.1-7.12. Murthy *et al* (2003) (Table 1.1 page 12) presented more than 40 applications of the Weibull distribution with references. Moreover, as an extreme value distribution it has been used to model climate and weather data such as rainfall (Revfeim, 1983), floods (Boes *et al*, 1989 and Heo *et al*, 2001) and wind speeds (Conradsen *et al*, 1984 and Carroll 2003).

The two-parameter Weibull distribution that is specified by the CDF in (1.2) has been extensively modified using different methods by different authors and there is a discrete version. The aim of this chapter is to provide a review of the modifications of the Weibull distribution. The continuous modifications of the continuous Weibull distribution are reviewed in Section 2.2. The discrete modifications of the discrete Weibull distribution are reviewed in Section 2.3. Maximum likelihood estimation, goodness of fit and information criteria are discussed in Section 2.4-2.7.

2.2 Continuous modifications of the Weibull distribution

In this section, some modifications of the continuous Weibull distribution are presented. Because of space limitations, we have not presented all of the known modifications of the continuous Weibull distribution. Some important modifications not presented include those due to Elbatal (2011), Provost *et al.* (2011), Razaq and Memon (2011), and Flaih *et al.* (2012b).

2.2.1 Inverse Weibull distribution

If $Y = \frac{1}{X}$ and X follows the Weibull distribution, then Y has the inverse Weibull (IW) distribution. The CDF, the PDF, and the HF of an IW distribution are

$$F(y;\alpha,\theta) = e^{-\alpha y^{-\theta}}, \quad y > 0, \tag{2.1}$$

$$f(y;\alpha,\theta) = \alpha\theta y^{-\theta-1} e^{-\alpha y^{-\theta}}, \quad y > 0$$

and

$$h(y;\alpha,\theta) = \alpha\theta y^{-\theta-1}, \quad y > 0,$$

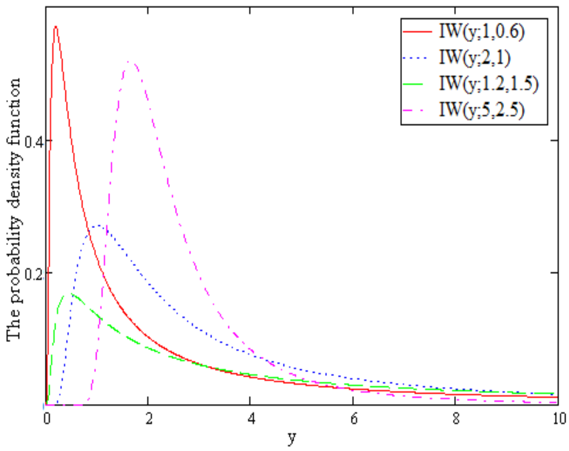
where $\alpha, \theta > 0$. The HF of the IW distribution has a unimodal shape, see Figure 2.1. This distribution is also called the reverse Weibull distribution (Simiu and Heckert, 1996), the complementary Weibull distribution (Drapella, 1993) and the reciprocal Weibull distribution (Mudholkar and Kollia, 1994). Some properties of order statistics of the IW distribution are derived in Razaq and Memon (2011).

A generalization of the IW distribution referred to as the Kumaraswamy IW, (KumIW), distribution was studied by Shahbaz *et al.* (2012). Its PDF and CDF are given by

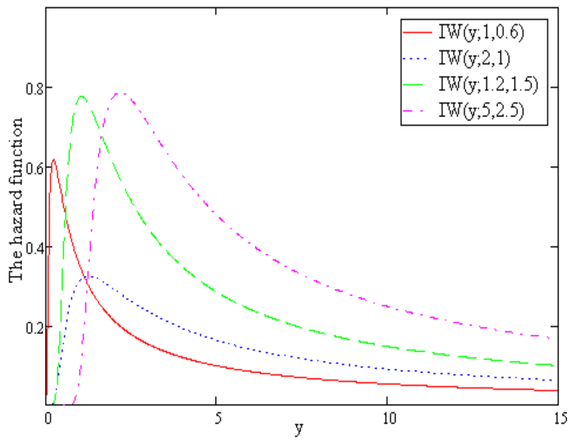
$$F(y) = 1 - \left[1 - e^{-ay^{-\theta}}\right]^{b}, \quad y > 0$$

and

$$f(y) = ab\theta y^{-\theta-1} e^{-ay^{-\theta}} \left[1 - e^{-ay^{-\theta}}\right]^{b-1}, \quad y > 0,$$



(a)



(b)

Figure 2.1: PDF and HF of the inverse Weibull distribution

where $\theta, a, b > 0$, The particular case for $a = b = 1$ is the IW distribution.

A generalization of the IW distribution referred to as the beta inverse Weibull (BIW) distribution was studied by Hanook *et al.* (2013). Its CDF and PDF are given by

$$F(y) = I_{e^{-y^{-\theta}}}(a, b), \quad y > 0$$

and

$$f(y) = \frac{\theta}{B(a,b)} y^{-(\theta+1)} e^{-ay^{-\theta}} \left[1 - e^{-y^{-\theta}}\right]^{b-1}, \quad y > 0,$$

where $\theta, a, b > 0$, $B(a, b)$ denotes the beta function defined by

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt,$$

and $I_x(a, b)$ denotes the incomplete beta function ratio defined by

$$I_x(a, b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt.$$

The particular case for $a = b = 1$ is the IW distribution. The particular case of for $b = 1$ is the exponentiated IW distribution due to Flaih *et al.* (2012a). Hanook *et al.* (2013) showed that the BIW distribution can exhibit monotonic increasing and unimodal HFs.

De Gusmao *et al.* (2011) claimed to have proposed another generalization of the IW distribution. But it turns out to be a simple reparameterization of the IW distribution. Sultan *et al.* (2007) proposed a mixture of two IW distributions.

2.2.2 Log-Weibull distribution

Let X be a Weibull random variable with parameters α and θ. Then $\frac{Y-a}{b} = \log\left(\alpha X^{\theta}\right)$ has the log-Weibull (LogW) distribution (extreme-value distribution Type I). It is due to Gumbel (1958). Its CDF and PDF are

$$F(y;\theta,a,b) = 1 - \exp\left\{-e^{\left(\frac{y-a}{b}\right)}\right\}, \quad -\infty < y < \infty$$

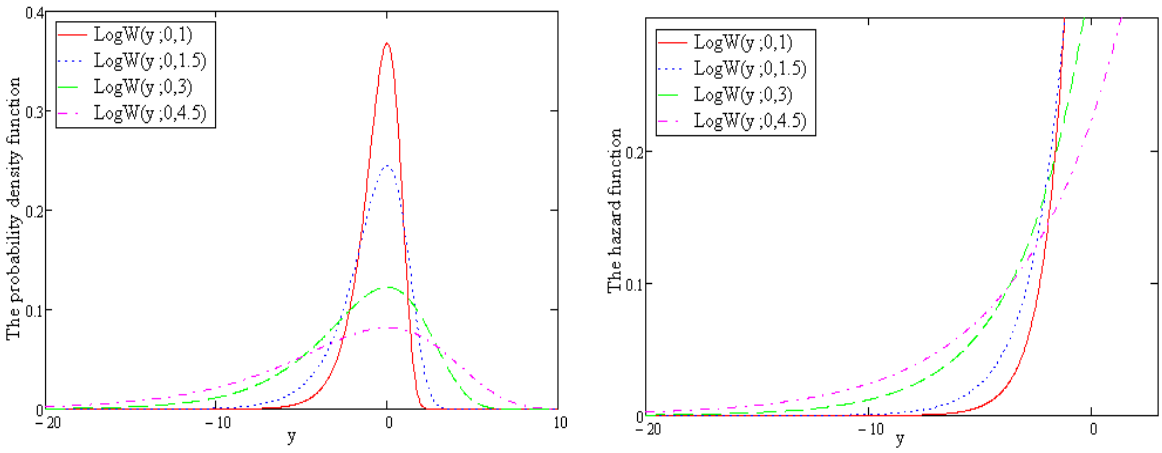
and

$$f(y;\theta,a,b) = \frac{1}{b}e^{\left(\frac{y-a}{b}\right)}\exp\left\{-e^{\left(\frac{y-a}{b}\right)}\right\}, \quad -\infty < y < \infty,$$

where $-\infty < a < \infty$ and $b > 0$. This distribution is also called the type I extreme value distribution. The HF of the LogW distribution is

$$h(y;\theta,a,b) = \frac{1}{b}e^{\left(\frac{y-a}{b}\right)},$$

an increasing function of y. The PDFs and the HFs of the LogW distribution for $a = 0$ and $b = 1, 1.5, 3, 4.5$ are shown in Figure 2.2.



(a) (b)

Figure 2.2: PDF and HF of the log-Weibull distribution

White (1969) derived expressions for the moments of a LogW random variable.

2.2.3 Reflected Weibull distribution

By considering the transformation $Y = -X$, where X is a Weibull random variable, Cohen (1973) introduced the reflected Weibull (RefW) distribution. The CDF and the PDF of the RefW distribution are given by

$$F(y;\alpha,\theta) = e^{-\alpha(-y)^{\theta}}, \quad -\infty < y < 0$$

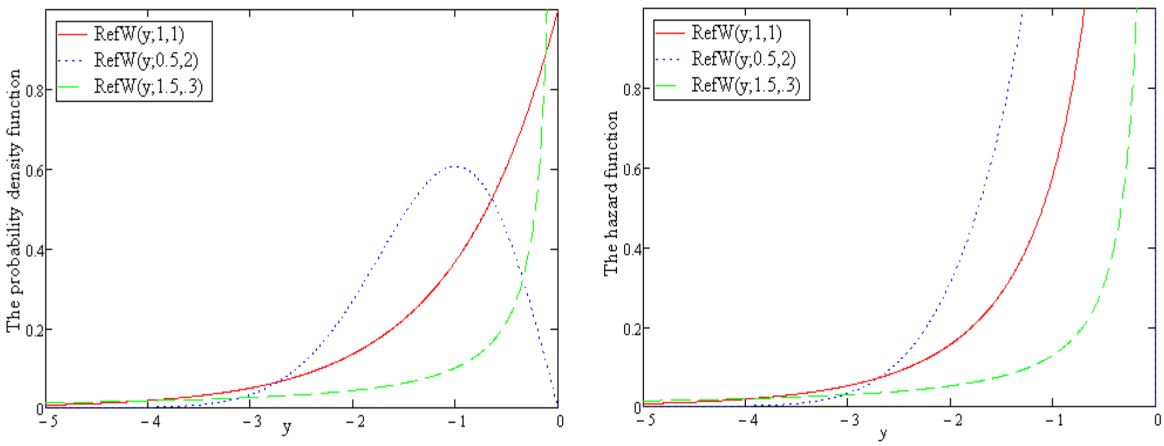
and

$$f(y;\alpha,\theta) = \alpha\theta(-y)^{\theta-1}e^{-\alpha(-y)^{\theta}}, \quad -\infty < y < 0,$$

where $\alpha, \theta > 0$. The HF of the RefW distribution is

$$h(y;\alpha,\theta) = \frac{\alpha\theta(-y)^{\theta-1}e^{-\alpha(-y)^{\theta}}}{1-e^{-\alpha(-y)^{\theta}}},$$

an increasing function of y. Figure 2.22 shows the PDF and the HF of the RefW distribution.



(a) (b)

Figure 2.3: PDF and HF of the reflected Weibull distribution

2.2.4 Gamma Weibull distribution

A modification of the Weibull distribution due to Stacy (1962) uses the gamma distribution. It is a three-parameter generalized gamma distribution. Its PDF and CDF are

$$f(x) = \frac{\theta\alpha^{k}}{\Gamma(k)}x^{k\theta-1}e^{-\alpha x^{\theta}}, \; x > 0$$

and

$$F(x) = \frac{\gamma\left(k, \alpha x^{\theta}\right)}{\Gamma(k)}, \; x > 0,$$

where $\alpha, \theta, k > 0$ and $\gamma(a, x)$ denotes the incomplete gamma function defined by

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt.$$

The particular case for $k = 1$ gives the Weibull distribution. The particular case for $\theta = 1$ gives the gamma distribution. The particular case for $k = \frac{r}{\theta} + 1$ is the weighted Weibull distribution due to Tzavelas and Panagiotakos (2013).

Stacy and Mihram (1965) introduced a further generalization of the gamma distribution by letting θ to take any real value. The PDF and the CDF of this distribution are

$$f(x) = \frac{|\theta|\, \alpha^k}{\Gamma(k)} x^{k|\theta|-1} e^{-\alpha x^{|\theta|}}, \ x > 0$$

and

$$F(x) = \frac{\gamma\left(k, \alpha x^{|\theta|}\right)}{\Gamma(k)}, \ x > 0,$$

where $\alpha, k > 0$. The particular case for $k = 1$ is the Weibull distribution. Other particular cases include the exponential, gamma, chi-squared and Rayleigh distributions. The latest studies about the generalized gamma distribution include Noufaily and Jones (2013a) and Noufaily and Jones (2013b).

Harter (1967) obtained a four-parameter generalized gamma distribution by adding a location parameter to the generalized gamma distribution of Stacy (1962). Its PDF and CDF are

$$f(x) = \frac{\theta \alpha^k}{\Gamma(k)} (x - \gamma)^{k\theta - 1} e^{-\alpha (x-\gamma)^\theta}, \ x > \gamma \geq 0$$

and

$$F(x) = \frac{\gamma\left(k, \alpha (x - \gamma)^\theta\right)}{\Gamma(k)}, \ x > \gamma \geq 0,$$

where $\alpha, \theta, k > 0$ and $-\infty < \gamma < \infty$.

Cordeiro *et al.* (2011a) introduced the four-parameter exponentiated generalized gamma distribution given by the PDF

$$f(x) = \frac{\lambda \beta}{\alpha \Gamma(k)} \left(\frac{x}{\alpha}\right)^{\beta k - 1} \exp\left[-\left(\frac{x}{\alpha}\right)^\beta\right] \left\{\gamma_1\left(k, \left(\frac{x}{\alpha}\right)^\beta\right)\right\}^{\lambda - 1}, \ x > 0,$$

where $\alpha, \beta, k, \lambda > 0$, and $\gamma_1(a, x)$ is the incomplete gamma ratio defined by $\gamma_1(a, x) = \gamma(a, x)/\Gamma(a)$. It allows for bathtub shaped, monotonically decreasing, monotonically increasing, and upside down bathtub shaped HFs.

Cordeiro *et al.* (2013a) introduced a five-parameter beta generalized gamma distribution. In addition to the exponentiated generalized gamma distribution, it contains many other distributions as particular cases. These include the gamma, chi-square, exponential, Weibull, Rayleigh, Maxwell, folded normal, log-normal, exponentiated gamma, exponentiated chi-square, exponentiated exponential, exponentiated Weibull, exponentiated Rayleigh, exponentiated Maxwell, exponentiated folded normal, exponentiated log-normal, beta gamma, beta chi-square, beta exponential, beta Weibull, beta Rayleigh, beta Maxwell, beta folded normal and beta log-normal distributions.

Pascoa *et al.* (2011) introduced the five-parameter Kumaraswamy-generalized gamma distribution given by the PDF

$$f(x) = \frac{\lambda\psi\tau}{\alpha\Gamma(k)} \left(\frac{x}{\alpha}\right)^{\tau k-1} \exp\left[-\left(\frac{x}{\alpha}\right)^{\tau}\right] \left\{\gamma_1\left(k, \left(\frac{x}{\alpha}\right)^{\tau}\right)\right\}^{\lambda-1} \left(1-\left\{\gamma_1\left(k, \left(\frac{x}{\alpha}\right)^{\tau}\right)\right\}^{\lambda}\right)^{\psi-1},$$

where $x, \alpha, \lambda, k, \psi, \tau > 0$. It allows for constant, bathtub shaped, monotonically decreasing, monotonically increasing, and upside down bathtub shaped HFs.

Al-Saleh and Agarwal (2006) proposed an extended Weibull type distribution given by the PDF

$$f(x) = \frac{mc^m x^{m-1} e^{-(cx)^m}}{\Gamma_\delta(1,1)\left[(cx)^m+1\right]^\delta}\ x > 0,$$

where $m, c > 0$, $-\infty < \delta < \infty$ and $\Gamma_\delta(m,n)$ denotes the generalized gamma function defined by

$$\Gamma_\delta(m,n) = \int_0^\infty \frac{x^{m-1}e^{-x}}{(x+n)^\delta} dx.$$

The corresponding CDF and HF are

$$F(x) = \Gamma_{(cx)^m,\delta}(1,1),\ x > 0$$

and

$$h(x) = \frac{mc^m x^{m-1} e^{-(cx)^m}}{\left[1-\Gamma_{(cx)^m,\delta}(1,1)\right]\left[(cx)^m+1\right]^\delta}\ x > 0,$$

where $\Gamma_{x,\delta}(j,1)$ denotes the incomplete generalized gamma function defined by

$$\Gamma_{x,\delta}(j,1) = \int_0^x \frac{u^{j-1}e^{-u}}{(u+1)^\delta} du.$$

The Weibull distribution is contained as the particular case for $\delta = 0$. Al-Saleh and Agarwal (2006) have shown that the HF can exhibit unimodal and bathtub shapes.

2.2.5 Kies and Phani's modified Weibull distributions

In some applications in the area of material science like strength of brittle materials such as glass and fiber, the strength values are limited where the Weibull distribution requires unlimited value. So, to address the limitation, Kies (1958) introduced a modification of the Weibull distribution by adding lower and upper limits to the random variable. Its CDF is

$$F(x;\alpha,\beta,a,b) = 1 - \exp\left[-\alpha\left(\frac{x-a}{b-x}\right)^{\beta}\right],$$

where $0 < a \leq x \leq b < \infty$ and $\alpha, \beta, \gamma > 0$. The Kies's distribution is limited because it has just one shape parameter. To give more flexibility, Phani (1987) proposed a distribution by adding another shape parameter β_2. Its CDF is given by

$$F(x;\alpha,\beta_1,\beta_2,a,b) = 1 - \exp\left[-\alpha\frac{(x-a)^{\beta_1}}{(b-x)^{\beta_2}}\right],$$

where $0 < a \leq x \leq b < \infty$ and $\alpha, \beta_1, \beta_2 > 0$. The corresponding PDF and HF are

$$f\left(x;\alpha,\beta_1,\beta_2,a,b\right) = \frac{\alpha(x-a)^{\beta_1-1}\left[\left(b\beta_1 - a\beta_2\right) + \left(\beta_2-\beta_1\right)x\right]}{(b-x)^{\beta_2+1}} \exp\left\{\alpha\frac{(x-a)^{\beta_1}}{(b-x)^{\beta_2}}\right\}$$

and

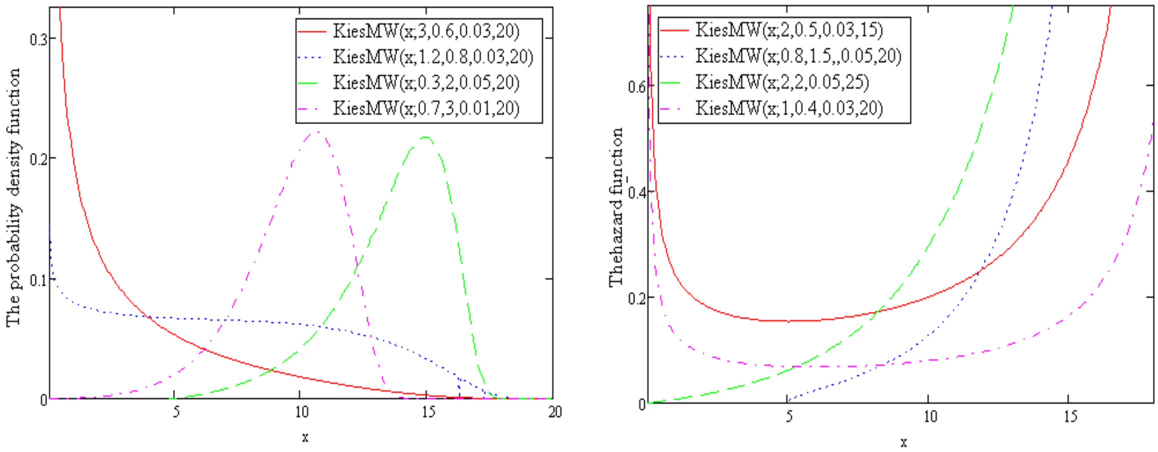
$$h\left(x;\alpha,\beta_1,\beta_2,a,b\right) = \frac{\alpha(x-a)^{\beta_1-1}\left[\left(b\beta_1 - a\beta_2\right) + \left(\beta_2-\beta_1\right)x\right]}{(b-x)^{\beta_2+1}},$$

where $0 < a \leq x \leq b < \infty$. The HF has a bathtub shape when $0 < \beta_1, \beta_2 < 1$ and increases otherwise. Figures 2.5 and 2.4 show the PDF and the HF of the distributions due to Kies and Phani.

2.2.6 Exponentiated Weibull distribution

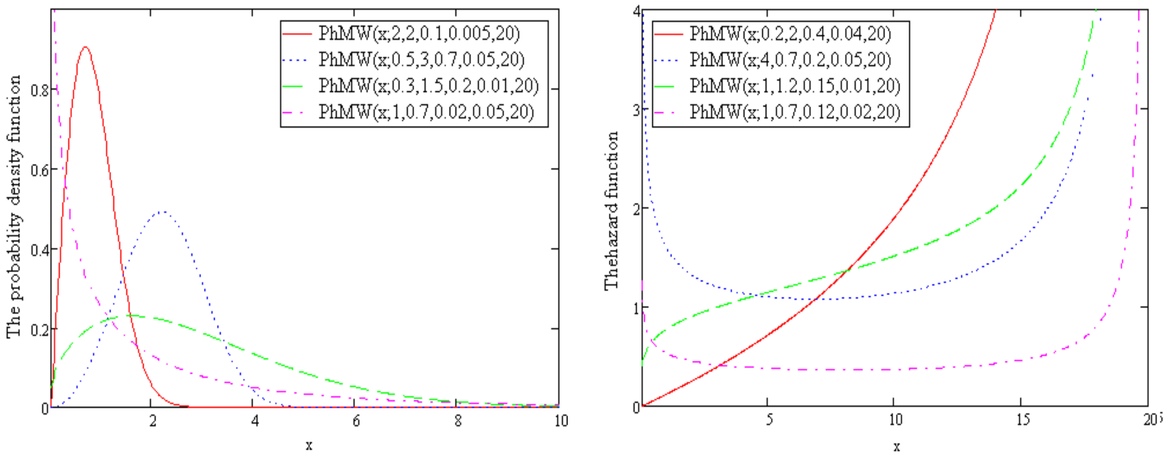
A simple modification of the two-parameter Weibull distribution, called the exponentiated Weibull, EW, distribution, was proposed by Mudholkar and Srivastava (1993) and further studied by Mudholkar *et al.* (1995) and Mudholkar and Hutson (1996) with applications to bus-motor failure data and flood data. They added a new shape parameter λ to the Weibull distribution. The PDF and the CDF of the EW distribution are

$$f(x;\alpha,\theta,\lambda) = \lambda\alpha\theta x^{\theta-1}e^{-\alpha x^{\theta}}\left(1-e^{-\alpha x^{\theta}}\right)^{\lambda-1}, \quad x > 0$$



(a) (b)

Figure 2.4: PDF and HF of Kies distribution



(a) (b)

Figure 2.5: PDF and HF of Phani distribution

and

$$F(x;\alpha,\theta,\lambda) = \left(1 - e^{-\alpha x^{\theta}}\right)^{\lambda}, \quad x > 0, \tag{2.2}$$

where $\alpha, \theta, \lambda > 0$, α is a scale parameter and θ and λ are shape parameters. The particular case for $\lambda = 1$ is the Weibull distribution. The particular case for $\theta = 2$ is the generalized Rayleigh distribution, GR, due to Surles and Padgett (2001). The particular case for $\theta = 1$ is generalized exponential distribution, GE, due to Kundu and Gupta (1999). The particular cases for $\lambda = 1$ and $\theta = 1, 2$ are the exponential and Rayleigh distributions. The HF of the EW distribution is

$$h(x;\alpha,\theta,\lambda) = \alpha\theta\lambda x^{\theta-1} e^{-\alpha x^{\theta}} \left(1 - e^{-\alpha x^{\theta}}\right)^{-1},$$

which is monotonically increasing when $\theta > 1$ and $\theta\lambda > 1$, monotonically decreasing when $\theta < 1$ and $\theta\lambda < 1$, bathtub shaped when $\theta > 1$ and $\theta\lambda < 1$ and unimodal shaped when $\theta < 1$ and $\theta\lambda > 1$. Figures 2.6 (a) and (b) show possible shapes of the PDF and the HF of the EW distribution.

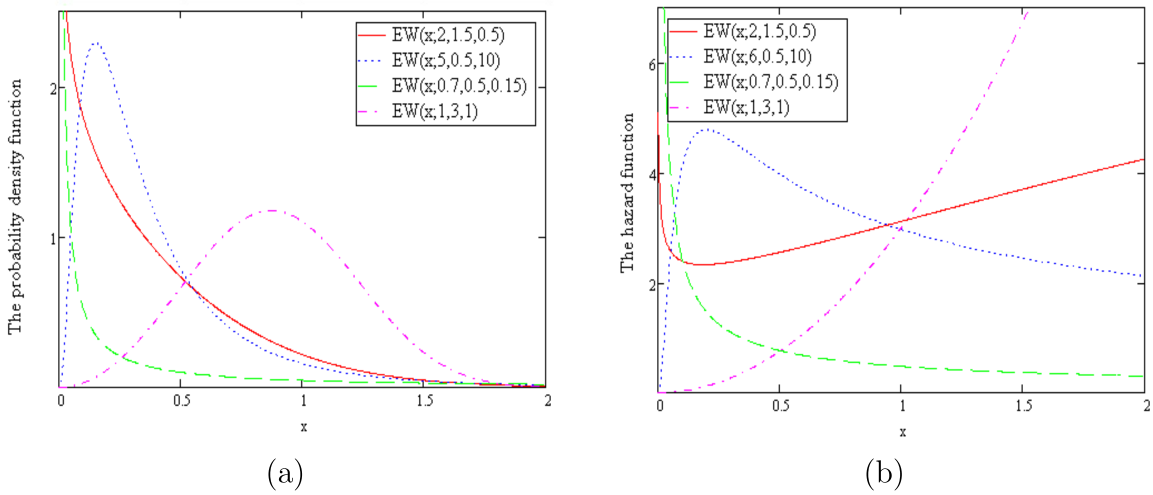


Figure 2.6: PDF and HF of the exponentiated Weibull distribution

Pal *et al.* (2006) re-introduced the EW distribution in more details. Aryal and Tsokos (2011) proposed an extension of the Weibull distribution and compare its fit with the EW distribution. Unfortunately, however, its hazard function cannot exhibit non-monotonic shapes.

2.2.7 Generalized Weibull distribution

Mudholkar and Kollia (1994) extended the Weibull distribution to a generalized Weibull GW distribution by adding a new parameter to the quantile function. Mudholkar *et al.* (1996) used this distribution to model two real survival data sets with bathtub shaped and unimodal shaped failure rates. The quantile function of the GWF distribution is

$$Q(u) = \left[\frac{1-(1-u)^{\lambda}}{\alpha\lambda}\right]^{\frac{1}{\theta}},$$

where $-\infty < \lambda < \infty$. If $\lambda \to 0$ we obtain the quantile function of the Weibull distribution. The CDF of the GWF distribution is

$$F(x;\alpha,\theta,\lambda) = 1 - \left[1-\alpha\lambda x^{\theta}\right]^{\frac{1}{\lambda}}, \quad x > 0,$$

where $\alpha, \theta > 0$ and $-\infty < \lambda < \infty$. The domain of the CDF is $(0,\infty)$ for $\lambda \le 0$ and $(0, (\alpha\lambda)^{-\frac{1}{\theta}})$ for $\lambda > 0$. The PDF of the GWF distribution is

$$f(x;\alpha,\theta,\lambda) = \alpha\theta x^{\theta-1}\left[1-\alpha\lambda x^{\theta}\right]^{\frac{1}{\lambda}-1} \quad x > 0.$$

Its HF is

$$h(x;\alpha,\theta,\lambda) = \alpha\theta x^{\theta-1}\left[1-\alpha\lambda x^{\theta}\right]^{-1} \quad x > 0.$$

The HF can take the following shapes:

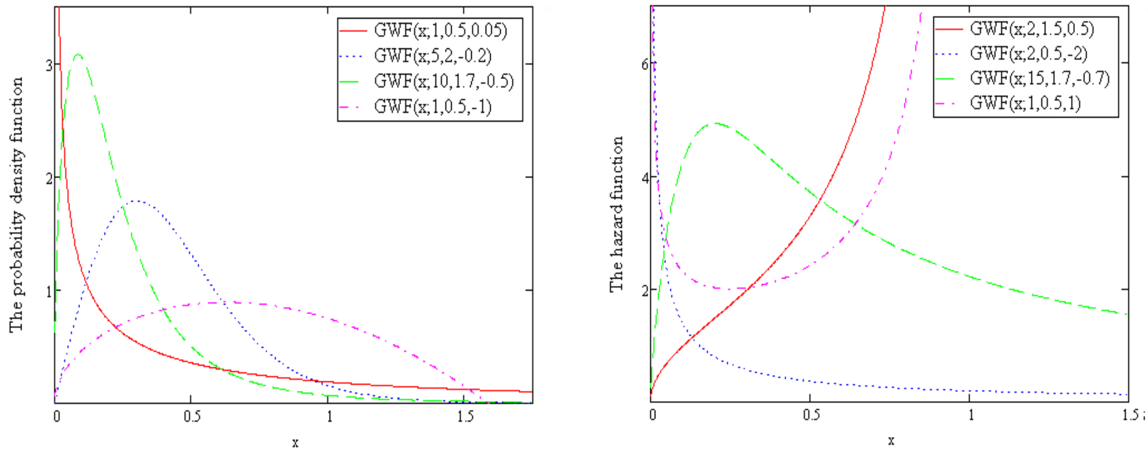
- Increasing if $\theta \ge 1$ and $\lambda > 0$;
- Decreasing if $\theta \le 1$ and $\lambda \le 0$;
- Bathtub shaped if $\theta < 1$ and $\lambda > 0$;
- Unimodal shaped $\theta > 1$ and $\lambda < 0$.

Figure 2.7 shows the different shapes of the PDF and the HF of the GWF distribution.

2.2.8 Additive Weibull distribution

Xie and Lai (1996) introduced the additive Weibull distribution, AddW, with bathtub shaped HF obtained as the sum of two HFs of the Weibull distribution. The PDF, the CDF and the HF of the AddW distribution are

$$f(x;\alpha,\beta,\theta,\gamma) = \left(\alpha\theta x^{\theta-1} + \beta\gamma x^{\gamma-1}\right)e^{-\alpha x^{\theta}-\beta x^{\gamma}}, \quad x > 0,$$



(a) (b)

Figure 2.7: PDF and HF of the exponentiated Weibull distribution

$$F(x;\alpha,\beta,\theta,\gamma) = 1 - e^{-\alpha x^{\theta} - \beta x^{\gamma}}, \quad x > 0 \tag{2.3}$$

and

$$h(x;\alpha,\beta,\theta,\gamma) = \alpha\theta x^{\theta-1} + \beta\gamma x^{\gamma-1}, \quad x > 0,$$

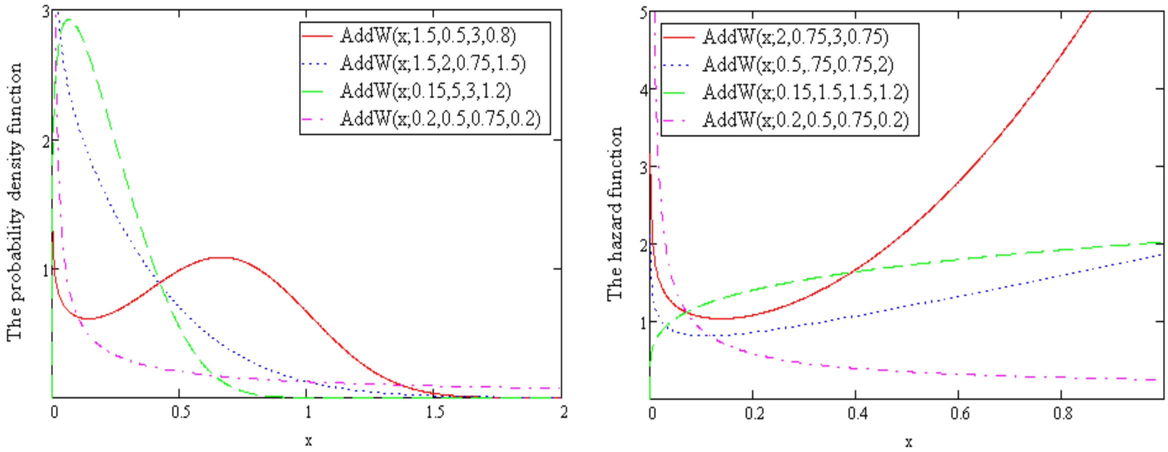
where α, β, $\theta > 0$ and $\gamma < 1$. The HF increases when the shape parameters, θ and γ, are greater than one. The HF decreases when the shape parameters are less than one. The particular case for $\alpha = 0$ or $\beta = 0$ is the Weibull distribution. The particular case for $\gamma = 2$ is the linear failure rate distribution due to Bain (1974). The particular case for $\theta = 1$ and $\gamma > 0$ is the modified Weibull distribution of Sarhan and Zaindin (2009) to be discussed later.

Some possible shapes of the HF and the PDF of the AddW distribution are shown in Figure 2.8.

Bebbington *et al.* (2006) considered a particular case of the AddW distribution with real data applications. A discrete analog of the AddW distribution is proposed in Bebbington *et al.* (2012).

2.2.9 Extended Weibull distribution

Marshall and Olkin (1997) proposed a general method to modify any distribution by adding a new parameter. Let $F_w(\cdot)$ denote the CDF of a two-parameter Weibull distribution with scale parameter α and shape parameter θ. Using Marshall and Olkin (1997)'s method, Zhang and Xie (2007) proposed an extended Weibull (ExW)



(a) (b)

Figure 2.8: PDF and HF of the additive Weibull distribution

distribution specified by the CDF

$$\begin{aligned} F(x;\alpha,\theta,\lambda) &= \frac{F_w(x)}{1-\overline{\lambda}\overline{F}_w(x)} \\ &= \frac{1-e^{-\alpha x^{\theta}}}{1-\overline{\lambda}e^{-\alpha x^{\theta}}}, \quad x>0, \end{aligned} \tag{2.4}$$

where $\overline{\lambda} = 1-\lambda$ and $\overline{F}_w(x) = 1 - F_w(x)$. The WG distribution of Barreto-Souza *et al.* (2011) is a particular case of this distribution for $0 < \lambda < 1$. The PDF and the HF of the ExW distribution are

$$f(x;\alpha,\theta,\lambda) = \frac{\alpha\theta\lambda x^{\theta-1}e^{-\alpha x^{\theta}}}{\left[1-\overline{\lambda}e^{-\alpha x^{\theta}}\right]^2}, \quad x>0 \tag{2.5}$$

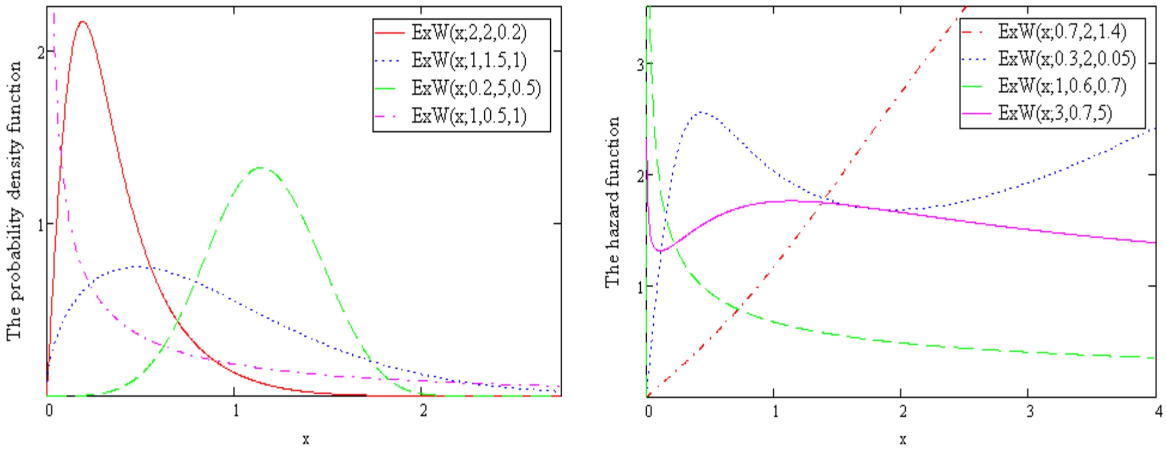
and

$$h(x;\alpha,\theta,\lambda) = \frac{\alpha\theta\lambda x^{\theta-1}}{1-\overline{\lambda}e^{-\alpha x^{\theta}}} \quad x>0. \tag{2.6}$$

The HF in (2.6) increases when $\theta,\lambda \geq 1$, decreases when $\theta,\lambda \leq 1$, unimodal followed by increasing when $\theta > 1$ and decreasing followed by unimodal when $\theta < 1$. The different shapes of the PDF and the HF of the ExW distribution are shown in Figure 2.9.

Ghitany *et al.* (2005) suggested an extension of Marshall and Olkin (1997)'s method for generating new distributions. For a given CDF $F_w(\cdot)$, they suggest defining

$$F(x;\alpha,\lambda,\beta) = \frac{\alpha\overline{F}_w(x)}{1-\overline{\alpha}\overline{F}_w(x)},$$



(a) (b)

Figure 2.9: PDF and HF of the extended Weibull distribution

where $\alpha > 0$ and $\overline{\alpha} = 1 - \alpha$. By taking $F_w(\cdot)$ to be a two-parameter Weibull CDF, Ghitany *et al.* (2005) generalized (2.4), (2.5) and (2.6) to

$$F(x; \alpha, \lambda, \beta) = 1 - \frac{\alpha e^{-(\lambda x)^\beta}}{1 - \overline{\alpha} e^{-(\lambda x)^\beta}}, \quad x > 0,$$

$$f(x; \alpha, \lambda, \beta) = \frac{\alpha \beta \lambda (\lambda x)^{\beta - 1} e^{-(\lambda x)^\beta}}{\left[1 - \overline{\alpha} e^{-(\lambda x)^\beta}\right]^2}, \quad x > 0$$

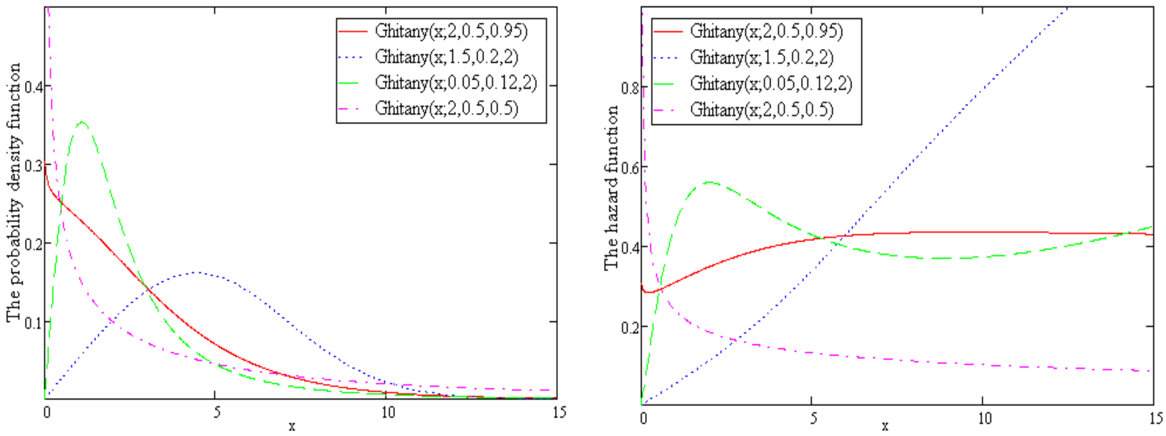
and

$$h(x; \alpha, \lambda, \beta) = \frac{\beta \lambda (\lambda x)^{\beta - 1}}{1 - \overline{\alpha} e^{-(\lambda x)^\beta}}, \quad x > 0,$$

respectively, where $\alpha, \beta, \lambda > 0$. Ghitany *et al.* (2005) established that the following shapes are possible for the HF:

- Increasing if either $\alpha \geq 1$ and $\beta \geq 1$ or $\alpha \leq 1$, $\beta > 1$ and $\beta - 1 - \overline{\alpha} e^{-1/\beta} \geq 0$;
- Decreasing if either $\alpha \leq 1$ and $\beta \leq 1$ or $\alpha \geq 1$, $\beta < 1$ and $\beta - 1 - \overline{\alpha} \beta e^{-1/\beta} \leq 0$;
- Increasing followed by decreasing followed by increasing if $\alpha \leq 1$, $\beta > 1$ and $\beta - 1 - \overline{\alpha} e^{-1/\beta} < 0$;
- Decreasing followed by increasing followed by decreasing if $\alpha \geq 1$, $\beta < 1$ and $\beta - 1 - \overline{\alpha} \beta e^{-1/\beta} > 0$.

Particular cases of this distribution include Gompertz (1825)'s distribution for $\beta = 1$ and Chen (2000)'s distribution for $\lambda = 1$. Some possible shapes of the PDF and the HF of Ghitany *et al.*'s distribution are shown in Figure 2.10.



(a) (b)

Figure 2.10: PDF and HF of Ghitany *et al.*'s distribution

2.2.10 Power Lindley distribution

Ghitany *et al.* (2013) proposed the power Lindley (PL) distribution specified by the CDF

$$F(x;\alpha,\beta) = 1 - \left(1 + \frac{\beta}{\beta+1}x^{\alpha}\right) e^{-\beta x^{\alpha}}, \quad x > 0,$$

where $\alpha > 0$ and $\beta > 0$. The corresponding PDF and HF are

$$f(x;\alpha,\beta) = \frac{\alpha\beta^2}{\beta+1}\left(1 + x^{\alpha}\right) x^{\alpha-1} e^{-\beta x^{\alpha}}, \quad x > 0$$

and

$$h(x;\alpha,\beta) = \frac{\left(1 + x^{\alpha}\right) x^{\alpha-1}}{\beta + 1 + \beta x^{\alpha}}, \quad x > 0.$$

This distribution was obtained by power transformation of a Lindley random variable (Lindley, 1958). It can also be obtained as a mixture of Weibull and generalized gamma distributions. The Weibull distribution arises as the limiting case for $\beta \to 0$.

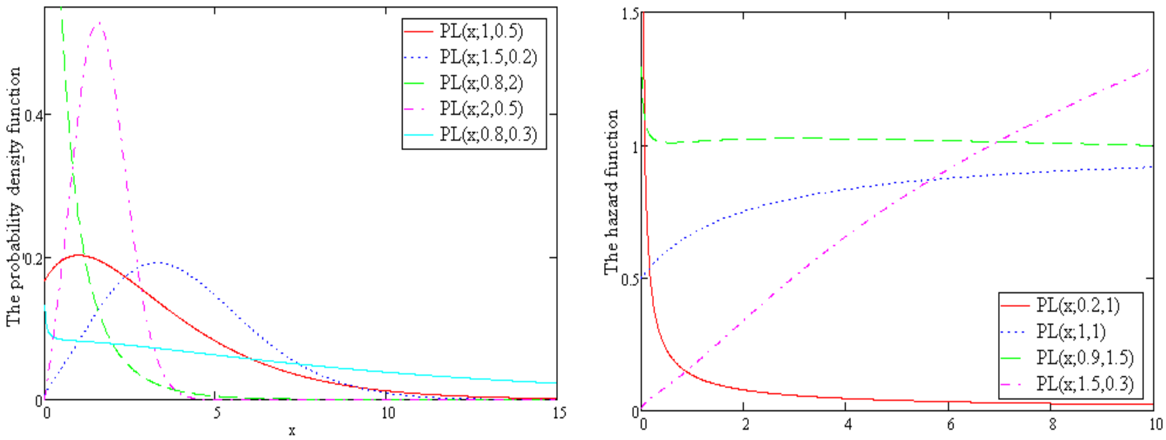
Ghitany *et al.* (2013) established the following shapes for the PDF of the PL distribution:

- Decreasing if either $0 < \alpha \leq 1/2$ and $\beta > 0$ or $1/2 < \alpha < 1$ and $\beta \geq \left\{1 - 2\sqrt{\alpha(1-\alpha)}\right\}/\alpha$ or $\alpha = 1$ and $\beta \geq 1$;
- Unimodal if either $\alpha = 1$ and $0 < \beta < 1$ or $\alpha > 1$ and $\beta > 0$;
- Decreasing followed by increasing followed by decreasing if $1/2 < \alpha < 1$ and $0 < \beta < \left\{1 - 2\sqrt{\alpha(1-\alpha)}\right\}/\alpha$.

Moreover, the following shapes were established for the HF:

- Decreasing if either $0 < \alpha \leq 1/2$ and $\beta > 0$ or $1/2 < \alpha < 1$ and $\beta \geq (2\alpha - 1)^2 / \{4\alpha(1-\alpha)\}$;
- Increasing if either $\alpha \geq 1$ and $\beta > 0$;
- Decreasing followed by increasing followed by decreasing if $1/2 < \alpha < 1$ and $0 < \beta < (2\alpha - 1)^2 / \{4\alpha(1-\alpha)\}$.

Some possible shapes of the PDF and the HF of the PL distribution are shown in Figure 2.11.



(a) (b)

Figure 2.11: PDF and HF of the power Lindley distribution

2.2.11 Modified Weibull distribution

Lai *et al.* (2003) proposed a modification by multiplying the Weibull cumulative HF, αx^{θ}, by $e^{\lambda x}$. This three-parameter distribution is known as modified Weibull distribution, MW. Lai *et al.* (2003) studied some of its properties and estimation by maximum likelihood and Weibull probability plot methods. The CDF of the MW distribution is

$$F(x; \beta, \gamma, \lambda) = 1 - e^{-\beta x^{\gamma} e^{\lambda x}}, \quad x > 0, \tag{2.7}$$

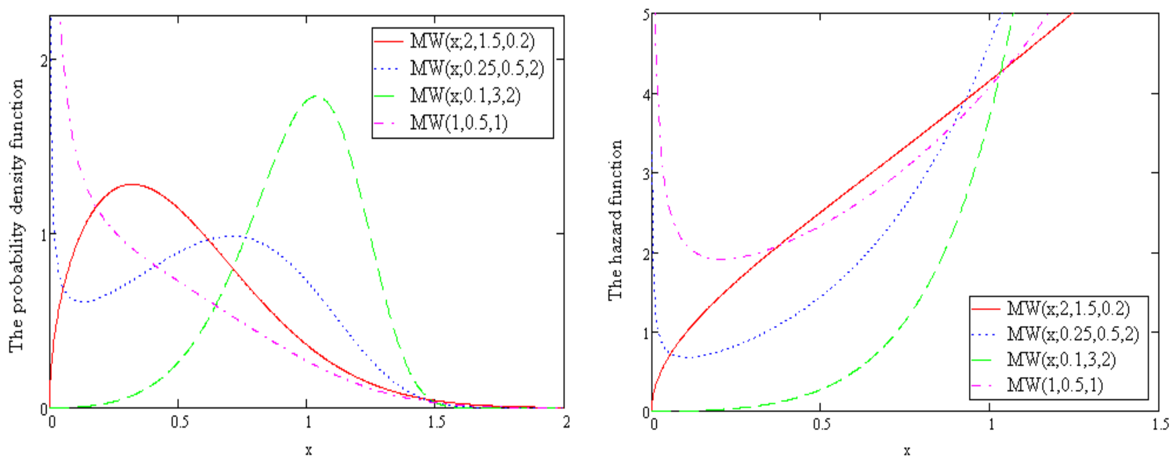
where $\beta > 0$, $\gamma, \lambda \geq 0$ and at most one of γ, λ is equal to zero. The particular case for $\lambda = 0$ is the Weibull distribution. The particular case for $\gamma = 0$ is the type I extreme value distribution. The PDF of the MW distribution is

$$f(x;\beta,\gamma,\lambda) = \beta\,(\gamma+\lambda x)\,x^{\gamma-1}e^{\lambda x}e^{-\beta x^{\gamma}e^{\lambda x}}, \quad x>0. \tag{2.8}$$

The corresponding HF is

$$h(x;\beta,\gamma,\lambda) = \beta\,(\gamma+\lambda x)\,x^{\gamma-1}e^{\lambda x}, \quad x>0. \tag{2.9}$$

The PDF of the MW distribution can be decreasing, unimodal or decreasing followed by unimodal. The HF can be increasing or bathtub shaped. Some possible shapes of the PDF and the HF of the MW distribution are shown in Figure 2.12.



(a) (b)

Figure 2.12: PDF and HF of the modified Weibull distribution

The MW distribution is one of the most important modifications of the Weibull distribution and also among the lifetime distributions. Many authors have proposed generalizations of the MW distribution: the generalized modified Weibull distribution due to Carrasco *et al.* (2008); the beta modified Weibull distribution due to Silva *et al.* (2010); and the Kumaraswamy modified Weibull distribution due to Cordeiro *et al.* (2014b). Many authors have also studied estimation issues of the MW distribution: based on progressively type-II censored data, Ng (2005) and Jiang *et al.* (2010) studied estimation of the parameters of the MW distribution using maximum likelihood and least-squares methods; Carrasco *et al.* (2008) proposed a regression model called log-regression modified Weibull model for lifetime data following the MW distribution and affected by some explanatory variables; Preda *et al.* (2010) obtained Bayes estimators of the MW distribution based on the Lindley approximation method; Soliman *et al.*

(2012) obtained Bayesian estimators of the MW distribution for progressive censored data by using Markov chain Monte Carlo (MCMC).

2.2.12 Generalized power Weibull distribution

A three-parameter distribution, called a generalized power Weibull (GPW) distribution, was proposed by Nikulin and Haghighi (2006). Its CDF and PDF are

$$F(x;\alpha,\theta,\lambda) = 1 - \exp\left\{1 - \left(1+\alpha x^{\theta}\right)^{\frac{1}{\lambda}}\right\}, \quad x > 0$$

and

$$f(x;\alpha,\theta,\lambda) = \alpha\theta\lambda^{-1}x^{\theta-1}\left(1+\alpha x^{\theta}\right)^{\frac{1}{\lambda}-1}\exp\left\{1 - \left(1+\alpha x^{\theta}\right)^{\frac{1}{\lambda}}\right\}, \quad x > 0,$$

where $\alpha, \theta, \lambda > 0$. The particular case for $\lambda = 1$ is the Weibull distribution. The HF of the GPW distribution is

$$h(x;\alpha,\theta,\lambda) = \alpha\theta\lambda^{-1}x^{\theta-1}\left(1+\alpha x^{\theta}\right)^{\frac{1}{\lambda}-1}, \quad x > 0.$$

It has the following possible shapes:

- Increasing if $\theta \geq 1$ and $\theta > \lambda$;
- Decreasing if $\theta \leq 1$ and $\theta < \lambda$;
- Bathtub shaped if $0 < \lambda < \theta < 1$;
- Unimodal shaped $\lambda > \theta > 1$.

Some possible shapes of the PDF and the HF of the GPW distribution are shown in Figure 2.13.

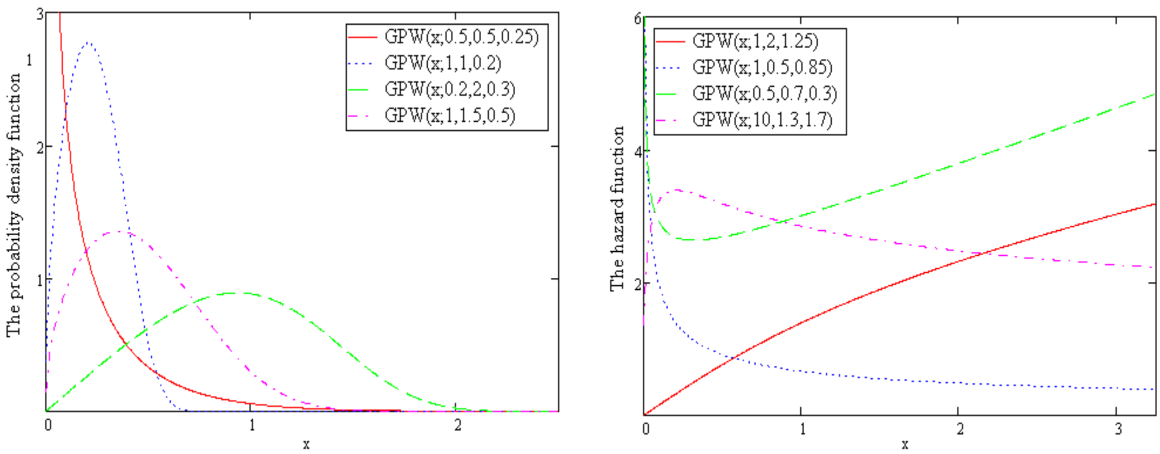
2.2.13 Modified Weibull extension

The modified Weibull extension (MWEx) with bathtub shaped failure rate function was proposed by Xie *et al.* (2002). The PDF and the CDF of this distribution are

$$f(x;\alpha,\theta,\lambda) = \lambda\alpha^{\left(\frac{\theta-1}{\theta}\right)}x^{\theta-1}e^{\alpha x^{\theta}}\exp\left[\lambda\alpha^{-\frac{1}{\theta}}\left(1-e^{-\alpha x^{\theta}}\right)\right], \quad x > 0$$

and

$$F(x;\alpha,\theta,\lambda) = 1 - \exp\left[\lambda\alpha^{-\frac{1}{\theta}}\left(1-e^{-\alpha x^{\theta}}\right)\right], \quad x > 0,$$



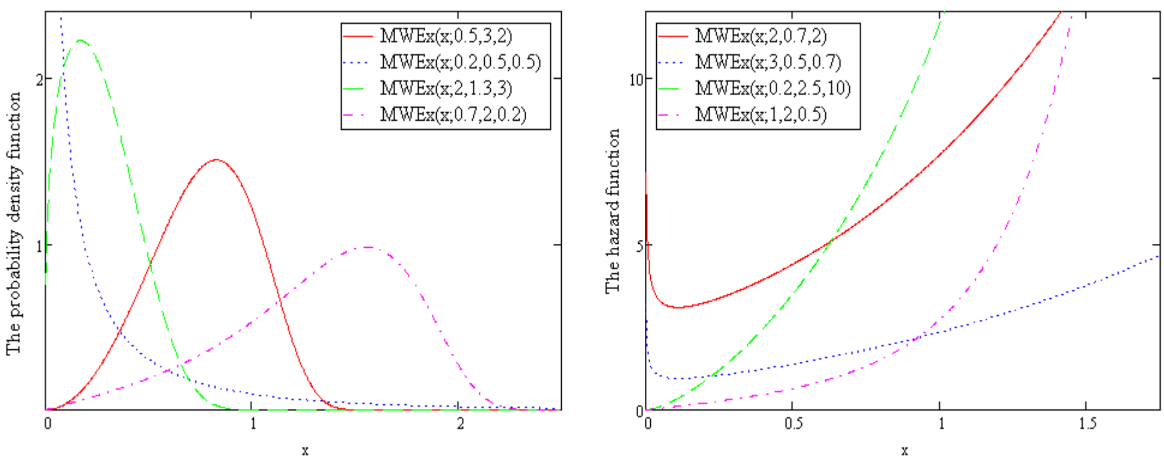
(a) (b)

Figure 2.13: PDF and HF of the generalized power Weibull distribution

where $\alpha, \theta, \lambda > 0$. The Weibull distribution can be obtained as a special case of this distribution when α is so small that $(1 - e^{-\alpha x^\theta})$ is approximately equal to $-\alpha x^\theta$. The particular case of the MWEx distribution for $\alpha = 1$ is Chen (2000)'s distribution. The HF of the MWEx distribution is

$$h(x; \alpha, \theta, \lambda) = \lambda \alpha^{\left(\frac{\theta-1}{\theta}\right)} x^{\theta-1} e^{\alpha x^\theta},$$

which depends only on the shape parameter θ. The HF increases if $\theta \geq 1$ and is bathtub shaped if $\theta < 1$. Some possible shapes of the PDF and the HF of the MWEx distribution are shown in Figure 2.14.

