MATHEMATICS IN ANCIENT EGYPT.

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This article is an amplification of a lecture given in the Rylands Library on February 11th, 1931. It was pointed out at the time that the subject lent itself to two somewhat different kinds of treatment. The first of these—and the only one possible in a lecture of an hour's duration—consists in examining and describing the actual processes used by the Egyptian mathematician in solving the problems which confronted him. The second consists in the attempt to analyse the mental processes which underlie the actual operations and in showing how far these agree with or differ from our own. These two methods of treatment correspond to two successive stages in the history of the study of Egyptian mathematics. The publication of the Rhind Papyrus (see below) by Eisenlohr\(^1\) in 1877 gave to science what was practically its first glimpse of pre-Greek mathematics, and the discussion which followed was mainly, though not entirely, concentrated on the external methods of the Egyptian mathematician. When the papyrus was republished\(^2\) in 1923 the history of mathematics had advanced considerably, and the new edition provoked a series of valuable writings not so much on the concrete methods of the Egyptians as on the mental processes which lay behind them.

In the space at my disposal in this Bulletin it will be possible to treat the subject shortly from both points of view.

\(^1\) August Eisenlohr, *Ein mathematisches Handbuch der alten Aegypter*, Leipzig, 1877.

Sources of our Knowledge.

Our knowledge of Egyptian mathematics is mainly derived from papyri and fragments of papyri which have come down to us from ancient Egyptian times. The most important of these is the Rhind Mathematical Papyrus, now in the British Museum. This document was copied by a scribe called Ahmose in the thirty-third year of King Apepi, who ruled somewhere between 1788 and 1580 B.C., from an older papyrus dating from the reign of King Nemare (Amenemhet III), who reigned from 1849 to 1801 B.C. It contains eighty-four sums more or less completely worked out, together with a table intended for use in the multiplication of fractions, to be described below.

The contents of the papyrus show a certain amount of orderly arrangement. The table of fractions comes first. Then follow simple problems in number. Next we have problems on the volume of simple solids and the content of corn-bins, followed by the geometry of plane figures, and questions on the height and slope of pyramids. The papyrus ends with a number of miscellaneous problems in number.

The Moscow Papyrus,¹ which comes next in importance, shows no arrangement whatsoever. It contains twenty-five sums dealing with subjects as wide apart as volumes, areas, and the exchange of bread for beer; no attempt has been made to group together even those sums whose content is almost precisely identical.

Some fragments of papyrus found in 1889 at El-Lähûn in Egypt contain a short table of fractions similar to that of Rhind, and six, possibly seven, other sums of varying content.²

The Egyptian Museum at Berlin possesses a short papyrus, No. 6619, which bears parts of three problems, two of which will be referred to below.³

Two wooden tablets in the Cairo Museum, Nos. 25367-8, bear complicated-looking calculations which are in effect nothing more than

the expression of certain fractions (a third, a seventh, etc.) of a hekat or gallon (dry measure) in terms of the dimidiated parts, \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \) up to \( \frac{1}{64} \), of the gallon which were used in ordinary transactions.\(^1\)

A leather roll in the British Museum,\(^2\) recently unrolled, contains a number of resolutions of various aliquot parts\(^3\) into the sum of two or more smaller aliquot parts, e.g. \( \frac{1}{3} = \frac{1}{6} + \frac{1}{6} \).

**Notation.**

The Egyptians had at a very early period developed a notation which enabled them to write numbers running into millions. Their system was decimal. The units from one to nine were written with from one to nine vertical strokes, e.g., \( \text{I}, \text{I}, \text{II}, \text{III} \). Ten was \( \text{I} \), twenty \( \text{II} \), and so on up to ninety. Each hundred was represented by \( \text{C} \), each thousand by \( \text{M} \), each ten-thousand by \( \text{X} \), each hundred-thousand by \( \text{G} \) (the tadpole), and each million by \( \text{X} \). The number 1,234,357 was written thus: \( \text{X} \text{CXXXCLVII} \).

The ability to count up to such high numbers, and the devising of so admirable a means of expressing them, would in themselves stamp the Egyptian mathematical consciousness as being far ahead of that of the ordinary savage. Only one improvement could be suggested in the system, namely the device of positional notation. It can be no reproach to the Egyptians of 2000 B.C., however, that they missed what even the Greeks failed to invent, and what was left to the Indian mathematicians to devise and develop.\(^4\)

**The Four Simple Rules.**

There is only one essential and elementary process in arithmetic, namely that of counting. When I say that 8 and 7 make 15

\(^1\) Just as we might be asked to reduce one-seventh of a ton to hundred-weights, quarters, pounds and so on, with the difference that in the Egyptian system each measure was half of the one above it; compare the modern decimal system, where each measure is one tenth of the next above it. See *Journ. Eg. Arch.*, ix, pp. 91 ff.


\(^3\) An aliquot part is a fraction whose numerator is 1.

\(^4\) Positional notation requires among other things the invention of a sign for zero. The Egyptians had none.
I commit an act of memory, which can only be verified by putting down two lots of 8 and 7 objects respectively and counting them. The same may be said of subtraction. It goes without saying that a people who could count beyond a million had no difficulty about the addition and subtraction of whole numbers.

But multiplication and division, too, are, at bottom, forms of counting. I may happen to remember that 8 times 7 is 56, but if I had forgotten there would be nothing to do but to lay out 8 lots of 7 objects and count them. And that is the reason why, as young children, we learn a multiplication table, running usually up to 12 times 12, and sometimes much higher. Now the only multiplication table which an Egyptian learned was 2-times; in other words, he could only multiply directly by 2, and if he wished to multiply by a larger number he had to do it by a series of doublings. Thus, to multiply 7 by 13 he did as follows:—

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Total 91

First he wrote down the number to be multiplied, with the unit 1 in front of it. By doubling he saw that $2 \times 7$ was 14, and by doubling again that $4 \times 7$ was 28, and so on. He then observed that the multipliers of the first, third and fourth lines, namely 1, 4 and 8, made up the required multiplier 13. He ticked off these lines and added up the products in them, which gave him 91 as his answer.

It will easily be seen that a little ingenuity in playing with the multipliers 2, 4, 8, etc., would enable any required multiplier to be made up, and the process was facilitated by the fact that, the notation being decimal, it was possible to multiply by 10 without any working or memory at all; for to multiply a number, say 30 $\text{n}$, by 10, you had merely to change each $\text{n}$ into the next higher decimal unit, namely 5, giving 300 $\text{e}$.

The process of division presented on paper the same appearance as that of multiplication. To divide 91 by 7 the Egyptian “counted” with 7, by the usual process of doubling, thus:—
At this point he saw that the products in the first, third and fourth lines added up to 91. He therefore ticked off those lines and added up the multipliers in them, whose sum gave him the required quotient 13.

**Fractions.**

With the exception of $\frac{8}{1}$, for which a special sign existed, the Egyptian could write no fractions except those whose numerator was unity, i.e., $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and so on; in other words, his fractional notation was limited to "aliquot parts." These were expressed by writing the figure which formed the denominator under the sign $\text{勐}$ (the mouth), perhaps an old word meaning a "part," reduced in hieratic\(^1\) to a mere dot. Thus $\frac{1}{11}$ stands for $\frac{1}{6}$.

To a modern reader the limits of this notation seem very narrow. He is inclined to ask whether they denote a similar limit in the Egyptian's conception of fractional quantities. The answer must depend on exactly what we mean here by conception. We, for instance, have a clear conception of $\frac{3}{4}$ not only as the result of dividing an object into 7 parts and taking three of them, or of dividing a mass made of 3 units into 7 parts, but also as a new unit in itself, which can be treated like a whole number and subjected to whatsoever mathematical operations we may choose. Now, the Egyptian, who was thoroughly at home with aliquot parts, can only have reached them by dividing unity into so many parts. And, that being so, having divided an object into 7 parts he must have further seen that these parts could be grouped together 1 against 6, 2 against 5, or 3 against 4. To this extent he must have had the conception of $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$ and $\frac{5}{7}$, but only as collections of 2, 3, 4, 5 and 6 units respectively, each consisting of $\frac{1}{7}$. He did not make the further step which we make when we regard each of these as a separate entity, $\frac{1}{7}$, $\frac{2}{7}$, etc., capable of being treated as a new unit.

