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Journal of Cleaner Production 7 (1999) 101–119

Journal of Cleaner Production

Allocation of environmental burdens in multiple-function systems

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Received 29 October 1997; received in revised form 29 September 1998; accepted 6 October 1998

Abstract

Allocation of environmental burdens is a recognised methodological problem in Life Cycle Assessment (LCA). It is the process of assigning to each of the functions of a multiple-function system only those environmental burdens and impacts that each function generates. It is argued in this paper that allocation is an artifact of applying LCA to individual products rather than to the whole productive system. To solve this problem, a new "marginal allocation" approach is proposed, based on whole system modelling. Marginal allocation is applicable when marginal changes about some defined state of the product system are to be considered and when the functional outputs can be varied independently. The specific approach developed here is based on representing the system by a model in the Linear Programming (LP) format. The allocation coefficients are equivalent to the marginal values calculated at the solution of the LP model. Marginal values represent a realistic description of the causal relationships between burdens and functional outputs and thus reflect the behaviour of the system. Changes in the system behaviour can also be modelled by LP. The approach is illustrated on three simple examples of multiple-function systems: combined waste treatment, co-production and recycling. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Allocation; Life Cycle Assessment; Linear Programming; Marginal Values; System Analysis

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Nomenclature

a_c	Input/output coefficients of a process or activity
b	Emissions of dioxin per tonne of waste
b_{max}	Maximum emission of dioxins per tonne of waste
b_H	Marginal change of dioxin emissions with H, for constant M, L, and T
b_M	Marginal change of dioxin emissions with M, for constant H, L, and T
b_L	Marginal change of dioxin emissions with L, for constant M, H, and T
b_T	Marginal change of dioxin emissions with T, for constant M, H, and L
$b_{j,l}$	Marginal allocated product-related burden
$b_{j,z}$	Burden allocation coefficient for a functional output y_z
$bc_{j,i}$	Environmental burden coefficients
B	Total emissions of dioxin
B_j	Environmental burden
e_c	Right hand side coefficients or parameters of the constraints
$ec_{k,j}$	Environmental impact coefficients
E_k	Environmental impact
f_i	Coefficients in the economic objective function
F	(Economic) Objective function
h	Chlorine fraction in waste
H	Total chlorine content in the waste
l	Specific calorific value of the waste
L	Total calorific value of the waste
M	Total mass of waste processed in the waste incinerator
P_l; P_2	Output of Product l; Product 2
Q	Heat requirement in the system
T	Combustion temperature in the incinerator
u_n	Material-(or product-) related parameter
$U_{j,n}$	Material-(or product-) related partial derivative
v_m	Process-related parameter
$V_{j,m}$	Process-related partial derivative
x_i	Output from a process or activity (operation level)
$\lambda_{j,c}$	Dual or marginal values relating burden j

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PII: S0959-6526(98)00046-8

to process or material properties

$\mu_{k,c}$ Dual or marginal values relating impact k to process or material properties

1. Introduction

Allocation of environmental burdens is one of the continuing methodological problems in Life Cycle Assessment (LCA). It refers to the problem of associating environmental burdens, such as resource depletion, emissions to air and water and solid waste, to each functional input or output of a multiple-function system. There are three types of multiple-function systems, as shown schematically in Fig. 1, where allocation of environmental burdens can be relevant:

1. Multiple-input systems (waste treatment processes),
2. Multiple-output systems (co-production), and
3. Multiple-use or "cascaded use" systems ("open-loop recycling").

In multiple-input systems, such as combined waste treatment processes, a number of different materials are treated in the same system. These input materials have different composition and therefore properties which determine the total environmental burdens from the system. The allocation problem in these systems is thus related to allocating the burdens between different inputs into the system. For example, if waste PVC is incinerated, the emissions of chlorinated organic compounds (including dioxins) depend not only on the input of PVC but also on other parameters, such as the calorific value

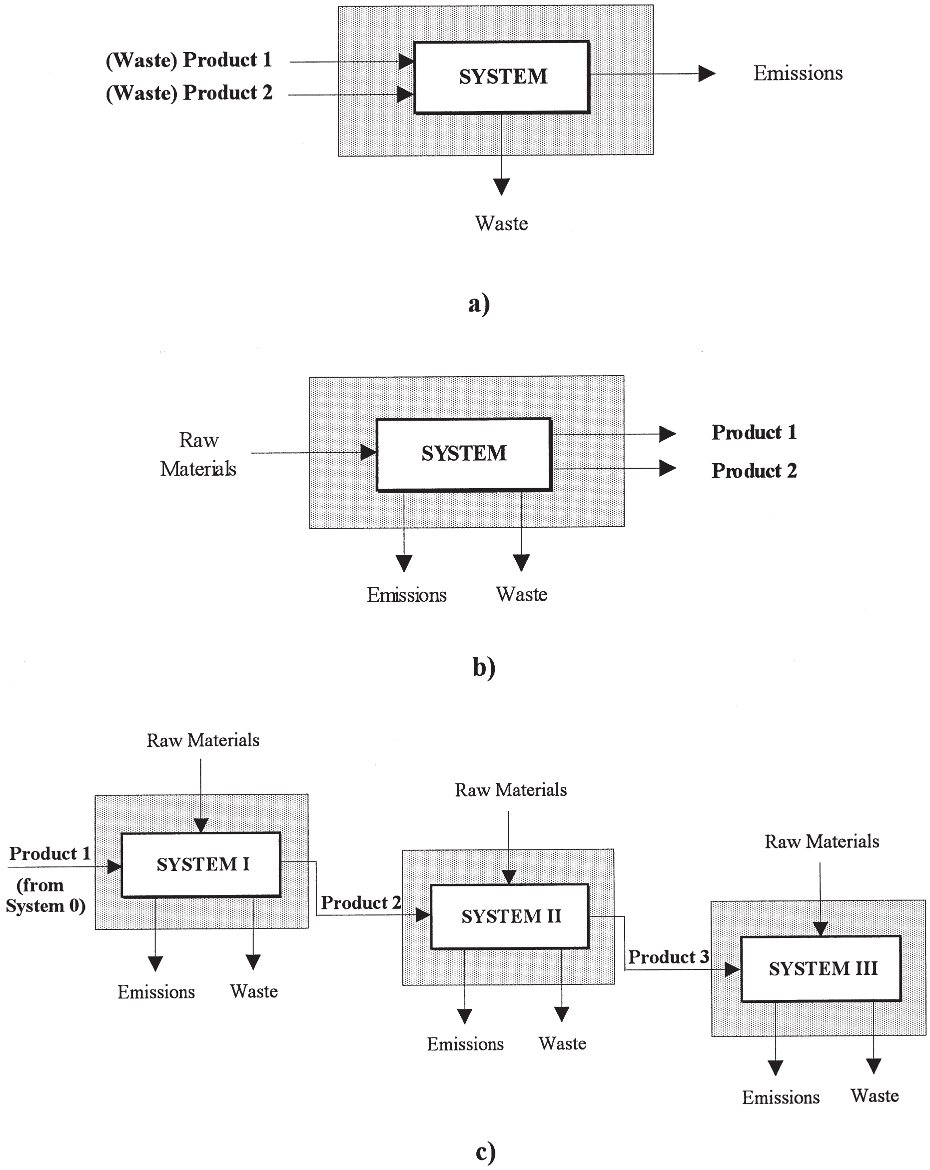

Fig. 1. Multiple-function systems. (a) Multiple-input system: combined waste treatment. (b) Multiple-output system: co-production. (c) Multiple-use or "cascaded use" systems: "open-loop recycling".

of the waste. Similar problems occur in multiple-output or co-product systems, which produce more than one functional output. An example of a co-product system is a naphtha cracker producing ethylene, propylene, butenes and pyrolysis gasoline. The problem of allocation is then to find a procedure to assign to each of the products only those environmental burdens which each product generates. The situation is even more complicated in multiple-use or "cascaded use" systems, where products can be reprocessed and reused in other systems; in LCA, this is termed "open-loop recycling". For instance, broken PET bottles can be melted and reused for manufacture of another plastic container (e.g. a crate) which is subsequently reprocessed and used as a raw material for carpet fibres. Here, the problem is to allocate the environmental burdens among the PET bottle, the crate and the carpet systems so as to reflect both use and production of recycled materials.

There are two general ways to deal with the allocation problem: it can either be avoided by expanding system boundaries or disaggregating the system, or solved by one of the many methods proposed by previous authors. Both ways are reviewed and discussed in the following section; however, first some definitions are introduced.

2. Foreground and background systems

It is useful to distinguish between "foreground" and "background" systems (or, strictly, subsystems) in setting the system boundaries. The foreground system is defined as the set of processes directly affected by the study [1], delivering a functional unit specified in Goal and Scope Definition. The background system is that which supplies energy and materials to the foreground system, usually via a homogeneous market so that individual plants and operations cannot be identified. A schematic representation of background and foreground systems is shown in Fig. 2.

Differentiation between foreground and background systems is also important for deciding on what kind of data should be used. The foreground system should be described by specific process data, while the background is normally represented by data for a mix or a set of mixes of different technologies or processes [1,2].