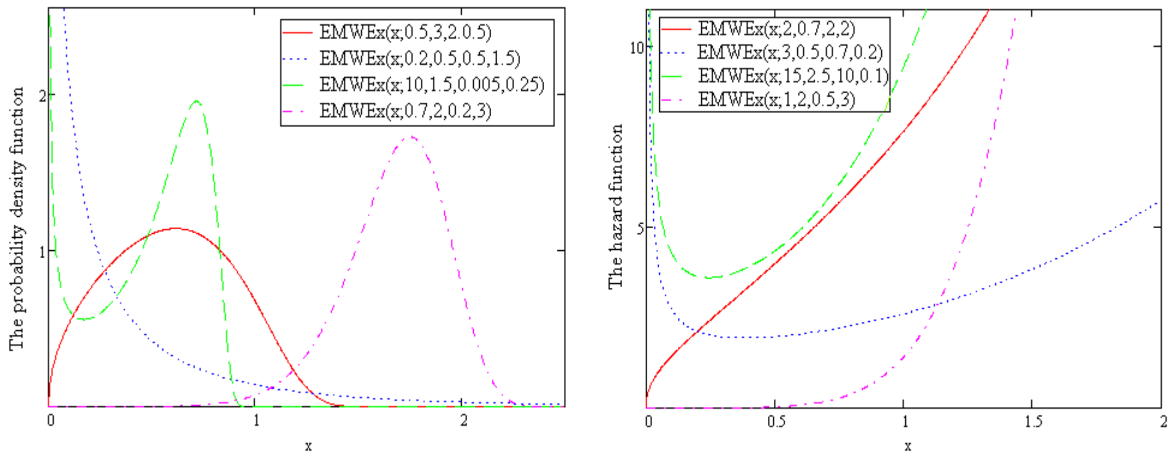


(a) (b)

Figure 2.14: PDF and HF of the modified Weibull extension

Sarhan and Apaloo (2013) generalized the MWEx distribution via exponentiation and referred to the generalization as exponentiated modified Weibull extension, EMWEx. Sarhan and Apaloo's distribution admits increasing and bathtub shaped HFs. It can also be shown to have a bathtub shaped HF with long useful period. Figure 2.15 shows possible shapes of the PDF and the HF of the EMWEx distribution.

Much researches have been performed with respect to the MWEx distribution, see, for example, Xie *et al.* (2002), Tang *et al.* (2003), Yong (2004), Nadarajah (2005), Elshahat (2007), Gupta *et al.* (2008), and El-Fotouh and Nassar (2011).



(a) (b)

Figure 2.15: PDF and HF of the exponentiated modified Weibull extension

2.2.14 Beta Weibull distributions

The four-parameter beta Weibull (BW) distribution was proposed by Famoye *et al.* (2005). Its PDF and CDF are

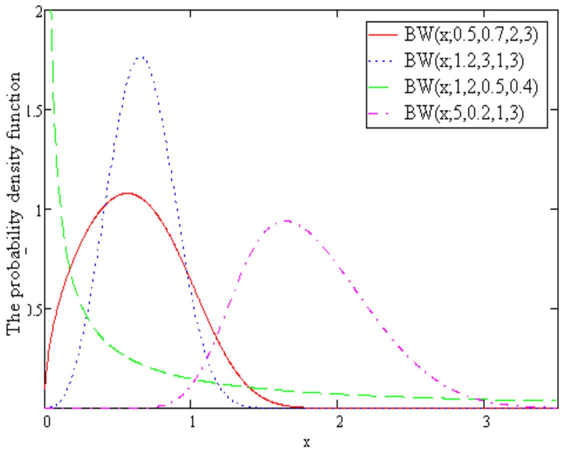
$$f(x; a, b, \alpha, \theta) = \frac{1}{B(a,b)} \alpha\theta x^{\theta-1} \left(1 - e^{-\alpha x^{\theta}}\right)^{a-1} e^{-\alpha b x^{\theta}}, \quad x > 0$$

and

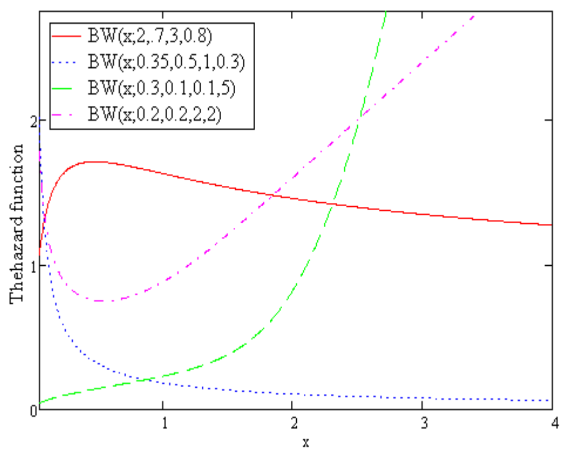
$$F(x; a, b, \alpha, \theta) = I_{1-e^{-\alpha x^{\theta}}}(a, b), \quad x > 0,$$

where $\alpha, \theta, a, b > 0$. The particular case for $a = b = 1$ is the Weibull distribution. Other particular cases include the beta exponential distribution due to Maynard (2003) and Nadarajah and Kotz (2006a), the GE distribution due to Gupta and Kundu (1999) and the EW distribution due to Mudhalkar and Srivastava (1993). The HF of the beta Weibull distribution increases if $\theta \geq 1$ and $a\theta \geq 1$, decreases if $\theta \leq 1$ and $a\theta \leq 1$, bathtub shaped if $\theta > 1$ and $a\theta < 1$ and unimodal shaped if $\theta < 1$ and $a\theta > 1$. The

PDF and the HF of the BW distribution for some selected parameter values are shown in Figure 2.16.



(a)



(b)

Figure 2.16: PDF and HF of the beta Weibull distribution

The distribution of Famoye *et al.* (2005) was re-introduced by Cordeiro *et al.* (2009) and Wahed *et al.* (2009) without acknowledging the original source. Lee *et al.* (2007) applied the BW distribution to censored data sets. Cordeiro *et al.* (2011b) derived a closed form expressions for the moments of a BW random variable. Cordeiro *et al.* (2013c) derived further properties, including closed form expressions for moment generating function, the behaviors of the extreme values, mean deviations, Bonferroni curves and Lorenz curves. Ortega *et al.* (2013) introduced a regression model, the log-beta Weibull regression model, to analyze lifetime data following the BW distribution. Mahmoud and Mandouh (2012) studied parameter estimation of the BW distribution for censored data.

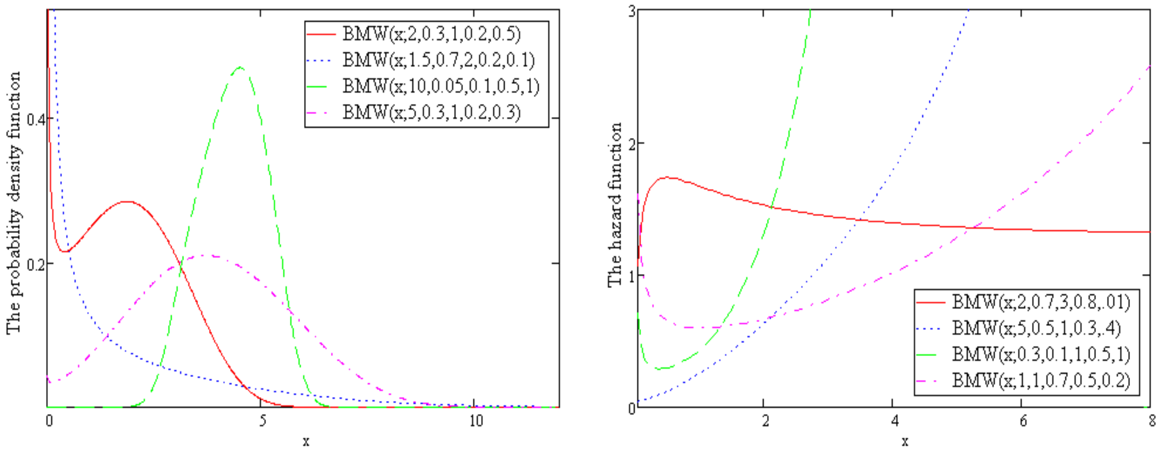
The beta modified Weibull distribution, BMW, with five parameters was introduced by Silva *et al.* (2010). The PDF and the CDF of this distribution are

$$f(x;a,b,\beta,\gamma,\lambda) = \frac{1}{B(a,b)} \frac{\beta\left(\gamma+\lambda x\right)x^{\gamma-1}e^{\lambda x}\left(1-e^{-\beta x^{\gamma}e^{\lambda x}}\right)^{a-1}}{e^{\beta b x^{\gamma}e^{\lambda x}}}, \quad x>0$$

and

$$F(x;a,b,\beta,\gamma,\lambda) = I_{1-e^{-\beta x^{\gamma}e^{\lambda x}}}(a,b), \quad x>0,$$

where $a,b,\beta,\gamma>0$ and $\lambda\geq 0$. The particular case for $a=b=1$ and $\lambda=0$ is the Weibull distribution. Other particular cases include the generalized modified Weibull distribution due to Carrasco *et al.* (2008) for $b = 1$, the MW distribution due to Lai *et al.* (2003) for $a = b = 1$, the BW distribution due to Famoye *et al.* (2005) for $\lambda = 0$, the EW distribution due to Mudholkar and Srivastava (1993) for $b = 1$ and $\lambda = 0$, and others. The BMW distribution allows the following shapes for its HF: increasing, decreasing, bathtub shape and unimodal shape. Figure 2.17 shows some possible shapes of the PDF and the HF of the BMW distribution.



(a) (b)

Figure 2.17: PDF and HF of the beta modified Weibull distribution

A detailed account of mathematical properties of the BMW distribution was given by Nadarajah *et al.* (2011). Closed form expressions were derived for moments, moment generating function, asymptotic distributions of the extreme values, mean deviations, Bonferroni and Lorenz curves, reliability and entropies. Procedures for estimation were also derived by the methods of moments and maximum likelihood.

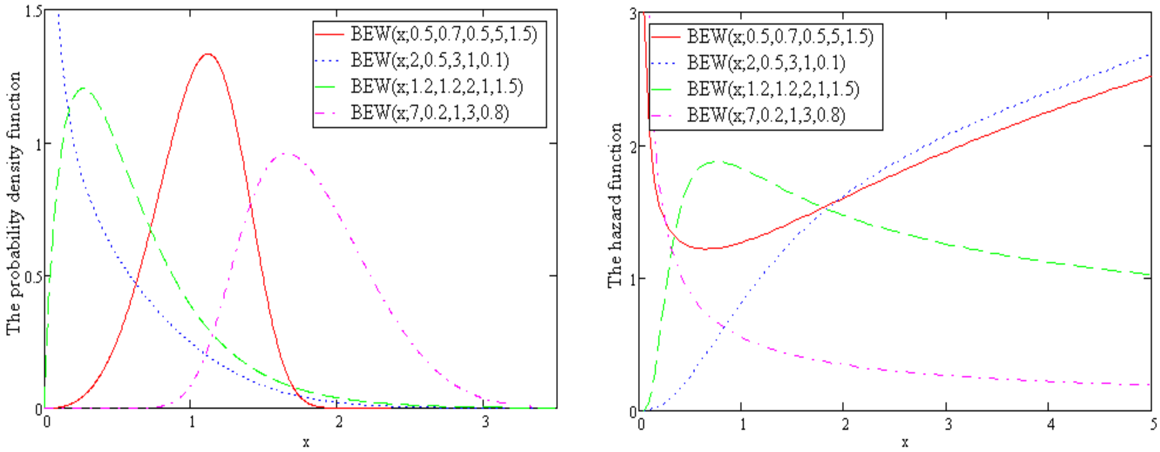
A five-parameter distribution called beta exponentiated Weibull (BEW) distribution was proposed independently by Alexander *et al.* (2012), Singla *et al.* (2012) and Cordeiro *et al.* (2013b). This distribution generalizes many well known distributions like the beta generalized exponential (BGE) distribution due to Barreto-Souza *et al.* (2010), the GE distribution, the BW distribution, the EW distribution and the GR distribution. The PDF and the CDF of the BEW distribution are

$$f(x; a, b, \alpha, \theta, \lambda) = \frac{1}{B(a, b)} \alpha\theta\lambda x^{\theta-1} \left(1 - e^{-\alpha x^\theta}\right)^{a\lambda-1} \left\{1 - \left(1 - e^{-\alpha x^\theta}\right)^\lambda\right\}^{b-1} e^{-\alpha x^\theta}, x > 0$$

and

$$F(x; a, b, \alpha, \theta, \lambda) = I_{\left(1-e^{-\alpha x^\theta}\right)^\lambda}(a, b), \ x > 0,$$

where $a, b, \alpha, \beta, \lambda > 0$. The HF of the BEW distribution can be increasing, decreasing, bathtub shaped or unimodal shaped, as illustrated in Figure 2.18.



(a) (b)

Figure 2.18: PDF and HF of the beta exponentiated Weibull distribution

A generalization of the BEW distribution is discussed in Cordeiro *et al.* (2014).

2.2.15 Odd Weibull distribution

Another modification of the Weibull distribution, referred to as the odd Weibull (OddW) distribution, was presented by Cooray (2006). It has three parameters and was obtained by considering the distribution of the odds of the Weibull and inverse Weibull distributions. The CDF of the OddW distribution is

$$F(x; \alpha, \theta, \lambda) = 1 - \left[1 + \left(e^{\alpha x^\theta} - 1\right)^\lambda\right]^{-1}, \ x > 0,$$

where $\alpha > 0$, $\theta\lambda > 0$, the shape parameters being θ and λ. The particular case for $\lambda = 1$ is the Weibull distribution. The particular case for $\lambda = -1$ is the IW distribution. The PDF and the HF of the OddW distribution are

$$f(x; \alpha, \theta, \lambda) = \alpha\theta\lambda x^{\theta-1} e^{\alpha x^\theta} \left(e^{\alpha x^\theta} - 1\right)^{\lambda-1} \left[1 + \left(e^{\alpha x^\theta} - 1\right)^\lambda\right]^{-2}, \ x > 0$$

and

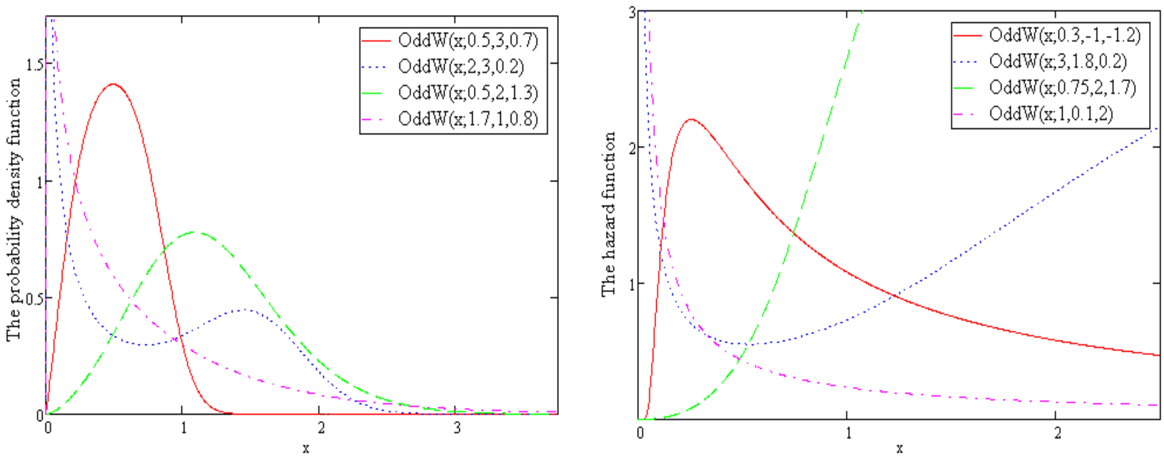
$$h(x; \alpha, \theta, \lambda) = \alpha\theta\lambda x^{\theta-1} e^{\alpha x^\theta} \left(e^{\alpha x^\theta} - 1\right)^{\lambda-1} \left[1 + \left(e^{\alpha x^\theta} - 1\right)^\lambda\right]^{-1}, \ x > 0.$$

The HF can take the following shapes:

- Increasing if $\theta > 1$ and $\theta\lambda > 1$;

- Decreasing if $\theta < 1$ and $\theta\lambda < 1$;
- Unimodal shaped if either $\theta, \lambda < 0$ or $\theta < 1$ and $\theta\lambda \geq 1$;
- Bathtub shaped if $\theta > 1$ and $\theta\lambda \leq 1$.

Figures 2.19 (a) and (b) show possible shapes of the PDF and the HF of the OddW distribution. Further studies about the OddW distribution can be found in Jiang *et al.* (2008) and Cooray (2012).



(a) (b)

Figure 2.19: PDF and HF of the odd Weibull distribution

2.2.16 Flexible Weibull extension

With a simple HF that can be increasing, bathtub shaped or unimodal shaped, Bebbington *et al.* (2007) proposed an interesting distribution with just two parameters. It is called the flexible Weibull extension (FlxWE). Its CDF, PDF and HF are

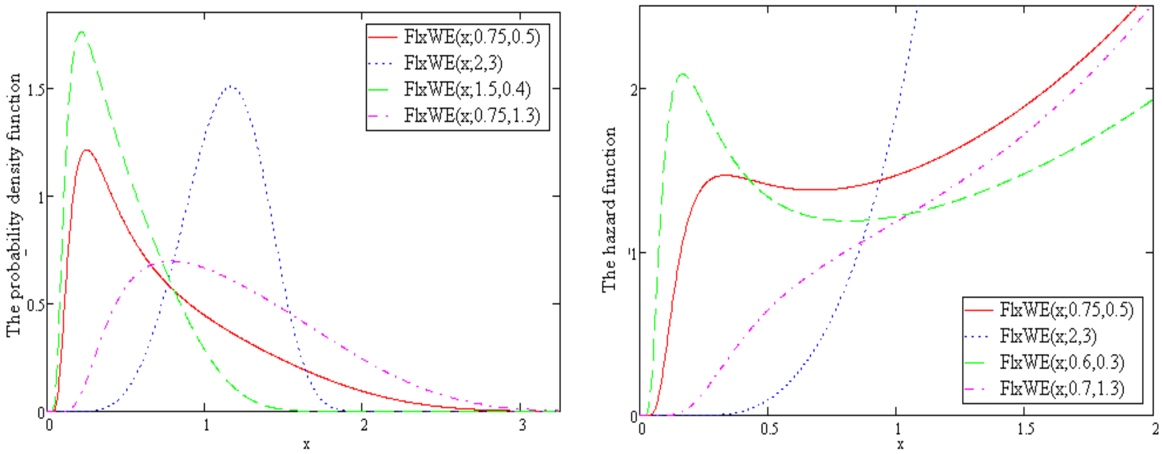
$$F(x; \alpha, \beta) = 1 - \exp\left[-e^{\alpha x - \frac{\beta}{x}}\right], \quad x > 0,$$

$$f(x; \alpha, \beta) = \left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}} \exp\left[-e^{\alpha x - \frac{\beta}{x}}\right], \quad x > 0$$

and

$$h(x; \alpha, \beta) = \left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}}, \quad x > 0,$$

where $\alpha, \beta > 0$. Some possible shapes of the PDF and the HF are illustrated in Figure 2.20.



(a) (b)

Figure 2.20: PDF and HF of the flexible Weibull extension

2.2.17 Generalized modified Weibull distribution

As mentioned before, Carrasco *et al.* (2008) generalized the MW distribution of Lai *et al.* (2003) by adding another shape parameter. We refered to the generalization as the GMW distribution. Its CDF is

$$F(x;\beta,\gamma,\lambda,\theta) = \left(1 - e^{-\beta x^{\gamma} e^{\lambda x}}\right)^{\theta}, \ x > 0, \tag{2.10}$$

where $\beta, \theta > 0$, $\gamma, \lambda \geq 0$ and at most one γ, λ is equal to zero. The GMW distribution has one scale parameter, β, one acceleration parameter, λ, and two shape parameters, γ and θ, giving it more flexibility. The particular case for $\lambda = 0$ and $\theta = 1$ is the Weibull distribution. The particular case for $\gamma = 0$ and $\theta = 1$ is the type I extreme value distribution. The particular case for $\theta = 1$ is the MW distribution. The particular case for $\lambda = 0$ is the EW distribution. The PDF and the HF of the GMW distribution are

$$f(x;\beta,\gamma,\lambda,\theta) = \beta\theta x^{\gamma-1}\left(\gamma + \lambda x\right) e^{\lambda x - \beta x^{\gamma} e^{\lambda x}} \left[1 - e^{-\beta x^{\gamma} e^{\lambda x}}\right]^{\theta-1}, \ x > 0 \tag{2.11}$$

and

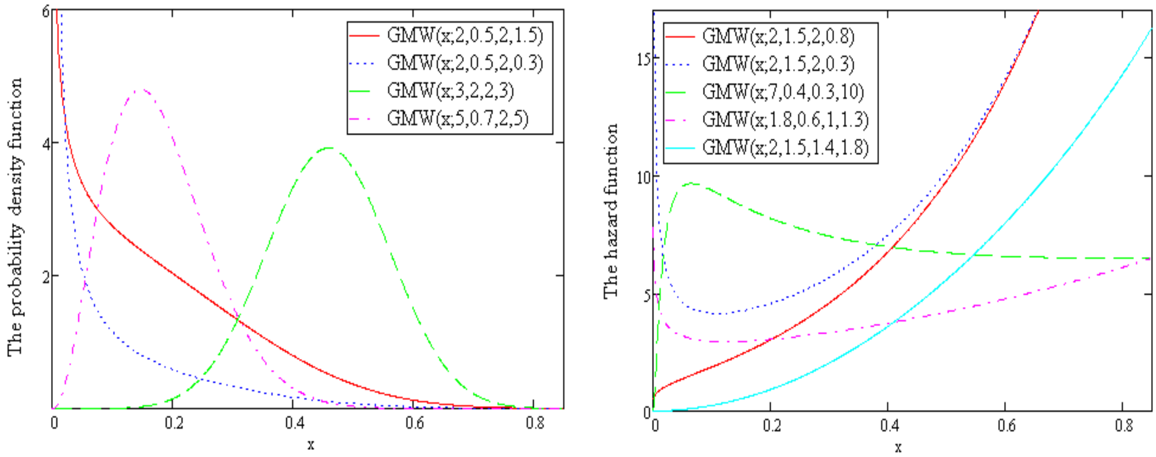
$$h(x;\beta,\gamma,\lambda,\theta) = \frac{\beta\theta x^{\gamma-1}\left(\gamma + \lambda x\right) e^{\lambda x - \beta x^{\gamma} e^{\lambda x}} \left[1 - e^{-\beta x^{\gamma} e^{\lambda x}}\right]^{\theta-1}}{1 - \left(1 - e^{-\beta x^{\gamma} e^{\lambda x}}\right)^{\theta}}, \ x > 0.$$

The HF exhibits the following shapes:

- Increasing if either $\gamma \geq 1$, $\theta < 1$ and $\gamma\theta > 1$ or $\gamma > 1$ and $\theta > 1$;

- Decreasing if $\gamma > 1, \lambda = 0$ and $\gamma\theta < 1$;
- Bathtub shaped if either $\gamma < 1$ and $\gamma\theta < 1$ or $\gamma \geq 1, \theta < 1$ and $\gamma\theta < 1$;
- Unimodal shaped if $\gamma < 1$ and $\theta \to \infty$.

Figure 2.21 shows possible shapes of the PDF and the HF of the GMW distribution.



(a) (b)

Figure 2.21: PDF and HF of the generalized modified Weibull distribution

Ortega *et al.* (2011) developed a regression model where the lifetimes follow the GMW distribution affected by some explanatory variables.

2.2.18 Sarhan and Zaindin's modified Weibull distribution

Sarhan and Zaindin (2009) introduced a three-parameter distribution, known as modified Weibull distribution, SZMW. This distribution is the same as the β distribution of Bousquet *et al.* (2006). Its PDF and CDF are

$$f(x;\alpha,\beta,\gamma) = \left(\alpha + \beta\gamma x^{\gamma-1}\right) e^{-\alpha x - \beta x^{\gamma}}, \quad x > 0$$

and

$$F(x;\alpha,\beta,\gamma) = 1 - e^{-\alpha x - \beta x^{\gamma}}, \quad x > 0,$$

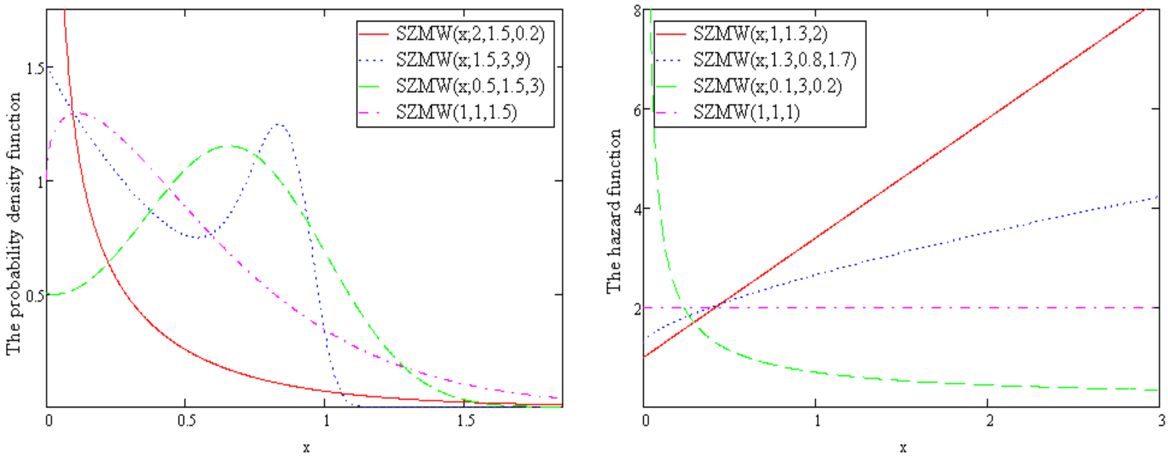
where $\alpha, \beta, \gamma > 0$. This distribution can be obtained as a particular case of the AddW distribution by setting one of its two shape parameters, θ and γ, in (2.3) to be one.

The exponential, Rayleigh, linear failure rate and Weibull distributions are particular cases of the SZMW distribution. The HF of the SZMW distribution is

$$h(x;\alpha,\beta,\gamma)=\alpha+\beta\gamma x^{\gamma-1},\quad x>0.$$

Unfortunately, the HF cannot exhibit non-monotonic shapes like the bathtub shape or the unimodal shape. It is monotonically increasing if $\gamma>1$ and monotonically decreasing if $\gamma<1$, as shown in Figure 2.22. The same distribution had proposed in (2006) by Bousquet *et al.*, it called the β distribution, Bousquet *et al.* (2006).

The SZMW distribution was generalized by Zaindin and Sarhan (2011), allowing the HF to exhibit increasing, decreasing, bathtub and unimodal shapes as well as giving the PDF more flexibility, as shown in Figure 2.23. Further studies of the SZMW distribution can be found in Zaindin and Sarhan (2009), Al-Hadhrami (2010), Zaindin (2010) and Gasmi and Berzig (2011).



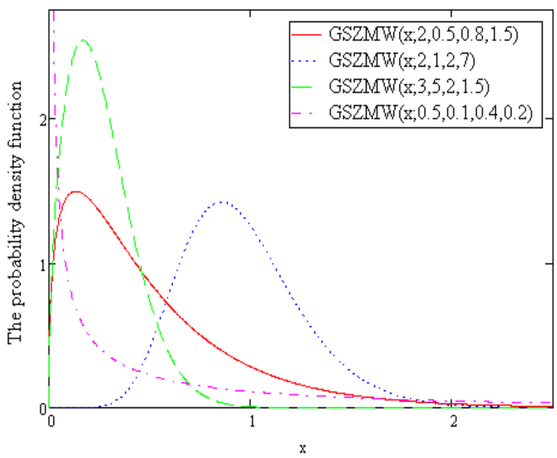
(a) (b)

Figure 2.22: PDF and HF of Sarhan and Zaindin's modified Weibull distribution

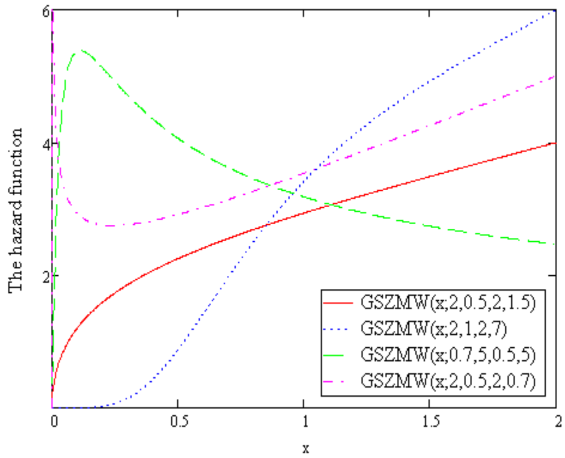
2.2.19 Kumaraswamy Weibull distribution

Cordeiro *et al.* (2010) introduced the Kumaraswamy Weibull (KumW) distribution. The KumW distribution has four parameters three of which are shape parameters, adding great flexibility. The CDF, the PDF and the HF of the KumW distribution are

$$F(x;\alpha,\theta,a,b)=1-\left[1-\left(1-e^{-\alpha x^{\theta}}\right)^{a}\right]^{b},\quad x>0,$$



(a)



(b)

Figure 2.23: PDF and HF of Sarhan and Zaindin's generalized modified Weibull distribution

$$f(x;\alpha,\theta,a,b) = ab\alpha\theta x^{\theta-1}e^{-\alpha x^{\theta}}\left[1-e^{-\alpha x^{\theta}}\right]^{a-1}\left[1-\left(1-e^{-\alpha x^{\theta}}\right)^{a}\right]^{b-1}, \quad x>0$$

and

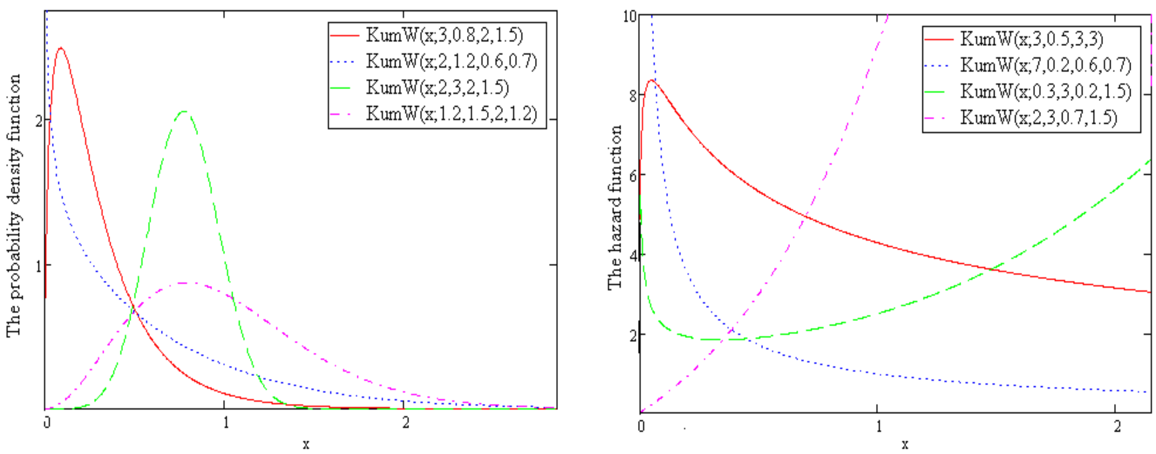
$$h(x;\alpha,\theta,a,b) = \frac{ab\theta x^{\theta-1}e^{-\alpha x^{\theta}}\left[1-e^{-\alpha x^{\theta}}\right]^{a-1}}{1-\left[1-e^{-\alpha x^{\theta}}\right]^{a}}, \quad x>0,$$

where $\alpha, \theta, a, b > 0$, the shape parameters being a, b and θ. The HF can be constant, increasing, decreasing, bathtub shaped and unimodal shaped. Particular case of the KumW distribution include the EW distribution due to Mudhalkar and Srivastava (1993), the GE distribution due to Gupta and Kundu (1999), the GR distribution due to Surles and Padgett (2001), the Kumaraswamy exponential (KumE) distribution, the Kumaraswamy Rayleigh distribution (KumR), the Weibull distribution, the Rayleigh distribution and the exponential distribution.

Lemonte *et al.* (2013) proposed the five-parameter exponentiated Kumaraswamy Weibull (EKum) distribution given by the CDF

$$F(x) = \left\{1-\left[1-\left(1-\exp\left(-\delta x^{\theta}\right)\right)^{\alpha}\right]^{\beta}\right\}^{\gamma}, \quad x>0,$$

where $\alpha, \beta, \gamma, \delta, \theta > 0$. This distribution contains the KumW distribution as the particular case for $\gamma = 1$.



(a) (b)

Figure 2.24: PDF and HF of Kumaraswamy Weibull distribution

2.2.20 Kumaraswamy modified Weibull distribution

Cordeiro *et al.* (2014b) proposed a modification of the Weibull distribution called Kumaraswamy modified Weibull (KumMW) distribution. Its CDF, PDF and HF are

$$F(x;\alpha,\theta,\lambda,a,b) = 1 - \left\{1 - \left[1 - \exp\left(-\alpha x^{\theta} e^{\lambda x}\right)\right]^{a}\right\}^{b},\ x > 0,$$

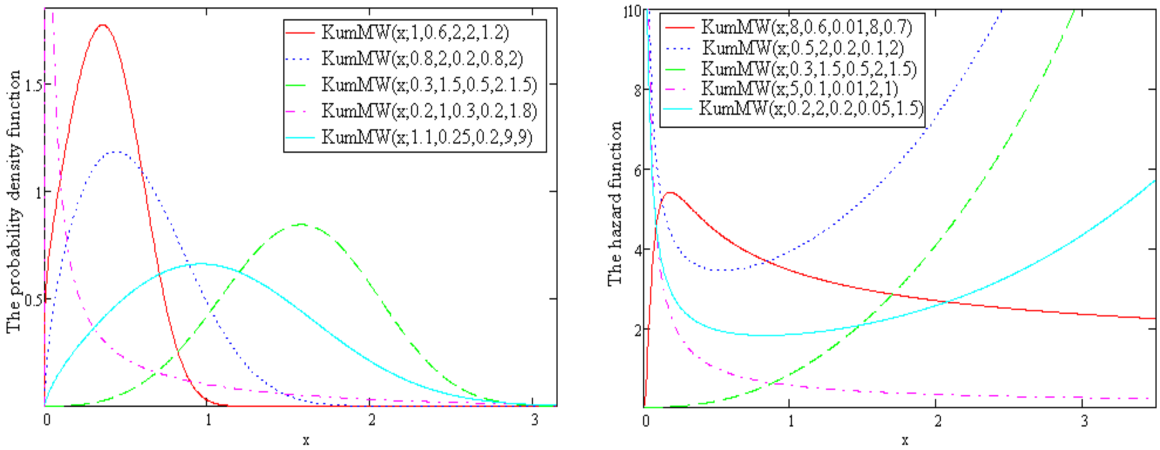
$$\begin{aligned} f(x;\alpha,\theta,\lambda,a,b) &= ab\alpha x^{\theta-1}(\theta+\lambda x)\exp\left(\lambda x - \alpha x^{\theta} e^{\lambda x}\right)\left[1 - \exp\left(-\alpha x^{\theta} e^{\lambda x}\right)\right]^{a-1} \\ &\quad \left\{1 - \left[1 - \exp\left(-\alpha x^{\theta} e^{\lambda x}\right)\right]^{a}\right\}^{b-1},\ x > 0 \end{aligned}$$

and

$$h(x;\alpha,\theta,\lambda,a,b) = \frac{ab\alpha x^{\theta-1}(\theta+\lambda x)\exp\left(\lambda x - \alpha x^{\theta} e^{\lambda x}\right)\left[1 - \exp\left(-\alpha x^{\theta} e^{\lambda x}\right)\right]^{a-1}}{\{1 - [1 - \exp(-\alpha x^{\theta} e^{\lambda x})]^{a}\}},\ x > 0,$$

where $\alpha, \theta, \lambda, a, b > 0$. Particular cases include the GMW distribution, the MW distribution, the LogW distribution and the KumW distribution. As shown in Figure 2.25, the PDF of the KumMW distribution can be very flexible and its HF can be increasing, decreasing, bathtub shaped and unimodal shaped.

Cordeiro *et al.* (2014b) provided a thorough study of the mathematical properties of the KumMW distribution. They derived expressions for its moments, moment generating function, order statistics, reliability, mean deviations, Bonferroni curve and Lorenz curve as well as considered maximum likelihood estimation and a log-KumMW regression model using censoring data.



(a) (b)

Figure 2.25: PDF and HF of Kumaraswamy modified Weibull distribution

2.3 Discrete Weibull distributions

This section reviews discrete analogues of the Weibull distribution and their modifications including shapes of their probability mass functions, PMFs, and HFs.

2.3.1 Discrete Weibull distribution I

The first discrete analogue of the two-parameter continuous Weibull distribution was introduced by Nakagawa and Osaki (1975). It was called type I discrete Weibull (DW(I)) distribution. The survival function (SF), the PMF and the HF of the DW(I) distribution are

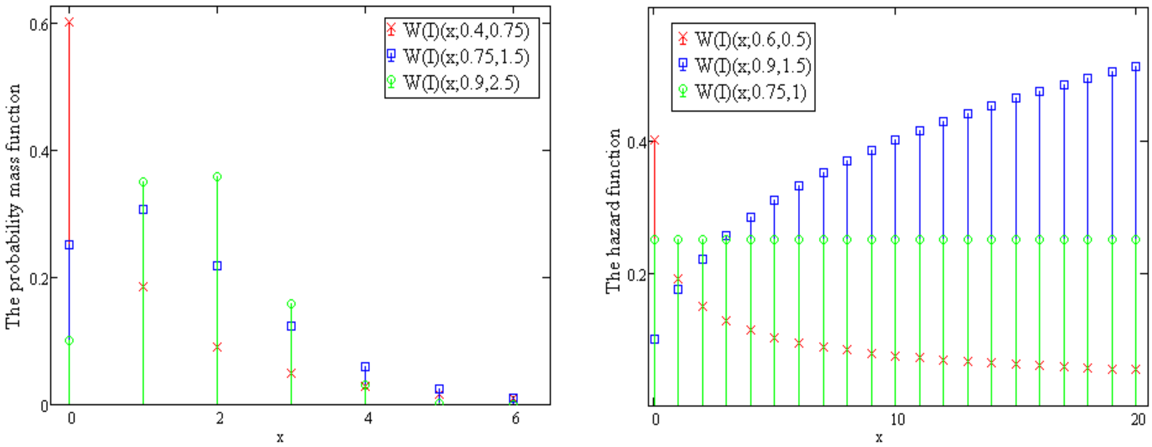
$$S(x; q, \theta) = q^{x^\theta},$$

$$p(x; q, \theta) = q^{x^\theta} - q^{(x+1)^\theta},$$

and

$$h(x; q, \theta) = 1 - q^{(x+1)^\theta - x^\theta}$$

for $x = 0, 1, \ldots$, $0 < q < 1$ and $\theta > 0$. The HF increases when $\theta > 1$, decreases when $\theta < 1$ and constant when $\theta = 1$. Figure 2.26 shows possible shapes of the PMF and the HF.



(a) (b)

Figure 2.26: PMF and HF of the discrete Weibull distribution I

2.3.2 Discrete Weibull distribution II

Stein and Dattero (1984) proposed another discrete Weibull distribution, later called type II discrete Weibull (DW(II)) distribution, by considering lifetimes as the integer part of the continuous Weibull distribution. The HF of the DW(II) distribution is

$$h(x;\alpha,\theta) = \begin{cases} \alpha\theta x^{\theta-1}, & \text{for} \quad \text{x=1,2, } \dots\text{, m,} \\ 0, & \text{for} \quad \text{x=0,} \end{cases}$$

where $h_x \leq 1$ and m is a positive integer defined as

$$m = \begin{cases} \text{int}\left\{\alpha^{-(\theta-1)^{-1}}\right\}, & \text{if} \quad \theta > 1, \\ +\infty, & \text{if} \quad \theta \leq 1. \end{cases}$$

The PMF and the HF of the DW(II) distribution are plotted for selected parameter values in Figure 2.27.

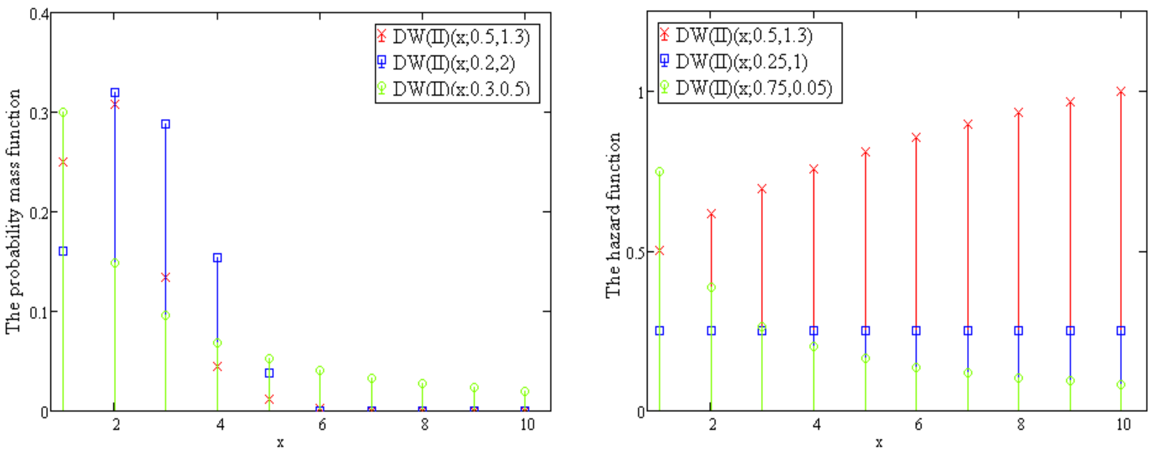
2.3.3 Discrete Weibull distribution III

The third discrete version of the Weibull distribution was suggested by Padgett and Spurrier (1985). It can exhibit increasing, decreasing and constant HFs. It is called type III discrete Weibull (DW(III)) distribution and its HF is

$$h(x;\alpha,\theta) = 1 - \exp\left\{-\alpha(x+1)^{\theta}\right\}, \qquad x = 0, 1, 2, \dots,$$

where $\alpha > 0$ and $-\infty < \theta < \infty$. The PMF is

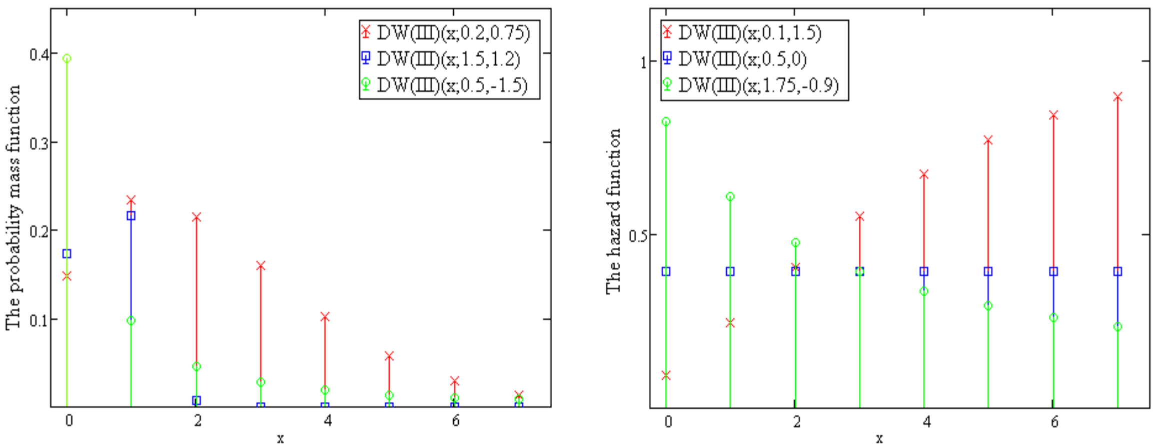
$$p(x;\alpha,\theta) = \left(1 - \exp\left\{-\alpha(x+1)^{\theta}\right\}\right) e^{-\alpha\sum_{j=1}^{x} j^{\theta}}, \qquad x = 0, 1, 2, \dots.$$



(a) (b)

Figure 2.27: PMF and HF of the discrete Weibull distribution II

The HF increases if $\theta > 0$ and decreases if $\theta < 0$. The distribution reduces to the geometric distribution if $\theta = 1$. Figures 2.28 (a) and (b) show the PMF and the HF for selected parameter values.



(a) (b)

Figure 2.28: PMF and HF of the discrete Weibull distribution III

2.3.4 Discrete inverse Weibull distribution

Jazi *et al.* (2010) introduced the discrete inverse Weibull (DIW) distribution with two parameters, by considering a characteristic property of the continuous inverse Weibull distribution. The CDF and the PMF of the DIW distribution are

$$F(x; q, \theta) = q^{x^{-\theta}}, \qquad x = 1, 2, \dots,$$

and

$$p(x; q, \theta) = \begin{cases} q, & \text{if} \quad x = 1, \\ q^{x^{-\theta}} - q^{(x-1)^{-\theta}}, & \text{if} \quad x = 2, 3, \ldots, \end{cases}$$

where $0 < q < 1$ and $\theta > 0$. The standard HF defined by

$$h_1(x; q, \theta) = \frac{\Pr(X = x)}{\Pr(X \geq x)}$$

takes the form

$$h_1(x; q, \theta) = \frac{q^{x^{-\theta}} - q^{(x-1)^{-\theta}}}{1 - q^{(x-1)^{-\theta}}}, \qquad x = 1, 2, \ldots.$$

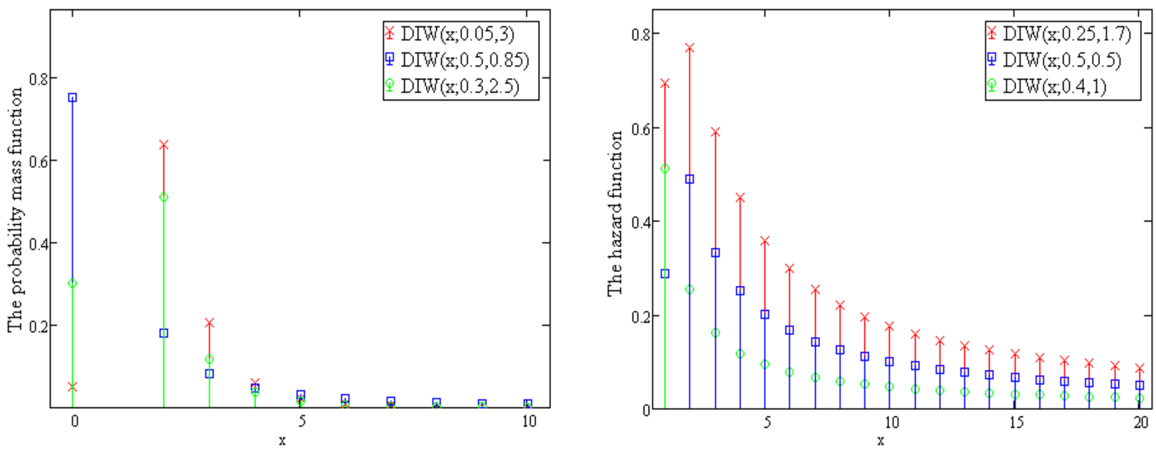
The alternative HF defined by

$$h_2(x; q, \theta) = \log\left[\frac{\Pr(X > x - 1)}{\Pr(X > x)}\right]$$

(see equation (8) in Jazi *et al.* (2010)) takes the form

$$h_2(x; q, \theta) = \log\left(\frac{1 - q^{(x-1)^{-\theta}}}{1 - q^{x^{-\theta}}}\right), \qquad x = 1, 2, \ldots.$$

If $(1-q)^{-1} \geq \sqrt{1 - q^{2^{-\theta}}}$, the alternative HF of the DIW distribution will be decreasing and otherwise it will take a unimodal shape. Figure 2.29 shows the PMF and the alternative HF for selected parameter values.



(a) (b)

Figure 2.29: PMF and HF of the discrete inverse Weibull distribution

2.3.5 Discrete modified Weibull distribution

An interesting discrete distribution related to the Weibull distribution is the discrete modified Weibull (DMW) distribution proposed by Nooghabi *et al.* (2011). This distribution is the discrete analogue of the modified Weibull distribution of Lai *et al.* (2003). The SF of the DMW distribution is

$$S(x;q,\theta,c) = q^{x^\theta c^x}, \qquad x = 0,1,2,\ldots$$

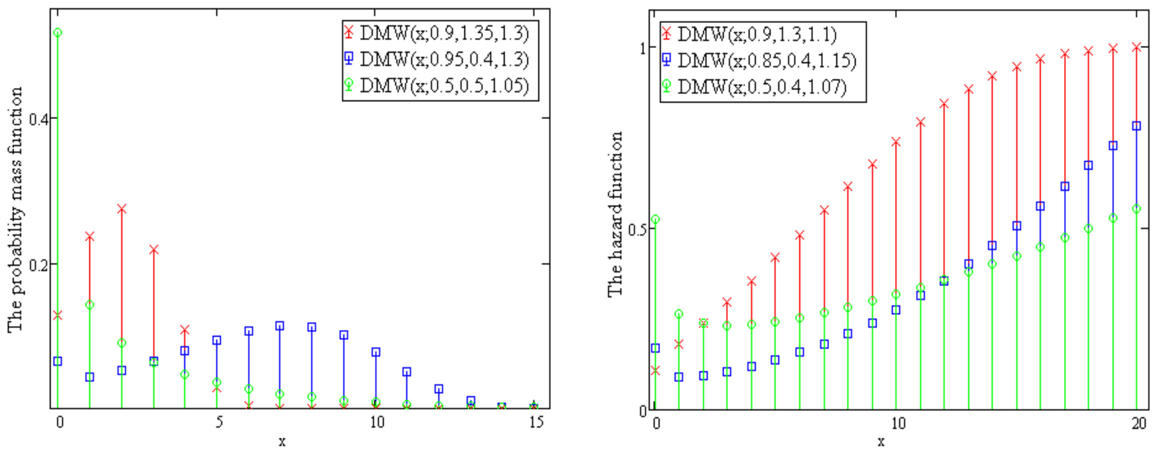
while its PMF and HF are

$$p(x;q,\theta,c) = q^{x^\theta c^x} - q^{(x+1)^\theta c^{x+1}}, \qquad x = 0,1,2,\ldots,$$

and

$$h(x;q,\theta,c) = 1 - q(x+1)^\theta c^{x+1} - x^\theta c^x,$$

where $0 < q < 1$, $\theta > 0$ and $c \geq 0$. The PMF of the DMW distribution can take different shapes: decreasing, unimodal or decreasing followed by unimodal. Its HF increases when $\theta < 1$ and $c2^{b-1} < 1$ and is bathtub shaped otherwise. The shapes of the PMF and HF are shown in Figure 2.30 for selected parameter values.



(a) (b)

Figure 2.30: PMF and HF of the discrete modified Weibull distribution

2.3.6 Discrete additive Weibull distribution

Bebbington *et al.* (2012) introduced the discrete additive Weibull (DAddW) distribution with four parameters. The SF, the PMF and the HF of the DAddW distribution are

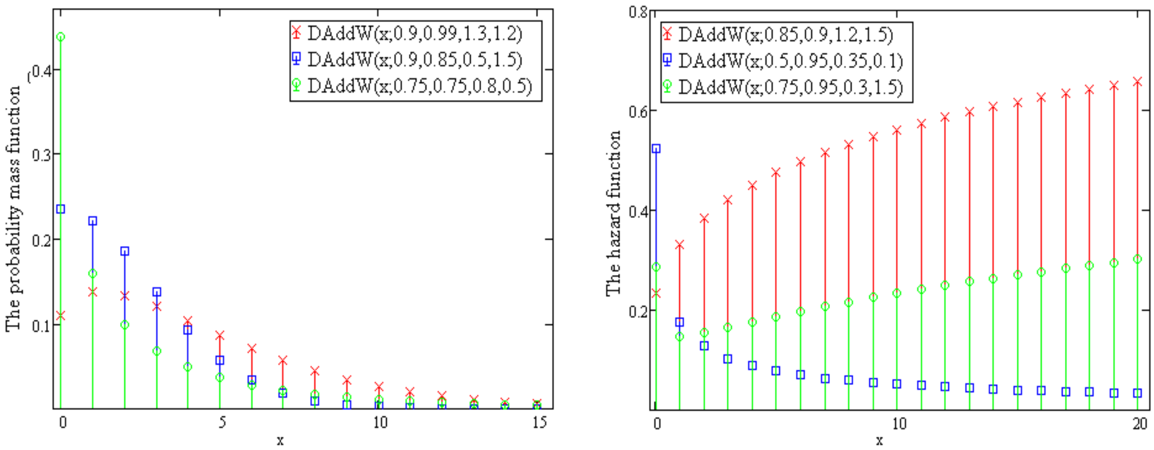
$$S\left(x;q_1,q_2,\theta,\gamma\right)=q_1^{x^{\theta}}q_2^{x^{\gamma}},\qquad x=0,1,2,\ldots,$$

$$p\left(x;q_1,q_2,\theta,\gamma\right)=q_1^{x^{\theta}}q_2^{x^{\gamma}}-q_1^{(x+1)^{\theta}}q_2^{(x+1)^{\gamma}},$$

and

$$h\left(x;q_1,q_2,\theta,\gamma\right)=1-q_1^{(x+1)^{\theta}-x^{\theta}}q_2^{(x+1)^{\gamma}-x^{\gamma}},$$

where $0<q_1,q_2<1$ and $\theta,\gamma>0$. The HF increases if $\theta\geq 1$ and $\gamma>1$ ($\theta>1$ and $\gamma\geq 1$), decreases if $\theta\leq 1$ and $\gamma<1$ ($\theta<$ and $\gamma\leq 1$) and is bathtub shaped if $\theta<1<\gamma$ ($\gamma<1<\theta$). Figure 2.31 shows the different shapes of the PMF and the HF.



(a) (b)

Figure 2.31: PMF and HF of the discrete additive Weibull distribution

2.4 Maximum likelihood estimation

Given a random sample $x_1,\ \ldots,\ x_n$ of n independent and identically distributed observations which coming from a lifetime distribution with PDF (PMF) $f(x;\underline{\phi})$, where $\underline{\phi}=(\phi_1,\phi_2,\ldots,\phi_k)$, the joint probability density (mass) function for the above sample is given by

$$f(x_1,\ldots,x_n|\underline{\phi})=\prod_{i=1}^{n}f(x_i;\underline{\phi}). \tag{2.12}$$

When $x_1, \ldots, x_n$ are observed the above joint probability density (mass) function becomes a function in $\underline{\phi}$, which is the likelihood function, denoted $L(\underline{\phi})$ and given by

$$L(\underline{\phi}|x_1,\ldots,x_n) = \prod_{i=1}^{n} f(x_i;\underline{\phi}). \tag{2.13}$$

It is more convenient in practice to deal with the logarithm of the likelihood function, the log-likelihood function, denoted as $\mathcal{L}$ and given by

$$\mathcal{L}(\underline{\phi}|x_1,\ldots,x_n) = \sum_{i=1}^{n} \log f(x_i;\underline{\phi}). \tag{2.14}$$

The likelihood equations are obtained by setting the first partial derivatives of $\mathcal{L}$ with respect to $\phi_1, \phi_2, \ldots, \phi_k$ to zero; that are,

$$\frac{\partial(\mathcal{L}(\underline{\phi}|x_1,\ldots,x_n))}{\partial\phi_i} = 0, \quad i = 1, 2, \ldots, k. \tag{2.15}$$

By solving the above system of nonlinear likelihood equations numerically for $\phi_1, \phi_2, \ldots \phi_k$, we can obtain maximum likelihood estimators of $\phi_1, \phi_2, \ldots, \phi_k$. This can be done using R (2013), Matlab and Mathcad, among other packages.