\(^1\) The cursive script, written in ink, usually on papyrus, as distinct from the hieroglyphic script in which the signs were carefully cut or painted on stone or wood, or engraved on metal.
and of undergoing mathematical operations. He could regard \( \frac{1}{4} \) as a unit, but not \( \frac{3}{4} \); that was for him merely the result of adding together 3 units each of \( \frac{1}{4} \). The only exceptions to this inability to conceive (in our sense) a fraction other than an aliquot part seem to have been the complementary fractions, i.e., those which are produced by subtracting an aliquot part from unity, such as \( \frac{2}{5} \), \( \frac{3}{5} \) and \( \frac{9}{10} \). These were, it is true, not in common use, but there is some evidence that a few such, especially \( \frac{2}{3} \), \( \frac{3}{4} \) and \( \frac{1}{5} \), existed, and could be expressed in writing.\(^1\)

Application of the Four Simple Rules to Fractions.

Fractions were multiplied in just the same way as whole numbers, i.e. by continued doublings. When a fraction with even denominator was to be doubled no difficulty was found, for the Egyptian knew that twice \( \frac{1}{4} \) was \( \frac{1}{2} \), and this was a fraction he could handle. But when he came to twice \( \frac{1}{8} \) or twice \( \frac{1}{16} \) his notation failed him, and, since he for some reason objected to writing the results as \( \frac{1}{8} + \frac{1}{16} \), \( \frac{1}{4} + \frac{1}{16} \), he was forced to find some other way out of the difficulty.

Now any fraction of the form \( \frac{2}{n} \), where \( n \) is an odd number, can be broken up into the sum of two or more fractions whose numerators are unity, e.g., \( \frac{2}{3} = \frac{1}{3} + \frac{1}{3} \). And this is just what the Egyptians did; they worked out for themselves a table of such resolutions, beginning with \( \frac{2}{3} \), \( \frac{2}{5} \), \( \frac{2}{7} \), and running up to \( \frac{2}{9} \) and \( \frac{2}{11} \). This table forms the opening portion of the Rhind Papyrus. The problem involved can be solved in a number of ways, and a modern mathematician would deal with it by some such formula as \( \frac{2}{n} = \frac{1}{a} + \frac{1}{na} \), where \( a = \frac{n + 1}{2} \), which would give a methodical series of results. The Egyptian table, however, shows that no general formula was used, but that the results were purely empirical and obtained by gradual collection.\(^2\) The method used was to take the number 2, and try to separate


\(^2\) For recent work on this table, which has attracted the attention of the mathematicians, see O. Neugebauer, *Die Grundlagen der ägyptischen Bruchrechnungen*, Berlin, 1926, pp. 18 ff., and K. Vogel, *Die Grundlagen der ägyptischen Arithmetik*, Munich, 1929, pp. 53 ff.
it into two or more parts each of which would divide without remainder into the denominator. If this was a multiple of 3 then the 2 was divided into $1\frac{1}{2} + \frac{1}{2}$. Thus $\frac{2}{3}$ became $\frac{1}{3} + \frac{1}{3}$ or $\frac{3}{6} + \frac{1}{6}$. When the denominator was a multiple of 5 the 2 was broken up into $1\frac{2}{5} + \frac{1}{5}$.

In more complicated cases simple devices of this sort failed, and the results were doubtless got in many cases by repeated trials. As an example of the more complicated results we may quote the resolution of $\frac{2}{3}$ into $\frac{1}{9} + \frac{2}{15} + \frac{1}{2} + \frac{1}{5}$.

With this table at their elbow, and with a process of multiplication which involved nothing more than doubling, the Egyptians handled the multiplication of fractions boldly and accurately. They were fully aware that to divide by, say, 7 was the same thing as to multiply by $\frac{1}{7}$, or, more simply, to take one-seventh, and they used such multipliers as $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}$ quite readily. One thing, however, astonishes us: they reached $\frac{1}{2}$ of a number not directly, but indirectly, by halving $\frac{2}{3}$ of it. We saw above that $\frac{2}{3}$ was the only fraction with numerator greater than unity for which there was a notation, and we now see that to take $\frac{2}{3}$ was in some way actually a more fundamental operation than taking $\frac{1}{2}$. This is not quite so paradoxical as might appear, for we may regard $\frac{2}{3}$ of $x$ as that number which when its half is added to it gives $x$. In this way we get at $\frac{2}{3}$ by the use only of the fraction $\frac{1}{3}$, which was more fundamental than $\frac{1}{2}$, since it only involved division by 2.

The addition of fractions is best explained by means of an actual example (R. 32), which will serve the further purpose of illustrating the multiplication of fractions by fractions. The task is to show that

$$(1 + \frac{1}{6} + \frac{1}{2} + \frac{1}{12} + \frac{1}{4} + \frac{1}{2} + \frac{1}{8})(1 + \frac{1}{3} + \frac{1}{6})$$

equals 2; the figures shown are as follows:

1 In reality the Egyptian seldom needed to add fractions; in the example before us it would have been in full accordance with his notation simply to write out the series of fractions in descending order, beginning with $1, \frac{1}{3}, \frac{1}{6}$ down to $\frac{1}{2}$. It was precisely the weakness of this notation that a complicated series like this might, as it does here, conceal a very simple fraction or even a whole number. This could only be ascertained by testing.

2 In quoting problems from the papyri R. is used for Rhind and M. for Moscow.
In the first three lines stand the multipliers on the left and the products on the right. These latter have now to be added together. The simpler quantities, namely \(1\frac{1}{3}, 3, \frac{1}{4}\), to the left of the vertical line, yield \(1\frac{1}{2} + \frac{1}{4}\), and, if the whole sum is to be 2, the remaining eleven fractions must come to \(\frac{1}{4}\). They are added by a process which looks very like that of the common denominator; for a number 912 is chosen—it happens to be the greatest of the denominators, though this is not always so—and each denominator is divided into it, and the results placed in red ink (here italics) under the respective fractions. These red numbers add up to 228, which is then shown by a simple division (bottom right) to be \(\frac{1}{4}\) of 912. Hence the eleven fractions total \(\frac{1}{4}\), and the whole product is 2.

The mental process which underlies this operation has been the subject of more discussion than it deserves.\(^1\) It is closely akin to our method of common denominator, with the difference that the Egyptian does not necessarily choose the L.C.M. for his denominator, and hence is liable to get fractions when the other denominators are divided into it. Now when we have to add \(\frac{1}{3}\) and \(\frac{1}{4}\) there is in reality only one way to proceed. The two are irreconcilable as they stand, and they can only be combined by expressing them in terms of some smaller fraction of unity, preferably the thirtieths (30 being the L.C.M.) or sixtieths or one-hundred-and-twentieths and so on.

way in which we write this process out shows that we know throughout exactly what we are doing. We say \( \frac{1}{5} \) is six thirtieths, and \( \frac{1}{3} \) is five thirtieths, total eleven thirtieths. Now the Egyptian, who had no notation for five thirtieths, could not show the mental process as clearly as we do in his writing out, yet it must have been the same as ours, for no other is possible. However we describe it we cannot get away from this. Chace,\(^1\) for instance, would say that the Egyptian "applied" his denominators in turn to the number 912, chosen as being suitable for the purpose, added the results, and found that the total 228 when "applied" to 912 gave \( \frac{1}{4} \). But though this is a fair enough description of the arithmetical operation performed it shirks completely the thought which alone could have prompted it. Just as the whole method of the table of resolutions of 2-fractions shows that the Egyptian was quite clear that to divide 2 by 5 was the same thing as to double a fifth, i.e., to add a fifth to a fifth, so here he is aware, when he divides 228 by 912 and gets \( \frac{2}{3} \), that this is the same thing as saying that if we divide unity into 912 parts and take 228 of them the result is \( \frac{1}{4} \) of unity; in other words, the red figures which add up to 228 represent in effect 912ths. Otherwise the process has no reason and could never have been devised.

**Other Arithmetical Processes.**

Squaring and taking the square root were both practised. The former, though distinguished by a special name, whose literal meaning is not certain (apparently connected with the verb sny, to "pass by"), was merely a special case of multiplication. The square root was expressed by "corner," a term clearly derived from the picture of a square. In the Berlin Papyrus 6619, the square roots of 6\( \frac{1}{2} \) and of 1\( \frac{1}{2} \) + \( \frac{1}{10} \) are correctly given. These were no doubt known from trials, and there is no evidence of the existence of a method for taking the square root.