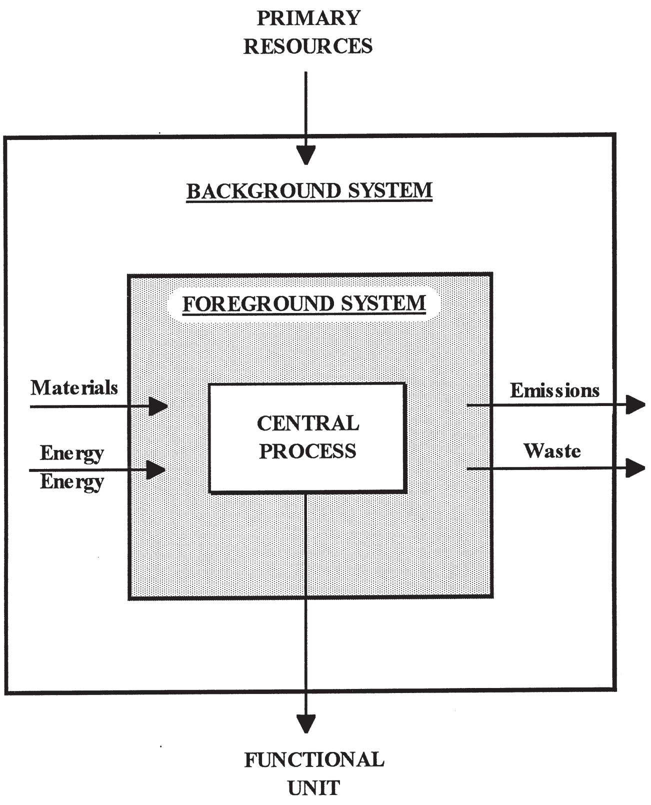

Fig. 2. Foreground and background system.

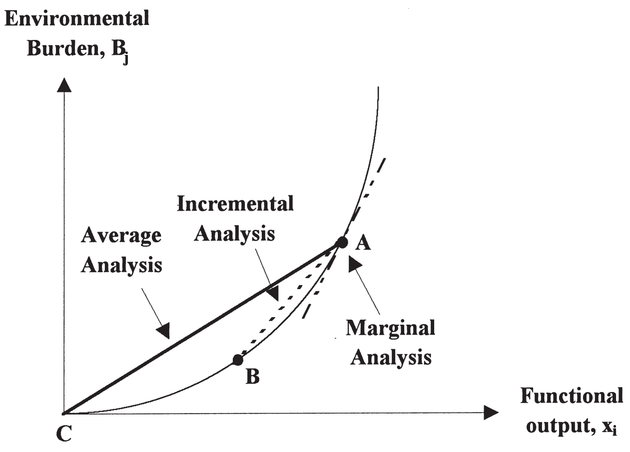

Fig. 3. Marginal, incremental and average analyses of changes in LCA [1,3].

3. Marginal, incremental and average changes

One of the main aims of LCA is to compare changes around an existing condition of the system, be it a small variation in a product composition or technology, a substantial change of feedstock or operating conditions, or a complete change to a different product or technology. Hence, changes in a system can either be marginal, incremental or average. Fig. 3 shows the distinction between the three types of changes for the very simple case of a single environmental burden which varies with the rate of provision of a single functional output [1,3]. Point A represents the "base case", usually current operation about which changes are to be considered.

Marginal changes represent infinitesimal variations about the existing operation, represented by the tangent to the curve at point A. Incremental analysis describes a change in system operation which corresponds to a shift to a new operating point (B). Average changes relate to a significant shift in the operation of the system; for instance, complete elimination of the functional output (point C). Incremental and average changes are together referred to as discrete changes [1].

To decide what basis for allocation is appropriate for a given study, it is necessary to decide what kind of change is to be considered:

1. Marginal changes describe changes which are sufficiently small to be approximated as infinitesimal and they are therefore always linear. Normally, marginal analysis can describe short-term variations in the output of a given system, or longer-term development of the technologies used in the system. An example of the former is comparing different waste management routes for a specific material which would comprise only a small fraction of the total waste stream [3]. As a further example, the transfers of materials and energy between the foreground and background systems normally represent a small part of the background activities, so that it is appropriate to describe the background system by marginal analysis. Similarly, long-term shifts in the mix of technologies and processes making up the background system can normally be described as marginal changes. Marginal analysis is also relevant in co-product and recycling systems, where outputs can be changed independently of each other and the effect of independent marginal changes in these outputs is of concern. Marginal changes are in effect very small and they do not cause a change in the way the system is operated. Therefore, as explored later in this paper, this approach is appropriate for most LCA studies, as it amounts to analysing and allocating burdens for an operating state of the system at a known set of conditions, be it current or any other operating state of interest. The marginal analysis approach has been proposed by the authors [3–5], and will be explained further below.
2. Incremental changes are applicable, for example, to comparing alternative products or wastes which represent a significant proportion of the output or waste processed, or to substantial changes in part of the product system. Examples include use of one or more different unit processes in an overall process tree or, in the case of a waste incinerator, changing combustion conditions or adding new emission control equipment, perhaps using different ancillary energy or materials. Incremental changes are also relevant if different processes or products with similar function are compared within an average technology mix; e.g. comparing different waste management options, such as incineration and recycling for a specific product or material.
3. Average changes are applicable when fundamental changes are considered that would influence or displace a large number of technologies. One such change would be a shift to a chlorine-free economy which would mean phasing out all products that contain chorine and introducing a completely new mix of technologies for producing alternative products, and would lead to discrete changes in emissions from processes such as mass-burn waste-to-energy plants.

4. Procedures for allocation in multiple-function systems

As noted above, there are two different generic ways to treat the problem of allocation [6,7]. The allocation can either be:

1. avoided by expanding the system boundaries or disaggregating the given process into different subprocesses, or
2. solved using a method based on the real behaviour of the product system; i.e. on *causal relationships*.

The current draft of the relevant International Standard [7] recommends that the former should be used in preference wherever possible.

4.1. Avoiding allocation in multiple-function systems

One of the procedures for dealing with the problem of allocation is to avoid it by broadening the system boundaries and introducing several functional units [8–11]. For instance, if System I produces products A and B and System II produces only product C, and A is to be compared with C (Fig. 4a), then allocation can be avoided in two ways. The system can be broadened so that an alternative way of producing B is added to System II. The comparison is now between System I producing A + B and Systems II and III producing C + B (see Fig. 4b).

An equivalent approach is to subtract burdens arising from the alternative way of producing B from System I, so that only A is now compared to C, as illustrated in Fig. 4c. This approach is also known as the “avoided burdens” or “avoided impacts” method, and has mostly been used for systems where a co-product can replace one or more other products, e.g. heat from co-generation to substitute heat from oil, or recovery of energy or material from a waste. The energy or materials produced or recovered substitute for activities in the background system, and so avoid the burdens associated with these activities [12,13]. The environmental burdens allocated to the main product or service are then calculated to include “credits” for the avoided environmental burdens by subtracting them from the total burdens in the system. In some cases the resulting burdens can be negative. For instance, Lindfors et al. [14] illustrate this approach for a refrigerator which produces heat during its life time

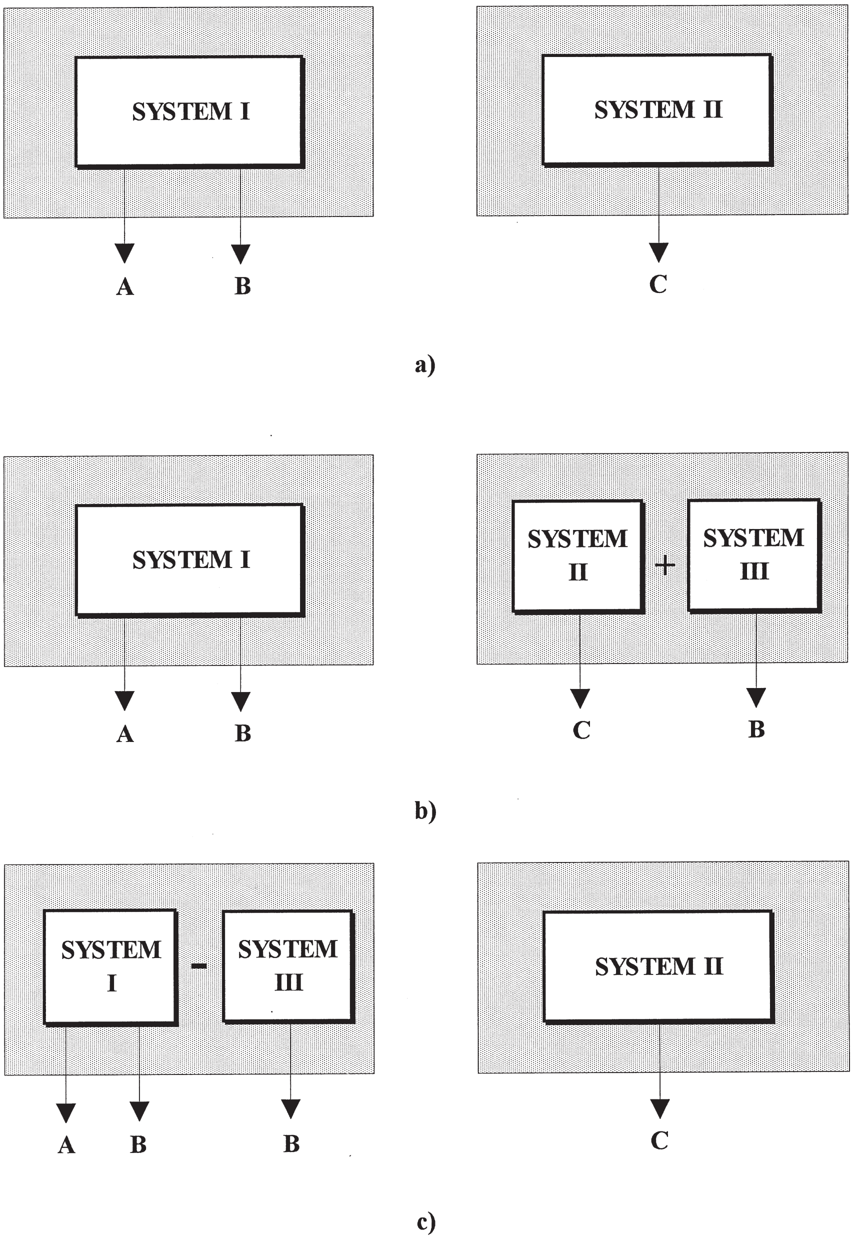

Fig. 4. Avoiding allocation by system enlargement. (a) Systems for comparison. (b) Expanding system boundaries. (c) Avoided burdens approach.

