EXTERIOR AND SYMMETRIC POWERS OF MODULES FOR CYCLIC 2-GROUPS

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Abstract. We prove a recursive formula for the exterior and symmetric powers of modules for a cyclic 2-group. This makes computation straightforward. Previously, a complete description was only known for cyclic groups of prime order.

1. Introduction

The aim of this paper is to provide a recursive procedure for calculating the exterior and symmetric powers of a modular representation of a cyclic 2-group. Let \( G \cong C_{2^n} \) be a cyclic group of order \( 2^n \) and \( k \) a field of characteristic 2. Recall that there are \( 2^n \) indecomposable \( kG \)-modules \( V_1, V_2, \ldots, V_{2^n} \) for which \( \dim V_r = r \).

**Theorem 1.1.** For all \( n \geq 1 \), \( r \geq 0 \) and \( 0 \leq s \leq 2^n-1 \) we have

\[
\Lambda^r(V_{2^n-1+s}) \cong \bigoplus_{i,j \geq 0} \Omega_{2^n}^{i+j}(\Lambda^i(V_s) \otimes_k \Lambda^j(V_{2^n-1-s})) \oplus tV_{2^n},
\]

where \( t \) is a non-negative integer chosen so that both sides have the same dimension.

Here \( \Omega_{2^n} \) is the syzygy or Heller operator over \( C_{2^n} \), so \( \Omega_{2^n}V_s = V_{2^n-s} \) for \( 1 \leq s \leq 2^n \). Since \( \Lambda(A \oplus B) \cong \Lambda(A) \otimes \Lambda(B) \), the left hand side of the formula is sufficient to cover all possible modules. On the right hand side, the two \( V \)'s that occur have dimension at most \( 2^n-1 \), so the group action factors through \( C_{2^n-1} \) and the formula for this smaller group can be applied. We also show that there is a simple recursive procedure for calculating tensor products, so we obtain a complete recursive procedure for calculating exterior powers. It is sufficiently efficient that it is easy to calculate even by hand far beyond the range that was previously attainable by machine computation.

For symmetric powers we use the following result from [21].

**Theorem 1.2.** For all \( n \geq 1 \), \( r \geq 0 \) and \( 0 \leq s \leq 2^n-1 \) we have

\[
S^r(V_{2^n-1+s}) \cong \text{ind} \Omega_{2^n}^{r'}\Lambda^{r'}(V_{2^n-1-s}),
\]

where \( 0 \leq r' < 2^n \) and \( r' \equiv r \pmod{2^n} \). Here the symbol \( \cong \text{ind} \) means up to direct summands induced from subgroups \( H \leq G \).

Thus a knowledge of the exterior powers determines the symmetric powers up to induced summands. In fact it is shown in [21] how such a formula determines the symmetric powers completely, using a recursive procedure.

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Formulas for the exterior and symmetric powers of a module for a cyclic group of prime order $p$ were given by Almkvist and Fossum [1] and Renaud [19]. These were extended to cyclic $p$-groups by Hughes and Kemper [14] provided that the power is at most $p-1$. A formula for $\Lambda^2$ in the case of cyclic $2$-groups was given by Gow and Laffey [11]. Also Kouwenhoven [15] obtained important results on exterior powers of modules for cyclic $p$-groups, including recursion formulas for $\Lambda(V_{qz1})$ where $q$ is a power of $p$. For $p=2$ these formulas are special cases or direct consequences of Theorem 1.1, so we obtain independent proofs for some of the results in [11, 15].

Our strategy is to consider $\Lambda(V_{2n-1+s})$ as the quotient of $S(V_{2n-1+s})$ by the ideal generated by the squares of elements of $V_{2n-1+s}$. It turns out that we need to consider an intermediate ring $\tilde{S}(V_{2n-1+s})$, in which we only quotient out the squares of the elements of $V_s \subseteq V_{2n-1+s}$. We show that $\tilde{S}^r(V_{2n-1+s}) \cong \text{ind}_V \Omega^r(V_{2n-1+s})$ for $r < 2^n$. But $\tilde{S}(V_{2n-1+s})$ can be resolved by the Koszul complex over $S(V_{2n-1+s})$ on the squares of the elements of a basis for $V_s$. We show that this Koszul complex is separated in the sense of [21], that is that the image of a boundary map is contained in a projective submodule. This leads to the formula

$$\tilde{S}^r(V_{2n-1+s}) \cong \text{proj}_{2t+j=r} \Omega_{2^n}^r(\Lambda^t(V_s) \otimes_k S^j(V_{2n-1+s})),$$

where the symbol $\cong_{\text{proj}}$ means up to projective summands. Using Theorem 1.2, the right hand side is easily seen to be equal to the right hand side of the formula in Theorem 1.1 modulo induced summands. This yields the formula of Theorem 1.1 modulo induced summands. The strengthening to an equality modulo just projective summands is a formal inductive argument.

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2. Koszul Complexes

Let $G$ be a finite group, $H$ a subgroup of $G$ and $k$ a field of characteristic $p > 0$. All tensor products will be over $k$ if not otherwise specified. We recall some general facts about chain complexes of $kG$-modules from [21, Section 3].

**Definition 2.1.** ([21, Definition 3.2]) A chain complex $C_\ast$ of $kG$-modules is called:

(a) *acyclic* if it is 0 in negative degrees and it only has homology in degree 0;

(b) *weakly induced* if each module is induced from $H$, and weakly induced except in degrees $I$ if each $C_i$, $i \notin I$, is induced from $H$;

(c) *separated at $C_i$* if $\text{Im}(d_{i+1}) \rightarrow C_i$ factors through a projective $kG$-module;

(d) *separated* if it is separated at each $C_i$.

Write $B_i$ for $\text{Im}(d_{i+1}) \subseteq C_i$. If the inclusion $B_i \hookrightarrow C_i$ factors through a projective then it factors through the injective hull of $B_i$, call it $P_i$ (injective is equivalent to projective for modular representations), and $P_i \hookrightarrow C_i$ is injective since it is so on the socle. Thus we can write $C_i = P_i \oplus C_i''$ and $B_i \subseteq P_i$.

**Lemma 2.2.** ([21, Lemma 3.9]) If the chain complexes $C_\ast, C'_\ast$ of $kG$-modules are separated then so is the (total) tensor product $C_\ast \otimes C'_\ast$. Similarly for a product of finitely many chain complexes.
Definition 2.6. Let $P_i$ be a projective module such that $\text{Im}(d_{i+1}) \subseteq P_i \subseteq C_i$ and similarly for $P'_i$. Then, summing over all degrees, $\text{Im}(d \otimes d') \subseteq P \otimes C' + C \otimes P'$. There is a short exact sequence $0 \to P \otimes P' \to P \otimes C' + C \otimes P' \to P \otimes C + C \otimes P \to 0$. The first two terms are projective, hence so is the third. \qed

We need to consider tensor-induced complexes. For details of the construction see [4, II 4.1].

Lemma 2.3. Suppose that every elementary abelian $p$-subgroup of $G$ is conjugate to a subgroup of $H$, and let $C_s$ be a complex of $kH$-modules that is separated. Then the tensor-induced complex $C_s \otimes^G_H$ is also separated.

Proof. By the proof of Lemma 2.2 above, the image $\text{Im}(d_{i+1}^G_H)$ is contained in

$$P \otimes C \otimes \cdots \otimes C + C \otimes P \otimes \cdots \otimes C + \cdots + C \otimes C \otimes \cdots \otimes P,$$

which is a $kG$-submodule of $C_s \otimes^G_H$. But the same proof shows that this module is projective on restriction to $H$, so it is projective, by Chouinard’s Theorem [7, Corollary 1.1]. \qed

The next two results comprise a variation on [21, Proposition 3.3] and have the same proof.

Proposition 2.4. Suppose that the complex $K_s : K_w \to \cdots \to K_0$ of $kG$-modules is:

(a) acyclic,
(b) weakly induced except in at most one degree and
(c) $K_s$ is separated on restriction to $H$.

Then $K_s$ is separated.

Recall that the Heller translate $\Omega V$ of a $kG$-module $V$ is defined to be the kernel of the projective cover $P(V) \to V$ and $\Omega^i V$ for $i \geq 1$ denotes $\Omega$ iterated $i$ times. Similarly $\Omega^{-1} V$ is the cokernel of the injective hull $V \to I(V)$ and $\Omega^{-i} V$ for $i \geq 1$ is its iteration. We let $\Omega^0 V$ denote $V$ with any projective summands removed. These have the properties that $\Omega \Omega^j V \cong \Omega^{i+j} V$ and that if $V$ is induced so is $\Omega V$.

Lemma 2.5. Suppose that the complex $K_s : K_w \to \cdots \to K_0$ of $kG$-modules is:

(a) acyclic with $H_0(K_s) = L$, and
(b) separated.

Then $L \cong \text{proj} \ K_0 \oplus \Omega^{-1} K_1 \oplus \Omega^{-2} K_2 \oplus \cdots \oplus \Omega^{-w} K_w$.

Let $V$ be a $kG$-module, finite-dimensional as a $k$-vector space, and $W$ a submodule of $V$. We write $S = S(V) = \bigoplus_{r=0}^{\infty} S^r(V)$ for the symmetric algebra on $V$ and $\Lambda(W) = \bigoplus_{r=0}^{\infty} \Lambda^r(W)$ for the exterior algebra on $W$. For $r < 0$ let $S^r(V)$ denote the 0 module.

Definition 2.6. Let $W$ be a submodule of a $kG$-module $V$ and let $W^p$ denote the $kG$-submodule of $S^p(V)$ spanned by the $p$-th powers of elements of $W$. Let $K(V, W^p)$ denote the Koszul complex of graded $kG$-modules:

$$\cdots \to d S(V) \otimes \Lambda^2(W^p) \to d S(V) \otimes \Lambda^1(W^p) \to d S(V) \otimes W^p \to d S(V),$$

where $d(f \otimes w_1^p \wedge \cdots \wedge w_r^p) = \sum_{j=1}^{i} (-1)^{i-j+1} f w_1^p \otimes w_j^p \wedge \cdots \wedge w_i^p \wedge \cdots \wedge w_r^p$ for $w_j \in W$ and $f \in S(V)$. We write $K^r(V, W^p)$ when we consider the complex $K(V, W^p)$ in grading-degree $r$. 

If \( k = \mathbb{F}_2 \) then the squaring map gives an isomorphism between \( W \) and \( W^2 \), so we can regard \( W^2 \) as a copy of \( W \) in degree 2 equipped with a squaring map into \( S^2(V) \). From this point of view, the boundary map is given by \( d(f \otimes w_1 \wedge w_2 \wedge \cdots \wedge w_i) = \sum_{j=1}^{i} f w_j^2 \otimes w_1 \wedge \cdots \wedge \widehat{w_j} \wedge \cdots \wedge w_i \) for \( w_j \in W \) and \( f \in S(V) \).

We will normally take the second point of view, so we will assume that \( k = \mathbb{F}_2 \) in a large part of this paper. Since any \( kC_{2^n} \)-module can be written in \( \mathbb{F}_2 \), this is not a significant restriction.