Proportion plays a very large part in Egyptian mathematics, though it is never explicitly formulated. An example (M. 11) will give readers an idea of its use:—

"Example of reckoning the work \(^2\) of a man in logs of wood. If

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\(^1\)Chace, *op. cit.*, pp. 7-10. A good criticism of this view by Vogel in *Archeion*, xii (1930), pp. 398-399.

\(^2\)This might mean "contribution," "tax."
they say to thee 'The work of a man in logs; the amount of his work is 100 logs of 5-handbreadth section.' He has brought them, however, in logs of 4-handbreadth section. (How many should there be?)

"You are to square these 5 handbreadths; result 25. You are to square the 4 handbreadths; result 16. You are to operate on this 16 to get 25; result $1\frac{1}{2} + \frac{1}{16}$ times. You are to take 100 this number of times; result $156\frac{1}{4}$. Then shall you say to him (i.e., to the scribe who sets the problem) 'Behold, this is the number of logs which he brought of 4-handbreath section. You will find it right.'"

A modern boy handling this sum would be expected to state the proportion $16 : 25 : : 100 : x$, and then either to multiply 100 by 25 and divide the result by 16, or, like the Egyptian, to divide 25 by 16 and multiply the result by 100.

Problems in Arithmetic.

The simplest of the arithmetical examples in the papyri are the sums (R. 1-6) which deal with the division of various numbers of loaves among 10 men. For the modern there is no problem here, for if 7 loaves are to be divided among 10 men the answer $\frac{7}{10}$ can be given directly from the data. For the Egyptian, however, who could not write the fraction $\frac{7}{10}$, there was a real problem involved, namely how to express $\frac{7}{10}$ as the sum of two or more aliquot parts. The answer is $\frac{3}{8} + \frac{1}{8}$ (remember that $\frac{3}{8}$ ranks with the aliquot parts). This answer is not worked out from the data; it is assumed, and proved to be correct by multiplying by 10. The whole reads as follows:

"To divide 7 loaves among 10 men.
You are to multiply $\frac{3}{8} + \frac{1}{8}$ by 10:
The doing of it:

1 $\frac{3}{8} + \frac{1}{8}$
2 $1\frac{1}{8} + \frac{1}{8}$
4 $2\frac{3}{8} + \frac{1}{8} + \frac{1}{8}$
8 $5\frac{1}{8} + \frac{1}{8}$

Total 7 loaves. This is it."

1 It matters not whether the logs are round or square in section. They are assumed to be constant in length.
This method of dealing with a problem, namely assuming the answer and proving that it gives the data ("This is it"), is typical of Egyptian mathematics, and we shall have to return to it later.

Among the simple problems in number the most interesting are perhaps those which correspond to what we now call equations of the first degree with one unknown, e.g., $x + \frac{x}{a} = b$. An example (R. 26) will make this clear.

"A quantity whose fourth part is added to it becomes 15. (What is the quantity?)

(Step A) Reckon with 4.
Make their quarter, namely 1. Total 5.

(Step B) Reckon with 5 to find 15.
\[ \begin{array}{c|c|c}
0 & 1 & 5 \\
\hline
-1 & 5 & 4 \\
-2 & 10 & 3 \\
\end{array} \]
The result is 3.

(Step C) Multiply 3 by 4
\[ \begin{array}{c|c|c|c|c|c|c|c|c|c}
8 & 1 & 3 \\
12 & 2 & 6 \\
16 & 4 & 12 \\
\hline
12 & : & 3 & Total 15. \\
\end{array} \]

(Proof). \[ 1 \frac{1}{4} × 3 \text{ Total 15.} \]

(Answer). The quantity is 12; its quarter is 3. Total 15."

The arithmetical operations here performed are obvious. The number 4 is taken, its quarter is added to it, giving 5. This 5 is divided into the given 15, and the resulting 3 is multiplied by 4, giving the correct answer 12.

But what is the thought-process behind this? Cantor thought that it was precisely that of the solution of the equation $1 \frac{1}{4} x = 15$; that in Step A $1 \frac{1}{4}$ was reduced to an improper fraction $\frac{5}{4}$, in Step B the numerator 5 was divided into 15, and in Step C the result multiplied by the denominator 4.

This solution does not fit in with what we know of the handling of fractional quantities by the Egyptians.\(^1\) It seems more probable that the method is that of trial. The number 4 with which Step A begins is a trial number, chosen because its fourth part involves no fractions. The result of performing the given operation on the trial number is 5.

\(^1\) See, however, p. 422, note 2.
But the given result is 15, and our trial number must, therefore, be multiplied by 3. This is precisely the method which we should follow if told to solve this sum without the use of the algebraical symbol $x$, and it involves no mathematical principle save that of proportion: if the trial number 4 gives a third of the required result then we must take three times 4.¹

Some problems of this type are solved directly by division. R. 30 is an example of this kind; it will also show the reader a favourite manner of wording problems in Egyptian:

"If a scribe" says to you

'10 has become $\frac{2}{3} + \frac{1}{10}$ of what?'

"Let him hear:

'Operate on $\frac{2}{3} + \frac{1}{10}$ to find 10,' etc.

As a result of the operation, which consists, as may be guessed, of trials by successive doubling, the answer is found to be $13\frac{1}{3}$, and the sum is proved by multiplying this number by $\frac{2}{3} + \frac{1}{10}$ and getting the expected 10 as a result.

¹ VOGEL, Archeion, xii (1930), pp. 138-140, objects that if this were the method used Step C ought to be not multiply 3 by 4 but multiply 4 by 3. In the strictly logical sense this is true; 4 is the original trial number and 3 the "proportional factor." At the same time I see no reason why the Egyptian, who was well aware of the Commutative Law ($a \times b = b \times a$) should not have preferred the very practical if slightly less logical method of taking the result of each step as the first premise of the next. Just as in Step B he takes as basis the 5 which resulted from Step A, so in Step C he starts off with the 3 which he has just obtained from Step B. Vogel's own solution of the problem is based on the conception of abstract number. He thinks that the Egyptian envisaged the quantity to be found as a 4-head consisting of 4 quarters (Rechnen mit einer Vierheit d.h. mache den Haufen zu einer Vierheit von 4 Häufchen, von 4 "Viertel-x"). This does, it is true, explain the steps as they stand, and above all shows why in Step C 3 is multiplied by 4 and not 4 by 3. But Vogel's claim that this avoids the use of a trial number is baseless, for what is his 4-head but a trial number?

² No one, so far as I know, has explained why the settler of the problem is always referred to as "the scribe." We know of no Egyptian word for a mathematician, and possibly there was no more specific word than scribe, though doubtless not all scribes were mathematicians. Scribe must have been in ancient Egypt almost synonymous with educated man, since the rare ability to write lay at the basis of all education. The use of this word here lends no colour to the widely accepted belief that mathematics, and indeed all the sciences, lay in the hands of the priesthood.
These “quantity” problems were sometimes worded in that concrete manner which appealed so strongly to the Egyptian mind. R. 35 reads:—

“I have gone three times into the gallon measure; my third part is then added to me and I return complete. What is it that says this?”

This wording is puzzling, but its meaning is explained by the working of the sum, which consists in dividing unity by $\frac{3}{4}$. To “return complete” must then mean to “make up a gallon,” and the problem is to find the quantity which when taken $3\frac{1}{2}$ times comes to a gallon (dry measure). The answer is found to be $\frac{1}{4} + \frac{1}{10}$ of a gallon, or 96 ro (320 ro = 1 gallon), or, expressed in the diminished portions of the gallon used in everyday transactions ($\frac{1}{4} + \frac{1}{3} + \frac{1}{6}$) gallon + 1 ro.

While the problems just treated correspond to our simple equations of the first degree with one unknown, there are two others which might similarly be said to correspond to equations of the second degree. The first is M. 6 (the same problem also occurs as part of one of the sums in the El-Lähûn fragments). Here we are to find the sides of a rectangle of area 12, given that one side is $\frac{3}{4}$ of the other. Stated in the form of an equation this would be $z_4^2 = 12$, where the sides are $x$ and $\frac{3}{4}x$. The Egyptian, however, uses no $x$ and approaches the question graphically (Fig. 1). He sees that had the figure been a square whose side is the longer of the two sides of the given rectangle it would be $\frac{3}{4}$ times as large, $\frac{4}{3}$ being the reciprocal of $\frac{3}{4}$. Such a square he proceeds to construct. To get the reciprocal of $\frac{3}{4}$ he divides unity by it; result $1\frac{1}{3}$. Then he multiplies the given area 12 by $1\frac{1}{3}$, getting 16 for the area of the square. The square root of this, namely 4, will be the longer side required, and the other will be got by taking $\frac{3}{4}$ of this.