and so reduces the demand for heat produced from other sources. The emissions and resource demand avoided through substitution of refrigerator heat for fuel are included in the system as a credit for the refrigerator. The analysis shows that the net emissions of CO_2, SO_2, NOx, CO, HC and particulates are negative; i.e. heat from the refrigerator is more beneficial than that from, for example, oil. The same authors illustrate the avoided burdens approach for open-loop recycling. If some parts of the refrigerator, e.g. steel and aluminium, are recycled and used in other products, the system boundaries can be expanded to include the life cycle of the products containing recycled metals from the refrigerator. A similar approach to allocation for open-loop recycling has also been proposed by Fava et al. [15], Vigon et al. [16], Fleischer [17] and others.

The avoided burdens method has also been applied to waste incineration [16,18]. Doig and Clift [12] have applied this approach to waste-to-energy systems, where avoided burdens are associated with the background activities; e.g. reduction in electricity generation by coal-fired plants through recovery of energy from waste in the foreground system.

Avoiding allocation either by broadening system boundaries or by the avoided burdens method is an appealing way to deal with allocation; however, there are some difficulties in applying this approach. Although broadening system boundaries will imply a more complete and accurate model of a system, its main drawback is that, by including other functional units, the system becomes more complicated. In addition, there must exist a realistic alternative process for producing a functional output added to the system. The avoided burdens approach has similar problem: it is suitable only if the co-product (or waste) can replace another product with an equivalent function. Another way of avoiding allocation is to disaggregate a process or system into a number of subprocesses, each of which contributes only to one functional output. However, this approach rarely avoids allocation completely because most multiple-function systems include processes which are common for some or all of its functional outputs so that some kind of allocation will still be necessary [5].

4.2. Solving allocation

If allocation cannot be avoided, then an appropriate method has to be chosen to allocate the burdens in a multiple-function system. Most of the approaches proposed allocate burdens in proportion to some physical property or economic value. Physical properties used as a basis for allocation include mass, energy or exergy content, volume and molecular mass [16,19–22]. Methods based on economic value usually include market value (gross sales value) of products or expected economic gain [23], where expected economic gain is equal to gross sales value minus total production and distribution costs so that the two methods are closely related. While in some cases these approaches may be appropriate, in many instances the allocation method has been chosen arbitrarily, without considering any causal relationships in the system.

The importance of causality in LCA is quite obvious: one of the main aims of LCA is comparison of either marginal, incremental or average changes around an existing condition of the system based on its real behaviour in response to the change. If there are causal relationships between the delivery of functional outputs and the environmental burdens, then a change in one of the outputs or system parameters with the other parameters held constant will cause changes in the burdens which must provide the basis for allocation. ISO 14041 recommends that, where allocation cannot be avoided, physical causality is to be used as the basis for allocation where possible. This means that the burdens should be partitioned between different functions of the system so as to reflect the underlying physical[1] relationships between them and will not necessarily be in proportion to a simple measure such as mass or energy content. An example where physical causation applies is a naphtha cracker [4] in which it is possible, by changing operating conditions, to change one functional output while the others remain unchanged. Physical causation is to be used regardless of whether the change in the product output is marginal, incremental, or average. However, as discussed above, the type of change considered will depend on the goal and scope of the study and the questions to be answered by LCA.

For physical causality to be used as the basis for allocation, that causality must be expressed quantitatively, usually via a mathematical model, i.e. a set of mathematical relationships, which describe the real behaviour of the product system. Formulating such a model requires understanding of the system and detailed data on the performance of unit processes, at least in the foreground system. In some cases allocation in proportion to a simple physical quantity, such as mass or energy content, will result from the physical causation embodied in the model. However, this is completely different from arbitrarily choosing some simple parameter as a basis for allocation.

However, there are some cases where physical relationships cannot be used to describe the effects of changing different functional units. For instance, the ratio between two or more functional units delivered by the system may be fixed so that the outputs cannot be changed independently. Examples of this arise in the chemical industry, where the ratio of sodium hydroxide (NaOH) and chlorine (Cl_2) produced by electrolysing brine is fixed by stoichiometry, and in agricultural production, where ratios are defined by the physical and chemical structure of a plant crop (e.g. rapeseed oil and residue) or an animal (e.g. beef and leather). Where there is no possibility of varying one functional output while keeping the other constant, allocation cannot be based on the physical causation principle, so that other relationships between the functional units must be used instead. ISO [7] in these cases recommends allocation on the basis of economic value. The argument for this is that economic relationships reflect the socio-economic demands which cause the multiple-function systems to exist at all [1].

This paper is concerned with the systems in which physical causality can be used as a basis for solving allocation. A general mathematical approach to allocation by physical causality is presented next.

5. Allocation by physical causation

System analysis in the context of LCA has previously been based, often without even implicit recognition of the fact, on linear homogeneous unconstrained models to describe system behaviour. This approach assumes that changes in the burdens and the resulting environmental

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[1] The term “physical relationships” has a broader meaning in this context and includes physical, chemical, biological and technical relationships.

impacts are directly proportional to changes in functional outputs, which are unconstrained by market demand, material availability, productive capacities or any other constraints. In a linear homogeneous model the J burdens, B_j, are related to the functional outputs, y_z, by a set of J equations of the form:

$$B_j = \sum_{z=1}^{Z} b_{j,z} y_z \quad (1)$$

A linear but inhomogeneous model describes a system for which the environmental burdens do not reduce to zero when the functional outputs are zero, and takes the form:

$$B_j = B_{j,0} + \sum_{z=1}^{Z} b_{j,z} y_z \quad (2)$$

An example of a linear homogenous system would be the resource consumption, i.e. hydrocarbons used as feedstock in a naphtha cracker, which is directly proportional to the amount of the co-products (i.e. ethylene, propylene, butenes and pyrolysis gasoline) produced. The same system is linear inhomogeneous with respect to the energy used to heat the cracker, some of which is consumed regardless of the amount of product ($B_{j,0}$ in Eq. (2)).

In reality, almost all systems are non-linear and subject to constraints. However, if the analysis is intended to investigate the effect of marginal changes about some known state of the system then, as discussed above, the system can be linearised about this state so that the use of a linear model is appropriate (see Eq. (4) below).

This paper presents a general method for allocation which can be used whenever physical causation can be represented by a linear model of the most general form. This approach can be used when marginal analysis is appropriate and under some circumstances when incremental or average changes are to be considered.

5.1. *Marginal allocation in multiple-function systems*

5.1.1. *Linearisation of the process model*

Total environmental burdens from a multiple-function system depend, in general, on the properties of the materials processed and on the design and operation of the processes in the system, i.e. on the operating state of the system. The material properties may include physical and chemical properties as well as a material throughput, while process properties may include capacity of the unit operations, operating pressure and temperature etc. In a system which can be described by a model based on physical causality, a change in either material or process properties will cause a change in the environmental burdens. If a change in a material property leads to a change in the state of the system which in turn causes a change in the total burden, then the burden is said to be material-related for a multiple-input system, or product-related for a multiple-output or multiple-use system [3,22]. An example of a material-related burden is the total emission of dioxins from a waste incinerator, which can increase with increasing total chlorine content in the waste. However, if the environmental burdens change as a result of a change in the process, the burdens are said to be process-related. For a waste incinerator, for example, a change in incineration temperature or residence time in the combustion zone can cause a change in the burdens. Thus the total burdens are, in general, related to the material (or product) and process properties by:

$$B_j = f\,[u_1,u_2,\ldots u_N;v_1,v_2,\ldots,v_M] \quad (3)$$

where B_j is environmental burden j and $u_1,u_2,...,u_N$ and $v_1,v_2,...v_M$, are the material (or product) and process properties, respectively. If marginal changes in the system are considered, then the corresponding changes in the environmental burdens[2] are given by:

$$\begin{aligned} \mathrm{dB}_j = {} & \left(\frac{\partial B_j}{\partial u_1}\right)_{u_2,\ldots,u_N,v_1\ldots,v_M} \mathrm{du}_1 \\ & + \left(\frac{\partial B_j}{\partial u_2}\right)_{u_1,u_3,\ldots,u_N,v_1,\ldots v_M} \mathrm{du}_2 + \ldots \\ & + \left(\frac{\partial B_j}{\partial u_N}\right)_{u_1,\ldots,u_{N-1},v_1,\ldots,v_M} \mathrm{du}_N \\ & + \left(\frac{\partial B_j}{\partial v_1}\right)_{u_1,\ldots,u_N,v_2,\ldots,v_M} \mathrm{dv}_1 \\ & + \left(\frac{\partial B_j}{\partial v_2}\right)_{u_1,\ldots,u_N,v_1,v_3,\ldots v_M} \mathrm{dv}_2 + \ldots \\ & + \left(\frac{\partial B_j}{\partial v_M}\right)_{u_1,\ldots,u_N,v_1,\ldots v_{M-1}} \mathrm{dv}_M \end{aligned} \quad (4)$$

provided that f in Eq. (3) has partial derivatives and at least one of them is continuous[3]. The partial derivatives:

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[2] B_j may be an intensive variable, i.e. burden per quantity of waste treated, and in this case the u must be intensive variables, such as waste composition or calorific value. If B_j is an extensive variable, e.g. total quantity of some emission, then the u must also be extensive variables, such as total mass, total calorific value or total chlorine content of the waste processed. For further details, see [3,5].