**Lemma 2.7.** In the context of Definition 2.6, the complex \( K(V, W^p) \) is acyclic and its homology in degree 0 is \( S(V)/(W^p) \), where \( (W^p) \) is the ideal generated by all elements \( w^p \), \( w \in W \).

**Proof.** If \( \{w_1, \ldots, w_r\} \) is a basis for \( W \) then \( \{w_1^p, \ldots, w_r^p\} \) is a regular sequence in \( S(V) \) and it spans \( W^p \). This is now a standard result about Koszul complexes. \( \square \)

**Lemma 2.8.** Let \( V, V' \) be \( kG \)-modules, finite-dimensional as \( k \)-vector spaces and let \( W, W' \) be submodules of \( V \) and \( V' \), respectively. The complex \( K(V \oplus V', (W \oplus W')^p) \) is isomorphic to the (total) tensor product \( K(V, W^p) \otimes K(V', W'^p) \) as a complex of graded \( kG \)-modules.

**Proof.** This is analogous to [21, Lemma 3.8]. \( \square \)

We also need to deal with tensor induction of graded modules and complexes.

**Lemma 2.9.** Let \( H \) be a subgroup of \( G \) and let \( V, W \) be \( kH \)-modules. Then \( S(V \uparrow_H^G) \cong S(V) \uparrow_H^{S_G}, \Lambda(V \uparrow_H^G) \cong \Lambda(V) \uparrow_H^{S_G} \) as graded \( kG \)-modules, and if the characteristic of \( k \) is 2 then \( K(V \uparrow_H^G, (W \uparrow_H^G)^p) \cong K(V, W^p) \uparrow_H^{S_G} \) as complexes of graded \( kG \)-modules.

Without the restriction on the characteristic of \( k \) we would have to deal with the sign convention that appears in the definition of the action of \( G \) on the tensor-induced complex.

**Proof.** Let \( \{g_i\} \) be a set of coset representatives for \( G/H \) and write \( V \uparrow_H^G = \oplus g_i \otimes V \). The formulas now follow from the usual formulas for \( S \) and \( \Lambda \) of a sum and the definition of the group action on a tensor induced module. \( \square \)

### 3. Modules for Cyclic 2-Groups

From now on, let \( G = \langle g \rangle \cong C_{2^n} \) be a cyclic group of order \( 2^n \), \( n \geq 1 \), and \( k \) a field of characteristic 2. We write \( \omega(G) \) for the Green ring of \( kG \)-modules. Up to isomorphism, there are \( 2^n \) indecomposable \( kG \)-modules \( V_1, V_2, \ldots, V_{2^n} \), and for convenience we write \( V_0 \) for the 0 module. We have \( \dim_k(V_i) = i \), and the generator \( g \in G \) acts on \( V_i \) with matrix a Jordan block with ones on the diagonal. Choose a \( k \)-basis \( \{x_1, x_2, \ldots, x_n\} \) of \( V_n \) such that \( gx_i = x_i + x_{i-1} \) for all \( 2 \leq i \leq n \) and \( gx_1 = x_1 \). Each element of \( S(V_i) \) can be written uniquely as a polynomial in \( x_1, \ldots, x_i \), and for \( j \leq i \), we can identify \( V_j \) with the \( kG \)-submodule of \( V_i \) spanned by \( x_1, x_2, \ldots, x_j \). Each \( V_i \) is uniserial with composition series \( 0 < V'_i < V'_2 < \cdots < V'_{i-1} < V'_i \). Note that for \( i \leq 2^{n-1} \) the kernel of \( V'_i \) is nontrivial and so \( V'_i \) can be identified with the \( i \)-dimensional indecomposable module for the quotient group \( C_{2^{n-1}} \).

Decompositions of tensor products into indecomposables have been studied by several authors, see for example [2, 3, 9, 13, 16, 17, 18, 20]. In our case, this decomposition can easily be computed using the Heller translate \( \Omega \). We write \( \Omega_{2^n} \) instead of \( \Omega \) when we want
Lemma 3.4. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $kG$-modules that is separated at $B$ on restriction to $H$ and such that $C \cong_{\text{ind}} B \oplus \Omega_{2^n-1}^{-1} A$ as $kG$-modules. Then the sequence is separated at $B$ (as a sequence of $kG$-modules).
Proof. The hypotheses imply that there are induced modules $X$ and $Y$ such that $C \oplus X \cong B \oplus \Omega^1_{2n}A \oplus Y$ and also $C \downarrow_H^G \cong \text{proj } B \downarrow_H^G \oplus \Omega^1_{2n-1}A \downarrow_H^G \cong \text{proj } (B \oplus \Omega^1_{2n}A) \downarrow_H^G$ by Lemma 2.5 applied to $0 \to A \downarrow_H^G \to B \downarrow_H^G \to 0$. It follows that $X \downarrow_H^G \cong \text{proj } Y \downarrow_H^G$, hence, by Lemma 3.2, $X \cong \text{proj } Y$ and then $C \cong \text{proj } B \oplus \Omega^1_{2n}A$.

Thus our short exact sequence is $0 \to A \xrightarrow{d} B \xrightarrow{e} B \oplus \Omega^1_{2n}A \to 0$, up to projective summands. Consider the long exact sequence for Tate Ext:

$$\cdots \to \text{Hom}_{kG}(A,A) \xrightarrow{d} \text{Hom}_{kG}(A,B) \xrightarrow{\text{Hom}_{kG}(B) \oplus \text{Hom}_{kG}(A,B)} \text{Hom}_{kG} \to \text{Ext}^1(A,A) \to \cdots,$$

where $\text{Hom}_{kG}$ denotes homomorphisms modulo those that factorize through a projective.

Since $\text{Ext}^1(A,A) \cong \text{Hom}_{kG}(A,\Omega^1_{2n}A)$, a dimension count shows that $e_*$ is injective and so $d_*=0$. But $d=d_*(\text{Id}_A)$, so $d$ factors through a projective, as required. 

The next lemma describes tensor induction from $H$ to $G$ modulo induced modules and gives information on the structure of the exterior algebra $\Lambda(V_{2j})$ as a $kG$-module in terms of the $kH$-module $\Lambda(V_j)$.

**Lemma 3.5.** Let $r, j$ be integers such that $r \geq 0$ and $1 \leq j \leq 2^{n-1}$. We consider $V_j$ as a $kH$-module and $V_{2j} = V_j \uparrow_H^G$ as a $kG$-module.

(a) Let $A$ and $B$ be $kH$-modules. Then $(A \oplus B) \uparrow_H^{\otimes G} \cong A \uparrow_H^{\otimes G} \oplus B \uparrow_H^{\otimes G} \oplus X$ for some induced $kG$-module $X$.

(b) There is an induced $kG$-module $X$ such that $\Lambda^{2r}(V_{2j}) \cong \Lambda^r(V_j)^{\uparrow_H^{\otimes G}} \oplus X$.

(c) If $r$ is odd, then the $kG$-module $\Lambda^r(V_{2j})$ is induced from $H$.

(d) If $j$ is even, then the $kG$-module $V_j \uparrow_H^{\otimes G}$ is induced from $H$.

(e) If $j$ is odd, then $V_j \uparrow_H^{\otimes G} \cong \text{ind } V_1$.

Proof. (a) follows from [4, I 3.15.2 (iii)].

(b) By the construction of induced modules, we have $V_{2j} = V_j \oplus gV_j$ as vector spaces and the action of the generator $g$ of $G$ on $V_{2j}$ is given by $g(v + gv') = g^2v' + gv$. So there is a natural isomorphism

$$\Lambda^{2r}(V_{2j}) = \Lambda^{2r}(V_j \oplus gV_j) \cong \bigoplus_{r', r'' \geq 0, r'+r''=2r} (\Lambda^{r'}(V_j) \otimes \Lambda^{r''}(gV_j))$$

of vector spaces, and thus

$$\Lambda^{2r}(V_{2j}) \cong (\Lambda^r(V_j) \otimes \Lambda^{r''}(gV_j)) \oplus \bigoplus_{0 \leq r', r'' \leq r', r'+r''=2r} ((\Lambda^{r'}(V_j) \otimes \Lambda^{r''}(gV_j)) \oplus g(\Lambda^{r'}(V_j) \otimes \Lambda^{r''}(gV_j))).$$

Via this isomorphism, the right hand side becomes a $kG$-module and from the action of $g$, we see that $\Lambda^{r'}(V_j) \otimes \Lambda^{r''}(gV_j) \oplus g(\Lambda^{r'}(V_j) \otimes \Lambda^{r''}(gV_j))$ is a $kG$-submodule isomorphic to $\Lambda^{r'}(V_j) \otimes \Lambda^{r''}(gV_j) \uparrow_H^{\otimes G}$ and $\Lambda^{r'}(V_j) \otimes \Lambda^{r''}(gV_j)$ is a submodule isomorphic to $\Lambda^{r'}(V_j) \uparrow_H^{\otimes G}$.

(c) The proof is similar to that of (b). Note that, if $r$ is odd, the summand corresponding to $r'' = r'$ which leads to the tensor induced submodule in (b) does not occur.

(d),(e) We say that a $kG$-module is induced except for possibly one trivial summand if it is isomorphic to $A \uparrow_H^{\otimes G} \oplus A \uparrow_H^{\otimes G} \oplus V_i$ for some $kH$-module $A$. We prove (d) and (e) simultaneously by showing that for all $1 \leq j \leq 2^{n-1}$ the $kG$-module $V_j \uparrow_H^{\otimes G}$ is induced except for possibly
one trivial summand. The claim then follows from the fact that \( \text{dim}_k(V_j \uparrow_H^G) \) is even if and only if \( j \) is even.

The proof is by induction on \( j \). Because \( V_1 \uparrow_H^G \cong V_1 \) we can assume \( j > 1 \). If \( j \) is even, then the \( kH \)-module \( V_j \) is induced from a proper subgroup of \( H \). So [4, I 3.15.2 (iv)] implies that \( V_j \uparrow_H^G \) is a direct sum of modules induced from \( H \) (even from proper subgroups of \( H \)). Assume that \( j \) is odd. So we can write \( j = 2m + j' \) with \( 1 \leq m < n - 1 \) and \( 1 \leq j' < 2m \).

First, we treat the case \( j' = 1 \). By the Mackey formula for tensor induction [4, I 3.15.2 (v)] we have \( V_j \uparrow_H^G \cong V_{2m+1} \otimes V_{2m+1} \cong V_1 \oplus (2^m - 2)V_{2m} \oplus 2V_{2m+1} \), and so \( V_j \uparrow_H^G \cong V_1 \oplus (2^{m-1} - 1)V_{2m+1} \oplus V_{2m+2} \), which is induced up to one trivial summand. Now assume \( j' > 1 \). Then \( V_j \otimes V_{2m+1} \cong V_j \oplus (j' - 1)V_{2m} \) as \( kH \)-modules. By [4, I 3.15.2 (i)] and (a) we get

\[
(1) \quad (V_j \uparrow_H^G) \otimes (V_{2m+1} \uparrow_H^G) \cong \text{ind}_H \left( V_j \uparrow_H^G \right) \oplus (j' - 1)(V_{2m} \uparrow_H^G).
\]

By induction and the case \( j' = 1 \), we know that the left hand side of (1) is induced except for possibly one trivial summand. Hence, \( V_j \uparrow_H^G \) is induced except for possibly one trivial summand. \( \square \)

We can now see that the symmetric and exterior powers of even dimensional indecomposable modules have a particularly restricted form.

