# ADVANCED MATHEMATICAL METHODS

J. S. B. Gajjar
j.gajjar@manchester.ac.uk
http://www.maths.manchester.ac.uk/∼gajjar

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# Contents

# Chapter 1

# Introduction

## 1.1 Scope of course

This course aims to introduce and discuss a number of commonly used mathematical methods techniques that graduate students will find useful in their research. In the subsequent lectures we will aim to cover the following topics:

- Advanced differential equations, series solution,classification of singularities. Properties near ordinary and regular singular points. Approximate behaviour near irregular singular points. Method of dominant balance. Airy, Gamma and Bessel functions.

- Asymptotic methods. Boundary layer theory. Regular and singular perturbation problems. Uniform approximations. Interior layes. LG approximation, WKBJ method.

- Generalised functions. Basic definitions and properties.

- Revision of basic complex analysis. Laurent expansions. Singularities. Cauchy's Theorem. Residue calculus. Plemelj formuale.

- Transform methods. Fourier transform. FT of generalised functions. Laplace Transform. Properties of Gamma function. Mellin Transform. Analytic continuation of Mellin transforms.

- Asymptotic expansion of integrals. Laplace's method. Watson's Lemma. Method of stationary phase. Method of steepest descent. Estimation using Mellin transform technique.

- Conformal mapping. Riemann-Hilbert problems.

Many of the above topics could easily be studied in detail over many lectures, but our motivation is to give a flavour of the particular topic rather than give an exhaustive treatment of the subject. There will be sufficient detail for the interested reader to follow up and investigate further if required. The course therefore proceeds at a fairly rapid pace and students are strongly advised to study the techniques, work through the examples covered and also attempt the set problems.

**Recommended Texts** It will be assumed that students have done basic courses on real and complex analysis. The following texts cover various topics discussed in the course, although no single book covers all the topics that we will be discussing.

1. C. M. Bender & S.A. Orszag, 'Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill.

2. N. Bleistein & R.A. Handelsman, 'Asymptotic Expansions of Integrals.'

3. F. W. J. Olver, 'Introduction to Asymptotics and Special Functions', Dover.

4. M. J. Ablowitz & A. S. Fokas 'Complex variables, introduction and applications', C.U.P.

5. M. J. Lighthill 'Introduction to Fourier analysis and generalised functions.', Dover.

The book by Bender & Orszag is particularly recommended and is one of my favourites. It contains a wonderful selection of worked examples and accompanying commentary. It is easy to read and well worth purchasing.

## 1.2 Important definitions and preliminaries

In this section we will introduce some of the definitions and notation which will be used extensively in later parts of the course.

### 1.2.1 Ordering symbols, 'O and 'o' notation

**Ordering symbols 'O' and 'o'**

**Definition of 'O':** *Let $\phi(x), \psi(x)$ be real or complex valued functions. Let $x_0$ be a limit point of a set $R$ not necessarily belonging to $R$. We write*

$$\psi = O(\phi) \quad in \quad R$$

*if $\exists$ a constant $A$ (independent of $x$) so that*

$$|\psi| \leq A|\phi| \quad \forall x \in R.$$

*Also* $\psi = O(\phi)$ *as* $x \to x_0$ *in some neighbourhood* $\Delta$*, if* $\exists A$ *such that*

$$|\psi| \leq A|\phi| \quad \forall x \in \Delta \cap R.$$

*If* $\phi \neq 0$ *in* $R$ *then* $\psi = O(\phi)$ *as* $x \to x_0$ *if* $\frac{\psi}{\phi}$ *is bounded in* $R$ *as* $x \to x_0$.

**Examples**

$$\sin x = O(x) \quad \text{as} \quad x \to 0.$$

$$\cos x = O(1) \quad \text{as} \quad x \to 0.$$

**Definition of 'o':** *We write* $\psi = o(\phi)$ *as* $x \to x_0$ *if for any given* $\epsilon > 0 \quad \exists$ *neighbourhood* $\Delta_\epsilon$ *of* $x_0$ *such that*

$$|\psi| \leq A\epsilon|\phi| \quad \forall x \in \Delta_\epsilon \cap R.$$

*Note that if* $\phi \neq 0$ *in* $R$ *then* $\psi = o(\phi)$ *as* $x \to x_0$ *if* $\frac{\psi}{\phi} \to 0$ *as* $x \to x_0$.

*Sometimes* $<<$ *used in place of o notation.*

If the functions involved depend on parameters, in general the constants $A$, and neighbourhoods $\Delta, \Delta_\epsilon$ will depend on the parameters. If however, $A, \Delta, \Delta_\epsilon$ are independent of the parameters, the order relation is said to hold uniformly in the parameters.

**Examples**

$$\sin x = o(1) \quad \text{as} \quad x \to 0.$$

**Examples**

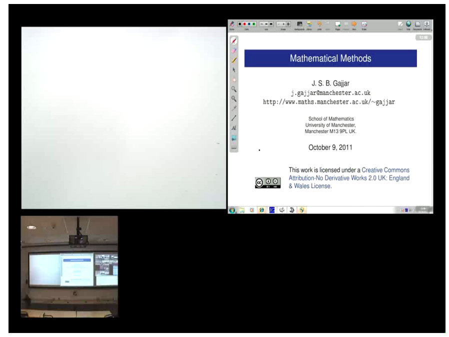
$$\sin(x + \epsilon) = O(1) \quad \text{uniformly as} \quad x \to 0.$$

$$\sqrt{x + \epsilon} - \sqrt{x} = O(\epsilon) \quad \text{nonuniformly as} \quad \epsilon \to 0.$$

$$\sin(x + \epsilon) = o(\epsilon^{-\frac{1}{2}}) \quad \text{uniformly as} \quad \epsilon \to 0.$$

### 1.2.2 Asymptotic sequences

Asymptotic sequences are extremely useful and will be used throughout this course.




Video clip of section on the $O$ and $o$ notation.Click here to open video clip in external player.

**Definition** *The sequence of functions $\{\phi_n\}$ is called an* **asymptotic sequence** *for $x \to x_0$ in $R$ if for each $n$, $\phi_n$ is defined in $R$ and*

$$\phi_{n+1} = o(\phi_n) \quad as \quad x \to x_0 \quad in \quad R.$$

*If the sequence is infinite and $\phi_{n+1} = O(\phi_n)$ uniformly in $n$, then the $\{\phi_n\}$ is said to be an asymptotic sequence uniformly in $n$. If the $\phi_n$ depend on parameters, and $\phi_{n+1} = o(\phi_n)$ in the parameters, the $\{\phi_n\}$ is an asymptotic sequence uniformly in the parameters.*

**Example** The following define asymptotic sequences

$$\{(x - x_0)^n\} \quad x \to x_0 \quad x \in C.$$

$$\{x^{-n}\}, \quad \text{as} \quad x \to \infty.$$

$$\{x^{-\lambda_n}\}, \quad \text{as} \quad x \to \infty,$$

where $\Re(\lambda_n) < \Re(\lambda_{n+1})$ for each $n$.

**Definition** *We say*

$$f(x) \sim g(x) \quad as \quad x \to x_0$$

*if*

$$\frac{f(x)}{g(x)} \to 1 \quad as \quad x \to x_0.$$

Observe that this implies

$$f(x) = (1 + o(1))g(x) \quad \text{as} \quad x \to x_0.$$

### 1.2.3 Asymptotic expansion

**Definition** *Let $\{\phi_n\}$ be an asymptotic sequence. The series*

$$\sum a_n \phi_n(x)$$

*is said to be an* **asymptotic expansion** *to $N$ terms of $f(x)$ as $x \to x_0$ if*

$$f(x) - \sum_{n=1}^{N} a_n \phi_n(x) = O(\phi_{N+1}) \quad as \quad x \to x_0.$$

*Sometimes this is written as*

$$f(x) \sim \sum a_n \phi_n(x) \quad to \quad N \quad terms\ as \quad x \to x_0 \quad in \quad R.$$

*If $N = \infty$ then*

$$f(x) \sim \sum a_n \phi_n(x)$$

*is called an asymptotic expansion.* An asymptotic expansion involving certain parameters is said to hold uniformly in the parameters if

$$f - \sum_{n=1}^{N} a_n \phi_n(x) = O(\phi_{N+1})$$

uniformly in the parameters for each sufficiently large $N$, (not necessarily uniformly in $N$). If $\phi_n = x^{-\lambda_n}$ where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \ldots$ then

$$\frac{\phi_{n+1}}{\phi_n} = \frac{x^{\lambda_n}}{x^{\lambda_{n+1}}} = \frac{1}{x^{\lambda_{n+1}-\lambda_n}} \to 0 \quad \text{as} \quad x \to \infty.$$

The above definitions stem from Poincaré's studies, Poincaré (1886), where he introduced asymptotic power series as a means for making divergent series more useful. In Poincare's definition the point $x_0$ is infinity and the asymptotic sequence is $z^{-n}$ where $z \to \infty$ in some section in the complex plane.

**Poincaré power series expansions** *A series $\sum_{n=0}^{\infty} a_n z^{-n}$ is called an asymptotic expansion of $f(z)$ in some sector $S$, $\alpha \leq \arg(z) \leq \beta$ if for each $N \geq 0$*

$$f(z) = \sum_{n=0}^{N} a_n z^{-n} + O(z^{-(N+1)}), \quad z \to \infty.$$

**Examples** Consider

$$\sqrt{x+\epsilon} = \sqrt{x}(1 + \frac{\epsilon}{x})^{\frac{1}{2}}.$$

This suggests

$$\sqrt{x+\epsilon} \sim \sqrt{x}\left[1 + \frac{\epsilon}{2x} - \frac{\epsilon^2}{8x^2} + \ldots\right].$$

Define

$$\phi_n(x,\epsilon) = \frac{\frac{1}{2}(\frac{1}{2}-1)\ldots(\frac{1}{2}-n+1)}{n!}\frac{\epsilon^n}{x^n},$$

with $x > 0$ and fixed $n$. Now

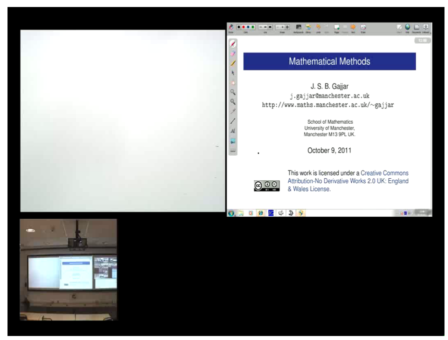
$$\frac{\phi_{n+1}}{\phi_n} = \frac{(\frac{1}{2} - n)}{(n+1)} \frac{\epsilon}{x} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Thus $\sum \phi_n$ is an asymptotic expansion.

Note that that the series

$$\sum_{n=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\ldots(\frac{1}{2}-n+1)}{n!} \frac{\epsilon^n}{x^n}$$

converges only for $|\epsilon| < |x|$. **Thus the series in an asymptotic expansion does not necessarily converge.**




Clip covering asymptotic expansions. Click here to open video clip in external player.

How do we know that $\sqrt{x+\epsilon} \sim x^{\frac{1}{2}} \sum \phi_n(x, \epsilon)$ above?

**Theorem** *The asymptotic expansion to a given number of terms of a given function is unique if the asymptotic sequence is given.*

**Proof**

If $f(x) \sim \sum a_n \phi_n(x)$ then

$$f(x) = \sum_{k=1}^{n} a_k \phi_k + R_n(x)$$

where $R_n(x) = o(\phi_n)$.

Hence

$$f(x) = \sum_{k=1}^{n-1} a_k \phi_k + a_n \phi_n + R_n(x)$$

Therefore

$$\left| \frac{f(x) - \sum_{k=1}^{n-1} a_k \phi_k}{\phi_n} - a_n \right| = \left| \frac{R_n}{\phi_n} \right| \to 0 \quad \text{as} \quad x \to x_0.$$

Hence $a_n$ is given uniquely by

$$a_n = \lim_{x\to x_0}\left(\frac{f(x) - \sum_{k=1}^{n-1} a_k\phi_k(x)}{\phi_n(x)}\right) \tag{1.2.1}$$

**Conversely,** *suppose we have $N+1$ functions $f(x), \phi_1(x), \ldots \phi_N(x)$ defined in $R$. Then if (1.2.1) holds and $a_m \neq 0$ for $m = 1, 2, \ldots, N$ then $\{\phi_n\}$ is an asymptotic sequence for $x \to x_0$ and $\sum a_n\phi_n$ is an asymptotic expansion to $N$ terms of $f(x)$ as $x \to x_0$.*

**Proof:** We have to show that $\phi_{n+1} = o(\phi_n)$ for $n = 1, 2, \ldots, N-1$. Now from (1.2.1)

$$f - \sum_{k=1}^{m} a_k\phi_k = o(\phi_m).$$

Replace $m$ by $m+1$ and we have

$$f - \sum_{k=1}^{m} a_k\phi_k = a_{m+1}\phi_{m+1} + o(\phi_{m+1}).$$

$$= a_{m+1}\phi_{m+1} + o(1)\phi_{m+1},$$

$$= (a_{m+1} + o(1))\phi_{m+1}.$$

Hence

$$(a_{m+1} + o(1))\phi_{m+1} = o(\phi_m).$$

Thus if $a_{m+1} \neq 0$ then $q_{m+1} + o(1) \neq 0$ for some $x$ in the neighbourhood of $x_0$ and dividing gives the result

$$\phi_{m+1} = o(\phi_m).$$

The same function may have different asymptotic expansions involving two different asymptotic sequences, or two different functions may have the same asymptotic expansion.

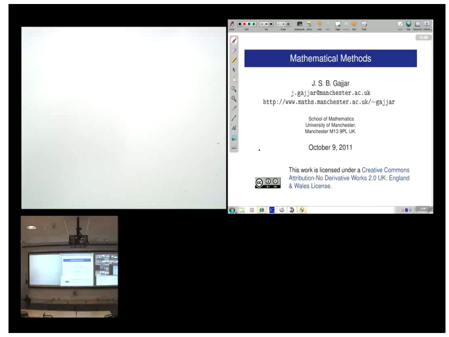
**Examples**

$$\frac{1}{x+1} = \frac{1}{x(1+\frac{1}{x})} \sim \sum_1^\infty \frac{(-1)^{n+1}}{x^n} \quad \text{as} \quad x \to \infty.$$

$$\frac{1}{x+1} = \frac{x-1}{x^2-1} \sim \sum_1^\infty \frac{(x-1)}{x^{2n}} \quad \text{as} \quad x \to \infty.$$

Also

$$\frac{1}{x+1} + e^{-x^2} \sim \sum_1^\infty \frac{(-1)^{n+1}}{x^n}.$$




Video clip covering above examples.Click here to open video clip in external player.

$\phi, \psi$ are said to be asymptotically equivalent as $x \to x_0$ if

$$f(x) = g(x)(1 + O(1)).$$

The usefulness of an asymptotic expansion arises from the fact that only a few terms of the series are required to give a good approximation to the function, whereas with a Taylor series expansion many terms are required for equivalent accuracy.

Note that from the definition of an asymptotic expansion, the remainder after $N$ terms is much smaller than the last term retained as $x \to x_0$.

**Example** Consider

$$\mathrm{Ei}(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt.$$

Put $t = x + z$ and then

$$\mathrm{Ei}(x) = \frac{e^{-x}}{x} \int_0^\infty \frac{e^{-z}}{1 + \frac{z}{x}}\, dz,$$

$$= \frac{e^{-x}}{x} \int_0^\infty e^{-z}\, dz \left[ 1 - \frac{z}{x} + \frac{z^2}{x^2} - + \cdots + \frac{(-1)^{n-1} z^{n-1}}{x^{n-1}} + \frac{(-1)^n z^n}{x^n (1 + \frac{z}{x})} \right].$$

Integrating term by term gives

$$\mathrm{Ei}(x) = S_n(x) + R_n(x)$$

where

$$S_n(x) = e^{-x} \sum_{j=1}^{n} \frac{(-1)^{j+1}(j-1)!}{x^j},$$

$$R_n(x) = (-1)^n \frac{e^{-x}}{x} \int_0^\infty \frac{e^{-z} z^n}{x^n (1 + \frac{z}{x})}\, dz = e^{-x} \int_0^\infty e^{-xt} \frac{(-1)^n t^n}{1 + t}\, dt.$$

We have

$$|R_n(x)| < e^{-x} \int_0^\infty e^{-tx} t^n \, dt = e^{-x} \frac{n!}{x^{n+1}}.$$

Thus for fixed $n$, $R_n = O(\frac{e^{-x}}{x^{n+1}})$ as $x \to \infty$. Hence $S_n$ is an asymptotic expansion for $\mathrm{Ei}(x)$ to $n$ terms as $x \to \infty$.