2.5 Goodness of Fit

There are different methods that can be used for testing whether a given random sample $x_1, x_2, ..., x_n$, of n observations, is coming from a specific distribution or for comparing the underlying distribution with other distributions for fitting a given data set. This section contains two different methods of goodness of fit. The first method is the likelihood ratio test and the second one is the Kolmogorov-Smirnov test.

2.5.1 The Log-likelihood Ratio Test

The log-likelihood ratio test (LRT) is one of the popular tools that can be used to compare two models where one of them is nested in the other one, and then determine which one is more appropriate for a given data.

Suppose that a given random sample $x_1, \ldots, x_n$ with probability distribution $f(x;\underline{\phi})$ and we are interested to test the following null and alternative hypotheses

$$H_0 \ : \ \underline{\phi} \in \Phi_0$$
$$H_1 \ : \ \underline{\phi} \in \Phi_1$$

where $\Phi_0 \subset \Phi_1$, Φ_0 is the parameter space of the reduced model and Φ_1 is the parameter space of the original (full) model. The log-likelihood ratio statistic (ω) is defined as following

$$\omega = -2\ln\left(\frac{L_0(\hat{\phi})}{L1(\hat{\phi})}\right) \tag{2.16}$$

where $L_0(\hat{\phi})$ is the likelihood function of the reduced model, while $L_1(\hat{\phi})$ is the likelihood function of the original model. Under the H_0, ω follows a chi square distribution with $(k_1 - k_0)$ degrees of freedom, where k_1 and k_0 are the number of parameters of the original and reduced models respectively. The full model could be preferred if $\omega > \chi^2_{0.95,k_1-k_0}$.

2.5.2 Kolmogorov−Smirnov test

Kolmogorov (1933) proposed the Kolmogorov−Smirnov test (K-S test) for testing while a given random sample $x_1, x_2, ..., x_n$ belongs to a population with a specific distribution. The K-S test calculates the distance between the empirical distribution function of the given sample and the estimated cumulative distribution function of the candidate distribution. The null and alternative hypotheses are H_0: sample follow the specific distribution versus $H_1 : H_0$ is false.

Let $F(x_i)$ denote the value of the cumulative distribution function of the candidate distribution at x_i and $\hat{F}(x_i)$ denote the value of the empirical distribution function at x_i. The value of the K-S test statistic is define by

$$\text{K-S} = \max\left\{|F(x_i) - \hat{F}(x_i)|, |F(x_i) - \hat{F}(x_{i-1})|\right\},$$

where $\hat{F}(x_i) = \frac{\#\{x_j : x_j \le x_i\}}{n}$.

The computed K-S statistic is then compared with the tabulated K-S at a significance level α to decide if the null hypothesis is not rejected. Moreover, if there are more than one distribution to be compared, the distribution with smaller K-S value will be more appropriate to fit the given sample.

2.6 Total time on test

Total time on test transformation, denoted as (TTT-transform), is a graphical technique to provide information about the shape of the hazard rate of a given data. It was proposed by Barlow and Doksum (1972) for statistical inference problems under order restrictions. For modelling a data set, Barlow and Campo (1975) used this test as a selection method. Aarset (1987) used the TTT-transform to test if a random sample belongs to a life distribution with bathtub shaped hazard rate.

The TTT-transform of a distribution with a CDF F is defined as

$$H^{-1}(p) = \int_0^{F^{-1}(p)} S(u)du, \qquad 0 \leq p \leq 1.$$

The scaled TTT-transform of the distribution is

$$\varphi_F(p) = \frac{H^{-1}(p)}{H^{-1}(1)}.$$

The curve $\varphi_F(p)$ versus $0 \leq p \leq 1$ is called the scaled TTT-transform curve. It has been proposed by Barlow and Campo (1975), using the scaled TTT-transform curve that the shape of the hazard function of the distribution can be classified as one of the following.

- If the scaled TTT-transform curve is concave above the 45° line, the hazard rate is increasing.
- If the scaled TTT-transform curve is convex below the 45° line, then the hazard rate is decreasing.
- If the scaled TTT-transform curve is first convex below the 45° line then concave above the line the shape of the hazard rate is a bathtub shaped.
- The shape of the hazard rate will be unimodal shaped if the scaled TTT-transform curve is first concave above the 45° line followed by convex below the 45° line.

Figure 2.32 summarizes the different shapes of the scaled TTT-transform curve for distributions with increasing, decreasing, bathtub and unimodal hazard rate functions.

For an ordered sample $0 = x_{0:n}, x_{1:n}, x_{2:n}, ..., x_{n:n}$, the total time one test statistics are given by

$$TTT_i = \sum_{j=1}^{i} (n - j + 1)\left(x_{j:n} - x_{j-1:n}\right), i = 1, 2, ..., n.$$

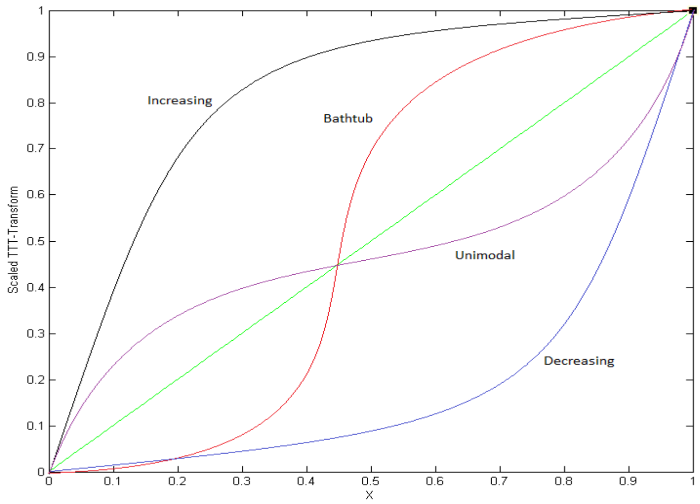


Figure 2.32: Scaled TTT-transform curve for different distributions.

The empirical scaled TTT-transform is

$$TTT_i^* = \frac{TTT_i}{TTT_n},$$

where $0 \leq TTT_n \leq 1$. The TTT-plot can be provided by plotting $\frac{i}{n}$ against TTT_i^*.

2.7 Information criterion

As a consequence of increasing the number of parameters, usually the fitting of the data sets will be improved and of course the likelihood will be increased. A model with too many parameters could be preferred in the comparison using the maximum likelihood. Then, an information criterion can be used to make a comparison between different statistical models which may have different numbers of parameters. The measures Akaike information criterion (AIC), Bayesian information criterion (BIC), corrected Akaike information criterion (AICc), and consistent Akaike information criterion (CAIC) are widely used information criterion for selecting the appropriate model among different others models. The Akaike information criterion due to Akaike (1974) is defined by

$$\text{AIC} = -2\mathcal{L}\left(\widehat{\underline{\phi}}; x_i\right) + 2k,$$

where $x_1, \ldots, x_n$ is the given random sample, $\widehat{\underline{\phi}}$ is the MLE of $\underline{\phi}$ and k is the length of the vector $\underline{\phi}$. The model which has smallest AIC could be the most appropriate model to fit the given data set.

When the sample size n is not large or if the number of parameters k is large, the probability to select the model with many parameters will be increased using AIC. Therefore, Hurvich and Tsai (1989) has proposed a corrected Akaike information criterion AICc that is defined as

$$\mathrm{AICc} = \mathrm{AIC} + \frac{2k(k+1)}{n-k-1}, \tag{2.17}$$

where k is the length of the vector $\underline{\phi}$ and n is the sample size. When the sample size n is too much larger than k, AICc converges to AIC. It is strongly recommended to use AICc instead of AIC if the sample size is not large or the model has too many parameters, (Burnham and Anderson, 2002).

Bozdogan (1987) provided consistent Akaike Information Criterion (CAIC), which is defined by

$$\mathrm{CAIC} = -2\mathcal{L} + k(\ln(n) + 1); \tag{2.18}$$

The Bayesian information criteria due to Schwarz (1978) is defined by

$$\mathrm{BIC} = k \ln n - 2\mathcal{L}\left(\widehat{\underline{\phi}}; x_i\right), \tag{2.19}$$

where k is the length of the vector $\underline{\phi}$ and n is the sample size. It is also called Schwarz Bayesian criterion (SBC) or (SBIC).

2.8 Summary

This chapter gave a comprehensive review of known discrete modifications and generalizations of the Weibull distribution, a comprehensive review of known continuous modifications and generalizations of the Weibull distribution and some tools and methods that will be used for data analysis in the next chapters. The review part contains over 100 references on modifications/generalizations of the Weibull distribution and more than 55 percent of the cited references appeared in the last 5 years.

Chapter 3

New Modified Weibull Distribution

3.1 Introduction

In this chapter we propose a new lifetime distribution by considering a serial system with one component following a Weibull distribution and another following a modified Weibull distribution. We shall refer to this distribution as *New Modified Weibull Distribution* (NMW). It contains several popular lifetime distributions as special sub-models. We study its mathematical properties including moments, moment generating function and order statistics. The shapes of the probability density function and the hazard function of the NMW, simulation methods and the estimation of parameters by maximum likelihood are discussed. It will be demonstrated that the proposed distribution fits two well-known data sets better than other modified Weibull distributions including the latest beta modified Weibull distribution. The model can be simplified by fixing one of the parameters whilst still providing a better fit than existing models.

Among the modifications of Weibull distributions, the additive Weibull distribution of Xie and Lai (1996) has a bathtub-shaped hazard function

$$h(x) = \alpha\theta x^{\theta-1} + \beta\gamma x^{\gamma-1},\ x > 0, \tag{3.1}$$

as it combines two Weibull hazard functions, an increasing function $\alpha\theta x^{\theta-1}$ with $\theta > 1$ and a decreasing function $\beta\gamma x^{\gamma-1}$ with $0 < \gamma < 1$. When $\theta = 1$ and $\gamma > 0$ it becomes the modified Weibull distribution of Sarhan and Zaindin (2009) (SZMW) which includes the linear failure rate distribution when $\gamma = 2$. The modified Weibull distribution of Lai *et al* (2003) multiplies the Weibull cumulative hazard function αx^{β} by

$e^{\lambda x}$. These modifications plus the beta modified Weibull distribution (BMW) proposed by Silva *et al* (2010) are the most important modifications of the Weibull because they are flexible and fit real data sets very well. Unfortunately, these distributions do not fit the last phase of the bathtub shape as well as the first and the middle phases. In this case the BMW fits this kind of the data well. This distribution was shown to be the best lifetime distribution to fit a popular and widely used data sets such as the Aarset data Aarset (1987), the voltage data of Meeker and Escobar (1998), the Kumar (LHD) data of Kumar *et al* (1989) and Serum-reversal data of Silva (2004). A disadvantage of the BMW is that its CDF does not have a closed form of expression and its hazard rate function is quite complicated.

3.2 Definition

This section provides the definition of the new model. We define a new modified Weibull distribution (NMW) by the following CDF

$$F(x) = 1 - e^{-\alpha x^{\theta} - \beta x^{\gamma} e^{\lambda x}}, \ x > 0, \tag{3.2}$$

where the parameters α, β, θ, γ and λ are non-negative, with θ and γ being shape parameters and α and β being scale parameters and λ is an acceleration parameter.

We propose a new lifetime distribution NMW which can fit all three parts of the data very well and at the same time its hazard rate function is simple and its CDF has a closed form. The survival function of this distribution is

$$S(x) = e^{-\alpha x^{\theta} - \beta x^{\gamma} e^{\lambda x}}, \tag{3.3}$$

and the hazard function is

$$h(x) = \alpha\theta x^{\theta-1} + \beta(\gamma + \lambda x) x^{\gamma-1} e^{\lambda x}, \ x > 0. \tag{3.4}$$

This hazard function can be interpreted as that of a serial system with two independent components, one of which follows the Weibull distribution with parameters α and θ, and the other follows the modified Weibull distribution of Lai *et al* (2003) with parameters β, γ and λ. This suggests that the distribution can be used when there are two types of failure, as shown in examples later.

The probability density function (PDF) is

$$f(x) = \left(\alpha\theta x^{\theta-1} + \beta(\gamma+\lambda x)x^{\gamma-1}e^{\lambda x}\right) e^{-\alpha x^{\theta}-\beta x^{\gamma}e^{\lambda x}}, \; x > 0. \tag{3.5}$$

It can be rewritten as

$$f(x) = [h_W(x;\alpha,\theta) + h_{MW}(x;\beta,\gamma,\lambda)]\, S_W(x,\alpha,\theta)S_{MW}(x;\beta,\gamma,\lambda), \tag{3.6}$$

where S_W, h_W, S_{MW} and h_{MW} are survival and hazard functions of the Weibull and modified Weibull distributions respectively, see equations 1.2, 1.4, 2.7 and 2.9.

Fig. 3.1 and 3.2 show the PDFs and the hazard functions for different parameter values. It is clear that the PDF and the hazard function have many different shapes, which allows this distribution to fit different types of lifetime data. The shapes of these functions will be discussed in more details later.

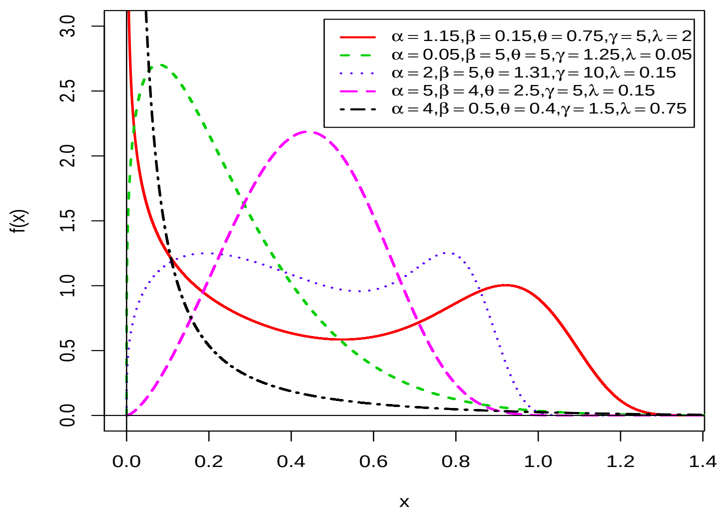


Figure 3.1: Probability density functions of the NMW.

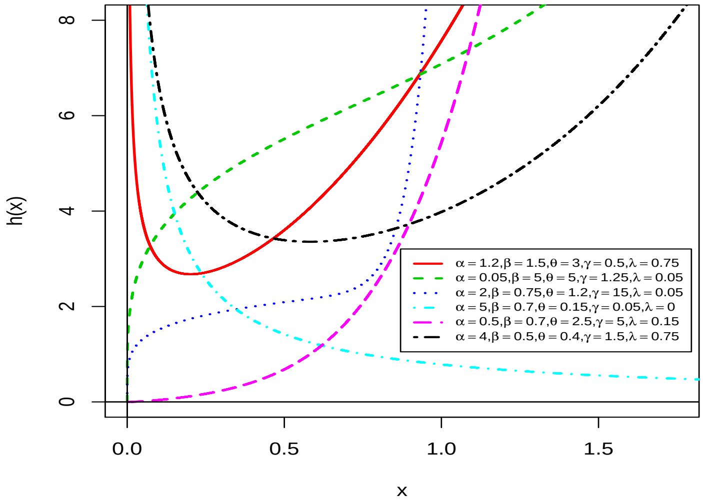


Figure 3.2: Hazard functions of the NMW.

3.3 Sub-models

This distribution includes some important sub models that are widely used in survival analysis. In particular, from (3.2) the NMW becomes:

1. **The AddW distribution**

   If $\lambda = 0$, the NMW distribution reduces to the additive Weibull distribution (Xie *et al*, 2006) with CDF
$$F(x) = 1 - e^{-\alpha x^{\theta} - \beta x^{\gamma}}.$$

2. **The MW distribution**

   When $\alpha = \theta = 0$, we get the modified Weibull of Lai *et al* (2003) with the following CDF
$$F(x) = 1 - e^{-\beta x^{\gamma} e^{\lambda x}}.$$

3. **The extreme-value distribution of Type I**

   The LogW (extreme-value distribution of Type I), Bain (1974), can be obtained from the NMW by setting $\alpha = \theta = \gamma = 0$ to give
$$F(x) = 1 - e^{-\beta e^{\lambda x}}.$$

4. **Linear failure rate distribution**

   Setting $\theta = 1, \gamma = 2$ and $\lambda = 0$, the NMW distribution reduces to the linear failure rate distribution of Bain (1974) with CDF

$$F(x) = 1 - e^{-\alpha x - \beta x^2}.$$

5. **The Weibull distribution**:
   If $\beta = \gamma = \lambda = 0$ or $\alpha = \theta = \lambda = 0$, we get the Weibull distribution with parameter α and θ or β and γ respectively. Moreover, the exponential distribution with parameter α, β or $(\alpha + \beta)$ can be obtained by setting $\beta = \gamma = \lambda = 0$ and $\theta = 1$, $\alpha = \theta = \lambda = 0$ and $\gamma = 1$ or $\theta = \gamma = 1$ and $\lambda = 0$ respectively. Also the Rayleigh distribution can be obtained if $\beta = \gamma = \lambda = 0$ and $\theta = 2$, $\alpha = \theta = \lambda = 0$ and $\gamma = 2$ or $\theta = \gamma = 2$ and $\lambda = 0$.

Table 3.1 shows a list of models that can be derived from the NMW distribution.

Distribution	α	β	θ	γ	λ
Additive Weibull	-	-	-	-	0
Modified Weibull	0	-	0	-	-
S-Z modified Weibull	-	-	1	-	0
Linear failure rate	-	-	1	2	0
Extreme-value	0	-	0	0	-
Rayleigh	-	0	2	0	0
Weibull	-	0	-	0	0
Exponential	-	0	1	0	0

Table 3.1: The sub-models of the NMW.

3.4 The Shape of the probability density function

In this section the shape of the probability density function of the NMW will be discussed. In order to discuss the shapes of the pdf of the NMW, we first differentiate $f(x)$ with respect to x. The CDF and PDF of the NMW can be rewritten as

$$F(x) = 1 - e^{-H(x)}, \; f(x) = h(x)e^{-H(x)}, \tag{3.7}$$

where $h(x)$ is the hazard function (3.4) and $H(x) = \alpha x^\theta + \beta x^\gamma e^{\lambda x}$ is the cumulative hazard.

Then

$$\begin{aligned} f'(x) &= \left[h'(x) - h^2(x)\right] e^{-H(x)} \\ &= f(x)\left[\frac{h'(x)}{h(x)} - h(x)\right], \end{aligned}$$

where

$$h'(x) = \alpha\theta(\theta-1)x^{\theta-2} - \beta x^{\gamma-2}((\gamma-1)(\gamma+\lambda x) + \lambda x(1+\gamma+\lambda x))e^{\lambda x}.$$

The modes of the PDF of the NMW are the roots of the equation.

$$h'(x) - h^2(x) = 0. \tag{3.8}$$

So, the modes of $f(x)$ at say $x = x_0$ are the roots of (3.8). The mode corresponds to a local maximum if $h'(x) - h^2(x) > 0$ for all $x < x_0$ and $h'(x) - h^2(x) < 0$ for all $x > x_0$. The mode corresponds to a local minimum if $h'(x) - h^2(x) < 0$ for all $x < x_0$ and $h'(x) - h^2(x) > 0$ for all $x > x_0$.

As we can see it is difficult to determine the shapes of the PDF of the NMW from (3.8), but as shown in Figure 3.1 the PDF of the NMW can be monotonically decreasing, unimodal, initially decreasing then increasing-decreasing, unimodal or bimodal.

The different shapes of the PDF of the NMW can be discuss from (3.6) by using the shapes of the MW and the Weibull distributions. From Lai *et al* (2003), the PDF of the MW is monotonically decreasing if $0 < \gamma < 1$ and $\frac{\beta}{\gamma^\lambda}$ is large, initially decreasing then increasing-decreasing if $0 < \gamma < 1$ and $\frac{\beta}{\gamma^\lambda}$ small or unimodal if $\gamma > 1$. It is well known that the PDF of the Weibull distribution is monotonically decreasing if $\theta \leqslant 1$ and unimodal if $\theta > 1$. The PDF of the NMW takes different shapes and Figure 3.3 and Figure 3.5 show the following cases:

- When $\theta, \gamma > 1$, both the PDF of the MW and the PDF of the Weibull are unimodal, then the PDF of the NMW is either unimodal, see Figure 3.3(a), or if one of the them dominated in the beginning and the other in the middle then the NMW can be bimodal with two different peaks, see Figure 3.4.

- When $\theta, \gamma < 1$, the Weibull is decreasing and the MW decreasing or initially decreasing then increasing-decreasing, then the NMW can be either decreasing or initially decreasing then increasing-decreasing, see Figure 3.3 (b), Figure 3.5 (a) and 3.5 (b).

- As shown in Figure 3.3 (c), Figure 3.5 (c) and 3.5 (d), if $\theta > 1$ and $\gamma < 1$, the shape of the PDF of the NMW can be the same shape as in previous case where the MW is unimodal or decreasing then increasing-decreasing and the Weibull unimodal.

- If $\theta < 1$ and $\gamma > 1$, the MW is unimodal and the Weibull is decreasing, then the PDF of the NMW can be decreasing, or decreasing then increasing-decreasing, see Figure 3.3 (d).

3.5 The shape of the hazard function

In order to discuss the shape of the hazard function of the NMW we will find the first derivative of $h(x)$,

$$h'(x) = \alpha\theta(\theta - 1)x^{\theta-2} + \beta x^{\gamma-2}((\gamma - 1)(\gamma + \lambda x) + \lambda x(1 + \gamma + \lambda x))e^{\lambda x}. \tag{3.9}$$

The modes of $h(x)$ at say $x = x_0$ are the roots of the equation $h'(x) = 0$. The mode corresponds to a local maximum if $h'(x) > 0$ for all $x < x_0$ and $h'(x) < 0$ for all $x > x_0$. The mode corresponds to a local minimum if $h'(x) < 0$ for all $x < x_0$ and $h'(x) > 0$ for all $x > x_0$. The mode corresponds to a point of inflexion if either $h'(x) > 0$ for all $x \neq x_0$ or $h'(x) < 0$ for all $x \neq x_0$.

The equation $h'(x) = 0$ cannot be solved analytically. The sign of $h'(x)$ in (3.9) can be determined by the values of θ and γ and whether if they are greater or less than 1. We discuss the shape of the hazard function of the NMW when $\theta, \gamma > 1$, $\theta, \gamma < 1$, $\theta > 1$ and $\gamma < 1$ or $\theta < 1$ and $\gamma > 1$. Also, it can be figured out using the shapes of the hazard function of the Weibull and the MW where

$$h(x) = \left[h_W(x; \alpha, \theta) + h_{MW}(x; \beta, \gamma, \lambda)\right].$$

It is well known that $h_W(x; \alpha, \theta)$ is increasing when $\theta > 1$ and decreasing when $\theta < 1$ and $h_{MW}(x; \beta, \gamma, \lambda)$ is increasing when $\gamma > 1$ and bathtub shaped otherwise. Regarding to Lai and Xie (2006), a distribution with a serial system of increasing hazard

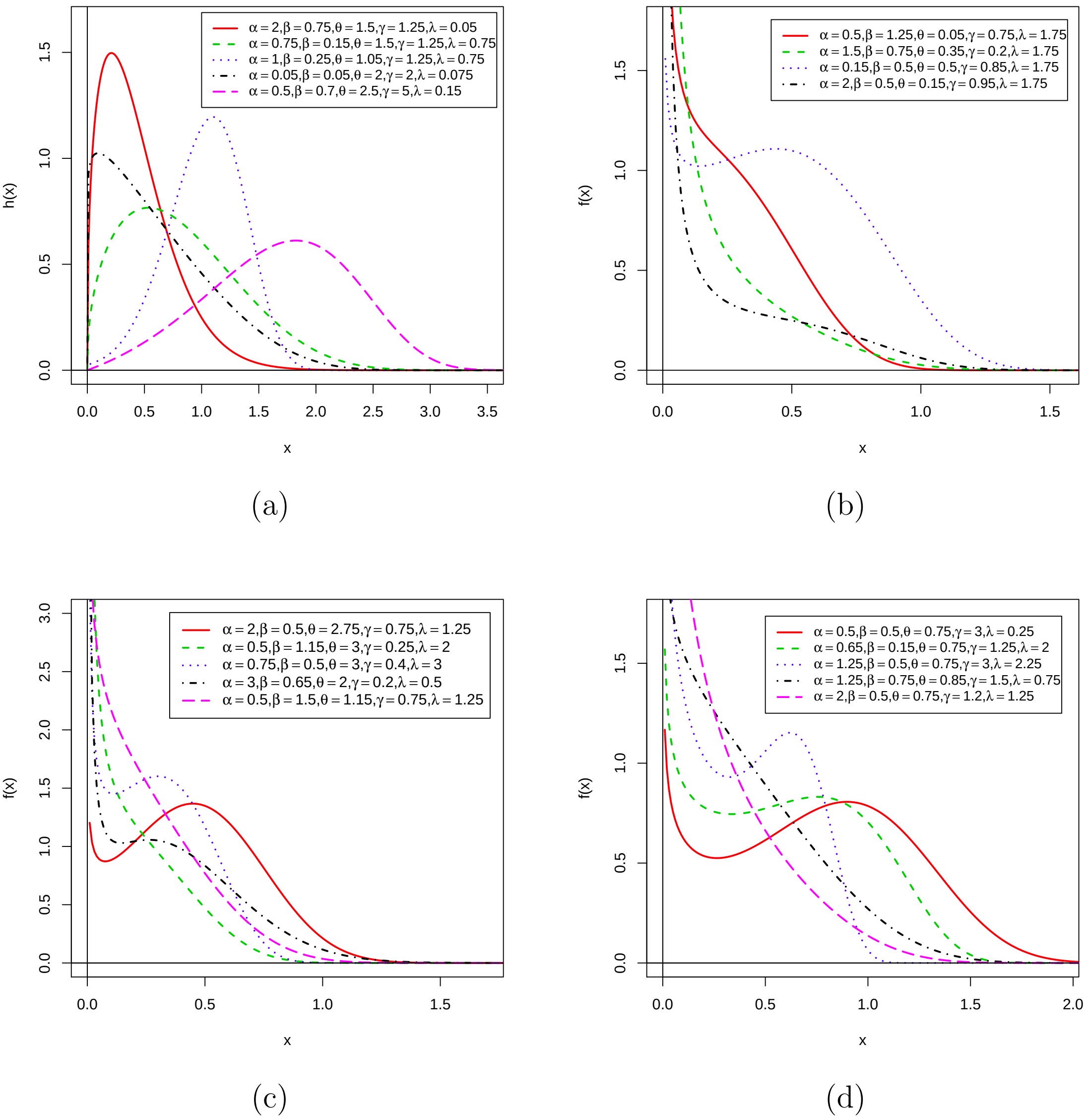
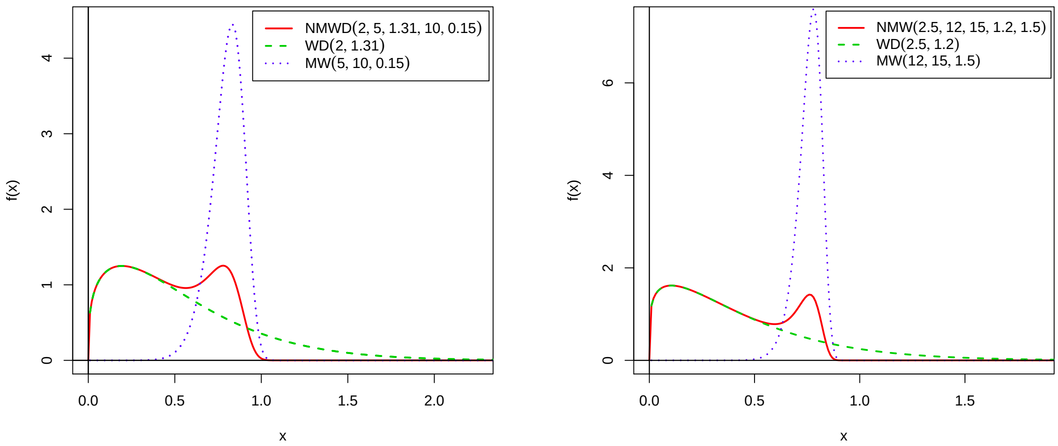


Figure 3.3: Probability density function of NMW: (a): $\theta, \gamma > 1$, (b): $\theta, \gamma < 1$, (c): $\theta > 1, \gamma < 1$, (d): $\theta < 1, \gamma > 1$.



(a) (b)

Figure 3.4: Probability density function of NMW when $\theta, \gamma > 1$.

rate components has an increasing hazard rate distribution. So if $\theta, \gamma \geq 1$, the hazard function of the NMW will be increasing. Figure 3.6 (a)-(d) show the hazard functions of the NMW when $\theta, \gamma > 1$, $\theta, \gamma < 1$, $\theta > 1$ and $\gamma < 1$ and $\theta < 1$ and $\gamma > 1$. It can be seen that the hazard function of the NMW is increasing if $\theta, \gamma \geq 1$ and bathtub shaped otherwise.

Many different applications in reliability and lifetime analysis require bathtub shaped hazard rate functions with a long useful life period with the middle period of the bathtub shape having a relatively constant hazard rate. For example, electric machine life cycles and electronic devices, cf. Kuo and Zuo (2001). A few distributions have this property, so does the NMW as shown in Figure 3.7.

3.6 Simulation from the NMW

The CDF of the NMW has a closed form which makes the simulation from this distribution easier. In this section two methods will be used to simulate from the NMW.

In order to simulate a random sample from this distribution, first 1000 random numbers are generated from the uniform distribution with range (0, 1), these generated random numbers are then transformed using the inverse CDF by solving the equation $F(x) = u$ numerically. Figure 3.8 shows exact probability density functions of the new modified Weibull distribution and histograms of four simulated data sets. It is clear

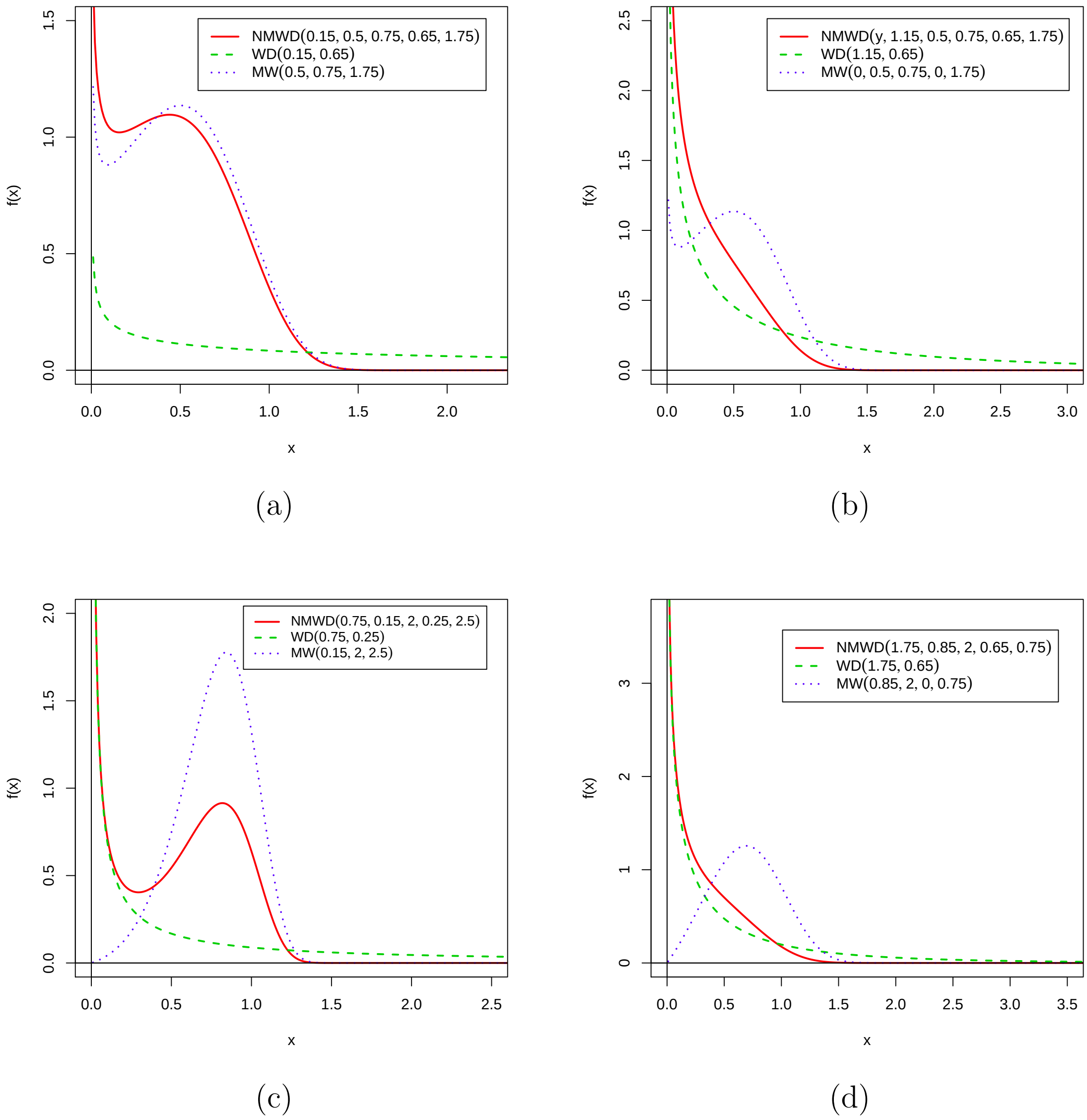


Figure 3.5: Probability density functions of NMW: (a): $\theta, \gamma > 1$, (b): $\theta, \gamma < 1$, (c): $\theta > 1, \gamma < 1$, (d): $\theta < 1, \gamma > 1$.

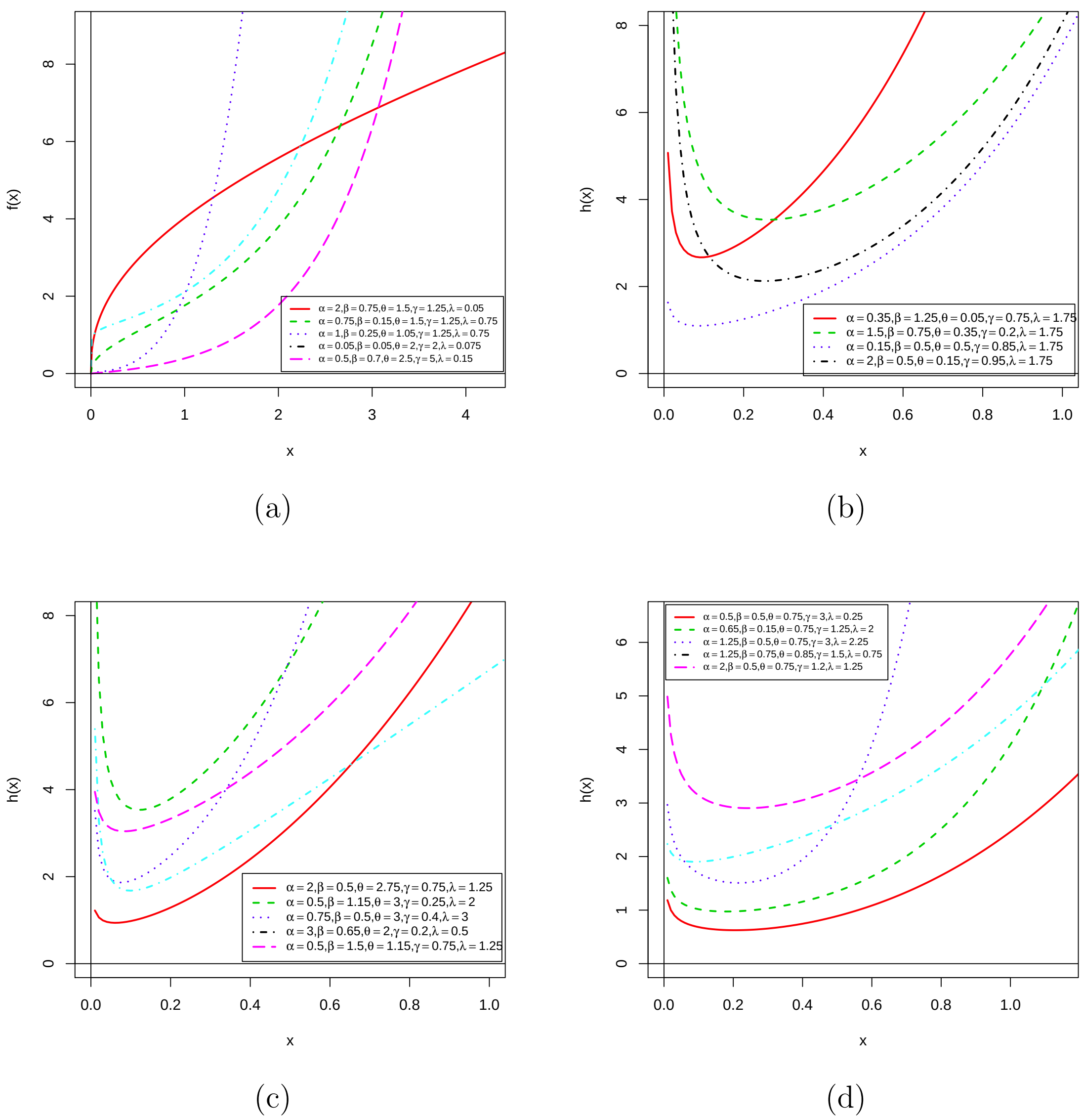


Figure 3.6: Hazard function of NMW: (a): $\theta, \gamma > 1$, (b): $\theta, \gamma < 1$, (c): $\theta > 1, \gamma < 1$, (d): $\theta < 1, \gamma > 1$.

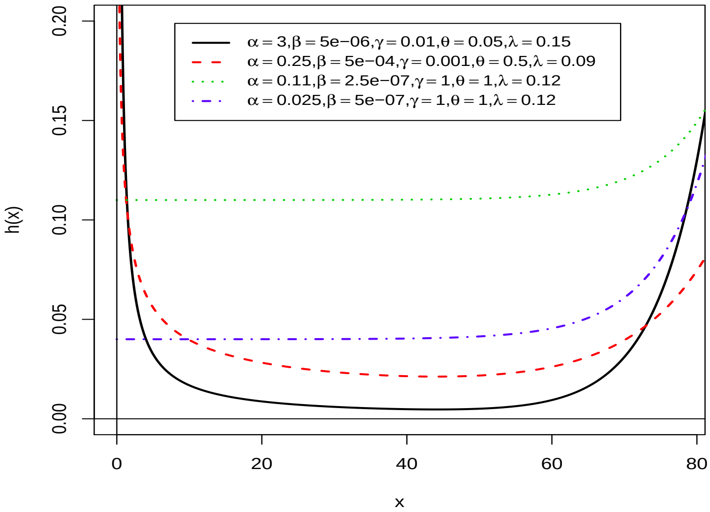


Figure 3.7: Hazard function of the NMW with a long useful life period.

that the simulated values are consistent with the new modified Weibull distribution.

Another method can be used to simulate a random sample from the proposed distribution. As shown this distribution can be used to describe a serial system of two independent components, one of which follows the Weibull distribution and the other one follows the MW. The simulation can be done using the following method. Two random samples, $Y_1, Y_2, ..., Y_n$ and $Z_1, Z_2, ..., Z_n$, are generated from the Weibull with parameters α and θ and the MW with parameters β, γ and λ respectively, where n=1000. Then, a random sample is given by $X_i = min\{Y_i, Z_i\}, i = 1, ..., n$. The output sample of this method, $X_1, X_2, ..., X_n$, follows the NMW with parameters $\alpha, \beta, \gamma, \theta$ and λ. Using the same selected parameters as in the previous method, four random samples are generated, Figure 3.9 shows exact probability density functions of the new modified Weibull distribution and histograms of four simulated data sets using this method.

3.7 The moments

This section derives the theoretical moments of the NMW. The following theorem gives the non-central moments of the distribution.

Theorem 1. *The non-central moments of the NMW are given by*

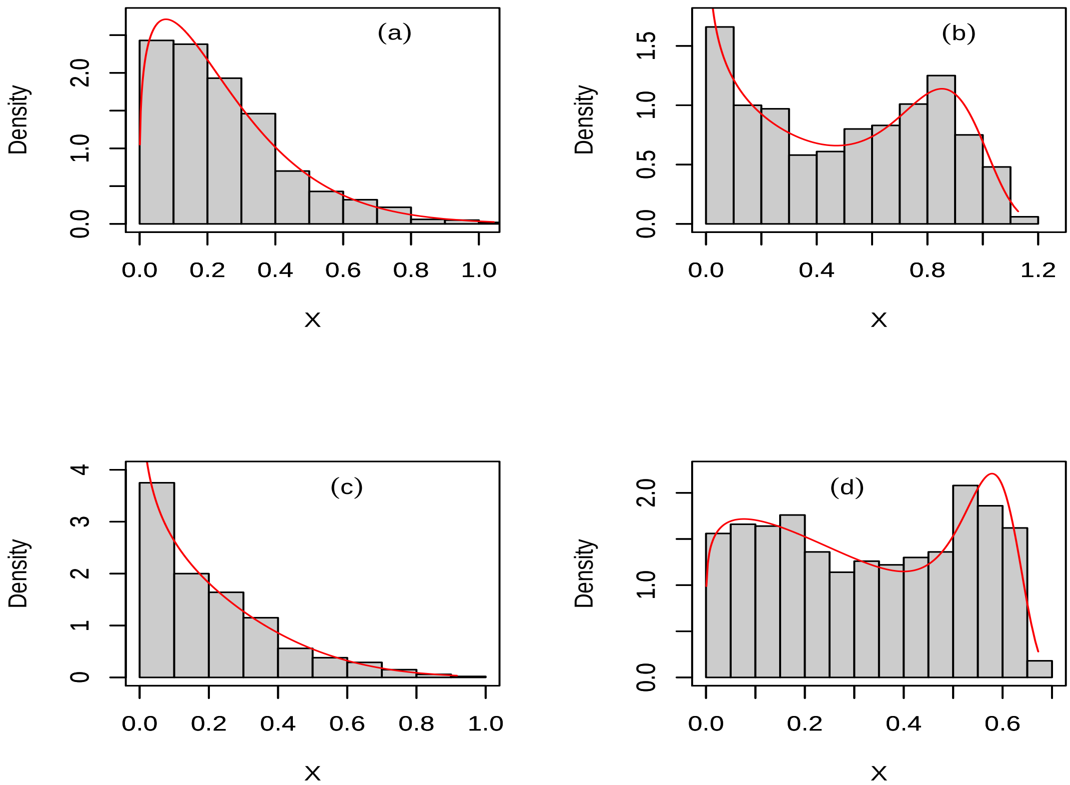


Figure 3.8: PDF of the NMW and the histograms of simulated data (Method 1, n=1000) where
(a): α=0.05, β=5, γ=1.25, θ=2 and λ=0.05, (b): α=1.15, β=0.25, γ=5, θ=0.8 andλ=2, (c): α=1.5, β=3, γ=0.85, θ=2.5 and λ=0.5, (d): α=2.5, β=5, γ=9, θ=1.15 and λ=5.

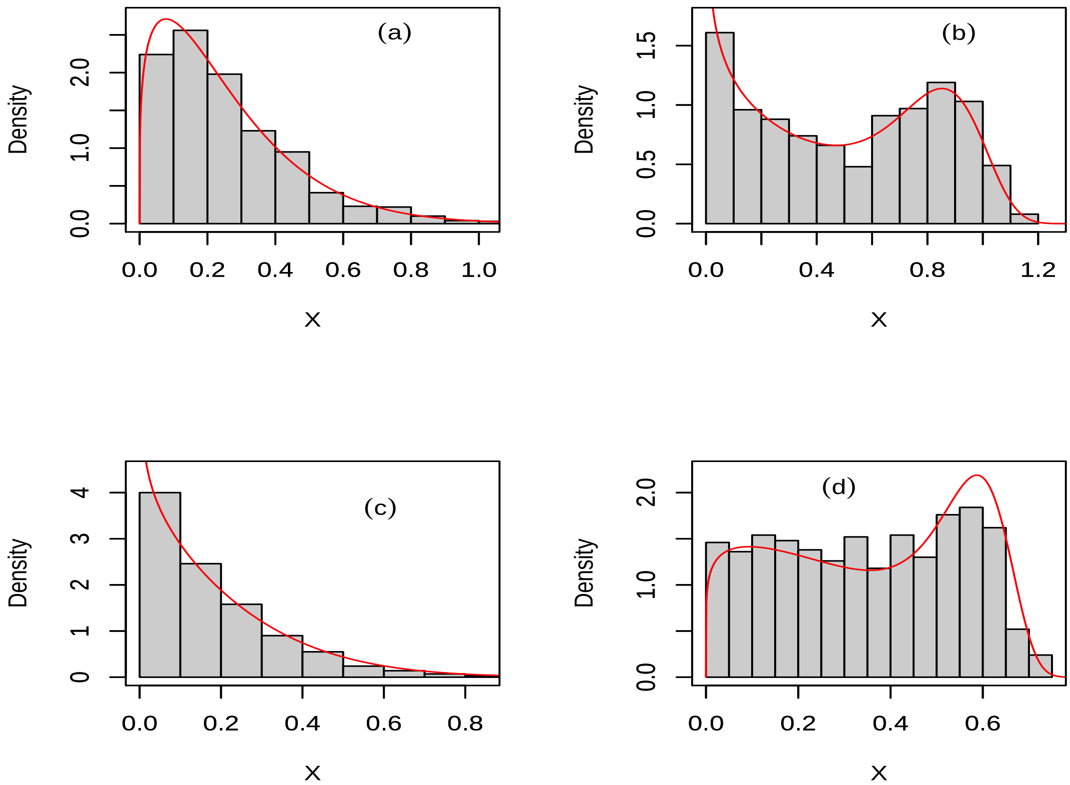


Figure 3.9: Probability density functions of the NMW and the histograms of simulated data (Method 2, n=1000) where
(a):α=0.05, β=5, γ=1.25, θ=2 and λ=0.05, (b):α=1.15, β=0.25, γ=5, θ=0.8 andλ=2, (c):α=1.5, β=3, γ=0.85, θ=2.5 and λ=0.5, (d):α=2.5, β=5, γ=9, θ=1.15 and λ=5.

$$\mu'_r = \frac{r}{\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\beta)^n(\lambda n)^m}{n!m!}\alpha^{-\frac{n\gamma+m+r}{\theta}}\Gamma\left(\frac{n\gamma+m+r}{\theta}\right), \tag{3.10}$$

for $r = 1, 2, \ldots,$ where $\Gamma(\cdot)$ is the gamma function.

Proof: Using the identities

$$\int_0^{\infty} x^{\beta-1}e^{-\alpha x^{\theta}}\mathrm{d}x = \alpha^{-\beta/\theta}(1/\theta)\Gamma(\beta/\theta),$$

and $e^x = \sum_{n=0}^{\infty} x^n/n!$, the r-th order non-central moment of the new modified Weibull distribution is

$$\begin{aligned}
\mu'_r &= \int_0^{\infty} x^r \mathrm{d}F(x) \\
&= -\int_0^{\infty} x^r \mathrm{d}e^{-\alpha x^{\theta}-\beta x^{\gamma}e^{\lambda x}} \\
&= \int_0^{\infty} rx^{r-1}e^{-\alpha x^{\theta}-\beta x^{\gamma}e^{\lambda x}}\mathrm{d}x \\
&= \int_0^{\infty} rx^{r-1}e^{-\alpha x^{\theta}}\sum_{n=0}^{\infty}\frac{(-\beta x^{\gamma}e^{\lambda x})^n}{n!}\mathrm{d}x \\
&= \int_0^{\infty} rx^{r-1}e^{-\alpha x^{\theta}}\sum_{n=0}^{\infty}\frac{(-\beta)^n x^{n\gamma}}{n!}\sum_{m=0}^{\infty}\frac{(\lambda n x)^m}{m!}\mathrm{d}x \\
&= \int_0^{\infty} rx^{r-1}e^{-\alpha x^{\theta}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\beta)^n(\lambda n)^m x^{n\gamma+m}}{n!m!}\mathrm{d}x \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\beta)^n(\lambda n)^m}{n!m!}\int_0^{\infty} rx^{r-1}x^{n\gamma+m}e^{-\alpha x^{\theta}}\mathrm{d}x \\
&= \frac{r}{\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\beta)^n(\lambda n)^m}{n!m!}\alpha^{-\frac{n\gamma+m+r}{\theta}}\Gamma\left(\frac{n\gamma+m+r}{\theta}\right).
\end{aligned}$$

3.8 The moment generating function

The moment generating function of the NMW is provided in the next theorem.

Theorem 2. *The moment generating function of the NMW is*

$$M_X(t) = 1 + \frac{1}{\theta}\sum_{n,m,k=0}^{\infty}\frac{(-\beta)^n(n\lambda)^m t^{k+1}}{n!m!k!}\left[\alpha^{-\frac{\gamma n+m+k+1}{\theta}}\Gamma\left(\frac{\gamma n+m+k+1}{\theta}\right)\right], \tag{3.11}$$

where $\Gamma(\cdot)$ *is the gamma function.*

Proof: The moment generating function of the NMW is given by

$$\begin{aligned}
M_X(t) &= E(e^{tx}) \\
&= \int_0^\infty e^{tx} f(x)dx \\
&= \int_0^\infty e^{tx} dF(x) \\
&= 1 + \int_0^\infty te^{tx} e^{-\alpha x^\theta - \beta x^\gamma e^{\lambda x}} dx \\
&= 1 + \int_0^\infty te^{tx} e^{-\alpha x^\theta} \sum_{n,m=0}^\infty \frac{(-\beta)^n (\lambda n)^m x^{n\gamma+m}}{n!m!} dx \\
&= 1 + \int_0^\infty e^{-\alpha x^\theta} \sum_{n,m,k=0}^\infty \frac{(-\beta)^n (\lambda n)^m t^{k+1} x^{n\gamma+m+k}}{n!m!k!} dx \\
&= 1 + \sum_{n,m,k=0}^\infty \frac{(-\beta)^n (n\lambda)^m t^{k+1}}{n!m!k!} I,
\end{aligned}$$

where $I = \int_0^\infty x^{\gamma n+m+k} e^{-\alpha x^\theta} dx$.

Using gamma-integral formula, we have

$$I = \frac{1}{\theta \alpha^{\frac{\gamma n+m+k+1}{\theta}}} \Gamma\left(\frac{\gamma n+m+k+1}{\theta}\right),$$

then

$$M_X(t) = 1 + \frac{1}{\theta} \sum_{n,m,k=0}^\infty \frac{(-\beta)^n (n\lambda)^m t^{k+1}}{i!j!k!} \left[\alpha^{-\frac{\gamma n+m+k+1}{\theta}} \Gamma\left(\frac{\gamma n+m+k+1}{\theta}\right)\right].$$

3.9 Order statistics

It will also be useful to derive the pdf of the rth order statistic $X_{(r)}$ of a random sample $X_1, \ldots, X_n$ drawn from the NMW with parameters α, β, θ, γ and λ. From Arnold *et al.* (2008), the PDF of $X_{(r)}$ is given by

$$f_{r:n}(x) = \frac{F(x)^{r-1}(1 - F(x))^{n-r} f(x)}{B(r, n-r+1)}, \tag{3.12}$$

where $B(.,.)$ is the beta function.

From (3.7),

$$(1-F(x))^{n-r} = e^{-(n-r)H(x)}, \tag{3.13}$$

and

$$F(x)^{r-1} = \left(1-e^{-H(x)}\right)^{r-1} = \sum_{\ell=0}^{r-1}\binom{r-1}{\ell}(-1)^{\ell}e^{-\ell H(x)}. \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12), we get

$$\begin{aligned}
f_{r:n}(x) &= \frac{1}{B(r,n-r+1)}\sum_{\ell=0}^{r-1}\binom{r-1}{\ell}(-1)^{\ell}h(x)e^{-(n+\ell+1-r)H(x)}, \\
&= n\binom{n-1}{r-1}\sum_{\ell=0}^{r-1}\binom{r-1}{\ell}(-1)^{\ell}h(x)e^{-(n+\ell+1-r)H(x)}, \\
&= n\binom{n-1}{r-1}\sum_{\ell=0}^{r-1}\binom{r-1}{\ell}(-1)^{\ell}\left(\alpha\theta x^{\theta-1}+\beta(\gamma+\lambda x)x^{\gamma-1}e^{\lambda x}\right) \\
&\quad e^{-(n+\ell+1-r)(\alpha x^{\theta}+\beta x^{\gamma}e^{\lambda x})}, \\
&= n\binom{n-1}{r-1}\sum_{\ell=0}^{r-1}\binom{r-1}{\ell}\frac{(-1)^{\ell}}{(n+\ell+1-r)}f(x;\ \alpha_{\ell},\beta_{\ell},\theta,\gamma,\lambda),
\end{aligned}$$

where $f(x;\ \alpha_{\ell},\beta_{\ell},\theta,\gamma,\lambda)$ is the PDF of the NMW with parameters $\alpha_{\ell} = (n+\ell+1-r)\alpha$, $\beta_{\ell} = (n+\ell+1-r)\beta$, θ, γ and λ.

Using (3.10), the kth non-central moment of the rth order statistic $X_{(r)}$ is

$$\begin{aligned}
\mu_k^{'(r:n)} &= \frac{nk}{\theta}\binom{n-1}{r-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\ell=0}^{r-1}\binom{r-1}{\ell}\frac{(-1)^{\ell+i}\beta^{i}(i\lambda)^{j}}{(n+\ell+1-r)^{\frac{i\gamma+j+k}{\theta}+1-i}\alpha^{\frac{i\gamma+j+k}{\theta}}} \\
&\quad \Gamma\left(\frac{i\gamma+j+k}{\theta}\right).
\end{aligned} \tag{3.15}$$

3.10 Parameter estimation

Given a random sample $x_1,\ldots,x_n$ from the NMW with parameters $(\alpha,\beta,\theta,\gamma,\lambda)$, the usual method of estimation is by maximum likelihood, cf. Fisher (1922). Other possible approaches include Bayesian estimation using Lindley approximation (Lindley, (1980)) or MCMC, cf. Soliman *at el* (2012) and Upadhyay and Gupta (2010).