**Corollary 3.6.** Suppose that we have non-negative integers \( j, s, t, u \) with \( u, j \) odd and \( s \geq 1 \). Furthermore, assume that \( 2^u < 2^n \) and \( 2^s j \leq 2^n \). Then \( \Lambda^{2^u}(V_{2s}) \) and \( S^{2^u}(V_{2s}) \) are both induced unless \( t \geq s \). If \( t \geq s \) then \( \Lambda^{2^u}(V_{2s}) \cong mV_1 \oplus X \) and \( S^{2^u}(V_{2s}) \cong m'V_1 \oplus Y \), where \( X, Y \) are induced modules and \( m \) and \( m' \) are the numbers of non-induced indecomposable summands in \( \Lambda^{2^u}(V_j) \) and \( S^{2^u}(V_{2^n-s-j}) \), respectively.

**Proof.** Using Lemma 3.5 (a),(b),(d) we see that, up to induced direct summands, \( \Lambda^{2^u}(V_{2s}) \) is tensor-induced from a subgroup of index \( 2^\min(s,t) \). If \( t < s \) then, up to induced direct summands, it is tensor-induced from \( \Lambda^s(V_{2^n-t-j}) \) and thus is induced, by part (c) of the same lemma. If \( t \geq s \) then, again up to induced direct summands, it is tensor-induced from \( \Lambda^{2^u}(V_{2s}) \); the description given is then seen to be valid using parts (a), (d) and (e). The case of \( S^{2^u}(V_{2s}) \) reduces to that of \( \Lambda^{2^u}(V_{2^n-s-j}) \), by Theorem 1.2. \( \square \)

**Corollary 3.7.** If \( X \) is a \( kG \)-module such that every direct summand has dimension divisible by 4 then \( S^2(X) \) is induced.

**Proof.** By the identity \( S^2(A \oplus B) \cong S^2(A) \oplus S^2(B) \oplus A \otimes B \), we may assume that \( X \) is indecomposable, say \( X = V_{4u} \). The claim now follows from Corollary 3.6. \( \square \)

In the proof of our main result we will often have information only modulo induced direct summands. The following definition and lemmas deal with the splitting of maps in such situations.

Recall that a map \( f : A \rightarrow B \) of \( kG \)-modules is split (injective), if there is a map \( g : B \rightarrow A \) of \( kG \)-modules such that \( g \circ f = \text{Id}_A \). For maps \( f : A \rightarrow B \), \( f' : A \rightarrow B' \) of \( kG \)-modules we write \((f, g) : A \rightarrow B \oplus B', a \mapsto (f(a), f'(a)) \).

**Definition 3.8.** Let \( f : A \rightarrow B \) be a map of \( kG \)-modules. We say that \( f \) is split modulo induced summands if there exists an induced \( kG \)-module \( X \) and a map \( f' : A \rightarrow X \) of \( kG \)-modules such that \((f, f') : A \rightarrow B \oplus X \) is split.
Split modulo induced summands behaves in much the same way as split.

**Lemma 3.9.** Given maps \( f : A \to B, g : B \to C \) and \( h : D \to E \) of \( kG \)-modules:

(a) if \( f \) and \( g \) are split modulo induced summands then so is \( g \circ f \),

(b) if \( g \circ f \) is split modulo induced summands then so is \( f \),

(c) if \( f \) and \( h \) are split modulo induced summands then so is \( f \otimes h : A \otimes D \to B \otimes E \).

**Proof.** (a) By assumption, we have induced modules \( X, Y \) and maps \( f' : A \to X, u : B \to A, \)
\( u' : X \to A, g' : B \to Y, v : C \to B, v' : Y \to B \) such that \( u \circ f + u' \circ f' = \text{Id}_A \) and
\( v \circ g + v' \circ g' = \text{Id}_B \). We define \( (g \circ f)' : A \to X \otimes Y, a \mapsto (f'(a), g' \circ f(a)), \)
\( w : C \to A, c \mapsto u \circ v(c) \) and \( w' : X \oplus Y \to A, (x, y) \mapsto u'(x) + u \circ v'(y) \). Then \( w \circ (g \circ f) + w' \circ (g \circ f)' = \text{Id}_A \).

Parts (b) and (c) are proved in a similar way; the proofs are left to the reader. \( \Box \)

**Lemma 3.10.** Let \( f : A \to B \) be a map of \( kG \)-modules and write \( A = A' \oplus A'' \), where \( A' \) has only non-induced summands and \( A'' \) has only induced summands. Let \( i \) denote the inclusion of \( A' \) in \( A \). Then \( f \) is split modulo induced summands if and only if \( f \circ i \) is split.

**Proof.** Suppose that \( f \) is split modulo induced summands; we want to show that \( f \circ i \) is split.
By Lemma 3.9 (a), the map \( f \circ i \) is split modulo induced summands, so we can assume that \( A = A' \) and we have to show that \( f \) is split.

Since \( f \) is split modulo induced summands we have an induced module \( X \) and maps \( f' : A \to X, u : B \to A, u' : X \to A \) such that \( u \circ f + u' \circ f' = \text{Id}_A \). Since \( X \) and \( A \) have no summands in common, we know that \( u' \circ f' \) lies in the radical of \( \text{End}_{kG}(A) \) (note that if \( A = \bigoplus A_i \) with \( A_i \) indecomposable and we write elements of \( \text{End}_{kG}(A) \) as matrices with entries in \( \text{Hom}_{kG}(A_i, A_j) \) then the radical consists of the morphisms for which no component is an isomorphism). Thus \( u \circ f \) is surjective, hence an automorphism of \( A \), and \( f \) is split.

Conversely, suppose that \( f \circ i : A' \to B \) is split, so there is a map \( g : B \to A' \) such that \( g \circ (f \circ i) = \text{Id}_{A'} \). Let \( j \) denote the inclusion of \( X := A'' \) in \( A \) and \( f' \) the projection of \( A \) onto \( A'' \). We define \( v := i \circ g : B \to A \), and \( v' : X \to A, x \mapsto -(i \circ g \circ f \circ j)(x) + j(x) \). Then \( v \circ f + v' \circ f' = \text{Id}_{A'} \), so \( f \) is split modulo induced summands. \( \Box \)

**Remark.** The proof above shows that the induced module \( X \) in Definition 3.8 can always be chosen in such a way that \( X \) only contains indecomposable direct summands that also occur in \( A \).

**Remark.** Definition 3.8 makes sense for any finite group and any class of indecomposable modules and Lemmas 3.9(a,b) and 3.10 remain true.

It will turn out that certain symmetric and exterior powers of modules for cyclic 2-groups are contained in the \( \mathbb{Z} \)-submodule \( c(G) \) of the Green ring \( a(G) \) spanned by the indecomposable modules \( V_r \) for \( r \) satisfying \( r \equiv 2 \pmod{4} \). We describe some properties of \( c(G) \).

**Lemma 3.11.** The submodule \( c(G) \) is

(a) a subring of \( a(G) \) and

(b) closed under \( \Omega_2^r \).

**Proof.** Part (b) is clear from the definitions.

For part (a) we need to show that \( V_i \otimes V_j \in c(G) \) for all \( 0 \leq i, j \leq 2^n, i, j \not\equiv 2 \pmod{4} \).
For \( n = 1 \) we only have \( V_1 \otimes V_1 = V_1 \in c(G) \). Suppose that \( n > 1 \). By the remarks on the computation of tensor products at the beginning of this section, we have \( V_i \otimes V_j = \).
Theorem 4.1. Let \( n \geq 1 \) and \( 0 \leq s \leq 2^n - 1 \). For \( s < 2^n - 1 \) we have \( \tilde{S}(V_{2^n-1+s}) \cong \text{ind}_{k[a]} \tilde{S}^{\leq 2^n}(V_{2^n-1+s}) \) as graded \( kG \)-modules. For \( s = 2^n - 1 \) we have \( \tilde{S}(V_{2^n}) \cong \text{ind}_{k[a]} (\tilde{S}^{< 2^n}(V_{2^n}) \oplus \tilde{b}) \). In both cases the isomorphism from right to left is induced by the product in \( \tilde{S}(V_{2^n-1+s}) \).

(c) (Splitting) The short exact sequence of graded \( kG \)-modules

\[
0 \rightarrow \tilde{N}(V_{2^n-1+s}) \rightarrow \tilde{S}(V_{2^n-1+s}) \rightarrow \Lambda(V_{2^n-1+s}) \rightarrow 0
\]

is split and \( \tilde{N}(V_{2^n-1+s}) = \tilde{a}\tilde{S}(V_{2^n-1+s}) \oplus \tilde{I} \), where \( \tilde{I} \) is a \( kG \)-module induced from \( H \).

(d) (Exterior powers) For each \( r \geq 0 \) we have the following isomorphism of graded \( kG \)-modules

\[
\Lambda^r(V_{2^n-1+s}) \cong \text{proj} \bigoplus_{i+j=r, 2i+j \geq 0} \Omega_{2^n}^{i+j}(\Lambda^i(V_s) \otimes \Lambda^j(V_{2^n-1-s})).
\]

The case \( s = 0 \) is a little unnatural, but we need it for the induction, because the restriction of \( V_{2^n-1} \) is \( V_{2^n-2} \oplus V_{2^n-2} \).

It is sometimes more succinct to consider Hilbert series with coefficients in the Green ring (possibly modulo projectives or induced modules). For more details see [12]. In particular,
we consider the following series associated to a $kG$-module $V$:

$$
\lambda_t(V) = \sum_{r=0}^{\infty} \Lambda^r(V)t^r, \quad \sigma_t(V) = \sum_{r=0}^{\infty} S^r(V)t^r, \quad \tilde{\sigma}_t(V) = \sum_{r=0}^{\infty} \tilde{S}^r(V)t^r, \quad \lambda_t^\Omega(V) = \sum_{r=0}^{\infty} \Omega^r(V)t^r.
$$

The last of these requires $G$ to be specified in order for the $\Omega$ to be determined; it is naturally considered modulo projectives. They all commute with restriction and turn direct sums of modules into products of series. This is an easy consequence of the corresponding properties of the corresponding functors on modules, except perhaps for $\lambda_t^\Omega(V \oplus W)$, where we need the formula $\Omega^rV \otimes \Omega^sW \cong_{\text{proj}} \Omega^{r+s}(V \otimes W)$.

Many of our statements about modules imply Hilbert series versions.

\begin{equation}
\begin{align*}
\sigma_t(V_{2n-1+s}) &= \text{ind } \lambda_t^\Omega(V_{2n-1-s})(1 - t^{2n})^{-1} \quad \text{Theorem 1.2} \\
\tilde{\sigma}_t(V_{2n-1+s}) &= \text{ind } \lambda_t^\Omega(V_s) \sigma_t(V_{2n-1+s}) \quad \text{Separation 4.1(a)} \\
\tilde{\sigma}_t(V_{2n-1+s}) &= \text{ind } \lambda_t(V_{2n-1-s})(1 - t^{2n})^{-1} \quad \text{Splitting and periodicity 4.1(b),(c)} \\
\lambda_t(V_{2n-1+s}) &= \text{proj } \lambda_t^\Omega(V_s) \lambda_t^\Omega(V_{2n-1-s}) \quad \text{Exterior powers 4.1(d),}
\end{align*}
\end{equation}

where the symbols $=\text{ind}$ and $=\text{proj}$ mean that we consider the coefficients only modulo induced or projective direct summands, respectively. The first and last of the above identities are, in fact, equivalent to the original versions. The second identity follows from Theorem 4.1(a), Lemma 2.5 and Lemma 2.7 (once the theorem is proved).