If we take $x = 10$ then

$$S_1(10) = 0.1 * e^{-10} \quad , |R_1(10)| < 0.01 * e^{-10}.$$

$$S_4(10) = 0.0914 * e^{-10}, \quad |R_4(10)| < 0.00024 * e^{-10}.$$

### 1.2.4 Additional notes

In general it is not permissible to differentiate asymptotic expansions.

**Example** If

$$f(x) = x + \sin x, \quad f'(x) = 1 + \cos x,$$

then

$$f(x) \sim x \quad \text{as} \quad x \to \infty$$

but it is **not true** that

$$f'(x) \sim 1 \quad \text{as} \quad x \to \infty.$$

**Example** If

$$f(x) = e^{-x} \cos(e^x)$$

and $x$ is real, then

$$f(x) \sim 0 + \frac{0}{x} + \frac{0}{x^2} + \dots \quad \text{as} \quad x \to \infty,$$

but

$$f'(x) = -\sin(x) - e^{-x} \cos(e^x)$$

oscillates as $x \to \infty$.

Differentiation is ok when it is known that $f'(x)$ is continuous and its asymptotic expansion exists. Also if $f(z)$ is an analytic function of $z$ and has a Poincaré type of asymptotic power series expansion ie

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots \quad \text{to} \quad N \quad \text{terms as} \quad z \to \infty$$

uniformly in $\arg(z)$ in some sector $S$, then the expansion can be differentiated ie

$$f'(z) \sim -\frac{a_1}{z^2} + \frac{2a_2}{z^3} + \dots \quad \text{to} \quad N-1 \quad \text{terms as} \quad z \to \infty$$

uniformly in $\arg(z)$ in some sector $S'$ contained in $S$.

Integration is usually ok. Additional properties and proofs concerning asymptotic expansions may be found in Erdélyi (1956), Olver (1974).

# Bibliography

Erdélyi, A. 1956 *Asymptotic Expansions*. Dover (reprint).

Olver, F. W. J. 1974 *Introduction to Asymptotics and Special Functions*. Dover.

Poincaré, H. 1886 Sur les intégrales irréguliéres des équations linéaires. *Acta Math.* **8**, 259–344.

# Chapter 2

# Basic complex analysis, review of important results

In the section we will review some fundamantal concepts and theorems of complex analysis which are used in later sections. A good reference text is the book on *Complex Variables* by Ablowitz & S. (2003).

## 2.1 Singularities of complex functions

**Definition** *An* **isolated singular point** *is a point where a (single-valued or a single branch of a multivalued) function $f(z)$ is not analytic, ie near $z = z_0$ the derivative of the function $f'(z_0)$ does not exist.*

*In the neighbourhood of an isolated singular point the function may be represented by a* **Laurent expansion***:*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n. \tag{2.1.1}$$

*An isolated singularity of a function $f(z)$ at $z = z_0$ is* **a pole of order** $N$**,** *where $N \geq 1$ is a positive integer, if*

$$f(z) = \frac{\phi(z)}{(z - z_0)^N}$$

*where $\phi(z)$ is analytic in a neighbourhood of $z = z_0$ and $\phi(z_0) \neq 0$. A* **simple pole** *is when $N = 1$.*

**Example** The function

$$f(z) = \frac{(z-2)^2}{z(z+i)^3}$$

has a simple pole at $z = 0$ and a pole of order 3 at $z = -i$.

It may turn out that the singularity is **removable** as for example with

$$f(z) = \frac{\sin z}{z}$$

where one could define $f(0)$ to be 1.

**Definition** *An isolated singularity that is neither removable nor a pole is called an* **essential singular point**. *An essential singular point has a full Laurent expansion in that given*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

*then for any $M > 0$ there exists an $m < -M$ such that $c_m \neq 0$.*

**Example** $f(z) = e^{-\frac{1}{z}}$ has an essential singular point at $z = 0$ and

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! z^n}.$$

**Multivalued Functions** A function such as $f(z) = z^{\frac{1}{2}}$, $f(z) = \log(z)$ are multivalued functions with a **branch point**.

*A point is* **a branch point** *if the multivalued function is discontinuous after traversing a small circuit around the point.*

**Example** Consider

$$f(z) = (z - 1)^{\frac{1}{2}}$$

and consider the circuit $z = 1 + re^{i\theta}$ as $\theta$ ranges from $\theta = 0$ to $\theta = 2\pi$. The argument of the function when $\theta = 0$ is zero but when $\theta = 2\pi$ the argument of $f(z)$ is $\pi$.

We can work with a single-valued **branch** of a multivalued function if we work in a restricted region of the complex plane with **branch cuts**.

**Example** Consider

$$f(z) = (z^2 + 1)^{\frac{1}{2}}.$$

This has branch points at $z = \pm i$. We can define $f(z) = (r_1 r_2)^{\frac{1}{2}} e^{i(\theta_1 + \theta_2)/2}$ where $z = -i + r_1 e^{i\theta_1}, \quad -3\pi/2 < \theta_1 < \pi/2$ and $z = i + r_2 e^{i\theta_2}, \quad -\pi/2 < \theta_2 < 3\pi/2$.

This makes the function continuous in the region $z = iy, \quad |y| < 1$ and discontinuous for $|y| > 1$. Hence we have branch cuts as shown in figure 2.1.

**Example** Consider

$$f(z) = (z^2 + 1)^{\frac{1}{2}}.$$

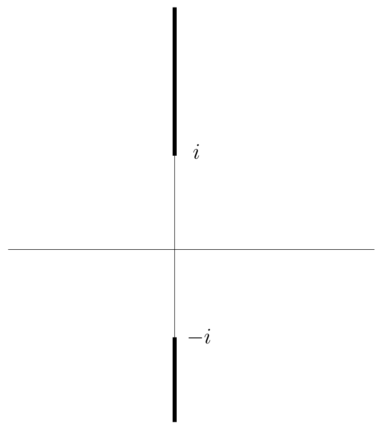



Figure 2.1: Solid lines indicate branch cuts for $f(z) = (z^2 + 1)^{1/2}$.

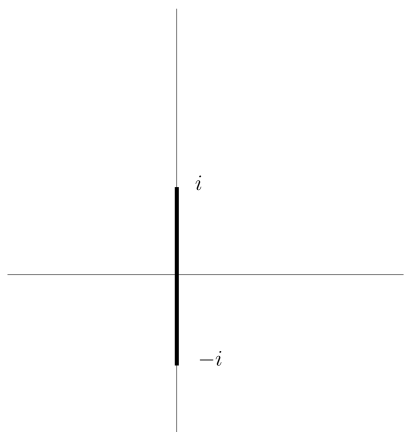



Figure 2.2: Solid lines indicate branch cuts for $f(z) = (z^2 + 1)^{1/2}$.

Alternatively we can define $f(z) = (r_1 r_2)^{\frac{1}{2}} e^{i(\theta_1+\theta_2)/2}$ where $z = -i + r_1 e^{i\theta_1}, \quad -\pi/2 < \theta_1 < 3\pi/2$ and $z = i + r_2 e^{i\theta_2}, \quad -\pi/2 < \theta_2 < 3\pi/2$.

This makes the function discontinuous in the region $z = iy, \quad |y| < 1$ and continuous for $|y| > 1$. We then have branch cuts as shown in figure 2.2.

## 2.2 Cauchy's residue and other important theorems

The following results are heavily used in our work later. Proofs of the theorems may be found in standard texts.

**Theorem (Cauchy)** *Suppose $f(z)$ is analytic in a simply connected domain $\mathcal{D}$,* *and if $\mathcal{C}$ is a closed contour in $\mathcal{D}$ then*

$$\oint_{\mathcal{C}} f(z)\, dz = 0.$$

**Theorem- Cauchy's integral formula.** *If $f(z)$ is analytic in $\mathcal{D}$ and on a closed contour $\mathcal{C}$ then all the derivatives $f^{(k)}(z), k = 1, 2, \ldots$ exist in $\mathcal{D}$ and*

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_{\mathcal{C}} \frac{f(\xi)}{(\xi - z)^{k+1}}\, d\xi.$$

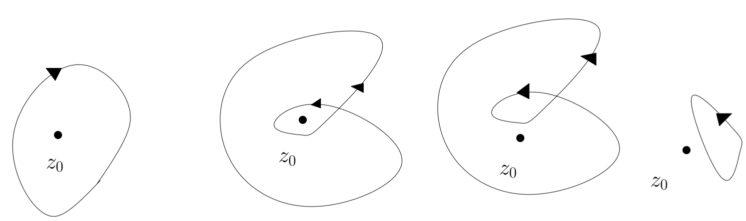
Let $f(z)$ be analytic in region $\mathcal{D}$ except for an isolated singular point at $z = z_0$, and suppose that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n.$$

*The coefficient $c_{-1}$ is called the* **residue** *of $f(z)$ at $z = z_0$.*

**Definition** *Let $\mathcal{C}$ be a closed curve in region $\mathcal{D}$ containing the point $z = z_0$ and $z_0$ does not lie on the curve. The* winding number *or* index *of $\mathcal{C}$ with respect to $z_0$ is defined by*

$$I(\mathcal{C}, z_0) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{dz}{z - z_0}.$$




$I = -1$ $I = 2$ $I = 1$ $I = 0$

Figure 2.3: Winding numbers $I$ for different circuits around $z = z_0$.

**Cauchy's Residue Theorem** *Let $f(z)$ be analytic inside and on a closed contour $\mathcal{C}$ except for a finite number of isolated singular points $z_1, z_2, \ldots z_N$ inside $\mathcal{C}$. Then*

$$\oint_{\mathcal{C}} f(z)\, dz = 2\pi i \sum_{k=1}^{N} a_k I(\mathcal{C}, z_k),$$

*where $a_k$ is the residue of $f(z)$ at $z = z_k$, and $I(\mathcal{C}, z_k)$ is winding number of $\mathcal{C}$ with respect to $z_k$.*

**Example** Consider the different circuits as shown in figure 2.3. The winding numbers are as shown for the different circuits.

**Proof of Cauchy's theorem:** This is a sketch proof for a simple closed curve, see figure 2.4 and apply Cauchy's theorem to contour $C' = C + L_1 + c_1 + L_1' + ... + L_n + c_n + L_1'$ and use fact that integrals over $L_k$ and $L_k'$ cancel out, and integrals around $-c_k$ give residues of $f(z)$ around $z = z_k$.

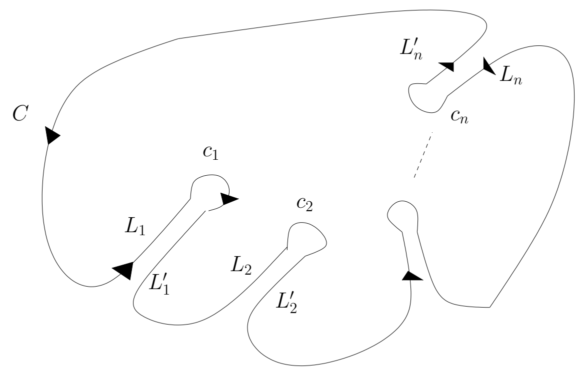



Figure 2.4: Contour for proof of Cauchy;s residue theorem.

## 2.3 Use of Cauchy's residue theorem in evaluation of integrals

Cauchy's residue theorem is very useful when evaluating certain integrals, inverting transforms etc. We will study a few examples.

**Example** Suppose $f(z)$ is analytic in $\mathcal{C}$ except for a finite number of poles which do not lie on the real axis. Also suppose that there exists an $M, R$ and $k > 1$ such that for $|z| > R$

$$|f(z)| \leq \frac{M}{|z|^k},$$

then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \{res(f(z) \text{ in upper-half plane}\},$$

$$\int_{-\infty}^{\infty} f(x)\,dx = -2\pi i \sum \{res(f(z) \text{ in lower-half plane }\}.$$

**Proof** Consider $\oint_\Gamma f(z)\,dz$ where $\Gamma$ is as shown in fig. 2.5 Cauchy's theorem gives

$$\oint_\Gamma = 2\pi i \sum \{res(f(z) \text{ in upper-half plane}\}.$$

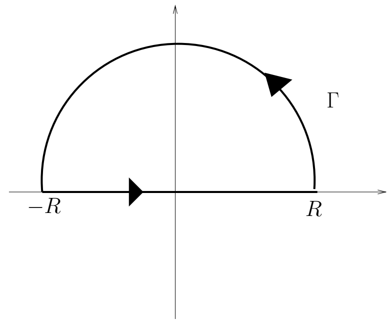



Figure 2.5: Contour for example taken in upper-half plane

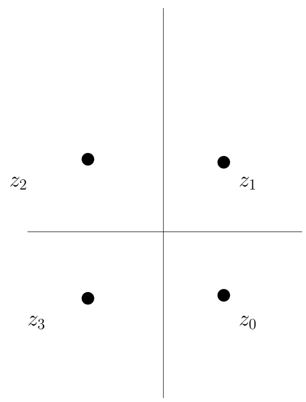



Figure 2.6: Poles of $1/(1+z^4)$.

But

$$\oint_\Gamma = \int_{-R}^{R} f(x)\,dx + \int_0^\pi f(Re^{i\theta})iRe^{i\theta}\,d\theta.$$

Now

$$\left|\int_0^\pi f(Re^{i\theta})iRe^{i\theta}\,d\theta\right| \le \pi\frac{M}{R^k}R = \frac{\pi M}{R^{k-1}}$$

Hence as $R \to \infty$ the integral $\int_0^{2\pi} f(Re^{i\theta})iRe^{i\theta}\,d\theta \to 0$ and we obtain the result.

A similar result is obtained by taking a contour in the lower half plane.

**Example** Consider

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

The function $f(z) = 1/(z^4+1)$ has simple poles at $z = z_k = e^{\frac{2ik\pi - i\pi}{4}}, \quad k = 0,1,2,3$ and $z = z_1, z_2$ lie in the upper-half plane, see fig. 2.6.

Hence

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\pi i \sum_{k=1,2} Res[\frac{1}{1+z^4}, z_k].$$

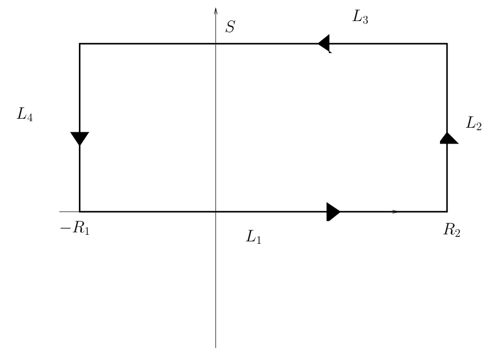



Figure 2.7

Now

$$Res[\frac{1}{1+z^4}, z_k] = \frac{1}{4z_k^3} = -\frac{z_k}{4}.$$

Hence

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\pi i \left[ -\frac{1}{4}(e^{\frac{i\pi}{4}} + e^{\frac{3i\pi}{4}}) \right]$$

$$= -\frac{\pi i}{2} \left[ \frac{1+i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}} \right] = \frac{\pi}{\sqrt{2}}.$$

Suppose $f(z)$ is analytic in $\mathcal{C}$ except for a finite number of poles which do not lie on the real axis. Also suppose that there exists an $M, R$ such that for $|z| > R$

$$|f(z)| \leq \frac{M}{|z|},$$

then for any $a > 0$

$$\int_{-\infty}^{\infty} e^{iax} f(x)\, dx = 2\pi i \sum \{res[f(z)e^{iaz}] \text{ in upper-half plane}\}.$$

Note by taking real and imaginary parts we can use this result to work out integrals of the form $\int_{-\infty}^{\infty} f(x) \cos ax\, dx$ and $\int_{-\infty}^{\infty} f(x) \sin ax\, dx$.

