EXIT TIMES FOR INTEGRATED RANDOM WALKS

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Abstract. We consider a centered random walk with finite variance and investigate the asymptotic behaviour of the probability that the area under this walk remains positive up to a large time \( n \). Assuming that the moment of order \( 2 + \delta \) is finite, we show that the exact asymptotics for this probability are \( n^{-1/4} \). To show these asymptotics we develop a discrete potential theory for integrated random walks.

1. Introduction, main results and discussion

1.1. Background and motivation. Let \( X, X_1, X_2, \ldots \) be independent identically distributed random variables with \( \mathbb{E}[X] = 0 \). For every starting point \((x, y)\) define

\[
S_n = y + X_1 + X_2 + \ldots + X_n, \quad n \geq 0.
\]

and

\[
S_n^{(2)} = x + S_1 + S_2 + \ldots + S_n = x + ny + nX_1 + (n-1)X_2 + \ldots + X_n.
\]

Sinai [13] initiated the study of the asymptotics of the probability of the event

\[
A_{n} := \left\{ S_{k}^{(2)} > 0 \text{ for all } k \leq n \middle| S_{0} = S_{0}^{(2)} = 0 \right\}.
\]

Assuming that \( S_n \) is a simple symmetric random walk he showed that

\[
C_{1} n^{-1/4} \leq \mathbb{P}(A_{n}) \leq C_{2} n^{-1/4}.
\]

The same bounds were obtained for some other special cases in [14].

Aurzada and Dereich [2] have shown that if \( \mathbb{E}e^{|X|} < \infty \) for some positive \( \beta \) then

\[
C_{*} n^{-1/4} \log^{-\gamma} n \leq \mathbb{P}(A_{n}) \leq C^{*} n^{-1/4} \log^{\gamma} n
\]

with some positive constants \( C_{*}, C^{*} \) and some finite \( \gamma \). Bounds (2) are just a special case of the results in [2] for \( q\)-times integrated random walks and Levy processes. Dembo, Ding and Gao [5] have recently shown that (1) is valid for all random walks with finite second moment.

Exact asymptotics for \( \mathbb{P}(A_{n}) \) are known only in some special cases. Vysotsky [15] have shown that if, in addition to the second moment assumption, \( S_n \) is either right-continuous or right-exponential then

\[
\mathbb{P}(A_{n}) \sim C n^{-1/4}.
\]

(Here and throughout \( a_{n} \sim b_{n} \) means that \( \frac{a_{n}}{b_{n}} \to 1 \) as \( n \to \infty \).)

\[
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\]

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It is natural to expect that (3) holds for all driftless random walks with finite variance.

If one replaces the second moment condition by the assumption that $X$ belongs to the normal domain of attraction of the spectrally positive $\alpha$-stable law with some $\alpha \in (1, 2]$, then (1) and (3) remain valid with $n^{-(\alpha-1)/2\alpha}$ instead of $n^{-1/4}$, see [5] and [15].

The methods used in the above mentioned papers are quite different. It is not clear what is the most natural tool for this problem. Here we propose another approach to this problem. More precisely, we develop a potential theory for integrated random walks, which allows one to determine the exact asymptotic behaviour of $P(A_n)$. It can be seen as a continuation of our studies of exit times of multidimensional random walks, see [6, 7].

It is clear that the sequence $\{S_n^{(2)}\}_{n \geq 1}$ is non-markovian. This fact complicates the analysis of the integrated random walk. However, it is possible to obtain the markovian property by increasing the dimension of the process. More precisely, we consider the process

$$Z_n := (S_n^{(2)}, S_n).$$

Then, the first time when $S_n^{(2)}$ is not positive coincides with the following exit time of $Z_n$

$$\tau := \min\{k \geq 1 : Z_k \notin \mathbb{R}_+ \times \mathbb{R}\}.$$

In our recent paper [7] we suggested a method of studying random walks conditioned to stay in a cone. Similarly in the case of the integrated random walks we have a (quite simple) cone $\mathbb{R}_+ \times \mathbb{R}$, but the process $Z_n$ is 'really' Markov, i.e. the increments are not independent. We show that the method from [7] can be adapted to the case of Markov chain $Z_n$, and this adaptation allows one to find asymptotics of $P_z(\tau > n)$ for every starting point $z = (x, y)$.

1.2. Main result. Our approach essentially relies on a strong normal approximation and corresponding results for the integrated Brownian motion. Hence we will start with results and notation for the integrated Brownian motion. This process is also known as the Kolmogorov diffusion.

Let $B_t$ be a standard Brownian motion and consider a two-dimensional process $(\int_0^t B_s ds, B_t)$. Since this process is gaussian, one can obtain by computing correlations that the transition density of $(\int_0^t B_s ds, B_t)$ is given by

$$g_t(x, y; u, v) = \frac{\sqrt{3}}{\pi t^2} \exp \left\{ - \frac{6(u - x - ty)^2}{t^3} + \frac{6(u - x - ty)(v - y)}{t^2} - \frac{2(v - y)^2}{t} \right\}.$$

Let

$$x^{bm} := \min\left\{ t > 0 : x + yt + \int_0^t B_s ds \leq 0 \right\}.$$

The behaviour of the killed at leaving $\mathbb{R}_+ \times \mathbb{R}$ version of $(\int_0^t B_s ds, B_t)$ was studied by many authors. Here we will follow a paper by Groeneboom, Jongbloed and Wellner [9], where one can also find a history of the subject and corresponding references. In particular they found the positive harmonic function for this process,
which is given by the following relations:

\[
    h(x, y) = \begin{cases} 
        \left(\frac{2}{3}\right)^{1/6} \frac{y}{x^{1/6}} U \left(\frac{1}{6}, \frac{4}{3}, \frac{2y^3}{9x}\right), & y \geq 0 \\
        -\left(\frac{2}{3}\right)^{1/6} \frac{1}{6x^{7/6}} e^{2y^3/9x} U \left(\frac{7}{6}, \frac{4}{3}, -\frac{2y^3}{9x}\right), & y < 0
    \end{cases}
\]  

(4)

where \( U \) is the confluent hypergeometric function. Function \( h(x, y) \) is harmonic in the sense that \( D h = 0 \) on \( \mathbb{R}^+ \times \mathbb{R} \), where \( D = y \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial y^2} \) is the generator of \( \left( \int_0^t B_s ds, B_t \right) \).

Using the explicit density of \( P_{(0,1)}(\tau_{bm} > t) \) found in [11], they derived asymptotics

\[
    P_{(x,y)}(\tau_{bm} > t) \sim \kappa h(x, y) t^{-1/4}, \quad t \to \infty, \quad (5)
\]

where \( \kappa = \frac{3\Gamma(1/4)}{2^{3/4} \pi^{3/2}} \).

The harmonic function \( h \) defined in (4) helps us to construct the corresponding harmonic function for the killed integrated random walk. As function \( h \) is defined only for \( z \in \mathbb{R}^+ \times \mathbb{R} \), we extend it to \( \mathbb{R}^2 \) by putting \( h = 0 \) outside \( \mathbb{R}^+ \times \mathbb{R} \).

Function \( h \) is harmonic for the killed integrated Brownian motion but not for the killed integrated random walk. To overcome this difficulty we introduce a corrector function for \( z = (x, y) \in \mathbb{R}^2 \),

\[
    f(x, y) = E_z h(Z(1)) - h(z).
\]

(6)

This function is well defined since we have extended \( h \) to the whole plane. Now we are in position to define the harmonic function for the killed integrated random walk. For \( z \in \mathbb{R}^+ \times \mathbb{R} \) let

\[
    V(z) = h(z) + E_z \sum_{l=1}^{\tau-1} f(Z_k).
\]

(7)

This function is harmonic for the killed integrated random walk in the sense that

\[
    E_z [V(Z_1), \tau > 1] = V(z) \text{ for all } z \in \mathbb{R}^+ \times \mathbb{R}. \quad (8)
\]

It is not at all clear that function \( V \) in (7) is well-defined and positive. In fact this is the most difficult part of the proof and it is done in Section 2.

Our main result is the following theorem.