Remark. An easy calculation shows that, for fixed $n$ and $s$, the last of the formulas in (2) follows formally from the first three if we are satisfied with only $=\text{ind}$.

Remark. The proof of Theorem 1.2 in [21] actually gives a more precise formula than the first one in (2). It works by showing that the complex $K(V_{2n}, V_{2n-1-s})$ defined in [21] is separated and then applying Lemma 2.5; note that the definition of $K(V_{2n}, V_{2n-1-s})$ in [21] is different from our Definition 2.6. The result is that $\sigma_t(V_{2n-1+s}) = \text{proj } \sigma_t(V_s) \lambda_t^\Omega(V_{2n-1-s})$. Since $V_{2n}$ can be given a basis that is permuted by $G$, each $S^r(V_{2n})$ has a monomial basis that is permuted. For small $n$, the decomposition of $\sigma_t(V_{2n})$ can be calculated by hand; in general the calculation can be organized using [21, Proposition 2.2]. Alternatively, [21, Proposition 2.2] can be applied directly to $\sigma_t(V_{2n-1+s})$.

The next six sections are devoted to the proof of Theorem 4.1 by induction on $n$.

5. The Case $n = 1$

In this section we start the inductive proof of Theorem 4.1. Suppose that $n = 1$, so we have to prove the statements in Theorem 4.1 for $s \in \{0,1\}$. With these assumptions on $n$ and $s$, parts (b)-(d) of Theorem 4.1 can easily be verified by a direct calculation. In fact, in (b) one obtains isomorphisms of $kG$-modules (not only modulo induced summands), and in (c) one gets $\tilde{t} = 0$. Separation for $s = 0$ is trivial.

Let us consider part (a) for $n = s = 1$. We have to show that for each $r > 0$ the short exact sequence $0 \to S^{-2}(V_2) \to S^r(V_2) \to \tilde{S}^r(V_2) \to 0$ of $kG$-modules is separated at $S^r(V_2)$. If $r$ is odd, then $S^r(V_2)$ is induced by Theorem 1.2, hence projective, and so separation is obviously true. Separation is trivial for $r = 0$. For even $r > 0$, a direct calculation and
Theorem 1.2 show that $\tilde{S}^r(2V) \cong V_1 \oplus V_1 \cong_{\text{ind}} S^r(2V) \oplus \Omega^1_2 S^{r-2}(2V)$, and so separation follows from Lemma 3.4.

Sections 6-10 comprise the inductive step in the proof of Theorem 4.1. In these sections we always assume that $n > 1$ is an integer and that Theorem 4.1 holds for all smaller values of $n$. Throughout these sections the notation remains the same as in Sections 3 and 4; thus $G = \langle g \rangle \cong C_{2^n}$ is a cyclic group of order $2^n$, $k = \mathbb{F}_2$ is a field with two elements and $s$ is an integer such that $0 \leq s \leq 2^{n-1}$.

6. Periodicity

In this section we prove part (b) of Theorem 4.1, assuming that parts (a)-(d) of the theorem hold for all smaller values of $n$.

Let $H$ be the unique maximal subgroup of $G$ and let $\{x_1, x_2, \ldots, x_{\text{top}}\}$ be a $k$-basis of $V_{2^n-1+s}$ as in Section 4. We choose $G$-invariant elements $a \in S^{2^i}(V_{2^n-1+s})$ and $\tilde{a} \in \tilde{S}^{2^i}(V_{2^n-1+s})$ as in Section 4. Let $T(V_{2^n-1+s})$ be the $kG$-submodule of $S(V_{2^n-1+s})$ spanned by the monomials in $x_1, \ldots, x_{\text{top}}$ that are not divisible by $x_{\text{top}}$. We have $S(V_{2^n-1+s}) \cong k[a] \otimes T(V_{2^n-1+s})$ as $kG$-modules; see [21, Lemma 1.1]. So $T^{<2^n}(V_{2^n-1+s}) = S^{<2^n}(V_{2^n-1+s})$. Notice that the periodicity of $S(V_{2^n-1+s})$ in [21, Theorem 1.2] is equivalent to $T^{<2^n}(V_{2^n-1+s})$ being induced. In fact, we know something stronger from [21, Corollary 3.11], namely that $T^{<2^n-1}(V_{2^n-1+s})$ is induced.

We can make the same construction for $\tilde{S}(V_{2^n-1+s})$, obtaining $\tilde{S}(V_{2^n-1+s}) \cong k[\tilde{a}] \otimes \tilde{T}(V_{2^n-1+s})$ as $kG$-modules.

Define $L(V_{2^n-1+s}, V_s^2)$ to be the subcomplex of $K(V_{2^n-1+s}, V_s^2)$ defined using $T(V_{2^n-1+s})$ instead of $S(V_{2^n-1+s})$, that is

$$
\cdots \xrightarrow{d} T(V_{2^n-1+s}) \otimes \Lambda^2(V_s) \xrightarrow{d} T(V_{2^n-1+s}) \otimes V_s \xrightarrow{d} T(V_{2^n-1+s}),
$$

where the boundary morphisms are as in Definition 2.6 (this can be done since the $x_{\text{top}}$ used in the definition of $T(V_{2^n-1+s})$ is not contained in $V_s$). Thus $L(V_{2^n-1+s}, V_s^2)$ is a complex of graded $kG$-modules; it is exact except in degree 0, where the homology is $H_0(L(V_{2^n-1+s}, V_s^2))$, which is isomorphic to $\tilde{T}(V_{2^n-1+s})$ as a $kG$-module. Notice that, by construction, the complexes $K(V_{2^n-1+s}, V_s^2)$ and $k[a] \otimes L(V_{2^n-1+s}, V_s^2)$ of $kG$-modules are isomorphic. In particular, note for later use that one of them is separated (over $G$ or over $H$) if and only if the other is so too.

From now on we fix $s$ and abbreviate the notation to just $S$, $T$, $K$, $L$, etc.

Suppose that $s < 2^{n-1}$. We claim that $L'_i = T^{r-2i} \otimes \Lambda^i(V_s)$ is induced for all $i \geq 0$, $r \geq 2^n$. We may assume that $i \leq s$. Then $r - 2i \geq 2^n - 2s > 2^{n-1} - s$ and so $T^{r-2i}$ is induced. Thus $L'$ is a complex of induced $kG$-modules for each $r \geq 2^n$.

Consider the restriction of the complex $K$ to the subgroup $H$. It decomposes as a tensor product of two complexes, by Lemma 2.8. Each of these is separated, by induction and Theorem 4.1 (a), hence so is their product, by Lemma 2.2. It follows that for each $r \geq 0$, the complex $L'$ is separated on restriction to $H$. We have just seen that $L'$ is a complex of induced modules for all $r \geq 2^n$. Thus, for each $r \geq 2^n$, the complex $L'$ is separated, by Proposition 2.4. Now Lemma 2.5 shows that $H_0(L^{>2^n})$ is induced. But this is exactly $\tilde{T}^{>2^n}$, so $\tilde{S}(V_{2^n-1+s}) \cong k[\tilde{a}] \otimes \tilde{T} \cong_{\text{ind}} k[\tilde{a}] \otimes \tilde{T}^{<2^n} = k[\tilde{a}] \otimes \tilde{S}^{<2^n}$ is periodic if $s < 2^{n-1}$. 

Now suppose that $s = 2^{n-1}$. By the same argument as for $s < 2^{n-1}$, we see that $\tilde{T}^{\geq 2^n}$ is induced. To complete the proof of Theorem 4.1 (b) we have to show that $\tilde{S}^n \cong_{\text{ind}} k\tilde{a} \oplus kb$.

Set $y_i := g^{2^{n-1}}x_{2^n}$ for $i = 1, 2, \ldots, 2^n$, so $\{y_1, \ldots, y_{2^n}\}$ is a $k$-basis of $V_{2^n}$ which is permuted by $G$. A basis for $V_{2^n-1} < V_{2^n}$ is given by $g^{2^{n-1}}y_i - y_i = y_{i+2^n-1} - y_i$ for $i = 1, 2, \ldots, 2^{n-1}$.

Write $\tilde{y}_i$ for the image of $y_i$ in $\tilde{S}(V_{2^n})$, so $\tilde{y}_i^2 = \tilde{y}_{i+2^n-1}$ for $i = 1, \ldots, 2^{n-1}$. The set consisting of all monomials of degree $2^n$ in all the $\tilde{y}_i$ such that $\tilde{y}_1, \ldots, \tilde{y}_{2^n-1}$ only occur to the power at most 1 forms a $k$-basis for $\tilde{S}(V_{2^n})$. The group $G$ permutes these monomials and it is straightforward to check that there are two invariant monomials, namely $\tilde{y}_1 \tilde{y}_2 \cdots \tilde{y}_{2^n} = \tilde{b}$ and $\tilde{y}_{2^n-1+1} \tilde{y}_{2^n-1+2} \cdots \tilde{y}_{2^n} = \tilde{a}$; the rest span induced submodules. This completes the proof of periodicity.

7. Splitting

In this section we prove part (c) of Theorem 4.1, assuming the whole of the theorem for smaller $n$.

Let $H$ be the unique maximal subgroup of $G$ and $\{x_1, x_2, \ldots, x_{\top}\}$ a $k$-basis of $V_{2^n-1+s}$ as in Section 4. As in Theorem 4.1 we write $\tilde{N}(V_{2^n-1+s})$ for the kernel of the natural surjection $\tilde{S}(V_{2^n-1+s}) \to \Lambda(V_{2^n-1+s})$. The following proposition deals with the structure of $\tilde{S}(V_{2^n-1+s})$ in degrees less than $2^n$.

**Proposition 7.1.** For any integer $s$ such that $0 \leq s \leq 2^{n-1}$, the short exact sequence

$$0 \to \tilde{N}^{< 2^n}(V_{2^n-1+s}) \to \tilde{S}^{< 2^n}(V_{2^n-1+s}) \to \Lambda^{< 2^n}(V_{2^n-1+s}) \to 0$$

of graded $kG$-modules is split, and $\tilde{N}^{< 2^n}(V_{2^n-1+s})$ is induced from $H$.