[3] When $z = f(x,y)$ has partial derivatives f_x, f_y and one of them at least is continuous, the function $f(x,y)$ is said to be differentiable and the total differential dz is defined by the formula $\mathrm{d}z = f_x\mathrm{d}x + f_y\mathrm{d}y$ [24]

$$U_{j,n} = \left(\frac{\partial B_j}{\partial u_n}\right)_{u_1,\ldots,u_{n-1},u_{n+1},\ldots,u_N,v_1,\ldots,v_M} \tag{5}$$

$$V_{j,m} = \left(\frac{\partial B_j}{\partial v_m}\right)_{u_1,\ldots,u_N,v_1,\ldots,v_{m-1},v_{m+1},\ldots,v_M} \tag{6}$$

are defined in the usual way: they represent the change in burden B_j resulting from a marginal change in one of the material or process properties, while all other parameters are held constant. For instance, if B_j is the total dioxin emission and u_n is the chlorine content in the waste material being incinerated, then derivative (5) represents the change in dioxin emissions resulting from a marginal change in the total chlorine content in the waste, without changing any other properties of the material or the process. Similarly, for the process-related burdens, if derivative (6) is related to the temperature (v_m) in the waste incinerator, then $V_{j,m}$ relates the change of total dioxin emissions to the change in combustion temperature only, with all other parameters kept constant. Naturally, in cases where it is not possible to change the system properties independently, the derivatives $U_{j,n}$ and $V_{j,m}$ are not defined, and Eq. (4) is not applicable. In such a case, physical causality cannot be used for allocation and some other basis, such as economic value, must be applied instead.

For marginal changes, the derivatives (5) and (6) are constant, i.e. the properties and the operating state of the system do not change. On integration, Eq. (4) then yields:

$$\begin{aligned} B_j = {} & \left(\frac{\partial B_j}{\partial u_1}\right)_{u_2,\ldots u_N,v_1,\ldots,v_M} u_1 + \left(\frac{\partial B_j}{\partial u_2}\right)_{u_1,u_3,\ldots,u_N,v_1,\ldots,v_M} u_2 \\ & + \ldots + \left(\frac{\partial B_j}{\partial u_N}\right)_{u_1,\ldots,u_{N-1},v_1,\ldots,v_M} u_N \\ & + \left(\frac{\partial B_j}{\partial v_1}\right)_{u_1,\ldots,u_N,v_2,\ldots,v_M} v_1 + \left(\frac{\partial B_j}{\partial v_2}\right)_{u_1,\ldots,u_N,v_1,v_3,\ldots,v_M} v_2 \\ & + \ldots + \left(\frac{\partial B_j}{\partial v_M}\right)_{u_1,\ldots,u_N,v_1,\ldots,v_{M-1}} v_M \end{aligned} \tag{7}$$

The constant of integration can be neglected here if the function B_j is linear and homogeneous to degree one [1,5]. Alternatively, Eq. (4) can be derived from Eq. (7) by Taylor's theorem, showing that Eqs. (4) and (7) represent local linearisation in the general form of Eq. (3).

In simplified notation, Eq. (7) can be written as:

$$B_j = \sum_{n=1}^{N} U_{j,n} u_n + \sum_{m=1}^{M} V_{j,m} v_m \tag{8}$$

Eq. (8) relates total burdens in the system to the material and process properties through the marginal allocation coefficients, U_n and V_m, defined in Eqs. (5) and (6). If the system is modelled by Linear Programming (LP) with B_j defined as the objective function [25,26], these coefficients are equal to the marginal or dual values at the solution of the LP model, as explained below.

5.1.2. System modelling in the linear programming format

For present purposes, a model means a set of mathematical relationships which describe the operation of the unit processes forming a product system. LCA is based on linear homogeneous models of human economic activities and their effect on the environment. In other words, environmental burdens and their impacts are in LCA assumed to be directly proportional to the number of functional units produced [27,28]. The approach to solving allocation by physical causation set out here uses Linear Programming (LP) to model the behaviour of a linear system or a system that can be approximated as linear.

Linear Programming is not a new modelling technique: it has been used routinely for over forty years to describe different productive and economic systems, and to solve problems in scheduling and distribution. Although somewhat more complex than the other linear techniques, such as Regression and Input-Output analysis, it was accepted readily by industry because it was able to account for the internal structure of the system and it helped to improve the efficiency of industrial operations rather than merely describing their performance. The main characteristic of this kind of modelling is that it is based on physical and technical relationships between the inputs and outputs and environmental interventions of the system. Therefore, a LP model describes the underlying physical causation in the system and thus lends itself naturally to allocation according to the procedure recommended by ISO 14041 [7]. Moreover, because LP modelling describes complex interactions between different parts of the system, it can describe changes in the operating state of the system and associated environmental interventions, resulting from changes in material or process properties. This approach therefore reveals how environmental burdens and impacts—and their allocation between different functions—change as the operation of the system is changed. These features are particularly useful in the Inventory and Impact Assessment phases, as discussed below. In addition, the LP approach is valuable in Improvement Assessment, as developed by Azapagic [5] and Azapagic and Clift [26,29].

A LP model of a system takes the general form:

Maximise

$$F = \sum_{i=1}^{I} f_i x_i \tag{9}$$

subject to

$$\sum_{i=1}^{I} a_{c,i} x_i = e_c \quad c = 1,2,\ldots,D \tag{10}$$

$$\sum_{i=1}^{I} a_{c,i} x_i \le e_c \quad c = D+1,\ldots,C \tag{11}$$

and

$$x_i \ge 0 \quad i = 1,2,\ldots,I \tag{12}$$

where Eq. (9) represents an objective function, usually a measure of economic performance, and Eqs. (10)–(12) are the constraints in the system, describing material and energy balance relationships, productive capacity, raw material availabilities, market demand and so on. The constraints are, therefore, related to the material and process properties of the system and the right hand side coefficients, e_c, represent their limiting values. The variables, x_i, often referred to as activities, represent quantitative measures of material and energy flows including inputs, flows within the economic system, and outputs. The model defined above is usually referred to as "primal", to distinguish it from its corresponding "dual" model, described in Appendix A.

LP is most commonly used to find the optimum system operation, defined as operation that maximises profit and uses the optimum amount of resources, subject to the constraints. The optimum operating point is defined by the "active" constraints, i.e. by the constraints that are satisfied as equalities at the solution of the LP model. The other, non-active constraints do not influence the solution of the system and usually are associated with "slack" or unused resources. At the solution, each active constraint has a dual or marginal value, which shows the change in the objective function resulting from a change in the right hand side coefficient of that constraint, e_c. The following analysis shows that the dual values, calculated by representing the system in the form of a LP model, represent the allocation coefficients which describe the physical causality in the system. The general mathematics of dual values is explained in Appendix A, while more detailed accounts of LP modelling and its application to LCA can be found in Azapagic and Clift [26,29].

In the context of LCA, the LP model defined by Eqs. (10)–(12) takes the same form, with the constraints (10) and (11) encompassing all activities from extraction of the primary materials from the earth through processing to final disposal. In addition, the functional outputs are treated as activities. However, the objective functions at the Inventory level are now the environmental burdens which are to be minimised, rather than an economic objective [25,26]. This is represented by:

Minimise

$$B_j = \sum_{i=1}^{I} bc_{j,i} x_i \tag{13}$$

where $bc_{j,i}$ is burden j from an activity x_i. The objective functions may also be defined at the Impact Assessment level as the environmental impacts:

Minimise

$$E_k = \sum_{j=1}^{J} ec_{k,j} B_j \tag{14}$$

where $ec_{k,j}$ represents the relative contribution of burden B_j to impact E_k, as defined by the "problem-oriented" approach [27]. For example, in the specific case of global warming, the $ec_{k,j}$ are the relative greenhouse warming potentials of atmospheric emissions. It should be noted that optimisation is not performed at this stage; the LP model is solved for Eqs. (13) and (14) to calculate the total burdens and impacts, as a part of Inventory Analysis. At the solution, the marginal values, equivalent to the marginal allocation coefficients, are also calculated. Optimisation is performed at the Improvement Assessment level to identify the best possibilities for system improvements, as described by Azapagic [5] and Azapagic and Clift [26,29].

To simplify the explanation, the following discussion will deal with allocation of the burdens only. However, exactly the same approach can be applied to allocation of environmental impacts, given that a LP model relating burdens to functional outputs inevitably leads to LP relationships between environmental impacts and functional outputs.