**Theorem 1.** Assume that \( \mathbb{E}X = 0, \mathbb{E}[X^2] = 1 \) and \( \mathbb{E}|X|^{2+\delta} < \infty \) for some \( \delta > 0 \). Then the function \( V \) from (7) is well-defined and strictly positive on

\[
    K_+ := \{ z : P_z(Z_n \in \mathbb{R}^+ \times \mathbb{R}^+, \tau > n) > 0 \text{ for some } n \geq 0 \}.
\]

Moreover,

\[
    P_z(\tau > n) \sim \kappa \frac{V(z)}{n^{1/4}} \quad \text{as } n \to \infty \quad (9)
\]

and

\[
    P_z \left( \left( \frac{S_n^{(2)}}{n^{3/2}}, \frac{S_n}{n^{1/2}} \right) \in \cdot \bigg| \tau > n \right) \to \mu \quad \text{weakly}, \quad (10)
\]

where \( \mu \) has density

\[
    Ch(x, y)g_1(0, 0; x, y), \quad (x, y) \in \mathbb{R}^+ \times \mathbb{R}.
\]

From (9) and the total probability formula we obtain
Corollary 2. For every random walk satisfying the conditions of Theorem 1 holds

\[ P_0(A_n) \sim C \frac{n^{1/4}}{n} \]

with

\[ C = E[V((X, X)), X > 0]. \]

1.3. Local asymptotics for integrated random walks. Caravenna and Deuschel [4] have proven a local limit theorem for \( Z_n \) under the assumption that the distribution of \( X \) is absolutely continuous. Using similar arguments one can show that if \( X \) is \( Z \)-valued and aperiodic then

\[ \sup_{\tilde{z}} \left| n^{2+1/4} P_{\tilde{z}}(Z_n = \tilde{z}, \tau > n) - \mathbb{E}[h\left((\frac{n^{3/2}}{n^{1/2}}, \frac{\tilde{y}}{n^{1/2}})\right)] \right| \rightarrow 0. \]  

(11)

Combining this unconditioned local limit theorem with (10) one can derive a conditional local limit theorem:

\[ \sup_{\tilde{z}} \left| n^{2+1/4} P_{\tilde{z}}(Z_n = \tilde{z}, \tau > n) - \mathbb{E}[h\left((\frac{n^{3/2}}{n^{1/2}}, \frac{\tilde{y}}{n^{1/2}})\right)] \right| \rightarrow 0. \]  

(12)

Furthermore, for every fixed \( \tilde{z} \in K_+ \),

\[ \lim_{n \to \infty} n^{2+1/2} P_{\tilde{z}}(Z_n = \tilde{z}, \tau > n) = V(z) V'(z) \]  

(13)

with some positive function \( V' \).

The proof of (12) and (13) repeats virtually word by word the proof of local asymptotics in [7], see Subsection 1.4 and Section 6 there. For this reason we do not give a proof of these statements.

Having (12) one can easily show that

\[ P_0(A_n | Z_{n+2} = 0) \sim C \frac{n^{1/2}}{n} \]

with some positive constant \( C \). A slightly weaker form of this relation was conjectured by Caravenna and Deuschel [4, equation (1.22)].

Aurzada, Dereich and Lifshits [3] have recently obtained lower and upper bounds for the integrated simple random walk,

\[ cn^{-1/2} \leq P_0(S_{1(2)}^{(2)} \geq 0, \ldots, S_{4n}^{(2)} \geq 0 | S_4 = 0, S_{4n} = 0) \leq Cn^{-1/2}. \]

1.4. Organisation of the paper. In [7] we have suggested a method of investigating exit times from cones for random walks. In the present paper we have a Markov chain instead of a random walk with independent increments. But it turns out that this fact is not important, and the method from [7] works also for Markov processes.

The first step consists in construction of a harmonic function \( V_0(z) \). As in [7] we start from the harmonic function for the corresponding limiting process. Obviously,

\[ \left( \frac{S_{n1}^{(2)}}{n^{3/2}}, \frac{S_{n1}}{n^{1/2}} \right) \Rightarrow \left( \int_0^t B_s ds, B_t \right). \]

We then define for every \( z \in \mathbb{R}_+ \times \mathbb{R} \)

\[ V_0(z) = \lim_{n \to \infty} \mathbb{E}_z[h(Z_n), \tau > n]. \]  

(14)
The justification of this formal definition is the most technical part of our approach. It is worth mentioning that we cannot just repeat the proof from [7]. There we used a certain a-priori information on the behaviour of first exit times. (It was some moment inequalities, which were already known in the literature.) For integrated random walks we do not have such information and, therefore, should find an alternative way of justification of (14). This is done in Section 2.

Having constructed harmonic function for $Z_n$ we follow our approach in [6, 7] and apply the KMT-coupling to obtain the asymptotics for $\tau$. (This explains our moment condition in Theorem 1.) For details see Section 3. We omit the proof of (10), since it is again a repetition of the corresponding arguments in [7]. For integrated random walks a strong approximation was used in Aurzada and Dereich [2] to obtain (2). This formula shows that a direct, without use of potential theory, application of coupling produces a superfluous logarithmic terms even under exponential moment assumption.

Finally we show that the definitions (7) and (14) are equivalent, that is, $V(z) = V_0(z)$.

1.5. Conclusion. In our previous works [6, 7] we showed that Brownian asymptotics for exit times can be transferred to exit times for multidimensional random walks. In the present work we consider an integrated random walk which can be viewed as a two-dimensional Markov chain. We study exit times from a half-space and transfer the corresponding results for the Kolmogorov diffusion. These examples make plausible the following hypothesis.

Let $X_n$ be a Markov chain, $D$ be an unbounded domain and $\tau_D := \min\{n \geq 1 : X_n \notin D\}$. Assume that this Markov chain, properly scaled, converges as a process to a diffusion $Y_t, t \geq 0$. Assume also that the exit time of this diffusion $T_D := \min\{t \geq 0 : Y_t \notin D\}$ has the following asymptotics

$$\mathbb{P}_y(T_D > t) \sim \frac{h(y)}{t^p}, \quad t \to \infty,$$

where $h(y)$ is the corresponding harmonic function of the killed diffusion $Y_{t \wedge T_D}$. Then, there exists a positive harmonic function $V(x)$ for the killed Markov chain $X_{n \wedge \tau_D}$ such that

$$\mathbb{P}_x(\tau_D > n) \sim \frac{V(x)}{n^p}, \quad n \to \infty.$$

Naturally, this general theorem will require some moment assumptions and some assumptions on the smoothness of the unbounded domain $D$. Since we have a convergence of processes the domain $D$ should have certain scaling properties. Hence it seems natural for the domain $D$ to be a cone, at least asymptotically.

2. Construction of harmonic function

This section is devoted to the construction of the harmonic function $V_0$. Let

$$Y_0 = h(z),$$

$$Y_{n+1} = h(Z_{n+1}) - \sum_{k=0}^{n} f(Z_k), \quad n \geq 0. \quad (15)$$

Lemma 3. The sequence $Y_n$ defined in (15) is a martingale.
Proof. Clearly,
\[
\begin{align*}
\mathbf{E}_z [Y_{n+1} - Y_n | \mathcal{F}_n] &= \mathbf{E}_z [(h(Z_{n+1}) - h(Z_n) - f(Z_n)) | \mathcal{F}_n] \\
&= -f(Z_n) + \mathbf{E}_z [(h(Z_{n+1}) - h(Z_n)) | Z_n] \\
&= -f(Z_n) + f(Z_n) = 0,
\end{align*}
\]
where we used the definition of the function \( f \) in (6).

Before proceeding any further we need to study some properties of the functions
\( h(x,y) \) and \( f(x,y) \).

**Lemma 4.** Function \( h \) has the following partial derivatives,
\[
\frac{\partial^i h(x,y)}{\partial x^i} = \begin{cases} 
C_i \left( \frac{2}{9} \right)^{1/6} \frac{y}{x^{1/6+i}} U \left( \frac{1}{6} + i, \frac{4}{3}, \frac{2y^3}{9x} \right), & x \geq 0, y \geq 0 \\
- \left( \frac{2}{9} \right)^{1/6} \frac{y}{x^{1/6+i}} e^{2y^3/9x} U \left( \frac{7}{6} - i, \frac{4}{3}, \frac{2y^3}{9x} \right), & x \geq 0, y < 0,
\end{cases} \tag{16}
\]
for \( i \geq 0 \) and
\[
\frac{\partial^{i+1} h(x,y)}{\partial x^i \partial y} = \begin{cases} 
C_i \left( \frac{2}{9} \right)^{1/6} \frac{3(y/6+i)}{x^{1/6+i+1}} U \left( \frac{1}{6} + i, \frac{1}{3}, \frac{2y^3}{9x} \right), & x \geq 0, y \geq 0 \\
\left( \frac{2}{9} \right)^{1/6} \frac{y}{x^{1/6+i}} e^{2y^3/9x} U \left( \frac{5}{6} - i - 1, \frac{4}{3}, \frac{2y^3}{9x} \right), & x \geq 0, y < 0.
\end{cases} \tag{17}
\]
Here, \( C_0 = 1 \) and \( C_{i+1} = -C_i (i + 1/6) (i - 1/6) \) for \( i \geq 0 \).

Proof. We will prove (16) by induction. The base of induction \( i = 0 \) corresponds to the definition of \( h \). Now suppose that (16) is true for \( i \) and prove it for \( i + 1 \).

Consider first \( y \geq 0 \). By the induction hypothesis,
\[
\frac{\partial^{i+1} h(x,y)}{\partial x^{i+1}} = C_i \left( \frac{2}{9} \right)^{1/6} \frac{\partial}{\partial x} \left[ \frac{y}{x^{1/6+i}} U \left( \frac{1}{6} + i, \frac{4}{3}, \frac{2y^3}{9x} \right) \right]
\]
\[
= -C_i \left( \frac{2}{9} \right)^{1/6} \frac{y}{x^{1/6+i+1}} \frac{1}{6} U \left( \frac{1}{6} + i, \frac{4}{3}, \frac{2y^3}{9x} \right) + \frac{2y^3}{9x} U \left( \frac{1}{6} + i, \frac{4}{3}, \frac{2y^3}{9x} \right)
\]
\[
= -C_i \left( \frac{2}{9} \right)^{1/6} \frac{y}{x^{1/6+i+1}} (i + 1/6) (i - 1/6) \frac{1}{6} U \left( \frac{1}{6} + i, \frac{4}{3}, \frac{2y^3}{9x} \right),
\]
where we applied (13.4.23) of [1] in the last step. Recalling the definition of \( C_{i+1} \) we see that (16) holds for \( i + 1 \) and positive \( y \).