This solution involves no algebra, nor even the use of a trial number, and the only assumption made is that if we have a rectangle, and we multiply one of its sides by $k$, leaving the other side constant, the area will also be multiplied by $k$—a theorem which follows at once from the formula for the area of a rectangle, i.e., from the conception of square measure (see below, p. 430).
In the next problem we again have what corresponds to an equation of the second degree. The problem occurs, much damaged, in Papyrus Berlin 6619. Here, despite the difficulties of restoring a consistent text, which are even greater than any writer hitherto has realised, there seems no reason to doubt that the problem was to divide 100 into two parts, one of whose square roots is $\frac{3}{2}$ that of the other.

The method is that of trial, unity being taken for one of the


2 Neugebauer in a recent publication, *Arithmetik und Rechentechnik der Ägypter* (Quellen u. Studien zur Gesch. d. Math., Abt. B, Bd. I, Heft 3), pp. 305 ff., denies the use of a trial number, be it unity or any other, in all these problems, and returns to Cantor’s theory that they are solved as equations, in the modern manner, by multiplying the absolute term (on the right in our modern arrangement) by the inverse of the coefficient of $x$ (on the left). The unknown $x$ is, of course, the ‘$\frac{3}{2}$’ or “quantity,” and in one case, M. 25, where the equation is $2x + x = 9$, Neugebauer believes that this unknown is explicitly operated on under the name ‘$\frac{3}{2}$’. The words are “Add the quantity to the 2; result 3. Divide the 3 into 9; result 3 times. 3 is the number required.” At the same time, even if we accept the curious wording of the text here in spite of the suspicion thrown on it by the occurrence of a vital omission in the setting of the sum (the preposition ‘$n'$ is followed by no object 1), and agree that it involves the explicit use of an unknown and the solution of an equation in the modern style, it does not follow that the same method was used in other cases. Indeed it is by no means a merit in Neugebauer’s hypothesis that it assumes uniformity of treatment in all these problems, even including those which correspond to equations of the second degree, for the outstanding characteristic of Egyptian mathematics is precisely the lack of any such uniformity.

Thus while I should be sorry to deny outright the possibility of Neugebauer’s being right in regard to such problems as are solved by what I have called direct division (R. 30-38, M. 19, 25), yet I still think that in R. 24-27, where the coefficient of $x$ (speaking in modern terms) is 1 plus an aliquot part, e.g., $1 \frac{1}{7}$ (R. 24), the method used was one of trial, the trial number chosen being, for obvious reasons, in each case the denominator of the aliquot part, e.g., in the case quoted, 7. In this example, if Neugebauer were right, and the process was that of simply dividing the 19 by the coefficient of $x$, namely $1 \frac{1}{7}$, the method would have been that of R. 31-34, namely to operate on $1 + \frac{1}{7}$ to find 19. Yet he asks us to believe that the Egyptian turned $1 \frac{1}{7}$ into $\frac{8}{7}$ (an improper fraction, be it noted), and then multiplied the 19 by the inversion of this. Where in the papyri can we find a justification for such a procedure? What Egyptian ever took seven-eighths of 19 by dividing it by 8 and multiplying the result by 7, which are the two arithmetical operations actually performed in this problem.
quantities, and consequently $\frac{3}{4}$ for the other. These are squared and added, result $1\frac{1}{2} + 1\frac{1}{8} = 1\frac{1}{8}$. The square root of this is taken, namely $1\frac{1}{4}$. Next the square root of 100 is taken, viz., 10. This is 8 times $1\frac{1}{2}$, and accordingly the trial numbers taken, namely 1 and $\frac{2}{3}$, must be multiplied by 8. Result 8 and 6 for the numbers sought.

The equation to which this problem corresponds is $x^2 + y^2 = 100$, where $y$ is $\frac{3}{4}x$, or, in other words, $x^2 + \frac{3}{4}y^2 = 100$. The steps taken in the Egyptian solution are arithmetically exactly those which we should take in solving this equation by "algebra," for the simple reason that there are no other steps by which the answer can be obtained. But the Egyptian method is not algebraic, for no symbol is operated on, and the method is clearly that of trial.

The problems with which we have just been dealing bring us face to face with the question whether the Egyptians made use in any sense of algebra in their mathematics. In so far as algebra consists of the employment of symbols accompanied by the performance of mathematical operations upon them, our reply to the question will largely depend upon the view we have taken of these problems which correspond to modern equations of the first and second degree. If we accept Cantor's and Neugebauer's view of them, then algebra, in the sense above indicated, is involved. If their view be not accepted—and the writer is far from being convinced by it—there is in the papyrus no instance, either implicit or explicit, of the use of a symbol for an unknown.\footnote{Even if Neugebauer's supposition be correct he can only point to one use of a symbol (M. 25, see p. 422, n. 2), and that in a problem whose text is not above suspicion.}

The fact that the problem from the Berlin Papyrus dealt with above speaks of the two unknowns as "the one quantity" and "the other quantity" does not tell against this; it is impossible to state a problem about two quantities without referring to them by some name, and the names or symbols are not operated on in the problems.\footnote{Contrary to the opinion of Vogel, Die Algebra der Ägypter des mittleren Reiches, in Archeion, xii, p. 152, who has, I suspect, been misled by an inaccurate restoration and translation of the problem. The expressions "the one quantity" and "the other quantity" are never operated on, but only occur in the twice repeated explanatory phrase, "since the one quantity is $\frac{3}{4}$ of the other."} The case for the use of algebra in this sense in the papyrus is thus very far from convincing.
Algebra in the modern sense, however, has a wider meaning than this, and may be present even when no symbols are employed. It has been defined as the "performing of arithmetical operations upon combinations of number- or space-units of every kind."\(^1\) In this sense, too, algebra seems to be absent from the Egyptian papyri. Whether the application of a general formula of any kind, such as that of R. 616, where we are told that to make \(\frac{2}{3}\) of a fraction we must take its half plus its sixth part, in itself constitutes algebra is for the philosophers of mathematics rather than for the historians to decide.

Two problems involve arithmetical progression. In R. 64 we are asked to divide 10 gallons of barley among 10 men in such a way that the excess of each man over his neighbour is \(\frac{1}{4}\) of a gallon. In modern terms, find a series of 10 numbers in arithmetical progression whose sum is 10 and whose common difference is \(\frac{1}{4}\). The method of solution is as follows: Take the mean share, namely 1 gallon. Now take 1 from 10, result 9. Halve the common difference, giving \(\frac{1}{16}\), and multiply it by 9; result \(\frac{9}{2} + \frac{1}{16}\). Add this to the mean share to obtain the highest share. Subtract \(\frac{1}{2}\) each time to find the succeeding shares. The answer is \(1, \frac{9}{16}, \frac{7}{16}, \frac{5}{16}\) and so on down to \(\frac{1}{16}\) gallons.

Though this is not the modern method it is a perfectly sound way of looking at arithmetical progression. If the number of terms is odd the mean share will be actually the middle term, and to get the last term we must add on \(\frac{n-1}{2}\) times the common difference, where \(n\) is the number of terms. That the Egyptian added half the common difference multiplied by \(n-1\) merely argues his acquaintance with the Law of Commutation, which is evident over and over again in his mathematics.

It will be observed that if the number of terms is even the common share does not actually correspond with any of the shares, but a little thought will show that the Egyptian's rule remains valid. It would be interesting to know whether he had realised this. If he had, it would seem that he had evolved a general formula for the treatment of the arithmetical progression.

\(^1\) Thus to say \((8 - 2)^2 = 6^2\) is arithmetic, but to say \((8 - 2)^2 = 8^2 - 2 \cdot 8 \cdot 2 + 2^2\) is algebra. See, further, in this connection, O. Gillain, *La Science égyptienne: L'Arithmétique au Moyen Empire*, Brussels, 1927, pp. 245-250.
R. 40 also involves arithmetical progression. It runs: "A hundred loaves to 5 men, one seventh of the first three men to the two last. What is the difference of share?"

Here 100 loaves are to be divided among 5 men, the two last men together receiving one seventh of the combined shares of the first three. A further condition, not explicit in the setting, but perhaps implicit in the use of the word for "difference of share," is that the five shares are to be in arithmetical progression.