**Proof** Consider $\int_{L_1+L_2+L_3+L_4} e^{iaz} f(z)\, dz$ as shown in figure 2.7 where $L_1, L_2, L_3, L_4$ denote the sides of the rectangle $-R_1 \leq \Re(z) \leq R_2$, and $0 \leq \Im(z) \leq S$ and $R_1, R_2, S$ are such that all the zeros of the function in the upper-half plane are contained in the rectangle.

Using Cauchy's theorem

$$\int_{L_1+L_2+L_3+L_4} e^{iaz} f(z)\, dz = 2\pi i \sum [Res[f(z)e^{iaz}, \text{ in upper-half plane}].$$

Consider

$$I_2 = \int_{L_2} e^{iaz} f(z)\, dz = \int_0^S e^{ia(R_2+iy)} f(R_2+iy) i\, dy.$$

We have

$$|I_2| \le \int_0^S e^{-ay}|f(R_2+iy)|\,dy \le \frac{M_2}{R_2}\int_0^S e^{-ay}\,dy = \frac{M_2(1-e^{-aS})}{aR_2},$$

i.e

$$|I_2| \le \frac{M_2}{aR_2}.$$

Similarly if

$$I_4 = \int_{L_4} e^{iaz}f(z)\,dz = -\int_0^S e^{ia(-R_1+iy)}f(-R_1+iy)i\,dy.$$

We have

$$|I_4| \le \int_0^S e^{-ay}|f(-R_1+iy)|\,dy \le \frac{M_4}{R_1}\int_0^S e^{-ay}\,dy = \frac{M_4(1-e^{-aS})}{aR_1}.$$

Thus

$$|I_4| \le \frac{M_4}{aR_1}.$$

Next if

$$I_3 = \int_{L_3} e^{iaz}f(z)\,dz = -\int_{-R_1}^{R_2} e^{ia(iS+x)}f(iS+x)\,dx$$

then

$$|I_3| \le \int_{-R_1}^{R_2} e^{-aS}|f(iS+x)|\,dx \le \frac{M_3}{S}\int_{-R_1}^{R_2} e^{-aS}\,dx.$$

Thus

$$|I_3| \le M_3 e^{-aS}\frac{R_2+R_1}{S}.$$

Note that

$$I_1 = \int_{-R_1}^{R_2} e^{iax}f(x)\,dx.$$

Taking the limits $S\to\infty$, $R_1\to\infty$, $R_2\to\infty$ independently, and noting that $I_2, I_3, I_4 \to 0$ gives the required result

$$\int_{-\infty}^{\infty} e^{iax}f(x)\,dx = 2\pi i\sum[Res[f(z)e^{iaz},\ \text{in the upper-half plane}].$$

## 2.4 Jordan's Lemma

**Jordan's Lemma** is also useful when evaluating contour integrals. The conditions on the function are slightly weaker than in the previous result.

*Consider*

$$I = \int_\Gamma e^{iaz} f(z)\, dz$$

*where $a > 0$ and $\Gamma$ is the semicircle in the upper-half plane centered on the origin and of radius $R$. f $|f(Re^{i\theta})| \leq G(R)$ and $G(R) \to 0$ as $R \to \infty$ then*

$$\lim_{R\to\infty} I = 0.$$

**Proof of Jordan's Lemma** Now

$$I = \int_0^\pi ie^{iaR(\cos\theta + i\sin\theta)} f(Re^{i\theta}) Re^{i\theta}\, d\theta.$$

If we make use of the result that $0 \leq 2\theta/\pi \leq \sin\theta$ for $0 \leq \theta \leq \pi/2$ then

$$|I| \leq \int_0^\pi e^{-aR\sin\theta} RG(R)\, d\theta$$

$$\leq 2\int_0^{\pi/2} RG(R) e^{-2aR\theta/\pi}\, d\theta = \frac{\pi G(R)}{a}(1 - e^{-aR}).$$

Let $R \to \infty$ and the result follows as $G(R) \to 0$ as $R \to \infty$.

**Example** Show that

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}\, dx = \frac{\pi}{2}(1 - e^{-1}).$$

Consider

$$\oint_C \frac{e^{iz}}{z(z^2+1)}\, dz$$

We take a contour as shown in figure 2.8 with a semicircular path of large radius $R$ and a small semicircular path of radius $\delta$ around the origin. The integrand has a simple pole at $z = i$ inside $C$ Applying Cauchy's Theorem gives

$$\oint_C \frac{e^{iz}}{z(z^2+1)}\, dz = 2\pi i Res[\frac{e^{iz}}{z(z^2+1)}; i] = 2\pi i[\frac{e^{-1}}{2i^2}] = -\pi i e^{-1}.$$

Now

$$\oint_c = \int_{L_1} + \int_{C_R} + \int_{L_2} + \int_{C_\delta}$$

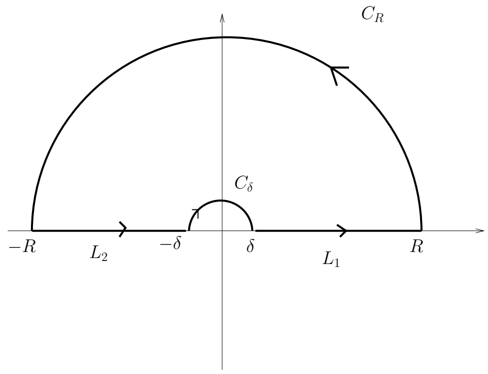



Figure 2.8: Contour $C$. Here $C_R$ is a semi-circle of radius $R$ and $C_\delta$ a semi-circle of radius $\delta$.

and using Jordan's Lemma $\int_{C_R} \to 0$ as $R \to \infty$. Also

$$\int_{C_\delta} = \int_\pi^\delta \frac{e^{i\delta e^{i\theta}} i\delta e^{i\theta}}{\delta e^{i\theta}(\delta^2 e^{2i\theta}+1)}\, d\theta = i(\delta - \pi) + O(\delta^2)$$

as $\delta \to 0$.

Note that

$$\int_{L_1} + \int_{L_2}$$

$$= \int_\delta^R \frac{e^{ix}}{x(x^2+1)}\, dz + \int_{-R}^{-\delta} \frac{e^{ix}}{x(x^2+1)}\, dx = 2i \int_\delta^R \frac{\sin x}{x(x^2+1)}\, dx.$$

Hence taking the limit as $R \to \infty$ and $\delta \to 0$ gives

$$2i \int_0^\infty \frac{\sin x}{x(x^2+1)}\, dx - i\pi = -\pi i e^{-1},$$

and thus

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}\, dx = \frac{\pi}{2}(1 - e^{-1}).$$

Let $f(z)$ be analytic in $\mathcal{C}$ except for a finite number of poles, none of which lie on the positive real axis. Suppose $a > 0$ and $a$ is not an integer. Suppose that (i) there exist constants $M, R > 0$ and $b > a$ such that $|f(z)| \le M/|z|^b$ for $|z| > R$ and (ii) and constants $S, W > 0$ and $0 < d < a$ such that for $0 < |z| \le S$, $|f(z)| \le W/|z|^d$. Then $\int_0^\infty x^{a-1} f(x)\, dx$ is absolutely integrable and

$$\int_0^\infty x^{a-1} f(x)\, dx =$$

$$-\frac{\pi e^{-\pi a i}}{\sin(a\pi)} \sum \{res[z^{a-1} f(z)] \text{ at the poles of} f(z), z \neq 0\}$$

and the branch $z^{a-1} = e^{(a-1)\log(z)} \quad 0 < \arg(z) < 2\pi$ is used.

**Proof** Note that for $0 < x \leq S$, we have

$$|x^{a-1}f(x)| \leq Wx^{a-d-1},$$

and for large positive $x > R$ we have

$$|x^{a-1}f(x)| \leq Mx^{a-b-1}$$

and thus the integral exists and is absolutely convergent. To evaluate the integral consider the integral of $z^{a-1}f(z)$ around the contour $\Gamma = C_1 + C_2 + C_3 + C_4$ as shown in the figure 2.9. Applying Cauchy's theorem gives

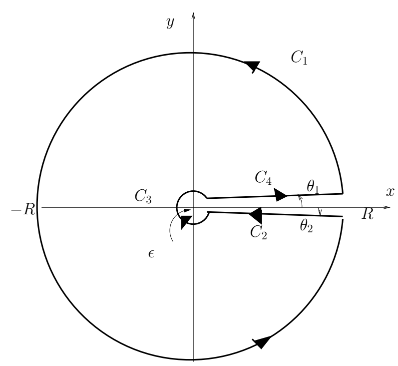



Figure 2.9

$$I_\Gamma = \int_{C_1+C_2+C_3+C_4} z^{a-1}f(z)\,dz =$$

$$2\pi i \sum \{res[z^{a-1}f(z)] \text{ at the poles of} f(z), z \neq 0\}.$$

Now let $I_n = \int_{C_n} z^{a-1}f(z)\,dz$ and note that

$$|I_1| = \left| \int_{\theta_1}^{2\pi-\theta_2} (Re^{i\theta})^{a-1} f(Re^{i\theta}) iRe^{i\theta}\,d\theta \right|$$

$$\leq M \int_{C_1} \frac{|R|^{a-1}}{|R|^b} R\,d\theta < 2\pi MR^{a-b}$$

independent of $\theta_1, \theta_2$. Hence $I_1 \to 0$ as $R \to \infty$. Similarly

$$|I_3| = \left| \int_{2\pi-\theta_2}^{\theta_1} (\epsilon e^{i\theta})^{a-1} f(\epsilon e^{i\theta}) i\epsilon e^{i\theta}\,d\theta \right| \leq W \int_{\theta_1}^{2\pi-\theta_2} \frac{\epsilon^{a-1}}{\epsilon^d} \epsilon\,d\theta$$

$$< 2\pi W\epsilon^{a-d}.$$

Hence $I_3 \to 0$ as $\epsilon \to 0$.

Next

$$I_4 = \int_{C_4} z^{a-1} f(z)\, dz = \int_\epsilon^R (ye^{i\theta_1})^{a-1} f(ye^{i\theta_1}) e^{i\theta_1}\, dy$$

and

$$I_2 = \int_{C_2} z^{a-1} f(z)\, dz = -\int_\epsilon^R (ye^{2i\pi - i\theta_2})^{a-1} f(ye^{2i\pi - i\theta_2}) e^{2i\pi - i\theta_2}\, dy$$

$$= -\int_\epsilon^R y^{a-1} e^{2\pi i(a-1)} e^{-i(a-1)\theta_2} e^{-i\theta_2} f(ye^{-i\theta_2})\, dy.$$

Letting $R \to \infty, \epsilon \to 0, \theta_{1,2} \to 0$ shows that

$$I_2 + I_4 \to \int_0^\infty y^{a-1}(1 - e^{2i\pi(a-1)}) f(y)\, dy.$$

Hence putting it all together shows that

$$-\int_0^\infty y^{a-1} e^{ia\pi} 2i \sin(\pi a) f(y)\, dy$$

$$= 2\pi i \sum \{res[z^{a-1} f(z)] \text{ at the poles of} f(z), z \neq 0\}.$$

Thus

$$\int_0^\infty x^{a-1} f(x)\, dx =$$

$$-\frac{\pi e^{-\pi a i}}{\sin(a\pi)} \sum \{res[z^{a-1} f(z)] \text{ at the poles of} f(z), z \neq 0\}$$

**Example** Consider

$$\int_0^\infty \frac{x^{s-1}}{1+x}\, dx$$

with $0 < \Re(s) < 1$. Applying the previous result gives

$$\int_0^\infty \frac{x^{s-1}}{1+x}\, dx =$$

$$-\frac{\pi e^{-\pi s i}}{\sin(s\pi)} \sum \{res[z^{s-1} \frac{1}{1+z}] \quad \text{at} \quad z = -1\},$$

$$= \frac{\pi}{\sin(\pi s)}.$$

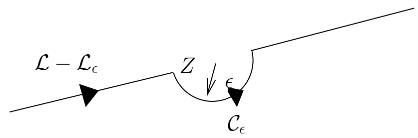



Figure 2.10: Contour $\mathcal{L}$ and $\mathcal{C}_\epsilon$.

## 2.5 Plemlj formulae

Suppose $\mathcal{L}$ is a smooth contour (which may be closed or open) and suppose $\phi(z)$ is continuous at $z$. Consider

$$\Phi(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\phi(\xi)}{\xi - z}\, d\xi.$$

If $z$ lies on $\mathcal{L}$ then the integral may not exist in the normal sense and we have to work with the Cauchy principal value integral.

Consider the limit as $\epsilon \to 0+$ along the curve $\mathcal{L}-\mathcal{L}_\epsilon$ and $\mathcal{C}_\epsilon$ as shown in the figure 2.1. The convention is that $\epsilon \to 0+$ implies the region on the left in the positive direction of $\mathcal{L}$.

Thus

$$\Phi^+(z) = \lim_{\epsilon\to 0+}\frac{1}{2\pi i}\int_{\mathcal{L}-\mathcal{L}_\epsilon} \frac{\phi(\xi)}{\xi - z}\, d\xi + \lim_{\epsilon\to 0+}\frac{1}{2\pi i}\int_{\mathcal{C}_\epsilon} \frac{\phi(\xi)}{\xi - z}\, d\xi.$$

The first integral reduces to

$$\frac{1}{2\pi i}\fint_{\mathcal{L}} \frac{\phi(\xi)}{\xi - z}\, d\xi$$

and for the second put $\xi = z + \epsilon e^{i\theta}$ to obtain

$$\lim_{\epsilon\to 0+}\frac{1}{2\pi i}\int_{\mathcal{C}_\epsilon} \frac{\phi(z+\epsilon e^{i\theta})}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}\, d\theta = \frac{1}{2}\phi(z).$$

Hence

$$\Phi^+(z) = \frac{1}{2}\phi(z) + \frac{1}{2\pi i}\fint_{\mathcal{L}} \frac{\phi(\xi)}{\xi - z}\, d\xi. \tag{2.5.1}$$

Similarly

$$\Phi^-(z) = -\frac{1}{2}\phi(z) + \frac{1}{2\pi i}\fint_{\mathcal{L}} \frac{\phi(\xi)}{\xi - z}\, d\xi. \tag{2.5.2}$$

The formulae (2.5.1,2.5.2) are known as the Plemlj formulae Plemelj (1908).

They have important applications in many Riemann-Hilbert problems (see later).

**Example** Consider the following problem which arises in a (triple-deck) application.

$$u_x + v_y = 0, \quad u_x = -p_x, \quad v_x = -p_y,$$

for $0 \leq y < \infty, \quad -\infty < x < \infty$, and

$$u(x, y = 0+) = -P(x), \quad v(x, y = 0+) = -A'(x).$$

In terms of the complex velocity $u - iv$ and $z = x + iy$ we have

$$u - iv = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{m(\xi)}{z - \xi} d\xi.$$

Here $m(\xi)$ is a suitable distribution of sources (to be found).

With $z = x + iy$ let $y \to 0+$ and use the Plemlj formula. This gives This gives

$$u(x, 0+) - iv(x, 0+) = i \left[ \frac{1}{2\pi i} ⨍_{-\infty}^{\infty} \frac{m(\xi)}{\xi - x} \, d\xi + \frac{1}{2} m(x) \right].$$

Using the conditions on $y = 0$ gives

$$m(x) = 2A'(x),$$

and

$$P(x) = -\frac{1}{2\pi} ⨍_{-\infty}^{\infty} \frac{m(\xi)}{\xi - x} \, d\xi.$$

Hence

$$P(x) = -\frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{A'(\xi)}{\xi - x} \, d\xi,$$

ie the interaction law in subsonic flow.