Before starting with the proof of Proposition 7.1 we introduce some further notation. As described at the beginning of Section 3, we have $V_{2^n-1+s} \cong_{\text{ind}} V_{2^n-2+s'} \oplus V_{2^n-2+s''}$ where $0 \leq s', s'' \leq 2^{n-2}$ and $s' = s''$ or $s' = s'' + 1$. The $kH$-submodule $V_{2^n-2+s'}$ of $V_{2^n-1+s}$ has the $k$-basis $\{x_{\top}, x_{\top-2}, x_{\top-4}, \ldots\}$ and the $kH$-submodule $V_{2^n-2+s''}$ has the $k$-basis $\{x_{\top-1}, x_{\top-3}, x_{\top-5}, \ldots\}$. We write $\widetilde{S}(V_{2^n-2+s'})$ and $\widetilde{S}(V_{2^n-2+s''})$ for $S(V_{2^n-2+s'})/(V_2^2)$ and $S(V_{2^n-2+s''})/(V_2^2)$, respectively. So the $x_i$ with odd $i$ and the $x_i$ with even $i$ provide natural embeddings $\widetilde{S}(V_{2^n-2+s'}) \to \widetilde{S}(V_{2^n-1+s})$ and $\widetilde{S}(V_{2^n-2+s''}) \to \widetilde{S}(V_{2^n-1+s})$ of $kH$-modules, and we have $\tilde{S}(V_{2^n-1+s}) \cong \tilde{S}(V_{2^n-2+s'}) \otimes \tilde{S}(V_{2^n-2+s''})$ as $kH$-modules, where the isomorphism is given by $f_1 \otimes f_2 \mapsto f_1 \cdot f_2$.

Choose $a' \in S(V_{2^n-2+s'})$, $\tilde{a}' \in \widetilde{S}(V_{2^n-2+s'})$ according to the description preceding Theorem 4.1, but working over $H$. Thus $a'$ is homogeneous of degree $2^{n-1}$, has degree $2^{n-1}$ when considered as a polynomial in $x_{\top}$. Furthermore, $a'$ and $\tilde{a}'$ are invariant under the action of $H$ and $\tilde{a}'$ has image 0 in $\Lambda(V_{2^n-2+s'})$. Similarly, choose $a'' \in S(V_{2^n-2+s''})$ and $\tilde{a}'' \in \widetilde{S}(V_{2^n-2+s''})$. So $a''$ is homogeneous of degree $2^{n-1}$, has degree $2^{n-1}$ when considered as a polynomial in $x_{\top-1}$ and $a''$ and $\tilde{a}''$ are invariant under the action of $H$ and $\tilde{a}''$ has image 0 in $\Lambda(V_{2^n-2+s''})$.

By induction, $a'$ and $a''$ are periodicity generators of $\tilde{S}'(V_{2^n-2+s'})$ and $\tilde{S}''(V_{2^n-2+s''})$, respectively. That is, we have $\tilde{S}'(V_{2^n-2+s'}) \cong_{\text{ind}} k[a'] \otimes \tilde{S}^{< 2^{n-1}}(V_{2^n-2+s''})$ and $\tilde{S}''(V_{2^n-2+s''}) \cong_{\text{ind}} k[a''] \otimes \tilde{S}^{< 2^{n-1}}(V_{2^n-2+s''})$ or the variant with $\tilde{b}'$ or $\tilde{b}''$ if $s' = 2^{n-2}$ or $s'' = 2^{n-2}$.
Lemma 7.2. Let $s$ be an integer such that $0 \leq s \leq 2^{n-1}$ and let $\tilde{a}'$ be a periodicity generator for $\tilde{S}'(V_{2n-2+s})$ as above. Then
\[
\tilde{S}^{<2^n}(V_{2n-1+s})^{G}_{V_{2n-1+s}} = \tilde{S}^{<2^n-1}(V_{2n-1+s}) \oplus \tilde{a}' \tilde{S}^{<2^n-1}(V_{2n-1+s}) \oplus (g\tilde{a}') \tilde{S}^{<2^n-1}(V_{2n-1+s}) \oplus \tilde{X}
\]
as $kH$-modules, where the $kH$-submodule $\tilde{X}$ is generated as a $k$-vector space by the images of all monomials $x \in \bigoplus_{r=2^{n-1}}^{2^n-1} S^r(V_{2n-1+s})$ such that $x$ has degree strictly less than $2^{n-1}$ when considered as a polynomial in $x_{top}$ and $x$ has degree strictly less than $2^{n-1}$ when considered as a polynomial in $x_{top-1}$.

Proof. We give all monomials in $S(V_{2n-1+s})$ the lexicographic order with $x_{top-1} > x_{top} > x_{top-2} > \cdots > x_1$. Note that $\tilde{a}'$ has leading term $x_{top}^{2^n-1}$ and only involves monomials in $x_{top}$. Then $x_{top-1}, x_{top-2}, \ldots$, while $g\tilde{a}'$ has leading term $x_{top-1}^{2^n-1}$ and does involve $x_{top}$. Now standard methods apply.

We are now ready to prove Proposition 7.1.

Proof. (of Proposition 7.1) We study the restriction of the sequence (3) to the maximal subgroup $H$. By Lemma 7.2, the middle term is
\[
0 \rightarrow \tilde{J} \rightarrow \bigoplus_{r=2^{n-1}}^{2^n-1} \tilde{S}^r(V_{2n-1+s}) \rightarrow \tilde{X} \rightarrow 0,
\]
where $\tilde{X} := \bigoplus_{r=2^{n-1}}^{2^n-1} \tilde{S}^r(V_{2n-1+s})/\tilde{J}$. We know from Lemma 7.2 that the sequence (4) is split when restricted to $H$ and $\tilde{X}^{G}_{V_{2n-1+s}} \cong \tilde{X}$ as $kH$-modules. Since $\tilde{J}$ is induced from $H$ it is relatively $H$-injective, and so the sequence (4) splits over $kG$ (see [8, Theorem (19,2)]). Thus $\tilde{J}$ is a direct summand of $\tilde{S}^{<2^n}(V_{2n-1+s})$ over $kG$, so there is a $kG$-submodule $\tilde{J}'$ of $\tilde{S}^{<2^n}(V_{2n-1+s})$ such that $\tilde{S}^{<2^n}(V_{2n-1+s}) = \tilde{J} \oplus \tilde{J}'$. Since $\tilde{J} \subseteq \tilde{N}^{<2^n}(V_{2n-1+s})$, it follows that $\tilde{N}^{<2^n}(V_{2n-1+s}) = \tilde{J} \oplus \tilde{J}'$, where $\tilde{J}' := \tilde{J}' \cap \tilde{N}^{<2^n}(V_{2n-1+s})$. We have
\[
\tilde{J}^{G}_{V_{2n-1+s}} = \tilde{a}' \tilde{S}^{<2^n-1}(V_{2n-1+s})^{G}_{V_{2n-1+s}} \oplus (g\tilde{a}') \tilde{S}^{<2^n-1}(V_{2n-1+s})^{G}_{V_{2n-1+s}}.
\]
Because $\tilde{S}^{<2^n-1}(V_{2n-1+s})^{G}_{V_{2n-1+s}} \cong (\tilde{S}'(V_{2n-2+s'}) \otimes \tilde{S}''(V_{2n-2+s'}))^{<2^n-1}$, we have, by induction and ignoring the grading,
\[
\tilde{J}^{G}_{V_{2n-1+s}} \cong \text{ind} (\Lambda' \otimes \Lambda'')^{<2^n-1} + (\Lambda' \otimes \Lambda'')^{<2^n-1}.
\]
Here we write $\Lambda' \otimes \Lambda''$ for the graded $kH$-module $\Lambda(V_{2n-2+s'}) \otimes \Lambda(V_{2n-2+s'})$. Restricting the sequence (3) to $H$, we obtain the sequence
\[
0 \rightarrow \tilde{N}^{<2^n}(V_{2n-1+s})^{G}_{V_{2n-1+s}} \rightarrow \tilde{S}'(V_{2n-2+s'}) \otimes \tilde{S}''(V_{2n-2+s'})^{<2^n} \rightarrow (\Lambda' \otimes \Lambda'')^{<2^n} \rightarrow 0
\]
of $kH$-modules, which is split by induction. Thus, by induction again, we obtain
\[
\tilde{N}^{<2^n}(V_{2n-1+s})^{G}_{V_{2n-1+s}} \cong \text{ind} \tilde{a}' (\Lambda' \otimes \Lambda'')^{<2^n-1} \oplus \tilde{a}''(\Lambda' \otimes \Lambda'')^{<2^n-1}.
\]
Equations (5) and (6) imply that $\tilde{J}^G_H \oplus \tilde{J}'^G_H \cong (\tilde{J} \oplus \tilde{J}')^G_H \cong \tilde{N}^{<2^n}(V_{2^n-1+s})^G_H \cong_{\text{ind}} \tilde{J}^G_H$. It follows that $\tilde{J}^G_H$ is induced from proper subgroups of $H$. By Lemma 3.1 the $kG$-module $\tilde{J}'$ is induced from $H$, and hence $\tilde{N}^{<2^n}(V_{2^n-1+s}) = \tilde{J} \oplus \tilde{J}'$ is induced from $H$. We have just seen that sequence (3) is split on restriction to $H$; since $\tilde{N}^{<2^n}(V_{2^n-1+s})$ is relatively $H$-injective the sequence must split over $kG$. This completes the proof of Proposition 7.1. 

The following corollary provides a connection between $\tilde{S}(V_{2^n-1+s})$ and the exterior powers of $V_{2^n-1+s}$ in degrees less than 2n.

**Corollary 7.3.** For $r$ and $s$ integers such that $0 \leq s \leq 2^{n-1}$ and $0 \leq r < 2^n$, the map $f$ induces an isomorphism of $kG$-modules modulo induced summands

$$\tilde{S}^r(V_{2^n-1+s}) \cong_{\text{ind}} \Lambda^r(V_{2^n-1+s}).$$

**Proof.** This is clear from Proposition 7.1 (for $n > 1$) and Section 5 (for $n = 1$). 

We can now prove Theorem 4.1 (c). For $s < 2^{n-1}$ we have $\tilde{S}^{<2^n}(V_{2^n-1+s}) \cong \Lambda(V_{2^n-1+s}) \oplus X$, where $X$ is induced, so part (c) of Theorem 4.1 follows from part (b). For $s = 2^{n-1}$ we have $\tilde{S}^{<2^n}(V_{2^n-1+s}) \oplus k\tilde{b} \cong \Lambda(V_{2^n-1+s}) \oplus X'$, where $X'$ is induced. Note that $\tilde{b}$ maps to a generator of $\Lambda^2(V_{2^n-1+s})$. Again, part (c) of Theorem 4.1 is a consequence of (b).

8. Preparation for Separation

In this section we prepare for the proof of part (a) of Theorem 4.1, assuming the whole of the theorem for smaller $n$.

Let $H$ be the unique maximal subgroup of $G$ and let $\{x_1, x_2, \ldots, x_{\text{top}}\}$ be a $k$-basis of $V_{2^n-1+s}$ as in Section 4. The main goal of this section is to develop useful criteria for the complex $K(V_{2^n-1+s}, V^2)$ to be separated.

**Lemma 8.1.** Let $r$, $s$ be non-negative integers such that $0 \leq s \leq 2^{n-1}$. Suppose that for each $0 \leq r' < r$ with $r' \equiv r \mod 2$, the complex $K^{r'}(V_{2^n-1+s}, V^2)$ is separated at $K^{r'}_i(V_{2^n-1+s}, V^2) = S^{r'}(V_{2^n-1+s})$. Then $K^r(V_{2^n-1+s}, V^2)$ is separated at $K^r_i(V_{2^n-1+s}, V^2) = S^{r-2i}(V_{2^n-1+s}) \oplus \Lambda^i(V_s)$ for all $i \geq 1$. The same is true when $K$ and $S$ are replaced by $L$ and $T$ from Section 6.