The log-likelihood function is given by

$$\mathcal{L} = \sum_{i=1}^{n}\ln\left(\beta(\gamma+\lambda x_i)x_i^{\gamma-1}e^{\lambda x_i}+\alpha\theta x_i^{\theta-1}\right) - \alpha\sum_{i=1}^{n}x_i^{\theta} - \beta\sum_{i=1}^{n}x_i^{\gamma}e^{\lambda x_i}. \tag{3.16}$$

Setting the first partial derivatives of $\mathcal{L}$ with respect to $\alpha, \beta, \theta, \gamma$ and λ to zero, the likelihood equations are

$$\sum_{i=1}^{n} \frac{\theta x_i^{\theta-1}}{h(x_i;\ \alpha,\beta,\gamma,\theta,\lambda)} - \sum_{i=1}^{n} x_i^{\theta} \;=\; 0, \tag{3.17}$$

$$\sum_{i=1}^{n} \frac{(\gamma+\lambda x_i)x_i^{\gamma-1}e^{\lambda x_i}}{h(x_i;\ \alpha,\beta,\gamma,\theta,\lambda)} - \sum_{i=1}^{n} x_i^{\gamma}e^{\lambda x_i} \;=\; 0, \tag{3.18}$$

$$\sum_{i=1}^{n} \frac{\alpha x_i^{\theta-1}(1+\theta\ln(x_i))}{h(x_i;\ \alpha,\beta,\gamma,\theta,\lambda)} - \alpha\sum_{i=1}^{n} x_i^{\theta}\ln(x_i) \;=\; 0, \tag{3.19}$$

$$\sum_{i=1}^{n} \frac{x_i^{\gamma-1}e^{\lambda x_i}((\gamma+\lambda x_i)\ln(x_i)+1)}{h(x_i;\ \alpha,\beta,\gamma,\theta,\lambda)} - \sum_{i=1}^{n} x_i^{\gamma}e^{\lambda x_i}\ln(x_i) \;=\; 0, \tag{3.20}$$

$$\sum_{i=1}^{n} \frac{(1+\gamma+\lambda x_i)x_i^{\gamma}e^{\lambda x_i}}{h(x_i;\alpha,\beta,\gamma,\theta,\lambda)} - \sum_{i=1}^{n} x_i^{\gamma+1}e^{\lambda x_i} \;=\; 0. \tag{3.21}$$

The maximum likelihood estimates can be obtained by solving the nonlinear equations numerically for α, β, θ, γ and λ. The relatively large number of parameters can cause problems especially when the sample size is not large. A good set of initial values is essential.

We have also obtained all the second partial derivatives of the log-likelihood function for the construction of the Fisher information matrix, so that standard errors of the parameter estimates can be obtained in the usual way. These are in Appendix A.

3.11 Applications

In this section we provide results of fitting the NMW to two well-known data sets and compare its goodness-of-fit with other modified Weibull distributions using Kolmogorov-Smirnov, K-S statistic, as well as Akaike information criterion (Akaike, 1974), Bayesian information criterion (Schwarz, 1978), corrected Akaike information criterion (Hurvich and Tsai, 1989) and consistent Akaike information criterion due to Bozdogan (1987).

3.11.1 Aarset data

The data below represent the lifetimes of 50 devices (in weeks) (Aarset, 1987).

.1	.2	1	1	1	1	1	2	3	6	7	11
12	18	18	18	18	18	21	32	36	40	45	46
47	50	55	60	63	63	67	67	67	67	72	75
79	82	82	83	84	84	84	85	85	85	85	86

Many authors have analysed this data set, including Mudholkar and Srivastava (1993), Xie *et al* (2006), Lai *et al* (2003), Sarhan and Zaindin (2009), and Silva *et al* (2010).

The data are known to have a bathtub-shaped failure rate as shown by the scaled TTT-Transform plot, which has a convex shape followed by a concave shape, see Fig 3.10.

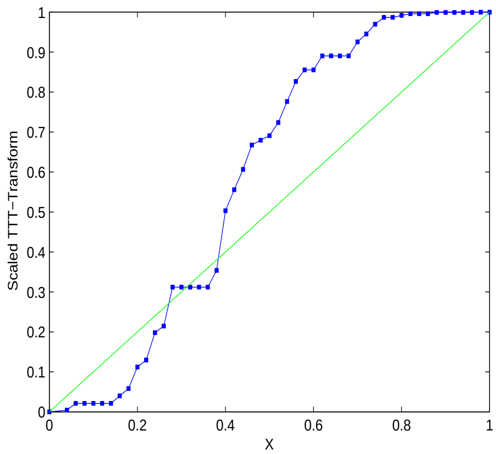


Figure 3.10: TTT-transform plot for the Aarst.

Table 3.2 gives MLEs of parameters of the NMW and sub-models with standard errors in brackets. Goodness of fit statistics are in Table 3.3. We find that the NMW distribution with the same number of parameters provides a better fit than the beta modified Weiull distribution (BMW) which was the best in Silva *et al* (2010). The BMW had the largest likelihood, and the smallest K-S, AIC, BIC, AICc and CAIC values among those considered in that paper. It is clear in Figure 3.11c that the NMW fits the left and right peaks in the histogram better and its survival function follows the Kaplan-Meier estimate more closely, see Figure 3.11 (d).

Table 3.2: MLEs of parameters and corresponding standard errors in brackets for the Aarset data.

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$
NMW	0.071	7.015×10^{-8}	0.016	0.595	0.197
	(0.031)	(1.501×10^{-7})	(3.602)	(0.128)	(0.184)
BMW	$a = 0.198$	0.0002	1.377	$b = 0.165$	0.054
Silva *et al* (2010)	(0.046)	(6.693×10^{-5})	(0.339)	(0.083)	(0.016)
MW		0.062	0.356		0.023
$(\alpha = 0,\ \theta = 0)$		(0.027)	(0.113)		(4.845×10^{-3})
AddW	1.133×10^{-8}	0.086	0.477	4.214	
$(\lambda = 0)$	(5.183×10^{-8})	(0.036)	(0.102)	(1.033)	
SZMW	0.013	8.408×10^{-9}	4.224		
$(\theta = 1,\ \lambda = 0)$	(2.819×10^{-3})	(4.204×10^{-8})	(1.140)		

Table 3.3: Log-likelihood, K-S statistic, AIC, BIC, AICc and CAIC values of models fitted to Aarst data for comparison with beta modified Weibull (Silva *et al* (2010)).

Model	Log-lik	K-S	AIC	BIC	AICc	CAIC
NMW	-212.90	0.088	435.8	445.4	437.2	450.4
MW	-227.16	0.129	460.3	466.0	460.8	469.1
AddW	-221.51	0.127	451.0	458.7	451.9	462.7
SZMW	-229.88	0.151	465.8	471.5	466.3	474.5
BMW	-220.80	0.127	451.6	461.2	453.0	466.2

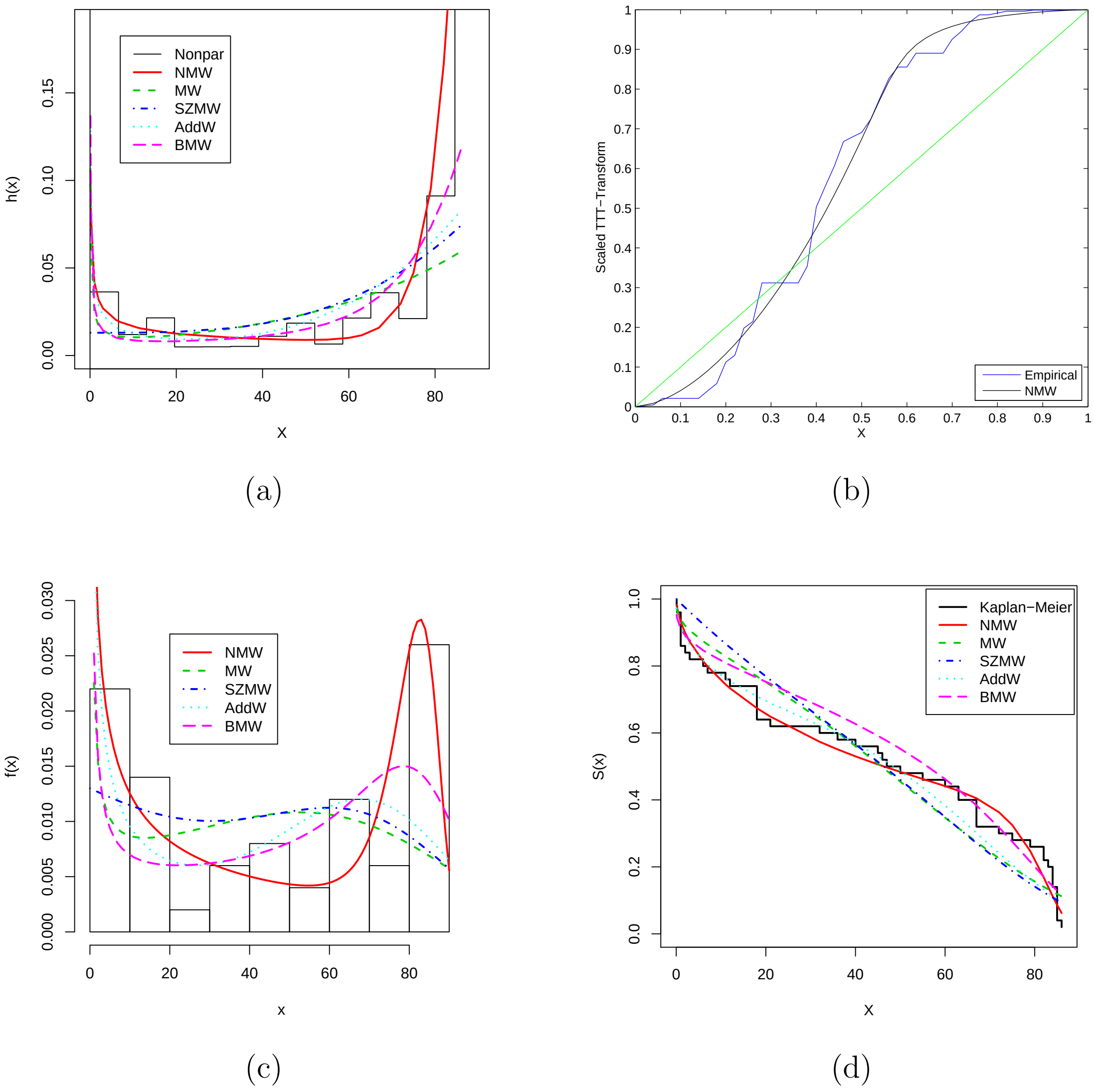


Figure 3.11: For Aarst data: (a) hazard function (b) TTT-Transform plot (c) pdf and (d) survival function using NMW plus sub models, and beta modified Weibull.

3.11.2 Meeker and Escobar data

The data are failure and running times of a sample of 30 devices (Meeker and Escobar (1998), p. 383). Two types of failures were observed for this data. It was shown by Nadarajah *et al* (2011) to be best fit by the beta modified Weibull distribution.

The data have a bathtub shaped failure rate as its empirical TTT-plot has an s-shape, see Figure 3.12. Again the NMW distribution (Table 3.4) provides a better fit than the BMW, as can be seen from Table 3.5. Figures 3.13 (a) to (d) show the hazard function, TTT-Transform plot, PDF and the survival function using NMW, its sub models and the BMW distribution.

The NMW distribution has the largest likelihood, and the smallest K-S, AIC, BIC, AICc and CAIC values comparing to the BMW and the all other modified Weibull distributions which support that the NMW provides a better fit than all considered distributions.

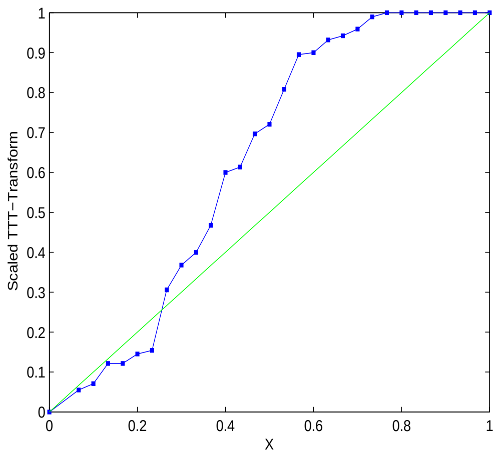


Figure 3.12: TTT-transform plot for the Mekeer data.

Table 3.4: MLEs of parameters and corresponding standard errors in brackets for the Meeker and Escobar data (1998).

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$
NMW	0.024	5.991×10^{-8}	0.012	0.629	0.056
	(0.019)	(8.164×10^{-8})	(1.29)	(0.15)	(0.024)
BMW	$a = 0.068$	4.9×10^{-17}	4.266	$b = 0.099$	0.0528
Silva *et al* (2010)	(0.016)	(6.693×10^{-5})	(0.011)	(0.049)	(0.002)
MW		0.018	0.454		7.133×10^{-3}
$(\alpha = 0,\ \theta = 0)$		(0.018)	(0.220)		(2.113×10^{-3})
AddW	1.320×10^{-7}	0.019	0.604	2.830	
$(\lambda = 0)$	(7.435×10^{-7})	(0.018)	(0.197)	(0.974)	
SZMW	2.939×10^{-3}	1.497×10^{-9}	3.585		
$(\theta = 1,\ \lambda = 0)$	(9.290×10^{-4})	(1.114×10^{-8})	(1.314)		

Table 3.5: Log-likelihood, K-S statistic, AIC, BIC, AICc and CAIC values of models fitted to Meeker and Escobar data (1998)

Model	Log-lik	K-S	AIC	BIC	AICc	CAIC
NMW	-166.18	0.148	342.4	349.4	344.9	354.4
MW	-178.06	0.182	362.1	366.3	363.1	369.3
AddW	-178.11	0.191	364.2	369.8	365.8	373.8
SZMW	-177.90	0.186	361.8	366.0	362.7	369.0
BMW	-167.55	0.161	345.1	352.1	347.6	357.2

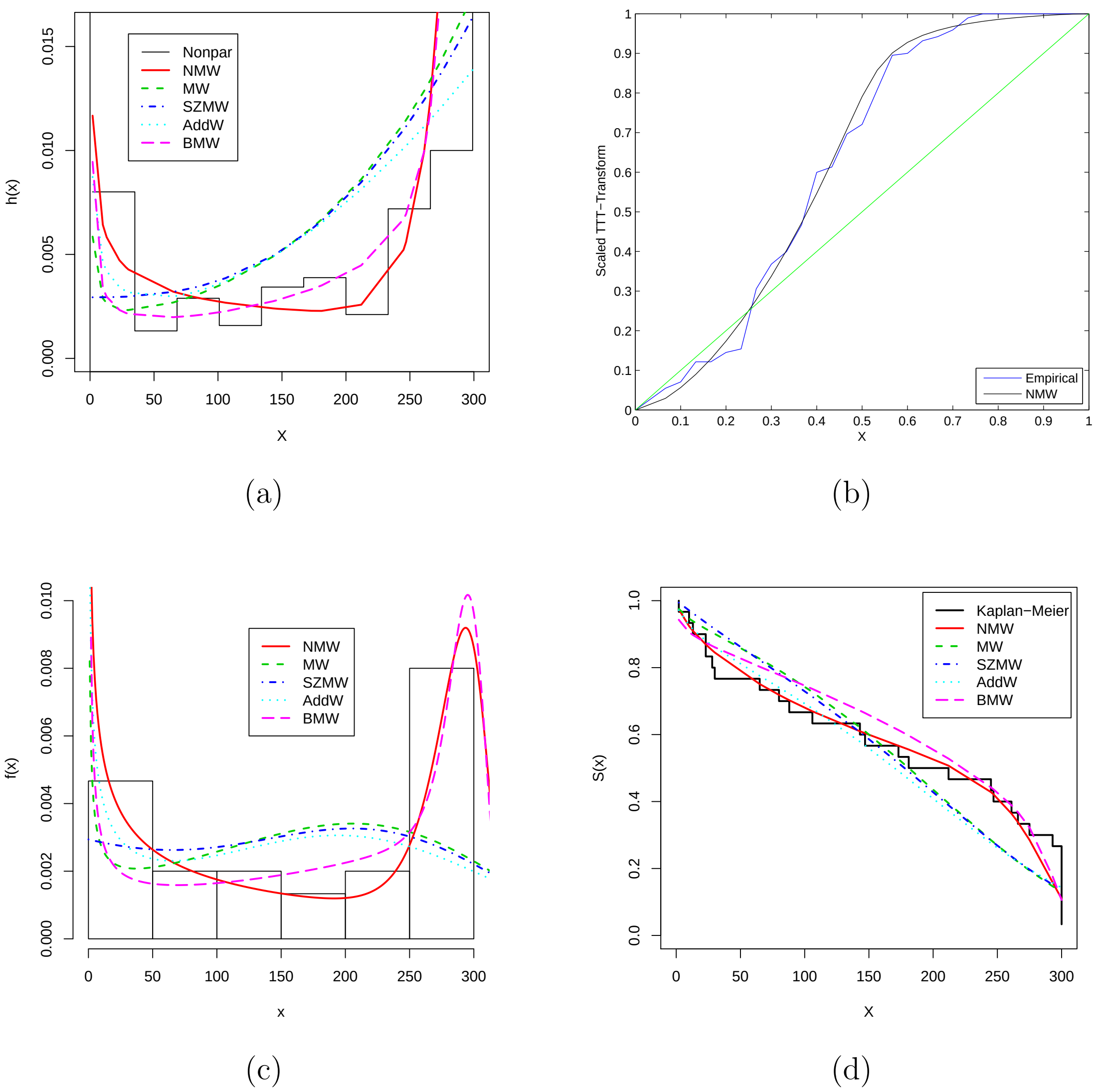


Figure 3.13: For Meeker and Escobar data: (a) hazard function (b) TTT-Transform plot (c) pdf and (d) survival function using NMW plus sub models, and beta modified Weibull.

3.12 Sub-model of the NMW with $\gamma = 1$

To simplify the statistical inference, it is always a good idea to reduce the number of parameters of any distribution and investigate how that affects the ability of the simplified model to fit the data. In this section we reduce the number of parameters from five to four, by setting $\gamma = 1$. We test the reduced model (H_0: $\gamma = 1$) against the original model (H_a: $\gamma \neq 1$). For each data set, Table 3.6 shows MLEs of the four parameter NMW, the log-likelihood value under H_0, likelihood ratio statistic, LRT with P-value in brackets, AIC, K-S statistic with P-value in brackets .

The likelihood ratio statistics against the full model with five parameters are 1.31 (P-value = 0.252) and 2.45 (P-value=0.118) respectively on 1 d.f.. Therefore we can choose the reduced model with 4 parameters. The likelihood and AIC value also points to this model when the modified beta distribution is included in the comparison. Figure 3.14 shows the reduced model is nearly as good as the full model for both data sets.

Table 3.6: Results of fitting NMW with $\gamma = 1$ to both data sets.

Data	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	Log-lik	AIC	LRT (P-value)	K-S
Aarset	0.092	2.2×10^{-8}	0.531	0.160	-213.56	435.1	1.31	0.105
	(0.039)	(2.1×10^{-8})	(0.104)	(0.011)			(0.252)	
Meeker	0.017	3.5×10^{-8}	0.675	0.039	-167.40	342.8	2.45	0.153
	(0.013)	(4.2×10^{-8})	(0.141)	(4.0×10^{-3})			(0.118)	

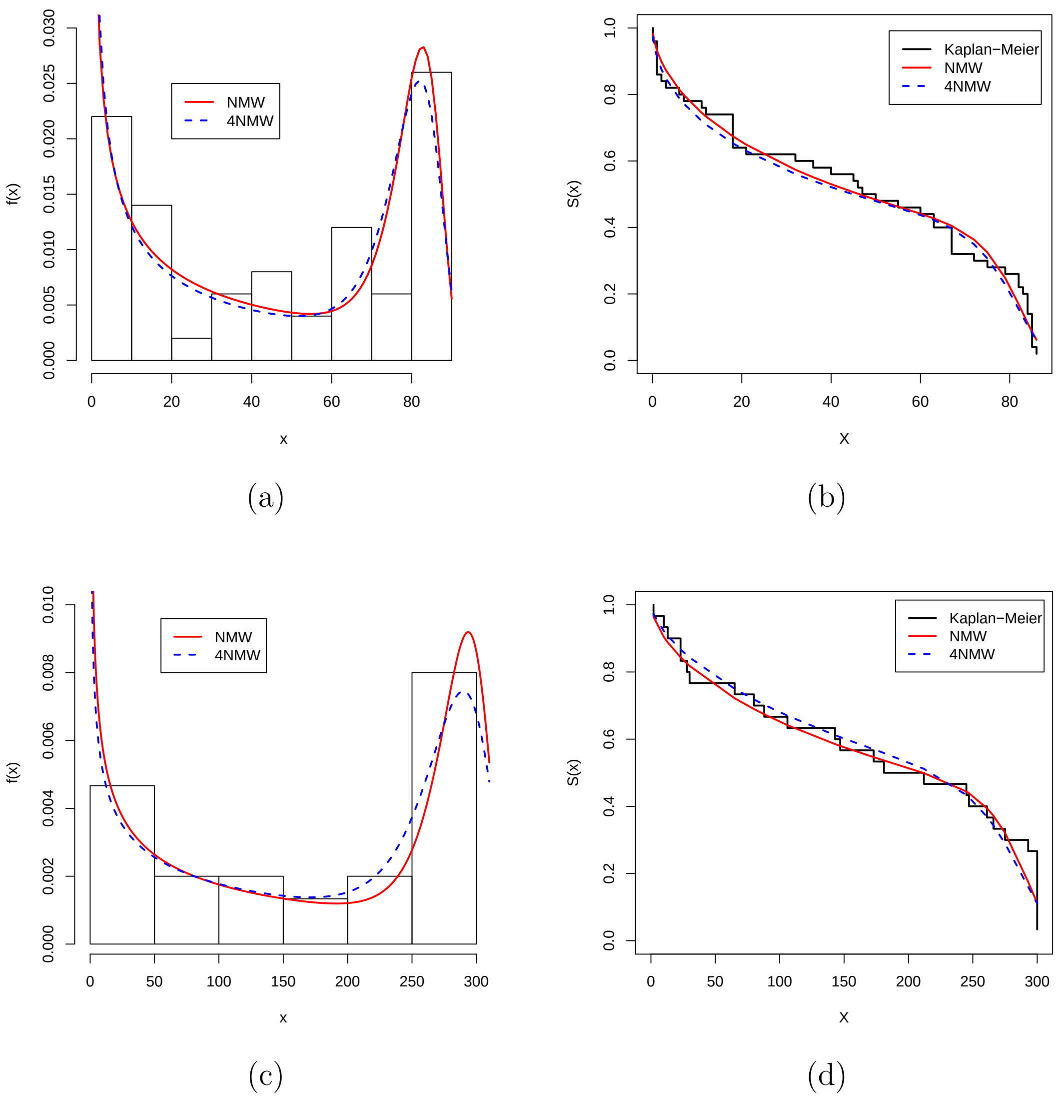


Figure 3.14: (a) and (b): fitted pdf and survival functions for Aarst data, (c) and (d): those for Meeker and Escobar data, 5 parameters (solid lines) vs 4 parameters (dotted lines).

3.13 Summary

Chapter 3 introduced a new lifetime distribution by considering a serial system with one component following a Weibull distribution and another following a modified Weibull distribution. We studied its mathematical properties including moments and order statistics. The estimation of parameters by maximum likelihood was discussed. We demonstrated that the proposed distribution fits two well-known data sets better than other modified Weibull distributions including the latest beta modified Weibull distribution. The model can be simplified by fixing one of the parameters, whilst still provides a better fit than existing models.

Chapter 4

Reduced New Modified Weibull Distribution

4.1 Introduction

In chapter 3 we proposed a new modification of the Weibull distribution NMW which generalises several commonly used distributions in reliability and lifetime data analysis. This distribution has five parameters and we have studied some of its properties and estimation of its parameters by using the maximum likelihood method and applied it to real data sets. The NMW could be one of the most important distributions in reliability and life testing due to its properties, its sub-models, its flexibility in fitting real data sets and the interpretation of its survival and hazard functions. It has also been shown to be the best lifetime distribution to date in terms of fitting data sets such as the Aarset data of Aarset (1987) and the voltage data of Meeker and Escobar (1998).

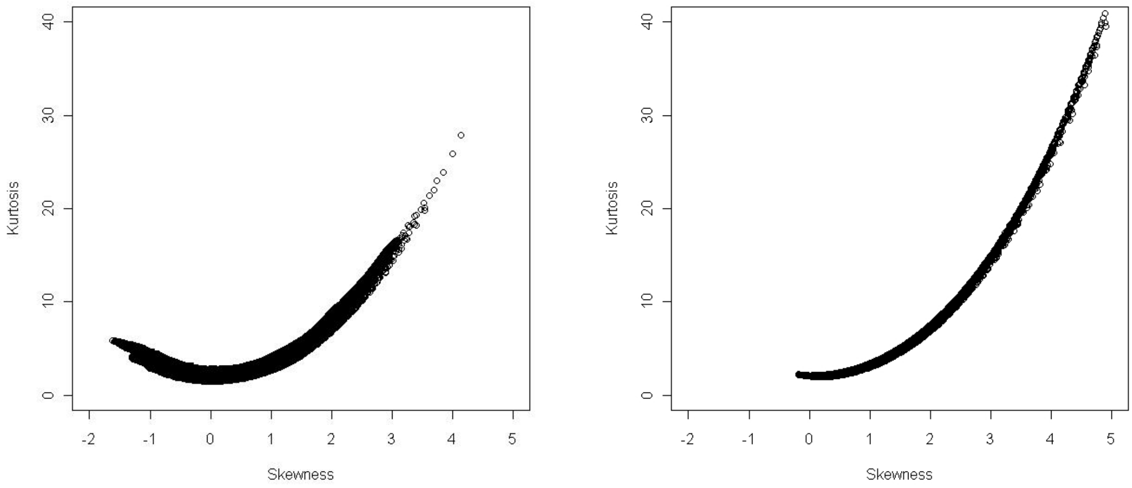
Although distributions with four or more parameters are flexible and exhibit bathtub shaped hazard rates, they are also complex (Nelson, (1990)) and cause estimation problems as a consequence of the number of parameters, especially when the sample size is not large. The main purpose of this chapter is to reduce the number of parameters of the NMW so as to address these problems while maintaining the same flexibility and ability to fit data so well. We show this can be achieved by choosing the shape parameters $\gamma = \theta = \frac{1}{2}$. The new distribution will be called *Reduced New Modified Weibull Distribution* and be denoted as RNMW.

There are various tools to assess the flexibility of a given univariate distribution. One commonly used tool is the skewness-kurtosis plot. Values of (skewness, kurtosis) plotted on the (x, y) plane for all possible values of the parameters of the distribution give what is referred to as the skewness-kurtosis plot. The area or the range covered by the skewness-kurtosis plot is a measure of flexibility of the distribution. If we have more than one distribution, the distribution with the widest range of skewness-kurtosis will be more flexible than others. The skewness-kurtosis plot can be used to choose which distribution is appropriate to fit a given data. Also the relationships between distributions can be seen via a skewness-kurtosis plot, for example the exponential distribution is located at the point of intersection of the gamma and Weibull distributions and it is well known as a special case of both distributions, see Cox (1984) and Hartless and Leemis (1996).

We now show the flexibility of the particular case of the NMW, when $\theta = \gamma = \frac{1}{2}$, by means of skewness-kurtosis plot. The skewness-kurtosis plot for the NMW distribution is drawn on the left hand side of Figure 4.1. The values of (skewness, kurtosis) were computed over $\theta = 0.1, 0.2, \ldots, 5$, $\gamma = 0.1, 0.2, \ldots, 5$, $\theta = \gamma \neq \frac{1}{2}$, $\alpha = 0.1, 0.2, \ldots, 2$, $\beta = 0.1, 0.2, \ldots, 2$ and $\lambda = 0.1, 0.2, \ldots, 2$. The skewness-kurtosis plot for the reduced distribution is drawn on the right hand side of Figure 4.1. The values of (skewness, kurtosis) were computed over $\alpha = 0.1, 0.2, \ldots, 2$, $\beta = 0.1, 0.2, \ldots, 2$ and $\lambda = 0.1, 0.2, \ldots, 2$.

We can see that the particular case $\theta = \gamma = \frac{1}{2}$ is a flexible member of the NMW distribution. The range of kurtosis values is the widest for the case $\theta = \gamma = \frac{1}{2}$. The range of positive skewness values is also widest for the case $\theta = \gamma = \frac{1}{2}$, but some negative skewness values are not accommodated by this case.

The rest of this chapter is organized as follows. Sections 4.2-4.7 consider the mathematical properties of the RNMW, including its pdf, hazard function, moments, Renyi entropy and the estimation of its reliability. The distribution of order statistics and moments of data from this distribution are derived in section 4.8. Section 4.9 discusses the maximum likelihood estimates of the unknown parameters. Four real data sets, uncensored and censored, are analyzed in section 4.10. Section 4.11 summarizes the chapter.



(a) (b)

Figure 4.1: Kurtosis-skewness plot for the NMW distribution (left) and its reduced version (right).

4.2 The reduced distribution

The variable X has a reduced new modified Weibull distribution if its CDF given by

$$F(x) = 1 - e^{-\alpha\sqrt{x} - \beta\sqrt{x}e^{\lambda x}}, \ x > 0. \tag{4.1}$$

where α, β and λ are non-negative, with α and β being scale parameters and λ an acceleration parameter. This reduced version of the new modified Weibull (NMW) distribution has a bathtub-shaped hazard function, as will be shown later, is called reduced new modified Weibull and is denoted by RNMW.

From the CDF of the RNMW in (4.1), the corresponding survival function and density function are

$$S(x) = e^{-\alpha\sqrt{x} - \beta\sqrt{x}e^{\lambda x}}, \tag{4.2}$$

and

$$f(x) = \frac{1}{2\sqrt{x}} \left(\alpha + \beta(1 + 2\lambda x)e^{\lambda x}\right) e^{-\alpha\sqrt{x} - \beta\sqrt{x}e^{\lambda x}}, \ x > 0, \tag{4.3}$$

where α, β and $\lambda > 0$. Figure 4.2 shows that the reduced distribution is nearly as flexible as the NMW distribution.

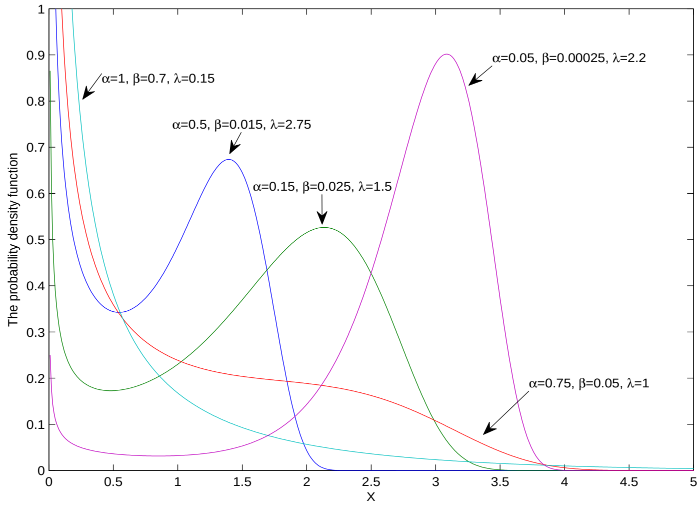


Figure 4.2: Probability density function of the reduced distribution.

4.3 The hazard rate function

The hazard rate function of the RNMW distribution is

$$h(x) = \frac{1}{2\sqrt{x}}\left[\alpha + \beta(1+2\lambda x)e^{\lambda x}\right],\ x > 0. \tag{4.4}$$

To derive the shape of $h(x)$, we obtain the first derivative of $\log\{h(x)\}$:

$$\frac{d}{dx}\log\{h(x)\} = -\frac{1}{2x} + \frac{\beta\lambda(2\lambda x+3)e^{\lambda x}}{\alpha+\beta(2\lambda x+1)e^{\lambda x}}. \tag{4.5}$$

Setting this to zero, we have

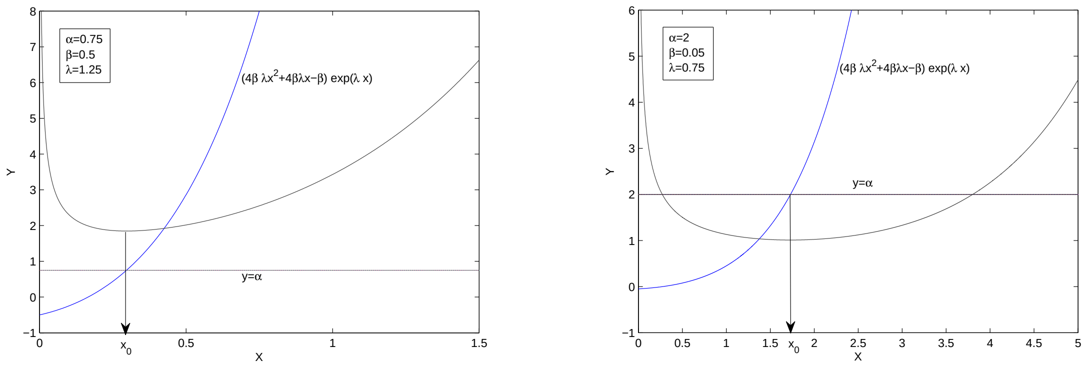
$$-\frac{1}{2x} + \frac{\beta\lambda(2\lambda x+3)e^{\lambda x}}{\alpha+\beta(2\lambda x+1)e^{\lambda x}} = 0. \tag{4.6}$$

Let x_0 denote the root of (4.6). Using the limit of (4.5) when $x \to 0$ and $x \to \infty$, it can be deduced that $\frac{d}{dx}\log\{h(x)\} < 0$ for $x \in (0, x_0)$, $\frac{d}{dx}\log\{h(x_0)\} = 0$, and $\frac{d}{dx}\log\{h(x)\} > 0$ for $x > x_0$. So, $h(x)$ initially decreases before increasing. Hence, we have a bathtub shape. Where x_0 denote the solution of (4.6); that is, the solution of

$$\left(4\beta\lambda x^2 + 4\beta\lambda x - \beta\right)e^{\lambda x} = \alpha. \tag{4.7}$$

The left hand side of (4.7) is $e^{\lambda x}$ multiplied by the quadratic function $4\beta\lambda x^2+4\beta\lambda x-\beta$.

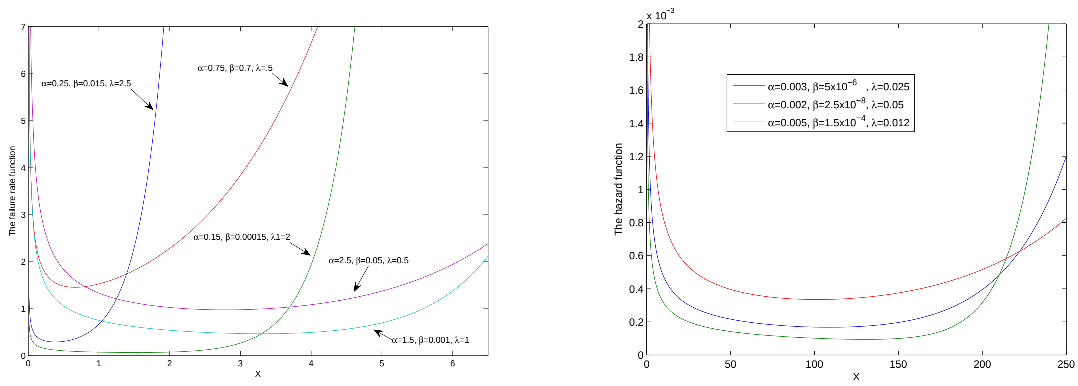
Then, x_0 can be obtained from the vertical projection from the intersection point between $y = (4\beta\lambda x^2 + 4\beta\lambda x - \beta)\,e^{\lambda x}$ and $y = \alpha$, hence the value x_0 is unique and positive as shown in Figure 4.3.



(a) (b)

Figure 4.3: The hazard function of the RNMW and the root of $\frac{d}{dx}\log\{h(x_0)\} = 0$.

Plots of the hazard rate function of the RNMW distribution are shown in Figure 4.4 (a). As mentioned in the previous chapter the bathtub shaped hazard function of the NMW distribution has shown to have a long useful life period, and so does the RNMW distribution, as Figure 4.4(b) shows.



(a) (b)

Figure 4.4: Hazard function of the RNMW distribution.

4.4 The moments

The r-th moment of the RNMW is represented in this section.

Corollary 1. *The non-central r-th moment of a random variable X that follows the reduced distribution $RNMW(\alpha, \beta, \lambda)$ is given by*

$$\mu_r' \quad = \quad 2r \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-\beta)^n (\lambda n)^m}{n!m!} \frac{\Gamma\left(n + 2(m+r)\right)}{\alpha^{(n+2(m+r))}}, \tag{4.8}$$

for $r = 1, 2, \ldots$.

The first eight moments of the RNMW $\alpha = 2, \beta = 0.15$ and $\lambda = 7.4 \times 10^{-4}$ are obtained using (4.8) and the numerical integration. Table 4.1 shows that the results of both of them, given to eight decimal places, are the same; but using (4.8) we need just around 0.17 seconds for each moment to calculate while the moment using numerical integration require more than 2.1 seconds. Matlab code is used to calculate the moments of the proposed distribution.

r	μ_r' Numerical Inte.	Time(sec)	μ_r' Using (4.8)	Time(sec)
1	0.28475750	7.3	0.28475750	0.18
2	0.48630879	2.4	0.48630879	0.18
3	2.07478272	2.6	2.07478272	0.17
4	16.50652427	2.4	16.50652427	0.17
5	210.77542527	2.3	210.77542527	0.18
6	3941.05391664	2.2	3941.05391664	0.17
7	101406.81407589	2.2	101406.81407589	0.17
8	3433153.21006252	2.1	3433153.21006251	0.17

Table 4.1: The first 8-th moments of the RNMW using (4.8) and numerical integration.

4.5 The moment generating function

The moment generating function of the RNMW is provided in the this section.

Theorem 3. *If X is a random variable following the RNMW, then the moment generating function is given by*

$$M_X(t) = 1 + 2 \sum_{n,m,k=0}^{\infty} \frac{(-\beta)^n (n\lambda)^m t^{k+1}}{n!m!k!} \left[\frac{\Gamma\left(n + 2(m+k) + 2\right)}{\alpha^{n+2(m+k)+2}} \right]. \tag{4.9}$$

Proof: The moment generating function is defied by

$$\begin{aligned}
M_X(t) &= E(e^{tx}) \\
&= \int_0^\infty e^{tx} f(x)dx \\
&= \int_0^\infty e^{tx} dF(x) \\
&= 1 + \int_0^\infty te^{tx} e^{-\alpha\sqrt{x}-\beta\sqrt{x}e^{\lambda x}} dx \\
&= 1 + \sum_{n,m,k=0}^{\infty} \frac{(-\beta)^n (n\lambda)^m t^{k+1}}{n!m!k!} I, \qquad (4.10)
\end{aligned}$$

where $I = \int_0^\infty x^{\frac{n}{2}+m+k} e^{-\alpha\sqrt{x}} dx$ and using gamma-integral formula the moment generating function the RNMW is given by

$$M_X(t) = 1 + 2 \sum_{n,m,k=0}^{\infty} \frac{(-\beta)^n (n\lambda)^m t^{k+1}}{n!m!k!} \left[\frac{\Gamma\left(n + 2(m+k) + 2\right)}{\alpha^{n+2(m+k)+2}} \right].$$

Using (4.9), the first four moments of the RNMW is

$$\begin{aligned}
M'_X(0) &= \mu'_1 = 2 \sum_{n,m=0}^{\infty} \frac{(-\beta)^n (n\lambda)^m}{n!m!} \left[\frac{\Gamma\left(n+2m+2\right)}{\alpha^{n+2m+2}} \right], \\
M''_X(0) &= \mu''_2 = 4 \sum_{n,m=0}^{\infty} \frac{(-\beta)^n (n\lambda)^m}{n!m!} \left[\frac{\Gamma\left(n+2m+4\right)}{\alpha^{n+2m+4}} \right], \\
M^{(3)}_X(0) &= \mu^{(3)}_3 = 6 \sum_{n,m=0}^{\infty} \frac{(-\beta)^n (n\lambda)^m}{n!m!} \left[\frac{\Gamma\left(n+2m+6\right)}{\alpha^{n+2m+6}} \right], \\
M^{(4)}_X(0) &= \mu^{(4)}_2 = 8 \sum_{n,m=0}^{\infty} \frac{(-\beta)^n (n\lambda)^m}{n!m!} \left[\frac{\Gamma\left(n+2m+8\right)}{\alpha^{n+2m+8}} \right].
\end{aligned}$$

This is consistent with the formal of the moment in (4.8).

4.6 Renyi entropy

In information theory, if X is a random variable with density function $f(x)$, the Renyi entropy is a measure of the uncertainty of the random variable.

Theorem 4. *For X $RNMW(\alpha, \beta, \lambda)$, the Renyi entropy is given by*

$$H(p) = \frac{1}{1-p} \log \left(\sum_{l=0}^{p} \sum_{m=0}^{l} \sum_{i,j=0}^{\infty} \binom{p}{l} \binom{l}{m} \frac{\beta^{l+i} \lambda^m (-p)^i (\lambda(l+i))^j}{2^{p-m-1} i! j!} \frac{\Gamma(2(j+m)+i-p+2)}{(p\alpha)^{2(j+m-p+1)+i+l)}} \right), \qquad (4.11)$$

where $\Gamma(\cdot)$ is the gamma function.

Proof: The Renyi entropy is given by

$$H(p) = \frac{1}{1-p} \log \left\{ \int_0^\infty \{f(x;\alpha,\beta,\lambda)\}^p \, dx \right\}. \tag{4.12}$$

Using the Taylor and binomial expansions,

$$\begin{aligned} [f(x)]^p &= \left\{ \frac{1}{2\sqrt{x}} (\alpha + \beta(1+2\lambda x)e^{\lambda x}) e^{-\alpha\sqrt{x} - \beta\sqrt{x}e^{\lambda x}} \right\}^p, \\ &= \sum_{l=0}^{p} \sum_{m=0}^{l} \sum_{i,j=0}^{\infty} \binom{p}{l} \binom{l}{m} \frac{\beta^{l+i} \lambda^m (-p)^i (\lambda(l+i))^j}{(2^{p-m}\alpha^{l-p}) i! j!} x^{j+m+\frac{i-p}{2}} e^{-\alpha p \sqrt{x}}. \end{aligned} \tag{4.13}$$

Substituting (4.13) into the integral part in (4.12), we have

$$\begin{aligned} \int_0^\infty [f(x)]^p \, dx &= \int_0^\infty \sum_{l=0}^{p} \sum_{m=0}^{l} \sum_{i,j=0}^{\infty} \binom{p}{l} \binom{l}{m} \frac{\beta^{l+i} \lambda^m (-p)^i (\lambda(l+i))^j}{(2^{p-m}\alpha^{l-p}) i! j!} x^{j+m+\frac{i-p}{2}} e^{-\alpha p \sqrt{x}} dx, \\ &= \sum_{l=0}^{p} \sum_{m=0}^{l} \sum_{i,j=0}^{\infty} \binom{p}{l} \binom{l}{m} \frac{\beta^{l+i} \lambda^m (-p)^i (\lambda(l+i))^j}{(2^{p-m}\alpha^{l-p}) i! j!} I, \end{aligned}$$

where $I = \int_0^\infty x^{j+m+\frac{i-p}{2}} e^{-\alpha p \sqrt{x}} dx$.

Using the Gamma integral

$$\int_0^\infty x^{b-1} e^{-ax^c} \mathrm{d}x = a^{-b/c} (1/c) \Gamma(b/c),$$

$$I = \frac{2\Gamma(2(j+m)+i-p+2)}{(p\alpha)^{2(j+m)+i-p+2)}}.$$

So, the Renyi entropy is

$$H(p) = \frac{1}{1-p} \log \left(\sum_{l=0}^{p} \sum_{m=0}^{l} \sum_{i,j=0}^{\infty} \binom{p}{l} \binom{l}{m} \frac{\beta^{l+i} \lambda^m (-p)^i (\lambda(l+i))^j}{2^{p-m-1} i! j!} \frac{\Gamma(2(j+m)+i-p+2)}{(p\alpha)^{2(j+m-p+1)+i+l)}} \right),$$

4.7 Reliability

The estimation of reliability is important in stress-strength models. If X is the strength of a component and Y is the stress, the component fails when $Y \geqslant X$. Then, the estimation of the reliability of the component R is $Pr(Y < X)$.

Theorem 5. *If X and Y are independent random variables following the same distribution $RNMW(\alpha, \beta, \lambda)$, then the estimation of reliability R is given by*

$$R = 1 - \sum_{i=0}^{\infty} \frac{(-2\beta)^i}{i!} \left(\alpha \sum_{j=0}^{\infty} \frac{(i\lambda)^j}{j!} \frac{\Gamma(2j+i+1)}{(2\alpha)^{2j+i+1}} + \beta \sum_{k=0}^{\infty} \frac{(\lambda(i+1))^k}{k!} \left(\frac{\Gamma(2k+i+1)}{(2\alpha)^{2k+i+1}} + \frac{2\lambda\Gamma(2(k+1)+i+1)}{(2\alpha)^{2(k+1)+i+1}} \right) \right). \tag{4.14}$$

Proof: The reliability R can be written as

$$R = \int_0^{\infty} f(x)F(x)dx. \tag{4.15}$$

Then,

$$\begin{aligned} R &= 1 - \int_0^{\infty} f(x)S(x)dx, \\ &= 1 - \sum_{i=0}^{\infty} \frac{(-2\,\beta)^i}{2(i!)} \left\{ \alpha \sum_{j=0}^{\infty} \frac{(i\lambda)^j}{j!} I_1 + \beta \sum_{k=0}^{\infty} \frac{(\lambda(i+1))^k}{k!} \times (I_2 + 2\lambda I_3) \right\}, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^{\infty} x^{\frac{i-1}{2}+j} e^{-2\alpha\sqrt{x}} dx, \\ I_2 &= \int_0^{\infty} x^{\frac{i-1}{2}+k} e^{-2\alpha\sqrt{x}} dx, \\ I_3 &= \int_0^{\infty} x^{\frac{i-1}{2}+k+1} e^{-2\alpha\sqrt{x}} dx. \end{aligned}$$

Using the Gamma integral,

$$\begin{aligned} I_1 &= \frac{2\Gamma\,(2j+i+1)}{(2\,\alpha)^{2j+i+1}}, \\ I_2 &= \frac{2\Gamma\,(2k+i+1)}{(2\,\alpha)^{2k+i+1}}, \\ I_3 &= \frac{2\Gamma\,(2k+i+3)}{(2\,\alpha)^{2k+i+3}}, \end{aligned}$$

then

$$R = 1 - \sum_{i=0}^{\infty} \frac{(-2\beta)^i}{i!} \left(\alpha \sum_{j=0}^{\infty} \frac{(i\lambda)^j}{j!} \frac{\Gamma(2j+i+1)}{(2\alpha)^{2j+i+1}} + \beta \sum_{k=0}^{\infty} \frac{(\lambda(i+1))^k}{k!} \left(\frac{\Gamma(2k+i+1)}{(2\alpha)^{2k+i+1}} + \frac{2\lambda\Gamma(2(k+1)+i+1)}{(2\alpha)^{2(k+1)+i+1}} \right) \right).$$

4.8 Order statistics

This section discusses order statistics, which have many applications in reliability and lifetime analysis. We derive the probability density function of the order statistic $X_{(r)}$, and its moments, where $X_1, \ldots, X_n$ are drawn from the $RNMW(\alpha, \beta, \lambda)$.

If $X_1, X_2, \cdots, X_n$ are a random sample from the $RNMW(\alpha, \beta, \lambda)$ with CDF (4.1) and pdf (4.3), and $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the order statistics from this sample, then, from Arnold *el at* (2008), the pdf of the r-th statistic $X_{(r)}$ is given by

From (3.12), (3.13) and (3.14), where $h(x)$ is the hazard function (4.4) and $H(x) = \alpha\sqrt{x} + \beta\sqrt{x}e^{\lambda x}$ is the cumulative hazard. we get

$$
\begin{aligned}
f_{r:n}(x) &= \frac{1}{B(r, n-r+1)} \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} (-1)^\ell h(x) e^{-(n+\ell+1-r)H(x)}, \\
&= n\binom{n-1}{r-1} \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} (-1)^\ell h(x) e^{-(n+\ell+1-r)H(x)}, \\
&= n\binom{n-1}{r-1} \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} (-1)^\ell \left(\frac{1}{2\sqrt{x}}\left(\alpha + \beta(1+2\lambda x)\right) e^{\lambda x}\right) \times \\
&\quad e^{-(n+\ell+1-r)(\alpha\sqrt{x}+\beta\sqrt{x})}, \\
&= n\binom{n-1}{r-1} \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} \frac{(-1)^\ell}{(n+\ell+1-r)} f(x;\ \alpha_\ell, \beta_\ell, \lambda),
\end{aligned}
$$

where $f(x;\ \alpha_\ell, \beta_\ell, \lambda)$ is the PDF of the RNMW with parameters $\alpha_\ell = (n+\ell+1-r)\alpha$, $\beta_\ell = (n+\ell+1-r)\beta$ and λ.

Using (4.8), the kth non-central moment of the rth order statistic is then

$$
\mu_k^{'(r:n)} = 2nk\binom{n-1}{r-1} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\ell=0}^{r-1} \binom{r-1}{\ell} \frac{(-1)^{\ell+i}\beta^i (i\lambda)^j \Gamma\left(i+2(j+k)+1\right)}{\alpha^i(\alpha\left(n+\ell+1-r\right))^{2(j+k)+1}}. \tag{4.16}
$$

.

4.9 Parameter estimation

In this section, point and interval estimators of the unknown parameters of the RNMW distribution are derived using the maximum likelihood method. We consider both complete data and censored data, then assess the finite sample performance of the MLEs with respect to sample size n.

4.9.1 Complete data

The PDF of the RNMW distribution can be rewritten as

$$
f(x) = h\left(x; \underline{\vartheta}\right) e^{-\alpha\sqrt{x}-\beta\sqrt{x}e^{\lambda x}}
$$

for $x > 0$, where $h\left(x; \underline{\vartheta}\right)$ is the hazard rate function in (4.4) and $\underline{\vartheta} = (\alpha, \beta, \lambda)$ is a vector of parameters.

Let $x_1, \ldots, x_n$ denote a random sample of complete data from the RNMW distribution. Then, the log-likelihood function is

$$\mathcal{L}\left(\underline{\vartheta}\right) = \sum_{i=1}^{n} \left[\log\left(h\left(x_i; \underline{\vartheta}\right)\right) - \alpha\sqrt{x_i} - \beta\sqrt{x_i}e^{\lambda x_i}\right].$$

The likelihood equations are obtained by setting the first partial derivatives of ℓ with respect to α, β and λ to zero; that is,

$$\sum_{i=1}^{n} \frac{1}{h\left(x_i; \alpha, \beta, \lambda\right)\left(2\sqrt{x_i}\right)} - \sum_{i=1}^{n} \sqrt{x_i} = 0, \tag{4.17}$$

$$\sum_{i=1}^{n} \frac{\left(0.5 + \lambda x_i\right) e^{\lambda x_i}}{h\left(x_i; \alpha, \beta, \lambda\right)\sqrt{x_i}} - \sum_{i=1}^{n} e^{\lambda x_i}\sqrt{x_i} = 0, \tag{4.18}$$

$$\sum_{i=1}^{n} \frac{\sqrt{x_i}\left(\frac{3}{2} + \lambda x_i\right) e^{\lambda x_i}}{h\left(x_i; \alpha, \beta, \lambda\right)} - \sum_{i=1}^{n} e^{\lambda x_i}\sqrt{x_i^3} = 0. \tag{4.19}$$

4.9.2 Censored data

Here, we consider maximum likelihood estimation for censored data without replacement. Let X_i and C_i denote the lifetime and the censoring time for tested individual i, $i = 1, \ldots, n$. Suppose X_i and C_i are independent random variables. The failure times are $x_i = \min(X_i, C_i)$, $i = 1, \ldots, n$. Then, the log-likelihood function is

$$\mathcal{L}\left(\underline{\vartheta}\right) = \sum_{i=1}^{d} \log\left[h\left(x_i; \underline{\vartheta}\right) - \alpha\sqrt{x_i} - \beta\sqrt{x_i}e^{\lambda x_i}\right] - \sum_{i \in C}\left[\alpha\sqrt{x_i} + \beta\sqrt{x_i}e^{\lambda x_i}\right],$$

where d is the number of failures and C contains indices of the censored observations.

Setting the first partial derivatives of $\mathcal{L}\left(\underline{\vartheta}\right)$ with respect to α, β and λ to zero, the likelihood equations are obtained as

$$\sum_{i=1}^{d} \left\{\frac{1}{h\left(x_i; \alpha, \beta, \lambda\right)\left(2\sqrt{x_i}\right)} - \sqrt{x_i}\right\} - \sum_{i \in C} \sqrt{x_i} = 0, \tag{4.20}$$

$$\sum_{i=1}^{d} \left\{\frac{\left(0.5 + \lambda x_i\right) e^{\lambda x_i}}{h\left(x_i; \alpha, \beta, \lambda\right)\sqrt{x_i}} - e^{\lambda x_i}\sqrt{x_i}\right\} - \sum_{i \in C} \sqrt{x_i}e^{\lambda x_i} = 0, \tag{4.21}$$

$$\sum_{i=1}^{d} \left\{\frac{\sqrt{x_i}\left(\frac{3}{2} + \lambda x_i\right) e^{\lambda x_i}}{h\left(x_i; \alpha, \beta, \lambda\right)} - e^{\lambda x_i}\sqrt{x_i^3}\right\} - \sum_{i \in C} e^{\lambda x_i}\sqrt{x_i^3} = 0. \tag{4.22}$$

By solving the systems of nonlinear likelihood equations, (4.17), (4.18), (4.19) and (4.20), (4.21), (4.22), numerically for α, β and λ, we can obtain maximum likelihood estimates for complete and censored data.

In order to find the interval estimation of α, β and λ, the observed information matrix is obtained since the expected information matrix is very complicated. The observed information matrix $J(\underline{\vartheta})$ is

$$J(\underline{\vartheta}) = -\begin{bmatrix} \mathcal{L}_{\alpha\alpha} & \mathcal{L}_{\alpha\beta} & \mathcal{L}_{\alpha\lambda} \\ & \mathcal{L}_{\beta\beta} & \mathcal{L}_{\beta\lambda} \\ & & \mathcal{L}_{\lambda\lambda} \end{bmatrix},$$

where the elements of this matrix are given in Appendix B

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}\left(\widehat{\vartheta} - \vartheta\right)$ is $N_3(0, I(\underline{\vartheta})^{-1})$, where $I(\underline{\vartheta})$ is the expected information matrix. This asymptotic behavior is valid if $I(\underline{\vartheta})$ is replaced by $J(\widehat{\underline{\vartheta}})$, i.e., the observed information matrix evaluated at $\widehat{\vartheta}$. The asymptotic multivariate normal $N_3(0, J(\widehat{\underline{\vartheta}})^{-1})$ distribution can be used to construct approximate confidence intervals and confidence regions for the individual parameters.

4.9.3 Simulation study

Here, we assess the performance of the MLEs with respect to sample size n. The assessment is based on a simulation study:

1. First, ten thousand samples of size n are generated from (4.1). We used the inversion method to generate samples, which was done for the RNMW distribution by solving

$$\alpha\sqrt{X} + \beta\sqrt{X}e^{\lambda X} = -\ln(1-U),$$

   where $U \sim U(0,1)$ is a uniform variate on the unit interval.

2. Obtain the MLEs for the ten thousand generated samples, $\left(\widehat{\alpha}_i, \widehat{\beta}_i, \widehat{\lambda}_i\right)$ for $i = 1, 2, \ldots, 10000$.

3. For the generated samples, compute the standard errors, say $\left(s_{\widehat{\alpha}_i}, s_{\widehat{\beta}_i}, s_{\widehat{\lambda}_i}\right)$ for $i = 1, 2, \ldots, 10000$. The standard errors are computed by using the observed information matrices.

4. The biases and mean squared errors of the MLEs can be computed as the following,

$$\mathrm{bias}_h(n) = \frac{1}{10000}\sum_{i=1}^{10000}\left(\widehat{h}_i - h\right),$$
$$\mathrm{MSE}_h(n) = \frac{1}{10000}\sum_{i=1}^{10000}\left(\widehat{h}_i - h\right)^2$$

for $h = \alpha, \beta, \lambda$.