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Plemelj, J. 1908 Riemannsche funktionenscharen mit gegebener monodromiegruppe. *Monatsh. Math. Phys.* **19**, 211–245.

# Chapter 3

# Approximate solution of linear differential equations using Frobenius series

## 3.1 Introduction

A large number of special functions are defined in terms of an ordinary differential equation. It is useful to be able to predict solution properties just by examining the coefficents of the differential operator. Fortunately, there exist powerful methods for predicting the local behaviour of the solutions near a point $x = x_0$ without needing to solve the full differential equation. In many cases the dominant behaviour can extracted without too much work, In this chapter we will discuss the Frobenius Frobenius (1873) method for obtaining series solution of linear ordinary equations about ordinary and regular singular points. In a later chapter we will extend the ideas to obtain apprioximate solutions near irregular singular points. The material discussed here follows closely the book by Bender & Orsag Bender & Orszag (1999) which contains many wonderful, fully worked examples.

## 3.2 Classification of singularities

Consider a homogeneous linear differential equation.

$$\mathcal{L}y = 0,$$

where

$$\mathcal{L} \equiv \frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots p_1(x)\frac{d}{dx} + p_0(x). \tag{3.2.1}$$

**Definition - Ordinary Point** *The point $x = x_0(\neq \infty)$ is called an* ordinary point *of (3.2.1) if $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ are analytic in a neighbourhood of $x_0$.*

**Definition - regular singular point**

*The point $x = x_0 (x_0 \neq \infty)$ is a* regular singular point *of (3.2.1) if all of $(x - x_0)^n p_0(x), (x - x_0)^{n-1} p_1(x), \ldots, (x - x_0) p_{n-1}(x)$ are analytic in a neighbourhood of $x = x_0$.*

**Definition - irregular singular point** *The point $x = x_0 (\neq \infty)$ is called an* irregular singular point *of (3.2.1) if it is neither an ordinary point or a regular singular point. To classify the point at infinity, put $x = 1/t$ and rewrite the differential equation in terms of $t$. Then the point at $\infty$ is either an ordinary point, regular singular point, or an irregular singular point, if $t = 0$ is an ordinary point, regular singular point, or irregular singular point respectively.*

**Examples**

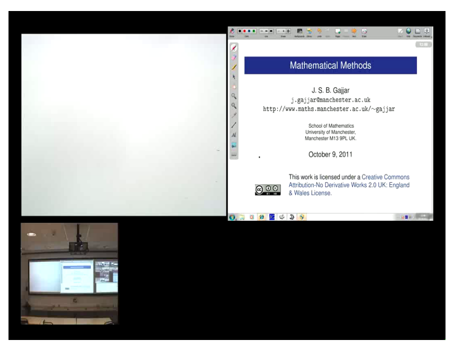
1. $y''(x) = (1 + x^2) y(x)$. Every point $x = x_0 (\neq \infty)$ is an ordinary point.

2. $xy'''(x) - y'(x) + y = 0$. Every point $x = x_0$ with $x_0 \neq 0$ or $x_0 \neq \infty$ is an ordinary point.

3. $(x - 1) y'''(x) + xy(x) = 0$ All points $x = x_0$, with $x_0 \neq 1$ or $\infty$ are ordinary points. $x_0 = 1$ is a regular singular point.

4. $x^3 y''(x) - y = 0$. The point $x = 0$ is not an ordinary point or a regular singular point.

Video clip of last section.Click here to open video clip in external player

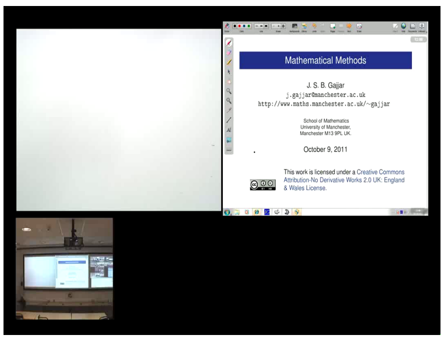



## 3.3 Properties near ordinary and regular singular points

All $n$ linearly independent solutions of (3.2.1) are analytic in a neighbourhood of an ordinary point, Fuchs (1866). The radius of convergence of a Taylor series of a solution about $x = x_0$ is at least as large as the distance to the nearest singularity of the coefficient functions. Near a regular singular point, the form of the $n^{th}$ solution is at worst of the form,

$$y(x) = (x - x_0)^\gamma \sum_{k=0}^{n-1} [\log(x - x_0)]^k A_k(x)$$

where $A_k(x)$ is analytic at $x_0$, and $\gamma$ is an *indicial exponent.*




Video clip discussing properties near ordinary and regular singular points.Click here to open video clip in external player.

**Example** Consider Airy's equation

$$y'' = xy. \tag{3.3.1}$$

Here every point ($\neq \infty$) is an ordinary point and the solution can be expressed as a Taylor series expansion.

Seek a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and substitution into the equation (3.3.1) and equating coefficients of like powers of $x$ leads to

$$a_n n(n-1) = 0, n = 0, 1, 2, \quad a_n n(n-1) = a_{n-3}, \quad n = 3, 4, \ldots$$

Thus $a_1, a_2$ are arbitrary, $a_2 = 0$ and

$$a_{3n} = \frac{a_0 \Gamma(\frac{2}{3})}{3^{2n} n! \Gamma(n + \frac{2}{3})}, \quad a_{3n+1} = \frac{a_1 \Gamma(\frac{4}{3})}{3^{2n} n! \Gamma(n + \frac{4}{3})}, \quad a_{3n+2} = 0.$$

The Gamma function $\Gamma(z)$ used above satisfies $\Gamma(z+1) = z\Gamma(z)$. Hence we have obtained two linearly independent solutions

$$y_1(x) = C_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{9^n n! \Gamma(n + \frac{2}{3})},$$

and

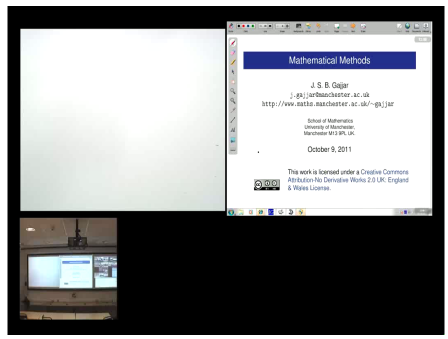
$$y_2(x) = C_1 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n! \Gamma(n + \frac{4}{3})}.$$

The radius of convergence of both series is infinity, the distance to the nearest singularity. By convention the two linearly independent solutions of Airy's equation, the Airy functions $\mathrm{Ai(x)}, \mathrm{Bi(x)}$ are defined by

$$\mathrm{Ai(x)} = 3^{-\frac{2}{3}} \sum_{\mathrm{n=0}}^{\infty} \frac{\mathrm{x}^{3\mathrm{n}}}{9^{\mathrm{n}}\mathrm{n}!\Gamma(\mathrm{n}+\frac{2}{3})} - 3^{-\frac{4}{3}} \sum_{\mathrm{n=0}}^{\infty} \frac{\mathrm{x}^{3\mathrm{n}+1}}{9^{\mathrm{n}}\mathrm{n}!\Gamma(\mathrm{n}+\frac{4}{3})},$$

$$\mathrm{Bi(x)} = 3^{-\frac{1}{6}} \sum_{\mathrm{n=0}}^{\infty} \frac{\mathrm{x}^{3\mathrm{n}}}{9^{\mathrm{n}}\mathrm{n}!\Gamma(\mathrm{n}+\frac{2}{3})} + 3^{-\frac{5}{6}} \sum_{\mathrm{n=0}}^{\infty} \frac{\mathrm{x}^{3\mathrm{n}+1}}{9^{\mathrm{n}}\mathrm{n}!\Gamma(\mathrm{n}+\frac{4}{3})}.$$

Video clip with worked example for Airy's equation.Click here to open video clip in external player.




## 3.4 Frobenius solution for 2nd order odes

Near a regular singular point the solution can be obtained as a Frobenius series in the form

$$y(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^{k+\gamma}.$$

Here $a_0 \neq 0$ and $\gamma$ is an indicial exponent (to be found), see below. Consider the equation

$$y''(x) + \bar{p}_1(x)y'(x) + \bar{p}_0(x)y(x) = 0, \tag{3.4.1}$$

and

$$\bar{p}_1(x) = \frac{1}{(x-x_0)} \sum_{n=0}^{\infty} p_n (x-x_0)^n, \quad \bar{p}_0(x) = \frac{1}{(x-x_0)^2} \sum_{n=0}^{\infty} q_n (x-x_0)^n.$$

If we seek a solution in Frobenius form

$$y(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^{k+\gamma},$$

then substitution into the equation (3.4.1) gives:

$$[\gamma^2 + (p_0 - 1)\gamma + q_0]a_0 = 0, \tag{3.4.2}$$

$$[(\gamma+n)^2+(p_0-1)(\gamma+n)+q_0]a_n = -\sum_{k=0}^{n-1}[(\gamma+k)p_{n-k}+q_{n-k}]a_k, \quad n=1,2,\ldots. \tag{3.4.3}$$

From (3.4.2) since $a_0 \neq 0$ we obtain the *indicial equation*

$$P(\gamma) \equiv \gamma^2+(p_0-1)\gamma+q_0 = 0.$$

This gives two roots $\gamma_1, \gamma_2$, and we will assume that $\Re(\gamma_1) \leq \Re(\gamma_2)$. Then $P(\gamma_2+n) \neq 0$ for $n = 1,2,,\ldots$.

From (3.4.3) solving for $a_n$ gives

$$a_n = -\frac{\sum_{k=0}^{n-1}[(\gamma+k)p_{n-k}+q_{n-k}]a_k}{P(\gamma+n)} \tag{3.4.4}$$

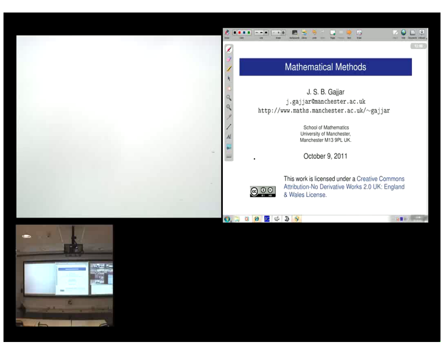
The expression (3.4.4) together with the fact that $P(\gamma_2+n) \neq 0$ shows that we can obtain at least one solution in Frobenius form with the $a_n$ given by (3.4.4) in terms of $a_0$ and $\gamma = \gamma_2$. Whether a second solution of this form exists or not, depends on whether the indicial roots differ by an integer or not. If $\gamma_2-\gamma_1 \neq$ integer, then $P(\gamma+n) \neq 0$ and a second solution of Frobenius form also exists with $a_n$ given by (3.4.4) in terms of $a_0$ and $\gamma = \gamma_1$. If $\gamma_2-\gamma_1 = N$, where $N$ is a positive integer then note that from (3.4.3) we obtain

$$P(\gamma_1+N)a_N = -\sum_{k=0}^{N-1}[(\gamma_1+k)p_{N-k}+q_{N-k}]a_k. \tag{3.4.5}$$

But $\gamma_1+N = \gamma_2$ and thus the left hand side of(3.4.5) is

$$P(\gamma_2)a_N = 0.$$

If the right hand side of (3.4.5) equals zero then $a_N$ is indeterminate and a second linearly independent solution of Frobenius type exists with $\gamma = \gamma_1$.




Video clip of solution properties near a regular singular point. Open video clip in external player.

**Example** Consider Rayleigh's equation which arises in Hydrodynamic Stability Theory, see MAGIC014.

$$\phi'' - (\alpha^2 + \frac{U''}{U-c})\phi = 0, \quad 0 < x < \infty,$$

where $\alpha$ and $c$ are constants and $U = U(x)$. Suppose $c$ is real and there exists $x_c$ such that $U(x_c) = c$, and near $x = x_c$

$$U(x) = c + (x - x_c)U'(x_c) + \frac{1}{2}(x - x_c)^2 U''(x_c) + \ldots .$$

Here $x = x_c$ is a regular singular point because in terms of our earlier notation in (3.4.1) $\bar{p}_1(x) = 0$ and

$$\bar{p}_0(x) = (\alpha^2 + \frac{U''}{U - c}) = \frac{q_1}{(x - x_c)} + q_2 + \ldots, \quad \text{and} \quad q_1 = \frac{U''(x_c)}{U'(x_c)}.$$

The indicial equation is

$$\gamma(\gamma - 1) = 0, \qquad \Longrightarrow \gamma = 0, 1$$

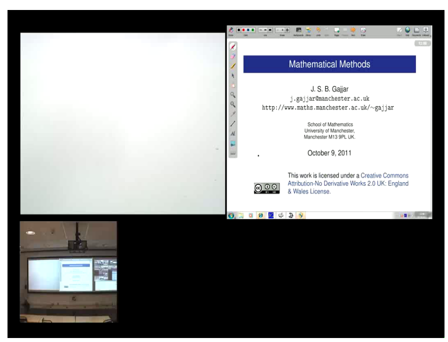
and the roots differ by an integer.

Also the condition from (3.4.5) with $N = 1$ reduces to

$$q_1 = \frac{U''(x_c)}{U'(x_c)} = 0.$$

Thus if $U''(x_c) = 0$ then we have two linearly independent solutions of Frobenius type.

Video clip of worked example for Rayleigh's equation. Click here to open video clip in external player.




## 3.5 Roots differ by an integer, $\gamma_2 - \gamma_1 = N$

Let

$$y(x, \gamma) = \sum_{n=0}^{\infty} a_n(\gamma)(x - x_0)^{\gamma+n}.$$

Now

$$\mathcal{L}y = a_0 P(\gamma)(x - x_0)^{\gamma-2} + \sum_{n=1}^{\infty} \left[ a_n P(\gamma + n) + \sum_{j=0}^{n-1} (p_{n-j}(\gamma + j)a_j + q_{n-j}a_j) \right] (x - x_0)^{\gamma+n-2}. \tag{3.5.1}$$

Now let $a_0$ be arbitrary and choose $a_n(\gamma), \quad n = 1, 2, ...$ so that

$$a_n(\gamma) = -\frac{\sum_{k=0}^{n-1}[(\gamma+k)p_{n-k} + q_{n-k}]a_k}{P(\gamma+n)}$$

and assume that $P(\gamma + n) \neq 0$ for $n = 1, 2, ...$

**Roots differ by an integer,** $\gamma_2 - \gamma_1 = N$

Then from (3.5.1) we have

$$\mathcal{L}y = a_0 P(\gamma)(x - x_0)^{\gamma-2}. \tag{3.5.2}$$

We can see that if $\gamma$ is chosen to be $\gamma_2$ the right hand side of (3.5.2) is zero and we have the solution $y(x, \gamma_2)$ obtained earlier.

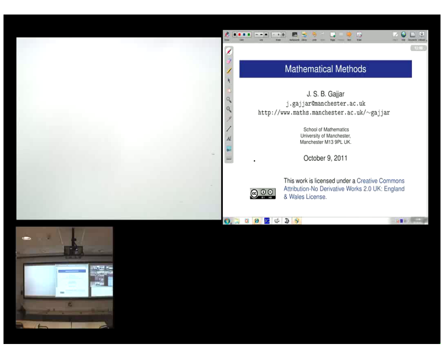
**Roots differ by an integer,** $\gamma_2 - \gamma_1 = 0$

Suppose we differentiate both sides of (3.5.2) with respect to $\gamma$ and then set $\gamma = \gamma_2$. Then

$$\begin{aligned}\mathcal{L}(\frac{\partial y}{\partial \gamma})|_{\gamma=\gamma_2} &= a_0((\gamma_2 - 2)\log(x - x_0)(x - x_0)^{\gamma_2-2}P(\gamma_2) \\ &+ (x - x_0)^{\gamma_2-2}P'(\gamma)).\end{aligned} \tag{3.5.3}$$

If the roots are equal ie $\gamma_2 - \gamma_1 = 0$ then $P'(\gamma_2) = 0$ and we see that the right hand side of (3.5.3) is zero. Therefore when we have equal roots a second linearly independent solution is

$$\frac{\partial y}{\partial \gamma}|_{\gamma=\gamma_2} = y(x, \gamma_2)\log(x - x_0) + \sum_{n=0}^{\infty} \frac{\partial a_n(\gamma)}{\partial \gamma}|_{\gamma=\gamma_2}(x - x_0)^{\gamma_2+n}.$$




Video clip of working for two equal roots. Click here to open video clip in external player.