**Proof.** We only demonstrate the proof for $K$ and $S$; the proof for $L$ and $T$ is analogous.

We write $V := V_{2^n-1+s}$ and $W := V_s$ for short. Fix $i \geq 1$ and consider the boundary morphism $d_{i+1} : S^{r-2i-2}(V) \oplus \Lambda^{i+1}(W) \to S^{r-2i}(V) \otimes \Lambda^i(W)$ in $K^r(V, W^2)$. We have to show that $\text{Im}(d_{i+1}) \to S^{r-2i}(V) \otimes \Lambda^i(W)$ factors through a projective $kG$-module. Since $K^{r-2i}(V, W^2)$ is separated at $K^{r-2i}_i(V, W^2)$ the inclusion $(W^2)^{r-2i} \to S^{r-2i}(V)$ factors through a projective $kG$-module $P^{r-2i}$. We can write the inclusion $\text{Im}(d_{i+1}) \to S^{r-2i}(V) \otimes \Lambda^i(W)$ as a composition of inclusions

$$\text{Im}(d_{i+1}) \to (W^2)^{r-2i} \otimes \Lambda^i(W) \to S^{r-2i}(V) \otimes \Lambda^i(W),$$

where the last map factors through the projective $kG$-module $P^{r-2i} \otimes \Lambda^i(W)$. 

**Lemma 8.2.** Let $s$ and $r$ be integers such that $0 \leq s \leq 2^{n-1}$ and $0 < r < 2^n$, and suppose that the complex $K^i(V_{2^n-1+s}, V^2_s)$ is separated for all $0 \leq i < r$. Then the following statements are equivalent:

(a) $K^r(V_{2^n-1+s}, V^2_s)$ is separated,

(b) The natural map $S^r(V_{2^n-1+s}) \to \Lambda^r(V_{2^n-1+s})$ is split modulo induced summands,
(c) $\Lambda^r(V_{2^n-1+s}) \cong_{\text{ind}} \bigoplus_{i \geq 1, j \geq 0} \Omega^{i+j}_{2^n}(\Lambda^i(V_s) \otimes \Lambda^j(V_{2^n-1-s})).$

**Proof.** We write $S^r := S^r(V_{2^n-1+s})$, $\tilde{S}^r := \tilde{S}^r(V_{2^n-1+s})$ and $K^i := K^i(V_{2^n-1+s}, V_s^2)$. The conditions on $K^i$ and Lemma 8.1 show that $K^r$ is separated except, perhaps, at $K_0 = S^r$. The restriction of the complex $K^r$ to $H$ decomposes as a tensor product of two complexes, by Lemma 2.8. Each of these is separated by our continuing induction hypothesis, hence so is their product, by Lemma 2.2, and so $K^r$ is separated on restriction to $H$. Thus the short exact sequence

\[ 0 \to \text{Im}(d_1) \xrightarrow{i} S^r \xrightarrow{j} \tilde{S}^r \to 0 \]  

from $K^r$ is separated at $S^r$ on restriction to $H$ (the maps $i$, $j$ should not be confused with the indices in part (c) of the lemma). The separation of $K^r$ in positive (complex-) degrees and Lemma 2.5 yield the formula $\text{Im}(d_1) \cong_{\text{proj}} \bigoplus_{i \geq 1, j \geq 0} \Omega^{i+j}_{2^n}(\Lambda^i(V_s)) \otimes S^j(V_{2^n-1+s})$. Theorem 1.2 now shows that

\[ \text{Im}(d_1) \cong_{\text{ind}} \bigoplus_{i \geq 1, j \geq 0} \Omega^{i+j-1}_{2^n}(\Lambda^i(V_s) \otimes \Lambda^j(V_{2^n-1-s})). \] 

(a) $\Rightarrow$ (b) Let $f : \tilde{S}^r \to \Lambda^r$ be the natural surjection, so $g = f \circ j$. By Proposition 7.1, the map $f$ is split modulo induced summands, and, by Lemma 3.9, it is enough to show that $j$ is split modulo induced summands. By assumption, $S^r = X \oplus M$ for some submodules $X$ and $M$ of $S^r$ such that $X$ is projective and $\text{ker}(j) = \text{Im}(d_1) \subseteq X$. Let $j' : S^r \to X$ be the projection onto $X$ and $u' : X \to S^r$ the natural embedding. Define $u : \tilde{S}^r = j(X) \oplus j(M) \to S^r, j(x) + j(m) \mapsto m$ (note that the restriction of $j$ to $M$ is injective). Then $u \circ j + u' \circ j' = \text{Id}_{\tilde{S}^r}$ and so $j$ is split modulo induced summands.

(b) $\Rightarrow$ (c) Assume (b). The factorization $g = f \circ j$ and Lemma 3.9 (b) imply that $j$ is also split modulo induced summands. Write $S^r = A' \oplus A''$, where $A'$ has only non-induced summands and $A''$ is induced. By Lemma 3.10, the restriction of $j$ to $A'$ is split, so $j$ maps $A'$ injectively into $\tilde{S}^r$ and $j(A')$ is a direct summand of $\tilde{S}^r$. Factoring out $A'$ and $j(A')$ in (7) we obtain the short exact sequence $0 \to \text{Im}(d_1) \xrightarrow{i} S^r/A' \xrightarrow{j} \tilde{S}^r/j(A') \to 0$.

As we have seen at the beginning of the proof, $i$ factors through a projective on restriction to $H$, and so the same is true for $i$. Thus the complex $\text{Im}(d_1) \to S^r/A'$ is separated on restriction to $H$. Because $S^r/A' \cong A''$ is induced, the complex is separated, by Proposition 2.4. Lemma 2.5 yields $\tilde{S}^r/j(A') \cong_{\text{proj}} S^r/A' \oplus \Omega_{2^n} \text{Im}(d_1) \cong_{\text{ind}} \Omega_{2^n} \text{Im}(d_1)$. Using (8) we obtain

\[ \tilde{S}^r/j(A') \cong_{\text{ind}} \bigoplus_{i \geq 1, j \geq 0} \Omega^{i+j}_{2^n} (\Lambda^i(V_s) \otimes \Lambda^j(V_{2^n-1-s})). \]

Theorem 1.2 implies $j(A') \cong A' \cong_{\text{ind}} S^r \cong_{\text{ind}} \Omega_{2^n} \Lambda^r(V_{2^n-1-s})$. Adding the summand $j(A')$ to both sides of (9) and using Corollary 7.3 gives us the formula in (c).

(c) $\Rightarrow$ (a) Assume that (c) holds. From Corollary 7.3, Theorem 1.2 and (8) we get $\tilde{S}^r \cong_{\text{ind}} S^r \oplus \Omega_{2^n} \text{Im}(d_1)$. Separation of $K^r$ now follows from applying Lemma 3.4 to the short exact sequence (7). \[ \square \]

Separation of $K^r(V_{2^n-1+s}, V_s^2)$ for $r = 0, 1$ is trivial. We will now prove it for $r = 2$. Notice that if a non-zero map $V_a \to V_b$ of $kG$-modules is to factor through a projective over $C_{2^n}$,
then we must have \( a + b > 2^n \). This is because the map must factor through the projective cover \( V_{2^n} \to V_b \), which has kernel \( V_{2^n-b} \), into which \( V_a \) will certainly be mapped if \( a \leq 2^n - b \).

**Lemma 8.3.** For any integer \( s \) such that \( 0 \leq s \leq 2^{n-1} \) the complex \( K^2(V_{2^n-1+s}, V^2_s) \) is separated.

**Proof.** The complex in question is \( V_s \to S^2(V_{2^n-1+s}) \). By induction, the map factors through a projective on restriction to \( H \). Write \( S^2(V_{2^n-1+s}) = A' \oplus A'' \), where \( A' \) has only non-induced summands and \( A'' \) has only induced summands. The component \( V_s \to A' \) factors through a projective, by Proposition 2.4.

We claim that the component \( V_s \to A' \) must be 0. From Theorem 1.2, we know that \( S^2(V_{2^n-1+s}) = \Lambda^2(V_{2^n-1-s}) \); but \( V_{2^n-1-s} \) is a module for \( C_{2^n-1} \), and it follows that \( A' \) contains only summands of dimension \( \leq 2^{n-1} \). Let \( V_t \) be such a summand, so \( t \leq 2^{n-1} \) and suppose that there is a non-zero component \( V_s \to V_t \). It must factor through a projective on restriction, where it is a map \( V_t \oplus V_{s'} \to V_t \oplus V_{s'} \), with \( s', s'', t', t'' \leq 2^{n-2} \). By the discussion above, none of the components can factor through a projective module over \( C_{2^n-1} \) unless they are 0.

We can readily prove separation when \( s \) is even.

**Lemma 8.4.** For any even integer \( s \) such that \( 0 \leq s \leq 2^{n-1} \), the complex \( K(V_{2^n-1+s}, V^2_s) \) of \( kG \)-modules is separated.

**Proof.** Write \( s = 2s' \). From Lemma 2.9 we know that \( K(V_{2^n-1+s}, V^2_s) \cong K(V_{2^n-2+s'}, V^2_{s'}) \uparrow \otimes^G \). The right hand side is separated by Lemma 2.3 and our induction hypothesis.

In view of this lemma, we assume now that \( s \) is odd.

**Lemma 8.5.** Let \( s \) be an odd integer such that \( 0 < s < 2^{n-1} \). Then (given our induction hypothesis):

(a) \( \Lambda^r(V_{2^n-1-s}) \in c(G) \) for all \( r \geq 0 \) and
(b) \( S^r(V_{2^n-1+s}) \in c(G) \) for all \( 0 \leq r < 2^{n-1} \).

Here \( c(G) \) is the subgroup of the Green ring in Lemma 3.11.

**Proof.** For part (a), the dimension \( d = 2^{n-1} - s \) of the module is in the range where we know that our formula for exterior powers (see Theorem 4.1 (d)) is valid by our continuing induction hypothesis. The statement is clearly true for \( d = 1 \) and we can employ induction on \( d \), using the formula and the properties of \( c(G) \) in Lemma 3.11.

For part (b) we use the formula \( \sigma_t(V_{2^n-1+s}) = \text{proj} \sigma_t(V_{2^n}) \Lambda^r_t(V_{2^n-1-s}) \) from the remark at the end of Section 4 and part (a). The summands of \( S(V_{2^n}) \) are permutation modules on a monomial basis, so are in \( c(G) \) unless the stabilizer of a monomial is of index 2. But this first happens in degree \( 2^{n-1} \), because if a monomial fixed by a subgroup of order \( 2^{n-1} \) contains \( y_i \), it must also contain all \( 2^{n-1} \) elements of the orbit of \( y_i \).