At the solution of the primal LP model, the total burden B_j arising from the activities in the system is calculated [4,5] using Eq. (13). However, by solving the dual LP model it can be shown (see Appendix A) that the total burden is also equal to:

$$B_j = \sum_{c=1}^{C} \lambda_{j,c} e_c \tag{15}$$

where $\lambda_{j,c}$ is the marginal or dual value of the *c*th constraint defined as (see Appendix A):

$$\lambda_{j,c} = \left(\frac{\partial B_j}{\partial e_c}\right)_{e_1, e_2, \ldots, e_{c-1}, u_{c+1}, \ldots, e_C} \tag{16}$$

These marginal values show how the total environmental burden would change with a change in one right hand side coefficient, e_c, while other coefficients are held constant. As the constraints describe either the material or process properties of the system, the right hand side coefficients of the corresponding constraints also represent these properties; therefore:

$$e_c = u_n \text{ or } e_c = v_m \tag{17}$$

It follows from Eq. (16) that:

$$\lambda_{j,c} = \left(\frac{\partial B_j}{\partial u_n}\right)_{u_1, \ldots, u_{n-1}, u_{n+1}, \ldots, u_N, v_1, \ldots, v_M} \tag{18}$$

or

$$\lambda_{j,c} = \left(\frac{\partial B_j}{\partial v_m}\right)_{u_1, \ldots, u_N, v_1, \ldots, v_{m-1}, v_{m+1}, \ldots, v_M} \tag{19}$$

Comparison with Eqs. (5) and (6) shows that $\lambda_{j,c}$ is equivalent to U_n or V_m, while B_j is defined by Eq. (8).

This is, indeed, the most important link between marginal allocation and LP—dual values evaluated at the solution of the LP model represent the marginal allocation coefficients, which relate changes in the burden to the marginal changes in one of the material or process properties while all other properties of the system are held constant. Thus formulating the system model in LP format leads to allocation coefficients which reflect causal relationships in the system.

The above analysis shows that the burdens can be both material- (or product-) and process-related. This kind of analysis is applicable where the LCA study is concerned with the response of the system to marginal changes in both material and process properties. However, in some cases, the analysis will be limited to the changes in material or product properties only with the process parameters kept constant, or vice versa. Eq. (8) then reduces to:

$$B_j = \sum_{n=1}^{N} U_n u_n = \sum_{c=1}^{C} \lambda_{j,c} e_c \tag{20}$$

or

$$B_j = \sum_{m=1}^{M} V_m v_m = \sum_{c=1}^{C} \lambda_{j,c} e_c \tag{21}$$

for the material- (or product-) and process-related burdens, respectively. As already pointed out, the same kind of analysis applies to allocation of environmental impacts, so that in general, Eq. (15) can be written as:

$$E_k = \sum_{c=1}^{C} \mu_{k,c} e_c \tag{22}$$

with

$$\mu_{k,c} = \left(\frac{\partial E_k}{\partial u_n}\right)_{u_1, \ldots, u_{n-1}, u_{n+1}, \ldots, u_N, v_1, \ldots, v_M} \tag{23}$$

or

$$\mu_{k,c} = \left(\frac{\partial E_k}{\partial v_m}\right)_{u_1, \ldots, u_N, v_1, \ldots, v_{m-1}, v_{m+1}, \ldots, v_M} \tag{24}$$

i.e. marginal coefficient $\mu_{k,c}$ now describes the change in environmental impact E_k resulting from a change in system property u or v.

Marginal allocation of the environmental burdens based on the physical causality principle will now be illustrated by examples of different multiple-function systems.

6. Examples of marginal allocation

6.1. Allocation in multiple-input systems

Multiple-input processes, typically found in waste treatment systems, represent a case where allocation of environmental burdens can become a particular problem, because the burdens have to be allocated between different input streams and their properties. This problem can be solved by modelling the effects of marginal changes in the multiple-input system parameters. This kind of allocation is appropriate in studies of independent marginal changes around an existing operating point, for example to compare alternative ways of managing a waste which forms a small fraction of the total waste stream. The specific example of waste incineration has already been developed by Clift and Azapagic [3], but it is set out here to place it in the more general context of this paper.

The independent parameters used to describe a waste incineration process are taken to be: the total mass of waste processed (*M*), total chlorine content in the mass *M* of waste (*H*), lower calorific value of the mass *M* of waste (*L*) and the combustion temperature (*T*). The examples developed concentrate on the case where one environmental burden—emission of dioxin (*B*)—is critical. However, in general, many burdens can be considered, including both emissions and resource usages.

As a limitation to the analysis for the purpose of this example, an unconstrained system in which all the independent parameters can in principle be subject to independent marginal changes is considered. This would exclude, for example, analysis of an incinerator which is already working at maximum throughput (i.e. *M* cannot be increased) but where the interest is in the effects of changing the characteristics of the waste passing through it.

The functional unit in this example is one tonne of waste processed. The values of the intensive parameters are, therefore, expressed per tonne of waste processed:

$$b = B/M;\ h = H/M;\ 1 = L/M \tag{25}$$

The total emission is related to the total waste processed by:

$$B = f[M,H,L,T] \tag{26}$$

Consider now marginal changes in the system and the corresponding changes in dioxin emission, which can be described by the total differential[4] as in Eq. (4):

$$\mathrm{dB} = \left(\frac{\partial B}{\partial M}\right)_{H,L,T}\mathrm{dM} + \left(\frac{\partial B}{\partial H}\right)_{L,T,M}\mathrm{dH} + \left(\frac{\partial B}{\partial L}\right)_{T,M,H}\mathrm{dL} + \left(\frac{\partial B}{\partial T}\right)_{M,H,L}\mathrm{dT} \tag{27}$$

The partial derivatives in Eq. (27) are defined in the way described in Section 5. Thus $(\partial B/\partial M)_{H,L,T}$ represents the change in dioxin emission resulting from a marginal change in mass of waste processed, without changing the total chlorine in the waste (*H*) or its total calorific value (*L*) or the processing conditions (*T*). It corresponds, in physical terms, to the effect of adding a small quantity of inert chlorine-free non-combustible solid—for example a glass container—to the waste processed. Similarly, $(\partial B/\partial H)_{L,T,M}$ describes the effect on emission of changing the chlorine content without changing the mass or calorific value or the operating conditions: e.g. substituting a piece of PVC for an equal mass of a chlorine-free waste with equal calorific value. $(\partial B/\partial L)_{T,M,H}$ describes the effect of changing the calorific value without changing mass or total chlorine—replacing a fragment of inert glass by an equal mass of chlorine-free combustible, for example. Finally, $(\partial B/\partial T)_{M,H,L}$ represents the effect of changing the combustion temperature but still treating exactly the same waste. To simplify the notation, the following symbols will be used:

$$b_{\mathrm{M}} \equiv \left(\frac{\partial B}{\partial M}\right)_{H,L,T};\ b_{\mathrm{H}} \equiv \left(\frac{\partial B}{\partial H}\right)_{L,T,M};\ b_{\mathrm{P}} \equiv \left(\frac{\partial B}{\partial L}\right)_{T,M,H};\ b_{\mathrm{T}} \equiv \left(\frac{\partial B}{\partial T}\right)_{M,H,L} \tag{28}$$

By substituting Eq. (28) into Eq. (27):

$$\mathrm{dB} = b_{\mathrm{M}}\mathrm{dM} + b_{\mathrm{H}}\mathrm{dH} + b_{\mathrm{L}}\mathrm{dL} + b_{\mathrm{T}}\mathrm{dT} \tag{29}$$

Thus the parameters b_{M}, b_{H}, b_{L}, and b_{T} are the marginal allocation coefficients, corresponding to the dual values in a LP model, which relate changes in the dioxin emission to marginal changes in the waste stream and the process conditions. For a marginal change in the system, the marginal allocation coefficients will remain constant, so that, by analogy with Eq. (4), Eq. (29) after integration becomes:

$$B = b_{\mathrm{M}}M + b_{\mathrm{H}}H + b_{\mathrm{L}}L + b_{\mathrm{T}}T \tag{30}$$

The total dioxin emission is therefore allocated to both material and process properties, i.e. the burden is in general both material- and process-related.

Consider now a case in which the waste processing technology and its operating conditions are kept unchanged, i.e. $\mathrm{d}T = 0$. Eq. (30) then becomes:

$$\mathrm{dB} = b_{\mathrm{M}}\mathrm{dM} + b_{\mathrm{H}}\mathrm{dH} + b_{\mathrm{L}}\mathrm{dL} \tag{31}$$

or after integration:

$$B = b_{\mathrm{M}}M + b_{\mathrm{H}}H + b_{\mathrm{L}}L \tag{32}$$

If the dioxin emission is expressed per functional unit, i.e. per tonne of waste processed, then by substituting the terms in Eq. (25) into Eq. (32):

$$b = b_{\mathrm{M}} + b_{\mathrm{H}}h + b_{\mathrm{L}}l \tag{33}$$

Eqs. (32) and (33) show that the total dioxin emission is allocated to the properties of the waste stream, i.e. the burden is material-related. Furthermore, because the locally linearised model in Eq. (27) is homogenous, this approach leads to complete allocation between relevant independent variables [1].

To take this analysis further, consider the case where, for constant treatment conditions (*T*) and specific calorific value (*l*), the dioxin emission per tonne of waste processed (*b*) varies with the chlorine fraction in the waste (*h*) as shown schematically in Fig. 5 [22]. When the chlorine content is large, so that chlorine is present in excess and does not limit dioxin emissions, *b* approaches an asymptotic value $b_{\max}$ which depends on the process used, i.e. on the type of combustion plant, combustion

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[4] It may be noted that the form of the analysis follows exactly the formulation of quantities such as partial molar properties in chemical thermodynamics.