The solution is disappointing. A trial "difference of share" of $5\frac{1}{2}$ is taken, and in addition a trial highest share of 23. The shares are found to be 23, $17\frac{1}{2}$, 12, $6\frac{1}{2}$, and 1, which do indeed fulfil the condition stated, namely that the sum of the two last is one seventh of that of three first. They add up to 60, and, since the given 100 is $1\frac{2}{3}$ times 60, the trial shares have each to be multiplied by $1\frac{2}{3}$, giving $38\frac{1}{3}$, $29\frac{1}{3}$, 20, $10\frac{1}{3}$, $1\frac{2}{3}$, for the shares.\footnote{1 The writer forgets that his problem was to find the common difference.}

The answer is right, but the method is fraudulent, for two trial numbers are taken—a wholly illogical procedure. Clearly it had been observed that in the actual series taken, 23, etc., the last two terms were one seventh of the first three, and a problem based on this observation had been devised.

The Rhind Papyrus has preserved for us a question which manifestly deals with what we should call a geometrical progression. It appears in the following form:

"An inventory of a household:

\[
\begin{array}{ll}
1 & \text{2801} \\
2 & \text{5602} \\
4 & \text{11204} \\
\text{Total} & \text{19607}
\end{array}
\]

\[
\begin{array}{ll}
7 \text{ houses} \\
49 \text{ cats} \\
343 \text{ mice} \\
2401 \text{ wheat} \\
16807 \text{ gallons}
\end{array}
\]

Total 19607."

Remembering the children's rhyme which begins "As I was going to St. Ives, I met a man with seven wives," we shall have no difficulty in interpreting the Egyptian sum as follows: "There were 7 houses, each house had seven cats, each cat killed 7 mice, each mouse would have eaten 7 grains of wheat, and each grain of wheat would have
produced 7 gallons of corn." The total is that of all the things mentioned. But the interest lies in the fact that it is not reached by mere addition of the second column, for in the first column stands what we at once recognise as the multiplication of 2801 by 7, giving the correct total, 19607. This multiplication corresponds to the modern formula for summing a geometric series in which the first term is the same as the common difference, namely \( \frac{l - 1}{r - 1} \), where \( l \) is the last term and \( r \) the common difference. Thus our series

\[
7 \quad 49 \quad 343 \quad 2401 \quad 16807,
\]

where the last term is 16807 and the common difference is 7, would give \( 7 \times 2801 \).

It is tempting here to suppose that the Egyptians were using a general formula. Neugebauer, however, has pointed out\(^1\) that if we take off the last term and put on another in front, namely 1, we get a series which, when multiplied by 7, would give the present one. This multiplication would appear as follows:—

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<tr>
<td>–1</td>
<td>1</td>
<td>7</td>
<td>49</td>
<td>343</td>
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<tr>
<td>–2</td>
<td>2</td>
<td>14</td>
<td>98</td>
<td>686</td>
</tr>
<tr>
<td>–4</td>
<td>4</td>
<td>28</td>
<td>196</td>
<td>1372</td>
</tr>
<tr>
<td>–7</td>
<td>7</td>
<td>49</td>
<td>343</td>
<td>2401</td>
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In the top line stands the series to be multiplied, and on the right, after the vertical line, is its sum. In the next line the series is doubled, as is also its sum. The third line shows the re-doubling of these again, and the last line gives the addition of the three lines, which is equivalent to 7 times the top line. The figure 19607 is thus seen to be not only the sum of the figures to the left of it in its own line but also 7 times 2801.

The principle underlying this is simply that the sum of any geometric series from its second to its \( n \)th term is equal to the sum from the first to the \((n - 1)\)th term multiplied by the common ratio. There is nothing to prove that the Egyptians had grasped this general principle, and in any case it is of little practical use in the summing of

\(^1\) Neugebauer, Die Grundlagen, etc., p. 14, note 4; Arithmetik u. Rechentechnik der Ägypter, pp. 316-317.
series; to add from the 2nd up to the \( n \)th term is little more trouble than adding from the 1st up to the \((n - 1)\)th.

Before we pass on to the subject of Egyptian geometry the reader may be interested to have some specimens of what may be called general arithmetical questions. These, it will be noticed, are all set in concrete form; they deal with problems which might arise in everyday life. For instance, in a country where there was no coinage and all trade was by barter, it was necessary to have some basis on which exchanges could be made. In the case of two very common necessities of life, bread and beer, this was particularly simple, for both were made from grain, and the exchange could be calculated on the basis of the amount of grain used to make the one or the other. If four equal loaves had been made from a gallon of barley the \( \text{pefsu} \) ("cooking figure," from \( \text{pefs} \), "to cook") of the loaves was said to be 4; each loaf would actually contain \( \frac{1}{4} \) gallon of grain. In the case of beer, if 2 \( \text{des} \) (let us say "pints") were made from a gallon of barley, the \( \text{pefsu} \) of the beer was 2. Notice, however, that while the \( \text{pefsu} \) gave the actual size of loaves it gave not the quantity of beer but the strength.

A single example (R. 78) will show how these sums were worked:

"Example of exchanging bread for beer. If they say to you: 'A hundred loaves of \( \text{pefsu} \) 10, exchanged for a quantity of beer of \( \text{pefsu} \) 2. (Find the number of pints.)'"

"You are to turn the 100 loaves of \( \text{pefsu} \) 10 back into flour, \text{i.e.,} 10 gallons. Multiply by 2; the result thereof is 20. Then shall you say 'This is their exchange.'"

Another set of examples drawn from everyday life is R. 82-84, which deal with the food of the domestic animals. These sums are unfortunately full of mistakes and contain other obscurities. Their calculations are very simple, and their importance is rather for the archaeologist than for the mathematician.

M. 21 shows how to find the "average" of two lots of loaves, 20 containing each \( \frac{1}{8} \) gallon of flour, and 40 containing each \( \frac{1}{10} \) gallon. The 20 are shown to contain in all \( 2 \frac{1}{2} \) gallons, and the 40 also \( 2 \frac{1}{2} \) gallons. The total flour is 5 gallons, and this, when made into 60 loaves \((20 + 40)\), will allow \( \frac{1}{12} \) gallon for each loaf. It is only when we reach the answer that we fully understand what was meant by the
"average"; the total number of loaves is to remain the same, but all the loaves are to be of the same size.

R. 63 is an interesting example because the solution is completely on modern lines:

"Example of dividing 700 loaves among four men, \( \frac{3}{8} \) to one, \( \frac{1}{4} \) to another, \( \frac{1}{8} \) to another, and \( \frac{1}{4} \) to another. Let me know the share of each."

The 700 loaves are to be divided in the proportions \( \frac{3}{8}, \frac{1}{4}, \frac{1}{8}, \frac{1}{4} \). The proportions are first added; total \( 1\frac{1}{4} \) (1\( \frac{1}{2} \)). This is now inverted, or, as an Egyptian says, 1 is divided by it; result \( \frac{1}{2} + \frac{1}{14} \). This fraction (\( \frac{1}{2} \) in our notation) is now taken of 700, the result being 400. The 400 is lastly multiplied by each of the proportions \( \frac{3}{8} \) etc. in turn, and the results are 266\( \frac{3}{8} \), 200, 133\( \frac{3}{8} \), 100. The result is proved by simple addition.

The procedure here used can be found in any modern elementary primer of arithmetic: "Add the proportions, divide the number to be distributed by their sum; multiply the result by each of the proportions in turn."

Our last example of arithmetic is R. 66, which illustrates an important point:

"Ten gallons of fat have been issued for a year. What is the daily ration thereof?"

The 10 gallons are turned into \( \frac{1}{20} \) (320 per gallon), giving 3200. The year is turned into 365 days, and 3200 is divided by 365, giving \( 8\frac{3}{8} + \frac{1}{10} + \frac{1}{181} \) 20 per day. The example ends with the words: "You may do similarly for any example like this which is put to you."

The interest of these last words lies in the fact that only twice, here and in R. 61\( \delta \), do we find it stated that a method is of general application. In the present case the words mean no more than that whenever a year's supply of any commodity is to be apportioned out into days it must be divided by 365. This must have been fairly obvious, and the phrase hardly deserves the name formula.