## 9. Separation

First we make some general constructions related to symmetric and exterior powers of vector spaces. It is convenient to do this integrally first and then reduce modulo 2. Let \( U \) be a free module over the integers localized at 2, \( \mathbb{Z}_{(2)} \). For \( r \geq 0 \) set

\[
T^r(U) = U \otimes_{\mathbb{Z}_{(2)}} \cdots \otimes_{\mathbb{Z}_{(2)}} U \quad (r \text{ times}).
\]
Let the symmetric group $\Sigma_r$ act on $T^r(U)$ by permuting the factors. Factoring out the action of $\Sigma_r$ we get $S^r(U) = T^r(U)/\Sigma_r = T^r(U) \otimes_{\mathbb{Z}(2)} \Sigma_r \mathbb{Z}(2)$. We can also let $\Sigma_r$ act on $T^r(U)$ by permuting the factors and multiplying by the signature of the permutation, in which case we write $T^r(U)_\sigma$. Similarly, on factoring out the action of $\Sigma_r$ we obtain $\Lambda^r(U) = T^r(U)_{\sigma}/\Sigma_r$.

For any subset $I \subseteq \{1, \ldots, r\}$ we set $\Sigma_I := \{\pi \in \Sigma_r \mid \pi(i) = i \text{ for all } i \not\in I\}$ (so $\Sigma_I$ is a subgroup of $\Sigma_r$ isomorphic to $S_{|I|}$). For $r \geq 2$, write $r = 2^p + t$ with $1 \leq t \leq 2^p$. Consider the subgroup

$$Q_r := \left\{ \Sigma_{\{1, \ldots, 2^p\}} \times \Sigma_{\{2^p+1, \ldots, r\}} \quad \text{if } t < 2^p, \right.$$\[\left. (\Sigma_{\{1, \ldots, 2^p\}} \times \Sigma_{\{2^p+1, \ldots, 2^p+t\}}) \times \langle \tau \rangle \quad \text{if } t = 2^p \right\}

of $\Sigma_r$, where $\tau \in \Sigma_r$ is the involution mapping $i \mapsto 2^p + i$ for $i = 1, 2, \ldots, 2^p$. The importance of $Q_r$ lies in the fact that the index $|\Sigma_r : Q_r|$ is odd. This can be seen as follows: $|\Sigma_{2^p+1} / (\Sigma_{2^p} \cdot \Sigma_I)| = \left(\begin{smallmatrix} 2^p+t \\ t \end{smallmatrix}\right)$, which is equal to the coefficient of $x^t$ in $(1 + x)^{2^p+t} = (1 + x^2)(1 + x)^t \pmod{2}$; also $(1 + x)^{2^p+1} = (1 + x^2 + 2X)^2 \equiv 1 + 2x^2 + x^{2^p+1} \pmod{4}$.

Define $L_S^r(U) := T^r(U)/Q_r$ and $L_L^r(U) := T^r(U)_\sigma/Q_r$. There are natural quotient maps $q_S : L_S^r(U) \to S^r(U)$ and $q_L : L_L^r(U) \to \Lambda^r(U)$, which have sections $\text{tr}_S : S^r(U) \to L_S^r(U)$ and $\text{tr}_L : \Lambda^r(U) \to L_L^r(U)$ given by $\text{tr}_x := \frac{1}{|\Sigma_r|/Q_r} \sum_{\pi \in \Sigma_r/Q_r} \pi x$. These have the property that $q_S \circ \text{tr}_S = \text{Id}_{S^r(U)}$ and $q_L \circ \text{tr}_L = \text{Id}_{\Lambda^r(U)}$. These maps are all natural transformations of functors on free $\mathbb{Z}(2)$-modules.

Writing $r = 2^p + t$ as before, we see from the description of $Q_r$ that

$$L_S^r(U) \cong \begin{cases} S^{2^p}(U) \otimes S^t(U) & \text{if } t < 2^p, \\ (S^{2^p}(U) \otimes S^{2^p}(U))/C_2 \cong S^2(S^{2^p}(U)) & \text{if } t = 2^p. \end{cases}$$

Similarly, if $r \geq 3$ we have

$$L_L^r(U) \cong \begin{cases} \Lambda^{2^p}(U) \otimes \Lambda^t(U) & \text{if } t < 2^p, \\ (\Lambda^{2^p}(U) \otimes \Lambda^{2^p}(U))/C_2 \cong S^2(\Lambda^{2^p}(U)) & \text{if } t = 2^p, \end{cases}$$

because the involution $\tau$ has signature 1, provided that $p \geq 1$.

Now let $V$ be an $F_2$-vector space and let $U$ be a free $\mathbb{Z}(2)$-module such that $V \cong F_2 \otimes_{\mathbb{Z}(2)} U$. Let $L'$ denote one of the functors $S^r$, $\Lambda^r$, $L_S^r$, $L_L^r$ above and use it to define a functor with the same name on $F_2$-vector spaces by $L'(V) = F_2 \otimes_{\mathbb{Z}(2)} L'(U)$. This gives the expected result for $S^r(V)$ and $\Lambda^r(V)$.

In order to verify that $L'$ is really a functor on vector spaces, notice that if $U$ and $U'$ are two free $\mathbb{Z}(2)$-modules then the natural map $\text{Hom}_{\mathbb{Z}(2)}(U, U') \to \text{Hom}_{F_2}(F_2 \otimes_{\mathbb{Z}(2)} U, F_2 \otimes_{\mathbb{Z}(2)} U')$ is surjective, so all maps of vector spaces lift. Furthermore, a map in the kernel has image in $2U'$, so factors through multiplication by 2 on $U'$. But multiplication by 2 on $U'$ induces multiplication by $2^2$ on $T^r(U')_\sigma$, thus it induces 0 on $F_2 \otimes_{\mathbb{Z}(2)} L^r(U')$.

It follows that the formulas above are also valid for $F_2$-vector spaces. A difference is that we now have natural transformations $e^* : S^r \to \Lambda^r$ and $\Lambda^r : L_S^r \to L_L^r$ induced by reducing modulo squares.

The above functors induce functors on modules for a group in the obvious way.

**Remark.** Any representation of $G$ over a field of characteristic 2 can be written in $F_2$, so this is sufficient for our purposes. If we really needed functors on vector spaces over a bigger field, this could be achieved by starting with a larger ring than $\mathbb{Z}(2)$.
In the rest of this section we prove part (a) of Theorem 4.1, assuming the whole of the theorem for smaller \( n \). We use the same notation as before.

**Lemma 9.1.** Suppose that \( V \) is a \( kG \)-module, \( r \geq 2 \) and \( L^r_e : L^r_S(V) \to L^r_A(V) \) is split modulo induced summands. Then \( e^r : S^r(V) \to \Lambda^r(V) \) is split modulo induced summands.

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
S^r(V) & \xrightarrow{e^r} & \Lambda^r(V) \\
\downarrow \text{tr}_S & & \downarrow \text{tr}_A \\
L^r_S(V) & \xrightarrow{L^r_e} & L^r_A(V).
\end{array}
\]

The map \( \text{tr}_S \) is split, so if \( L^r_e \) is split modulo induced summands then so is \( L^r_e \circ \text{tr}_S \), by Lemma 3.9 (a). But this is equal to \( \text{tr}_A \circ e^r \) and Lemma 3.9 (b) shows that \( e^r \) is split modulo induced summands. \( \square \)

**Lemma 9.2.** If \( s \) is an odd integer such that \( 0 < s < 2^{n-1} \) and \( r \) is an integer such that \( 0 \leq r < 2^n \), then \( e^r : S^r(V_{2^{n-1}+s}) \to \Lambda^r(V_{2^{n-1}+s}) \) is split modulo induced summands.

**Proof.** We use induction on \( r \). The cases \( r = 0, 1 \) are trivial and \( r = 2 \) is covered by Lemma 8.3 combined with Lemma 8.2. Let \( r \geq 3 \) and write \( r = 2t + s \) with \( 1 \leq t \leq 2^n \).

Abbreviate \( V_{2^{n-1}+s} \) to \( V \). By Lemma 9.1, it is sufficient to check that \( L^r_e : L^r_S(V) \to L^r_A(V) \) is split modulo induced summands.

If \( t < 2^s \) then \( L^r_e = e^{2s} \otimes e^t : S^{2s}(V) \otimes S^t(V) \to \Lambda^{2s}(V) \otimes \Lambda^t(V) \). This is split modulo induced summands by induction and Lemma 3.9 (c).

If \( t = 2^s \) then \( L^r_e = e^{2s} : S^t(S^{2s}(V)) \to S^t(\Lambda^{2s}(V)) \). By induction, \( e^{2s} \) is split modulo induced summands, so it extends to a split map \( M : S^{2s}(V) \to \Lambda^{2s}(V) \oplus X \) with left inverse \( N \), where \( X \) is induced. By the remark after Lemma 3.10, we may assume that \( X \) only contains summands that are also summands of \( S^{2s}(V) \); by Lemma 8.5 and the assumption that \( r < 2^n \) (and hence \( 2^{s} < 2^{n-1} \), since \( t = 2^s \)), these are of dimension divisible by 4. Applying \( S^2 \), we see that \( S^2(e^{2s}) \) extends to

\[
S^2(M) : S^2(S^{2s}(V)) \to S^2(\Lambda^{2s}(V)) \oplus S^2(X) \oplus (\Lambda^{2s}(V) \otimes X),
\]

with left inverse \( S^2(N) \). Certainly \( \Lambda^{2s}(V) \otimes X \) is induced, and \( S^2(X) \) is induced, by Corollary 3.7. Thus \( L^r_e \) and thus \( e^r \) are split modulo induced summands. \( \square \)

Again, let \( s \) be an odd integer such that \( 0 < s < 2^{n-1} \). It follows from Lemmas 8.1, 8.2 and 9.2 that the complex \( K^r(V_{2^{n-1}+s}, V_s^2) \) is separated for all \( 0 \leq r < 2^n \).

Recall that \( K(V_{2^{n-1}+s}, V_s^2) \) is separated if and only if the complex \( L(V_{2^{n-1}+s}, V_s^2) \) from Section 6 is separated. For the rest of this section we will write just \( K, L \) etc. Now \( L^r \) is separated for all \( 0 \leq r < 2^n \), because it coincides with \( K^r \) in this range. We will show that \( L^r \) is separated for \( r \geq 2^n \) by induction on \( r \), so let \( r \geq 2^n \) and assume that the complex is separated in all lower degrees.

By Lemma 8.1, we can also assume that \( L^r \) is separated in positive (complex-)degrees, so it is enough to prove that the short exact sequence

\[
0 \to \text{Im}(d_1^r) \to T^r \to \tilde{T}^r \to 0
\]

is separated at \( T^r \). By Lemma 2.8, the restriction of \( K \) to the maximal subgroup \( H \) of \( G \) decomposes as a tensor product of two complexes, and each of these is separated, by our
continuing induction hypothesis and Theorem 4.1(a). Their product is also separated, by Lemma 2.2, hence so is $L'$. It follows that the sequence (10) is separated at $T'$ restriction to $H$.

But $T'$ is induced for this range of $r$. Separation of (10) follows immediately from Proposition 2.4 applied to $\text{Im}(d'_1) \rightarrow T'$.

This proves that the complex $K'(V_{2n-1+s}, V_2^2)$ is separated for all $r \geq 0$, and part (a) of Theorem 4.1 follows.