The above steps were repeated for $n = 10, 11, \ldots, 100$ with $\alpha = 1$, $\beta = 1$ and $\lambda = 1$, then, we computed $\mathrm{bias}_h(n)$ and $\mathrm{MSE}_h(n)$ for $h = \alpha, \beta, \lambda$ and $n = 10, 11, \ldots, 100$.

Figure 4.5 shows the biases of the MLEs of (α, β, λ) versus $n = 10, 11, \ldots, 100$ when $(\alpha, \beta, \lambda) = (0.1, 0.1, 0.1)$. It can be seen that the biases for the three parameters are small and vary with respect to the samples sizes. The mean squared errors of the MLEs of each parameters are shown in Figure 4.6, which are vary with respect to the samples sizes. The broken line in Figure 4.5 indicates to the distance between the biases and the zero. The broken line in Figure 4.6 indicates to the distance between the mean squared errors and the zero.

From these figures the following observations can be made. The biases of the MLEs of the parameters α and β are generally negative; the biases of the MLEs of the parameter λ are generally positive; the biases of the MLEs of the parameter β are the smallest; the biases of the MLEs of the parameter λ are the largest; the biases of the MLEs of the parameters α and β increase to zero as $n \to \infty$ where the biases for the parameter λ decrease to zero as $n \to \infty$; the mean squared errors of the MLEs of the parameter β are the smallest; the mean squared errors of the MLEs of the parameter α are the the largest; the mean squared errors for the three parameters decrease to zero as $n \to \infty$; The above observations are observed by choosing $(\alpha, \beta, \lambda) = (0.1, 0.1, 0.1)$. A similar results were found for other choices.

4.10 Applications

This section provides four applications, two of them are for complete (uncensored) data sets and the others are for censored data sets, to show how the RNMW distribution can be applied in practice. It has been shown in chapter 3 that the NMW distribution

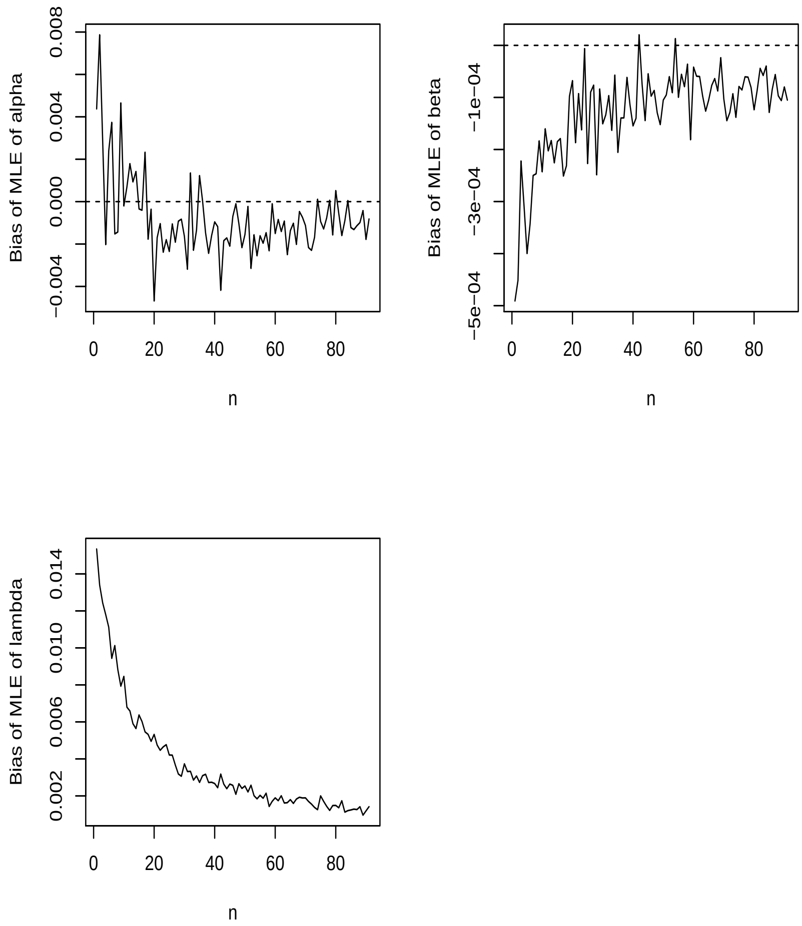


Figure 4.5: Biases of the MLEs of (α, β, λ) versus $n = 10, 11, \ldots, 100$ $((\alpha, \beta, \lambda) = (0.1, 0.1, 0.1))$.

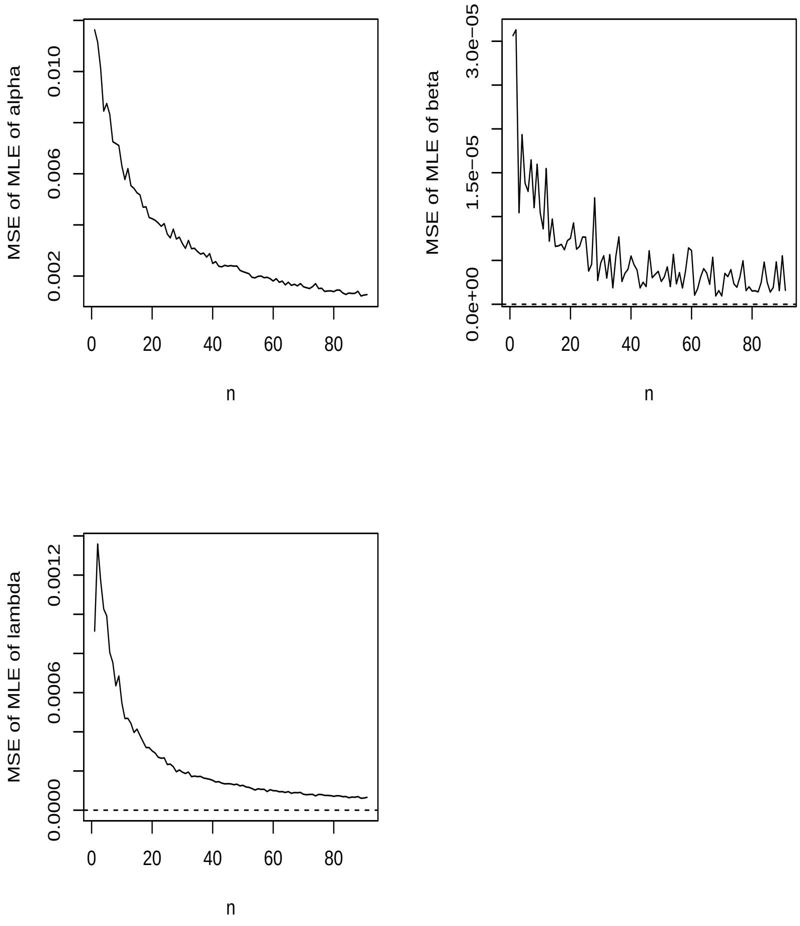


Figure 4.6: Mean squared errors of the MLEs of (α, β, λ) versus $n = 10, 11, \ldots, 100$ $(\alpha, \beta, \lambda) = (0.1, 0.1, 0.1)$.

fits several data sets better than existing modifications of the Weibull distribution like the BMW distribution, the AddW distribution, the MW distribution and the SZMW distribution. So, we shall compare the fits of the RNMW and NMW distributions to see if the former can perform as well as the NMW distribution. The Kolmogorov-Smirnov test, AIC, BIC, AICc and CAIC are used to compare the candidate distributions. The log-likelihood ratio test is used to compare the NMW and RNMW distributions by testing the hypotheses: $H_0 : \theta = \gamma = \frac{1}{2}$ versus $H_1 : H_0$ is false. The likelihood ratio test statistic for testing H_0 against H_1 is $\omega = 2(\mathcal{L}_{NMW} - \mathcal{L}_{RNMW})$, which follows a χ^2 distribution with two degrees of freedom under H_0.

This section presents four real data applications. The sample size for the first data set is fifty. The sample size for the second data set is forty four. The sample size for the third data set is eighty two. The sample size for the fourth data set is one hundred and forty eight. Hence, the biases for $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\lambda}$ can be expected to be less than 0.003, 2×10^{-4} and 0.003, respectively, for the first two data sets and to be less than 0.002, 1×10^{-4} and 0.002, respectively, for the other two data sets. The mean squared errors for $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\lambda}$ can be expected to be less than 0.003, 5×10^{-6} and 3×10^{-4} respectively, for first two data sets and be less than 0.002 5×10^{-6} and 2×10^{-4} respectively, for the other two data sets.

4.10.1 Complete data

In this section, we show how the RNMW distribution can be applied in practice for two complete (uncensored) real data sets.

Aarset data

The Aarset data (Aarset, 1987), consisting of lifetimes of fifty devices, which has been presented in the previous chapter, exhibits a bathtub shaped hazard rate. Both the NMW and RNMW distributions are now fitted to this data set. Table 4.2 gives the MLEs of the parameters, the corresponding standard errors, AIC, BIC, AICc and CAIC. Table 4.3 provides the K-S test statistics. Figures 4.7a and 4.7b show the histogram of the data, PDFs of the fitted NMW and RNMW distributions, the empirical survival function, and the survival functions of the fitted NMW and RNMW distributions.

Table 4.2: MLEs of parameters, standard errors, AIC, BIC, AICc and CAIC for Aarset data.

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\gamma}$	$\widehat{\theta}$	$\widehat{\lambda}$	AIC	BIC	AICc	CAIC
NMW	0.071	7.015×10^{-8}	0.016	0.595	0.197	435.8	445.4	437.2	450.4
	(0.031)	(1.501×10^{-7})	(3.602)	(0.128)	(0.184)				
RNMW	0.102	3.644×10^{-8}	$\frac{1}{2}$	$\frac{1}{2}$	0.180	433.1	439.0	433.8	442.0
	(0.019)	(6.089×10^{-8})	–	–	(0.020)				

Table 4.3: K-S statistics for models fitted to Aarset data.

Model	K-S
NMW	0.088
RNMW	0.092

It is clear that both the NMW and RNMW distributions provide adequate fits. Both have very small K-S values (0.088 and 0.092, respectively). The NMW distribution has the larger log-likelihood of -212.9. However, the RNMW distribution has the smaller values for AIC, BIC, AICc and CAIC. The likelihood ratio test statistic for testing $H_0 : \theta = \gamma = \frac{1}{2}$ versus $H_1 : H_0$ is false takes the value 1.436 and the corresponding p-value is 0.488, so there is no significant evidence to reject H_0. Hence, the NMW distribution does not improve significantly on the fit of the RNMW distribution.

The plots of the empirical TTT-transform, TTT-transforms of the fitted NMW and RNMW distributions, the nonparametric hazard rate function, and the hazard rate functions of the fitted NMW and RNMW distributions are shown in Figures 4.7c and 4.7d. It is clear that the RNMW distribution provides as good a fit as the NMW distribution.

The variance-covariance matrix for the estimated parameters of the RNMW may be approximated as J^{-1},

$$J^{-1} = \begin{bmatrix} 5.203 \times 10^{-4} & -3.697 \times 10^{-9} & 1.214 \times 10^{-3} \\ -3.697 \times 10^{-9} & 9.078 \times 10^{-14} & -2.994 \times 10^{-8} \\ 1.214 \times 10^{-3} & -2.994 \times 10^{-8} & 9.882 \times 10^{-3} \end{bmatrix}.$$

So, approximate 95 percent confidence intervals for the parameters α, β and λ are $[0.064, 0.14]$, $[0, 1.558 \times 10^{-7}]$ and $[0.219 \times 10^{-5}, 0.142]$, respectively.

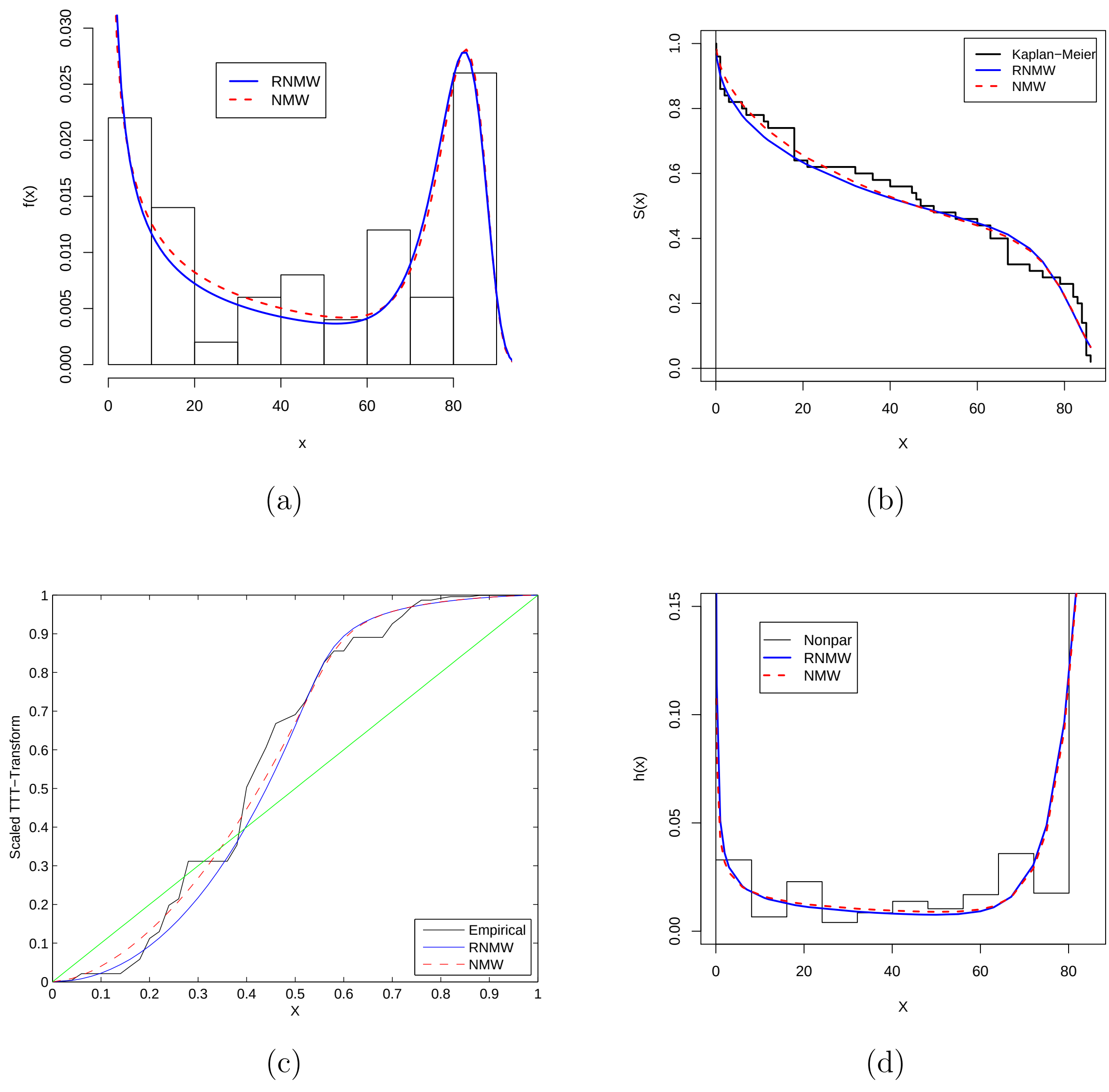


Figure 4.7: For Aarst data: (a) Histogram and fitted PDFs; (b) Empirical and fitted survival functions; (c) Empirical and fitted TTT-transforms; (c) Nonparametric and fitted hazard rate functions.

Table 4.4: MLEs of parameters, standard errors, AIC, BIC, AICc and CAIC for Kumar data.

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\gamma}$	$\widehat{\theta}$	$\widehat{\lambda}$	AIC	BIC	AICc	CAIC
NMW	0.034	4.588×10^{-4}	0.423	0.657	0.071	410.0	418.9	411.6	423.9
	(0.022)	(3.794×10^{-3})	(2.351)	(0.18)	(0.034)				
RNMW	0.055	5.852×10^{-4}	$\frac{1}{2}$	$\frac{1}{2}$	0.065	406.9	412.2	407.5	415.2
	(0.017)	(6.523×10^{-4})	–	–	(0.012)				

Kumar data

Kumar *et al.* (1989) presented data consisting of times between failures (TBF) in days of load-haul-dump machines (LHD) used to pick up rock or waste. As shown in Figure 4.8, the scaled TTT-Transform plot of this data has a convex shape followed by a concave shape. That corresponds to a bathtub shaped hazard function for this data set.

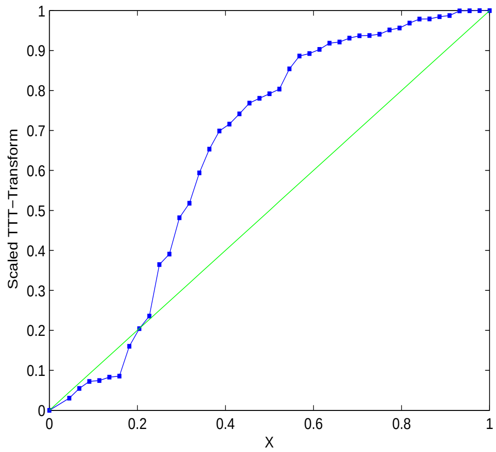


Figure 4.8: TTT-transform plot for the Kumar.

Tables 4.4 and 4.5 show the MLEs of the parameters, corresponding standard errors, AIC, BIC, AICc, CAIC and the K-S test statistic. Both distributions (NMW and RNMW) provide adequate fits. The log-likelihood value is larger for the NMW distribution. That for the RNMW distribution is only slightly smaller. The K-S statistic is larger for the NMW distribution, and for the reduced model it is very small at 0.055. However, the RNMW distribution has the smaller AIC, BIC, AICc and CAIC values.

Table 4.5: K-S statistics for models fitted to Kumar data.

Model	K-S
NMW	0.068
RNMW	0.061

Figures 4.9 (a) and (d) show that both distributions fit the data adequately. However, the log-likelihood ratio statistic for testing $H_0 : \theta = \gamma = \frac{1}{2}$ versus $H_1 : H_0$ is false is $\omega = 0.895$ with the corresponding p-value of 0.639. Hence, again there is no evidence that the NMW distribution provides a better fit than the RNMW distribution.

The variance-covariance matrix for the estimated parameters of the RNMW may be approximated as J^{-1},

$$J^{-1} = \begin{bmatrix} 2.839 \times 10^{-4} & -3.968 \times 10^{-6} & 6.731 \times 10^{-5} \\ -3.968 \times 10^{-6} & 4.254 \times 10^{-7} & -7.836 \times 10^{-6} \\ 6.731 \times 10^{-5} & -7.836 \times 10^{-6} & 1.498 \times 10^{-4} \end{bmatrix}.$$

So, approximate 95 percent confidence intervals for the parameters α, β and λ are $[0.022, 0.088]$, $[0, 1.864 \times 10^{-3}]$ and $[0.041, 0.089]$, respectively.

4.10.2 Censored data

In this section, we show how the RNMW distribution can be applied in practice for two real censored data sets, one of which is presented here for the first time.

Drug data

This data set was collected from a prison in the Middle East in 2011. It represents a sample of eighty two prisoners convicted of using or selling drugs. They were all released as part of a general amnesty for prisoners. We consider the time from release to re-offending to be the failure time. Of the eighty two prisoners, sixty six were arrested again for abuse or sale of drugs. After one hundred and eleven weeks, the others were considered to be censored. Again, regarding to the TTT-transform plot in Figure 4.10, the data has a bathtub shaped hazard rate.

Both the NMW and RNMW distributions were fitted to the data. Tables 4.6 and 4.7 show the MLEs of the parameters, corresponding standard errors, AIC, BIC, AICc,

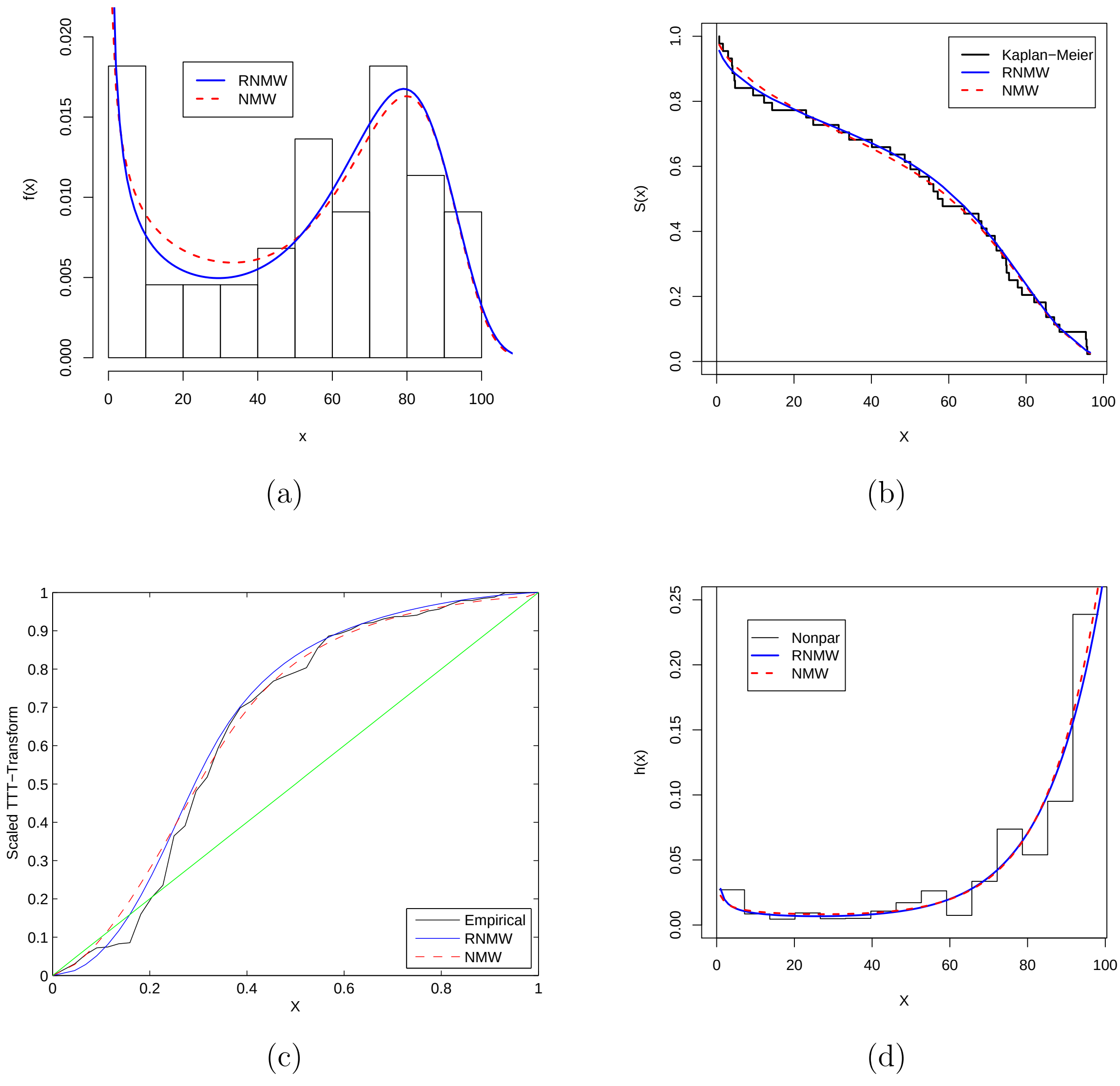


Figure 4.9: For Kumar data: (a) Histogram and fitted PDFs; (b) Empirical and fitted survival functions; (c) Empirical and fitted TTT-transforms; (c) Nonparametric and fitted hazard rate functions.

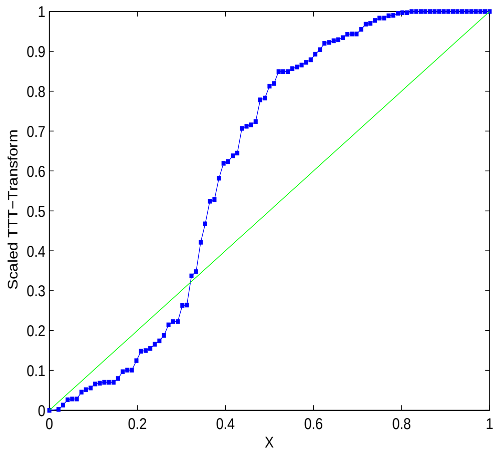


Figure 4.10: TTT-transform plot for the drug data.

Table 4.6: MLEs of parameters, standard errors, AIC, BIC, AICc and CAIC for drug data.

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\gamma}$	$\widehat{\theta}$	$\widehat{\lambda}$	AIC	BIC	AICc	CAIC
NMW	0.019	2.954×10^{-3}	0.484	0.735	0.031	700.9	713.0	701.7	718.0
	(9.426×10^{-3})	(3.656×10^{-3})	(0.504)	(0.159)	(0.019)				
RNMW	0.038	5.863×10^{-3}	$\frac{1}{2}$	$\frac{1}{2}$	0.026	698.4	705.6	698.7	708.7
	(0.017)	(6.584×10^{-3})	–	–	(9.319×10^{-3})				

CAIC and the K-S test statistics. We see that the RNMW distribution has the smaller AIC, BIC, AICc and CAIC values. The K-S statistic values for both distributions are approximately equal to 0.055.

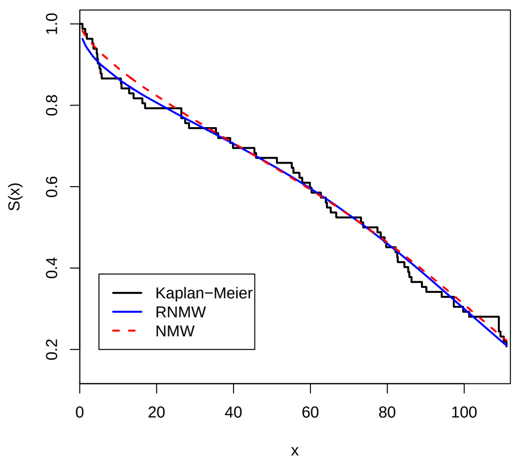
Figures 4.11 (a)-(d) show the histogram of the data, PDFs of the fitted NMW and RNMW distributions, the empirical survival function, the survival functions of the fitted NMW and RNMW distributions, the empirical TTT-transform, TTT-transforms of the fitted NMW and RNMW distributions, the nonparametric hazard rate function, and the hazard rate functions of the fitted NMW and RNMW distributions. We can see that the RNMW distribution fits the data as well as the NMW distribution.

The log-likelihood ratio statistic for testing $H_0 : \theta = \gamma = \frac{1}{2}$ versus $H_1 : H_0$ is false is $\omega = 1.496$ with the corresponding p-value of 0.473. Hence, again there is no evidence that the NMW distribution provides a better fit than the RNMW distribution.

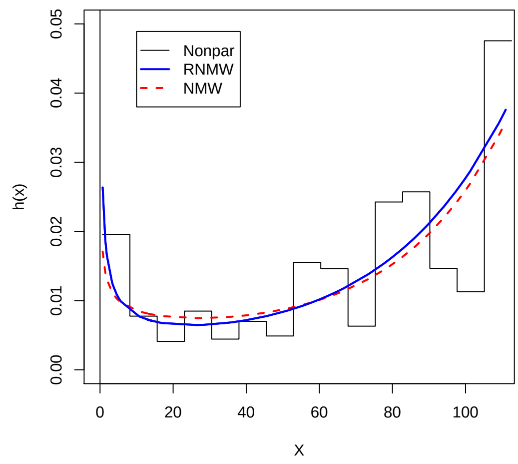
The variance-covariance matrix for the estimated parameters of the RNMW may

Table 4.7: K-S statistics for models fitted to the drug data.

Model	K-S
NMW	0.0550
RNMW	0.0553



(a)



(b)

Figure 4.11: For drug data: (a) Empirical and fitted survival functions; (b) Nonparametric and fitted hazard rate functions.

be approximated as J^{-1},

$$J^{-1} = \begin{bmatrix} 2.801\times10^{-4} & -8.519\times10^{-5} & 1.138\times10^{-4} \\ -8.519\times10^{-5} & 4.335\times10^{-5} & -6.037\times10^{-5} \\ 1.138\times10^{-4} & -6.037\times10^{-5} & 8.684\times10^{-5} \end{bmatrix}.$$

So, approximate 95 percent confidence intervals for the parameters α, β and λ are $[5.553\times10^{-3}, 0.071]$, $[0, 0.019]$ and $[8.137\times10^{-3}, 0.045]$, respectively.

Serum-reversal data

The Serum-reversal data consists of serum-reversal time in days of one hundred and forty eight children contaminated with HIV from vertical transmission at the university hospital of the Ribeiro Preto School of Medicine (Hospital das Clnicas da Faculdade de Medicina de Ribeiro Preto) from 1986 to 2001, cf. Perdona (2006) and Silva (2004).

The scaled TTT-Transform plot for this data is shown in Figure 4.12, which takes a convex shape followed by a concave shape. This corresponds to a bathtub shaped

Table 4.8: MLEs of parameters, standard errors, AIC, BIC, AICc and CAIC for serum-reversal data.

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\gamma}$	$\widehat{\theta}$	$\widehat{\lambda}$	AIC	BIC	AICc	CAIC
NMW	1.74×10^{-3}	6.141×10^{-4}	0.542	0.438	0.015	783.8	798.8	784.4	803.8
	(3.115×10^{-3})	(2.274×10^{-3})	(0.507)	(0.763)	(3.764×10^{-3})				
RNMW	1.799×10^{-3}	5.955×10^{-4}	$\frac{1}{2}$	$\frac{1}{2}$	0.014	780.1	789.1	780.3	792.1
	(1.971×10^{-3})	(4.174×10^{-4})	–	–	(2.061×10^{-3})				

Table 4.9: K-S statistics for models fitted to serum-reversal data.

Model	K-S
NMW	0.115
RNMW	0.107

HF.

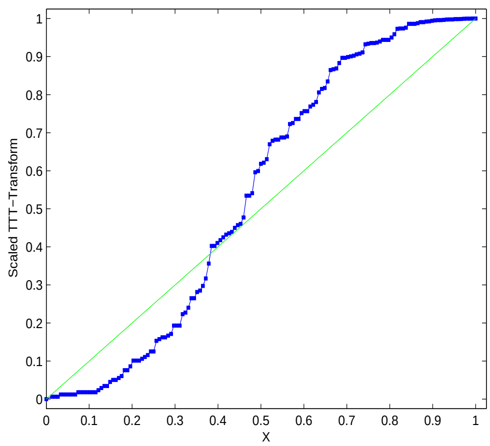
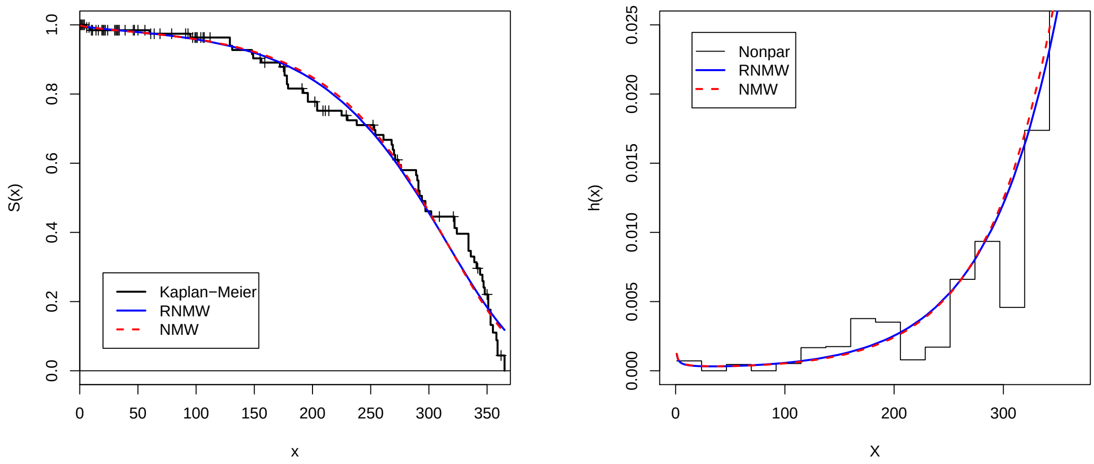


Figure 4.12: TTT-transform plot for the serum-reversal data.

Tables 4.8 and 4.9 show the MLEs of the parameters, corresponding standard errors, AIC, BIC, AICc, CAIC and the K-S test statistics. We see that the RNMW distribution has smaller values for AIC, BIC, AICc and CAIC. The K-S statistic is only slightly larger for the NMW distribution.

Figures 4.13a-d show that both distributions fit the data adequately. However, the log-likelihood ratio statistic for testing $H_0 : \theta = \gamma = \frac{1}{2}$ versus $H_1 : H_0$ is false is $\omega = 0.194$ with the corresponding p-value of 0.908. Hence, again there is no evidence that the NMW distribution provides a better fit than the RNMW distribution.

The variance-covariance matrix for the estimated parameters of the RNMW may



(a) (b)

Figure 4.13: For serum data: (a) Empirical and fitted survival functions; (b) Nonparametric and fitted hazard rate functions.

be approximated as J^{-1},

$$J^{-1} = \begin{bmatrix} 3.886 \times 10^{-6} & -4.245 \times 10^{-7} & 1.990 \times 10^{-6} \\ -4.245 \times 10^{-7} & 1.742 \times 10^{-7} & -8.445 \times 10^{-7} \\ 1.990 \times 10^{-6} & -8.445 \times 10^{-7} & 4.246 \times 10^{-6} \end{bmatrix}.$$

So, approximate 95 percent confidence intervals for the parameters α, β and λ are $[0, 5.662 \times 10^{-3}]$, $[0, 1.414 \times 10^{-3}]$ and $[0.010, 0.080]$, respectively.

4.11 Summary

In this chapter, we reduced the number of parameters of the new modified Weibull distribution introduced in chapter 3 from five to three parameters to simplify the distribution. We studied the mathematical properties of the reduced distribution, including the hazard function, moments, order statistics and Renyi entropy. The estimation of the reliability $P(X < Y)$ was studied. The maximum likelihood estimation of the parameters was discussed. Two applications of complete and censoring data were presented and it has shown that the reduced distribution fits the data as good as the original new modified Weibull distribution, but better than its sub-models.

Chapter 5

Exponentiated Reduced Modified Weibull Distribution

5.1 Introduction

Chapter 4 introduced a three-parameter modified Weibull distribution called the reduced new modified Weibull (RNMW) distribution. Its CDF, PDF and HF are given in (4.1), (4.3) and (4.4) respectively.

The RNMW distribution can exhibit bathtub shaped HFs but not more other complicated shapes. The aim of this chapter is to introduce a four-parameter generalization that can accommodate monotonically increasing, bathtub shaped, unimodal and modified unimodal HFs. The new distribution will be called the *Exponentiated Reduced Modified Weibull Distribution* and abbreviated ERMW.

The contents of this chapter are organized as follows. The new distribution is introduced in Section 5.2. Some special cases of the ERMW distribution are presented in Section 5.3. The shapes of the PDF and the HF of the ERMW distribution are studied in Section 5.4. The moments of the ERMW distribution are given in Section 5.5. Section 5.6 and 5.7 discuss order statistics and maximum likelihood estimates of the unknown parameters of the ERMW distribution including a simulation study to assess the performance of the MLEs. Two real data applications of the ERMW distribution are given in Sections 5.8. One of these data sets has a bathtub shaped HF and the other has a unimodal HF. Section 5.9 summaries the chapter.

5.2 The ERMW distribution

Let X denote a random variable having the ERMW distribution. Its CDF is defined by exponentiating the CDF of the RNMW distribution. That is, we take the CDF of X as

$$F(x) = \left[1 - e^{-\alpha\sqrt{x} - \beta\sqrt{x}e^{\lambda x}}\right]^{\theta}, \ x > 0 \tag{5.1}$$

where $\alpha > 0$, $\beta > 0$, $\lambda > 0$ and $\theta > 0$. Here, α and β are scale parameters, λ is an acceleration parameter and θ is a shape parameter. The corresponding PDF and HF are

$$f(x) = \frac{\theta}{2\sqrt{x}}\left[\alpha + \beta(1 + 2\lambda x)e^{\lambda x}\right]\left[1 - e^{-\alpha\sqrt{x} - \beta\sqrt{x}e^{\lambda x}}\right]^{\theta - 1} e^{-\alpha\sqrt{x} - \beta\sqrt{x}e^{\lambda x}}, \ x > 0 \tag{5.2}$$

and

$$h(x) = \frac{\theta\left[\alpha + \beta(1 + 2\lambda x)e^{\lambda x}\right]\left[1 - e^{-\alpha\sqrt{x} - \beta\sqrt{x}e^{\lambda x}}\right]^{\theta - 1} e^{-\alpha\sqrt{x} - \beta\sqrt{x}e^{\lambda x}}}{2\sqrt{x}\left\{1 - \left[1 - e^{-\alpha\sqrt{x} - \beta\sqrt{x}e^{\lambda x}}\right]^{\theta}\right\}}, \ x > 0 \tag{5.3}$$

where $\alpha > 0$, $\beta > 0$, $\lambda > 0$ and $\theta > 0$.

If the parameter θ is a positive integer, the ERMW distribution can be interpreted as the lifetime distribution of a parallel system of θ independent and identically distributed components, where each component is a serial system with two independent components: one of which follows the Weibull distribution with parameters α and $\frac{1}{2}$, and the other follows the modified Weibull distribution of Lai et al. (2003) with parameters β, $\frac{1}{2}$ and λ.

Using the series expansion of $(1 - x)^{\theta - 1}$, we have

$$\left(1 - e^{-\alpha\sqrt{x} - \beta\sqrt{x}e^{\lambda x}}\right)^{\theta - 1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\theta)}{\Gamma(\theta - j) j!} e^{-j\alpha\sqrt{x} - j\beta\sqrt{x}e^{\lambda x}}, \tag{5.4}$$

the PDF in (5.2) can be expressed as

$$f(x) = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\theta + 1)}{\Gamma(\theta - j)(j + 1)!} f_{\mathrm{RNMW}}(x; \alpha_{j+1}, \beta_{j+1}, \lambda), \tag{5.5}$$

for $x > 0$, where $\phi_{j+1} = (j + 1)\phi$ and $f_{\mathrm{RNMW}}(x; \alpha, \beta, \lambda)$ is the PDF of the RNMW in (4.3). So, the PDF of the ERMW distribution is a linear combination of the PDFs of the RNMW distribution.

The PDF in (5.2) can also be expressed as

$$f(x) = h_{\mathrm{RNMW}}(x)\, e^{-\alpha\sqrt{x}-\beta\sqrt{x}e^{\lambda x}} \left(1 - e^{-\alpha\sqrt{x}-\beta\sqrt{x}e^{\lambda x}}\right)^{\theta-1}, \tag{5.6}$$

for $x > 0$, where $h_{\mathrm{RNMW}}(x)$ is the HF of the RNMW distribution in (4.4).

Figure 5.1 shows possible shapes of the PDF in (5.2). We can see that monotonically decreasing, unimodal and monotonically decreasing followed by unimodal shapes are possible for the PDF. Monotonically decreasing shapes correspond to small values of θ. Unimodal shapes correspond to large values of θ. The location and the magnitude of the mode become larger as θ becomes larger.

Figure 5.2 shows possible shapes of the HF in (5.3). We can see that monotonically increasing, bathtub, unimodal and modified unimodal shapes are possible for the HF.

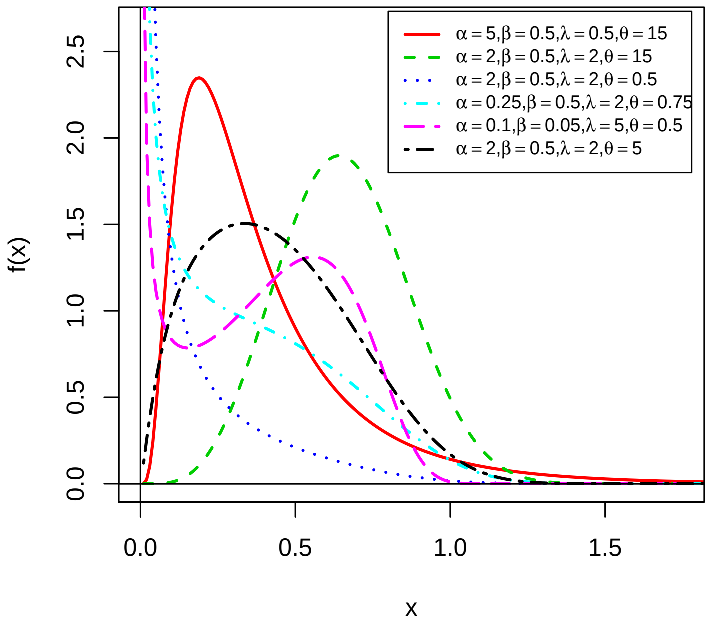


Figure 5.1: PDF of the ERMW distribution for selected α, β, λ and θ.

5.3 Special cases

The ERMW distribution includes as special cases distributions which are special cases of several distributions widely used in survival analysis:

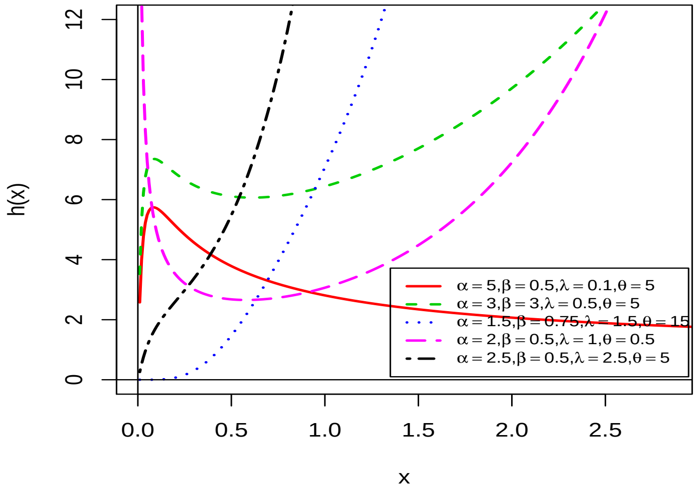
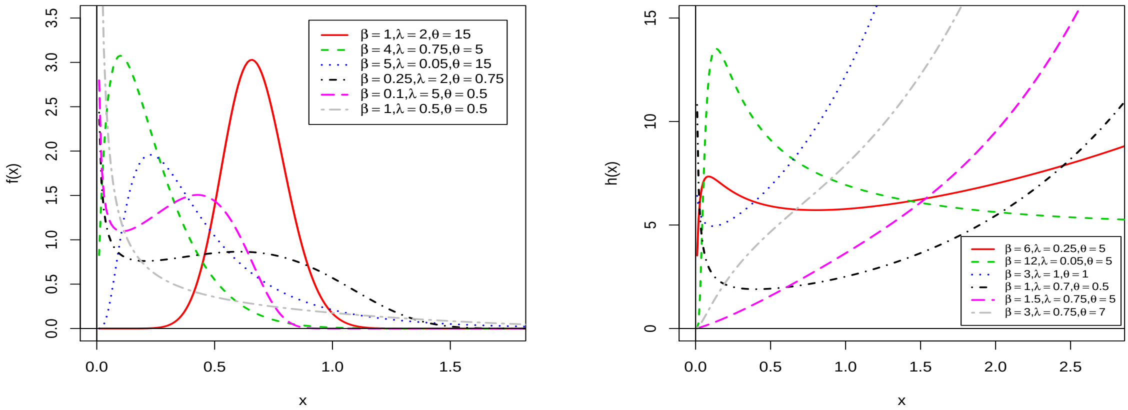


Figure 5.2: The HF of the ERMW distribution.

1. **The reduced generalized modified Weibull (RGMW) distribution**:
   If $\alpha = 0$, the ERMW distribution reduces to the RGMW distribution specified by the CDF
   $$F(x) = \left(1 - e^{-\beta\sqrt{x}e^{\lambda x}}\right)^{\theta},$$
   a special case of the GMW distribution of Carrasco et al. (2008) when its first shape parameter is equal to $\frac{1}{2}$. The RGMW distribution has just three parameters. Its HF can be monotonically increasing, bathtub shaped, unimodal or unimodal following by increasing. Not many distributions have this property, especially with three parameters. Figure 5.3 shows possible shapes of the PDF and the HF of the RGMW distribution.

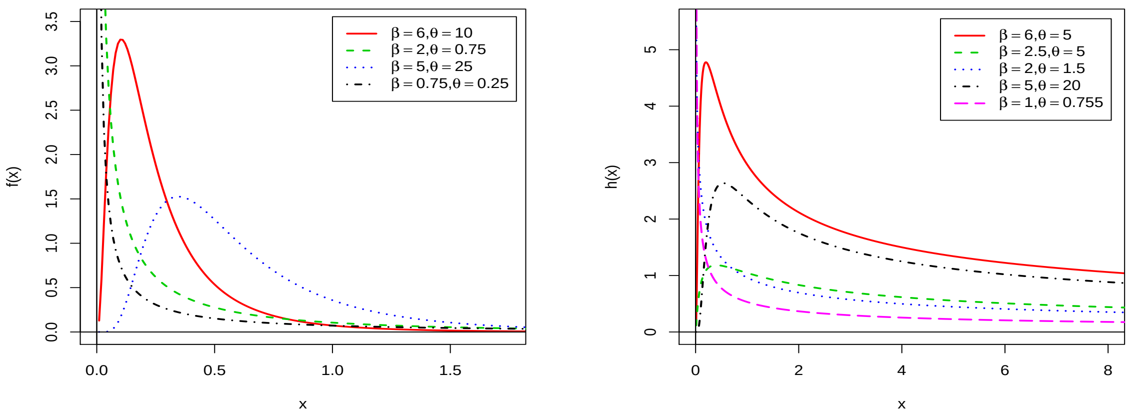
2. **The reduced exponentiated Weibull (REW) distribution**:
   When $\alpha = \lambda = 0$ or $\beta = \lambda = 0$, the ERMW distribution reduces to the REW distribution specified by the CDF
   $$F(x) = \left(1 - e^{-\alpha\sqrt{x}}\right)^{\theta},$$
   a special case of the EW distribution due to Mudholkar and Srivastava (1993). The PDFs and the HFs of the REW distribution are shown in Figure 5.4.

3. **The reduced Lai et al.'s modified Weibull (RMW) distribution**:
   For $\alpha = 0$ and $\theta = 1$, the ERMW distribution reduces to the RLMW distribution



(a) (b)

Figure 5.3: (a) The PDF of the RGMW distribution, (b) The HF of the RGMW distribution.



(a) (b)

Figure 5.4: (a) The PDF of the REW distribution, (b) The HF of the REW distribution.

specified by the CDF

$$F(x) = 1 - e^{-\beta\sqrt{x}e^{\lambda x}},$$

a special case of the MW distribution due to Lai et al. (2003).

4. **The reduced Weibull (RW) distribution**:

   By setting $\beta = \lambda = 0$ and $\theta = 1$, we can reduce the ERMW distribution to the RW distribution specified by the CDF

$$F(x) = 1 - e^{-\alpha\sqrt{x}}.$$

   We can also get a RW distribution by setting $\lambda = 0$ and $\theta = 1$.

5.4 Shape

Here, we discuss shapes of the PDF in (5.2) and the HF in (5.3).

The derivative of $\log f(x)$ with respect to x is

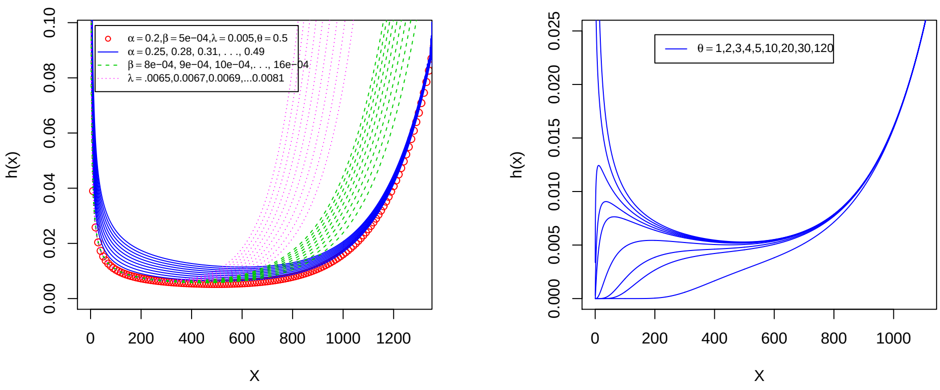
$$\begin{aligned}\frac{d\log f(x)}{dx} &= \frac{\beta\lambda(3+2\lambda x)e^{\lambda x}}{\alpha+\beta(1+2\lambda x)e^{\lambda x}} + \frac{(\theta-1)\left[\alpha+\beta(1+2\lambda x)e^{\lambda x}\right]}{2\sqrt{x}\left(e^{\alpha\sqrt{x}+\beta\sqrt{x}e^{\lambda x}}-1\right)} \\ &\quad -\frac{1}{2x} - \frac{\alpha}{2\sqrt{x}} - \frac{\beta(1+2\lambda x)e^{\lambda x}}{2\sqrt{x}}.\end{aligned}$$

So, the modes of $f(x)$ at say $x = x_0$ are the roots of $\frac{d\log f(x)}{dx} = 0$. A mode corresponds to a local maximum (see Figure 5.1) if $\frac{d\log f(x)}{dx} > 0$ for all $x < x_0$ and $\frac{d\log f(x)}{dx} < 0$ for all $x > x_0$. A mode corresponds to a local minimum (see Figure 5.1) if $\frac{d\log f(x)}{dx} < 0$ for all $x < x_0$ and $\frac{d\log f(x)}{dx} > 0$ for all $x > x_0$.

The derivative of $\log h(x)$ with respect to x is

$$\frac{d\log h(x)}{dx} = \frac{d\log f(x)}{dx} + \frac{\theta\left[\alpha+\beta(1+2\lambda x)e^{\lambda x}\right]\left(1-e^{-\alpha\sqrt{x}-\beta\sqrt{x}e^{\lambda x}}\right)^{\theta-1}}{2\sqrt{x}e^{\alpha\sqrt{x}+\beta\sqrt{x}e^{\lambda x}}\left[1-\left(1-e^{-\alpha\sqrt{x}-\beta\sqrt{x}e^{\lambda x}}\right)^{\theta}\right]}.$$

So, the modes of $h(x)$ at say $x = x_0$ are the roots of $\frac{d\log h(x)}{dx} = 0$. The mode corresponds to a local maximum (see, for example, the top left plot in Figure 5.6) if $\frac{d\log h(x)}{dx} > 0$ for all $x < x_0$ and $\frac{d\log h(x)}{dx} < 0$ for all $x > x_0$. The mode corresponds to a local minimum (see, for example, Figure 5.5) if $\frac{d\log h(x)}{dx} < 0$ for all $x < x_0$ and $\frac{d\log h(x)}{dx} > 0$ for all $x > x_0$. The mode corresponds to a point of inflexion (see, for example, the bottom



(a) (b)

Figure 5.5: The HF of the ERMW distribution as α changes.

right plot in Figure 5.6) if either $\frac{d \log h(x)}{dx} > 0$ for all $x \neq x_0$ or $\frac{d \log h(x)}{dx} < 0$ for all $x \neq x_0$.

The property of a long useful lifetime period, exhibited by the bathtub shaped HF seen in the NMW and RNMW distributions, which are presented in Chapter 3 and Chapter 4, also holds for the ERMW distribution, see Figure 5.5 (a).

Figure 5.5 (a) shows how the bathtub shaped HF of the ERMW distribution is affected by the parameters β, λ and α. The curve of open red circles is the bathtub shaped HF when $\alpha = 0.2, \beta = 5 \times 10^{-4}, \lambda = 0.005, \theta = 0.5$. The blue curves in this figure are the HF at the same values of α, β and λ, while α increases from 0.25 to 0.49 in steps of 0.03. Increasing the value of this parameter has the effect of scaling the bathtub shape slightly up. The green dash curves and the pink dot curves are, respectively, the HF when $\beta = 8 \times 10^{-4}, 9 \times 10^{-4}, \ldots, 16 \times 10^{-4}$ and $\lambda = 0.0065, 0.0067, \ldots, 0.0081$. Increasing the value of these parameters has the effect of shortening the useful life period. That is, increasing the value of β and λ reduces the survival time of the system.

Figure 5.5 (b) shows the role of the shape parameter θ and how it turns the HF from being bathtub shaped to unimodal and to increasing. The HF is bathtub shaped when $\theta \leq 2$. It is unimodal when $\theta > 2$. The HF increases for large values of θ.

Figures 5.6 (a), (b), (c), (d) show how the unimodal HFs of the ERMW distribution are affected by the parameters β, λ, θ and α. (Here, by unimodal HF we mean HF taking a unimodal shape followed by a monotonic increasing shape.) The curve of open

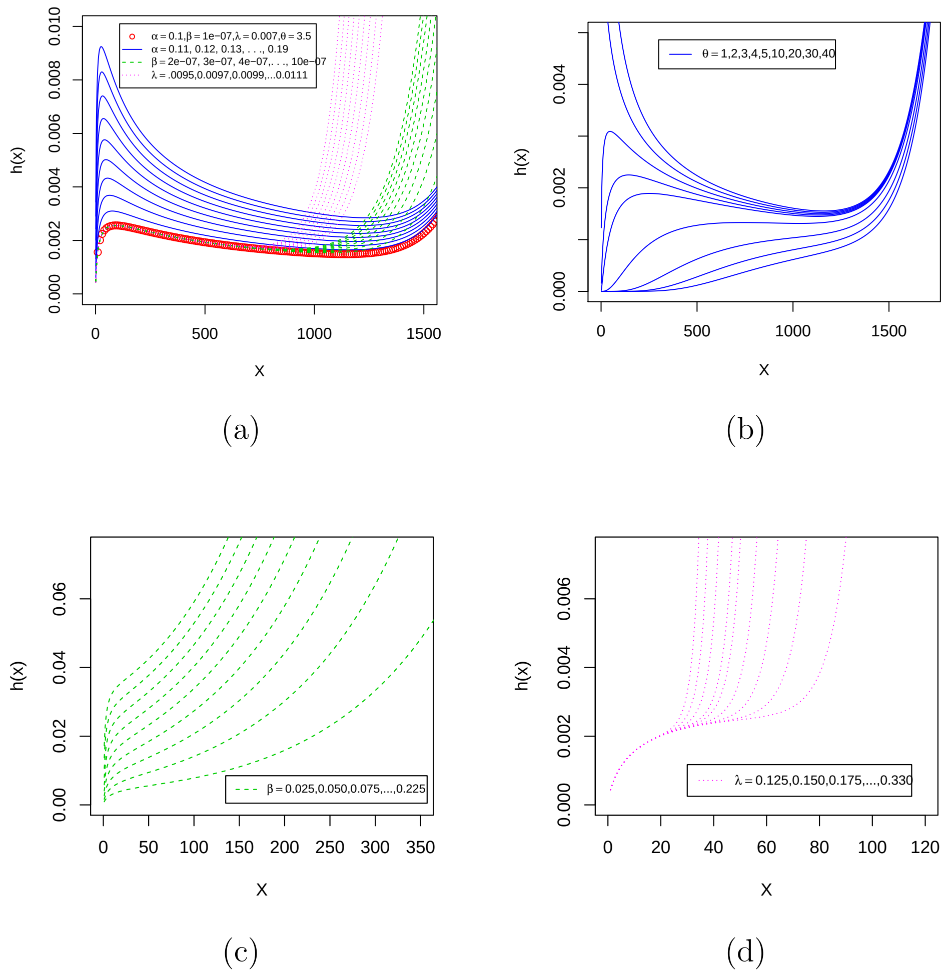


Figure 5.6: The HF of the ERMW distribution as θ changes.

red circles in Figure 5.6 (a) is a unimodal HF of the ERMW distribution. Increasing the value of α has the effect of scaling the unimodal shape. A slight increase of β and λ has the effect giving an earlier start for the monotonic increasing part of the HF. That is, the survival time of the system decreases as β and λ increase slightly. A sharp increase of β and λ has the effect of turning the unimodal HF into an increasing HF, see Figures 5.6 (c) and 5.6 (d). Increasing values of θ have the effect of changing the HF from being bathtub shaped to unimodal to increasing, see Figure 5.6 (b).