## 3.6 Roots differ by an integer $\gamma_2 - \gamma_1 = N > 0$

From (3.5.3) note that is we set $\gamma = \gamma_2$ the right hand side is equal to

$$a_0(x-x_0)^{\gamma_2-2}P'(\gamma_2) = a_0(x-x_0)^{\gamma_1+N-2}P'(\gamma_2),$$

and is not zero. However, consider

$$\mathcal{L}\left[(\frac{\partial y}{\partial \gamma})|_{\gamma=\gamma_2} + \sum_{n=0}^{\infty} b_n(x-x_0)^{\gamma_1+n}\right],$$

$$\begin{aligned} &= a_0(x-x_0)^{\gamma_1+N-2}P'(\gamma_2) + b_0P(\gamma_1)(x-x_0)^{\gamma_1-2} \\ &+ \sum_{n=1}^{\infty}[P(\gamma_1+n)b_n + \sum_{j=0}^{n-1}(p_{n-j}b_j + q_{n-j}b_j)](x-x_0)^{\gamma_1+n-2}. \end{aligned} \tag{3.6.1}$$

Equating powers of $(x-x_0)$ to zero gives:

$$P(\gamma_1)b_0 = 0, \tag{3.6.2}$$

$$P(\gamma_1+n)b_n + \sum_{j=0}^{n-1}(p_{n-j}(\gamma_1+j)+q_{n-j})b_j = 0, \quad n = 1,2,..,N-1,$$

$$P(\gamma_1+n)b_n + \sum_{j=0}^{n-1}(p_{n-j}(\gamma_1+j)+q_{n-j})b_j = 0, \quad n = N+1,\ldots \tag{3.6.3}$$

$$P(\gamma_1+N)b_N + \sum_{j=0}^{n-1}(p_{n-j}(\gamma_1+j)+q_{n-j})b_j = a_0P'(\gamma_2). \tag{3.6.4}$$

From (3.6.2) since $P(\gamma_1) = 0$ we see that $b_0$ is undetermined.

But from (3.6.4) since $P(\gamma_1+N) = P(\gamma_2)$ we have an expression which determines $a_0$ in terms of $b_0, b_1, \ldots, b_{N-1}$.

The term $b_N$ is undetermined, but a non-zero $b_N$ just replicates a multiple of the $y(x,\gamma_2)$ solution. Hence a second linearly independent solution is obtained in the form

$$y_1 = \frac{\partial y}{\partial \gamma}|_{\gamma=\gamma_2} + \sum_{n=0}^{\infty} b_n(x-x_0)^{\gamma_1+n}.$$

This can be expressed as

$$y_1 = k\log(x-x_0)y_2(x,\gamma_2) + \sum_{n=0}^{\infty} c_n(x-x_0)^{\gamma_1+n}. \tag{3.6.5}$$

Note that if the right-hand side of (3.4.5) is zero, $a_0$ is zero and the coefficient $k$ of the logarithmic term in (3.6.5) is zero.

**Example** Consider again Rayleigh's equation which we met in an earlier example:

$$\phi'' - (\alpha^2 + \frac{U''}{U-c})\phi = 0, \quad 0 < x < \infty,$$

where $\alpha$ and $c$ are constants and $U = U(x)$. Suppose $c$ is real and there exists $x_c$ such that $U(x_c) = c$, and near $x = x_c$

$$U(x) = c + (x - x_c)U'(x_c) + \frac{1}{2}(x - x_c)^2 U''(x_c) + \dots .$$

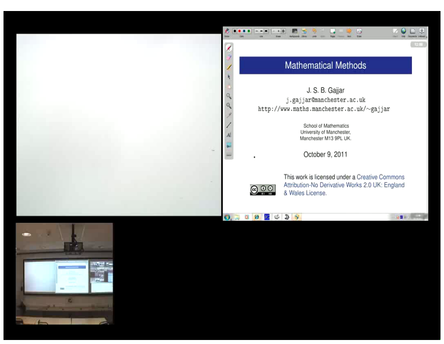
Here $x = x_c$ is a regular singular point because in terms of our earlier notation as in (3.4.1) $\bar{p}_1(x) = 0$ and

$$\bar{p}_0(x) = (\alpha^2 + \frac{U''}{U-c}) = \frac{q_1}{(x-x_c)} + q_2 + \dots, \quad \text{and} \quad q_1 = \frac{U''(x_c)}{U'(x_c)}.$$

The indicial equation gives two roots $\alpha = 0$ and 1 differing by an integer. The Frobenius method gives two linearly independent solutions of the form

$$\begin{aligned}\phi_1(x) &= (x - x_c) + a_2(x - x_c)^2 + a_3(x - x_c)^3 + \dots, \\ \phi_2(x) &= 1 + b_1(x - x_c) + b_2(x - x_c)^2 + b_3(x - x_c)^3 + \dots \\ &\quad + \frac{U''(x_c)}{U'(x_c)}\phi_1(x)(x - x_c)\log(x - x_c) \quad x > x_c.\end{aligned}$$

The presence of the logarithmic branch point raises questions about what happens for $x < x_c$.




Video clip of worked example for Rayleigh's equation. Click here to open video clip in external player.

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# Chapter 4

# Solution properties of linear differential equations near an irregular singular point

## 4.1 Introduction

We have seen how to construct the local solution properties near ordinary points and regular singular points. The more interesting case is to estimate behaviours near irregular singular points.

There is a powerful technique developed by Carlini (1817), Liouville (1837), Green (1837) based on the method of dominant balance. This is explained clearly with lots of illustrative examples in Bender & Orszag (1999). Carlini's (1817) work concerned a problem in planetary motion. He introduced what is now known as the WKB expansion (see later in the course) and obtained an asymptotic expansion for a Bessel function of the first kind for large values of the parameter. Almost 20 years later Liouville (1837) used a similar WKB type expansion for a problem in heat conduction, and Green (1837) for a problem concerning waves in a fluid. The technique is more popularly known as the WKBJ after Wentzel (1926), Kramers (1926), Brillouin (1926), and Jeffreys (1924). A historical account of the development of the WKBJ method can be found in Pike (1964), and Fröman & Fröman (2002).

Note that Frobenius type solutions do not work near irregular singular points. One example will suffice to illustrate this.

**Example** Consider

$$x^4 y'' = y,$$

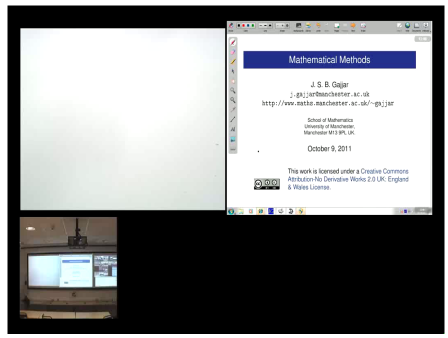
and we see that $x = 0$ is an irregular singular point. If we look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+\gamma}, \quad (a_0 \neq 0),$$

then we obtain

$$\sum_{n=0}^{\infty} a_n (n+\gamma)(n+\gamma-1) x^{n+\gamma+2} = \sum_{n=0}^{\infty} a_n x^{n+\gamma}.$$

The coefficient of $x^\gamma$ gives $a_0 = 0$ which is a contradiction and therefore no solution of this type exists near $x = 0$.




Video clip of worked example.Click here to open video clip in external player.

## 4.2 Method of Dominant balance

In this section we will discuss the method of dominant balance which is described in Bender & Orszag (1999) and which is used to obtain approximate solutions to linear ordinary differential equations about irregular singular points. We will study this technique by working through some examples.

The method of dominant balance relies on looking for local solutions of the form $y = e^{S(x)}$, as $x \to x_0$. The various steps are as follows.

- Substitute into the equation and retain only the dominant terms.
- Solve asymptotically for $S(x)$.
- Continue like this until the full leading order behaviour is obtained.
- Check that any assumptions made in the working are consistent.

We will illustrate the technique with an example.

**Example** Consider the equation

$$x^4 y'' = y \tag{4.2.1}$$

and look for a solution as $x \to 0$ of the form

$$y = e^{S(x)}.$$

Now

$$y'(x) = S'(x)e^{S(x)}, \quad y''(x) = (S'^2(x) + S''(x))e^{S(x)}. \tag{4.2.2}$$

Substitution of (4.2.2) into the equation (4.2.1) gives

$$x^4(S'^2 + S'') - 1 = 0. \tag{4.2.3}$$

We have to solve this for $S(x)$ as $x \to 0$. Let us assume that

$$S'(x) = cx^\alpha + \dots, \quad S''(x) = c\alpha x^{\alpha-1} + \dots.$$

Substitution into (4.2.3) gives

$$x^4(c^2x^{2\alpha} + c\alpha x^{\alpha-1}) \sim 1. \tag{4.2.4}$$

By balancing the various terms in (4.2.4) there appears to be various possibilities for choosing $\alpha$. For example

- $c^2x^{4+2\alpha} \sim 1 \quad \implies \alpha = -2.$
- $c^2x^{4+2\alpha} \sim -c\alpha x^{4+\alpha-1} \quad \implies \alpha = -1.$
- $c\alpha x^{4+\alpha-1} \sim 1 \quad \implies \alpha = -3.$

Only the first possibility is self consistent because choosing $\alpha = -1$ or 3 implies that the term omitted is larger than the ones retained for the balancing as $x \to 0$. Thus with $\alpha = -2$ and retaining the dominant terms gives

$$c^2 = 1, \quad \implies c = \pm, 1.$$

We can continue in this manner and set

$$S'(x) = cx^{-2} + A_1(x), \tag{4.2.5}$$

where $A_1(x) = o(x^{-2})$. Substitution into (4.2.3) gives

$$x^4(c^2x^{-4} + 2cx^{-2}A_1 + A_1^2) + x^4(-2cx^{-3} + A_1') \sim 1,$$

or

$$2cx^2A_1 + x^4A_1^2 - 2cx + x^4A' \sim 0.$$

Again looking for a term of the form $A_1(x) = c_1x^\beta$ and looking for a dominant balance suggests that

$$\beta = -1, \quad c_1 = 1.$$

Other possibilities lead to inconsistencies.

Thus

$$S'(x) = cx^{-2} + x^{-1} + A_2(x), \quad A_2(x) = o(x^{-1}).$$

The equation for $A_2$ is

$$x^2(1 + 2cA_2) + 2x^3 A_2 + x^4 A_2^2 - x^2 + x^4 A_2' \sim 0. \tag{4.2.6}$$

Note that (4.2.6) is identically satisfied by $A_2 = 0$ (not typical) giving

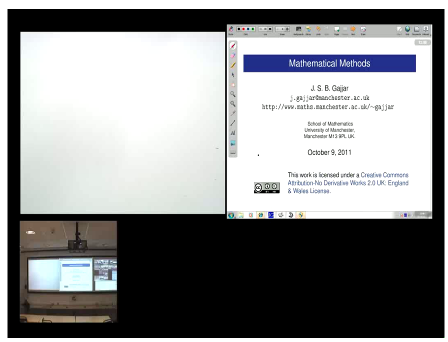
$$S'(x) = cx^{-2} + x^{-1}, \quad S(x) = -cx^{-1} + \log(x) + S_0.$$

Hence

$$y(x) = e^{S(x)} = Kxe^{\pm\frac{1}{x}}.$$

It can be verified that this satisfies the equation $x^4 y'' = y$ exactly.

The previous example was unusual in that the expansion for $S(x)$ terminated after a finite number of terms. This is not typical.




Video clip of worked example for $y'' - x^4 y = 0$'.
Click here to open video clip in external player.

**Example** Consider the equation

$$x^3 y'' - y = 0. \tag{4.2.7}$$

Note that $x = 0$ is an irregular singular point of (4.2.7). We will seek a solution of the form $y = e^{S(x)}$ as $x \to 0$. This gives

$$x^3(S'^2 + S'') = 1. \tag{4.2.8}$$

A dominant balance gives (with $c = \pm 1$)

$$S'(x) = cx^{-\frac{3}{2}} + A(x), \quad A(x) = o(x^{-\frac{3}{2}}). \tag{4.2.9}$$

Substituting (4.2.9) into (4.2.8) gives

$$x^3(c^2 x^{-3} + 2cx^{-\frac{3}{2}} A + A^2 - \frac{3}{2} cx^{-\frac{5}{2}} + A') \sim 1,$$

ie

$$2cx^{\frac{3}{2}}A + x^3A^2 - \frac{3c}{2}x^{\frac{1}{2}} + x^3A' \sim 0.$$

A dominant balance gives

$$A \sim \frac{3}{4x}.$$

Hence

$$S'(x) = cx^{-\frac{3}{2}} + \frac{3}{4}x^{-1} + B(x), \quad B(x) = o(x^{-1}). \tag{4.2.10}$$

The equation for $B$ after substituting (4.2.10) into (4.2.8) is

$$\frac{9}{16}x + 2cBx^{\frac{3}{2}} + \frac{3}{2}x^2B + x^3B^2 - \frac{3}{4}x + x^3B' \sim 0.$$

This gives

$$B(x) = \frac{3}{32c}x^{-\frac{1}{2}} + o(x^{-\frac{1}{2}}).$$

Hence

$$S'(x) = cx^{-\frac{3}{2}} + \frac{3}{4}x^{-1} + \frac{3}{32c}x^{-\frac{1}{2}} + \dots,$$

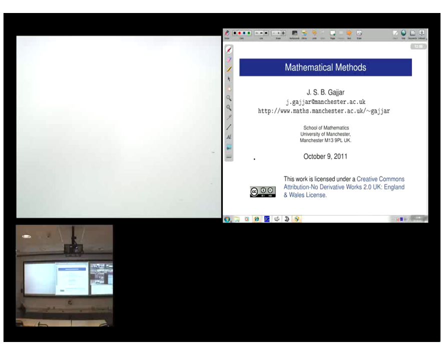
giving

$$S(x) = -2cx^{-\frac{1}{2}} + \frac{3}{4}\log(x) - \frac{3}{16c}x^{\frac{1}{2}}.$$

Thus the leading order behaviour of $y(x)$ as $x \to 0$ is

$$y \sim e^{S(x)} \sim x^{\frac{3}{4}}e^{-2cx^{-\frac{1}{2}}}U(x),$$

where $c = \pm 1$ and $U(x) = 1 + o(x^{\frac{1}{2}})$.




Video clip showing leading order behaviour of $x^3y'' - y = 0$ as $x \to \infty$. Click here to open video clip in external player.

The above gives the leading order asymptotic behaviour of the the solutions. The full asymptotic behaviour for $y(x)$ requires more work. To do this we first set

$$y(x) = e^{2cx^{-\frac{1}{2}}}W(x), \quad W(x) \sim \sum_{n=0}^{\infty} a_n x^{n\alpha + \frac{3}{4}}, \tag{4.2.11}$$

where $\alpha$ is to be found. We have

$$y' = [-cx^{-\frac{3}{2}}W + W']e^{2cx^{-\frac{1}{2}}},$$

$$y'' = [c^2x^{-3}W - 2cx^{-\frac{3}{2}}W' + \frac{3c}{2}x^{-\frac{5}{2}}W + W'']e^{2cx^{-\frac{1}{2}}}.$$

Substituting into the equation $x^3y'' - y = 0$ gives

$$W'' - 2cx^{-\frac{3}{2}}W' + \frac{3c}{2}x^{-\frac{5}{2}}W = 0. \tag{4.2.12}$$

If we seek an asymptotic expansion for $W(x)$ as $x \to 0$ in the form

$$W(x) \sim \sum_{n=0}^{\infty} a_n x^{n\alpha+\frac{3}{4}},$$

with ($a_0 \neq 0$) then substitution into (4.2.12) gives

$$\sum_{n=0}^{\infty} a_n(n\alpha+\frac{3}{4})(n\alpha+\frac{3}{4}-1)x^{n\alpha+\frac{3}{4}-2} - 2c\sum_{n=0}^{\infty} a_n(n\alpha+\frac{3}{4})x^{n\alpha+\frac{3}{4}-\frac{5}{2}} + \frac{3c}{2}\sum_{n=0}^{\infty} a_n x^{n\alpha+\frac{3}{4}-\frac{5}{2}} \sim 0. \tag{4.2.13}$$

Note that the coefficient of the dominant term $x^{\frac{3}{4}-\frac{5}{2}}$ in (4.2.13) is

$$a_0(-\frac{3c}{2} + \frac{3c}{2}) = 0,$$

which is satisfied identically leaving $a_0$ undetermined. Balancing the next terms in (4.2.13) suggests that

$$x^{-2} \sim x^{\alpha-\frac{5}{2}}$$

giving $\alpha = \frac{1}{2}$.