Consider second the case \( y < 0 \). By the induction hypothesis,
\[
\frac{\partial^{i+1} h(x,y)}{\partial x^{i+1}} = - \left( \frac{2}{9} \right)^{1/6} \frac{1}{6} \frac{\partial}{\partial x} \left[ \frac{e^{2y^3/9x} y}{x^{1/6+i}} U \left( \frac{7}{6} - i, \frac{4}{3}, \frac{2y^3}{9x} \right) \right]
\]
\[
= - \left( \frac{2}{9} \right)^{1/6} \frac{1}{6} \frac{y}{x^{1/6+i+1}} e^{2y^3/9x} \left( -(1/6 + i) U \left( \frac{7}{6} - i, \frac{4}{3}, \frac{2y^3}{9x} \right) \right)
\]
\[
+ \frac{2y^3}{9x} U \left( \frac{7}{6} - i, \frac{4}{3}, \frac{2y^3}{9x} \right) + \frac{2y^3}{9x} U \left( \frac{7}{6} - i, \frac{4}{3}, \frac{2y^3}{9x} \right)
\]
\[
= - \left( \frac{2}{9} \right)^{1/6} \frac{1}{6} \frac{y}{x^{1/6+i+1}} e^{2y^3/9x} U \left( \frac{7}{6} - i, \frac{4}{3}, \frac{2y^3}{9x} \right),
\]
Lemma 5. There exist positive constants $c$ and $C$ such that
\[ c \sqrt{\alpha(z)} \leq h(z) \leq C \sqrt{\alpha(z)}, \quad z \in \mathbb{R}^2_+. \] (19)
Furthermore, the upper bound is valid for all $z$. Function $h$ is at least $C^3$ continuous except the half-line $\{ z : x = 0, y \geq 0 \}$.

For the derivatives we have
\[ |h_x(x, y)| \leq C \alpha(x, y)^{-2.5}, \quad |h_{xx}(x, y)| \leq C \alpha(x, y)^{-5.5}, \quad |h_{xxx}(x, y)| \leq C \alpha(x, y)^{-8.5}, \]
\[ |h_y(x, y)| \leq C \alpha(x, y)^{-0.5}, \quad |h_{yx}(x, y)| \leq C \alpha(x, y)^{-3.5}, \quad |h_{yxx}(x, y)| \leq C \alpha(x, y)^{-6.5}, \]
\[ |h_{yy}(x, y)| \leq C \alpha(x, y)^{-4.5}, \quad |h_{yyy}(x, y)| \leq C \alpha(x, y)^{-1.5}, \quad |h_{yxy}(x, y)| \leq C \alpha(x, y)^{-2.5}. \]

Here and throughout the text we denote as $C, c$ some generic constants.

Proof. The estimates will follow from Lemma 4 and the following properties of the confluent hypergeometric function, see (13.1.8), (13.5.8) and (13.5.10) of [1],
\[ U(a, b, s) \sim s^{-a}, \quad s \to \infty, \] (20)
\[ U(a, b, s) \sim \frac{\Gamma(b - 1)}{\Gamma(a)} s^{1-b}, \quad s \to 0, b \in (1, 2), \] (21)
\[ U(a, b, s) \sim \frac{\Gamma(1-b)}{\Gamma(1-a-b)}, \quad s \to 0, b \in (0, 1). \] (22)
Asymptotics (20), (21) and the definition of $h$ immediately imply (19).

Function $h$ is obviously infinitely differentiable when $x < 0$ or $x > 0$. The only problematic zone is $x = 0, y < 0$. Since $h(x, y) = 0$ for $x < 0, y < 0$ all derivatives are equal to 0. Using the expressions for derivatives found in Lemma 4 one can immediately see that derivatives of $h(x, y)$ go to 0 as $x \to 0$ for $y < 0$ thanks to the exponent $e^{2y^3/9x}$.

We continue with partial derivatives with respect to $x$. First, using (16) and (20) for sufficiently large $A > 0$ and $y^3/x > A$,

\[
\left| \frac{\partial^i h(x, y)}{\partial x^i} \right| \leq C \frac{y}{x^{1/6+i}} \left( \frac{2y^3}{9x} \right)^{-1/6-i} \leq C y^{1/2-3i}, \quad i \geq 0.
\]

For $y < 0$, sufficiently large $A$ and $-y^3/9x > A$, the same inequality hold since $e^{2y^3/9x}$ is decreasing much faster than any power function as $y^3/x \to -\infty$. Next, using (16) and (21) for sufficiently small $\varepsilon > 0$ and $y : |y^3/x| \leq \varepsilon$,

\[
\left| \frac{\partial^i h(x, y)}{\partial x^i} \right| \leq C \frac{y}{x^{1/6+i}} \left( \frac{2y^3}{9x} \right)^{-1/3} \leq C x^{1/6-i}.
\]

Finally, when $y^3/x \in (\varepsilon, A)$,

\[
\left| \frac{\partial^i h(x, y)}{\partial x^i} \right| \leq C.
\]

We can summarise this in one formula

\[
\left| \frac{\partial^i h(x, y)}{\partial x^i} \right| \leq C \alpha(x, y)^{-1/2-3i},
\]

where $\alpha(x, y)$ is defined in (18). This proves the first line of estimates.

To prove the second line we use a similar approach. First, using (17) and (20) for sufficiently large $A > 0$ and $y^3/x > A$,

\[
\left| \frac{\partial^{i+1} h(x, y)}{\partial x^i \partial y} \right| \leq C \frac{1}{x^{1/6+i}} \left( \frac{2y^3}{9x} \right)^{-1/6-i} \leq C y^{-1/2-3i}, \quad i \geq 0.
\]

For $y < 0$, sufficiently large $A$ and $-y^3/9x > A$, the same inequality hold since $e^{2y^3/9x}$ is decreasing much faster than any power function as $y^3/x \to -\infty$. Next, using (16) and (22) for sufficiently small $\varepsilon > 0$ and $y : |y^3/x| \leq \varepsilon$,

\[
\left| \frac{\partial^{i+1} h(x, y)}{\partial x^i \partial y} \right| \leq C \frac{1}{x^{1/6+i}} = C x^{-1/6-i}.
\]

Finally, when $y^3/x \in (\varepsilon, A)$,

\[
\left| \frac{\partial^{i+1} h(x, y)}{\partial x^i \partial y} \right| \leq C.
\]

As a result we have

\[
\left| \frac{\partial^{i+1} h(x, y)}{\partial x^i \partial y} \right| \leq C \alpha(x, y)^{-1/2-3i}.
\]

This proves the second line of estimates.

To prove the third line we are using fact that $h_{yy} + 0.5y h_x = 0$. Hence,

\[
|h_{yy}(x, y)| \leq C |y| |h_x(x, y)| \leq C \alpha(x, y) \alpha(x, y)^{-2.5} \leq C \alpha(x, y)^{-1.5}.
\]
Next, \[|h_{yyz}(x, y)| \leq C|y||h_{zx}(x, y)| \leq C\alpha(x, y)\alpha(x, y)^{-5.5} \leq C\alpha(x, y)^{-4.5}.\]

Finally, \[|h_{yyyy}(x, y)| \leq C|h_{xx}(x, y)| + C|y||h_{xy}(x, y)| \leq C\alpha(x, y)^{-2.5} + C\alpha(x, y)\alpha(x, y)^{-3.5} \leq C\alpha(x, y)^{-2.5}.

The proof is complete. \(\Box\)

Next we derive an upper bound for \(f(x, y)\).

**Lemma 6.** Let the assumptions of Lemma 5 hold and let the moment assumptions hold. Then
\[|f(x, y)| \leq C\min(1, \alpha(x, y)^{-3/2-\delta}), \quad (x, y) \in \mathbb{R}_+ \times \mathbb{R},\]
for some \(\delta > 0\).

**Proof.** Let \(A\) be a large constant. Then for \((x, y)\) such that \(\alpha(x, y) \leq A\) using the fact that function \(h\) is bounded on any compact we have \(|f(x, y)| \leq C\). In the rest of the proof we consider the case \(\alpha(x, y) > A\) where \(A\) is sufficiently large.

According to Lemma 5 function \(h\) is at least \(C^3\) smooth except the line \((x = 0, y \geq 0)\). Then, for \(t: |t| \leq \frac{1}{2}\alpha(x, y),\) by the Taylor formula,
\[
\begin{align*}
|h(x + y + t, y + t) - h(x, y) - (y + t)h_x(x, y) + th_y(x, y) &+ \frac{1}{2}h_{xx}(x, y)(y + t)^2 + h_{xy}(x, y)(y + t)t + \frac{1}{2}h_{yy}(x, y)(y + t)^2| \\
&\leq \sum_{i+j=3} \max_{\theta:|\theta| \leq \frac{1}{2}\alpha(x, y)} |\partial^{i+j}h(x + y + \theta, y + \theta)/(\partial x^i\partial y^j)(y + t)^t| := r(x, y, t)
\end{align*}
\]

To ensure that the Taylor formula is applicable we need to check that the set \(\{(x+y+t, y+t) : |t| \leq \frac{1}{2}\alpha(x, y)\}\) is sufficiently far away from the half-line \(\{x = 0, y > 0\}\), where the derivatives of the function \(h(x, y)\) are discontinuous. First, if \(y < 0\) and \(\alpha(x, y) = |y|\), then \(y + t \leq -\frac{1}{2}|y|\) for any \(t\) with \(|t| \leq \frac{1}{2}|y|\). Therefore \(|y + t| \geq \frac{1}{2}A\) in this case. Second, if \(y > 0\) and \(\alpha(x, y) = y\), then \(x + y + t \geq x + y/2 \geq \frac{1}{2}A\).