In R. 61\( \delta \), however, the words used have more force. The problem is to make \( \frac{\pi}{\text{tjt gebt}} \), which probably means a fraction with numerator 1 and denominator an odd number.\(^1\) The solution is

\(^1\) GUNN'S suggestion in Journ. Eg. Arch., xii, p. 134, is undoubtedly right. Obviously \( \frac{\pi}{\text{tjt gebt}} \) of a fraction with even denominator could be—and, as the papyrus itself shows, was—taken directly.
to take $\frac{1}{2} + \frac{1}{3}$, and the scribe adds "Behold it is done likewise in the case of any tyt gebt which may occur." Here we have certainly a general formula, the only one which is explicitly stated in the whole literature of Egyptian mathematics. Writers on ancient mathematics have been inclined to reproach the Egyptians with this failure to lay down general rules in their mathematical papyri, and in consequence these have been regarded as collections of examples rather than as instructional treatises. This reproach is removed completely, to my mind, by Gunn’s admirable suggestion that the examples are intended to serve as formulæ, but formulæ in which, either for convenience or of necessity, numbers have to take the place of symbols. Thus the writer of M. 14, wishing to give the formula for the volume of a truncated pyramid whose top and bottom are squares of sides a and b respectively, and whose height is h, cannot write, as we do, $V = \frac{h}{3} (a^2 + ab + b^2)$, for he has evolved no symbols. If he were to write out "Square the side of the upper bounding square, add to it the product of the sides of both bounding squares, and the square of the side of the lower bounding square; divide this sum by 3, and multiply by the height," his formula would be exceedingly clumsy, if indeed it remained intelligible at all. What he does, therefore, is to give to the three dimensions the simple numbers 2, 4, and 6, and to write "Square 2, multiply 2 by 4, square 4. Add the results and multiply by one third of 6." All this is just as clear as if a, b, and c had been used for the data, and the example serves not only as a specific case but also as a general formula.

Geometry of two Dimensions.

The Egyptians had explored the field of two-dimensional geometry with considerable success. They had correctly determined the area of the rectangle and (in the writer’s opinion) of the triangle, and found a remarkably good approximation to the area of the circle.

2 It is indeed not altogether certain that "example of" is the correct translation for the words $\text{tp n}$, with which most of these sums are introduced, although in the last line of R. 66 quoted above, p. 428, the meaning "example" seems more suitable than any other.
The Rectangle.

That the Egyptians had found the correct expression for the area of the rectangle amounts to no more than saying that they had the conception of square measure, for this in itself involves the ability to divide up a rectangular figure by means of two sets of parallel lines at right-angles the one set to the other into squares whose side is the unit of length and which themselves constitute the unit of area. An interesting application of the formula—if such it can be called—for the area of a rectangle is to be seen in M. 6, where we are given the area, 12 acres, of a rectangle and asked to find the length and breadth, supposing that the latter is \( \frac{3}{4} \) of the former. See above, p. 421.

The Triangle.

The writer believes that the Egyptians had found the correct formula, half the base multiplied by the vertical height, for the scalene triangle, though he doubts whether a rigid logical proof of this can be given from the material at our disposal. What they did was to multiply half the tep-ro, "mouth," by the meryet, whatever this may mean. About the tep-ro there is no possible doubt; the figures which accompany the problems on triangles are all, for some reason which no one has explained, drawn with one angle very small, and the "mouth" is the short side opposite to this. In other words, if we call the sharp point the apex, then the mouth is the base. What then is the meryet? Meryet is a common Egyptian word for the edge of a river or sea, more particularly an artificial edge, a quay. Some writers, laying stress on the meaning "edge," have argued that it must mean the side, i.e., one of the long sides, and, in order to meet the obvious question Which of the two sides?, have further supposed that in the examples where meryet appears (R. 51, 52; M. 4) the triangle was regarded as being isosceles. This would make it necessary to believe that the Egyptians made the mistake of multiplying half the base by the slant instead of the vertical height, an error which might be palliated by the fact that in the examples actually illustrated, where the angle at the apex is small, the error would not be very significant. Some colour is lent to this belief by the fact that in Ptolemaic times and later the area of four-sided fields was obtained by the formula

\[
\text{Area} = \frac{a+c}{2} \times \frac{b+d}{2},
\]
where \(a, b, c,\) and \(d\) are the sides; when the field happened to be
triangular \(d\) was made zero and the formula became \(\frac{a + c}{2} \cdot \frac{b}{2}\), which
gave \(\frac{1}{2}ab\) when \(a\) and \(c\) were equal, i.e., where the triangle was
isosceles. But this field formula was only an approximation for
taxation purposes, as is shown by the neglect of small fractions in its
application, and the existence of such a formula in late times does not
prove that the Egyptian mathematician of 2000 B.C. did not possess a
better one.

The supposition that by meryet the Egyptians meant the slant
height of an equilateral triangle has not met with general acceptance.
Struve, the recent editor of the Moscow papyrus, advances a modified
version of it. He believes, as Eisenlohr did, that the triangles in the
textbooks in which the meryet is mentioned are isosceles, but that the
meryet is not one of the long sides but the vertical height, i.e., the
perpendicular from apex to base. This perpendicular, he remarks,
would divide the triangle into two equal halves, between which it would
form a "limit," thus explaining, according to him, the choice of the
technical term meryet. This explanation, however, is not very con-
vincing, for the word meryet does not mean "a limit" or "division"
between two things, but the "edge" of one thing. It is just worth while
 remarking, too, that the illustrations in the papyri do not wholly bear
out his supposition. Though the triangle in R. 53 is approximately
isosceles, that in R. 51 is definitely right-angled.\(^1\) The evidence of
these roughly drawn figures is of very uncertain value, but it must either
be rejected altogether or respected when it tells against, just as much as
when it tells in favour of, a theory.

Let us now turn to the examples themselves and see whether
a case cannot be made out for the belief that meryet is the vertical
height, and that the triangle dealt with is meant to be scalene, and
the solution consequently general.

(a) The word meryet is definitely connected with the edge of
a river or sea, and its use for the side of a triangle regarded as an
edge therefore seems out of place. If we take its very common mean-
ing of quay, i.e., a horizontal structure built (in Egypt at least) over
a sloping bank of the river to make a landing-place, it is not hard to

\(^1\) Struve's restoration of the damaged triangle in M. 4 as isosceles seems
to me very doubtful.
see how it might have been applied by mathematicians to the perpen-
dicular AD in Fig. 2. The Egyptians drew their triangles lying, as
we should say, on their sides, with the apex to right or left and the
tep-ro, or base, roughly horizontal at the opposite side. Gunn may
even be right in supposing that the conception of a quay also included
its vertical edge and that this also helped out the simile, AD being
the quay, with an edge DC at right
angles to it, and AC the river bank.

(δ) In R. 5 1 we have to find
the area of a triangle of base 4 and
meryet 10, and we are told to take
half of the 4 "to get its\(^1\) rectangle"
(or possibly "to make it\(^1\) rectangu-
lar"). It is clear from this that the
Egyptian regarded his triangle as
equal in area to a rectangle on half its base. It is of course conceivable
that the other side of this rectangle was one of the long sides of the
triangle, but in this case we must suppose the triangle isosceles to avoid
a double solution, and we must also suppose that the method applied
only to triangles with very sharp apexes, for in others the error would
be too patent to escape notice. It seems much more natural, however,
to accept the hint of a graphic solution offered by this reference to

\begin{center}
\includegraphics[width=0.5\textwidth]{fig2.png}
\end{center}

\textbf{Fig. 2.}

\begin{center}
\includegraphics[width=0.5\textwidth]{fig3.png}
\end{center}

\textbf{Fig. 3.}

a rectangle on half the base, and to suppose that the Egyptians had
rightly solved the scalene triangle by means of some such figure as
Fig. 3. This belief is strengthened by R. 5 2, where a truncated
triangle is proposed for solution. Here we halve the sum of the
parallel sides "in order to get its rectangle" and multiply by the
meryet. Surely the idea of halving the sum of the parallel sides can
only have come from a graphic solution such as that shown in Fig. 4,

\(^1\) The gender shows that the triangle is meant.
and with the truth so clearly in front of their eyes it is not possible to believe that the Egyptians were so silly as to multiply half this sum not by the correct vertical height of the figure but by one of the slant sides (assumed in this case to be equal). Surely meryet can here only mean the vertical height, and, if here, so also in the case of the triangle.

(c) Although the measurement of the meryet is actually written in R. 51 and 52 above the middle of one of the long sides (so also in M. 4) this is by no means an unnatural position for the length of the perpendicular from apex to base, which, in a triangle drawn in the Egyptian fashion with base vertical, is a horizontal measurement. But what is more, in R. 53 a triangle whose base is marked in as $2\frac{1}{4}$ (reels-of-thread) and whose area of $7\frac{3}{8}$ acres is written inside it (the usual place for an area) has the meryet of 7 (reels) written at the apex, a position totally unsuitable if the figure (7) gave the length of a side.