The coefficient of $x^{\frac{3}{4}+n\alpha-2}$ in (4.2.13) shows that

$$a_n(n\alpha + \frac{3}{4})(n\alpha + \frac{3}{4} - 1) - 2ca_{n+1}(n+1)\alpha = 0, \quad n = 0, 1, 2, \ldots$$

giving

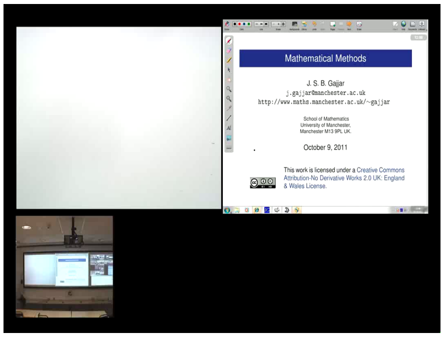
$$a_{n+1} = \frac{(2n+3)(2n-1)}{16c(n+1)}a_n, \quad n = 0, 1, 2, \ldots.$$

Hence an asymptotic expansion of the solution to

$$x^3y'' - y = 0$$

as $x \to 0$ is

$$y \sim Ae^{2cx^{-\frac{1}{2}}}x^{\frac{3}{4}}(1 - \frac{3}{16c}x^{\frac{1}{2}} + \ldots a_n x^{\frac{n}{2}} + \ldots),$$




Video clip of example showing full asymptotic expansion for $x^3y'' = y$. as $x \to \infty$. Click here to open video clip in external player.

where $A$ is an arbitrary constant, $c = \pm 1$ and $a_0 = 1$

$$a_{n+1} = \frac{(2n+3)(2n-1)}{16c(n+1)} a_n, \quad n = 0, 1, 2, \ldots .$$

**Example** Consider the equation

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0. \tag{4.2.14}$$

Note that $x = \infty$ is an irregular singular point. The leading order behaviour is easily obtained to be

$$y(x) \sim Cx^{-\frac{1}{2}}e^{\pm x} \quad \text{as} \quad x \to \infty.$$

We will obtain the full asymptotic behaviour as $x \to \infty$. Write

$$y(x) = Ae^{cx}W(x),$$

where

$$W(x) = x^{-\frac{1}{2}}(1 + o(x)), \quad \text{as} \quad x \to \infty,$$

$A$ is an arbitrary constant and $c = \pm 1$. If we substitute into the equation (4.2.14) we find that $W$ satisfies

$$x^2W'' + (2cx^2 + x)W' + (cx - \nu^2)W = 0.$$

We seek an asymptotic expansion of $W(x)$ as

$$W(x) \sim \sum_{n=0}^{\infty} a_n x^{n\alpha - \frac{1}{2}},$$

with $a_0 \neq 0$. Substitution into the equation for $W$ gives

$$\sum_{n=0}^{\infty} a_n(n\alpha - \frac{1}{2})(n\alpha - \frac{3}{2})x^{n\alpha-\frac{1}{2}} + \sum_{n=0}^{\infty} a_n(n\alpha - \frac{1}{2})(2cx^{n\alpha+\frac{1}{2}} + x^{n\alpha-\frac{1}{2}}) + \sum_{n=0}^{\infty} a_n(cx^{n\alpha+\frac{1}{2}} - \nu^2x^{n\alpha-\frac{1}{2}}) \sim 0. \tag{4.2.15}$$

The coefficient of the $a_0$ term in (4.2.15) is zero. The dominant balance in (4.2.15) suggests that

$$x^{-\frac{1}{2}} \sim x^{\alpha+\frac{1}{2}} \qquad \Longrightarrow \alpha = -1.$$

Equating the coefficients of $x^{n\alpha-\frac{1}{2}}$ in (4.2.15) to zero gives

$$(n\alpha - \frac{1}{2})(n\alpha - \frac{3}{2})a_n + a_{n+1}((n+1)\alpha - \frac{1}{2})2c + a_n(n\alpha - \frac{1}{2}) + a_{n+1}c - \nu^2 a_n = 0.$$

Hence

$$a_n[(n + \frac{1}{2})^2 - \nu^2] - 2c(n+1)a_{n+1} = 0, \quad n = 0, 1, \ldots\ldots,$$

giving

$$a_{n+1} = \frac{(n+\frac{1}{2})^2 - \nu^2}{2c(n+1)}, \quad n = 0, 1, \ldots. \tag{4.2.16}$$

The solutions of (4.2.14) therefore have the behaviour

$$y(x) \sim Ax^{-\frac{1}{2}}e^{cx}(1 + \sum_{n=1}^{\infty} a_n \frac{1}{x^n})$$

as $x \to \infty$, with $a_0 = 1$ and $a_n$ given by (4.2.16).

Note that the series terminates if

$$\nu = \pm(n + \frac{1}{2}) \quad n = 0, 1, \ldots,$$

in which case we have an exact solution of the equation.

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# Chapter 5

# Some special functions

## 5.1 Airy Functions

Airy functions arise often in asymptotic expansions and in the theory of differential equations. We will look at a few properties.

Ai$(z)$, Bi$(z)$ are called Airy functions and are two linearly independent solutions of

$$y'' - zy = 0. \tag{5.1.1}$$

Note that every point $\neq \infty$ is an ordinary point of the differential equation, and if we look for a Taylor series solution we find

$$y = \sum_{n=0}^{\infty} a_n z^n, \quad a_0 \neq 0$$

and

$$a_{3n} = \frac{\Gamma(\frac{2}{3})}{9^n n! \Gamma(n+\frac{2}{3})} a_0, a_{3n+1} = \frac{\Gamma(\frac{4}{3})}{9^n n! \Gamma(n+\frac{4}{3})} a_1, a_{3n+2} = 0, \tag{5.1.2}$$

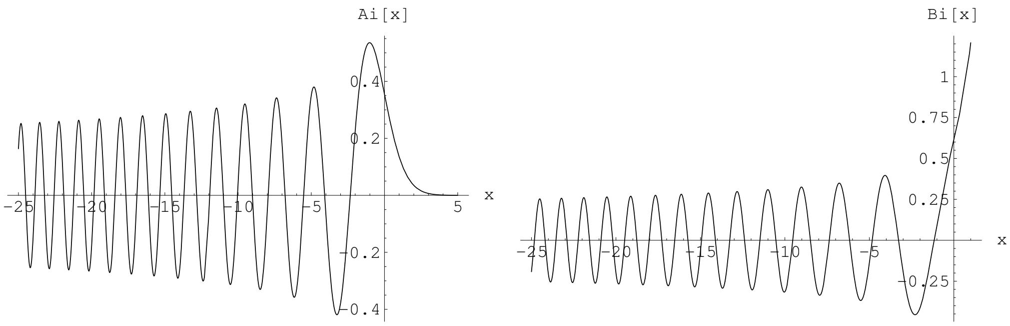
where $a_0, a_1$ are arbitrary constants - see lecture 2. Thus

$$y(z) = a_0 \Gamma(\frac{2}{3}) \sum_{n=0}^{\infty} \frac{z^{3n}}{9^n n! \Gamma(n+\frac{2}{3})} + a_1 \Gamma(\frac{4}{3}) \sum_{n=0}^{\infty} \frac{z^{3n+1}}{9^n n! \Gamma(n+\frac{2}{3})}.$$

The radius of convergence of the series is infinite since all points are ordinary points. We define Ai$(z)$, Bi$(z)$ by

$$\text{Ai(z)} = 3^{-2/3} \sum_{n=0}^{\infty} \frac{z^{3n}}{9^n n! \Gamma(n+\frac{2}{3})} - 3^{-4/3} \sum_{n=0}^{\infty} \frac{z^{3n+1}}{9^n n! \Gamma(n+\frac{4}{3})} \tag{5.1.3}$$

$$\text{Bi}(z) = 3^{-1/6} \sum_{n=0}^{\infty} \frac{z^{3n}}{9^n n! \Gamma(n+\frac{2}{3})} + 3^{-5/6} \sum_{n=0}^{\infty} \frac{z^{3n+1}}{9^n n! \Gamma(n+\frac{4}{3})}. \tag{5.1.4}$$




(a) (b)

Figure 5.1: Sample plots of the Airy function (a) $\mathrm{Ai}(x)$ and (b) $\mathrm{Bi}(x)$ on the real line. Notice the highly oscillatory behaviour for large negative $x$. $\mathrm{Ai}(x)$ decays exponentially for large positive $x$ and $\mathrm{Bi}(x)$ grows exponentially for large positive $x$.

For large $x \to \infty$

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}},$$
$$\mathrm{Bi}(x) \sim \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{\frac{2}{3}x^{\frac{3}{2}}}.$$

A study of the large $x$ behaviour of the differential equation yields

$$y(x) \sim C x^{-\frac{1}{4}} e^{\pm x^{\frac{3}{2}}},$$

but the constants appropriate to $\mathrm{Ai}(x), \mathrm{Bi}(x)$ can only be determined from an integral representation of the functions. For $x \to -\infty$ we can look for a solution of the form $y = e^{S(x)}$ as before. This leads to

$$S'' + S'^2 - x = 0. \tag{5.1.5}$$

Hence

$$S'(x) \sim \pm i(-x)^{\frac{1}{2}}, \quad S(x) \sim \pm\frac{2}{3} i(-x)^{\frac{3}{2}} \quad \text{as} \quad x \to -\infty.$$

Writing

$$S = \pm i\frac{2}{3}(-x)^{\frac{3}{2}} + B(x), \quad B(x) = o((-x)^{\frac{3}{2}}),$$

we find that after substitution into (5.1.5) that

$$B(x) \sim -\frac{1}{4}\log(-x).$$

Hence

$$y(x) \sim C(-x)^{-\frac{1}{4}} e^{\pm\frac{2}{3}i(-x)^{\frac{3}{2}}} \quad \text{as} \quad x \to -\infty.$$

### 5.1.1 Airy functions, behaviour for large $x$.

Since $\mathrm{Ai}(x), \mathrm{Bi}(x)$ are real for real arguments $x$ the behaviour as $x \to -\infty$ must be a linear combination of the above solutions. Hence

$$y(x) \sim C_1(-x)^{-\frac{1}{4}} \sin(\frac{2}{3}(-x)^{\frac{3}{2}}) + C_2(-x)^{-\frac{1}{4}} \cos(\frac{2}{3}(-x)^{\frac{3}{2}}), \tag{5.1.6}$$

as $x \to -\infty$.

Further terms in the expansion may be obtained by writing

$$y(x) \sim C_1(-x)^{-\frac{1}{4}} w_1(x) \sin(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}) + C_2(-x)^{-\frac{1}{4}} w_2(x) \cos(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}), \tag{5.1.7}$$

where the $\pi/4$ factor is inserted for convenience. After substitution into the equation (5.1.1) one can find the behaviours of $w_1(x), w_2(x)$. It is convenient to introduce $t = -x$ and rewrite (5.1.7) as

$$y(x) \sim W_1(t) \sin(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4}) + W_2(t) \cos(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4}), \tag{5.1.8}$$

Then

$$\frac{dy}{dt} = [W_1 t^{\frac{1}{2}} + W_2'] \cos(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4}) + [W_1' - W_2 t^{\frac{1}{2}}] \sin(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4}).$$

$$\frac{d^2y}{dt^2} = [-t^{\frac{1}{2}}(W_1 t^{\frac{1}{2}} + W_2') + W_1'' - \frac{1}{2}t^{-\frac{1}{2}} W_2 - W_2' t^{\frac{1}{2}}] \sin(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4})$$
$$+[t^{\frac{1}{2}}(W_1' - W_2 t^{\frac{1}{2}}) + W_2'' + \frac{1}{2}t^{-\frac{1}{2}} W_1 + W_1' t^{\frac{1}{2}}] \cos(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4}).$$

Hence substituting into Airy's equation in terms of $t$ ie

$$\frac{d^2y}{dt^2} + ty = 0,$$

and equating the coefficents of the sine and cosine terms to zero leads to

$$W_1'' - 2t^{\frac{1}{2}} W_2' - \frac{1}{2}t^{-\frac{1}{2}} W_2 = 0, \tag{5.1.9}$$

$$W_2'' + 2t^{\frac{1}{2}} W_1' + \frac{1}{2}t^{-\frac{1}{2}} W_1 = 0. \tag{5.1.10}$$

Next seek asymptotic expansion solutions to these equations in the form

$$W_1(t) = \sum_{n=0}^{\infty} a_n t^{-n\alpha - \frac{1}{4}}, \quad W_2(t) = \sum_{n=0}^{\infty} b_n t^{-n\beta - \frac{1}{4}}.$$

The equation (5.1.9) for $W_1$ leads to

$$\sum_{n=0}^{\infty} a_n(n\alpha+\frac{1}{4})(n\alpha+\frac{1}{4}+1)t^{-n\alpha}+\sum_{n=0}^{\infty} b_n(2(n\beta+\frac{1}{4})-\frac{1}{2})t^{-n\beta+\frac{3}{2}}=0, \quad (5.1.11)$$

and the equation (5.1.10)for $W_2$ to

$$\sum_{n=0}^{\infty} b_n(-n\beta+\frac{1}{4})(n\beta+\frac{1}{4}+1)t^{-n\beta}+\sum_{n=0}^{\infty} a_n(-2(n\alpha+\frac{1}{4})+\frac{1}{2})t^{-n\alpha+\frac{3}{2}}=0, \quad (5.1.12)$$

The dominant terms in (5.1.11,5.1.12) show that

$$0.b_0=0, \quad 0.a_0=0$$

leaving $a_0, b_0$ arbitrary.

At next order we obtain $\beta=\alpha=\frac{3}{2}$ and

$$b_{n+1}=-\frac{(6n+1)(6n+5)}{48(n+1)}a_n, \quad a_{n+1}=\frac{(6n+1)(6n+5)}{48(n+1)}b_n.$$

The choice of the constants

$$a_0=\frac{1}{\sqrt{\pi}}, \quad b_0=0$$

represents the behaviour of $\mathrm{Ai}(x)$ as $x\to-\infty$. In this case the terms

$$a_{2n+1}=0, n=0,1,\ldots, \quad b_{2n}=0, n=0,1,\ldots.$$

We obtain

$$\mathrm{Ai}(x)\sim(-x)^{-\frac{1}{4}}w_1(x)\sin[\frac{2}{3}(-x)^{\frac{3}{2}}+\frac{\pi}{4}]+(-x)^{-\frac{1}{4}}w_2(x)\cos[\frac{2}{3}(-x)^{\frac{3}{2}}+\frac{\pi}{4}]$$

where

$$w_1(x)\sim\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}c_{2n}x^{-3n}, \quad x\to-\infty,$$

$$w_2(x)\sim-\frac{1}{\sqrt{\pi}}(-x)^{-\frac{3}{2}}\sum_{n=0}^{\infty}c_{2n+1}x^{-3n} \quad x\to-\infty,$$

and

$$c_n=\frac{1}{2\pi}\left(\frac{3}{4}\right)^n\frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!}.$$

The behaviour of $\mathrm{Bi}(x)$ is described by the choice

$$a_0=0, b_0=\frac{1}{\sqrt{\pi}}.$$

In this case the terms

$$a_{2n} = 0, \quad n = 0, 1, \ldots, \quad b_{2n+1} = 0, \quad n = 0, 1, \ldots.$$

We obtain

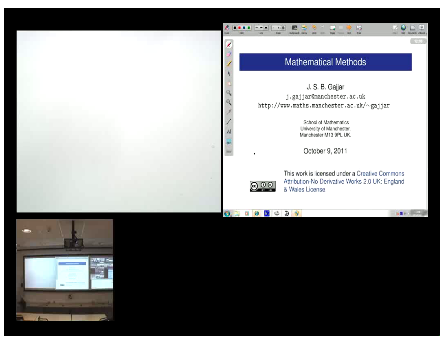
$$\mathrm{Bi}(x) \sim (-x)^{-\frac{1}{4}} w_1(x) \sin[\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}] + (-x)^{-\frac{1}{4}} w_2(x) \cos[\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}]$$

where

$$w_1(x) \sim \frac{1}{\sqrt{\pi}}(-x)^{-\frac{3}{2}} \sum_{n=0}^{\infty} c_{2n+1} x^{-3n}, \quad x \to -\infty,$$

$$w_2(x) \sim \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} c_{2n} x^{-3n} \quad x \to -\infty,$$

and $c_n$ are as before.