Third, if \(\alpha(x, y) = x^{1/3}\), then \(|x + y + t| \geq |x| - 1.5|x|^{1/3} \geq 0.5A\) for all sufficiently large \(A\). This shows that the Taylor formula is valid.

Then, \[
|\mathbb{E}h(x + y + X, y + X) - h(x, y)| \\
\leq \mathbb{E}\left[|h(x + y + X, y + X) - h(x, y); |X| > \frac{1}{2}\alpha(x, y)|\right] \\
+ \mathbb{E}\left[|h(x + y + X, y + X) - h(x, y); |X| \leq \frac{1}{2}\alpha(x, y)|\right].
\]
We can estimate the second term in the right-hand side using the Taylor formula above,

\[
\left| E \left[ h(x + y + X, y + X) - h(x, y); |X| \leq \frac{1}{2} \alpha(x, y) \right] \right|
\leq \left| E \left[ (y + X) h_x(x, y) + X h_y(x, y) + \frac{1}{2} h_{xx}(x, y)(y + X)^2 
+ h_{xy}(x, y)(y + X)X + \frac{1}{2} h_{yy}(x, y)X^2 \right] \right|
+ \left| E \left[ (y + X) h_x(x, y) + X h_y(x, y) + \frac{1}{2} h_{xx}(x, y)(y + X)^2 
+ h_{xy}(x, y)(y + X)X + \frac{1}{2} h_{yy}(x, y)X^2; |X| > \frac{1}{2} \alpha(x, y) \right] \right|
+ E \left[ r(x, y, X); |X| \leq \frac{1}{2} \alpha(x, y) \right]
\leq E_1(x, y) + E_2(x, y) + E_3(x, y).
\]

First, we can simplify the first term \( E_1(x, y) \) using the assumption \( E X = 0, E X^2 = 1 \). Then,

\[
E_1(x, y) = \left| y h_x(x, y) + \frac{1}{2} h_{yy}(x, y) + \frac{1}{2} h_{xx}(x, y)(y^2 + 1) + h_{xy}(x, y) \right|.
\]

Recalling that \( y h_x + \frac{1}{2} h_{yy} = 0 \), we obtain

\[
E_1(x, y) = \left| \frac{1}{2} h_{xx}(x, y)(y^2 + 1) + h_{xy}(x, y) \right|.
\]

Applying Lemma 5, we finally get

\[
E_1(x, y) \leq C \alpha(x, y)^{-5.5} \alpha(x, y)^2 + C \alpha(x, y)^{-3.5} \leq C \alpha(x, y)^{-3.5}. \tag{23}
\]

Second, noting that \( |X| > |y|/2 \) on the event \( |X| > \frac{1}{2} \alpha(x, y) \) and applying the Chebyshev inequality, we obtain

\[
E_2(x, y) \leq C E \left[ |X| (h_x(x, y) + h_y(x, y)) + X^2 (h_{xx}(x, y) + h_{xy}(x, y) + h_{yy}(x, y)); |X| > \frac{1}{2} \alpha(x, y) \right]
\leq C \frac{|h_x(x, y)| + |h_y(x, y)|}{\alpha(x, y)^{1+\delta}} + C \frac{|h_{xx}(x, y)| + |h_{xy}(x, y)| + |h_{yy}(x, y)|}{\alpha(x, y)^{\delta}}.
\]

Applying Lemma 5, we obtain

\[
E_2(x, y) \leq C \alpha(x, y)^{-1.5-\delta}. \tag{24}
\]
Third, applying Lemma 5 once again,
\[ E_3(x, y) \leq C \max_{\theta:|\theta| \leq \frac{1}{2} \alpha(x, y)} |h_{xxx}(x + y + \theta, y + \theta)| \mathbb{E} \left[ |y + X|^3; |X| \leq \frac{1}{2} \alpha(x, y) \right] \\
+ C \max_{\theta:|\theta| \leq \frac{1}{2} \alpha(x, y)} |h_{xyy}(x + y + \theta, y + \theta)| \mathbb{E} \left[ |y + X|^2|X|; |X| \leq \frac{1}{2} \alpha(x, y) \right] \\
+ C \max_{\theta:|\theta| \leq \frac{1}{2} \alpha(x, y)} |h_{xyy}(x + y + \theta, y + \theta)| \mathbb{E} \left[ |y + X|^2; |X| \leq \frac{1}{2} \alpha(x, y) \right] \\
\leq C \alpha(x, y)^{-8.5} \alpha(x, y) \mathbb{E} X^2 + C \alpha(x, y)^{-6.5} \alpha(x, y) \mathbb{E} X^2 \\
+ C \alpha(x, y)^{-4.5} \alpha(x, y) \mathbb{E} X^2 + C \alpha(x, y)^{-2.5} \alpha(x, y)^{1-\delta} \mathbb{E} |X|^{2+\delta} \\
\leq C \alpha(x, y)^{-1.5-\delta} \mathbb{E} X^{2+\delta}.
\]

We are left to estimate
\[ \mathbb{E} \left[ h(x + y + X, y + X) - h(x, y); |X| > \frac{1}{2} \alpha(x, y) \right] \]
\[ \leq C \mathbb{E} \left[ \alpha(|x + y + X|, |y + X|); |X| \leq \frac{1}{2} \alpha(x, y) \right] \\
\quad + h(x, y) \mathbb{P}(|X| > \frac{1}{2} \alpha(x, y)) \\
\leq C \mathbb{E} \left[ |X|; |X| > \frac{1}{2} \alpha(x, y) \right] + C \alpha(x, y)^{0.5} \mathbb{P}(|X| > \frac{1}{2} \alpha(x, y)) \\
\leq C \alpha(x, y)^{-1.5-\delta} \mathbb{E} |X|^{2+\delta},
\]
where we applied the Chebyshev inequality in the last step and Lemma 5 in the first step. This proves the statement of the lemma.

\[ \square \]

**Lemma 7.** There exists a constant $C$ such that
\[ \sup_{x, y} \mathbb{P} \left( |S^{(2)}_n - x| \leq 1, |S_n - y| \leq 1 \right) \leq \frac{C}{n^2}, \ n \geq 1 \]
and
\[ \sup_{x} \mathbb{P} \left( |S^{(2)}_n - x| \leq 1 \right) \leq \frac{C}{n^{3/2}}, \ n \geq 1. \]

**Proof.** In order to prove the first statement one has to apply Theorem 1.2 from Friedland and Sodin [8] with $\alpha_k = (k, 1)$ and to note that $\alpha$ from that theorem is not smaller than $cn$ for this special choice of vectors $\alpha_k$. The second inequality follows from Theorem 1.1 of the same paper.

Let
\[ K_{n, \varepsilon} = \{(x, y) : y > 0, x \geq n^{3/2-\varepsilon}\}. \]

**Lemma 8.** For any sufficiently small $\varepsilon > 0$ there exists $\gamma > 0$ such that for $k \leq n$ the following inequalities hold
\[ \mathbb{E}_z [h(Z_k); \tau > k] \leq (1 + \frac{\varepsilon}{n}) h(z), \ z \in K_{n, \varepsilon}, \] (25)
\[ \mathbb{E}_z [h(Z_k); \tau > k] \geq (1 - \frac{\varepsilon}{n}) h(z), \ z \in K_{n, \varepsilon}, \] (26)
Proof. First, using (15) we obtain,

$$
E_z[h(Z_k); \tau > k] = E_z[Y_k; \tau > k] + \sum_{l=0}^{k-1} E_z[f(Z_l); \tau > k]
$$

$$
= E_z[Y_k] - E_z[Y_k; \tau \leq k] + \sum_{l=0}^{k-1} E_z[f(Z_l); \tau > k].
$$

Since $Y_k$ is a martingale, $E_z[Y_k] = E_z[Y_0] = h(z)$ and

$$
E_z[Y_k; \tau \leq k] = E_z[Y_{\tau}; \tau \leq k].
$$

Using the definition of $Y_k$ once again we arrive at

$$
E_z[h(Z_k); \tau > k] = h(z) - E_z[h(Z_{\tau}); \tau \leq k]
$$

$$
+ E_z \left[ \sum_{l=0}^{\tau-1} f(Z_l); \tau \leq k \right] + \sum_{l=0}^{k-1} E_z[f(Z_l); \tau > k]
$$

$$
= h(z) + E_z \left[ \sum_{l=0}^{\tau-1} f(Z_l); \tau \leq k \right] + \sum_{l=0}^{k-1} E_z[f(Z_l); \tau > k],
$$

(27)

since $h(Z_{\tau}) = 0$.