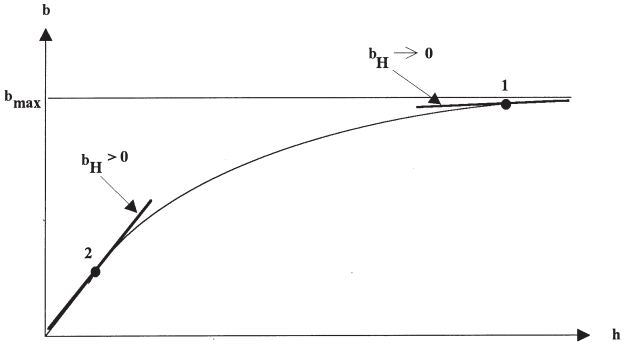

Fig. 5. Variation with waste composition of dioxin emitted per tonne waste (l = const. and T = const.).

temperature, residence time in the combustion chamber, etc. In practice, b_{max} is usually set by regulations on permissible emissions from the plant.

From Eq. (33), the dioxin to be allocated to chlorine content is given by the gradient of the curve in Fig. 5. Under conditions at point 1 (which corresponds to the current composition of municipal solid waste throughout Europe), the chlorine content is sufficiently high that incinerators are effectively operating at the asymptote. Thus changes in chlorine content have virtually no effect on dioxin emissions: the gradient is very small and $b_H \rightarrow 0$. Eqs. (32) and (33) then simplify to:

$$B \approx b_M M + b_L L \tag{34}$$

and

$$b \approx b_M + b_L l \tag{35}$$

Eggels and van der Van [22] have also argued that dioxin emissions depend on the (lower) calorific value of the waste rather than its mass; i.e. that b_M is also very small. Given the definition of b_M—see the first term in Eq. (28)—this conclusion is perhaps not surprising. It implies that adding inert non-combustible material to the waste has no effect on dioxin levels. The system model then reduces to:

$$B \approx b_L L \tag{36}$$

and

$$b \approx b_L l \approx b_{max} \tag{37}$$

From Eq. (37):

$$b_L \approx b_{max}/l \tag{38}$$

so that after substituting Eq. (38) into Eq. (37):

$$B \approx b_{max} M \tag{39}$$

because $M = L/l$ from Eq. (25). Eqs. (37) and (39) indicate that the dioxin emission is now a process-related burden because it depends primarily on b_{max} which in turn reflects the process technology and its operating conditions.

The situation is, however, quite different for conditions in the region of point 2 in Fig. 5. The gradient of the curve is now significant, i.e. b_H is no longer vanishingly small. Fig. 6 shows the variation of the total dioxin emission, B, with, for instance, the total lower heating value of the waste incinerated, L. When Eq. (39) applies, B simply varies linearly with L for all chlorine content, h, which corresponds to the conditions in the region of point 1 in Fig. 6. In the region of point 2, however, the total dioxin emission, B, now depends on both the total calorific value and the total chlorine content of the waste processed (or the average concentration in the waste) so that:

$$B \approx b_H H + b_L L \tag{40}$$

or

$$b \approx b_H h + b_L l \tag{41}$$

and the dioxin emission is now material-related.

By definition, allocation on a marginal basis is only appropriate if the system parameters can be changed independently. If this is not the case—for example, if the total mass of waste which can be processed in an interval of time is limited by the maximum possible plant throughput—then M, H and L (or u_N and v_M in general) cannot be varied independently. If the system conditions are described by a LP model, then its (optimum) operation always lies at the intersection of active constraints. System conditions can then only be changed by changing the values of constraints, for instance by modifying the plant to increase throughput, or by shifting to the intersection of a different set of constraints. In that case, the burdens are allocated to the active constraints at the operating point of interest, as will now be demonstrated for multiple-output systems.

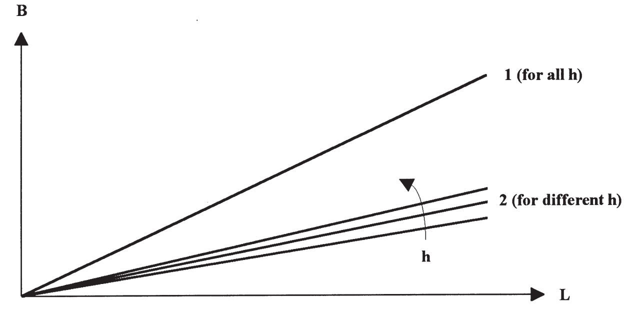

Fig. 6. Variation with waste properties of total dioxin emitted (T = const.).

6.2. Allocation in multiple-output systems

Multiple-output or co-product systems represent another case where the problem of allocation is encountered: the burdens have to be allocated between different functional outputs produced in the same system. This section presents an illustration of how the marginal approach to allocation can be applied to such a system. Again, the emphasis is on systems where the goal of the study is to consider marginal changes to a specific technology and where the functional outputs of the system can be changed independently.

The approach is illustrated by a hypothetical example of a system producing two products, Product 1 and Product 2. The system boundary is drawn to include all activities from extraction of primary resources through refining and transport to the production of the two products (Fig. 7). The use and the disposal phases of the life cycle are not considered here, i.e. the example considers "cradle-to-gate" rather than a "cradle-to-grave" approach. The functional units of the system are quantities of Product 1 and Product 2 and it is assumed that their output can be changed independently of each other using the existing process and equipment. The system is described in LP terms in the following manner.

Suppose that Product 1 and Product 2 are produced from two raw materials both of which can be used as alternative feedstock. The Raw material 1 and Raw material 2 inputs into the production stage are represented by activities x_1 and x_2, respectively. The outputs of the products are related to the two input activities, x_1 and x_2, by mass balance relationships which, for the purposes of this example, are taken to be:

$$\text{Product 1: } x_1 + 4x_2 = 70 \tag{42}$$

$$\text{Product 2: } x_1 + 0.16x_2 = 9 \tag{43}$$

where

$$x_1 \geq 0, x_2 \geq 0$$

The right-hand sides of the equality constraints Eqs. (42) and (43) represent production limitations for the products. Output of Product 1 is constrained to 70 (unspecified) units, while demand for Product 2 is 9 units.

In addition, suppose that the plant is subject to a processing capacity constraint of 100 units. This is represented by the following inequality constraint:

$$\text{Capacity: } 6x_1 + 2x_2 \leq 100 \tag{44}$$

To provide the energy requirements for the process, a maximum of 40 units of heat can be supplied, so that there is a heat supply constraint of the form:

$$\text{Heat supply: } 2x_1 + 1.6x_2 < 40 \tag{45}$$

As discussed in Section 5, in the context of LCA, the objective functions are defined as environmental burdens or impacts, depending on whether the analysis is performed at the Inventory or Impact Assessment level. To simplify the explanation, only two burdens are considered in this hypothetical example; one is associated with resource extraction or depletion:

Minimise

$$B_1 = x_1 + 2x_2 \tag{46}$$

while the other represents atmospheric emissions such as carbon dioxide (CO_2):

Minimise

$$B_2 = x_1 + 15x_2 \tag{47}$$

As pointed out earlier, the system is not optimised at this stage; the LP model is solved to calculate the values of the variables and the objective functions. The solution is found at the intersection of the constraints Eqs. (42) and (43), as represented by point B in Fig. 8. Therefore,

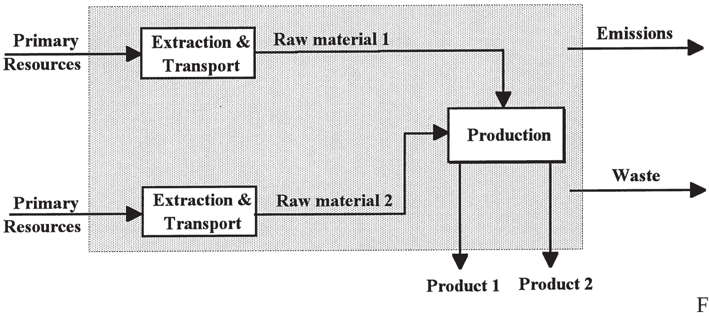

Fig. 7. Simplified LCA flow diagram for the co-product example.

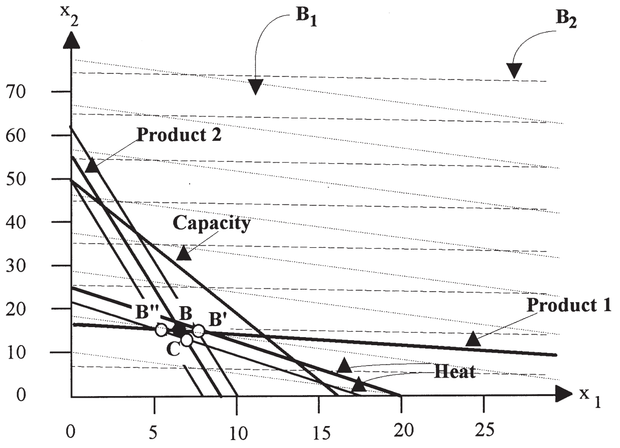

Fig. 8. Allocation by the marginal approach for the co-product example.

only two constraints—Product 1 and Product 2—are active at the solution; i.e. only these constraints have non-zero marginal or dual values (see Table 1). As discussed in Section 5, these dual values are equivalent to the marginal allocation coefficients and in the case of burden B_1 they are equal to 0.48 for Product 1 and 0.52 for Product 2. Therefore, the environmental burdens are fully allocated between Products 1 and 2; i.e. the burdens are product-related in this case. Process-related burdens are zero because the constraints that describe the process, i.e. processing capacity and heat supply, are non-active, so that they do not constrain the system operation. The total burdens are thus equal to:

$$B_1 = b_{1,1}P_1 + b_{1,2}P_2 \tag{48}$$

$$B_2 = b_{2,1}P_1 + b_{2,2}P_2 \tag{49}$$

It is obvious that Eqs. (48) and (49) are equivalent to Eq. (20), with the allocation coefficients b equivalent to Eqs. (16) and (18).