5.5 Moments

Let X denote an ERMW random variable. Using (5.5), the rth raw moment of X say $\mu_r^{'}$ can be expressed as

$$\mu_r^{'} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\theta+1)}{(j+1)!\Gamma(\theta-j)} \mu_{r,\mathrm{RNMW}(\alpha_{j+1},\beta_{j+1},\lambda)}^{'} \tag{5.7}$$

for $r \geq 1$, where $\mu_{r,\mathrm{RNMW}(\alpha,\beta,\lambda)}^{'}$ denotes the rth raw moment of a random variable having the RNMW distribution with parameters (α, β, λ). Using (5.7) and expressions for $\mu_{r,\mathrm{RNMW}(\alpha,\beta,\lambda)}^{'}$ given in (4.8), one can obtain expressions for the mean, variance, skewness and kurtosis of X.

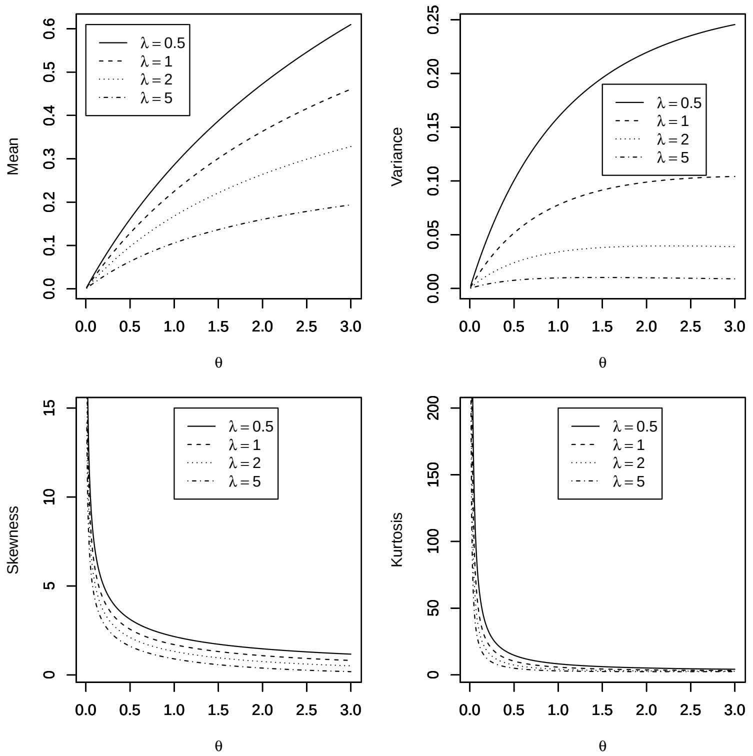


Figure 5.7: Mean, variance, skewness and kurtosis of the ERMW distribution versus θ for selected λ and $\alpha = \beta = 1$.

Figures 5.7 and 5.8 show how the mean, variance, skewness and the kurtosis of the ERMW distribution vary with respect to λ and θ for $\alpha = \beta = 1$. We can observe the following from the figures: mean is a monotonic increasing function of θ and a monotonic decreasing function of λ; variance is a monotonic increasing function of θ and a monotonic decreasing function of λ; skewness is a monotonic decreasing function of θ and a monotonic decreasing function of λ; kurtosis is a monotonic decreasing function of θ and a monotonic decreasing function of λ.

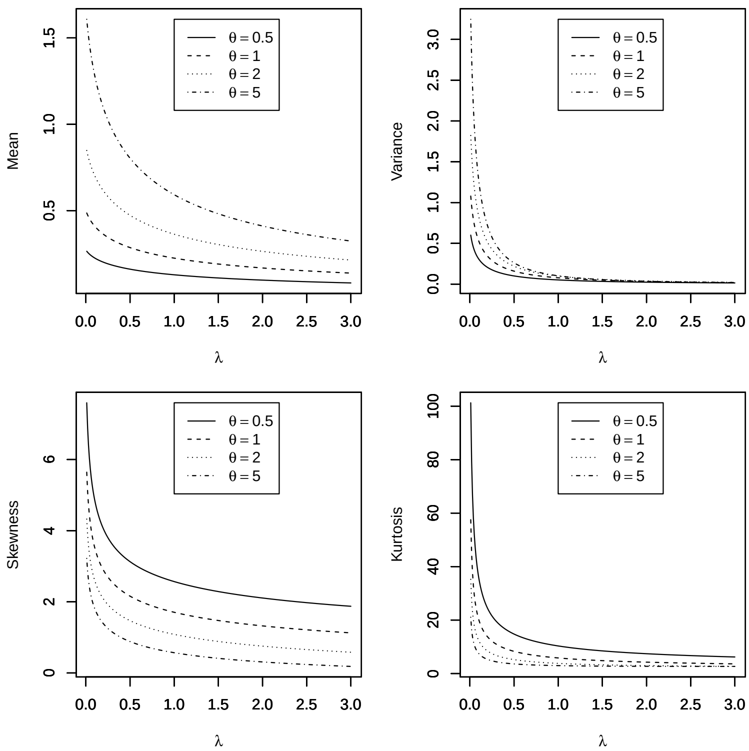


Figure 5.8: Mean, variance, skewness and kurtosis of the ERMW distribution versus λ for selected θ and $\alpha = \beta = 1$.

5.6 Order statistics

This section discusses the order statistic, which have many applications in reliability and lifetime analysis. We derive the probability density functions of the order statistics $X_{(r)}$, and its moments, for the ERMW distribution.

If $X_1, X_2, \cdots, X_n$ are a random sample from the ERMW with CDF (5.1) and pdf (5.2), and $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the order statistics from this sample, then, from Arnold *el at* (2008), using (3.12) the pdf of the r-th statistic $X_{(r)}$ is given by

$$\begin{aligned} f_{r:n}(x) &= \frac{1}{B(r, n-r+1)} \sum_{\ell=0}^{n-r} \binom{n-r}{\ell} (-1)^{\ell} f(x) F(x)^{\ell+r-1}, \\ &= n \binom{n-1}{r-1} \sum_{\ell=0}^{n-r} \binom{n-r}{\ell} \frac{(-1)^{\ell}}{(\ell+r)} f(x;\ \alpha, \beta, \lambda, \theta_{\ell+r}), \end{aligned}$$

where $B(.,.)$ is the beta function and $f(x;\ \alpha, \beta, \lambda, \theta_{\ell+r})$ is the pdf of the ERMW with parameters α, β, λ and $(\ell + r)\theta$.

Using (5.7), the kth non-central moment of the rth order statistic is then

$$\mu_k^{'(r:n)} = n\binom{n-1}{r-1}\sum_{\ell=0}^{n-r}\sum_{j=0}^{\infty}\binom{n-r}{\ell}\frac{(-1)^{\ell+j}\Gamma(\theta_{\ell+r})\mu'_{k,\mathrm{RNMW}(\alpha_{j+1},\beta_{j+1},\lambda)}}{(\ell+r)(j+1)!\Gamma(\theta_{\ell+r}-j)}, \tag{5.8}$$

where $\mu'_{k,\mathrm{RNMW}(\alpha,\beta,\lambda)}$ denotes the kth raw moment of a random variable having the RNMW distribution given in (4.8) and $\theta_{\ell+r} = (\ell+r)\theta$.

5.7 Estimation

Suppose $x_1, x_2, \ldots, x_n$ is a random sample from the ERMW distribution with unknown parameters $(\alpha, \beta, \lambda, \theta)$. Section 5.7.1 estimates these parameters by the method of maximum likelihood. Section 5.7.2 assesses the finite sample performance of the MLEs with respect to sample size n.

5.7.1 Maximum likelihood estimation

Given the data $x_1, \ldots, x_n$, the log-likelihood function of $(\alpha, \beta, \lambda, \theta)$ is

$$\begin{aligned}\mathcal{L}(\alpha,\beta,\lambda,\theta) &= n\log(\theta) + \sum_{i=1}^{n}\left[\log\left(h_{\mathrm{RNMW}}(x_i)\right) - \alpha\sqrt{x_i} - \beta\sqrt{x_i}e^{\lambda x_i}\right] \\ &\quad +(\theta-1)\sum_{i=1}^{n}\log\left[1-e^{-\alpha\sqrt{x_i}-\beta\sqrt{x_i}e^{\lambda x_i}}\right].\end{aligned}$$

By setting the first partial derivatives of $\mathcal{L}$ with respect to α, β, λ and θ to zero, we obtain the normal equations

$$\sum_{i=1}^{n}\frac{1}{h_{\mathrm{RNMW}}(x_i;\alpha,\beta,\lambda)\left(2\sqrt{x_i}\right)} - \sum_{i=1}^{n}\sqrt{x_i} + \sum_{i=1}^{n}\frac{(\theta-1)\sqrt{x_i}}{e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}-1} = 0, \tag{5.9}$$

$$\sum_{i=1}^{n}\frac{(0.5+\lambda x_i)e^{\lambda x_i}}{h_{\mathrm{RNMW}}(x_i;\alpha,\beta,\lambda)\sqrt{x_i}} - \sum_{i=1}^{n}e^{\lambda x_i}\sqrt{x_i} + \sum_{i=1}^{n}\frac{(\theta-1)\sqrt{x_i}e^{\lambda x_i}}{e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}-1} = 0, \tag{5.10}$$

$$\beta\sum_{i=1}^{n}\frac{\sqrt{x_i}\left(\frac{3}{2}+\lambda x_i\right)e^{\lambda x_i}}{h_{\mathrm{RNMW}}(x_i;\alpha,\beta,\lambda)} - \beta\sum_{i=1}^{n}e^{\lambda x_i}\sqrt{x_i^3} + \sum_{i=1}^{n}\frac{(\theta-1)\sqrt{x_i^3}e^{\lambda x_i}}{e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}-1} = 0, \tag{5.11}$$

$$\frac{n}{\theta} + \sum_{i=1}^{n}\log\left[1-e^{-\alpha\sqrt{x_i}-\beta\sqrt{x_i}e^{\lambda x_i}}\right] = 0. \tag{5.12}$$

The MLEs can be obtained by solving the nonlinear equations, (5.9)-(5.12), numerically for α, β, λ and θ.

To obtain confidence intervals for α, β, λ and θ, we use the observed information matrix since the expected information matrix is very complicated and the MLEs of the unknown parameters cannot be obtained analytically. The observed information matrix $J(\underline{\vartheta})$ is

$$J(\underline{\vartheta}) = - \begin{bmatrix} \mathcal{L}_{\alpha\alpha} & \mathcal{L}_{\alpha\beta} & \mathcal{L}_{\alpha\lambda} & \mathcal{L}_{\alpha\theta} \\ & \mathcal{L}_{\beta\beta} & \mathcal{L}_{\beta\lambda} & \mathcal{L}_{\beta\theta} \\ & & \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda\theta} \\ & & & \mathcal{L}_{\theta\theta} \end{bmatrix}.$$

where $\underline{\vartheta} = (\alpha, \beta, \lambda, \theta)$ and the explicit expressions for the elements of this matrix are given in the appendix C.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}\left(\widehat{\vartheta} - \vartheta\right)$ is $N_4(0, I(\underline{\vartheta})^{-1})$, where $I(\underline{\vartheta})$ is the expected information matrix. This asymptotic behavior is valid if $I(\underline{\vartheta})$ is replaced by $J(\underline{\widehat{\vartheta}})$, i.e., the observed information matrix evaluated at $\widehat{\vartheta}$. The asymptotic multivariate normal $N_4(0, J(\underline{\widehat{\vartheta}})^{-1})$ distribution can be used to construct approximate confidence intervals and confidence regions for the individual parameters.

5.7.2 Simulation study

Here, we assess the performance of the MLEs with respect to sample size n. The assessment is based on a simulation study:

1. First, ten thousand samples of size n are generated from (5.1). Using the inversion method samples were generated, that was done for the ERMW distribution by solving

$$\left(1 - e^{-\alpha\sqrt{X} - \beta\sqrt{X}e^{\lambda X}}\right)^{\theta} = U,$$

   where $U \sim U(0, 1)$ is a uniform variate on the unit interval.

2. Obtain the MLEs for the ten thousand generated samples, say $\left(\widehat{\alpha}_i, \widehat{\beta}_i, \widehat{\lambda}_i, \widehat{\theta}_i\right)$ for $i = 1, 2, \ldots, 10000$.

3. For the generated samples, compute the standard errors of the MLEs, say $\left(s_{\widehat{\alpha}_i}, s_{\widehat{\beta}_i}, s_{\widehat{\lambda}_i}, s_{\widehat{\theta}_i}\right)$ for $i = 1, 2, \ldots, 10000$. The standard errors computed by using the observed information matrices.

4. The biases and mean squared errors of the MLEs can be computed as the following,

$$\text{bias}_h(n) = \frac{1}{10000} \sum_{i=1}^{10000} \left(\widehat{h}_i - h\right),$$
$$\text{MSE}_h(n) = \frac{1}{10000} \sum_{i=1}^{10000} \left(\widehat{h}_i - h\right)^2,$$

for $h = \alpha, \beta, \lambda, \theta$.

The above steps were repeated $n = 10, 11, \ldots, 100$ with $\alpha = 1$, $\beta = 1$, $\lambda = 1$ and $\theta = 1$, so computing $\text{bias}_h(n)$ and $\text{MSE}_h(n)$ for $h = \alpha, \beta, \lambda, \theta$ and $n = 10, 11, \ldots, 100$.

Figure 5.9 shows the biases of the MLEs of $(\alpha, \beta, \lambda, \theta)$ versus $n = 10, 11, \ldots, 100$, and it can be seen that the biases for the four parameters are small and vary with respect to the samples sizes. The mean squared errors of the MLEs of each parameters are shown in Figure 5.10, which are vary with respect to the samples sizes. The broken line in Figure 5.9 indicates to the distance between the biases and the zero. The broken line in Figure 5.10 indicates to the distance between the mean squared errors and the zero.

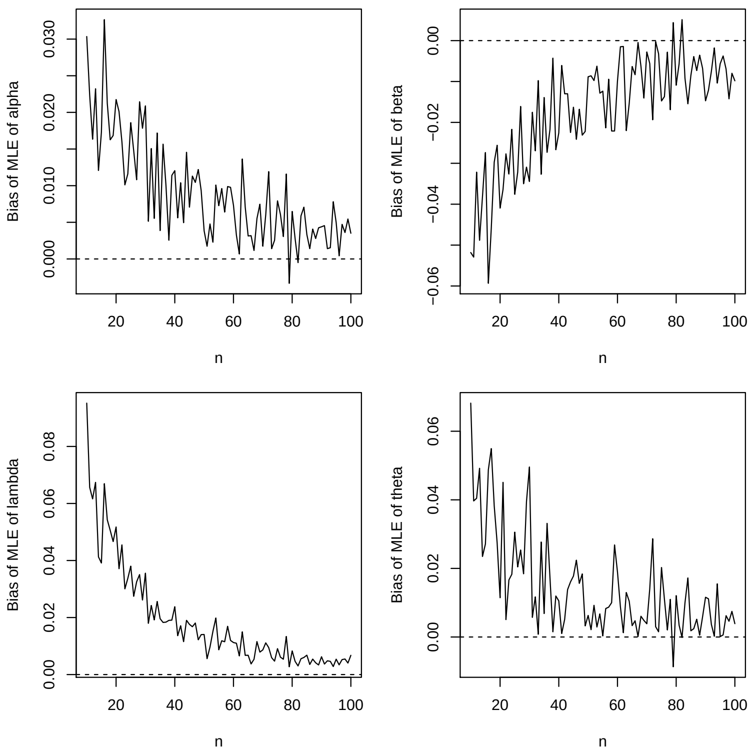


Figure 5.9: Biases of the MLEs of $(\alpha, \beta, \lambda, \theta) = (1, 1, 1, 1)$ versus $n = 10, 11, \ldots, 100$.

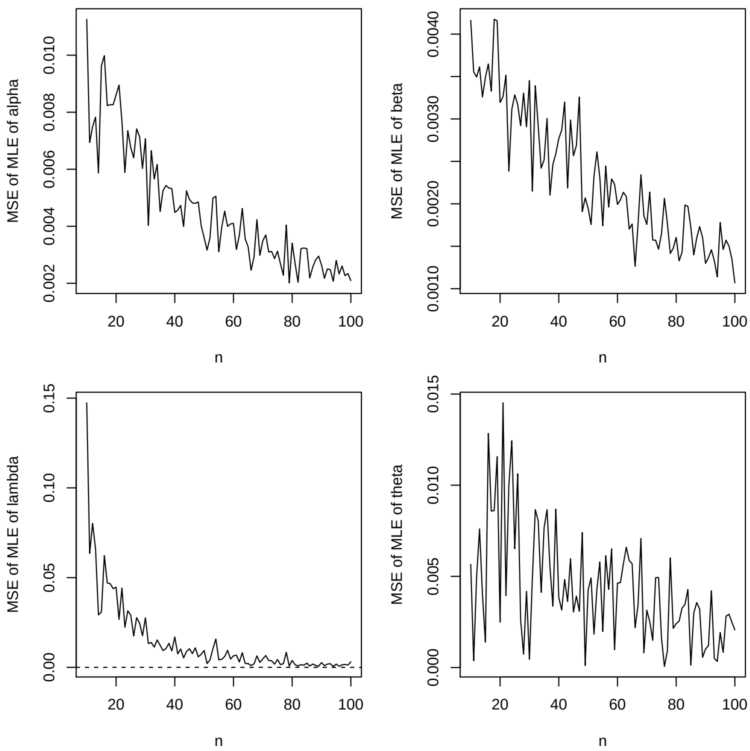


Figure 5.10: Mean squared errors of the MLEs of $(\alpha, \beta, \lambda, \theta) = (1, 1, 1, 1)$ versus $n = 10, 11, \ldots, 100$.

From these figures the following observations can be made. The biases of the MLEs of the parameters α, λ and θ are generally positive; the biases of the MLEs of the parameter β are generally negative; among the four biases, the biases of the MLEs of the parameter α are the smallest; the biases of the MLEs of the parameter λ are the largest; the biases of the MLEs of the parameters α, λ and θ decrease to zero as $n \to \infty$; the biases of the MLEs of the parameter β increase to zero as $n \to \infty$; the mean squared errors for the parameter β are the smallest; the mean squared errors for the parameter λ are the largest; the mean squared errors for all four parameters decrease to zero as $n \to \infty$; The above observations are observed by choosing $(\alpha, \beta, \lambda, \theta) = (1, 1, 1, 1)$. A similar results were found for other choices.

5.8 Applications

This section uses two well-known data sets to show how the proposed distribution can be applied in practice to provide good fits. The HF of the first data set is modified unimodal shape. That of the second data set is bathtub shaped. The proposed distribution will be compared to the four-parameter GMW distribution, the three-parameter RGMW distribution, the two-parameter REW distribution and the three-parameter EW distribution.

The sample size for the first data set is seventy two. The sample size for the second data set is forty four.

Hence, the biases for $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\lambda}$ and $\widehat{\theta}$ can be expected to be less than 0.01, 0.02, 0.02 and 0.02, respectively, for both data sets. The mean squared errors for $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\lambda}$ and $\widehat{\theta}$ can be expected to be less than 0.005, 0.003, 0.02 and 0.005, respectively, for both data sets. Hence, the point estimates given in Section 5.8 can be considered accurate enough for both data sets.

The likelihood ratio test will be used to compare the GMW distribution against the RGMW distribution by testing the hypotheses $H_0 : \gamma = \frac{1}{2}$ versus $H_1 : H_0$ is false. Under the null hypothesis, the likelihood ratio test statistic $\omega = 2\left(\mathcal{L}_{\rm GMW} - \mathcal{L}_{\rm RGMW}\right)$ has an asymptotic χ^2 distribution with one degree of freedom. The EW distribution will be compared against the REW distribution by testing the hypotheses $H_0 : \gamma = \frac{1}{2}$ versus $H_1 : H_0$ is false. Under the null hypothesis, the likelihood ration test statistic $\omega = 2\left(\mathcal{L}_{\rm EW} - \mathcal{L}_{\rm REW}\right)$ has an asymptotic χ^2 distribution with one degree of freedom.

5.8.1 Infected pigs data

The data in Table 5.1 are survival times (in days) of seventy two pigs infected by virulent tubercle bacilli (Greenwich, 1992). The TTT plot in Figure 5.11 shows a concave shape and then a convex shape followed by a concave shape. This corresponds to the HF being modified unimodal shape. Hence, the ERMW distribution is suitable for this data, see Section 5.2.

Table 5.1: Infected pigs data (in days).

43	45	53	56	56	57	58	66	67	73	74	79	80	80	81	81
81	82	83	83	84	88	89	91	91	92	92	97	99	99	100	100
101	102	102	102	103	104	107	108	109	113	114	118	121	123	126	128
137	138	139	144	145	147	156	162	174	178	179	184	191	198	211	214
243	249	329	380	403	511	522	598								

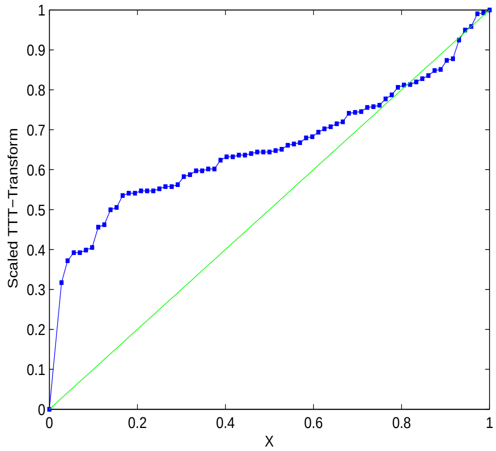


Figure 5.11: TTT-transform plot for the infected pigs data.

Table 5.2 shows the MLEs of the parameters, their standard errors, AIC values, BIC values, AICc values and CAIC values for the fitted ERMW, GMW, RGMW, REW and EW distributions.

The ERMW distribution has the smallest negative log-likelihood of 399.898. The negative log-likelihood of the GMW distribution is 400.350. The REW distribution has the smallest values for the AIC, the BIC the AICc and the CAIC. Given these observations, one could be tempted to say that the REW distribution provides the best fit.

Table 5.3 shows the K-S test statistics for the fitted distributions. The values of K-S statistics are nearly the same for all fitted distributions. All of them are about 0.11.

Figure 5.12 (a) shows the histogram of the data and the estimated PDFs. Figure 5.12 (b) shows empirical survival function of the data and the estimated ones. Figure 5.12 (d) shows the nonparametric HF of the data and the estimated HFs.

Figures 5.12a and 5.12b show that all of the distributions provide good fits. Figure 5.12d shows that all of the distributions provide good fits to the first and middle parts of the nonparametric HF. But the GMW, RGMW, REW and the EW distributions do not appear to capture the last part of the nonparametric HF well. Only the ERMW distribution appears to provide a good fit to the last part of the nonparametric HF. Since the differences in the likelihood and the differences in the AIC, BIC, AICc and CAIC measures are small among the fitted distributions, we conclude that the ERMW

Table 5.2: MLEs of parameters, standard errors, AIC, BIC, AICc and CAIC for the distributions fitted to the infected pigs data set.

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\gamma}$	$\widehat{\lambda}$	$\widehat{\theta}$	AIC	BIC	AICc	CAIC
ERMW	0.412	6.396×10^{-8}	–	0.022	57.562	807.8	816.9	808.4	820.9
	(0.042)	(7.109×10^{-7})	–	(0.019)	(22.924)				
RGMW	0	0.422	–	1.108×10^{-8}	63.403	807.2	814.0	807.5	817.0
	–	(0.108)	–	4.885×10^{-4}	(50.557)				
REW	0.423	0	–	0	36.517	805.2	809.7	805.3	811.7
	0.042	–	–	–	(25.683)				
GMW	–	0.496	0.470	4.639×10^{-9}	69.877	808.7	817.8	809.3	821.8
	–	(0.247)	(0.008)	4.573×10^{-4}	(50.012)				
EW	–	0.471	0.478	0	65.135	806.9	813.7	807.2	816.7
	–	(0.254)	0.081	–	(49.625)				

Table 5.3: K-S statistics for the distributions fitted to the infected pigs data set.

Model	K-S
ERMW	0.112
RGMW	0.110
REW	0.111
GMW	0.114
EW	0.110

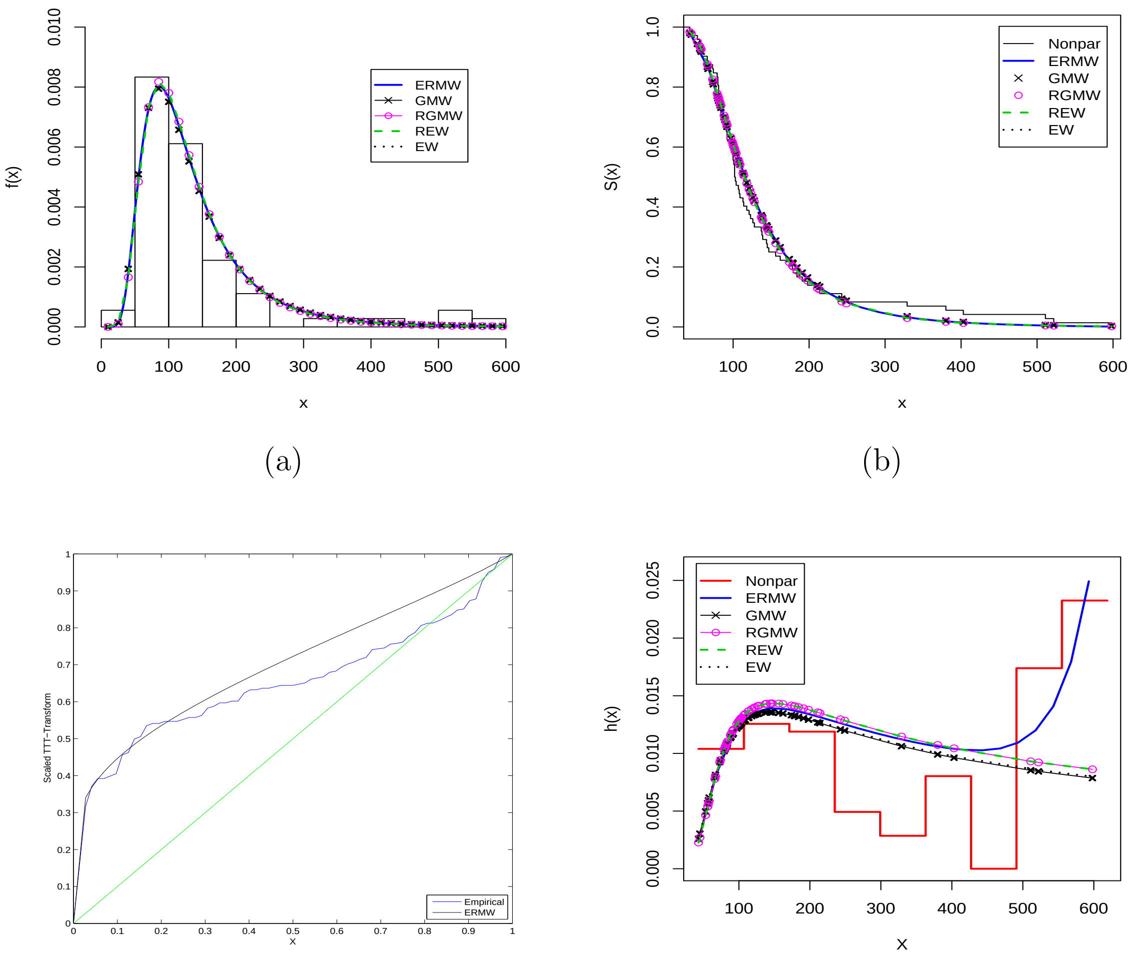
distribution gives the best fit. The ERMW distribution is the only one that captures the nonparametric HF well in all three parts.

The variance-covariance matrix for the fitted ERMW distribution may be approximated as J^{-1},

$$J^{-1} = \begin{bmatrix} 1.739 \times 10^{-3} & -4.108 \times 10^{-10} & 4.366 \times 10^{-6} & 0.913 \\ -4.108 \times 10^{-10} & 5.054 \times 10^{-13} & -1.351 \times 10^{-8} & -2.119 \times 10^{-7} \\ 4.366 \times 10^{-6} & -1.351 \times 10^{-8} & 3.655 \times 10^{-4} & 2.195 \times 10^{-3} \\ 0.913 & -2.119 \times 10^{-7} & 2.195 \times 10^{-3} & 525.529 \end{bmatrix}.$$

So, approximate ninety five percent confidence intervals for the parameters α, β, λ and θ are $[0.331, 0.494]$, $[0, 1.457 \times 10^{-6}]$, $[0, 0.06]$ and $[12, 102]$, respectively.

Finally, we give results of likelihood ratio tests. The log-likelihood ratio statistic for testing $H_0 : \gamma = \frac{1}{2}$ in (2.10) versus $H_1 : H_0$ (testing the GMW against the RGMW) is false is $\omega = 0.445$ and the corresponding p-value is 0.505. So, there is no evidence to reject the null hypothesis at any reasonable level of significance. Therefore, the RGMW distribution can be chosen as an alternative to the GMW distribution for modeling the data set.



(a) (b)

(c) (d)

Figure 5.12: For infected pigs data: (a) Histogram and fitted PDFs; (b) Empirical and fitted survival functions; (c) Empirical and fitted TTT-transforms; (d) Nonparametric and fitted HFs.

The log-likelihood ratio statistic for testing $H_0 : \gamma = \frac{1}{2}$ in (1.6) versus $H_1 : H_0$ (testing the EW against the REW) is false is $\omega = 0.294$ and the corresponding p-value is 0.588. So, the null hypothesis could not be rejected. Therefore, the REW distribution can be chosen as an alternative to the EW distribution for modeling the data set.

5.8.2 Kumar data

Kumar et al. (1989) presented data consisting of times between failures (TBF) in days of load-haul-dump machines (LHD) used to pick up rock or waste. The TTT plot in Figure 5.13 shows a convex shape followed by a concave shape. This corresponds to a bathtub shaped HF. Hence, the ERMW, GMW, RGMW and the EW distributions are appropriate for modeling this data. The REW distribution is not appropriate as it does not exhibit a bathtub shaped HF.

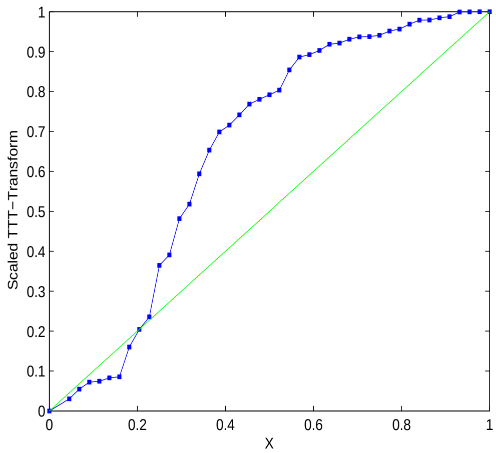


Figure 5.13: TTT-transform plot for Kumar data.

Tables 5.4 and 5.5 show the MLEs of the parameters, their standard errors, AIC values, BIC values, AICc values, CAIC values and K-S test statistics for the fitted ERMW, GMW, RGMW, REW and EW distributions.

The ERMW distribution has the smallest negative log-likelihood of -199.882, the smallest AIC value, the smallest BIC value, the smallest AIC value, the smallest CAIC value and the smallest K-S test statistic. So, the ERMW distribution provides the best fit with respect to these criteria. This observation is confirmed by the diagnostic plots in Figure 5.14.

Table 5.4: MLEs of parameters, standard errors, AIC, BIC AICc and CAIC for the distributions fitted to Kumar data.

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\gamma}$	$\widehat{\lambda}$	$\widehat{\theta}$	AIC	BIC	AICc	CAIC
ERMW	0.099	4.86×10^{-4}	–	0.067	1.476	407.8	414.9	408.8	418.9
	(0.049)	(7.593×10^{-4})	–	(0.017)	(0.034)				
RGMW	0	2.919×10^{-3}	–	0.046	0.521	414.9	420.3	415.5	423.3
	–	(6.530×10^{-3})	–	0.023	(0.270)				
REW	0.292	0	–	0	3.686	450.6	454.1	450.9	456.1
	0.041	(0)	–	–	(0.880)				
GMW	–	4.590×10^{-6}	1.578	0.060	0.266	412.7	419.8	413.7	423.8
	–	(7.693×10^{-6})	0.463	0.019	(0.061)				
EW	–	1.984×10^{-7}	3.460	0	0.310	423.9	429.3	424.5	432.3
	–	(3.158×10^{-7})	0.347	–	(0.061)				

Table 5.5: K-S statistics for the distributions fitted to Kumar data.

Model	K-S
ERMW	0.069
RGMW	0.094
REW	0.234
GMW	0.090
EW	0.175

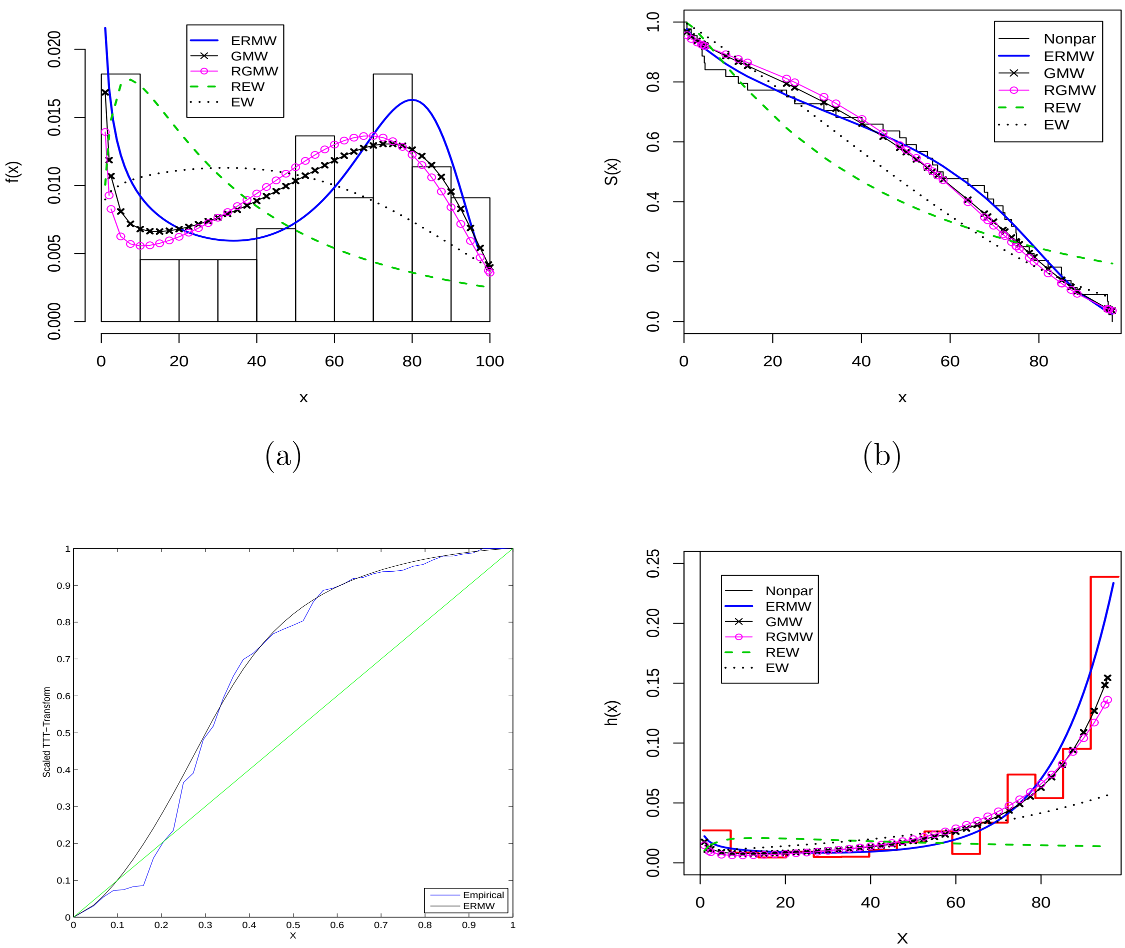
The variance-covariance matrix for the fitted ERMW distribution may be approximated as J^{-1},

$$J^{-1} = \begin{bmatrix} 2.43 \times 10^{-3} & -1.575 \times 10^{-5} & 3.267 \times 10^{-4} & 0.023 \\ -1.575 \times 10^{-5} & 5.766 \times 10^{-7} & -1.264 \times 10^{-5} & -1.043 \times 10^{-4} \\ 3.267 \times 10^{-4} & -1.264 \times 10^{-5} & 2.83 \times 10^{-4} & 2.176 \times 10^{-3} \\ 0.023 & -1.043 \times 10^{-4} & 2.176 \times 10^{-3} & 0.282 \end{bmatrix}.$$

So, approximate ninety five percent confidence intervals for the parameters α, β, λ and θ are $[2.075 \times 10^{-3}, 0.195]$, $[0, 1.974 \times 10^{-3}]$, $[0.034, 0.100]$ and $[0.436, 2.516]$, respectively.

Finally, we give results of likelihood ratio tests. The log-likelihood ratio statistic for testing $H_0 : \gamma = \frac{1}{2}$ in (2.10) versus $H_1 : H_0$ (testing the GMW against the RGMW) is false is $\omega = 4.227$ and the corresponding p-value is 0.04. So, there is evidence that the null hypothesis can be rejected at the five percent significance level. Therefore, the RGMW distribution cannot be chosen as an alternative to the GMW distribution for modeling the data set.

As reported before, the REW distribution does not exhibit bathtub shaped HFs.



(a) (b) (c) (d)

Figure 5.14: For Kumar data: (a) Histogram and fitted PDFs; (b) Empirical and fitted survival functions; (c) Empirical and fitted TTT-transforms; (d) Nonparametric and fitted HFs.

The following result of the likelihood ratio test is an outcome of this inability. The log-likelihood ratio statistic for testing $H_0 : \gamma = \frac{1}{2}$ in (1.6) versus $H_1 : H_0$ (testing the EW against the REW) is false is $\omega = 28.631$ and the corresponding p-value is 8.759×10^{-8}. So, the null hypothesis can be rejected at any level of significance greater than or equal 8.759×10^{-8}.

5.9 Summary

A new four-parameter distribution named the exponentiated reduced modified Weibull distribution has been introduced. It is a generalization of the reduced modified Weibull distribution of Chapter 4. The proposed distribution has the ability to capture monotonically increasing, unimodal and bathtub shaped hazard rates. We derived expressions for its mathematical properties and maximum likelihood estimation. We conducted a simulation study to assess the finite sample performance of the latter. Finally, we show that the proposed distribution gives the best fit for two well-known data sets (when compared to other distributions including those having four parameters).

Chapter 6

A New Discrete Modified Weibull Distribution

6.1 Introduction

In this chapter a three-parameter discrete distribution is introduced. It is based on a recent modification of the continuous Weibull distribution that is presented in Chapter 4. It is one of very few discrete distributions allowing for bathtub shaped hazard rate functions. We study some of its mathematical properties, discuss estimation by the method of maximum likelihood, and describe applications to four real data sets. The new distribution is shown to outperform at least three other models including the ones allowing for bathtub shaped hazard rate functions.

The new discrete distribution is based on a five-parameter modification of the continuous Weibull distribution proposed in Chapter 3. The five-parameter distribution is flexible, has a bathtub shaped hazard rate function and fits data better than many other modifications of the Weibull distribution. Chapter 4 developed a three-parameter version retaining much flexibility of the five-parameter modification.

This chapter proposes a discrete analogue of the three-parameter version. We shall refer to it as the *Discrete reduced modified Weibull* distribution and denoted as DRMW.

We shall show that the discrete reduced modified Weibull distribution allows for bathtub shaped hazard functions. We shall also show it fits data better than all of the known discrete distributions allowing for bathtub shaped hazard functions.

The contents of this chapter are organized as follows. The new distribution is introduced in Section 6.2. Its hazard function and shape are established in Section 6.3. Series expansions for its survival function, probability mass function and moments are given in Sections 6.4 and 6.5. The Order statistics and the extreme value behavior of the new distribution are established in Section 6.6 and 6.7. Estimation of its parameters by the method of maximum likelihood is presented in Section 6.8. Then, applications to four real data sets are described in Section 6.9. Finally, the chapter is summarized in Section 6.10.

6.2 The new distribution

6.2.1 Discrete reduced modified Weibull distribution

By retaining the survival function in (4.2) to integer valued x and setting $q = \exp\{-\alpha\}$, $b = \beta/\alpha$ and $c = \exp\{\lambda\}$, we can write the survival and the cumulative distribution functions of the DRMW distribution as

$$S(x) = q^{\sqrt{x}(1+bc^x)}$$

and

$$F(x) = 1 - q^{\sqrt{x}(1+bc^x)},\ x = 0, 1, \ldots \tag{6.1}$$

where $0 < q < 1$, $b > 0$ and $c \geq 1$.

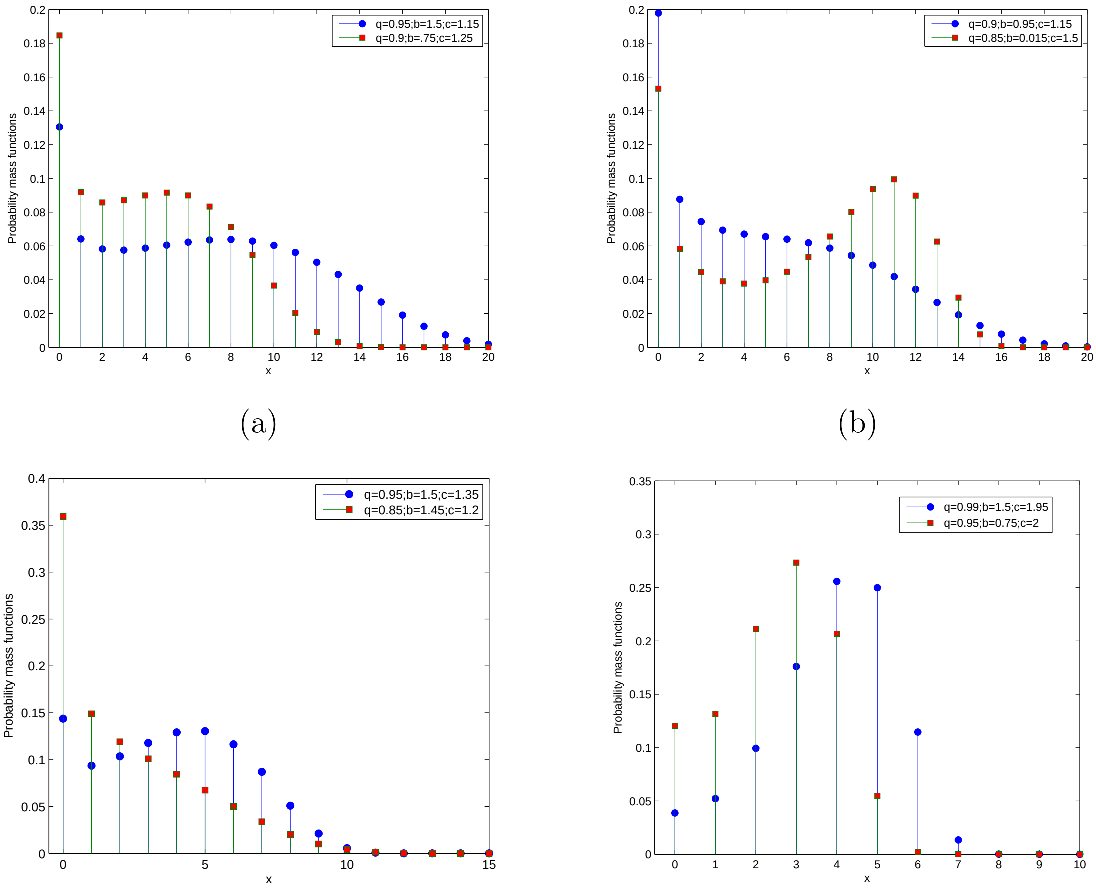
The particular case of the DRMW distribution with $b = 0$ is the discrete Weibull distribution DW due to Nakagawa and Osaki (1975). The DW distribution has an attractive physical feature: if $X_1, \ldots, X_n$ are independent and identical DW random variables then $\min(X_1, \ldots, X_n)$ is also a DW random variable. If $X_1, \ldots, X_n$ are the failure times of n independent and identical components of a series system then $\min(X_1, \ldots, X_n)$ will denote the failure time of the system.

The probability mass function (PMF) of the DRMW distribution is

$$P(x) = q^{\sqrt{x}(1+bc^x)} - q^{\sqrt{x+1}\left(1+bc^{x+1}\right)}$$

for $x = 0, 1, \ldots$.

As shown in Figure 6.1, the DRMW distribution is a flexible distribution. Its PMF can take one of the following shapes: i) a unimodal shape; ii) a monotonic deceasing shape; iii) a decreasing shape followed by an increasing shape followed by a decreasing shape.



(a) (b)

(c) (d)

Figure 6.1: PMFs of the DRMW distribution.

6.3 The hazard rate function

The hazard function of the DRMW distribution is

$$h(x) = 1 - q^{\sqrt{x+1}\left(1+bc^{x+1}\right)-\sqrt{x}(1+bc^{x})}$$

for $x = 0, 1, \ldots$. Theorem 1 discuses the shape of the hazard function of the DRMW distribution.

Theorem 6. *The hazard function of the DRMW distribution is increasing if $bc(c-\sqrt{2}) > \sqrt{2}-1$ and has bathtub shape otherwise.*

Proof: Define

$$\psi(x,c,b) = bc^x g(x,c) + k(x)$$

for $x = 0, 1, \ldots$, where

$$g(x,c) = c^2\sqrt{x+2} - 2c\sqrt{x+1} + \sqrt{x},$$

and

$$k(x) = \sqrt{x+2} - 2\sqrt{x+1} + \sqrt{x}.$$

It is clear that $k(x)$ is an increasing negative valued function. Its minimum value is $\sqrt{2} - 2$ at $x = 0$. When x is large $k(x) \longrightarrow 0$. Now, consider $g(x,c) = c^2\sqrt{x+2} - 2c\sqrt{x+1} + \sqrt{x}$ and define $\Delta_x(c)$ as

$$\Delta_x(c) = g(x,c) - g(x-1,c)$$

for $x = 1, 2, \ldots$. Then

$$\Delta_x(c) = c^2\delta_2(x) - 2c\delta_1(x) + \delta_0(x), \tag{6.2}$$

where

$$\begin{aligned} \delta_0(x) &= \sqrt{x} - \sqrt{x-1}, \\ \delta_1(x) &= \sqrt{x+1} - \sqrt{x}, \\ \delta_2(x) &= \sqrt{x+2} - \sqrt{x+1}. \end{aligned}$$

By differentiating $\Delta_x(c)$ with respect to c,

$$\frac{\partial}{\partial c}\Delta_x(c) = 2c\delta_2(x) - 2\delta_1(x). \tag{6.3}$$

Setting (6.3) to zero and solving for, we obtain

$$c_0 = \frac{\delta_1(x)}{\delta_2(x)}. \tag{6.4}$$

By substituting (6.4) into (6.2),

$$\begin{aligned} \Delta_x(c_0) &= \delta_0(x) - \frac{\delta_1^2(x)}{\delta_2(x)} \\ &= \frac{\delta_1^2(x)}{\delta_2(x)}\left(\frac{\frac{\delta_0(x)}{\delta_1(x)}}{\frac{\delta_1(x)}{\delta_2(x)}} - 1\right). \end{aligned}$$

Since $\frac{\delta_0(x)}{\delta_1(x)} > \frac{\delta_1(x)}{\delta_2(x)}$, we have $\Delta_x(c_0) > 0$. The value c_0 is the minimum of $\Delta_x(c)$, so $\Delta_x(c) > 0$ for $x = 1, 2, \ldots$.

So, $g(x, c)$ is an increasing and positive function for $x = 1, 2, \ldots$ and $g(0, c) > 0$ if $c > \sqrt{2}$. The function $\psi(x, c, b) > 0$ for all x if $bc\left(c - \sqrt{2}\right) > \sqrt{2} - 1$ (in other words, if $bc^t g(0, c) + k(0) > 0$). If $bc\left(c - \sqrt{2}\right) < \sqrt{2} - 1$ there an exist x_0 such that $\psi(x, c, b) < 0$ for $x = 0, 1, 2, \ldots, x_0$ and $\psi(x, c, b) > 0$ for $x = x_0 + 1, x_0 + 2, \ldots$. Now, define $v(x)$ as

$$v(x) = 1 - h(x),$$

then

$$\frac{v(x+1)}{v(x)} = q^{\psi(x,c,b)}.$$

The proof is complete.

Figures 6.2 (a)-6.2 (c) show possible bathtub shaped hazard rates while Figure 6.2d shows possible increasing hazard rates. Figure 6.3 shows that the DRMW distribution has a long useful life period which is desirable in lifetime data analysis.

6.4 A series expansion

Here, we derive a series expansion for the survival function, and hence for the probability mass function.

Using the series expansion for the exponential, we can write

$$\begin{aligned}
S(x) &= q^{\sqrt{x}(1+bc^x)} \\
&= q^{\sqrt{x}(1+b\exp(x\ln c))} \\
&= q^{\sqrt{x}\left(1+b\sum_{i=0}^{\infty}\frac{x^i\ln^i c}{i!}\right)} \\
&= \exp\left(\ln(q)\sqrt{x}\left(1+b\sum_{i=0}^{\infty}\frac{x^i\ln^i c}{i!}\right)\right) \\
&= \sum_{k=0}^{\infty}\frac{\ln^k(q)x^{\frac{k}{2}}}{k!}\left(1+b\sum_{i=0}^{\infty}\frac{x^i\ln^i c}{i!}\right)^k \\
&= \sum_{k=0}^{\infty}\frac{\ln^k(q)x^{\frac{k}{2}}}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}b^\ell\left(\sum_{i=0}^{\infty}\frac{x^i\ln^i c}{i!}\right)^\ell. \qquad (6.5)
\end{aligned}$$

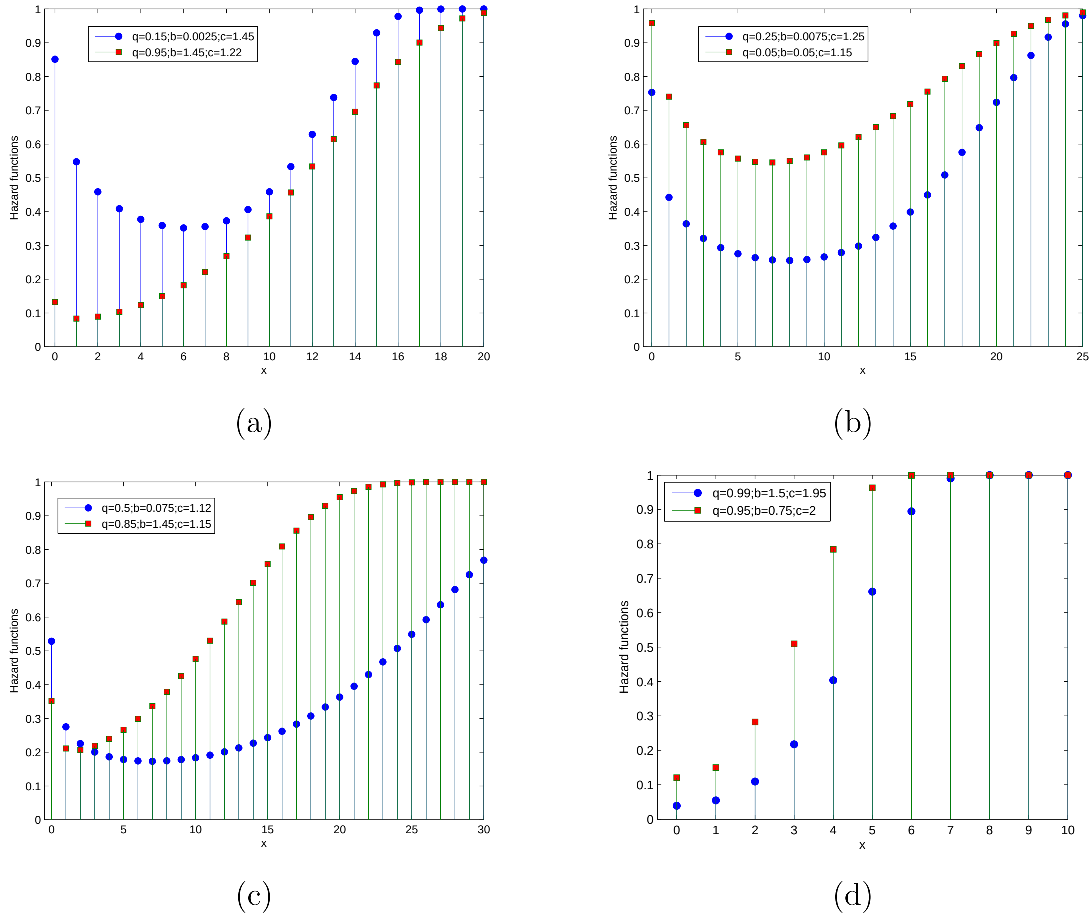


Figure 6.2: Hazard rate functions of the DRMW distribution.

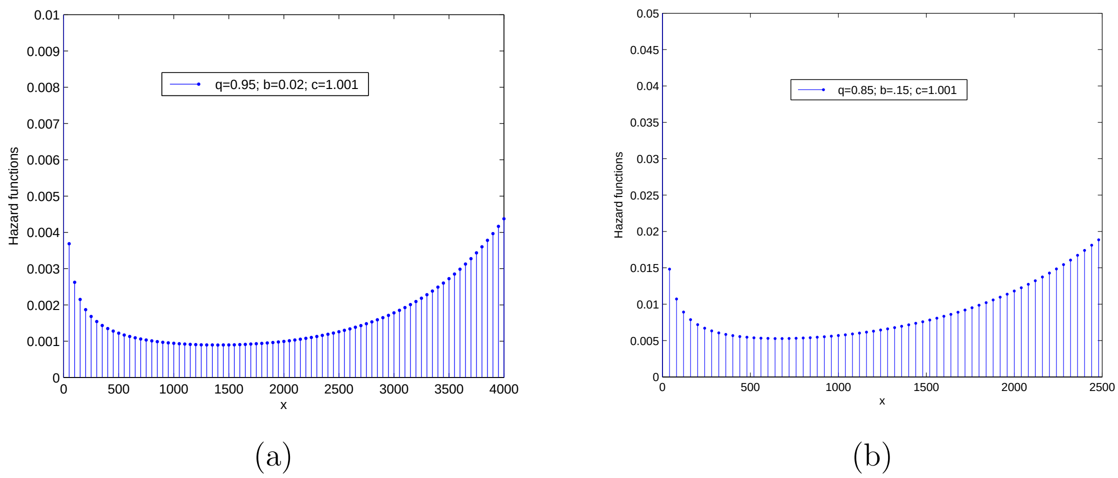


Figure 6.3: Hazard rate functions of the DRMW distribution with long useful life period.