For $k \leq n$ we have

$$
E_z \left[ \sum_{l=0}^{\tau-1} f(Z_l); \tau \leq k \right] + \sum_{l=0}^{k-1} E_z[f(Z_l); \tau > k] \leq \sum_{l=0}^{n-1} E_z[|f(Z_l)|].
$$

(28)

We split the sum in (28) into three parts,

$$
\sum_{l=0}^{n-1} E_z[|f(Z_l)|] = f(z) + E_z \sum_{l=1}^{n-1} \left[ |f(Z_l)|; \max(|S_{l}^{(2)}|, |S_l|) \leq 1 \right]
$$

$$
+ E_z \sum_{l=1}^{n-1} \left[ |f(Z_l)|; |S_{l}^{(2)}|^{1/3} > |S_l|, \max(|S_{l}^{(2)}|, |S_l|) > 1 \right]
$$

$$
+ E_z \sum_{l=1}^{n-1} \left[ |f(Z_l)|; |S_{l}^{(2)}|^{1/3} \leq |S_l|, \max(|S_{l}^{(2)}|, |S_l|) > 1 \right]
$$

$$
=: f(z) + \Sigma_1 + \Sigma_2 + \Sigma_3.
$$

First, using the fact that $|f(x, y)| \leq C$ for $|x|, |y| \leq 1$ and Lemma 7, we obtain

$$
\Sigma_1 \leq C \sum_{l=1}^{\infty} P_z(|S_{l}^{(2)}|, |S_l| \leq 1) \leq C \sum_{l=1}^{\infty} l^{-2} < C.
$$
Second, by Lemma 6,
\[
\Sigma_2 \leq C \sum_{l=1}^{n-1} E_z \left[ |S_l^{(2)}|^{1/2-\delta/3} \right]
\leq C \sum_{l=1}^{n-1} \sum_{j=1}^{\infty} E_z \left[ |S_l^{(2)}|^{1/2-\delta/3}; j \leq |S_l^{(2)}| \leq j+1 \right]
\leq C \sum_{l=1}^{n-1} \left( \sum_{j=1}^{l^{1/2}} \sum_{j=1}^{j^{1/2}} \sum_{j=1}^{j^{1/2}} \right) P_z \left( j \leq |S_l^{(2)}| \leq j+1 \right) + l^{3/2(1/2-\delta/3)} P_z \left( |S_l^{(2)}| > l^{3/2} \right).
\]

Now we use the second concentration inequality from Lemma 7 to get an estimate
\[
P_z \left( j \leq |S_l^{(2)}| \leq j+1 \right) \leq C l^{-3/2}.
\]
Then,
\[
\Sigma_2 \leq C \sum_{l=1}^{n-1} \left( l^{-3/2} \sum_{j=1}^{l^{1/2}} j^{-1/2-\delta/3} + l^{-3/4-\delta/2} \right)
\leq C \sum_{l=1}^{n-1} l^{-3/4-\delta/2} \leq C n^{1/4-\delta/2}.
\]

Similarly,
\[
\Sigma_3 \leq C \sum_{l=1}^{n-1} E_z \left[ |S_l|^{-3/2-\delta}; |S_l| \geq 1; |S_l| \geq |S_l^{(2)}|^{1/3} \right]
\leq C \sum_{l=1}^{n-1} \sum_{j=1}^{\infty} E_z \left[ |S_l|^{-3/2-\delta}; j \leq |S_l| \leq j+1; |S_l^{(2)}| \leq (j+1)^3 \right]
\leq C \sum_{l=1}^{n-1} \left( \sum_{j=1}^{l^{1/2}} \sum_{j=1}^{j^{1/2}} \sum_{j=1}^{j^{1/2}} \right) P_z \left( j \leq |S_l| \leq j+1; |S_l^{(2)}| \leq (j+1)^3 \right) + l^{-3/4-\delta/2} P_z \left( |S_l| > l^{1/2} \right).
\]

Using Lemma 7 once again, we get an estimate
\[
P_z \left( j \leq |S_l| \leq j+1; |S_l^{(2)}| \leq (j+1)^3 \right) \leq C \sum_{i=1}^{(j+1)^3} P_z \left( j \leq |S_l| \leq j+1; |S_l^{(2)}| \in (i, i+1) \right)
\leq C l^{-2} j^3.
\]
Then,
\[
\Sigma_3 \leq C \sum_{l=1}^{n-1} \left( \sum_{j=1}^{l^{1/2}} \sum_{j=1}^{j^{1/2}} \sum_{j=1}^{j^{1/2}} \right) (l^{-3/4-\delta/2} j^3 + l^{-3/4-\delta/2} P_z \left( |S_l| > l^{1/2} \right))
\leq C \sum_{l=1}^{n-1} \left( l^{5/4-\delta/2} + l^{-3/4-\delta/2} \right) \leq C n^{1/4-\delta/2}.
\]
Therefore,
\[ \sum_{l=0}^{n-1} E_z[|f(Z_l)|] \leq f(z) + C n^{1/4-\delta/2}. \]

Combining Lemma 5 and Lemma 6, we get
\[ |f(z)| \leq C \max(1, \alpha(z))^{-3/2-\delta} \leq C \frac{h(z)}{\alpha(z)}^{2+\delta} \]

Applying now the lower bound from Lemma 5, we see that
\[ h(z) \geq c(\alpha(z))^{1/2} \geq c n^{1/4-\epsilon/2}, \quad z \in K_{n,\epsilon}. \]

From these estimates we infer that
\[ \sum_{l=0}^{n-1} E_z[|f(Z_l)|] \leq C n^{1/4-\delta/2} \leq C h(z) n^{1/4-\delta/2} \leq C h(z) n^{-\gamma}, \quad (29) \]

where \( \gamma \) is positive for sufficiently small \( \epsilon \).

Let
\[ \nu_n := \min\{k \geq 0 : Z_k \in K_{n,\epsilon}\}. \]

**Lemma 9.** There exist a constant \( C \) such that for
\[ \sup_{z \in \mathbb{R}_+ \times \mathbb{R}} P_z(\nu_n \geq n^{1-\epsilon}, \tau > n^{1-\epsilon}) \leq C \exp\{-n^{\epsilon/4}\}. \]

**Proof.** Fix some integer \( A > 0 \) and put \( b_n := A[n^{1-2\epsilon/3}] \). Define also \( R_n := [n^{1-\epsilon}/b_n] \). It is clear that
\[ P_z(\nu_n > n^{1-\epsilon}, \tau > n^{1-\epsilon}) \leq P_z\left(S_{nb_n}^{(2)} \in [0, n^{3/2-\epsilon}] \text{ for all } j \leq R_n\right). \]

It follows from the definition of \( S_{nb_n}^{(2)} \) that
\[ S_{(j+1)b_n}^{(2)} = S_{jb_n}^{(2)} + b_n S_{jb_n} + \tilde{S}_{b_n}^{(2)}, \]

where \( \tilde{S}_{b_n}^{(2)} \) is an independent copy of \( S_{b_n}^{(2)} \) with starting point \((0,0)\). From this representation and the Markov property we conclude that
\[ P_z\left(S_{jb_n}^{(2)} \in [0, n^{3/2-\epsilon}] \text{ for all } j \leq R_n\right) \leq \ldots \leq \left(Q_{b_n}\left(n^{3/2-\epsilon}\right)\right)^{R_n}, \]

where
\[ Q_n(\lambda) := \sup_{x \in \mathbb{R}} P_{(0,0)}\left(S_{n}^{(2)} \in [x, x + \lambda]\right). \]

Using the second inequality in Lemma 7, we get
\[ Q_{b_n}\left(n^{3/2-\epsilon}\right) \leq C n^{3/2-\epsilon} A^{3/2(n^{1-2\epsilon/3})^{3/2}} = C A^{3/2}. \]
Choosing $A$ so large that $\frac{C}{A^{3/2}} \leq \frac{1}{2}$, we obtain
\[ P_z (\nu_n > n^{1-\varepsilon}, \tau > n^{1-\varepsilon}) \leq \left( \frac{1}{2} \right)^{R_n}. \]
Thus, the proof is finished. \hfill \Box 

**Lemma 10.** There exist a constant $C$ such that for $k \geq n^{1-\varepsilon}$,
\[ E_z [h(Z_n), \nu_n \geq k, \tau > n^{1-\varepsilon}] \leq C(1 + \alpha(z))^{1/2} \exp \{-n^{\varepsilon/8}\}. \]

**Proof.** Using the Cauchy-Schwartz inequality, we obtain
\[
E_z [h(Z_n), \nu_n \geq k] \leq (E_z [h^2(Z_n), \tau > n^{1-\varepsilon}])^{1/2} (P_z (\nu_n \geq k, \tau > n^{1-\varepsilon}))^{1/2} \\
\leq (E_z [h^2(Z_n), \tau > n^{1-\varepsilon}])^{1/2} (P_z (\nu_n \geq n^{1-\varepsilon}, \tau > n^{1-\varepsilon}))^{1/2}.
\]
Recalling that $h(z) \leq C(\alpha(z))^{1/2}$ for all $z \in \mathbb{R}_+ \times \mathbb{R}$, one can easily obtain the inequality
\[
E_z \left[ h^2(Z_n), \tau > n^{1-\varepsilon} \right] \leq CE_z \left[ \alpha(Z_n) \right] \leq \alpha(z) + E_0 \max(M_n^{1/3}, n) \\
\leq C(1 + \alpha(z))n^{3/2},
\]
where $M_n = \max_{0 \leq i \leq n} S_i$. Combining this with Lemma 9, we complete the proof. \hfill \Box 

**Lemma 11.** For any starting point $z$ there exists a limit
\[ V_0(z) = \lim_{n \to \infty} E_z [h(Z_n); \tau > n]. \tag{30} \]
Moreover, this limit is harmonic and strictly positive on $K_+$. 