Video clip for discussion of the properties of Airy's functions. Click here to open video clip in external player.

## 5.2 Stokes's Phenomenon

If we write

$$f(z) \sim g(z) \quad \text{as} \quad z \to z_0$$

then it is unclear which path we are specifying as $z \to z_0$ in the complex plane.

For the equation

$$\frac{d^2y}{dz^2} - zy = 0$$

we found that

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}}, \quad \mathrm{Bi}(z) \sim \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} e^{+\frac{2}{3}z^{\frac{3}{2}}}. \tag{5.2.1}$$

But $\mathrm{Ai}(z)$ is an enitre function and its Taylor series (reftayairyai) converges for all finite values of $|z|$ whereas the right-hand side of (5.2.1) is a multi-valued function with branch points. How is this resolved. Note that

$$f(z) \sim g(z)$$

holds only in a certain sector. Since $\mathrm{Bi}(z)$ grows exponentially along the real axis it suggests to restrict $z$ such that

$$|\arg(z^{\frac{3}{2}})| < \frac{\pi}{2}, \quad \implies |\arg(z)| < \frac{\pi}{3}.$$

Thus the sector of validity for $\mathrm{Bi}(z)$ to have the behaviour as in (5.2.1) is $|\arg(z)| < \pi/3$.

In general if

$$f(z) \sim g(z) \quad \text{as} \quad z \to z_0$$

then

$$f(z) - g(z) = o(g(z)) \quad \text{as} \quad z \to z_0.$$

Now

$$f(z) = g(z) + (f(z) - g(z)).$$

We say that when $z$ lies in a certain sector $g(z)$ is *dominant* and $f(z)-g(z)$ small or *subdominant*. As the edges of the sector, or wedge, are approached $f(z)-g(z)$ is not small. Outside the sector $f-g$ becomes larger than $g$. This exchange of identities is called *Stokes's Phenomenon* after Stokes (1857) who first observed it.

The edges of the sector or wedge where the difference in behaviour occurs on different sides are called *Stokes Lines.* For some of the second order equations studied earlier, we observed that

$$y \sim e^{S_{1,2}(x)} \quad \text{as} \quad z \to z_0.$$

Stokes lines are defined by

$$\Re(S_1(z) - S_2(z)) = 0.$$

and *anti-Stokes* lines by

$$\Im(S_1(z) - S_2(z)) = 0.$$

**Example** Consider the Airy functions. The Stokes lines are given by

$$Re(z^{\frac{3}{2}}) = 0$$

giving

$$\arg(z) = \pm\frac{\pi}{3}, \pi, \quad |z| \to \infty.$$

The function $\mathrm{Bi}(z)$ has the behaviour

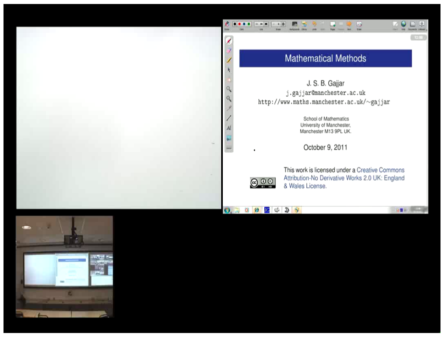
$$\mathrm{Bi}(z) \sim \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} e^{\frac{2}{3} z^{\frac{3}{2}}}$$

valid only in the sector $|\arg(z)| < \pi/3$.

However for $\mathrm{Ai}(z)$ it can be shown from the integral representation (see below) for $\mathrm{Ai}(z)$ that

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{\frac{3}{2}}},$$

as $z \to \infty$ holds in a much larger sector for $|\arg(z)| < \pi$.




Video clip for discussion of the Stokes' phenomenon.Click here to open video clip in external player.

## 5.3 Linear relations between Airy functions

In the equation

$$\frac{d^2y}{dz^2} - zy = 0$$

we can replace $zy$ by $\omega^3 zy$ where $\omega = e^{-2i\pi/3}$ is a cube root of unity.

Next put $t = \omega z$ and note that

$$\frac{d^2y}{dt^2} - ty = 0$$

so that $y = \mathrm{Ai}(\omega z)$ is also a solution of Airy's equation. Similarly $\mathrm{Ai}(z), \mathrm{Ai}(\omega z)$,$\mathrm{Ai}(\omega^2 z), \mathrm{Bi}(z)$ are all solutions of Airy's equation but we can only have two linearly independent solutions. Hence there exists $a, b$ such that

$$\mathrm{Ai}(z) = a\mathrm{Ai}(\omega z) + b\mathrm{Ai}(\omega^2 z).$$

From the Taylor series (5.1.3) for $\mathrm{Ai}(z)$ comparing the coefficents of the $z^0, z$ terms shows that

$$a + b = 1, \quad a\omega + b\omega^2 = 1.$$

Hence

$$a = -\omega, \quad b = -\omega^2.$$

Thus

$$\mathrm{Ai}(z) = -\omega\mathrm{Ai}(\omega z) - \omega^2\mathrm{Ai}(\omega^2 z), \tag{5.3.1}$$

and similarly

$$\mathrm{Bi}(z) = i\omega\mathrm{Ai}(\omega z) - i\omega^2\mathrm{Ai}(\omega^2 z). \tag{5.3.2}$$

These relations can be used to obtain asymptotic expansions for $\mathrm{Ai}(z), \mathrm{Bi}(z)$ valid in other sectors given that

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}} \sum_{n=0}^{\infty} (-1)^n c_n z^{-\frac{3n}{2}}, \quad |\arg(z)| < \pi. \tag{5.3.3}$$

with

$$c_n = \frac{1}{2\pi}\left(\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!}.$$

To use (5.3.3) with (5.3.1) we require that

$$-\pi < \arg(\omega z) < \pi, \quad \text{and} \quad -\pi < \arg(\omega^2 z) < \pi.$$

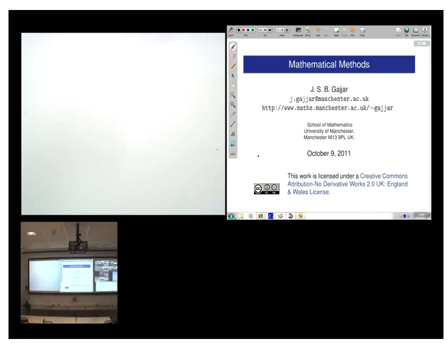
This implies that provided $\pi/3 < \arg(z) < 5\pi/3$, we can write

$$\mathrm{Ai}(z) \sim -\omega\left[\frac{1}{2\sqrt{\pi}}(\omega z)^{-\frac{1}{4}}e^{-\frac{2}{3}(\omega z)^{\frac{3}{2}}}\sum_{n=0}^{\infty}(-1)^n c_n(\omega z)^{-\frac{3n}{2}}\right]$$
$$-\omega^2\left[\frac{1}{2\sqrt{\pi}}(\omega^2 z)^{-\frac{1}{4}}e^{-\frac{2}{3}(\omega^2 z)^{\frac{3}{2}}}\sum_{n=0}^{\infty}(-1)^n c_n(\omega^2 z)^{-\frac{3n}{2}}\right].$$

Hence for $\pi/3 < \arg(z) < 5\pi/3$

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}}z^{-\frac{1}{4}}e^{-\frac{2}{3}z^{\frac{3}{2}}}\sum_{n=0}^{\infty}(-1)^n c_n z^{-\frac{3n}{2}} + \frac{i}{2\sqrt{\pi}}z^{-\frac{1}{4}}e^{\frac{2}{3}z^{\frac{3}{2}}}\sum_{n=0}^{\infty}(-1)^n c_n z^{-\frac{3n}{2}}.$$

Video clip for properties of Airy functions and asymptotic expansions across different sectors.Click here to open video clip in external player.




## 5.4 Integral represensations of Airy functions

Consider the Airy equation

$$\frac{d^2y}{dz^2} - zy = 0$$

and suppose we seek a solution in the form

$$y(z) = \int_c F(s)e^{sz}\,ds.$$

Substitution into the equation shows that

$$\int_c (s^2 - z)F(s)e^{sz}\,ds = 0.$$

Integrate by parts then

$$[-F(s)e^{sz}]_C + \int_C (s^2 F + \frac{dF}{ds}) e^{sz}\, ds = 0.$$

The first term above is to be evaluated at the endpoints of the curve $C$. Suppose we choose $F$ so that

$$\frac{dF}{ds} + s^2 F = 0,$$

ie

$$F(s) = e^{-\frac{s^3}{3}}.$$

For this to satisfy the equation we also need to choose a suitable contour $C$ so that

$$[F(s)e^{sz}]_C = [e^{-\frac{s^3}{3}+sZ}]_C = 0.$$

This gives rise to three sectors

$$-\frac{\pi}{6} < \arg(s) < \frac{\pi}{6}, \quad \frac{\pi}{2} < \arg(s) < \frac{5\pi}{6}, \quad -\frac{\pi}{2} < \arg(s) < -\frac{5\pi}{6}$$

where $|e^{-\frac{s^3}{3}}| \to 0$, provided the endpoints of the start and begin in these sectors. This gives rise to three functions

$$f_n = \frac{1}{2\pi i} \int_{C_n} e^{sz-\frac{s^3}{3}}\, ds,$$

where the curves are as in the fig. 5.2. The Airy function $Ai(z)$ is given by

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_{C_1} e^{sz-\frac{s^3}{3}}\, ds.$$

The Airy function of the second kind $\mathrm{Bi}(z)$ is given by

$$\mathrm{Bi}(z) = i[f_2(z) - f_3(z)].$$

## 5.5 Properties of parabolic cylinder functions

Consider the parabolic cylinder equations

$$y'' + (\nu + \frac{1}{2} - \frac{1}{4}z^2)y = 0.$$

All points except $z = \infty$ are ordinary points and one can readily obtain Taylor series solutions about $z = 0$. For $z \to \infty$ if we look for a solution of the form $y \sim e^{S(z)}$ we find that

$$y(x) \sim c_1 z^{-\nu-1} e^{\frac{z^2}{4}}, \quad \text{and} \quad y(x) \sim c_2 z^{\nu} e^{-\frac{z^2}{4}}, \quad z \to \infty.$$

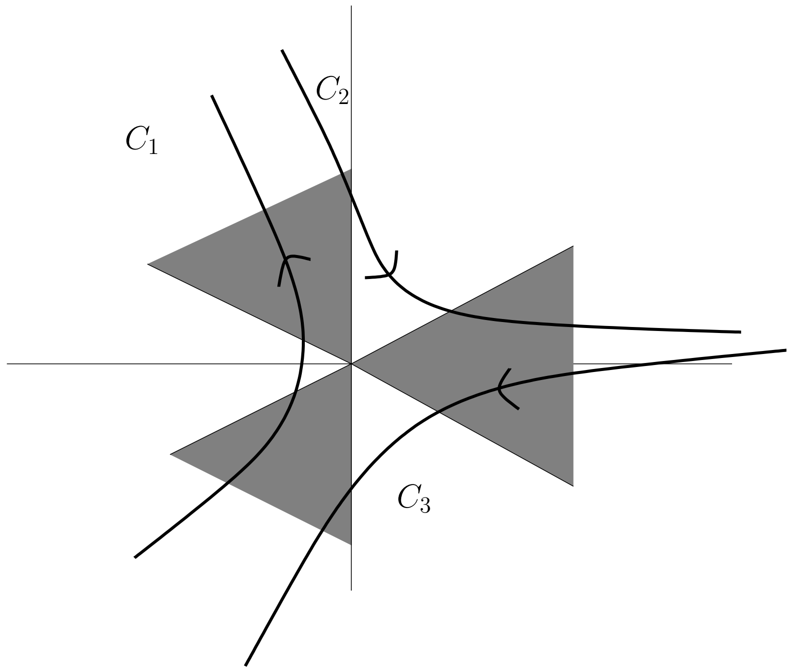
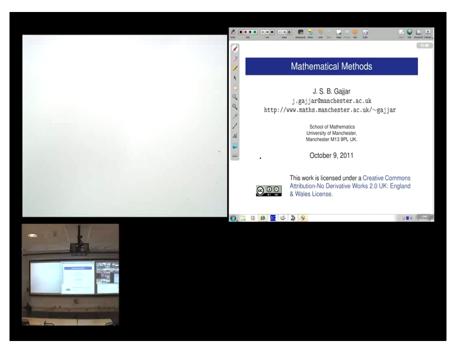



Figure 5.2: Various contours for solutions of the Airy equation in the integral representation

Video clip for discussion of the integral representation of Airy's functions.Click here to open video clip in external player.




The convention is to take $D_\nu(z)$ as the solution with the property that

$$y(z) \sim z^\nu e^{-\frac{z^2}{4}}, \quad z \to \infty.$$

Note that $D_\nu(-z)$ is also a solution and if we put $x = iz$ we find that the equation becomes

$$-\frac{d^2y}{dx^2} + (\nu + \frac{1}{2} + \frac{x^2}{4})y = 0,$$

ie

$$\frac{d^2y}{dx^2} + (-(\nu + \frac{1}{2}) - \frac{x^2}{4})y = 0,$$

or

$$\frac{d^2y}{dx^2} + (-(\nu + 1) + \frac{1}{2} - \frac{x^2}{4})y = 0.$$

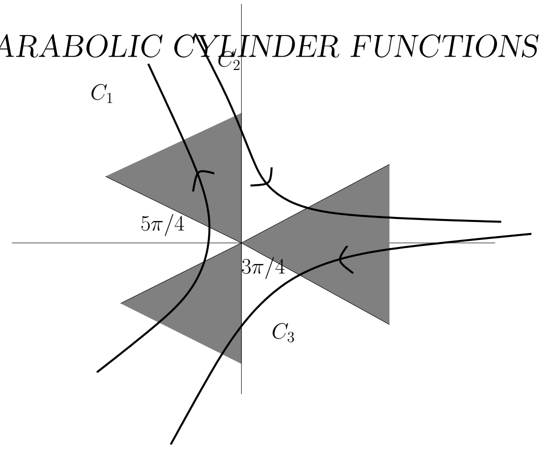



Figure 5.3: Stokes lines for parabolic cylinder function

Thus $y = D_{-\nu-1}(-iz)$ is also another (linearly independent) solution. A linear relationship must therefore exist between the three solutions and one can show that

$$D_\nu(z) = e^{i\nu z} D_\nu(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i(\nu+1)\frac{\pi}{2}} D_{-\nu-1}(-iz)$$

is valid for all $z$.