Now using the partial exponential Bell polynomial, $B_{rk}(\mathbf{x})$ (see Comtet, 1974), defined by

$$\left(\sum_{r=1}^{\infty} x_r t^r/r!\right)^k /k! = \sum_{r=k}^{\infty} B_{rk}(\mathbf{x}) t^r/r!$$

for $\mathbf{x} = (x_1, x_2, \ldots)$, we can rewrite (6.5) as

$$\begin{aligned} S(x) &= \sum_{k=0}^{\infty} \frac{\ln^k(q) x^{k/2}}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} b^\ell \ell! \sum_{m=\ell}^{\infty} B_{ml}\left(\ln c, \ln c, \ldots\right) \frac{x^m}{m!} \\ &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \sum_{m=\ell}^{\infty} \frac{\ln^k(q) b^\ell B_{ml}\left(\ln c, \ln c, \ldots\right)}{(k-\ell)! m!} x^{m+\frac{k}{2}}. \qquad (6.6) \end{aligned}$$

The representation in (6.6) can be used to derive similar expansions for the probability mass function as the following

$$\begin{aligned} p(x) &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \sum_{m=\ell}^{\infty} \frac{\ln^k(q) b^\ell B_{ml}\left(\ln c, \ln c, \ldots\right)}{(k-\ell)! m!} x^{\frac{k}{2}+m} \\ &\quad - \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \sum_{m=\ell}^{\infty} \frac{\ln^k(q) b^\ell B_{ml}\left(\ln c, \ln c, \ldots\right)}{(k-\ell)! m!} (x+1)^{\frac{k}{2}+m}. \qquad (6.7) \end{aligned}$$

Using the binomial expansion for non-integer powers:

$$(A+B)^p = \sum_{d=0}^{\infty} c(p,d) A^{p-d} B^p,$$

where $c(p,d) = \frac{p(p-1)(p-2)\cdots(p-d+1)}{d!}$, one can rewrite (6.7) as

$$\begin{aligned} p(x) &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \sum_{m=\ell}^{\infty} \frac{\ln^k(q) b^\ell B_{ml}\left(\ln c, \ln c, \ldots\right)}{(k-\ell)! m!} x^{\frac{k}{2}+m} \\ &\quad - \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \sum_{m=\ell}^{\infty} \sum_{p=0}^{\infty} \frac{c(\frac{k}{2}+m, p) \ln^k(q) b^\ell B_{ml}\left(\ln c, \ln c, \ldots\right)}{(k-\ell)! m!} x^{\frac{k}{2}+m-p}. \end{aligned}$$

6.5 The moments

If X is a discrete reduced modified Weibull distribution random variable, then the r-th moment of X is given by

$$\begin{aligned} E(X^r) &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \sum_{m=\ell}^{\infty} \sum_{x=0}^{\infty} \frac{\ln^k(q) b^\ell B_{ml}\left(\ln c, \ln c, \ldots\right)}{(k-\ell)! m!} x^{\frac{k}{2}+m+r} - \\ &\quad \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \sum_{m=\ell}^{\infty} \sum_{p=0}^{\infty} \sum_{x=0}^{\infty} \frac{c(\frac{k}{2}+m, p) \ln^k(q) b^\ell B_{ml}\left(\ln c, \ln c, \ldots\right)}{(k-\ell)! m!} x^{\frac{k}{2}+m+r-p}. \end{aligned} \qquad (6.8)$$

Using (6.6) the $r-th$ moment of X can be rewriten as

$$\begin{aligned} E(X^r) &= \sum_{x=1}^{\infty}(x^r-(x-1)^r)s(x), \\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=\ell}^{\infty}\sum_{x=1}^{\infty}\frac{\ln^k(q)b^\ell B_{ml}\left(\ln c, \ln c, \ldots\right)}{(k-\ell)!m!}\times \\ & \left(x^{m+\frac{k}{2}+r}-\sum_{n=0}^{r}\binom{r}{n}(-1)^{r-n}x^n\right). \end{aligned} \tag{6.9}$$

The first four moments of the DRMW are

$$\begin{aligned} E(x) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=\ell}^{\infty}\sum_{x=1}^{\infty}\frac{\ln^k(q)b^\ell B_{ml}\left(\ln c, \ln c, \ldots\right)}{(k-\ell)!m!}x^{m+\frac{k}{2}}, \\ E(x^2) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=\ell}^{\infty}\sum_{x=1}^{\infty}\frac{\ln^k(q)b^\ell B_{ml}\left(\ln c, \ln c, \ldots\right)}{(k-\ell)!m!}x^{m+\frac{k}{2}}(2x-1), \\ E(x^3) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=\ell}^{\infty}\sum_{x=1}^{\infty}\frac{\ln^k(q)b^\ell B_{ml}\left(\ln c, \ln c, \ldots\right)}{(k-\ell)!m!}x^{m+\frac{k}{2}}(3x^2-3x+1), \\ E(x^4) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=\ell}^{\infty}\sum_{x=1}^{\infty}\frac{\ln^k(q)b^\ell B_{ml}\left(\ln c, \ln c, \ldots\right)}{(k-\ell)!m!}x^{m+\frac{k}{2}}(4x^3-6x^2+4x-1). \end{aligned}$$

6.6 Order statistics

In this section provides the probability mass function, the cumulative distribution function and the moment of the i-th order statistics $X_{i:n}$.

Suppose that $X_1, X_2, \ldots, X_n$ are a random sample sizs n from the proposed discrete distribution, then the obtained order statistics from this sample are $X_{1:n}, X_{2:n} \ldots, X_{n:n}$.

The PMF of the i-th order statistics $X_{i:n}$ is given by

$$
\begin{aligned}
P(X_{i:n}=x) &= \frac{n!}{(i-1)!(n-i)!}\int_{F(x-1)}^{F(x)} u^{i-1}(1-u)^{n-i}du, \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j\int_{F(x-1)}^{F(x)} u^{i+j-1}du, \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{(i+j)}\times \\
&\quad \left[\left(1-q^{\sqrt{x}(1+bc^{x})}\right)^{i+j}-\left(1-q^{\sqrt{x-1}(1+bc^{x-1})}\right)^{i+j}\right] \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{(i+j)}\times \\
&\quad \left[\begin{array}{c} \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=\ell}^{\infty}\sum_{s=0}^{i+j}\frac{\binom{i+j}{s}(-1)^s\ln^k(q)s^k b^\ell B_{ml}(\ln c,\ln c,\ldots)}{(k-\ell)!m!}x^{m+\frac{k}{2}}- \\ \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=\ell}^{\infty}\sum_{s=0}^{i+j}\frac{\binom{i+j}{s}(-1)^s\ln^k(q)s^k b^\ell B_{ml}(\ln c,\ln c,\ldots)}{(k-\ell)!m!}(x-1)^{m+\frac{k}{2}} \end{array}\right] \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{(i+j)}\times \\
&\quad \left[\begin{array}{c} \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=\ell}^{\infty}\sum_{s=0}^{i+j}\frac{\binom{i+j}{s}(-1)^s\ln^k(q)s^k b^\ell B_{ml}(\ln c,\ln c,\ldots)}{(k-\ell)!m!}x^{m+\frac{k}{2}}- \\ \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=\ell}^{\infty}\sum_{s=0}^{i+j}\sum_{p=0}^{\infty}\frac{\binom{i+j}{s}(-1)^{s+p}\ln^k(q)s^k b^\ell B_{ml}(\ln c,\ln c,\ldots)}{(k-\ell)!m!}x^{m+\frac{k}{2}-p} \end{array}\right]
\end{aligned}
\tag{6.10}
$$

and the CDF of the $i-th$ order statistics $X_{i:n}$ is

$$
\begin{aligned}
P(X_{i:n} \leq x) &= \frac{n!}{(i-1)!(n-i)!}\int_{0}^{F(x)} u^{i-1}(1-u)^{n-i}du, \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j\int_{0}^{F(x)} u^{i+j-1}du, \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{(i+j)}\times \\
&\quad \left[\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=\ell}^{\infty}\sum_{s=0}^{i+j}\binom{i+j}{s}\frac{(-1)^s\ln^k(q)s^k b^\ell B_{ml}\left(\ln c,\ln c,\ldots\right)}{(k-\ell)!m!}x^{m+\frac{k}{2}}\right], \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=\ell}^{\infty}\sum_{s=0}^{i+j}\binom{n-i}{j}\binom{i+j}{s}\frac{(-1)^{j+s}}{(i+j)}\times \\
&\quad \frac{\ln^k(q)s^k b^\ell B_{ml}\left(\ln c,\ln c,\ldots\right)}{(k-\ell)!m!}x^{m+\frac{k}{2}}.
\end{aligned}
\tag{6.11}
$$

Then, from (6.10) the $r-th$ moment of the $i-th$ order statistics $X_{i:n}$ is obtained as

$$
\begin{aligned}
E(X^r) &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} \frac{(-1)^j}{(i+j)} \times \\
&\left[\begin{array}{c} \sum_{s=0}^{i+j} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \sum_{m=\ell}^{\infty} \sum_{x=0}^{\infty} \frac{\binom{i+j}{s}(-1)^s \ln^k(q) s^k b^\ell B_{ml}(\ln c, \ln c, \ldots)}{(k-\ell)!m!} x^{m+\frac{k}{2}+r} - \\
\sum_{s=0}^{i+j} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \sum_{m=\ell}^{\infty} \sum_{p=0}^{\infty} \sum_{x=0}^{\infty} \frac{\binom{i+j}{s}(-1)^{s+p} \ln^k(q) s^k b^\ell B_{ml}(\ln c, \ln c, \ldots)}{(k-\ell)!m!} x^{m+\frac{k}{2}+r-p} \end{array} \right] \\
&.
\end{aligned}
\tag{6.12}
$$

6.7 Extreme value behavior

Suppose $X_1, \ldots, X_n$ is a random sample from the DRMW distribution. If $\overline{X} = (X_1 + \cdots + X_n)/n$ denotes the mean of the random sample then by the usual central limit theorem $\sqrt{n}(\overline{X} - E(X))/\sqrt{Var(X)}$ approaches the standard normal distribution as $n \to \infty$ under suitable conditions. Sometimes one would be interested in the asymptotes of the extreme values $M_n = \max(X_1, \ldots, X_n)$ and $m_n = \min(X_1, \ldots, X_n)$.

By Corollary 2.4.1 in Galambos (1987), $(M_n - b_n)/a_n$ admits a non-degenerate limit for suitable constants $a_n > 0$, $b_n \in \mathbb{R}$ if and only if

$$\frac{P(x)}{S(x)} \to 0$$

as $x \to \infty$. For the DRMW distribution,

$$
\begin{aligned}
\lim_{x\to\infty} \frac{P(x)}{S(x)} &= \lim_{x\to\infty} \left\{ 1 - q^{\sqrt{x+1}\left(1+bc^{x+1}\right) - \sqrt{x}(1+bc^x)} \right\} \\
&= \lim_{x\to\infty} \left\{ 1 - q^{\left\{\sqrt{\frac{x+1}{x}} \frac{1+bc^{x+1}}{1+bc^x} - 1\right\}\sqrt{x}(1+bc^x)} \right\} \\
&= \begin{cases} 1, & \text{if } c < 1, \\ -\infty, & \text{if } c > 1, \\ 0, & \text{if } c = 1. \end{cases}
\end{aligned}
$$

Hence, $(M_n - b_n)/a_n$ admits a non-degenerate limit only if $c = 1$.

If $c = 1$ then $S(x) = q^{(1+b)\sqrt{x}}$. Note that

$$
\begin{aligned}
\lim_{x\to\infty} \frac{S\left(x + xg(t)\right)}{S(x)} &= \lim_{x\to\infty} \frac{q^{(1+b)\sqrt{t+xg(t)}}}{q^{(1+b)\sqrt{t}}} \\
&= \lim_{x\to\infty} q^{(1+b)\sqrt{t}\left[\sqrt{1+x\frac{g(t)}{t}} - 1\right]} \\
&= \lim_{x\to\infty} q^{\frac{(1+b)\sqrt{t}g(t)x}{2\sqrt{t}}} \\
&= \exp(-x)
\end{aligned}
$$

if $g(t) = -2\sqrt{t}/\{(1+b)\ln q\}$. It follows by Chapter 1 in Leadbetter *et al.* (1987) that the DRMW distribution for $c = 1$ belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n\to\infty} \Pr\left\{a_n\left(M_n - b_n\right) \le x\right\} = \exp\left\{-\exp(-x)\right\}$$

for $a_n = -2\sqrt{S^{-1}(1/n)}/\{(1+b)\ln q\}$ and $b_n = S^{-1}(1/n)$, where $S^{-1}(\cdot)$ denotes the inverse function of $S(\cdot)$.

6.8 Estimation

Suppose $x_1, x_2, \ldots, x_n$ is a random sample from the DRMW distribution with unknown parameters (q, b, c). Section 6.8.1 estimates these parameters by the method of maximum likelihood. Section 6.8.2 assesses the finite sample performance of the MLEs with respect to biases and mean squared errors.

6.8.1 Maximum likelihood estimation

The point and interval estimators of the unknown parameters of the DRMW distribution are derived using the maximum likelihood method.

Let $x_1, \ldots, x_n$ be a random sample from the DRMW distribution. Let $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3) = (q, b, c)$ denote the vector of unknown parameters. The log-likelihood function is

$$\mathcal{L}\left(\phi; x_i\right) = \sum_{i=1}^{n} \ln\left(q^{\sqrt{x_i}\left(1+bc^{x_i}\right)} - q^{\sqrt{x_i+1}\left(1+bc^{x_i+1}\right)}\right).$$

Now define

$$D_i(\boldsymbol{\phi}) = q^{A_{x_i}(b,c)} - q^{A_{x_i+1}(b,c)}$$

for $i = 1, 2, \ldots, n$, where $A_{x_i}(b,c) = \sqrt{x_i}(1 + bc^{x_i})$. So, the log-likelihood function $\mathcal{L}(\boldsymbol{\phi}; x_i)$ can be rewritten as

$$\mathcal{L}\left(\boldsymbol{\phi}; x_i\right) = \sum_{i=1}^{n} \ln D_i(\boldsymbol{\phi}).$$

The first order partial derivatives of $\mathcal{L}(\boldsymbol{\phi}; x_i)$ with respect to the three parameters are

$$\mathcal{L}_{\phi_j} = \sum_{i=1}^{n} \frac{D_{\phi_j}^{(i)}}{D_i(\boldsymbol{\phi})}$$

for $j = 1, 2, 3$, where $D_{\phi_j}^{(i)}$ is the first order partial derivative of $D_i(\boldsymbol{\phi})$ with respect to ϕ_j, $j = 1, 2, 3$ and

$$\begin{aligned}
D_q^{(i)} &= A_{x_i}(b,c) q^{A_{x_i}(b,c)-1} - A_{x_i+1}(b,c) q^{A_{x_i+1}(b,c)-1}, \\
D_b^{(i)} &= \ln(q) \left[A_{x_i}^{(b)} q^{A_{x_i}(b,c)} - A_{x_i+1}^{(b)} q^{A_{x_i+1}(b,c)} \right], \\
D_c^{(i)} &= \ln(q) \left[A_{x_i}^{(c)} q^{A_{x_i}(b,c)} - A_{x_i+1}^{(c)} q^{A_{x_i+1}(b,c)} \right],
\end{aligned}$$

where $A_{x_i}^{(b)} = c^{x_i}\sqrt{x_i}$ and $A_{x_i}^{(c)} = c^{x_i-1}\sqrt[3]{x_i}$ are the first order partial derivatives of $A_{x_i}(b,c)$ with respect to b and c, respectively.

The log-likelihood equations are

$$0 = \sum_{i=1}^{n} \frac{D_{\phi_j}^{(i)}}{D_i(\boldsymbol{\phi})} \tag{6.13}$$

for $j = 1, 2, 3$.

By solving the above system of three nonlinear equations, we can obtain maximum likelihood estimators (MLEs) of q, b and c. It is clear that the system cannot be solved in closed form.

To obtain confidence intervals for q, b and c, we use the observed information matrix since the expected information matrix is very complicated and the MLEs of the unknown parameters cannot be obtained analytically.

The observed information matrix $J(\underline{\vartheta})$ is

$$J(\underline{\vartheta}) = - \begin{bmatrix} \mathcal{L}_{qq} & \mathcal{L}_{qb} & \mathcal{L}_{qc} \\ & \mathcal{L}_{bb} & \mathcal{L}_{bc} \\ & & \mathcal{L}_{cc} \end{bmatrix}.$$

where $\underline{\vartheta} = (q, b, c)$ and the elements of this matrix are given in Appendix D.

If $\widehat{q}$, $\widehat{b}$ and $\widehat{c}$ are the MLEs of q, b and c, respectively, then under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, $\sqrt{n}\left(\widehat{q} - q, \widehat{b} - b, \widehat{c} - c\right)$ converges in distribution to a trivariate normal random vector with zero means and covariance matrix, I^{-1}, where $I(\underline{\vartheta})$ is the expected information matrix. For asymptotic normality, certain regularity conditions must be satisfied, see, for example, Ferguson (1996).

6.8.2 Simulation study

In this subsection, we assess the performance of the MLEs with respect to sample size n. The assessment is based on a simulation study:

1. First, ten thousand samples of size n are generated from (6.1). The inversion method used to generate samples, that was done for the DRMW distribution by solving

$$\sqrt{X}\left(1+bc^{X}\right)\ln q=\ln(1-U),$$

where $U\sim U(0,1)$ is a uniform variate on the unit interval.

2. Obtain the MLEs for the ten thousand generated samples, $\left(\widehat{q}_i,\widehat{b}_i,\widehat{c}_i\right)$ for $i=1,2,\ldots,10000$.

3. Compute the standard errors of the MLEs for the ten thousand samples, say $\left(s_{\widehat{q}_i},s_{\widehat{b}_i},s_{\widehat{c}_i}\right)$ for $i=1,2,\ldots,10000$. The standard errors computed by using the observed information matrices.

4. The biases and mean squared errors of the MLEs can be computed as the following,

$$\mathrm{bias}_h(n)=\frac{1}{10000}\sum_{i=1}^{10000}\left(\widehat{h}_i-h\right),$$
$$\mathrm{MSE}_h(n)=\frac{1}{10000}\sum_{i=1}^{10000}\left(\widehat{h}_i-h\right)^2$$

for $h=q,b,c$.

The above steps were repeated $n=10,11,\ldots,100$ with $q=\exp(-1)$, $b=1$ and $c=\exp(1)$, so computing $\mathrm{bias}_h(n)$ and $\mathrm{MSE}_h(n)$ for $h=q,b,c$ and $n=10,11,\ldots,100$.

Figure 6.4 shows the biases of the MLEs of (q,b,c) versus $n=10,11,\ldots,100$, and it can be seen that the biases for the four parameters are small and vary with respect to the samples sizes. The mean squared errors of the MLEs of each parameters are shown in Figure 6.5, which are vary with respect to the samples sizes. The broken line in Figure 6.4 indicates to the distance between the biases and the zero. The broken line in Figure 6.5 indicates to the distance between the mean squared errors and the zero.

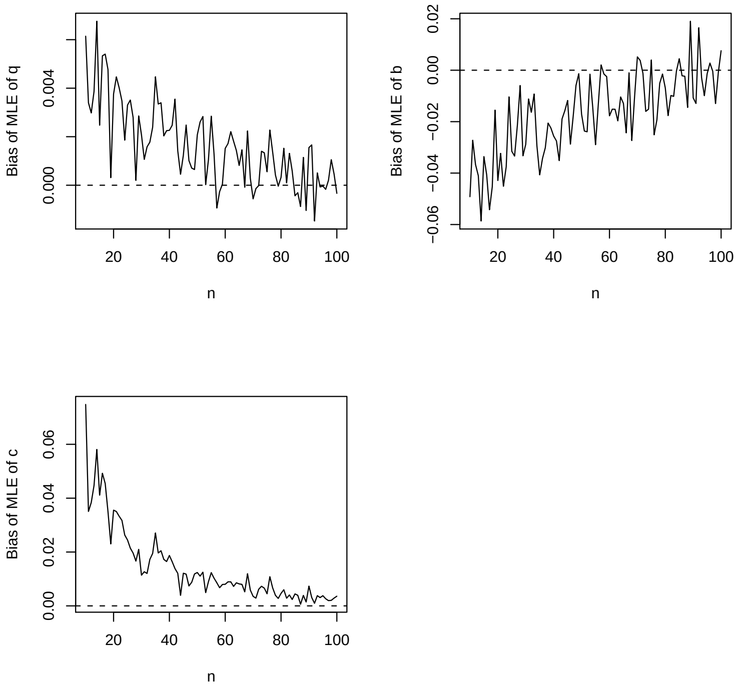


Figure 6.4: Biases of the MLEs of (q, b, c) versus $n = 10, 11, \ldots, 100$.

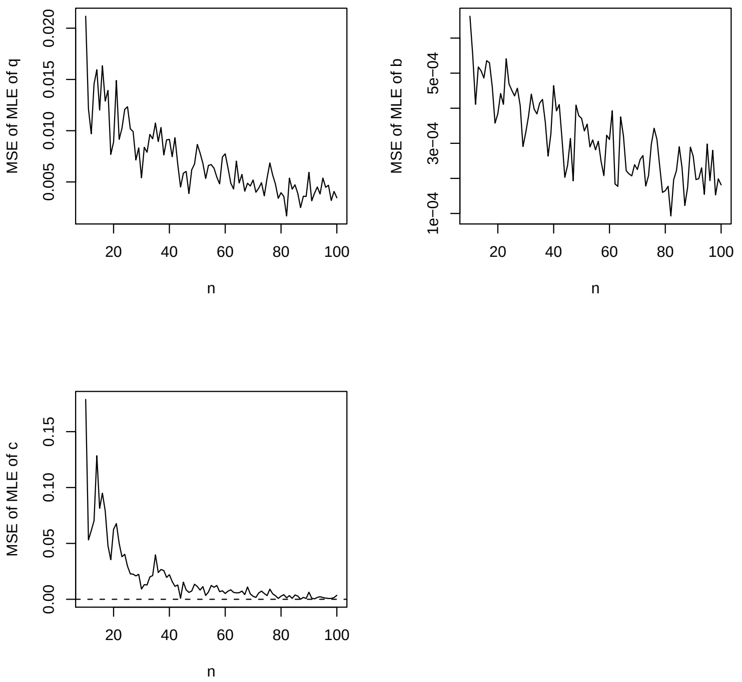


Figure 6.5: Mean squared errors of the MLEs of (q, b, c) versus $n = 10, 11, \ldots, 100$.

From these figures the following observations can be observed: the biases of the MLEs of the parameters q and c are generally positive; the biases of the MLEs of the parameter b are generally negative; the biases of the MLEs of the parameter q are the smallest; the biases of the MLEs of the parameter c are the largest; the biases for the parameters q and c decrease to zero as $n \to \infty$; the biases for the parameter$b$ increase to zero as $n \to \infty$; the mean squared errors of the MLEs of the parameter b are the smallest; the mean squared errors of the MLEs of the parameter c are the largest; the mean squared errors for all three parameter decrease to zero as $n \to \infty$; The above observations are observed by choosing $(q, b, c) = (\exp(-1), 1, \exp(1))$. A similar results were found for other choices.

6.9 Applications

In this section, we illustrate the flexibility of the proposed distribution using four real data sets. Three of these data sets are complete. The sample size for the first data set is fifty. The sample size for the second data set is eighteen. The sample size for the third data set is forty three. The sample size for the fourth data set is one hundred and ninety two.

Hence, the biases for $\widehat{q}$, $\widehat{b}$ and $\widehat{c}$ can be expected to be less than 0.004, 0.04 and 0.04, respectively, for all of the data sets. The mean squared errors for $\widehat{q}$, $\widehat{b}$ and $\widehat{c}$ can be expected to be less than 0.01, 0.0004 and 0.05, respectively, for all of the data sets.

The fourth one is censored. The fit of the proposed distribution will be compared with the DMW, DAddW and DW distributions, see Table 6.1. Note that the DMW and DAddW distributions are the known discrete distributions allowing for bathtub shaped hazard rate functions.

Table 6.1: The survival functions of the DMW, DAddW and DW distributions.

Model	S(x)
DMW	$q^{x^\theta c^x}$
DAddW	$q^{x^\theta + bx^\gamma}$
DW	q^{x^θ}

We will see that the proposed distribution gives the best fit for each data set in that it gives the smallest values for AIC, BIC, AICc and CAIC. We will also see that the proposed distribution gives the best fit for each data set in terms of the estimated PMF, the estimated hazard rate function, the estimated survival function, and the estimated Total Time on Test TTT-transform.

6.9.1 Complete data

Three real data sets (complete and uncensored) are considered in this section. Two of them have bathtub shaped hazard rate functions. The other one has an increasing hazard rate function.

Discrete Aarset data

The data are integer parts of the lifetimes of fifty devices, see section 3.11.1. The data are listed in Table 6.2. Figure 6.6 shows that the TTT-plot provides a convex shape followed by a concave shape which inducts that the hazard rate function of this data exhibits a bathtub-shape. Noughabi *et al.* (2011) have shown that the DMW distribution provides a good fit for this data.

Table 6.2: Aarset data (in weeks).

Time of failure	0	1	2	3	6	7	11	12	18	21	32	36	40	45	46
No of failure	2	5	1	1	1	1	1	1	5	1	1	1	1	1	1
Time of failure	47	50	55	60	63	67	72	75	79	82	83	84	85	86	
No of failure	1	1	1	1	2	4	1	1	1	2	1	3	5	2	

Table 6.3: MLEs of parameters for the discrete Aarset data, standard errors in brackets, and the measures AIC, BIC, AICc and CAIC.

Model	$\widehat{q}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\theta}$	$\widehat{\gamma}$	AIC	BIC	AICc	CAIC
DRMW	0.9137	6.8149×10^{-5}	1.1273	-	-	439.8	445.5	440.3	448.5
	(0.017)	(1.377×10^{-4})	(0.026)	-	-				
DMW	0.9403	-	1.0241	0.3450	-	464.2	469.9	464.7	472.9
	(0.026)	-	(0.005)	(0.126)	-				
DAddW	0.9216	6.0091×10^{-5}	-	0.4541	2.8387	464.4	472.0	465.3	476.1
	(0.034)	(2.181×10^{-4})	-	(0.119)	(0.807)				
DW	0.9805	-	-	1.0234	-	487.2	491.0	487.5	493.0
	(0.011)	-	-	(0.131)	-				

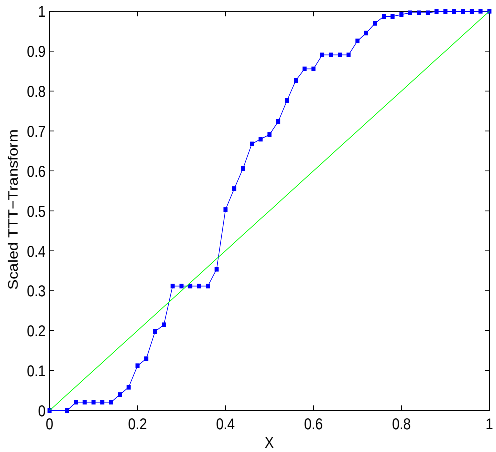


Figure 6.6: TTT-transform plot for the Aarst Discret.

The AIC, BIC, AICc and CAIC given in Table 6.3 are smallest for the DRMW distribution with AIC = 439.8, BIC = 445.5, AICc = 440.3 and CAIC = 448.5.

The estimated PMF, the estimated hazard rate function, the estimated survival function, and the estimated TTT-transform shown in Figure 6.7 are closest to the empirical versions for the DRMW distribution.

The approximate 95 percent confidence intervals for q, b and c are $[0.881, 0.946]$, $[0, 3.38 \times 10^{-4}]$ and $[1.076, 1.178]$, respectively.

Electronic devices

The bathtub shaped hazard rate function is required in many applications like failure of electronic components. In this example, times to failure of eighteen electronic devices Wang (2000) are used to show how the proposed distribution can be applied in practice. The data are listed in Table 6.4. The TTT-plot of this data exhibits a convex shape followed by a concave shape and that corresponds to a bathtub shaped hazard function, see Figure 6.8.

Table 6.4: Lifetimes of eighteen electronic devices (in days).

5	11	21	31	46	75	98	122	145
165	196	224	245	293	321	330	350	420

The AIC, BIC, AICc and CAIC given in Table 6.5 are smallest for the DRMW distribution with AIC = 223.9, BIC = 226.5, CAIc = 225.6 and CAIC = 229.5.

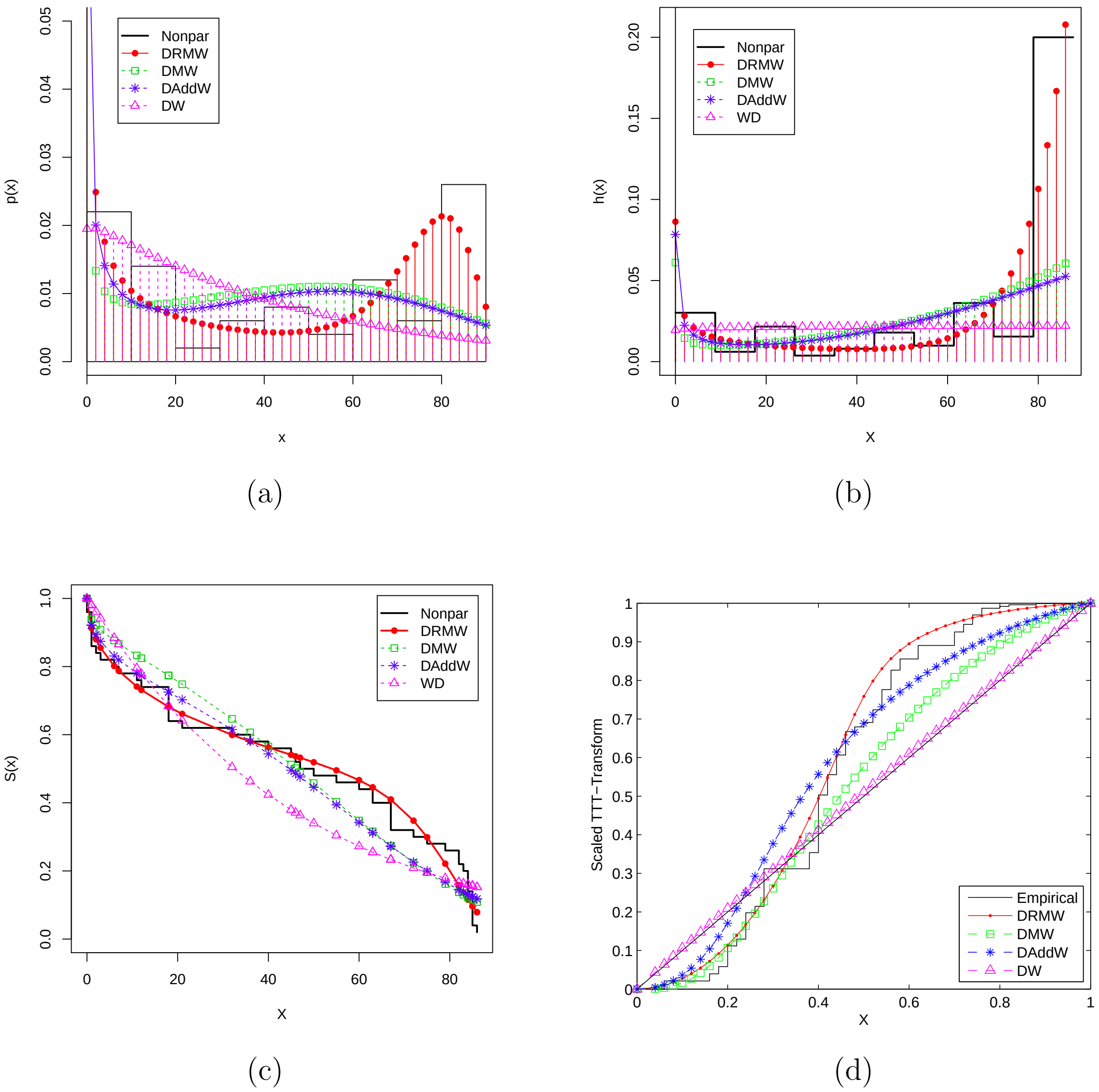


Figure 6.7: For the discrete Aarset data: (a) The histogram and the estimated PMFs; (b) The empirical and estimated hazard rate functions; (c) The empirical and estimated survival functions; (d) The empirical and fitted scaled TTT-transform plots.

Table 6.5: MLEs of parameters for data in Table 6.4, standard errors in brackets, and the measures AIC, BIC, AICc and CAIC.

Model	$\widehat{q}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\theta}$	$\widehat{\gamma}$	AIC	BIC	AICc	CAIC
DRMW	0.9658	0.1237	1.0086	-	-	223.9	226.5	225.6	229.5
	(0.019)	(0.236)	(3.804×10^{-3})	-	-				
DMW	0.9465	-	1.0055	0.3194	-	225.6	228.3	227.3	231.3
	(0.063)	-	(1.905×10^{-3})	(0.254)	-				
DAddW	0.9771	4.015×10^{-3}	-	0.5208	1.677	227.9	231.4	230.9	235.4
	(0.027)	(0.0187)	-	(0.34638)	(0.764)				
DW	0.9912	-	-	0.9222	-	226.1	227.9	226.9	229.9
	(9.257×10^{-3})	-	-	(0.189)	-				

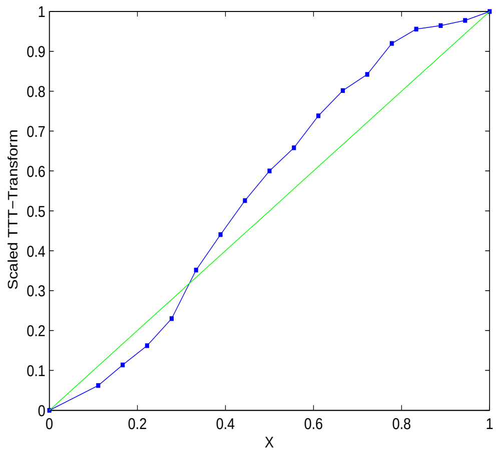


Figure 6.8: TTT-transform plot for the Discret 18.

The estimated PMF, the estimated hazard rate function, the estimated survival function, and the estimated TTT-transform shown in Figure 6.9 are closest to the empirical versions for the DRMW distribution.

The approximate 95 percent confidence intervals for q, b and c are $[0.928, 1]$, $[0, 0.587]$ and $[1.001, 1.016]$, respectively.

Leukemia data

The data set for this example is collected from the Ministry of Health Hospital in Saudi Arabia by Abouammoh *et al.* (1994). The data are lifetimes in days of forty three blood patients who had leukemia, see Table 6.6. The data set exhibits an increasing hazard rate. As the TTT-transform plot, in Figure 6.10, is concave then the data has an increasing hazard rate.

Table 6.6: The leukemia data.

115	181	255	418	441	461	516	739	743	789	807
865	924	983	1025	1062	1063	1165	1191	1222	1222	1251
1277	1290	1357	1369	1408	1455	1478	1549	1578	1578	1599
1603	1605	1696	1735	1799	1815	1852	1899	1925	1965	

The AIC, BIC, AICc and CAIC given in Table 6.7 are smallest for the DRMW distribution with AIC = 658.0, BIC = 663.2, CAIc= 658.6 and CAIC = 666.2.

The estimated PMF, the estimated hazard rate function, the estimated survival

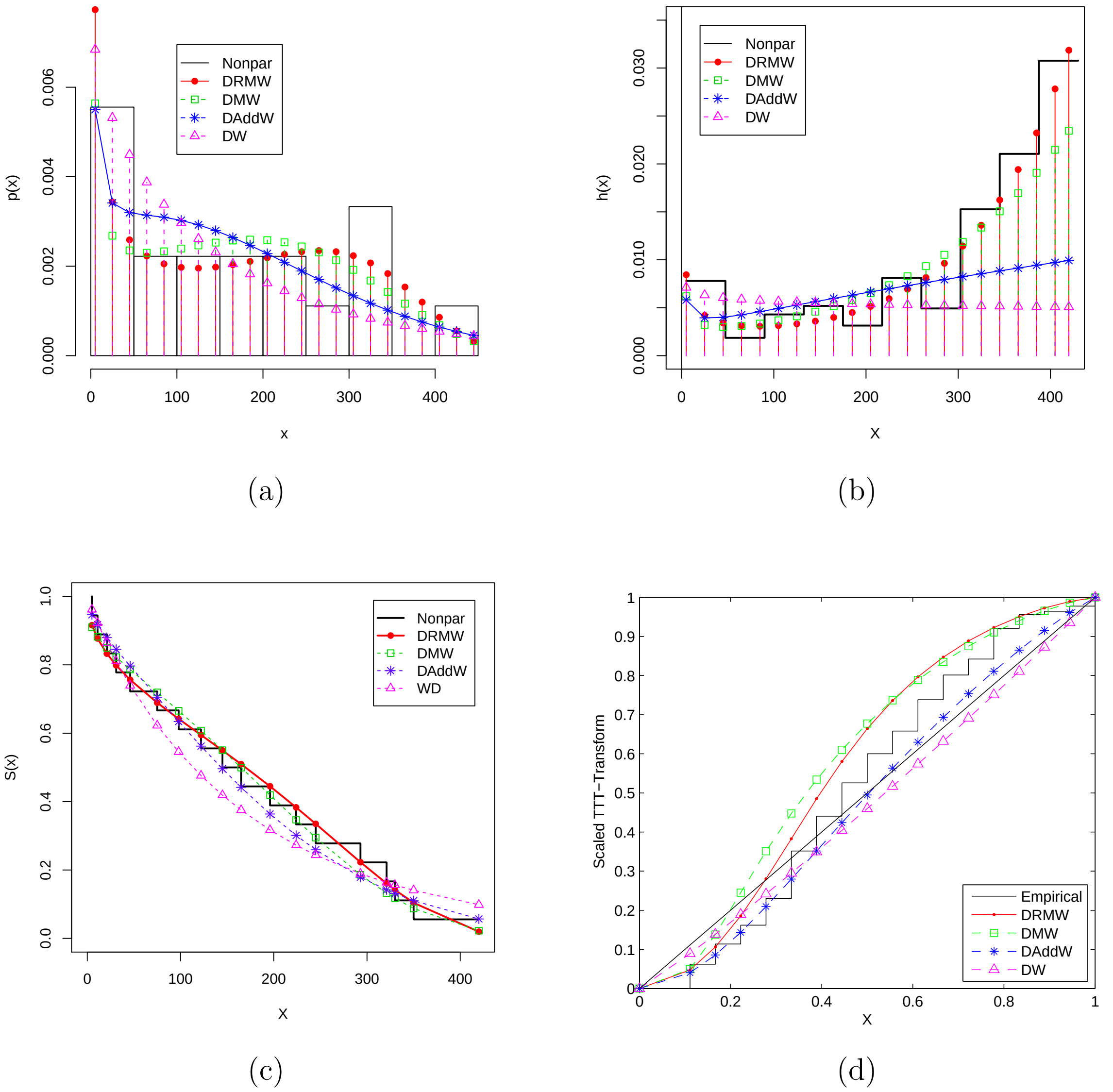


Figure 6.9: For the electronic devices data: (a) The histogram and the estimated PMFs; (b) The empirical and estimated hazard rate functions; (c) The empirical and estimated survival functions; (d) The empirical and fitted scaled TTT-transform plots.

Table 6.7: MLEs of parameters for leukemia data, standard errors in brackets, and the measures AIC, BIC, AICc and CAIC.

Model	$\widehat{q}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\theta}$	$\widehat{\gamma}$	AIC	BIC	AICc	CAIC
DRMW	0.9966	0.2189	1.0024	-	-	658.0	663.2	658.6	666.2
	(0.003)	(0.217)	(4.116×10^{-4})	-	-				
DMW	0.9729	-	1.0024	0.0118	-	660.0	665.3	660.3	668.3
	(0.031)	-	(3.915×10^{-4})	(0.203)	-				
DAddW	0.9981	1.127×10^{-5}	-	0.1680	2.4581	667.1	674.2	668.2	678.2
	(0.012)	(7.978×10^{-5})	-	(0.311)	(0.300)				
DW	0.9999	-	-	1.3545	-	690.7	694.3	691.0	696.3
	(2.984×10^{-5})	-	-	(0.072)	-				

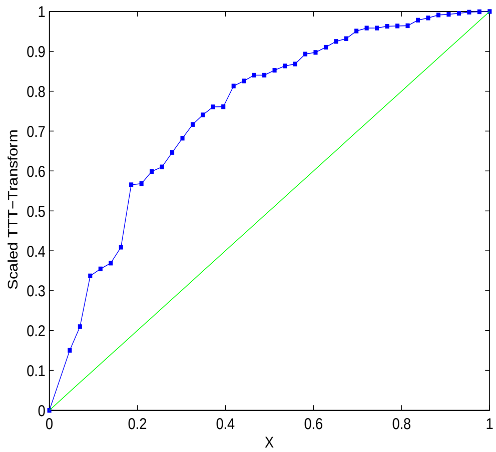


Figure 6.10: TTT-transform plot for the Leukamia.

function, and the estimated TTT-transform shown in Figure 6.11 are closest to the empirical versions for the DRMW distribution.

The approximate 95 percent confidence intervals for q, b and c are $[0.992, 1]$, $[0, 0.75]$ and $[1.002, 1.003]$, respectively.

6.9.2 Censored data

Here, we show how the proposed distribution can be applied in practice for censored (real) data sets. We use an original data presented here for the first time. The data relates to one hundred and ninety two prisoners who were imprisoned because of using or selling drugs. All of the prisoners were released at the same time. About seven tenths of them re-offended. The data set contains the times in months from release to re-imprisonment for the same crime or related one. The prisoners who did not re-offend were considered as censored.

The AIC, BIC, AICc and CAIC given in Table 6.8 are smallest for the DRMW distribution with AIC = 1353.9, BIC = 1363.6, CAIc= 1354.0 and CAIC = 1366.6.

The estimated hazard rate function and the estimated survival function shown in Figure 6.12 are closest to the empirical versions for the DRMW distribution.

The approximate 95 percent confidence intervals for q, b and c are $[0.882, 0.924]$, $[0, 0.018]$ and $[1.021, 1.116]$, respectively.

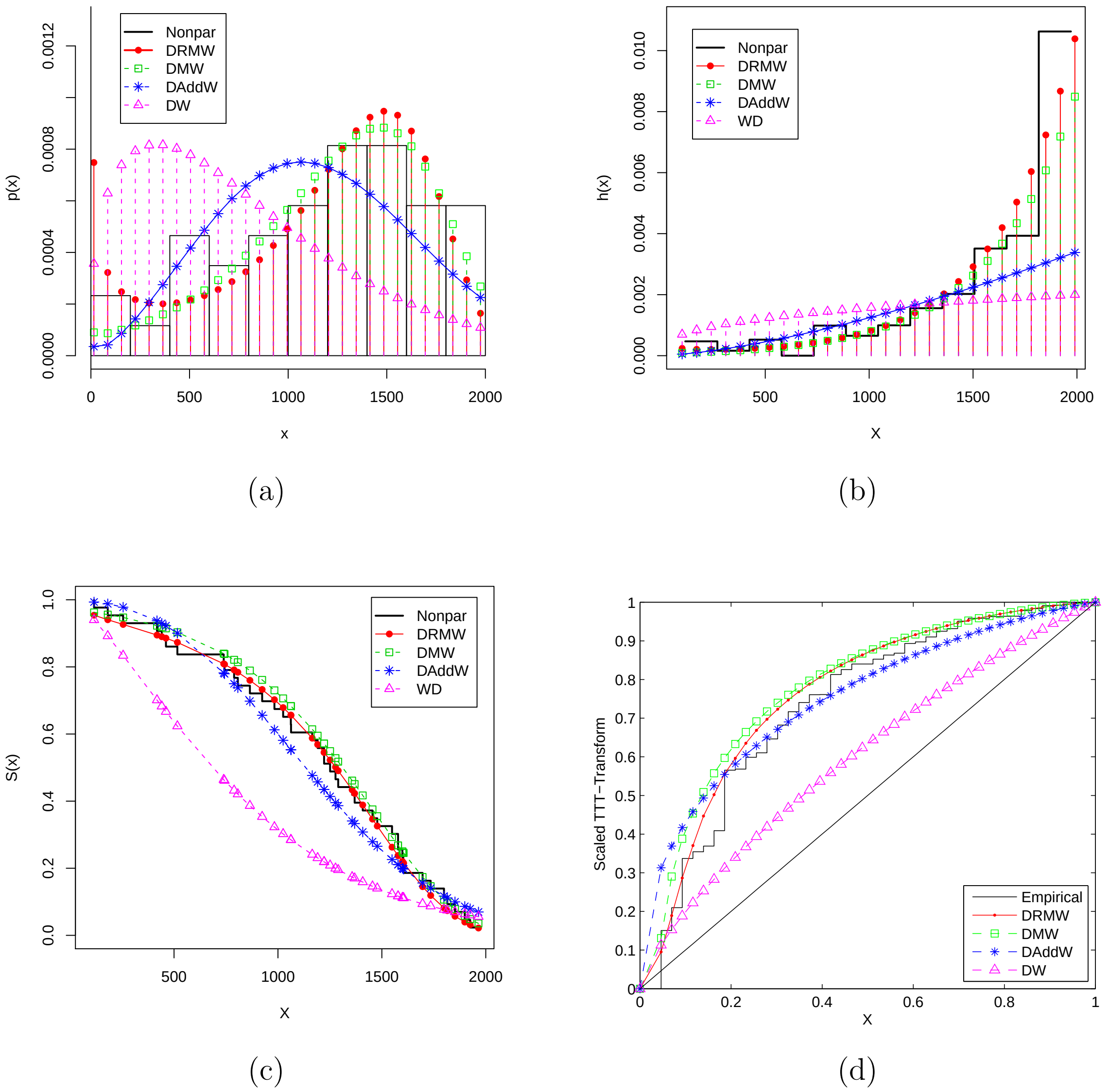


Figure 6.11: For leukemia data: (a) The histogram and the estimated PMFs; (b) The empirical and estimated hazard rate functions; (c) The empirical and estimated survival functions; (d) The empirical and fitted scaled TTT-transform plots.

Table 6.8: MLEs of parameters for the drug data, standard errors in brackets, and the measures AIC, BIC, AICc and CAIC.

Model	$\widehat{q}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\theta}$	$\widehat{\gamma}$	AIC	BIC	AICc	CAIC
DRMW	0.9029	0.0042	1.0689	-	-	1353.9	1363.6	1354.0	1366.6
	(0.011)	(7.352×10^{3})	(0.024)	-	-				
DMW	0.9410	-	1.0040	0.6445	-	1357.1	1366.8	1357.2	1369.8
	(0.015)	-	(2.867×10^{-3})	(0.087)	-				
DAddW	0.9372	2.8364×10^{-4}	-	0.6348	2.2669	1357.3	1370.3	1357.5	1374.3
	(0.015)	(1.085×10^{-3})	-	(0.07547)	(0.913)				
DW	0.9498	-	-	0.7456	-	1359.2	1369.0	1357.2	1372.0
	(0.012)	-	-	(0.057)	-				

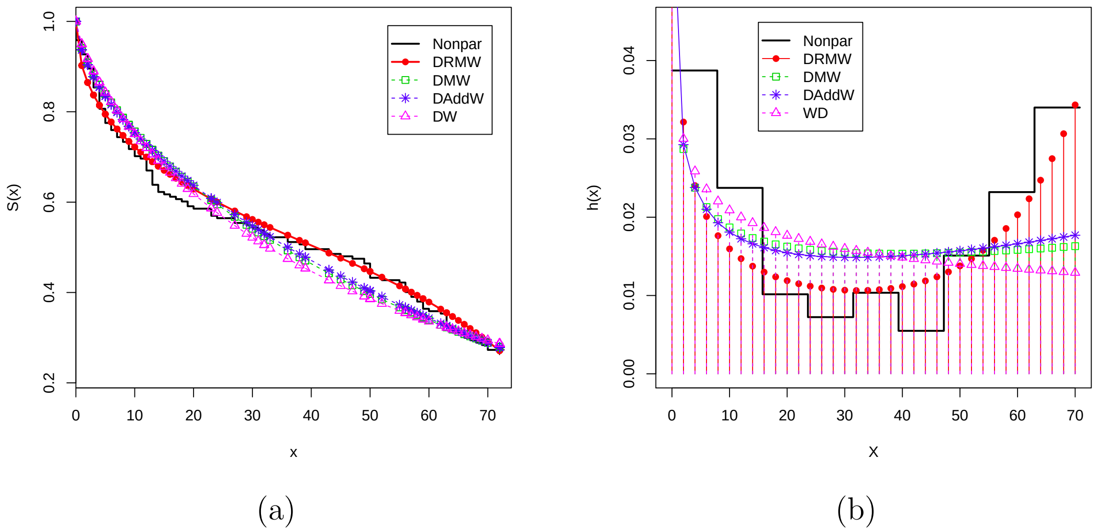


Figure 6.12: For drug data: (a) Fitted survival functions using the DRMW and other models; (b) Nonparametric and fitted hazard rate functions using the DRMW and other models.

6.10 Summary

A three-parameter discrete distribution is introduced in this chapter based on a recent modification of the continuous Weibull distribution which introduced in chapter 4. It is one of only three discrete distributions allowing for bathtub shaped hazard rate functions. We study some of its mathematical properties, discuss estimation by the method of maximum likelihood, and describe applications to four real data sets. The new distribution is shown to outperform at least three other models including the ones allowing for bathtub shaped hazard rate functions.

Chapter 7

Comparing the exponentiated and generalized modified Weibull distributions

7.1 Introduction

Among the various extensions and modified forms of the Weibull distribution (1951), the modified Weibull MW distribution of Lai *et al.* (2003) is one of the most important modifications of the Weibull distribution. It multiplies the Weibull cumulative hazard function αx^{β} by $e^{\lambda x}$. This distribution was later generalized to exponentiated form by Carrasco *et al.* (2008) by adding another shape parameter. Carrasco *et al.* (2008)'s modification is called the generalized modified Weibull GMW distribution.

To be consistent with paper of Carrasco *et al.* (2008), recalled the CDF of the GMW distribution (2) in Carrasco *et al.* (2008)

$$F(x) = \left(1 - e^{-\alpha x^{\gamma} e^{\lambda x}}\right)^{\beta} \tag{7.1}$$

for $x > 0$, $\alpha > 0$, $\gamma \geq 0$, $\lambda \geq 0$ and $\beta > 0$.

The sub-models of the GMW and the shapes of its HF are represented in Section 2.2.17. It is important to note for latter reasons that CDF, SF, PDF and HF of the GMW are not valid functions if $\lambda < 0$. For example, the CDF is not a monotonic increasing function of x if $\lambda < 0$. The PDF can take negative values if $\lambda < 0$. Also the HF can take negative values if $\lambda < 0$.

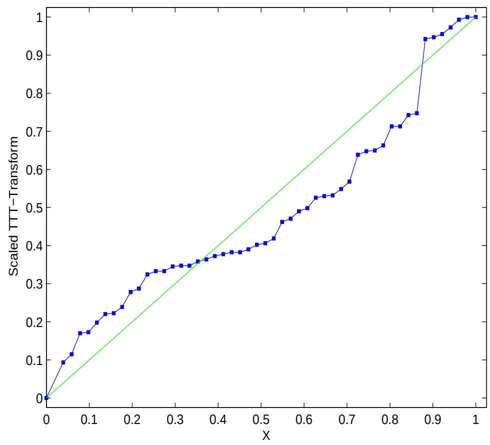


Figure 7.1: TTT-transform plot for the radiotherapy data.

Carrasco *et al.* (2008) applied the GMW distribution to two well-known censored data sets and compared its goodness-of-fit with its sub-models. The first data set is the serum-reversal data of Silva (2004) and Perdoná (2006). The TTT-plot for this data is shown in Figure 4.12, which takes a convex shape followed by a concave shape. This corresponds to a bathtub shaped HF.

The second data is a radiotherapy data. The TTT-plot for this data is shown in Figure 7.1, which takes a concave shape followed by a convex shape followed by a concave shape. Then, the HF of this data is modified unimodel shape (unimodal followed by increasing). Carrasco *et al.* (2008) mention that this corresponds to a unimodal HF. There are some other mistakes in the results of both applications in Carrasco *et al.* (2008).

According to Liddle (2004), usually, that will improve the fitting of the data and the maximised likelihood will increase. Sometimes, the new added parameter that is restricted to be positive dose not improve the maximum likelihood function, some models that are proposed by adding a new positive parameter dose not improve the maximum likelihood function. The generalized modified Weibull distribution of Carrasco *et al.* (2008) has this problem for some modified unimodal data sets. Also, it dose not provide a better fit than the the exponentiated Weibull distribution using this kind of data sets despite the exponentiated Weibull distribution is sub-model of the GMW.

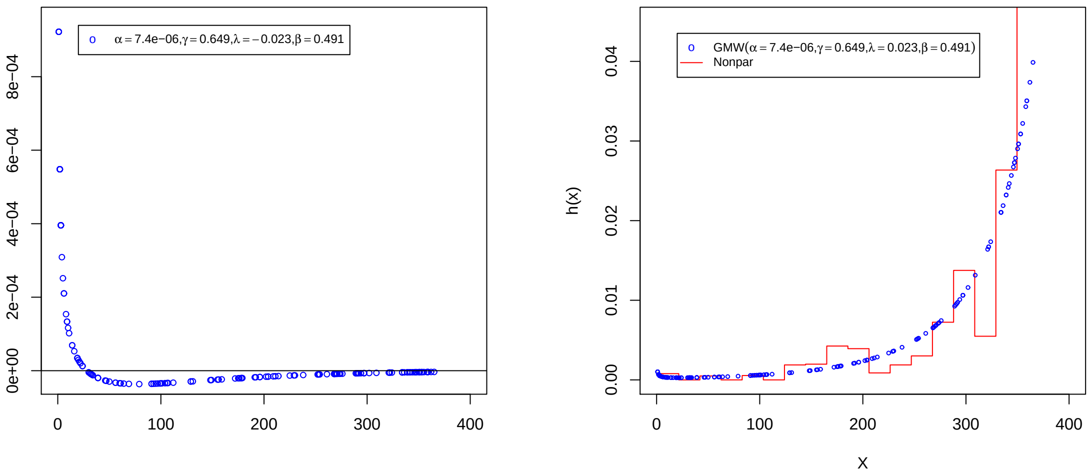
This chapter shows that no evidence that the generalized modified Weibull distribution can provide a better fit than the exponentiated Weibull distribution for data sets exhibiting the modified unimodal hazard function (the radiotherapy data and the infected pigs data that used in Chapter 5). Also, it shows the incorrect results of the paper of Carrasco *et al.* (2008) and presents the correct results.

7.2 Applications

This section uses three well-known data sets. The first data set has a bathtub shaped HF and the last data sets has a modified unimodal shaped HF. We will show that the GMW distribution can not provide better fits than the EW distribution for data sets exhibiting modified unimodal HFs. We will also point out incorrect results on the first two data sets presented in Carrasco *et al.* (2008). The fits are compared using the following measures: K-S statistic, AIC, BIC, CAIC and AICc.

7.2.1 Serum-reversal data

Table 1 in Carrasco *et al.* (2008) shows the maximum likelihood estimates (MLEs) of the parameters of the GMW distribution and its sub-models (MW, EW, EE, Weibull and GR distributions) for the serum-reversal data. All these results appear correct. It is clear that the GMW distribution presents a very good fit for this data with respect to AIC, BIC and CAIC values. We computed the Kolmogorov-Smirnov (K-S) statistic, the distance between the empirical CDF and the fitted CDF, for the GMW distribution and its sub-models. Again, the GMW distribution has the smallest K-S statistic with the value 0.117, whilst the K-S statistics for the MW, EW, EE, Weibull and GR distributions are 0.155, 0.169, 0.259, 0.182 and 0.247, respectively. The HF for the fitted GMW distribution is plotted in Figure 4c of Carrasco *et al.* (2008). This figure appears incorrect because it is well known that the HF must be nonnegative everywhere. It appears Carrasco *et al.* (2008) plotted the HF for the fitted GMW distribution using $\lambda = -0.023$, an invalid value for λ, see Section 1. The MLE of λ reported in Table 1 of Carrasco *et al.* (2008) is 0.023. The HF for the fitted GMW distribution when $\lambda = -0.023$ is plotted in Figure 7.2a, just to show that Figure 4c in Carrasco *et al.* (2008) was plotted using this negative value. Figure 7.2b presents the nonparametric HF of the data and the HF for the fitted GMW distribution using the MLEs in Table 1 of Carrasco *et al.* (2008).