**Proof.** Fix a large integer $n_0 > 0$ and put, for $m \geq 1$,
\[ n_m = \lfloor n_0^{(1-\varepsilon)m} \rfloor, \]
where $[r]$ denotes the integer part of $r$. Let $n$ be any integer. There exists unique $m$ such that $n \in (n_m, n_{m+1})$. We first split the expectation into 2 parts,
\[
E_z [h(Z_n); \tau > n] = E_1(z) + E_2(z) \\
= E_z [h(Z_n); \tau > n, \nu_n \leq n_m] + E_z [h(Z_n); \tau > n, \nu_n > n_m].
\]
By Lemma 10, since $n_m \geq n^{1-\varepsilon}$, the second term on the right-hand side is bounded by
\[ E_2(z) \leq E_z [h(Z_n); \tau > n, \nu_n > n_m] \leq C(1 + \alpha(z))^{1/2} \exp \{-Cn^{\varepsilon/8}\}. \]
Then,
\[
E_1(z) \leq \sum_{i=1}^{n_m} \int_{K_{n_i}} P_z \{ \nu_n = i, \tau > i, S_i^{(2)} \in da, S_i \in db \} E_{(a,b)}[h(Z_{n-i}); \tau > n - i].
\]
Then, by (25),
\[
E_1(z) \leq \left( 1 + \frac{C}{n^7} \right) \sum_{i=1}^{n_m} \int_{K_{n_i}} P_z \{ \nu_n = i, \tau > i, S_i^{(2)} \in da, S_i \in db \} h(a, b).
\]
Now noting that $K_{n,x} \subset K_{n,m,\varepsilon}$, we apply (26) to obtain

$$E_1(z) \leq \left( \frac{1 + C_{n_m}}{n_m} \right) \sum_{i=1}^{n_m} \int_{K_{n,x}} P_z \{ \nu_n = i, \tau > i, S_{i^{(2)}} \in da, S_i \in db \}$$

$$\times E_{(a,b)}[h(Z_{n_m-i}); \tau > n_m-i]$$

$$= \left( \frac{1 + C_{n_m}}{1 - C_{n_m}} \right) E_z[h(Z_{n_m}); \tau > n_m, \nu_n \leq n_m].$$

As a result we have

$$E_z[h(Z_n); \tau > n] \leq \left( \frac{1 + C_{n_m}}{1 - C_{n_m}} \right) E_z[h(Z_{n_m}); \tau > n_m] + C(1 + \alpha(z))^{1/2} \exp\{ -Cn_m^{z/8} \}.$$

Iterating this procedure $m$ times, we obtain

$$E_z[h(Z_n); \tau > n] \leq \prod_{j=0}^{m} \left( \frac{1 + \frac{C_{n_m}}{n_m^{(1-\varepsilon)^j}}}{1 - \frac{C_{n_m}}{n_m^{(1-\varepsilon)^j}}} \right)$$

$$\times \left( E_z[h(Z_{n_0}); \tau > n_0] + C(1 + \alpha(z))^{1/2} \sum_{j=0}^{m} \exp\{ -Cn_m^{z/8} \} \right).$$

First of all we immediately obtain that

$$\sup_n E_z[h(Z_n); \tau > n] \leq C(z) < \infty.$$

An identical procedure gives a lower bound

$$E_z[h(Z_n); \tau > n] \geq \prod_{j=0}^{m} \left( \frac{1 - \frac{C_{n_m}}{n_m^{(1-\varepsilon)^j}}}{1 + \frac{C_{n_m}}{n_m^{(1-\varepsilon)^j}}} \right)$$

$$\times \left( E_z[h(Z_{n_0}); \tau > n_0] - C(1 + \alpha(z))^{1/2} \sum_{j=0}^{m} \exp\{ -Cn_m^{z/8} \} \right).$$

For every positive $\delta$ we can choose $n_0$ such that

$$\left| \prod_{j=0}^{m} \left( \frac{1 - \frac{C_{n_m}}{n_m^{(1-\varepsilon)^j}}}{1 + \frac{C_{n_m}}{n_m^{(1-\varepsilon)^j}}} \right) - 1 \right| \leq \delta \quad \text{and} \quad \sum_{j=0}^{m} \exp\{ -Cn_m^{z/8} \} \leq \delta.$$
Consequently,
\[
\sup_{n>n_0} \mathbb{E}_z[h(Z_n); \tau > n] - \inf_{n>n_0} \mathbb{E}_z[h(Z_n); \tau > n] \\
\leq \delta \mathbb{E}_z[h(Z_{n_0}); \tau > n_0] + 2C(1 + \alpha(z))^{1/2} \delta.
\]

Taking into account (33) and that $\delta$ can be made arbitrarily small we arrive at the conclusion that the limit in (30) exists.

To prove harmonicity of $V_0$ note that by the Markov property
\[
\mathbb{E}_z[h(Z_{n+1}); \tau > n + 1] = \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{P}(z + Z \in dz') \mathbb{E}_{z'}[h(Z_{n+1}); \tau > n]
\]
Letting $n$ to infinity we obtain
\[
V_0(z) = \mathbb{E}_z[V_0(Z_1); \tau > 1].
\]
The existence of the limit in the right hand side is justified by the dominated convergence theorem and the above estimates for $\sup_{n>n_0} \mathbb{E}_z[h(Z_n); \tau > n]$.

Function $V_0$ has the following monotonicity property: if $x' \geq x$ and $y' \geq y$ then $V_0(x', y') \geq V_0(x, y)$. Indeed, first the function $h$ satisfies this property since $h_x \geq 0, h_y \geq 0$, see Lemma 4. Second it clear that the exit time $\tau' \geq \tau$, where $\tau'$ is the exit of time the integrated random walk started from $(x', y')$ and $\tau$ is the exit of time the integrated random walk started from $(x, y)$. Third,
\[
\bar{S}_n = y' + X_1 + X_2 + \ldots + X_n \geq y + X_1 + X_2 + \ldots + X_n = S_n,
\]
\[
\bar{S}_n^{(2)} = x' + \bar{S}_1 + \bar{S}_2 + \ldots + \bar{S}_n \geq S_n^{(2)}.
\]

Therefore, for any $n$,
\[
\mathbb{E}_{(x', y')}[h(Z_n); \tau > n] \geq \mathbb{E}_{(x, y)}[h(Z_n); \tau > n].
\]
Letting $n$ go to infinity, we obtain $V_0(x', y') \geq V_0(x, y)$.

It remains to show that $V_0$ is strictly positive on $K_+$. For every fixed $n_0$ we have $\mathbb{E}_{(x, y)}[h(Z_{n_0}); \tau > n_0] \sim h(x, y)$ as $x, y \to \infty$. Then, there exist $x_{n_0}, y_{n_0}$ such that
\[
\inf_{n>n_0} \mathbb{E}_z[h(Z_n); \tau > n] \geq (1 - \delta)^2 \left( h(z) - C(1 + \alpha(z))^{1/2} \delta \right).
\]
Taking into account (19), we conclude that $V_0(z)$ is positive for all $z$ with $x > x_{n_0}$, $y > y_{n_0}$. From every starting point $z \in \mathbb{R}_+^2$ our process visits the set $x > x_{n_0}$, $y > y_{n_0}$ before $\tau$ with positive probability. Then, using the equation $V_0(z) = \mathbb{E}_z[V_0(Z_1); \tau > 1]$, we conclude that $V_0(z) > 0$. The same argument shows that $V_0$ is strictly positive on $K_+$.  

3. Asymptotics for $\tau$

The proof of Theorem 1 goes along the same line as the proofs of conditional limit theorems in our earlier works [6, 7]. For that reason we give a proof of (9) only. (This allows us also to demonstrate all changes, which are needed in the case of integrated random walks.)
3.1. Coupling. We start with some properties of the integrated Brownian motion.

Lemma 12. There exists a finite constant $C$ such that
\[ P((x,y) > t) \leq C \frac{h(x,y)}{t^{1/4}}, \quad x, y > 0. \] (35)

Moreover,
\[ P((x,y) > t) \sim \frac{h(x,y)}{t^{1/4}}, \quad \text{as } t \to \infty, \] (36)
uniformly in $x, y > 0$ satisfying $x^{1/6} y^{1/2} \leq \theta t^{1/4}$ with some $\theta t \to 0$.

Proof. To prove this lemma we are going to use the scaling property of the Brownian motion, which immediately gives
\[ P((x,y) > t) = P((x^3 y, y^3) > t\lambda^2), \quad \lambda > 0. \] (37)
We start with (36). Consider first the case $x^{1/3} \geq y$. Putting $\lambda = x^{-1/3}$ in (37) we obtain
\[ P((x,y) > t) = P((1, yx^{-1/3}) > tx^{-2/3}). \]
In view of our assumption $tx^{-2/3} \geq \theta t^{-1/4} \to \infty$. We use the continuity of $h(1,u)$ in $u \in [0,1]$ and immediately obtain that the asymptotics
\[ P((1, yx^{-1/3}) > tx^{-2/3}) \sim \frac{h(1, yx^{-1/3})}{(tx^{-2/3})^{1/4}} \]
hold uniformly in $yx^{-1/3} \in [0,1]$. Then,
\[ P((x,y) > t) \sim \frac{h(1, yx^{-1/3})}{(tx^{-2/3})^{1/4}} = \frac{h(x,y)}{t^{1/4}}. \]
If $x^{1/3} \leq y$ then, choosing $\lambda = y^{-1}$ in (37), we obtain
\[ P((x,y) > t) = P((xy^{-3}, 1) > ty^{-2}). \]
The rest of the proof goes exactly the same way.