(d) The Egyptians found correctly the volume of a truncated pyramid, and evaluated the area of the circle with a very close measure of accuracy. We are within our rights when we ask ourselves whether these achievements are consistent with the belief that the area of a triangle was to be obtained by multiplying half its base by one of the other sides, or with an inability to find the area of any triangles save those which were either isosceles or right-angled. If we answer this question in the negative, as I believe we must, then meryet is the vertical height and the solution is general.

One other point in connection with the terminology of the triangle must be noticed here, though it does not help us to decide the vexed question of the meaning of meryet. In M. 7 and 17 we read of triangles whose "length" and "breadth" are to be found. Struve and Neugebauer both assume that these terms can only refer to right-angled triangles, and in support of this point to the illustration of M. 17 (M. 7 is not illustrated), a triangle which, though it is not exactly right-angled, may well have been intended to be, especially as it is very roughly drawn.

Now this view may be right. The avoidance of the usual terms tep-ro and meryet suggests that the triangle is here of a special kind;
and since a right-angled triangle is half a rectangle the transference to it of the ordinary terms "length" and "breadth," used to define a rectangle, seems reasonable. On the other hand, we must just bear in mind the possibility that the triangles are not right-angled and that the "length" meant is the perpendicular length AD in Fig. 2, i.e. the meryet, or what we should call the perpendicular height. The data are not sufficient to decide this point, and the other conditions and the working out of the two problems would suit either hypothesis. On the whole the writer is inclined to the view that the triangles are not necessarily right-angled.

In both problems the area is given and the "length" and "breadth" are to be found; In M. 17 we are told that the breadth is $\frac{3}{4} + \frac{1}{15}$ of the length; in M. 7, however, we are merely given that the ideb, "bank," is $2\frac{1}{2}$, and the context shows that the "bank" is the technical term for the ratio length divided by breadth.¹

The Circle.

The best achievement of the Egyptians in two-dimensional geometry is undoubtedly their close approximation to the area of the circle. They squared eight-ninths of its diameter, giving $\frac{25}{8} \times r^2$, where $r$ is the radius. Comparing this with our own $\pi r^2$ we get for the Egyptian value of $\pi \frac{25}{8}$ or $3\frac{1}{2}$, a very close approximation to the $3\frac{1}{2}$ which we find good enough for practical purposes. We have no idea how this result was obtained. The expression of the area as a square suggests a graphic solution.

Geometry of three Dimensions.

Just as the geometry of two dimensions was stimulated by the need to measure land, so that of three developed out of the necessity of determining the quantity of corn in a bin or the amount of stone needed to build a wall or a pyramid. The correct formula for the determination of a parallelepiped follows as a matter of course from the ability to conceive and measure three-dimensional units. The length is multiplied by the breadth, and the result by the height (R. 44-46). That it was fully realised that this was equivalent to multiplying the area of the base by the height is clear from the fact that the volume of

¹It would seem a just inference from M. 17 that the converse ratio, breadth divided by length, had no technical name.
a cylinder was got by squaring eight-ninths of the diameter of the circular base, thus determining its area, and multiplying by the height (R. 41-43).

Pyramids were an object of considerable interest, and a number of problems (R. 56-59) deal with their slope as determined by their vertical height and the length of a side of the square base. An example will make this clear (R. 56):

"Example of reckoning out a pyramid 360 in length of side and 250 in its vertical height. Let me know its batter."

"You are to take half of 360; result 180.
"You are to operate on 250 to find 180.
Result, \( \frac{1}{2} + \frac{1}{8} + \frac{1}{30} \) of a cubit.

Fig. 5 shows what is done in the opening lines. The angle of slope GFD is determined by what we now call its cotangent, namely \( \frac{GF}{GD} \), which in this case is \( \frac{1}{2} \frac{8}{8} \). This ratio is reduced to a fraction of

1 By batter is meant the slope of the four sides, not, of course, that of the four edges in which the pairs of sides meet.

2 The working has been omitted by the Egyptian scribe. The words "of a cubit" are, strictly speaking, illogical, but the practical reason for their introduction soon becomes evident. See below.
correct form, namely $\frac{1}{2} + \frac{1}{3} + \frac{1}{5}$. But this abstract ratio is of no use to the mason who has to shape the stones for the outer casing of the pyramid, for whom the calculation is clearly intended; and so it is treated as a fraction of a cubit and reduced to handbreadths, there being 7 handbreadths in a cubit.\textsuperscript{1} The result is $5\frac{2}{3}$ handbreadths, and all the mason has to do is to take his squared stone (Fig. 6), measure a cubit OP vertically on one of its edges, then $5\frac{2}{3}$ handbreadths horizontally, and draw a line\textsuperscript{2} from the point R thus found to the bottom corner O from which he started.

Egyptian solid geometry reaches its highest point in problem M. 14, in which the volume of a truncated pyramid, i.e. a pyramid with its top cut off, is correctly found. The working shown corresponds to the formula

$$V = \frac{h}{3} (a^2 + ab + b^2),$$

where $a$ and $b$ are the sides of the squares which bound the figure above and below, and $h$ is its vertical height. The word actually used for height is a άτριόξ ιεγόμενον, but there is no good reason for supposing that the slant height and not the vertical height was intended. Seeing that the slope of the Egyptian pyramids lies for the most part between 43° and 55°, the error caused by the use in this formula of slant height instead of vertical would be so enormous that it could never have escaped detection in a country one of whose main preoccupations must at all times have been the building of the reigning king's pyramid and tomb. Moreover, it is to credit the Egyptians with alternately too much and too little mathematical sense to suppose that the men who evolved the difficult factor $\frac{1}{3}(a^2 + ab + b^2)$ were then foolish enough to multiply it by the slant height.

Several attempts have been made to show how the Egyptians obtained this formula. Some suggest that it was found by cutting the truncated pyramid up into smaller and simpler solids,\textsuperscript{3} others that the

\textsuperscript{1} A cubit or forearm is about 20.6 inches, and a handbreadth, which is a seventh of this, contains four fingerbreadths, being measured across the fingers and so neglecting the thumb.

\textsuperscript{2} Such lines, drawn usually in red, are frequently found on the outer stones of pyramids and other sloped constructions.

element \( \frac{1}{3}(a^2 + ab + b^2) \) is an "average" of three areas,\(^1\) and yet others that the solid was treated as the difference between the original pyramid and the smaller one removed from its top.\(^2\) However this may be, the formula remains, a testimony to Egyptian genius of 2000 B.C. and earlier.

In the domain of solid geometry one more problem calls for notice. It is No. 10 of the Moscow papyrus. Struve in his publication of the papyrus maintains that this problem determines the area of the curved surface of a hemisphere—correctly, if the Egyptian value of \( \frac{256}{81} \) for \( \pi \) be accepted. If this interpretation of the problem were right our estimation of Egyptian mathematics would be much enhanced, for the very idea of the area of a curved surface other than one which, like that of the cylinder, can be transferred by rolling or unfolding to a plane surface, is a highly advanced mathematical conception. Unfortunately, however, as I have tried to show elsewhere,\(^3\) Struve’s translation and interpretation of the problem are inadmissible. Grammatical considerations make it quite clear that the scribe has omitted in copying a word and a figure which gave a second dimension, and this and other reasons make it impossible to maintain the hypothesis of a hemisphere.

General Character of Egyptian Mathematics.