Consider now the effect of increasing the output of Product 2 by one unit while the output of Product 1 is kept constant, for example in response to an increase in demand for Product 2. This corresponds to changing the right-hand side coefficient of the constraint Eq. (43) from 9 to 10. For a marginal change like this, the same constraints remain active and the solution of the system moves from point B to point B'. The two environmental burdens, B_1 and B_2, shown in Fig. 8 by sets of contours corresponding to their constant values, also change. In this case, B_1 increases from 38.23 units to 38.75; the corresponding marginal value of the burden, b_1, allocated to the output of Product 2 is positive and equal to 0.52 (Table 1). However, the same change causes B_2 to decrease from 244.74 to 241.87 units: the marginal value, b_2, allocated to the output of Product 2 is negative and equal to -2.87. This is possible because most of the burden B_2 arises from activity x_2 which is reduced by the increase in the Product 2 output. Similarly, if the output of Product 2 is decreased by the same marginal value, the environmental burden B_1 decreases while B_2 increases. This is represented by point B" in Fig. 8.

The above analysis shows that allocated environmental burdens, as determined by marginal values, can either be positive or negative. Clearly, in this example Product 2 contributes more to resource depletion than Product 1. The situation is reversed for emissions of CO_2: not only is the contribution of Product 2 less than that of Product 1, but its marginal value is also negative; i.e. increase in output of Product 2 would lead to a decrease in total CO_2 emissions. Thus, in addition to solving the problem of allocation, marginal analysis can also be useful in environmental management of a product system because it indicates possible places for system improvement [25,29].

Continuing with the analysis of marginal allocation in the co-product system, it is now interesting to observe what happens to the marginal values if the state of the system changes, i.e. if the system is operated in a different way. Suppose that the heat available to the process is reduced, perhaps due to fouling of heat exchange equipment or changes in the operation of ancillary plant, so that the heat supply is reduced from 40 to 35 units. Eq. (45) then becomes:

$$\text{Heat supply: } 2x_1 + 1.6x_2 = 35 \tag{50}$$

In addition, output of Product 1 can be less or equal to 70 units, which changes Eq. (42) into an inequality constraint. The system operation is now determined by a different set of active constraints, i.e. Product 2 and Heat, instead of Product 1 and Product 2. This is represented by point C in Fig. 8. Therefore, the marginal values of the constraints and hence the allocated burdens are now different; they are shown in Table 2. Since Product 1 and Capacity are non-active constraints their marginal values are equal to zero, so that they do not contribute to the total burdens from the system, in the sense that the burdens do not depend on either Product 2 output or total processing capacity. Thus, the burdens are allocated to output of Product 2 and to the heat requirements in the process, which means that burdens are both product- and process-related. They are equal to:

$$B_1 = b_{1,1}P_1 + b_{1,2}Q \tag{51}$$

$$B_2 = b_{2,1}P_1 + b_{2,2}Q \tag{52}$$

Eqs. (51) and (52) are equivalent to Eq. (8) with marginal allocation coefficients b corresponding to the formulation in Eqs. (18) and (19).

Table 1
Marginal allocation in the example of the co-product system ($x_1 = 6.46$; $x_2 = 15.89$)

Constraints	Value at optimum	b_1 ($B_1 = 38.23$)	b_2 ($B_2 = 244.74$)
Product 1	70.00	0.48	3.86
Product 2	9.00	0.52	−2.87
Capacity	70.52	0.00	0.00
Heat	38.33	0.00	0.00

Table 2
Change of marginal values with change in the state of the co-product system ($x_1 = 6.87$; $x_2 = 13.28$)

Constraints	Value at optimum	b_1 ($B_1 = 33.43$)	b_2 ($B_2 = 206.09$)
Product 1	60.00	0.00	0.00
Product 2	9.00	−1.88	−22.18
Capacity	67.82	0.00	0.00
Heat	35.00	1.44	11.59

This simple example illustrates the general point that the allocated environmental burdens depend on the state of the system, as defined by the way in which the system is operated. The approach to allocation developed here offers more accurate description of a product system because it reflects the consequences of changes in its operation. It has been shown to provide valuable information for real industrial multiple-product systems [5,26,29].

6.3. Allocation in multiple-use systems

At the end of their useful life, some products can be reprocessed and reused to fulfil the same function as before, or alternatively they can be reused in another productive system with a different function. In the former, closed-loop recycling systems, the problem of allocation does not occur because both recycled and virgin materials are used in the same system. However, in the latter, open-loop recycling systems, products (i.e. materials) are passed from one system to another, taking part of the burdens from the upstream to the downstream system in the cascade. Therefore, the burdens have to be allocated among these systems. The main problem here is to allocate the burdens so as to reflect the behaviour of the system in the most realistic way. Similar to other multiple-function systems, it is argued here that marginal changes in the behaviour of multiple-use systems can also be modelled by Linear Programming and the marginal values of the model can be used for allocation of the burdens. By describing the full cascade of uses in LP terms, the burdens are allocated among different uses so that they are “credited” or “penalised” for recycling, depending on the burdens associated with the reprocessing of recycled materials. Thus, allocation based on whole system modelling avoids double accounting of burdens, which may occur if burdens are attributed to both the product and the subsequent recycled material.

To illustrate the approach, consider a simplified open-loop recycling system with three cascaded uses, as shown schematically in Fig. 9. Product 1 (x_1) in the first system is produced from virgin materials (x_4) only, and at the end of its useful life 50% of it is collected and reprocessed to be reused in the second system for Product 2 (x_2). The rest of Product x_1 is landfilled as waste (x_7). For the purposes of this example it is therefore assumed that Product x_2 is made from 50% virgin material (x_5) and 50% material recovered from the first system (x_{10}). At the end of its useful life, 50% is recycled and used in System III while the rest is landfilled (x_8). It is also assumed that Product 3 (x_3) is made of 50% recycled product x_2 (x_{11}) and 50% virgin material (x_6) and after use is discarded as waste (x_9). Thus, for purposes of illustration, it is assumed that production of any individual product does not affect the proportion of recycled material in other products. However, as shown below, the general approach can be applied to other scenarios.

If total demand for each product is 100 units, then the LP model describing this system is defined by the following constraints:

$$x_1 = 100; x_2 = 100; x_3 = 100 \tag{53}$$

$$x_1 - x_4 = 0 \tag{54}$$

$$x_1 - x_7 - x_{10} = 0 \tag{55}$$

$$x_{10} - x_5 = 0 \tag{56}$$

$$x_2 - x_5 - x_{10} = 0 \tag{57}$$

$$x_2 - x_8 - x_{11} = 0 \tag{58}$$

$$x_{11} - x_6 = 0 \tag{59}$$

$$x_3 - x_6 - x_{11} = 0 \tag{60}$$

$$x_3 - x_9 = 0 \tag{61}$$

For simplicity, consider one burden only, e.g. CO_2, which is taken to be product-related; to keep the argument clear, process-related burdens are not considered. Suppose that each activity associated with the virgin materials generates 0.05 units of CO_2 per unit of virgin material. In addition, activities associated with the recovery and reprocessing of the recycled materials each produce 0.02 units CO_2/unit, so that the environmental objective function of the system is defined by:

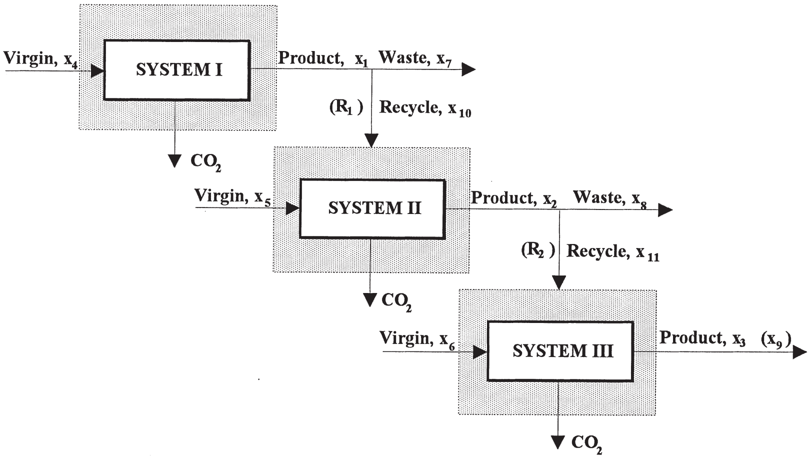

Fig. 9. Simplified LCA flow diagram for the open-loop recycling example.

$$B = 0.05 \cdot x_4 + 0.05 \cdot x_5 + 0.05 \cdot x_6 + 0.02 \cdot x_{10} + 0.02 \cdot x_{11} \quad (62)$$

At the solution of the LP model, given in Table 3, the marginal values of the active constraints, i.e. Eq. (53), represent the CO_2 emissions allocated between the three systems, i.e. the products. The marginal allocated burdens are equal to 0.050, 0.035 and 0.035 for products x_1, x_2 and x_3, respectively. This means that the first use in the cascade (System I) is allocated the CO_2 emissions that are equal to CO_2 generated by the virgin material used for the production of x_1. The first system, therefore, gets no credit in CO_2 emission for producing the recyclable material; however, its total waste is reduced by the amount of material being recycled. The other two uses in the cascade are both credited for using the recycled material: because they displace the production of virgin materials, they are taken to be "CO_2-free" and carry only the burdens arising from their recovery and reprocessing. Therefore, the CO_2 allocated to the second and third uses in the cascade is 0.035 per unit of product, compared with 0.025 (50% of the virgin material) if the burden from reprocessing of the re-used material is ignored. Because all the burdens associated with virgin material are allocated to the first use, there is no credit to subsequent uses, so that double accounting of the burdens is avoided.