To prove (35) first notice that the above proof showed that for sufficiently small $\varepsilon > 0$ and $t^{1/2} > \varepsilon^{-1} \max(x^{1/3}, y)$ the bound (35) holds. Hence, it is sufficient to consider $t^{1/2} \leq \varepsilon^{-1} \max(x^{1/3}, y)$. Using the lower bound in (19), we see that
\[ \frac{h(x,y)}{t^{1/4}} \geq \frac{c \max(x^{1/6}, y^{1/2})}{(\varepsilon^{-1} \max(x^{1/3}, y))^{1/2}} = c\varepsilon^2 > 0 \]
for $t^{1/2} \leq \varepsilon^{-1} \max(x^{1/3}, y)$. Therefore,
\[ P((x,y) > t) \leq 1 \leq \frac{h(x,y)}{c\varepsilon^2 t^{1/4}}. \]
This proves (35). \qed

We continue with the classical result (see, for example, [10]) on the quality of the normal approximation.

Lemma 13. If $E|X|^{2+\delta} < \infty$ for some $\delta \in (0,1)$, then for every $n \geq 1$ one can define a Brownian motion $B_t$ on the same probability space such that, for any $\gamma$ satisfying $0 < \gamma < \frac{\delta}{2(2+\delta)}$,
\[ P\left(\sup_{u \leq n} |S_u| - B_u| \geq n^{1/2-\gamma}\right) = o\left(n^{2\gamma+\gamma\delta+\delta/2}\right). \] (38)
Lemma 14. For all sufficiently small $\varepsilon > 0$,
\[
\mathbb{P}_z(\tau > n) = \varepsilon h(z) n^{-1/4} (1 + o(1)), \quad \text{as } n \to \infty
\] (39)
uniformly in $z \in K_{n,\varepsilon}$ such that $\max\{|x^{1/3}, y\} \leq \theta_n \sqrt{n}$ for some $\theta_n \to 0$. Moreover, there exists a constant $C$ such that
\[
\mathbb{P}_z(\tau > n) \leq C \varepsilon h(z) n^{1/4} \] (40)
uniformly in $z \in K_{n,\varepsilon}, n \geq 1$.

Proof. For every $z = (x, y) \in K_{n,\varepsilon}$ denote
\[
z = (x \pm n^{3/2-\gamma}, y).
\]
Note also that if we take $\gamma > \varepsilon$, then $y \in K_{n,\varepsilon'}$ for any $\varepsilon' > \varepsilon$ and all sufficiently large $n$.

Define
\[
A_n = \left\{ \sup_{u \leq n} |S_u - B_u| \leq n^{1/2-\gamma} \right\},
\]
where $B$ is the Brownian motion constructed in Lemma 13. Then, using (38), we obtain
\[
\mathbb{P}_z(\tau > n) = \mathbb{P}_z(\tau > n, A_n) + o(n^{-r})
\leq \mathbb{P}_z(\sigma_{bm} > n, A_n) + o(n^{-r})
= \mathbb{P}_z(\sigma_{bm} > n) + o(n^{-r})
\] (41)
where $r = r(\delta, \varepsilon) = \delta/2 - 2\gamma - \gamma \delta$. In the same way one can get
\[
\mathbb{P}_z(\tau_{bm} > n) \leq \mathbb{P}_z(\tau > n) + o(n^{-r})
\] (42)
By Lemma 12,
\[
\mathbb{P}_z(\sigma_{bm} > n) \sim \varepsilon h(z) n^{-1/4}.
\]
It follows from the Taylor formula and Lemma 5 that
\[
|h(z) - h(z^\pm)| \leq C n^{3/2-\gamma} \left( \alpha(x \pm n^{3/2-\gamma}, y) \right)^{-5/2} \leq C n^{1/4+5\varepsilon/6-\gamma}.
\] (43)
Furthermore, in view of (19),
\[
h(z) > c n^{1/4-\varepsilon/6}, \quad z \in K_{n,\varepsilon}.
\] (44)
From this bound and (43) we infer that
\[
h(z^\pm) = h(z)(1 + o(1)), \quad z \in K_{n,\varepsilon}.
\]
Therefore, we have
\[
\mathbb{P}_z(\tau_{bm} > n) = \varepsilon h(z) n^{-1/4} (1 + o(1)).
\]
From this relation and bounds (41) and (42) we obtain
\[
\mathbb{P}_z(\tau > n) = \varepsilon h(z) n^{-1/4} (1 + o(1)) + o(n^{-r}).
\]
Using (44), we see that $n^{-r} = o(h(z) n^{-1/4})$ for all $\varepsilon$ satisfying $r = \delta/2 - 2\gamma - 2\gamma \delta > \varepsilon/6$. This proves (39). To prove (40) it is sufficient to substitute (35) into (41). \qed
3.2. Asymptotic behaviour of \( \tau \). Applying Lemma 9, we obtain
\[
P_z(\tau > n) = P_z(\tau > n, \nu_n < n^{1-\varepsilon}) + P_z(\tau > n, \nu_n \geq n^{1-\varepsilon})
\]
\[
= P_z(\tau > n, \nu_n < n^{1-\varepsilon}) + O\left(e^{-n^{\varepsilon/4}}\right). \tag{45}
\]
Using the strong Markov property, we get for the first term the following estimates
\[
\int_{K_{n,\varepsilon}} P_z \left(Z_{\nu_n} \in d\tilde{z}, \tau > \nu_n, \nu_n < n^{1-\varepsilon}\right) P_{\tilde{z}}(\tau > n) \leq P_z(\tau > n, \nu_n < n^{1-\varepsilon})
\]
\[
\leq \int_{K_{n,\varepsilon}} P_z \left(Z_{\nu_n} \in d\tilde{z}, \tau > \nu_n, \nu_n < n^{1-\varepsilon}\right) P_{\tilde{z}}(\tau > n - n^{1-\varepsilon}). \tag{46}
\]
Applying now Lemma 14, we obtain
\[
P_z(\tau > n; \nu_n < n^{1-\varepsilon})
\]
\[
= \frac{\varepsilon + o(1)}{n^{1/4}} E_z \left[h(Z_{\nu_n}); \tau > \nu_n, |M_{\nu_n}| \leq \theta_n \sqrt{n}, \nu_n < n^{1-\varepsilon}\right]
\]
\[
+ O\left(\frac{1}{n^{1/4}} E_z \left[h(Z_{\nu_n}); \tau > \nu_n, |M_{\nu_n}| > \theta_n \sqrt{n}, \nu_n < n^{1-\varepsilon}\right]\right)
\]
\[
= \frac{\varepsilon + o(1)}{n^{1/4}} E_z \left[h(Z_{\nu_n}); \tau > \nu_n, \nu_n < n^{1-\varepsilon}\right]
\]
\[
+ O\left(\frac{1}{n^{1/4}} E_z \left[h(Z_{\nu_n}); \tau > \nu_n, |M_{\nu_n}| > \theta_n \sqrt{n}, \nu_n < n^{1-\varepsilon}\right]\right). \tag{47}
\]
where \( M_k := \max_{1 \leq k} |S_j| \).

We now show that the first expectation converges to \( V_0(z) \) and that the second expectation is negligibly small.

**Lemma 15.** Under the assumptions of Theorem 1,
\[
\lim_{n \to \infty} E_z \left[h(Z_{\nu_n}); \tau > \nu_n, \nu_n < n^{1-\varepsilon}\right] = V_0(z).
\]

**Proof.** Put \( T = \tau \wedge n^{1-\varepsilon} \). Since \( T \) is a bounded stopping time and \( Y_k \) is a martingale,
\[
E_z[Y_T] = E_z[Y_{\nu_n} \wedge T] = E_z[Y_{\nu_n}, \nu_n < T] + E_z[Y_T, \nu_n \geq T]
\]
and, consequently,
\[
E_z[Y_{\nu_n}, \nu_n < T] = E_z[Y_T, \nu_n < T].
\]

Using the definition of \( Y_k \), we infer from the last equality that
\[
E_z[h(Z_{\nu_n}), \nu_n < T] = E_z[h(Z_T), \nu_n < T] - E_z \left[\sum_{k=\nu_n}^{T-1} f(Z_k), \nu_n < T\right].
\]
Conditioning on \( Z_{\nu_n} \) and applying (29), we obtain
\[
\left|E_z \left[\sum_{k=\nu_n}^{T-1} f(Z_k), \nu_n < T\right]\right| \leq \frac{C}{n^{(1-\varepsilon)}} E_z[h(Z_{\nu_n}), \nu_n < \tau].
\]
From this inequality and Lemma 10 we conclude
\[
E_z[h(Z_{\nu_n}), \nu_n < T] = (1 + o(1)) E_z[h(Z_T), \nu_n < T] \quad \text{as } n \to \infty. \tag{48}
\]
Since \( h(Z_\tau) = 0 \), we have \( h(Z_T) = h(Z_{n^-}) 1\{\tau > n^{1-\varepsilon}\} \). Using Lemma 10 once again, we get
\[
\mathbb{E}_z[h(Z_T), \nu_n < T] = \mathbb{E}_z[h(Z_{n^-}), \nu_n < n^{1-\varepsilon}, \tau > n^{1-\varepsilon}]
\]
\[
= \mathbb{E}_z[h(Z_{n^-}), \tau > n^{1-\varepsilon}] + O(e^{-n^{1/8}}).
\]
And in view of Lemma 11,
\[
\lim_{n \to \infty} \mathbb{E}_z[h(Z_T), \nu_n < T] = V_0(z).
\]
Combining this relation with (48), we get the desired result. \(\square\)