10. Exterior Powers

In this section we prove part (d) of Theorem 4.1, assuming the whole of the theorem for smaller $n$.

Because we have already proved separation, periodicity and splitting we know that

$$\lambda_1(V_{2n-1+s}) =_{\text{ind}} \lambda^\wedge_1(V_s) \lambda^\wedge_1(V_{2n-1-s});$$

see the first remark at the end of Section 4. In order to obtain the formula with $=_{\text{proj}}$, we first consider the restriction to the subgroup $H$ of index $2$. Writing $V_{2n-1+s} \downarrow_H = V_{2n-2+s'} \oplus V_{2n-2+s''}$, the two sides of the formula become

$$\lambda_t(V_{2n-2+s'}) \lambda_t(V_{2n-2+s''}) \quad \text{and} \quad \lambda^\Omega_2(V_{s'}) \lambda^\Omega_2(V_{s''}) \lambda^\wedge_1(V_{2n-2-s'}) \lambda^\wedge_1(V_{2n-2-s''}).$$

But we know, by induction, that $\lambda_t(V_{2n-2+s'}) =_{\text{proj}} \lambda^\Omega_2(V_{s'}) \lambda^\wedge_1(V_{2n-2-s'})$ and similarly for $s''$. Thus, on restriction, the two sides are equal modulo projectives even before restriction. This finally completes the proof Theorem 4.1.

11. A Bound on the Number of Non-Induced Summands

The description of the tensor product given in Section 3 shows that the decomposition of $V_r \otimes V_s$ into indecomposable summands involves a summand of odd dimension if and only both $r$ and $s$ are odd, in which case it contains precisely one odd-dimensional summand.

Let us write $\text{sum}(V)$ for the number of indecomposable summands in the module $V$.

**Proposition 11.1.** The number of non-induced summands in $\Lambda(V)$ is at most

$$2^{\frac{1}{2} \text{dim}(V)+\text{sum}(V))} = \sqrt{2^{\text{sum}(V)} \text{dim}(\Lambda(V))}.$$ 

**Proof.** Let $f(V)$ denote the number of non-induced summands in $\Lambda(V)$. The comment about the tensor product above shows that $f(V \oplus W) = f(V) f(W)$. The proposed bound also turns sums into products, so it suffices to consider the case when $V = V_r$ is indecomposable and show that $f(V_r) \leq 2^{\frac{1}{2}(r+1)}$.

We use induction on $r$. Since the cases $r = 0, 1$ are trivial we can assume that $r \geq 2$, and we can write $r = 2^{n-1} + s$, where $1 \leq s \leq 2^{n-1}$. Setting $t = 1$ in the formula $\lambda_t(V_{2n-1+s}) =_{\text{proj}} \lambda^\Omega_2(V_s) \lambda^\wedge_1(V_{2n-1-s})$ and using induction we obtain

$$f(\Lambda(V_{2n-1+s})) = f(\lambda_1(V_{2n-1+s})) \leq 2^{\frac{1}{2}(2^{n-1-s}+1)} 2^{\frac{1}{2}(s+1)} = 2^{\frac{1}{2}(2^{n-1}+2)} \leq 2^{\frac{1}{2}(2^{n-1}+s+1)}.$$ 

\[\square\]
For an indecomposable $kG$-module $V_r$, if we assume that the group acts faithfully then $r > \frac{1}{2}|G|$, and the dimension of any direct summand of $\Lambda(V_r)$ is at most $|G|$. It follows that the dimension of the non-induced part of $\Lambda(V_r)$ divided by the dimension of the whole of $\Lambda(V_r)$ is at most $2^{1/(3-r)}r$.

12. Remarks

(a) As already mentioned in the introduction, the formula in Theorem 1.1 reduces the computation of $\Lambda^r(V_{2n-1+q})$ to the computation of tensor products of exterior powers of modules of smaller dimension. Since tensor products can easily be determined recursively (see Section 3), this gives an efficient recursive method for calculating the decomposition of exterior powers of modules for cyclic 2-groups into indecomposables. A program based on this recurrence relation was implemented in GAP [10] by the first author.

A restriction on the use of Theorem 1.1 is the growth of the multiplicities of direct summands of the form $V_{2n}$. For example, the multiplicity of $V_{128}$ as a direct summand of $\Lambda^8(V_{147})$ is $1.097519886357582458576803532720$. If one is only interested in the non-induced part of $\Lambda^r(V_{2n-1+q})$ the recurrence relation can be applied modulo induced summands to keep the multiplicities relatively small.

Together with the results in [21], the recurrence relation in Theorem 1.1 also provides an algorithm for computing the decomposition of the symmetric powers $S^r(V_m)$ into indecomposables for arbitrary $m$ and $r$.

**Example.** We determine the decomposition of $\Lambda^6(V_{13})$ into indecomposables:

$$
\Lambda^6(V_{13}) \cong_{\text{proj}} \Omega^6_{16}(\Lambda^6(V_5) \otimes \Lambda^6(V_3)) + \Omega^4_{16}(\Lambda^4(V_5) \otimes \Lambda^4(V_3)) + \\
\cong_{\text{proj}} \Omega^2_{16}(\Lambda^2(V_5) \otimes \Lambda^2(V_3)) + \Omega^0_{16}(\Lambda^0(V_5) \otimes \Lambda^0(V_3)).
$$

Furthermore $\Lambda^3(V_3) \cong \Lambda^2(V_3)$, by duality, and

$$
\Lambda^2(V_5) \cong \Omega^2_{16}(\Lambda^2(V_1) \otimes \Lambda^2(V_3)) + \Omega^0_{16}(\Lambda^0(V_1) \otimes \Lambda^0(V_3)) \cong V_3 \oplus \Omega_8(V_1) \cong V_3 \oplus V_7.
$$

Thus

$$
\Lambda^6(V_{13}) \cong_{\text{proj}} (V_3 \oplus V_7) \oplus \Omega_8(V_3) \oplus \Omega_{16}(V_3 \oplus V_7) \cong_{\text{proj}} (V_3 \oplus V_3) \oplus V_3 \oplus V_7 \oplus V_7
$$

$$
\cong_{\text{proj}} (V_1 \oplus 2V_4) \oplus (V_5 \oplus 2V_8) \oplus (V_{13} \oplus V_9) \cong (V_1 \oplus 2V_4 \oplus V_5 \oplus 2V_8 \oplus V_9 \oplus V_{13} \oplus 104V_{16}).
$$

Comparing dimensions, we obtain $\Lambda^6(V_{13}) \cong V_1 \oplus 2V_4 \oplus V_5 \oplus 2V_8 \oplus V_9 \oplus V_{13} \oplus 104V_{16}$.

(b) Obviously, Gow and Laffey’s formula for exterior squares [11, Theorem 2] is the special case $r = 2$ of Theorem 1.1. Furthermore, setting $s = 2^{n-1} - 1$ in Theorem 1.1 gives Kouwenhoven’s formula [15, Theorem 3.4] for $\Lambda^r(V_{q-1})$ when $q$ is a power of $2$ (for all $r$).

In [15, Theorem 3.5] Kouwenhoven proved the formula

$$
\lambda_t(V_{q+1} - V_{q-1}) = 1 + (V_{q+1} - V_{q-1})t + t^2
$$

in $a(C_q)[[t]]$, where $q$ is a power of a prime $p$. We will show how this can be derived from Theorem 1.1 in the case that $p = 2$. Note that, since the dimension series of the two sides match, it is sufficient to prove this modulo projectives. The theorem gives us:

$$
\lambda_t(V_{2n-1+1}) = (1 + V_{2n-1}t^2)\lambda_t^{\Omega_{2n}}(V_{2n-1+1})
$$
modulo $V_{2^n}$ and
\[ \lambda_{i}(V_{2^{n-1}-1}) = (1 + V_{2^{n-1}-1}t)\lambda_{i}^{\Omega_{2}^{n-1}}(V_{2^n-1}) \]
modulo $V_{2^n-1}$. The latter can be written as
\[ \lambda_{i}(V_{2^{n-1}-1}) = (1 + V_{2^{n-1}-1}t)(\lambda_{i}^{\Omega_{2}^{n-1}}(V_{2^n-1}) + V_{2^n-1}f(t)) \]
extactly (the last term can be written inside the parentheses, since $(1 + V_{2^{n-1}-1}t)$ is invertible). Applying $\Omega_{2}$, in odd degrees we obtain
\[ \lambda_{i}^{\Omega_{2}^{n}}(V_{2^{n-1}-1}) = (1 + V_{2^{n-1}+1}t)(\lambda_{i}^{\Omega_{2}^{n-1}}(V_{2^n-1}) + V_{2^n-1}f(t)). \]
Substituting into the left hand side of (11) yields
\[ (1 + V_{2^n-1}t^2)(1 + V_{2^n-1}+1t)(1 + V_{2^n-1-1}t)^{-1} \]
modulo $V_{2^n}$, and it is easy to verify that
\[ (1 + V_{2^n-1}t^2)(1 + V_{2^n-1}+1t) = (1 + V_{2^n-1-1}t)(1 + (V_{2^n-1} - V_{2^n-1-1})t + t^2) \]
modulo $V_{2^n}$.

(c) Theorem 1.1 can also be used to calculate the Adams operations on the Green ring $a(C_{2^n})$, as was shown to us by Roger Bryant and Marianne Johnson. For each $r > 0$ and $j \in \{1, \ldots, 2^n\}$, define an element $\psi^r_{\Lambda}(V_j) \in a(C_{2^n})$ by
\[ \psi^1_{\Lambda}(V_j) - \psi^2_{\Lambda}(V_j)t^2 - \cdots - \psi^r_{\Lambda}(V_j)t^r = \frac{d}{dt} \log \lambda_{i}(V_j). \]
By $\mathbb{Z}$-linear extension there is a map $\psi^r_{\Lambda} : a(C_{2^n}) \rightarrow a(C_{2^n})$, called the $r$th Adams operation defined by the exterior powers. It can be shown that if $r$ is odd, then $\psi^r_{\Lambda}$ is the identity map on $a(C_{2^n})$ and that $\psi^i_{\Lambda} = \psi^j_{\Lambda}$ for all $i \geq 1$, so all that remains is to describe $\psi^i_{\Lambda}$ for $i \geq 1$ (see [5] and [6] for details). For $j \geq 2$, write $j = 2^m + s$ with $m \geq 0$ and $1 \leq s \leq 2^m$; then
\[ \psi^i_{\Lambda}(V_{2^m+s}) = 2\psi^{2i-1}_{\Lambda}(V_s) + \psi^i_{\Lambda}(V_{2^m-s}) \]
for all $i \geq 2$ and
\[ \psi^2_{\Lambda}(V_{2^m+s}) = 2V_{2^m+s} - 2V_{2^m-1-s} + \psi^2_{\Lambda}(V_{2^m-s}). \]
This can be seen by applying the definition of the Adams operations to the Hilbert series form of Theorem 1.1, obtaining (in the obvious notation)
\[ \psi_{\Lambda,t}(V_{2^m+s}) \equiv_{\text{proj}} 2t\psi^{O}_{\Lambda,t}(V_s) + \psi^{O}_{\Lambda,t}(V_{2^m-s}). \]

REFERENCES


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