(a) (b)

Figure 7.2: For the serum-reversal data: (a) HF presented in Carrasco *et al.* (2008), (b) Nonparametric HF and the HF for the fitted GMW distribution using estimates in Table 1 of Carrasco *et al.* (2008).

7.2.2 Radiotherapy data

The fitted MLEs for this data are presented in Table 2 of Carrasco *et al.* (2008). Unfortunately, the MLEs for the GMW and MW distributions and the corresponding AIC, BIC and CAIC measures appear incorrect. We now explain the mistakes.

Modified Weibull distribution

The MLEs of the parameters of the MW distribution reported in Table 2 of Carrasco *et al.* (2008) are $\widehat{\alpha} = 0.001$, $\widehat{\gamma} = 1.245$ and $\widehat{\lambda} = 0.001$. But the reported values of AIC $= 594.4$, BIC $= 600.1$ and CAIC $= 594.9$ appear to have been computed using $\lambda = -0.001$ (an invalid value for λ). These values appear so close to the values of AIC, BIC and CAIC reported in Table 2 of Carrasco *et al.* (2008) for the GMW distribution. But the shape of the HF of the MW distribution can not be unimodal, so it is surprising that the MW and GMW distributions fit equally well for a data set exhibiting a unimodal HF.

For the MLEs of the MW distribution reported in Table 2 of Carrasco *et al.* (2008), the log-likelihood is -442.45. The log-likelihood is about -294.2 when $\widehat{\lambda} \approx 0.001$.

We computed the MLEs of the MW distribution and the corresponding measures AIC, BIC and CAIC. Table 1 shows the MLEs of the MW distribution reported by Carrasco *et al.* (2008) (when $\widehat{\lambda} = -0.001$ and when $\widehat{\lambda} = 0.001$), the MLEs we obtained (the corresponding standard errors in brackets) and the values of AIC, BIC and CAIC we obtained. The SFs for the fitted MW distribution using the MLEs in Table 2 of Carrasco *et al.* (2008) are shown in Figure 7.3a (red solid line for $\widehat{\lambda} = -0.001$ and blue dashed line for $\widehat{\lambda} = 0.001$). The corresponding HFs are plotted in Figure 7.3c (red solid line for $\widehat{\lambda} = -0.001$ and blue open circles for $\widehat{\lambda} = 0.001$). Figures 7.3b and 7.3d show the SF and the HF for the MW distribution we fitted.

Note from Table 1 that the GMW distribution we fitted is actually a EW distribution since the MLE of λ is zero (that is, the likelihood for the GMW distribution for the given data appears largest when $\lambda = 0$). Also the MW distribution we fitted is actually a Weibull distribution since the MLE of λ is zero (that is, the likelihood for the MW distribution for the given data appears largest when $\lambda = 0$). So, the added parameter λ does not improve the fit of the GW distribution or the fit of the Weibull distribution. This can happen sometimes when the parameter is restricted to be positive (Liddle, 2004).

Generalized modified Weibull distribution

According to Table 2 in Carrasco *et al.* (2008), the MLE of λ is 0.0002. But Figures 5b and 5c in Carrasco *et al.* (2008) appear to have used $\widehat{\lambda} = -0.0002$ (an invalid value for λ) to plot the SF and the HF for the fitted GMW distribution. Furthermore, the reported AIC, BIC and CAIC measures appear to have used the same negative value.

Figure 7.4a shows the SF for the fitted GMW distribution using the MLEs in Table 2 of Carrasco *et al.* (2008) (red solid line for $\widehat{\lambda} = -0.0002$ and blue dashed line for $\widehat{\lambda} = 0.0002$). The corresponding HFs are plotted in Figure 7.4c (red solid line for $\widehat{\lambda} = -0.0002$ and blue open circles for $\widehat{\lambda} = 0.0002$). The SF and the HF for the fitted GMW distribution with $\widehat{\lambda} = -0.0002$ (in red solid line and blue open circles) appear to be the same as Figures 5b and 5c in Carrasco *et al.* (2008).

We computed the MLEs of the GMW distribution and the corresponding measures

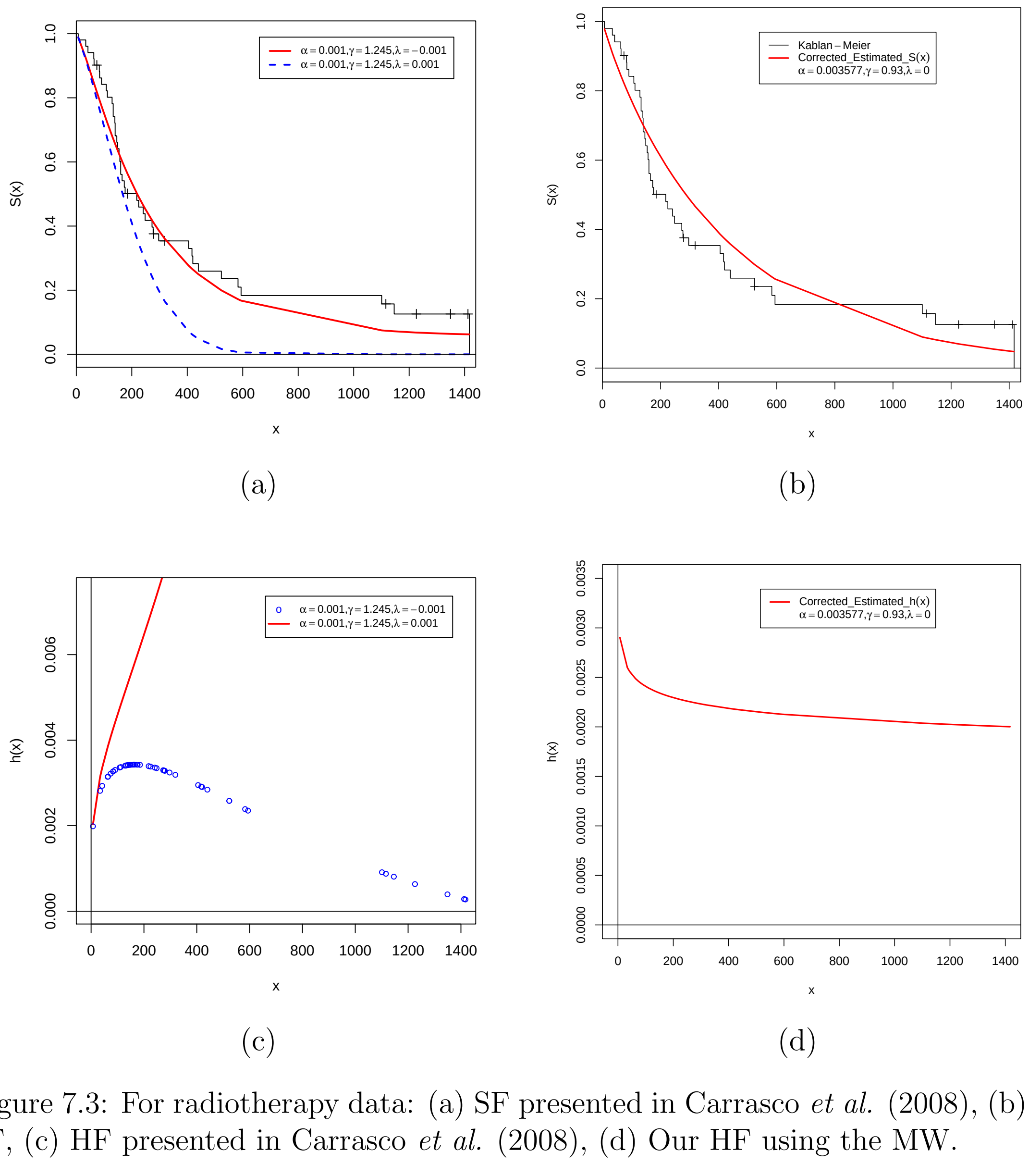


Figure 7.3: For radiotherapy data: (a) SF presented in Carrasco *et al.* (2008), (b) Our SF, (c) HF presented in Carrasco *et al.* (2008), (d) Our HF using the MW.

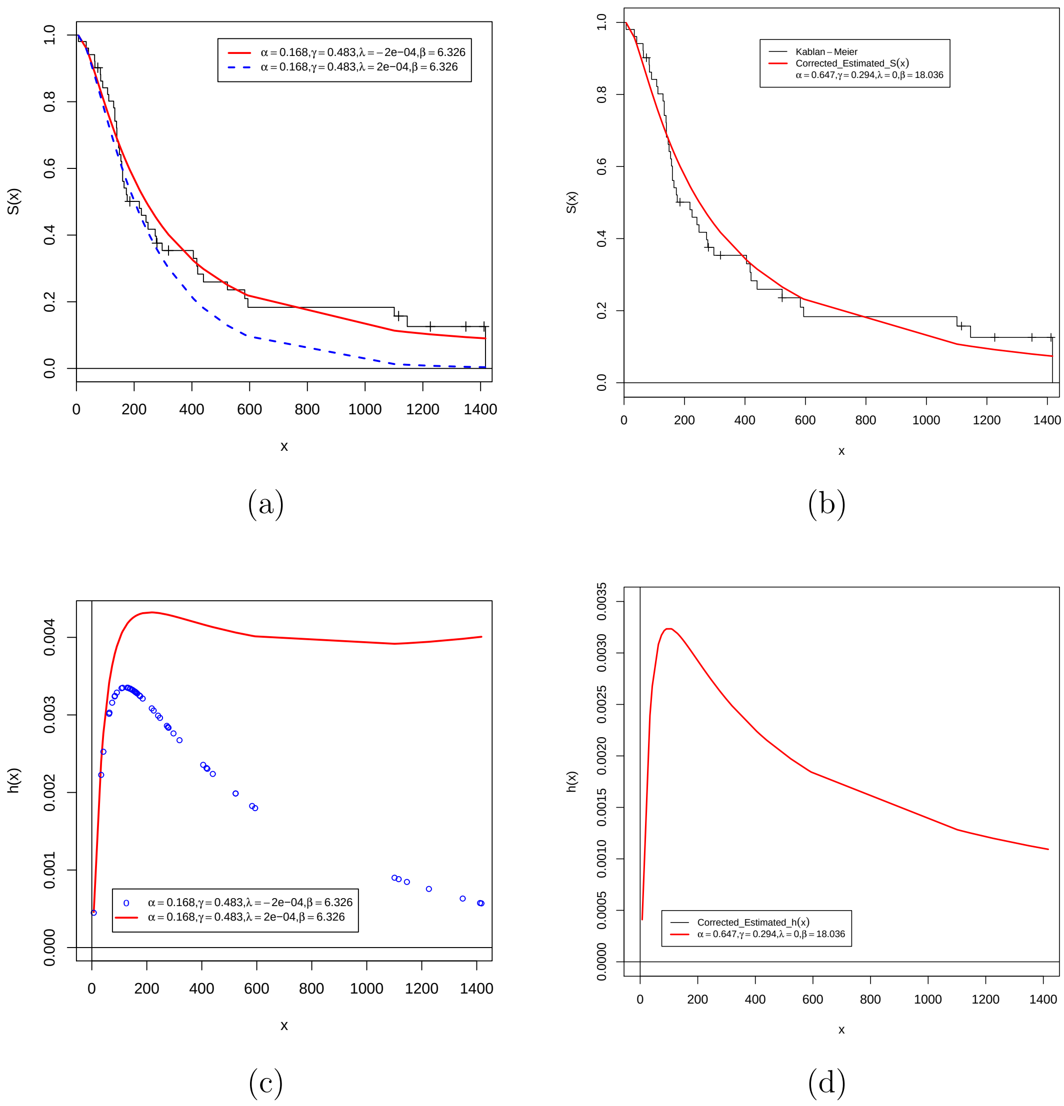


Figure 7.4: For the radiotherapy data: (a) SF presented in Carrasco *et al.* (2008), (b) Our SF, (c) HF presented in Carrasco *et al.* (2008), (d) Our HF using the GMW.

Table 7.1: MLEs of parameters, standard errors, AIC, BIC, AICc and CAIC for radiotherapy data.

Model	$\widehat{\beta}$	$\widehat{\gamma}$	$\widehat{\lambda}$	$\widehat{\alpha}$	AIC	BIC	AICc	CAIC	K-S
GMW of	6.326	0.483	-0.0002	0.168	593.3	601.0	594.1	605.0	0.114
Carrasco *et al.* (2008)	6.326	0.483	0.0002	0.168	611.5	619.3	612.4	623.2	0.170
	(2.662)	(0.297)	(0.0001)	(0.049)	–	–	–	–	–
Our GMW	18.036	0.294	0.000	0.647	594.1	601.8	595.0	605.8	0.119
	(32.480)	(0.165)	(1.593×10^{-4})	(0.934)	–	–	–	–	–
MW of	1	1.245	-0.001	0.001	594.4	600.1	594.9	603.1	0.130
Carrasco *et al.* (2008)	1	1.245	0.001	0.001	890.9	896.7	891.4	899.7	0.259
	–	(0.181)	(0.0002)	(0.0001)	–	–	–	–	–
Our MW	1	0.930	0.000	3.577×10^{-3}	599.8	605.6	600.3	608.63	0.143
	–	(0.171)	(1.277×10^{-4})	(3.345×10^{-2})	–	–	–	–	–

AIC, BIC and CAIC. Table 1 shows the MLEs of the GMW distribution computed by Carrasco *et al.* (2008) (when $\widehat{\lambda} = -0.0002$ and when $\widehat{\lambda} = 0.0002$), the MLEs we obtained (the corresponding standard errors in brackets) and the values of AIC, BIC and CAIC we obtained. The SF and the HF for the GMW distribution we fitted are plotted in Figures 7.4b and 7.4d.

7.2.3 Infected pigs data

As shown before, the infected pigs data that presented in Table 5.1 is shown to have modified unimodal shaped.

Table 7.2 shows the MLEs of the parameters, their standard errors, AIC values, BIC values, AICc values and CAIC values for the fitted GMW and EW distributions. Table 7.3 shows the K-S test statistics for the two fitted distributions.

The negative log-likelihood for the fitted GMW distribution is 400.350. The negative log-likelihood for the fitted EW distribution is 400.427. AIC values, BIC values, AICc values and CAIC values for the fitted GMW and EW distributions are 808.7, 817.8, 809.2, 821.8 and 806.9, 813.7, 807.2, 816.7, respectively. The values of the K-S statistic for the two distributions are 0.114 and 0.11.

The EW distribution has the smallest values for the AIC, the BIC the AICc, the CAIC and the K-S. The GMW distribution has the larger log-likelihood. But the likelihood ratio test statistic for testing $H_0 : \lambda = 0$ versus $H_1 : H_0$ is false is 0.154 and the corresponding p-value is 0.695, so there is no evidence to reject H_0. Hence, the GMW distribution does not improve significantly on the fit of the EW distribution.

Figure 7.5 (a) shows the histogram of the data and the fitted PDFs. Figure 7.5

Table 7.2: MLEs of parameters, standard errors, AIC, BIC, AICc and CAIC for the distributions fitted to the infected pigs data set.

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\gamma}$	$\widehat{\lambda}$	$\widehat{\theta}$	AIC	BIC	AICc	CAIC
GMW	–	0.496	0.470	4.639×10^{-9}	69.877	808.7	817.8	809.3	821.8
	–	(0.247)	(0.008)	4.573×10^{-4}	(50.012)				
EW	–	0.471	0.478	0	65.135	806.9	813.7	807.2	816.7
	–	(0.254)	0.081	–	(49.625)				

Table 7.3: K-S statistics for the distributions fitted to the infected pigs data set.

Model	K-S
GMW	0.114
EW	0.110

(b) shows the empirical SF of the data and the fitted ones. Figure 7.5 (d) shows the nonparametric HF of the data and the fitted HFs.

Figures 7.5a and 7.5b show that both distributions provide good fits. Figure 7.5d shows that both distributions provide good fits to the first and middle parts of the nonparametric HF. But neither of the distributions appear to capture the last part of the nonparametric HF well.

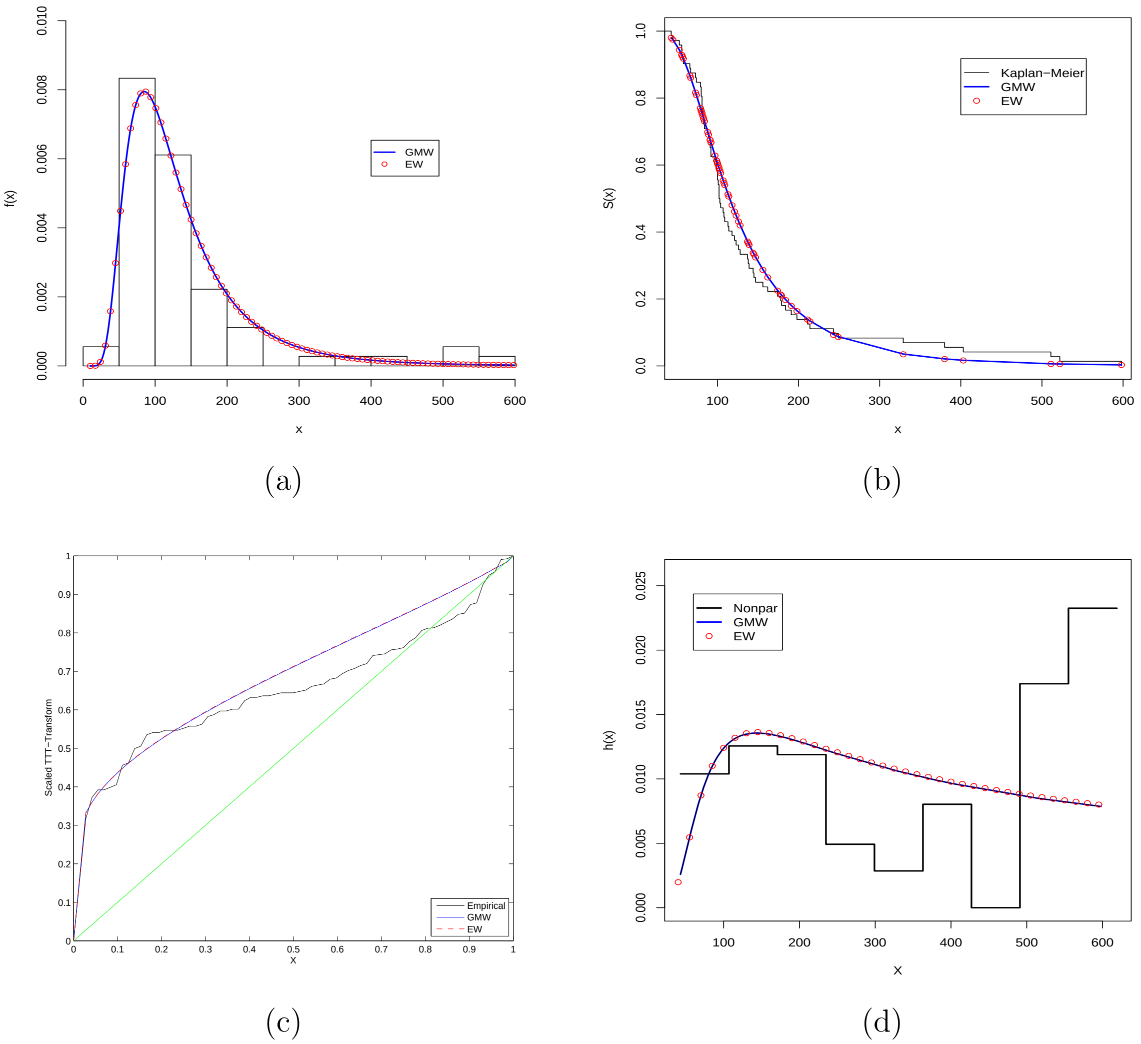


Figure 7.5: For infected pigs data: (a) Histogram and fitted PDFs; (b) Empirical and fitted SFs; (c) Empirical and fitted TTT-transforms; (d) Nonparametric and fitted HFs.

7.3 Summary

This chapter finds no evidence that the generalized modified Weibull distribution can provide a better fit than the exponentiated Weibull distribution for data sets exhibiting the modified unimodal hazard function. Also, it pointed out some of the incorrect results of the published paper of the GMW that proposed by Carrasco *et al.* (2008), and published by Computational Statistics and Data Analysis. The Figure of estimated hazard of Serum-reversal data set using GMW is incorrect because it is plotted using negative value of λ which should be restricted to positive value. In the radiotherapy data set, although Carrasco *et al.* (2008) reported positive value for the parameter λ, they used a negative value to plot the fitted SF and HF for the GMW and MW. Also, the AIC, BIC and AICc are calculated using the same negative value. The correct results and figures are presented with the estimated hazard rate function for the serum-reversal data. The added parameter λ to the Weibull distribution by Lai *et al.* (2003) did not improve the maximum likelihood function of the radiotherapy data for EW and the Weibull as sub-models of the GMW and MW respectively.

Chapter 8

Conclusion, discussion and future work

8.1 Conclusion

Chapter 2 reviewed known discrete and continuous modifications of the two-parameter Weibull distribution. For each modification, we have given expressions for the CDF, the PMF or the PDF and the HF. We have also discussed their shapes.

A new distribution, based on the Weibull and the modified Weibull distributions, has been proposed and its properties studied. The idea is to combine two components in a serial system, so that the hazard function is either increasing or more importantly, bathtub shaped. By using a modified Weibull component, the distribution has flexibility to model the second peak in a distribution. We have shown that the new modified Weibull distribution fits certain well-known data sets better than existing modifications of the Weibull distribution. Reducing the number of parameters to four by fixing one of the parameters still provides a better fit than existing models.

The new modified Weibull distribution introduced in Chapter 3 has been simplified with its five parameters reduced to three. The simplified distribution has been referred to as the RNMW distribution. We have studied several analytical properties of the reduced distribution and shown that it is a tractable distribution. We have also shown that the reduced distribution provides excellent fits to four real data sets: two of them are complete data sets and the other two are censored. By means of the likelihood ratio test, we have shown that the fit of the NMW distribution is not significantly better than that of the RNMW distribution. So, the RNMW distribution retains the same flexibility of the NMW distribution and yet the estimation for the former is much easier.

In chapter 6, We have generalized the reduced distribution, that presented in chapter 4, by exponentiation to accommodate unimodal and modified unimodal hazard function. The generalized distribution is referred to as the ERMW distribution. It includes as special cases several distributions like the RGMW, REW, RW, RLMW and RW distributions. We have discussed maximum likelihood estimation of the exponentiated distribution and presented two real data applications. One of the data sets has a unimodal HRF and the other has a bathtub shaped HRF. The exponentiated distribution was shown to give the best fit for both data sets.

We have introduced a three-parameter modification of the discrete Weibull distribution. We have shown that this distribution exhibits bathtub shaped hazard rates. There are only two other discrete distributions that exhibit bathtub shaped hazard rates. The flexibility of the proposed modification is illustrated using four real data sets: three of them complete and the other censored. For each data set, the proposed modification was shown to give better fit than several other competitors including the two known discrete distributions exhibiting bathtub shaped hazard rates.

Finally, we showed that the added parameter λ of the GMW distribution over the EW distribution did not improve the maximum likelihood function for the radiotherapy data. Also, the GMW distribution did not provide a better fit than the EW distribution for the infected pigs data. Both data sets have modified unimodal shaped HFs. Based on this, there is no evidence that the GMW distribution can provide a better fit than the EW distribution for data sets exhibiting modified unimodal HFs. We have pointed out some incorrect results in Carrasco *et al.* (2008). The fitted HF for the serum-reversal data set using the GMW distribution is incorrect because it is plotted using a negative value of λ. For the radiotherapy data set, although Carrasco *et al.* (2008) reported a positive value for λ, they used a negative value to plot the fitted SF and HF. Also, the AIC, the BIC, the AICc and the CAIC were calculated using the same negative value.

8.2 Discussion and future work

Some important questions with respect to the review part are: Which of the modifications is the "best" for a given data set? How do we select the "best" modification? Two common tools for selection among the modifications are the Weibull Probability Plot (WPP) and the Inverse Weibull Probability Plot (IWPP).

Let $t_1, t_2, \ldots, t_n$ denote a real data set. Let $t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n)}$ denote the sorted values in ascending order. Then, the WPP (for n not small) is a plot of

$$y_i = \log\left[-\log\left(1 - \frac{i}{n+1}\right)\right]$$

versus

$$x_i = \log\left(t_{(i)}\right)$$

for $i = 1, 2, \ldots, n$. The WPP for n small is a plot of

$$y_i = \log\left[-\log\left(1 - \frac{i - 0.3}{n + 0.4}\right)\right]$$

versus

$$x_i = \log\left(t_{(i)}\right)$$

for $i = 1, 2, \ldots, n$.

The IWPP for n not small is a plot of

$$y_i = \log\left[-\log\left(\frac{i}{n+1}\right)\right]$$

versus

$$x_i = \log\left(t_{(i)}\right)$$

for $i = 1, 2, \ldots, n$. The IWPP for n small is a plot of

$$y_i = \log\left[-\log\left(\frac{i - 0.3}{n + 0.4}\right)\right]$$

versus

$$x_i = \log\left(t_{(i)}\right)$$

for $i = 1, 2, \ldots, n$.

If the data are censored or grouped then appropriate changes should be made to the WPP and the IWPP. Sometimes changes are made to the plots depending on whether n is small, medium or large (Section 4.2, Murthy *et al.*, 2004). For details, we refer the readers to King (1971), Lawless (1982), Nelson (1982), and Meeker and Escobar (1998).

The WPP and the IWPP can exhibit a variety of shapes, including straight line, concave, concave with left asymptote vertical, convex, convex with right asymptote vertical, S shaped with parallel asymptotes, S shaped with vertical asymptotes, bell shaped and multiple inflection points, see Table 6 in Murthy *et al.* (2004).

Table 7 in Murthy *et al.* (2004) and Table 2 in Zhang and Dwight (2013) show how an appropriate modification can be chosen based on the shapes of the WPP and the IWPP. For example, a straight line in the WPP would mean appropriateness of the two-parameter Weibull distribution in (1.2). A convex shape of the WPP would mean appropriateness of the modified Weibull distribution in (2.7) due to Lai *et al.* (2003). An S shape of the WPP with vertical asymptotes would mean appropriateness of Kies (1958)'s modified Weibull distribution in (2.2). A concave shape of the WPP for $\theta < 1$ and a convex shape of the WPP for $\theta > 1$ would mean appropriateness of the extended Weibull distribution in (2.4) due to Zhang and Xie (2007). A convex shape of the WPP for $\lambda < 1$ and a concave shape of the WPP for $\lambda > 1$ would mean appropriateness of the exponentiated Weibull distribution in (2.2) due to Mudholkar and Srivastava (1993). An S shape of the WPP with parallel asymptotes would mean appropriateness of a mixture of two two-parameter Weibull distributions with the same shape parameter. A WPP with multiple inflection points would mean appropriateness of a mixture of two two-parameter Weibull distributions with different shape parameters. A straight line in the IWPP would mean appropriateness of the inverse Weibull distribution in (2.1). An S shape of the IWPP with parallel asymptotes would mean appropriateness of a mixture of two two-parameter inverse Weibull distributions with the same shape parameter. An IWPP with multiple inflection points would mean appropriateness of a mixture of two two-parameter inverse Weibull distributions with different shape parameters. For more on how a modification can be chosen based on the shapes of the WPP and the IWPP, we refer the readers to Jiang and Murthy (1995), Jiang *et al.* (2001a, 2001b), and Murthy *et al.* (2003).

It is possible that the WPP and the IWPP would imply appropriateness more than one modification. In this case, discrimination among the modifications can be based on the difference between the fitted and empirical estimates of the x-intercept of the WPP (Section 6.1, Murthy *et al.*, 2004), the difference between the fitted and empirical estimates of the y-intercept of the WPP (Section 6.1, Murthy *et al.*, 2004), the bootstrap and jackknife approaches as described in Section 6.2.1 of Murthy *et al.* (2004), the analysis of physics of failure as described in Section 4.1 of Zhang and Dwight (2013), the shape of the PDF, the shape of the PMF or the shape of the HF. The discrimination can also be performed using criteria like the sum of squares of residuals, the Akaike information criterion (Akaike, 1974), the Bayesian information criterion (Schwarz, 1978), the Hannan-Quinn information criterion (Hannan and Quinn, 1979) and the consistent Akaike information criterion (Hurvich and Tsai, 1989). The modification with smaller values of these criteria should be preferred.

A future work is to discuss possible applications of these modifications especially to reliability engineering. Other possible future works are to: i) develop tools other than the WPP and the IWPP for choosing among the modifications of the Weibull distribution; ii) provide a review of known modifications of other distributions commonly used in reliability, including the binomial, geometric, exponential, hypoexponential, hyperexponential, Gompertz-Makeham, Birnbaum-Saunders, gamma, Erlang, normal, lognormal, loglogistic, Pareto, Gumbel, Fréchet, and extreme value distributions; iii) provide a review of bivariate and multivariate versions of distributions commonly used in reliability; iv) provide a review of component mixture, one- or two-sided censored, truncated and transformed distributions commonly used in reliability; v) provide a review of reliability block diagram-based lifetime, fault tree-based lifetime and standby lifetime distributions commonly used in reliability.

The NMW distribution is a kind of competing risks model by considering two different distributions. In this case we can defined a new family of the competing risk models to be known as *Modified Weibull Competing Risks Model.*

Definition 8.2.1. *Modified Weibull competing risks model: Suppose* $X_1, X_2, \ldots, X_r$ *are independent random variables with a cumulative distribution functions* $F_i(x)$, $i = 1, 2 \ldots, r$, *where* $F_i(x)$ *are either the Weibull distribution or one of its modifications or related distributions. If* $Z = \min(X_1, \ldots, X_r)$ *then,* *the cumulative distribution functions of* Z *is given by*

$$F(x) = 1 - \prod_{i=0}^{r} \left[1 - F_i(x)\right], \tag{8.1}$$

the probability density function is given by

$$f(x) = \sum_{i=0}^{r} \left[h_i(x)\right] \prod_{i=0}^{r} S_i(x), \tag{8.2}$$

where $S_i(x) = 1 - F_i(x)$ *for* $i = 1, 2 \ldots, r$, *is the survival function,*
the the hazard function is given by

$$h(x) = \sum_{i=0}^{r} \left[h_i(x)\right]. \tag{8.3}$$

If $r = 2$, the above definition includes the NMW distribution, that was presented in chapter 3, and several other known distribution such as:

1. **The linear failure rate distribution**:
   The LFR involving exponential and Rayleigh distributions and its CDF is given by,

   $$F(x) = 1 - e^{-\alpha x - \beta x^2}, \quad x > 0 \tag{8.4}$$

2. **The additive Weibull model (AddW) of Xie and Lai (1996)**:
   That involves two Weibull distributions, with the following CDFk,

   $$F(x) = 1 - e^{-\alpha x^{\theta} - \beta x^{\gamma}}, \quad x > 0 \tag{8.5}$$

3. **The β distribution of Bousquet** *et al.* **(2006)**:
   The β distribution includes the exponential and the Weibull distributions,

   $$F(x) = 1 - e^{-\alpha x - \beta x^{\gamma}}, \quad x > 0 \tag{8.6}$$

   Sarhan and Zaindin (2009) introduced a three-parameter distribution called as modified Weibull (SZMW) distribution. It is the same as the β distribution of Bousquet *et al.* (2006).

A future work is to study the modified Weibull competing risks model further.

The reduced distribution has an exclusive bathtub shaped hazard rate function. Other hazard rates can be obtained from the NMW distribution. For example, setting $\gamma = \theta = 2$ we obtain

$$h(x) = 2\alpha x + \beta(2 + \lambda x)xe^{\lambda x},$$

for $x > 0$, which is an increasing function of x, see Figure 8.1. Also, a bivariate version of the reduced distribution can be obtained.

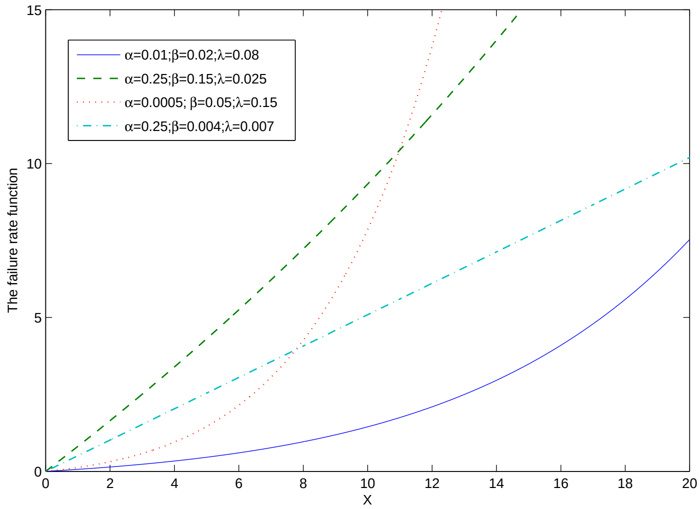


Figure 8.1: Hazard rate functions of the reduced distribution.

Future work includes MCMC methods with censored data, regression problems with covariates and parameter reduction. The drug data can be used in the regression problems. Alternative data which can be used as a lifetime data is that of telecom company data. One of largest mobile telecommunication network company in the Middle East is Saudi Telecom Company (STC). The mobile towers systems can be either serial or parallel systems. Our new lifetime models can be used to describe to fit failures times of the the serial system.

Progressively censored data is a type of censored data where after d_1 failures n_1 subjects are removed from the study then after d_2 failures n_2 removed from the study, until the end of the study. Based on progressively censored data the maximum likelihood estimates of parameters of the modified Weibull distributions can be computed using EM algorithm and Newton-Raphson algorithm.

A simulation study could be conducted in order to identify the appropriate operating conditions of the proposed modifications in terms of the sample size and proportion of censoring for example. We could also compare the proposed models and alternative ones on simulated data.

Sometimes extrapolation is more important than finding a best fitting model for the range of the data. So, investigation could also be done to see if the proposed distributions can provide reliable extrapolations.

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Appendix A

Appendix for chapter 3

The log-likelihood function of the $NMW(\alpha, \beta, \theta, \gamma, \lambda)$ can be written as

$$\mathcal{L}(\underline{\vartheta}) = \sum_{i=1}^{n} \left[\ln\left(h(x_i; \underline{\vartheta})\right) - \alpha x_i^{\theta} - \beta x_i^{\gamma} e^{\lambda x} \right],$$

where $h(x_i; \underline{\vartheta})$ is the hazard rate function (3.4) of the NMW and $\underline{\vartheta} = (\alpha, \beta, \theta, \gamma, \lambda)$ is the vector of parameters. The second partial derivatives are as follows

$$\begin{aligned}
\mathcal{L}_{\alpha\alpha} &= -\sum_{i=1}^{n} \left(\frac{h_{\alpha}(x_i; \underline{\vartheta})}{h(x_i; \underline{\vartheta})} \right)^2, \\
\mathcal{L}_{\alpha\beta} &= -\sum_{i=1}^{n} \frac{h_{\alpha}(x_i; \underline{\vartheta}) h_{\beta}(x_i; \underline{\vartheta})}{\left(h(x_i; \underline{\vartheta})\right)^2}, \\
\mathcal{L}_{\alpha\theta} &= \sum_{i=1}^{n} \left(\frac{h(x_i; \underline{\vartheta}) h_{\alpha\theta}(x_i; \underline{\vartheta}) - h_{\alpha}(x_i; \underline{\vartheta}) h_{\theta}(x_i; \underline{\vartheta})}{\left(h(x_i; \underline{\vartheta})\right)^2} - x_i^{\theta} \ln(x_i) \right), \\
\mathcal{L}_{\alpha\gamma} &= -\sum_{i=1}^{n} \frac{h_{\alpha}(x_i; \underline{\vartheta}) h_{\gamma}(x_i; \underline{\vartheta})}{\left(h(x_i; \underline{\vartheta})\right)^2}, \\
\mathcal{L}_{\alpha\lambda} &= -\sum_{i=1}^{n} \frac{h_{\alpha}(x_i; \underline{\vartheta}) h_{\lambda}(x_i; \underline{\vartheta})}{\left(h(x_i; \underline{\vartheta})\right)^2}, \\
\mathcal{L}_{\beta\beta} &= -\sum_{i=1}^{n} \left(\frac{h_{\beta\gamma}(x_i; \underline{\vartheta})}{h(x_i; \underline{\vartheta})} \right)^2, \\
\mathcal{L}_{\beta\theta} &= -\sum_{i=1}^{n} \frac{h_{\beta}(x_i; \underline{\vartheta}) h_{\theta}(x_i; \underline{\vartheta})}{\left(h(x_i; \underline{\vartheta})\right)^2}, \\
\mathcal{L}_{\beta\gamma} &= \sum_{i=1}^{n} \left(\frac{h(x_i; \underline{\vartheta}) h_{\beta\gamma}(x_i; \underline{\vartheta}) - h_{\beta}(x_i; \underline{\vartheta}) h_{\gamma}(x_i; \underline{\vartheta})}{\left(h(x_i; \underline{\vartheta})\right)^2} - x_i^{\gamma} e^{\lambda x_i} \ln(x_i) \right),
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{\beta\lambda} &= \sum_{i=1}^{n}\left(\frac{h(x_i;\underline{\vartheta})h_{\beta\lambda}(x_i;\underline{\vartheta})-h_{\beta}(x_i;\underline{\vartheta})h_{\lambda}(x_i;\underline{\phi})}{\left(h(x_i;\underline{\vartheta})\right)^2}-x_i^{\gamma+1}e^{\lambda x_i}\right),\\
\mathcal{L}_{\theta\theta} &= \sum_{i=1}^{n}\left(\frac{h(x_i;\underline{\vartheta})h_{\theta\theta}(x_i;\underline{\vartheta})-\left(h_{\theta}(x_i;\underline{\vartheta})\right)^2}{\left(h(x_i;\underline{\vartheta})\right)^2}-\alpha x_i^{\theta}\ln^2(x_i)\right),\\
\mathcal{L}_{\theta\gamma} &= -\sum_{i=1}^{n}\frac{h_{\theta}(x_i;\underline{\vartheta})h_{\gamma}(x_i;\underline{\vartheta})}{\left(h(x_i;\underline{\vartheta})\right)^2},\\
\mathcal{L}_{\theta\lambda} &= -\sum_{i=1}^{n}\frac{h_{\theta}(x_i;\underline{\vartheta})h_{\lambda}(x_i;\underline{\vartheta})}{\left(h(x_i;\underline{\vartheta})\right)^2},
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{\gamma\gamma} &= \sum_{i=1}^{n}\left(\frac{h(x_i;\underline{\vartheta})h_{\gamma\gamma}(x_i;\underline{\vartheta})-\left(h_{\gamma}(x_i;\underline{\vartheta})\right)^2}{\left(h(x_i;\underline{\vartheta})\right)^2}-\beta x_i^{\gamma}e^{\lambda x_i}\ln^2(x_i)\right),\\
\mathcal{L}_{\gamma\lambda} &= \sum_{i=1}^{n}\left(\frac{h(x_i;\underline{\vartheta})h_{\gamma\lambda}(x_i;\underline{\vartheta})-h_{\gamma}(x_i;\underline{\vartheta})h_{\lambda}(x_i;\underline{\vartheta})}{\left(h(x_i;\underline{\vartheta})\right)^2}-\beta x_i^{\gamma+1}e^{\lambda x_i}\ln^2(x_i)\right),\\
\mathcal{L}_{\lambda\lambda} &= \sum_{i=1}^{d}\left(\frac{h(x_i;\underline{\vartheta})h_{\lambda\lambda}(x_i;\underline{\vartheta})-\left(h_{\lambda}(x_i;\underline{\vartheta})\right)^2}{\left(h(x_i;\underline{\vartheta})\right)^2}-\beta x_i^{\gamma+2}e^{\lambda x_i}\right),
\end{aligned}$$

where

$$\begin{aligned}
h_{\alpha\theta}(x_i;\underline{\vartheta}) &= x_i^{\theta-1}(1+\theta\ln(x_i)),\\
h_{\beta\gamma}(x_i;\underline{\vartheta}) &= x_i^{\gamma-1}\left(1+(\gamma+\lambda x_i)\ln(x_i)\right)e^{\lambda x_i},\\
h_{\beta\lambda}(x_i;\underline{\vartheta}) &= x_i^{\gamma}\left(1+\gamma+\lambda x_i\right)\ln(x_i))\,e^{\lambda x_i},\\
h_{\theta\theta}(x_i;\underline{\vartheta}) &= \alpha x_i^{\theta-1}\left(2+\theta\ln(x_i)\right)\ln(x_i),\\
h_{\gamma\gamma}(x_i;\underline{\vartheta}) &= \beta x_i^{\gamma-1}\left(2+(\gamma+\lambda x_i)\ln(x_i)\right)\ln(x_i)e^{\lambda x_i},\\
h_{\gamma\lambda}(x_i;\underline{\vartheta}) &= \beta x_i^{\gamma}\left(1+(1+\gamma+\lambda x_i)\ln(x_i)\right)e^{\lambda x_i},\\
h_{\lambda\lambda}(x_i;\underline{\vartheta}) &= \beta x_i^{\gamma+1}\left(2+\gamma+\lambda x_i\right)\ln(x_i))\,e^{\lambda x_i},\\
h_{\alpha}(x_i;\underline{\vartheta}) &= \theta x_i^{\theta-1},\\
h_{\beta}(x_i;\underline{\vartheta}) &= x_i^{\gamma-1}(\gamma+\lambda x_i)\ln(x_i)e^{\lambda x_i},\\
h_{\theta}(x_i;\underline{\vartheta}) &= \alpha h_{\alpha\theta}(x_i;\underline{\vartheta}),\\
h_{\gamma}(x_i;\underline{\vartheta}) &= \beta h_{\beta\gamma}(x_i;\underline{\vartheta}),\\
h_{\lambda}(x_i;\underline{\vartheta}) &= \beta h_{\beta\lambda}(x_i;\underline{\vartheta}).
\end{aligned}$$

Appendix B

Appendix for chapter 4

The observed information matrix for complete data

The elements of the observed information matrix $J(\underline{\vartheta})$ for the $RNMW(\alpha,\beta,\lambda)$ for complete data are

$$\mathcal{L}_{\alpha\alpha} = -\sum_{i=1}^{d}\left[\frac{h_{\alpha}\left(x_i;\underline{\vartheta}\right)}{h\left(x_i;\underline{\vartheta}\right)}\right]^2,$$

$$\mathcal{L}_{\alpha\beta} = -\sum_{i=1}^{d}\frac{h_{\alpha}\left(x_i;\underline{\vartheta}\right)h_{\beta}\left(x_i;\underline{\vartheta}\right)}{h\left(x_i;\underline{\vartheta}\right)^2},$$

$$\mathcal{L}_{\alpha\lambda} = -\sum_{i=1}^{d}\frac{h_{\alpha}\left(x_i;\underline{\vartheta}\right)h_{\lambda}\left(x_i;\underline{\vartheta}\right)}{h\left(x_i;\underline{\vartheta}\right)^2},$$

$$\mathcal{L}_{\beta\beta} = -\sum_{i=1}^{d}\left[\frac{h_{\beta}\left(x_i;\underline{\vartheta}\right)}{h\left(x_i;\underline{\vartheta}\right)}\right]^2,$$

$$\mathcal{L}_{\beta\lambda} = -\sum_{i=1}^{d}\frac{h\left(x_i;\underline{\vartheta}\right)h_{\beta\lambda}\left(x_i;\underline{\vartheta}\right)-h_{\beta}\left(x_i;\underline{\vartheta}\right)h_{\lambda}\left(x_i;\underline{\phi}\right)}{h\left(x_i;\underline{\vartheta}\right)^2},$$

$$\mathcal{L}_{\lambda\lambda} = \sum_{i=1}^{d}\left[\frac{h\left(x_i;\underline{\vartheta}\right)h_{\lambda\lambda}\left(x_i;\underline{\vartheta}\right)-h_{\lambda}\left(x_i;\underline{\vartheta}\right)^2}{h\left(x_i;\underline{\vartheta}\right)^2}-\beta\sqrt{x_i^5}e^{\lambda x_i}\right],$$

where

$$h_\alpha\left(x_i;\underline{\vartheta}\right)=\frac{1}{2\sqrt{x_i}},$$
$$h_\beta\left(x_i;\underline{\vartheta}\right)=\frac{\left(0.5+\lambda x_i\right)e^{\lambda x_i}}{\sqrt{x_i}},$$
$$h_\lambda\left(x_i;\underline{\vartheta}\right)=\beta\sqrt{x_i}\left(\frac{3}{2}+\lambda x_i\right)e^{\lambda x_i},$$
$$h_{\beta\lambda}\left(x_i;\underline{\vartheta}\right)=\sqrt{x_i}\left(\frac{3}{2}+\lambda x_i\right)e^{\lambda x_i},$$
$$h_{\lambda\lambda}\left(x_i;\underline{\vartheta}\right)=\beta\sqrt{x_i}\left(\frac{5}{2}+\lambda x_i\right)e^{\lambda x_i}.$$

The observed information matrix for complete data censored data

The elements of the observed information matrix $J(\underline{\vartheta})$ for the $RNMW(\alpha,\beta,\lambda)$ for censored data are

$$\mathcal{L}_{\alpha\alpha}=-\sum_{i=1}^{d}\left[\frac{h_\alpha\left(x_i;\underline{\vartheta}\right)}{h\left(x_i;\underline{\vartheta}\right)}\right]^2,$$
$$\mathcal{L}_{\alpha\beta}=-\sum_{i=1}^{d}\frac{h_\alpha\left(x_i;\underline{\vartheta}\right)h_\beta\left(x_i;\underline{\vartheta}\right)}{h\left(x_i;\underline{\vartheta}\right)^2},$$
$$\mathcal{L}_{\alpha\lambda}=-\sum_{i=1}^{d}\frac{h_\alpha\left(x_i;\underline{\vartheta}\right)h_\lambda\left(x_i;\underline{\vartheta}\right)}{h\left(x_i;\underline{\vartheta}\right)^2},$$
$$\mathcal{L}_{\beta\beta}=-\sum_{i=1}^{d}\left[\frac{h_\beta\left(x_i;\underline{\vartheta}\right)}{h\left(x_i;\underline{\vartheta}\right)}\right]^2,$$
$$\mathcal{L}_{\beta\lambda}=-\sum_{i=1}^{d}\frac{h\left(x_i;\underline{\vartheta}\right)h_{\beta\lambda}\left(x_i;\underline{\vartheta}\right)-h_\beta\left(x_i;\underline{\vartheta}\right)h_\lambda\left(x_i;\underline{\phi}\right)}{h\left(x_i;\underline{\vartheta}\right)^2}-\sum_{i\in C}\sqrt{x_i^3}e^{\lambda x_i},$$
$$\mathcal{L}_{\lambda\lambda}=\sum_{i=1}^{d}\left[\frac{h\left(x_i;\underline{\vartheta}\right)h_{\lambda\lambda}\left(x_i;\underline{\vartheta}\right)-h_\lambda\left(x_i;\underline{\vartheta}\right)^2}{h\left(x_i;\underline{\vartheta}\right)^2}-\beta\sqrt{x_i^5}e^{\lambda x_i}\right]$$
$$-\beta\sum_{i\in C}\sqrt{x_i^5}e^{\lambda x_i},$$

where

$$
\begin{aligned}
h_{\alpha}\left(x_i;\underline{\vartheta}\right) &= \frac{1}{2\sqrt{x_i}},\\
h_{\beta}\left(x_i;\underline{\vartheta}\right) &= \frac{\left(0.5+\lambda x_i\right)e^{\lambda x_i}}{\sqrt{x_i}},\\
h_{\lambda}\left(x_i;\underline{\vartheta}\right) &= \beta\sqrt{x_i}\left(\frac{3}{2}+\lambda x_i\right)e^{\lambda x_i},\\
h_{\beta\lambda}\left(x_i;\underline{\vartheta}\right) &= \sqrt{x_i}\left(\frac{3}{2}+\lambda x_i\right)e^{\lambda x_i},\\
h_{\lambda\lambda}\left(x_i;\underline{\vartheta}\right) &= \beta\sqrt{x_i^3}\left(\frac{5}{2}+\lambda x_i\right)e^{\lambda x_i}.
\end{aligned}
$$

Appendix C

Appendix for chapter 5

The observed information matrix

The elements of the observed information matrix $J(\underline{\vartheta})$ in Section 5.7.1 are

$$\mathcal{L}_{\alpha\alpha} = -\sum_{i=1}^{n}\left[\frac{h_{\alpha}\left(x_i;\underline{\vartheta}\right)}{h\left(x_i;\underline{\vartheta}\right)}\right]^2 + (1-\theta)\sum_{i=1}^{n}\frac{x_i e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}}{\left(e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}-1\right)^2},$$

$$\mathcal{L}_{\alpha\beta} = -\sum_{i=1}^{n}\frac{h_{\alpha}\left(x_i;\underline{\vartheta}\right)h_{\beta}\left(x_i;\underline{\vartheta}\right)}{h\left(x_i;\underline{\vartheta}\right)^2} + (1-\theta)\sum_{i=1}^{n}\frac{x_i e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}+\lambda x_i}}{\left(e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}-1\right)^2},$$

$$\mathcal{L}_{\alpha\lambda} = -\sum_{i=1}^{n}\frac{h_{\alpha}\left(x_i;\underline{\vartheta}\right)h_{\lambda}\left(x_i;\underline{\vartheta}\right)}{h\left(x_i;\underline{\vartheta}\right)^2} + (1-\theta)\sum_{i=1}^{n}\frac{\beta x_i^2 e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}+\lambda x_i}}{\left(e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}-1\right)^2},$$

$$\mathcal{L}_{\alpha\theta} = \sum_{i=1}^{n}\frac{\sqrt{x_i}}{e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}-1},$$

$$\mathcal{L}_{\beta\beta} = -\sum_{i=1}^{n}\left[\frac{h_{\beta}\left(x_i;\underline{\vartheta}\right)}{h\left(x_i;\underline{\vartheta}\right)}\right]^2 + (1-\theta)\sum_{i=1}^{n}\frac{x_i e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}+2\lambda x_i}}{\left(e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}-1\right)^2},$$

$$\begin{aligned}\mathcal{L}_{\beta\lambda} = &-\sum_{i=1}^{n}\frac{h\left(x_i;\underline{\vartheta}\right)h_{\beta\lambda}\left(x_i;\underline{\vartheta}\right)-h_{\beta}\left(x_i;\underline{\vartheta}\right)h_{\lambda}\left(x_i;\underline{\phi}\right)}{h\left(x_i;\underline{\vartheta}\right)^2} - \sum_{i=1}^{n}x_i e^{\lambda x_i}\\ &+(\theta-1)\sum_{i=1}^{n}\frac{\sqrt{x_i^3}e^{\lambda x_i}\left\{\left(1-\beta\sqrt{x_i}e^{\lambda x_i}\right)e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}-1\right\}}{\left(e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}-1\right)^2},\end{aligned}$$

$$\mathcal{L}_{\beta\theta} = \sum_{i=1}^{n}\frac{\sqrt{x_i}e^{\lambda x_i}}{e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}-1},$$

$$
\begin{aligned}
\mathcal{L}_{\lambda\lambda} &= \sum_{i=1}^{n}\left[\frac{h\left(x_i;\underline{\vartheta}\right)h_{\lambda\lambda}\left(x_i;\underline{\vartheta}\right)-h_{\lambda}\left(x_i;\underline{\vartheta}\right)^2}{h\left(x_i;\underline{\vartheta}\right)^2}-\beta\sqrt{x_i^5}e^{\lambda x_i}\right] \\
&\qquad +(\theta-1)\sum_{i=1}^{n}\frac{\sqrt{x_i^5}e^{\lambda x_i}\left\{\left(1-\beta\sqrt{x_i}e^{\lambda x_i}\right)e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}-1\right\}}{\left(e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}-1\right)^2}, \\
\mathcal{L}_{\lambda\theta} &= \sum_{i=1}^{n}\frac{\sqrt{x_i^3}e^{\lambda x_i}}{e^{\alpha\sqrt{x_i}+\beta\sqrt{x_i}e^{\lambda x_i}}-1}, \\
\mathcal{L}_{\theta\theta} &= -\frac{n}{\theta^2},
\end{aligned}
$$

where

$$
\begin{aligned}
h_{\alpha}\left(x_i;\underline{\vartheta}\right) &= \frac{1}{2\sqrt{x_i}}, \\
h_{\beta}\left(x_i;\underline{\vartheta}\right) &= \frac{\left(0.5+\lambda x_i\right)e^{\lambda x_i}}{\sqrt{x_i}}, \\
h_{\lambda}\left(x_i;\underline{\vartheta}\right) &= \beta\sqrt{x_i}\left(\frac{3}{2}+\lambda x_i\right)e^{\lambda x_i}, \\
h_{\beta\lambda}\left(x_i;\underline{\vartheta}\right) &= \sqrt{x_i}\left(\frac{3}{2}+\lambda x_i\right)e^{\lambda x_i}, \\
h_{\lambda\lambda}\left(x_i;\underline{\vartheta}\right) &= \beta\sqrt{x_i}\left(\frac{5}{2}+\lambda x_i\right)e^{\lambda x_i},
\end{aligned}
$$

and $h\left(x_i;\underline{\vartheta}\right)$ is the HRF of the RNMW distribution.

Appendix D

Appendix for chapter 6

The observed information matrix

The elements of the observed information matrix $J(\underline{\vartheta})$ for the DRMW are

$$\begin{aligned}
D_{qq}^{(i)} &= q^{-2}\left[A_{x_i}(b,c)\left(A_{x_i}(b,c)-1\right)q^{A_{x_i}(b,c)} - A_{x_i+1}(b,c)(A_{x_i+1}(b,c)-1)q^{A_{x_i+1}(b,c)}\right],\\
D_{qb}^{(i)} &= q^{-1}\left[A_{x_i}^{(b)}\left[1+A_{x_i}(b,c)\ln(q)\right]q^{A_{x_i}(b,c)} - A_{x_i+1}^{(b)}\left[1+A_{x_i+1}(b,c)\ln(q)\right]q^{A_{x_i+1}(b,c)}\right],\\
D_{qc}^{(i)} &= q^{-1}\left[A_{x_i}^{(c)}\left[1+A_{x_i}(b,c)\ln(q)\right]q^{A_{x_i}(b,c)} - A_{x_i+1}^{(c)}\left[1+A_{x_i+1}(b,c)\ln(q)\right]q^{A_{x_i+1}(b,c)}\right],\\
D_{bb}^{(i)} &= \ln^2(q)\left[\left(A_{x_i}^{(b)}\right)^2 q^{A_{x_i}(b,c)} - \left(A_{x_i+1}^{(b)}\right)^2 q^{A_{x_i+1}(b,c)}\right],\\
D_{bc}^{(i)} &= \ln(q)\left[\left(A_{x_i}^{(bc)} + A_{x_i}^{(b)}A_{x_i}^{(c)}\ln(q)\right)q^{A_{x_i}(b,c)} - \left(A_{x_i+1}^{(bc)} + A_{x_i+1}^{(b)}A_{x_i+1}^{(c)}\ln(q)\right)q^{A_{x_i+1}(b,c)}\right],\\
D_{cc}^{(i)} &= \ln(q)\left[\left(A_{x_i}^{(cc)} + \left(A_{x_i}^{(c)}\right)^2\ln(q)\right)q^{A_{x_i}(b,c)} - \left(A_{x_i+1}^{(cc)} + \left(A_{x_i+1}^{(c)}\right)^2\ln(q)\right)q^{A_{x_i+1}(b,c)}\right],
\end{aligned}$$

where $A_{x_i}^{(\phi_p\phi_k)}$ is the second order partial derivative of $A_{x_i}(b,c)$ with respect to ϕ_p and ϕ_k for $p,k=2,3$. Also, $A_{x_i}^{(bc)} = \sqrt[3]{x_i}c^{x_i-1}$ and $A_{x_i}^{(cc)} = b(x_i-1)\sqrt[3]{x_i}c^{x_i-2}$.