**Lemma 16.** As \( n \to \infty \),
\[
\mathbb{E}_z \left[h(Z_{\nu_n}), \tau > \nu_n, \nu_n < n^{1-\varepsilon}, |M_{\nu_n}| > \theta_n \sqrt{n} \right] \to 0.
\]

**Proof.** On the event \( \nu_n \leq n^{1-\varepsilon} \),
\[
h(Z_{\nu_n}) \leq C\alpha(z) + C \max \left\{ \left( n^{1-\varepsilon} M_{n^{1-\varepsilon}} \right)^{1/3}, M_{n^{1-\varepsilon}} \right\}
\]
and, consequently,
\[
\mathbb{E}_z \left[h(Z_{\nu_n}), \tau > \nu_n, \nu_n < n^{1-\varepsilon}, |M_{\nu_n}| > \theta_n \sqrt{n} \right]
\]
\[
\leq C\alpha(z) \mathbb{P} \left( M_{n^{1-\varepsilon}} > \theta_n \sqrt{n} \right) + C \mathbb{E} \left[ M_{n^{1-\varepsilon}}, M_{n^{1-\varepsilon}} > \theta_n \sqrt{n} \right].
\]
(49)
Here we used the fact that if \( \theta_n \to 0 \) sufficiently slow, then
\[
\max \left\{ \left( n^{1-\varepsilon} M_{n^{1-\varepsilon}} \right)^{1/3}, M_{n^{1-\varepsilon}} \right\} = M_{n^{1-\varepsilon}}
\]
on the set \( \{ M_{n^{1-\varepsilon}} > \theta_n \sqrt{n} \} \).
Using now one of the Fuk-Nagaev inequalities, see Corollary 1.11 in [12], one can easily conclude that both summands on the right hand side of (49) vanish as \( n \to \infty.\) \(\square\)

In conclusion we prove that \( V \) and \( V_0 \) coincide.

**Lemma 17.** For any \( z \in K_+ \) we have
\[
V_0(z) = V(z)
\]
(50)

**Proof.** The proof follows closely the proof of Lemma 8. Recalling (27), we get
\[
\mathbb{E}_z[h(Z_{\tau}); \tau > n] = h(z) + \mathbb{E}_z \left[ \sum_{l=0}^{\tau-1} f(Z_l); \tau \leq n \right] + \sum_{l=0}^{n-1} \mathbb{E}_z[f(Z_l); \tau > n].
\]
Thus, it is sufficient to prove that
\[
\mathbb{E}_z \left[ \sum_{l=1}^{\tau-1} |f(Z(l))| \right] < \infty.
\]
(51)
Indeed, the dominated convergence theorem then implies that
\[
\mathbb{E}_z \left[ \sum_{l=0}^{\tau-1} f(Z(l)); \tau \leq n \right] \to \mathbb{E}_z \left[ \sum_{l=0}^{\tau-1} f(Z(l)) \right]
\]
and
\[
\sum_{l=0}^{n-1} \mathbb{E}_z[f(Z(l)); \tau > n] \leq \mathbb{E}_z \left[ \sum_{l=0}^{\tau-1} |f(Z(l))|; \tau > n \right] \to 0
\]
since $\tau$ is finite a.s. Then, as $n \to \infty$,
\[
E[z_h(Z_n); \tau > n] \to h(z) + E[z^{\tau-1}\sum_{l=0} f(Z_l)] = V(z),
\]
which proves (50).

To prove (51) we use the fact that we have already proved that
\[
P_z(\tau > n) \sim V_0(z) n^{-1/4}.
\]
We split (51) into three parts,
\[
E[z^{\tau-1}\sum_{l=0} |f(Z_l)|] = f(z) + \sum_{l=1}^\infty E[z(||f(Z_l)||; \tau > l)]
\]
\[
= f(z) + \sum_{l=1}^\infty E[z(||f(Z_l)||; |S_l^{(2)}|, |S_l| \leq 1, \tau > l)]
\]
\[
+ \sum_{l=1}^\infty E[z(||f(Z_l)||; |S_l^{(2)}|^{1/3} > |S_l|, \tau > l)]
\]
\[
+ \sum_{l=1}^\infty E[z(||f(Z_l)||; |S_l^{(2)}|^{1/3} \leq |S_l|, \tau > l)]
\]
\[
=: f(z) + \Sigma_1 + \Sigma_2 + \Sigma_3.
\]
First, using the fact that $|f(x,y)| \leq C$ for $|x|, |y| \leq 1$ and Lemma 7, we obtain
\[
\Sigma_1 \leq C \sum_{l=1}^\infty P_z(|S_l^{(2)}|, |S_l| \leq 1) \leq C \sum_{l=1}^\infty l^{-2} < C.
\]
Second, by Lemma 6,
\[
\Sigma_2 \leq C \sum_{l=1}^\infty E[z(||S_l^{(2)}||^{-1/2-\delta/3}, \tau > l)]
\]
\[
\leq C \sum_{l=1}^\infty P_z(\tau > l/2) \sup_{\tilde{z}} E\tilde{z}[|S_l^{(2)}|^{-1/2-\delta/3}]
\]
\[
\leq C(z) \sum_{l=1}^\infty l^{-1/4} \sum_{j=1}^{\infty} \sup_{\tilde{z}} E\tilde{z}[|S_l^{(2)}|^{-1/2-\delta/3}; j \leq |S_l^{(2)}| \leq j+1]
\]
\[
\leq C(z) \sum_{l=1}^\infty l^{-1/4} \left( \sum_{j=1}^{j^{1/2}} j^{-1/2-\delta/3} \sup_{\tilde{z}} P_{\tilde{z}}(j \leq |S_l^{(2)}| \leq j+1)
\]
\[
+ \tilde{j}^{3/2-1/2-\delta/3} \sup_{\tilde{z}} P_{\tilde{z}}(|S_l^{(2)}| > \tilde{j}^{1/2}) \right).
\]
Now we use the second concentration inequality from Lemma 7 to get an estimate
\[
\sup_{\tilde{z}} P_{\tilde{z}}(j \leq |S_l^{(2)}| \leq j+1) \leq Ct^{-3/2}.
\]
Then,
\[
\Sigma_2 \leq C(z) \sum_{l=1}^{\infty} l^{-1/4} \left( l^{-3/2} \sum_{j=1}^{l^{3/2}} j^{-1/2 - \delta/3} + l^{-3/4 - \delta/2} \right)
\]
\[
\leq C(z) \sum_{l=1}^{\infty} l^{-1-\delta/2} \leq C(z).
\]

Similarly,
\[
\Sigma_3 \leq C \sum_{l=1}^{\infty} \mathbb{P}_z(\tau > l/2) \sup_{\tilde{z}} \mathbb{E}_z \left[ |S_{l/2}|^{-3/2 - \delta}; |Y(l/2)| \geq 1; |S_{l/2}^{(2)}|^{1/3} \leq |S_{l/2}| \right]
\]
\[
\leq C(z) \sum_{l=1}^{\infty} l^{-1/4} \sum_{j=1}^{\infty} \sup_{\tilde{z}} \mathbb{E}_z \left[ |S_{l/2}|^{-3/2 - \delta}; j \leq |S_{l/2}| \leq j + 1; |S_{l/2}^{(2)}| \leq (j + 1)^3 \right]
\]
\[
\leq C(z) \sum_{l=1}^{\infty} l^{-1/4} \left( \sum_{j=1}^{l^{1/2}} \sum_{j=1}^{j^{3/2 - \delta}} \sup_{\tilde{z}} \mathbb{P}_z(j \leq |S_{l/2}| \leq j + 1; |S_{l/2}^{(2)}| \leq (j + 1)^3) + l^{-3/4 - \delta/2} \sup_{\tilde{z}} \mathbb{P}_z(|S_{l/2}| > l^{1/2}) \right).
\]

Using Lemma 7 once again, we get an estimate
\[
\mathbb{P}_z(j \leq |S_{l/2}| \leq j + 1; |S_{l/2}^{(2)}| \leq (j + 1)^3)
\]
\[
\leq C \sum_{i=1}^{(j+1)^3} \mathbb{P}_z(j \leq |S_{l/2}| \leq j + 1; |S_{l/2}^{(2)}| \in (i, i + 1)) \leq C(l + 1)^{-2} j^3
\]
uniformly in \(\tilde{z}\). Then,
\[
\Sigma_3 \leq C(z) \sum_{l=1}^{\infty} l^{-1/4} \left( \sum_{j=1}^{l^{1/2}} j^{-3/2 - \delta} l^{-2} j^3 + l^{-3/4 - \delta/2} \right)
\]
\[
\leq C(z) \sum_{l=1}^{\infty} l^{-1/4} \left( l^{-2/5 4/\delta 2} + l^{-3/4 - \delta/2} \right)
\]
\[
\leq C(z) \sum_{l=1}^{\infty} l^{-1-\delta/2} \leq CV_0(z).
\]

This proves that the sum (51) is finite.

\[
\square
\]

\textbf{References}