It is now of interest to investigate how the marginal burdens change with a change in the way the system is operated, e.g. with changing recycling ratios. For instance, if the percentage of the material recycled into the second cascade is increased from 50% to 90%, while all other parameters are kept constant, the marginal burdens allocated to this subsystem decrease from 0.035 to 0.023 (Table 3, Case 2), while the burden allocated to the other parts of the system remains the same. Again, it may be noted that System III is not credited for using the recycled material which has already been recycled in System II, so that double accounting is avoided. At the same time, the total emissions of CO_2 decrease from 12 units in Case 1 to 10.8. In this particular example, increasing the rate of recycling decreases the total burdens so that it is desirable to increase the total recycling rate as much as possible.

However, in some systems recycling may generate more burdens than the production of virgin materials. Suppose, for example, that in this hypothetical system, the virgin material x_5 can be replaced by an alternative virgin material $x_{5'}$ with unit emission of CO_2 equal to 0.035. However, this plant is situated in a remote area, so

Table 3
Marginal allocation in the open-loop recycling example

Constraints	Value at optimum	1. b_{CO2}[a] $R_1 = 50\%$; $R_2 = 50\%$ ($B_{CO2} = 12.0$)	2. b_{CO2}[a] $R_1 = 90\%$; $R_2 = 50\%$ ($B_{CO2} = 10.8$)	3. b_{CO2}[b] $R_1 = 90\%$; $R_2 = 50\%$ ($B_{CO2} = 12.4$)
Product 1	100	0.050	0.050	0.050
Product 2	100	0.035	0.023	0.039
Product 3	100	0.035	0.035	0.035

[a] Eq. (62).
[b] Eq. (63).

that the burden associated with transport of the recycled material x_{10} to the manufacturing site is increased to give a total emission of CO_2 from recycling x_{10} of 0.04. Eq. (62) now becomes:

$$B = 0.05 \cdot x_1 + 0.035 \cdot x_{5'} + 0.05 \cdot x_6 + 0.04 \cdot x_{10} + 0.02 \cdot x_{11} \tag{63}$$

If the recycling ratios are kept the same as in Case 2 (Table 3), then the burdens allocated to the first and the third subsystems remain the same, while the burden in the second increases from 0.023 to 0.039 units of CO_2 per unit of product. Since the burden associated with $x_{5'}$ is equal to 0.035, this means that emissions of CO_2 are higher with recycling than without.

This simple example illustrates again that marginal allocated burdens depend, in general, on the way the system is operated and not just on the internal structure of the system.

7. Conclusions

Allocation in LCA may be encountered wherever there is a system or process delivering more than one function. Depending on Goal and Scope definition, the allocation procedure should follow the recommendations of ISO 14041: it should be i) avoided by expanding system boundaries or disaggregating the process into different sub-processes; or ii) solved by a suitable allocation method. Where the latter is necessary, a self-consistent approach to allocation is essential.

Allocation on an arbitrary basis, such as mass or energy flow, must be avoided. Where physical causality between functional units and environmental burdens exists, allocation should always be based on these causal relationships. This means that it must be possible to change the value or delivery of any functional unit while keeping the delivery of other functions unchanged. The type of changes considered in the system can be marginal, incremental or average and they in general depend on the goal of the study and questions to be answered by LCA.

It is not always obvious what kind of causality is present in the system. In order to establish it, the system behaviour must be well understood and detailed data on the subprocesses in the system must be available. This approach to allocation requires the process or system to be described by a realistic system model. In some cases, allocation by causality using a system model may lead to a simple basis for allocation, such as mass flow. However, the basis must emerge from the model, rather than being an arbitrary a priori assumption.

As proposed in this paper, system behaviour can be described by whole system modelling using Linear Programming (LP). Given that LCA is based on linear models of human economic activities and the environment, LP is an appropriate tool for whole system modelling in LCA. In a system where physical causal relationships exist, and where marginal changes to a specific system are the goal of the study, the marginal values calculated at the solution of the LP model provide the allocation coefficients. Since the marginal values are a result of system modelling, they represent a realistic description of causal relationships and thus closely reflect changes in behaviour of the system. This approach amounts to analysing and allocating burdens for a known way of operating the product system, and is therefore appropriate for most LCA studies. Therefore, whole system modelling by LP serves as a tool for establishing allocation by physical causation in multiple-function systems.

The marginal allocation approach applies where marginal changes to a specific system are of interest; it cannot always be used to describe average or discrete changes in the system, because these may be nonlinear. In such a case, the system model has to be solved again to identify a new state of the system, but the same approach can be applied to allocate the burdens for the new state of the system.

Where the functions of a multiple-function system cannot be varied independently, allocation by physical causality cannot be implemented. Following the ISO approach, it is then necessary to allocate the burdens on the basis of socio-economic relationships, such as financial value of the functional outputs.

Acknowledgements

One of the authors (A.A.) wishes to thank the Committee of Vice-Chancellors and Principals (CVCP) of the UK and ClifMar Associates Ltd. for their financial support during this research.

Appendix A

Dual values in linear programming

Associated with every linear programming problem, called the “primal”, is another linear programming problem, called the “dual” problem. It is possible to use the dual LP problem to obtain a solution to the primal one. If a primal problem is defined as:

$$\text{Max } F = f_1x_1 + f_2x_2 + \ldots + f_1x_I$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1I}x_I \leq e_1 \tag{A1}$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2I}x_I \leq e_2$$

....

$$a_{J1}x_1 + a_{J2}x_2 + \ldots a_{JI}x_I \leq e_C$$

then its corresponding dual problem is created as follows:

$$\begin{aligned}&\text{Min } Z = e_1\lambda_1 + e_2\lambda_2 + \ldots + e_C\lambda_C\\&\text{subject to}\\&a_{11}\lambda_1 + a_{21}\lambda_2 + \ldots + a_{J1}\lambda_C \geq f_1 \qquad \text{(A2)}\\&a_{12}\lambda_1 + a_{22}\lambda_2 + \ldots + a_{J2}\lambda_C \geq f_2\\&\ldots\ldots\ldots\ldots\ldots\ldots\\&a_{1I}\lambda_1 + a_{2I}\lambda_2 + \ldots + a_{JI}\lambda_C \geq f_I\end{aligned}$$

The objective function is now minimised instead of maximised, and its coefficients are the right-hand sides of the primal problem. The constraints of the dual are formed by transposing coefficients in the constraints of the primal model and changing the direction of inequalities. If feasible solutions to the primal and dual systems exist, there exists an optimum solution for both systems and Min Z = Max F. For a more detailed account, see e.g. Dantzig [30].

The interpretation of a dual or marginal value is the effect of an incremental or marginal change in the right-hand side of the constraint on the optimal value of the objective function. The value of the objective function at the optimum is, therefore, a function of the right-hand side coefficients:

$$\text{Max } F = f[e_1,e_2,\ldots,e_C] \qquad \text{(A3)}$$

In the case of marginal changes in these coefficients, the corresponding marginal change in the objective function is equal to:

$$\text{dF} = \sum_{c=1}^{C}\left(\frac{\partial F}{\partial e_c}\right)_{e_1,e_2,\ldots,e_{c-1},e_{c+1},\ldots,e_C} \text{de}_c \qquad \text{(A4)}$$

where the partial derivative:

$$\lambda_c = \left(\frac{\partial F}{\partial e_c}\right)_{e_1,e_2,\ldots,e_{c-1},e_{c+1},\ldots,e_C} \text{e}_c \qquad \text{(A5)}$$

represents the dual or marginal value and is interpreted as a change in the optimum value of the objective function, F, with a marginal change in the right-hand side of one constraint, e_c, while the values of the right-hand sides of other constraints are held constant. This implies that coefficients or parameters e_c are independent, i.e. that they can in principle be subject to independent changes.

The most important characteristic of the dual values is that they are valid only for the optimal solution and for differential or marginal changes to that solution. The reason for this is that the dual values depend on which constraints are active or binding. By moving away from the optimal solution too far, a new set of constraints can become binding and hence change the dual values. Therefore, for a marginal change in the right-hand side coefficients, the marginal values will remain constant, so that Eq. A(4) can be integrated to give:

$$F = \sum_{c=1}^{C}\left(\frac{\partial F}{\partial e_c}\right)_{e_1,e_2,\ldots,e_{c-1},e_{c+1},\ldots,e_C} \qquad \text{(A6)}$$

or, substituting Eq. A(5) into Eq. A(6):

$$F = \sum_{c=1}^{C}\lambda_c e_c \qquad \text{(A7)}$$

which represents the objective function of the dual LP model corresponding to the primal model defined by Eqs. (9)–(12).

By moving away from the optimal solution too far, a new set of constraints can become binding and hence change the dual values. Therefore, it is only valid to interpret the dual values as referring to the effect of small changes in one right-hand side coefficient while all the others are kept constant; if two right-hand side coefficients are changed simultaneously, it does not necessarily follow that the effect on the objective function will be the sum of the dual values.

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