In conclusion I should like to deal with one or two general considerations regarding Egyptian mathematics. When in 1926 I published the Rhind papyrus I wrote "The outstanding feature of Egyptian mathematics is its intensely practical character." This general statement I see no reason to modify; the full publication of the Moscow papyrus merely serves to bear it out. At the same time I did perhaps go too far in saying that in the problems of Rhind "everything is expressed in concrete terms,"\(^4\) for this is not true of problems such as R. 24-34, which deal with abstract numbers, and

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1\(^{\text{VOGEL, in op. cit., xvi, pp. 242-249.}}\)
2\(^{\text{THOMAS, in op. cit., xvii, pp. 50-52.}}\)
3\(^{\text{Journ. of Eg. Arch., xvii, pp. 100-104. See too Neugebauer, Die Geometrie der ägyptischen mathematischen Texte in Quellen und Studien zur Geschichte der Mathematik, Abteilung B, Nr. 20 (1931).}}\)
4\(^{\text{My contention that the use of the concrete word 'h', "heap," for "quantity" is evidence for the concrete character of the system is, as several writers have pointed out, disproved by the fact that the word is here given the abstract determinative.}}\)
which explicitly set out to find a "quantity," not a number of men or a number of loaves. The realisation, too, that the examples in the papyrus are in themselves to be regarded rather as formulæ where simple numbers take the place of our \( x \) and \( y \) than as isolated sums forces us to take a more favourable view of the nature of Egyptian mathematics. There are, moreover, problems which, though stated in concrete terms, contain figures so fantastic or envisage cases so improbable that they could never arise in ordinary practical life. Thus the division of 100 loaves into five lots in arithmetical progression, such that one-seventh of the three first is equal to the two last, can hardly be called a practical problem. In the same way the summing of a geometric series, though it may serve a practical end in modern mathematics, certainly served none in Egyptian. Examples of this kind do suggest that, while mainly occupied with practical problems, the Egyptians occasionally allowed themselves to observe and even to record a result or a method which had no obvious and direct application to the concrete facts of life. But there is no sign that such things were regarded as more than idle curiosities.

There is a further question, which has by several writers been curiously confused with the last, but which in reality is totally distinct from it. Was Egyptian mathematics "scientific" in the sense that modern mathematics is? Among those who have written on this question the majority have declared themselves in favour of its scientific nature. The arguments in favour of this view are summed up very shortly by Vogel in a passage which may be quoted:—

"In concluding I must yet again draw attention to one point, namely to the gradual advance of scientific knowledge, which is not merely evidenced by the already existing conception of abstract number, by numerous problems of fanciful nature involving figures

\[1 \text{ See Gunn-Peet, Journ. Eg. Arch., xv., pp. 184-185.} \]

\[2 \text{ It is significant that of the two examples here quoted the first is "solved" by a method which is completely fraudulent and shows that the problem was set from the answer, while the second is nothing more than a piece of observation from a multiplication table (see above, pp. 425-6).} \]

\[3 \text{ See Kurt Vogel, Die Grundlagen der ägyptischen Arithmetik, pp. 52-53, 183-184; Wieleitner, War die Wissenschaft der alten Ägypter nur praktisch? in Isis, ix, pp. 21 ff.; Abel Rey, La Science orientale avant les Grecs, pp. 251 ff., 281 ff.; O. Gillain, La science égyptienne, L'Arithmétique au Moyen Empire, pp. 308-311.} \]
which could never occur in practice, by the grouping of the single examples according to the identity of their content and the division of the papyrus itself into groups which hang together, or by the gradual evolution of rules and laws, but which is above all incontrovertibly shown by the desire for a verification of the result, for a test, for a proof of the answer.”

Now how much does all this evidence prove? That the conception of abstract number existed is merely equivalent to saying that the Egyptians had passed beyond a certain primitive stage of thought where practically no mathematics is possible except such elementary operations as that of counting 8 sheep and 5 sheep and observing that together they count 13 sheep; to possess the concept of abstract number is an a priori condition of a mathematical system, not a proof of its scientific nature.

That problems occur which could not occur in practice might be urged as evidence for a certain non-practical speculative interest in mathematics, as we saw above, but nothing more. That sums of the same kind are grouped together proves nothing more than the existence of some elementary idea \(^1\) of orderly arrangement.

In the papyri themselves there is little evidence of the actual evolution of rules and laws. There are only two cases where a general formula is explicitly stated to be such (R. 61b and R. 66); and, even if the sums themselves are intended to serve as formulæ (see above, p. 429), the fact that the Egyptians had evolved no better means of stating a formula than that of giving three or four examples of its use is hardly a tribute to the scientific nature of their mathematics.

And lastly, the desire for a proof of the result does not in itself constitute scientific method in the modern sense. Obviously a system of mathematics which merely guessed its results would be no system but a chaos; it would be in conflict with experience at every hour of the day. It must defend itself against this by some kind of appeal to

\(^1\) The disgraceful chaos of the Moscow papyrus shows that this, elementary as it seems, was not universal. Let us hope that a mere scribe rather than a mathematician was to blame. The arrangement of Rhind itself is logically far from perfect, and, while we may be prepared to find excuses for this in the supposition that the collection was culled, somewhat at random, from other mathematical treatises, we cannot submit to its being held up as a model of consistent and logical arrangement.
logic. This may take the form of a proof *a posteriori*, *i.e.*, a formal testing of the result, showing that it does fulfil the conditions of the problem; or it may take the form of a demonstration *a priori*, beginning with the data and proceeding to the result which will satisfy them. The former, which is the Egyptian method, may be logically satisfactory in any particular case, or even in any group of similar cases. But this method can never constitute a scientific system in the strict sense. The conditions of such a system are "that every statement ... shall be open to common criticism and shall be protected against it from the outset by a system of demonstrations." Such is Neugebauer's definition, and he adds "The true mark of scientific method lies, to my mind, in the striving after an 'objective' demonstration of the statements made, *i.e.*, a proof independent of the subject." By such methods alone is it possible to build up a system of mathematics which forms a cogent, organic, and interconnected whole, and which can inspire full confidence in its users. Of this attitude of mind, with its demand for demonstration, there is no sign in Egyptian mathematics. The proofs given are the merest checkings of the figures and give no insight into the mathematical inwardness of the problems. Some of them are unsound in principle, and only "work" in the particular case owing to favourable circumstances. No desire is shown to obtain consistency in the proofs applied to similar cases. There is no attempt made to regularise the method of common denominator used in adding fractions. The geometrical problems are not proved at all. The table of the resolutions of 2-fractions into aliquot parts, with its acceptance of empirical results and its lack of any desire to achieve homogeneity or to view the problem as a single whole, is in itself a monument to the lack of the scientific attitude of mind in the Egyptians.

Are we then to damn Egyptian mathematics once and for all by attaching to it the epithet "unscientific" because it does not conform to our modern conception of scientific method? Not for a moment. The word unscientific conveys a reproach, and those who have studied what Egypt did for mathematics before 2000 B.C. are moved

1 *Die Grundlagen der äg. Arithmetik*, p. 93. In what follows I am under deep obligations to Neugebauer.

2 To suppose, as some seem to do, that the papyri which have survived are merely the products of stupid schoolboys, and that there existed really scientific treatises on mathematics, is surely to misunderstand completely the nature of the evidence.
to admiration rather than criticism. The true escape from the
dilemma which seems to impend over us here has been pointed out
by Neugebauer. He asks whether the question How far was
Egyptian mathematics scientific or unscientific? is legitimate. "No
attempt is made," he writes, "to answer the very necessary preliminary
question whether the application of our intellectual categories to these
early civilisations has any meaning, but we proceed at once to enquire
in exactly what percentage (however small it may be) they are to be
found in those times. The possibility that the intellectual structure of
these civilisations was of a fundamentally different order is not taken
into consideration." Heinrich Schäfer has shown that in the realm of
art it is meaningless to apply our categories to Egyptian work and to
say that the artist had no perspective or that his perspective was
wrong. His whole point of view was different, and the result was
different in consequence. So, too, in the case of mathematics it may
be doubted whether it is legitimate to ask to exactly what extent the
Egyptian method was scientific in the modern sense. Because the
Egyptian achieved results which are still acceptable to us we must
not assume that he did so, or ought to have done so, by a mental
attitude or by methods identical with ours. The subject matter was,
it is true, the same. Two and two already made four in Ancient Egypt,
and the fallacy of Undistributed Middle was as much a fallacy then as
it is now. When the Egyptian was logical he could only be logical
in the same way as ourselves. But we must not exclude the possi-
bility that there is much in his mathematics, as in his thought generally,
which was not illogical but un-logical; and when we find a piece of
logically sound arithmetic employed in what seems to us an un-logical
cause we must not be shocked. Still less must we blame.

Here, however, we are on the borders of philosophy, and our
concern is primarily with mathematics. Let it suffice to say in
conclusion that the Egyptians devised a workable and practical system
of notation, performed with ease simple and even complicated arith-
metical operations, explored with success the field of simple geometry
in two and three dimensions, and devised methods of dealing with
most of the mathematical problems of everyday life. That they did
not reach the conception of scientific mathematics and its dependence
on cogent a priori demonstration is merely another instance of the
vast debt which the world owes to the Greeks.