From the leading order asymptotic behaviour we see that the Stokes lines are given by

$$Re(z^2) = 0, \implies \arg(z) = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}.$$

The asymptotic behaviour for $D_\nu(z)$ as $z \to \infty$ can be obtained (see examples 2), and shows that

$$D_\nu(z) \sim z^\nu e^{-\frac{z^2}{4}} \sum_{n=0}^{\infty} (-1)^n a_n z^{-2n} \quad \text{as} \quad z \to \infty, \quad |\arg(z)| < \frac{3\pi}{4}. \qquad (5.5.1)$$

Here $a_0 = 1$, and

$$a_n = \frac{\nu(\nu-1)\dots(\nu-2n+1)}{2^n n!}.$$

Since $D_\nu(z)$ is subdominant in $|\arg(z)| < \pi/4$ the expression (5.5.1) is valid in the larger sector $|\arg(z)| < 3\pi/4$. [NB, as a rule of thumb this is generally true].

We can use the relation

$$D_\nu(z) = e^{i\nu z} D_\nu(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i(\nu+1)\frac{\pi}{2}} D_{-\nu-1}(-iz)$$

in conjunction with (5.5.1) to derive an expression valid for a larger sector.

Note that to use (5.5.1) with $D_\nu(-z)$ if we write $-z = |z|e^{-i\pi+i\theta}$ we require

$$-\frac{3\pi}{4} < -\pi + \theta < \frac{3\pi}{4}, \quad \implies \frac{\pi}{4} < \theta < \frac{7\pi}{4}.$$

Similarly to use (5.5.1) with $D_{-\nu-1}(-iz)$ if we write $-iz = |z|e^{-i\frac{\pi}{2}+i\theta}$ we require

$$\frac{\pi}{4} < \theta < \frac{5\pi}{4}.$$

Hence for $\pi/4 < \arg(z) < 5\pi/4$ using (5.5.1) we find that

$$D_\nu(z) \sim z^\nu e^{-\frac{z^2}{4}} \sum_{n=0}^{\infty} (-1)^n a_n z^{-2n} - \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)} e^{i\pi\nu} z^{-\nu-1} e^{\frac{z^2}{4}} \sum_{n=0}^{\infty} b_n z^{-2n}$$

$$\text{as} \quad z \to \infty, \quad \frac{\pi}{4} < \arg(z) < \frac{5\pi}{4}. \tag{5.5.2}$$

Here $b_0 = 1$, and

$$b_n = \frac{(\nu+1)(\nu+2)\dots(\nu+n)}{2^n n!}.$$

# Bibliography

Stokes, G. G. 1857 On the discontinuity of arbitrary constants which appear in divergent developments. *Trans. Camb. Philos. Soc.* **10**, 105–128.

# Chapter 6

# Properties of the Gamma function

Before we discuss Laplace and Mellin transforms we need a few properties of the Gamma function. Many of these can be found in standard texts such as the book by Olver (1974), or Whittaker & Watson (1927) *A course of modern analysis.*

## 6.1 Definition of the Gamma Function due to Weierstrass (1856)

*The Gamma function* $\Gamma(z)$ *is defined by the equation*

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\Pi_{n=1}^{\infty}\{(1+\frac{z}{n})e^{-\frac{z}{n}}\}, \tag{6.1.1}$$

*where the constant* $\gamma$ *is the Euler or Mascheroni constant*

$$\gamma = \lim_{m\to\infty}\{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}-\log m\} = 0.5772157\ldots.$$

The Gamma function was first defined by Euler in a different way (see below). Note that if

$$u_n = \int_0^1 \frac{t}{n(n+t)}\,dt = \frac{1}{n} - \log\frac{n+1}{n}$$

then

$$0 < u_n < \frac{1}{n^2}$$

and so $\sum_{n=0}^{\infty} u_n$ converges. Also

$$\begin{aligned}\gamma &= \lim_{m\to\infty}\{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}-\log m\}\\ &= \lim_{m\to\infty}\{\sum_{n=1}^{m} u_n + \log\frac{m+1}{m}\} = \sum_{n=0}^{\infty} u_n.\end{aligned}$$

Thus the constant $\gamma$ takes a finite value in the limit. Next consider (with the principal value of $\log(z)$, $\quad -\pi < \arg(z) < \pi$),

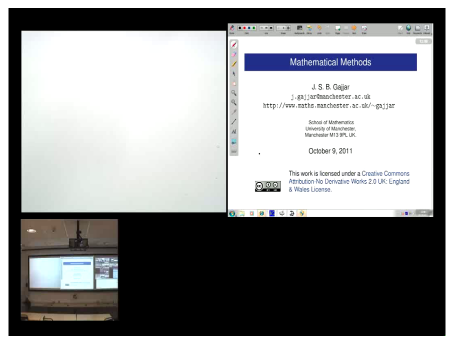
$$\left|\log(1+\frac{z}{n}) - \frac{z}{n}\right| = \left|-\frac{z^2}{2n^2} + \frac{z^3}{3n^3} - \dots\right|,$$

$$\leq \frac{|z|^2}{n^2}\left(1 + \frac{|z|}{n} + \frac{|z|^2}{n^2} + \dots\right).$$

Let integer $N$ be such that $|z| \leq N/2$ and then for $n > N$, we have $|z|/n < 1/2$ and so

$$\left|\log(1+\frac{z}{n}) - \frac{z}{n}\right| \leq \frac{1}{4}\frac{N^2}{n^2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots\right) \leq \frac{1}{2}\frac{N^2}{n^2}.$$

Video clip on Weierstrass definition of Gamma function. Click here to open video clip in external player.




Thus it follows that when $|z| \leq N/2$ the series

$$\sum_{n=N+1}^{\infty}\left(\log(1+\frac{z}{n}) - \frac{z}{n}\right)$$

is an absolutely and uniformly convergent series of analytic functions, and so its exponential

$$\Pi_{n=N+1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right]$$

is an analytic function. Thus

$$\Pi_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right]$$

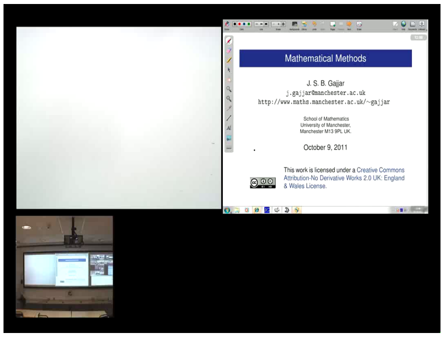
is an analytic function for all finite values of $z$. The Gamma function $\Gamma(z)$ defined by

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\Pi_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right]$$

is analytic except for the points $z = 0, -1, -2, \dots$ where it has simple poles.

It is easy to prove from this that

$$\Gamma(1) = 1, \quad \Gamma'(1) = -\gamma.$$




Video clip showing working for $\Gamma(1) = 1$. Click here to open video clip in external player.

## 6.2 Euler's (1729) definition of the Gamma function

Now

$$\frac{1}{\Gamma(z)} = z \lim_{m\to\infty} e^{\left(1+\frac{1}{2}+\ldots\frac{1}{m}-\log m\right)z} \left[\lim_{m\to\infty} \Pi_{n=1}^{m}\left[\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right]\right],$$

$$= z \lim_{m\to\infty} \left[e^{\left(1+\frac{1}{2}+\ldots\frac{1}{m}-\log m\right)z}\Pi_{n=1}^{m}\left[\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right]\right],$$

$$= z \lim_{m\to\infty} m^{-z}\Pi_{n=1}^{m}(1+\frac{z}{n}).$$

Thus except at the points $z = 0, -1, -2, \ldots$ we have

$$\Gamma(z) = \lim_{m\to\infty} \frac{1.2.\ldots(m)}{z(z+1)\ldots(z+m)} m^{z}.$$

This formula is due to Euler (1729). Note that if $z$ is not a negative integer, using Euler's formula

$$\frac{\Gamma(z+1)}{\Gamma(z)} =$$

$$\lim_{m\to\infty} \frac{1.2.\ldots(m)}{(z+1)(z+2)\ldots(z+m+1)} m^{z+1} \frac{z(z+1)\ldots(z+m)}{1.2.\ldots m} \frac{1}{m^{z}},$$

$$= \lim_{m\to\infty} \frac{mz}{z+m+1} = z.$$

Thus

$$\Gamma(z+1) = z\Gamma(z).$$

Using this for positive integer $n$, gives $\Gamma(n) = (n-1)!$.

## 6.3 Integral representation of the Gamma function

Consider

$$\Pi_n(z) = \int_0^n (1-\frac{t}{n})^n t^{z-1}\, dt,$$

where $\Re(z) > 0$. Note that

$$\Pi_n(z) = n^z \int_0^1 (1-t)^n t^{z-1}\, dz$$

and after integrating by parts a few times we obtain

$$\Pi_n(z) = \frac{n(n-1)\dots 1}{z(z+1)\dots(z+n-1)} \int_0^1 t^{n+z-1} dt,$$

ie

$$\Pi_n(z) = \frac{1.2.\dots n}{z(z+1)\dots(z+n)} n^z.$$

Taking the limit shows that $\Pi_n(z) \to \Gamma(z)$ as $n \to \infty$. Hence

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n (1 - \frac{t}{n})^n t^{z-1}\, dt.$$

Once can show that

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n (1 - \frac{t}{n})^n t^{z-1}\, dt = \int_0^\infty e^{-t} t^{z-1}\, dt.$$

where we have made use of the result that

$$e^t = \lim_{n\to\infty} (1 + \frac{t}{n})^n$$

## 6.4 Mittag-Leffler (1880) expansions and infinite products

An important identity is

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad z \neq 0, \pm 1, \pm 2, \dots.$$

From the Euler definition we have

$$\frac{1}{\Gamma(z)\Gamma(1-z)} =$$

$$\lim_{n\to\infty} \left\{ \frac{z(z+1)\dots(z+n)(1-z)(2-z)\dots(n+1-z)}{n! n^z n! n^{1-z}} \right\}$$

$$= z\Pi_{n=1}^\infty \left(1 - \frac{z^2}{n^2}\right) = \frac{\sin \pi z}{\pi}.$$

Put $z = \frac{1}{2}$ to obtain $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

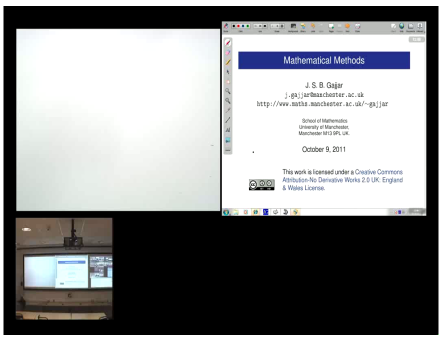
In the last section use is made of the more general result stated below.

*Suppose that $f(z)$ is an analytic function for all values of $z$ and which has simple zeros at $a_1, a_2, \ldots$ and $\lim_{n\to\infty} |a_n|$ is infinite. Then*

$$f(z) = f(0) e^{\frac{f'(0)}{f(0)} z} \Pi_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}} \right\}.$$

For the function $f(z) = \frac{\sin z}{z}$ we have $f(0) = 1$, $f'(0) = 0$ and the function has simple zeros at $z = \pm n\pi, \quad n = 1, 2, \ldots$. So

$$\frac{\sin z}{z} = \Pi_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{n\pi}\right) e^{\frac{z}{n\pi}} \right\} \left\{ \left(1 + \frac{z}{n\pi}\right) e^{-\frac{z}{n\pi}} \right\} = \Pi_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right).$$




Video clip showing proof of $\Gamma(z)\Gamma(1-z) = \pi \sin \pi z$. Click here to open video clip in external player.

The proof of the previous result requires the following theorem which shows how to construct a Mittag-Leffler expansion.

**Theorem**

*Suppose $g(z)$ is an analytic function whose only singularities are simple poles $a_1, a_2, \ldots$ where $|a_1| \le |a_2| \le \ldots$. Let $b_1, b_2, \ldots$ be the residues at these poles. We will assume that we can construct a sequence of circles $C_m$ with centre at $O$ not passing through the poles of $g(z)$ and such that $g(z)$ is bounded on $C_m$ ie $|g(z)| < M$ on $C_m$ and $M$ is independent of $m$.*

*Then if $x$ is not a pole of $g(z)$*

$$g(x) = g(0) + \sum_{n=1}^{\infty} b_n \left[ \frac{1}{x - a_n} + \frac{1}{a_n} \right].$$

**Proof**

We have from Cauchy's Theorem

$$\frac{1}{2\pi i} \int_{C_m} \frac{g(z)}{z - x}\, dz = g(x) + \sum_r \frac{b_r}{a_r - x}$$

the summation being over all poles inside $C_m$. But

$$\frac{1}{2\pi i} \int_{C_m} \frac{g(z)}{z - x}\, dz = \frac{1}{2\pi i} \int_{C_m} \frac{g(z)}{z}\, dz + \frac{x}{2\pi i} \int_{C_m} \frac{g(z)}{z(z - x)}\, dz$$

$$= g(0) + \sum_r \frac{b_r}{a_r} + \frac{x}{2\pi i} \int_{C_m} \frac{g(z)}{z(z-x)}\, dz.$$

Now

$$\left| \int_{C_m} \frac{g(z)}{z(z-x)} \right| \leq \frac{M}{R_m}$$

where $R_m$ is the radius of $C_m$. Let $m \to \infty$ and subtracting the two expansions gives the required result

$$g(x) = g(0) + \sum_{n=1}^{\infty} b_n \left[ \frac{1}{x-a_n} + \frac{1}{a_n} \right].$$

We apply the previous result to the function

$$g(z) = \frac{f'(z)}{f(z)}$$

where $f(z)$ is an analytic function for all $z$ and it has simple zeros at the points $a_1, a_2, \ldots,$ where $\lim_{m\to\infty} |a_m|$ is infinite.

Then $f'(z)$ is analytic and since

$$f(z) = (z-a_r)f'(a_r) + \frac{1}{2}(z-a_r)^2 f''(a_r) + \ldots,$$

$$f'(z) = f'(a_r) + (z-a_r)f''(a_r) + \ldots$$

the function $g(z) = \frac{f'(z)}{f(z)}$ has a simple pole at $z = a_r$ with residue 1.

Then if we can find a sequence of circles such that $f'(z)/f(z)$ is bounded on $C_m$ as $m \to \infty$ it follows that

$$\frac{f'(z)}{f(z)} = \frac{f'(0)}{f(0)} + \sum_{n=1}^{\infty} \left[ \frac{1}{z-a_n} + \frac{1}{a_n} \right].$$

Integrating and taking the exponential gives

$$f(z) = ce^{\frac{zf'(0)}{f(0)}} \Pi_{n=1}^{\infty} \left[ \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}} \right].$$

Putting $z = 0$ gives $c = f(0)$ and hence

$$f(z) = f(0)e^{\frac{zf'(0)}{f(0)}} \Pi_{n=1}^{\infty} \left[ \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}} \right].$$

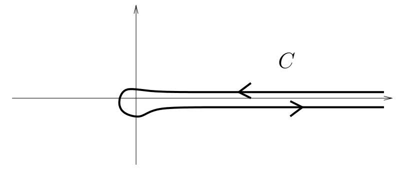



Figure 6.1: Loop contour for Hankel's integral representation of $\Gamma(z)$.

## 6.5 Hankel's (1864) loop integral for $\Gamma(z)$.

An alternative integral representation of $\Gamma(z)$ was given by Hankel and is

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi}\int_{\mathcal{C}} (-t)^{-z}e^{-t}\,dt$$

where $\mathcal{C}$ is the loop contour, see figure 6.1), which starts at $\infty + i0$ encircles the origin and tends to $\infty - i0$. Alternatively

$$I(z) = \frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\mathcal{D}} t^{-z}e^{t}\,dt \tag{6.5.1}$$

where $\mathcal{D}$ is the contour as in the figure 6.2.

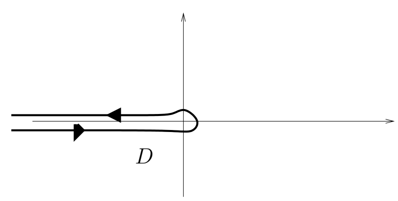



Figure 6.2: Loop contour for Hankel's integral representation of $\Gamma(z)$.

We will use the second form, and the branch of $t^{-z} = e^{-z\log t}$ is the principal branch of the log function, ie $-\pi < \arg(t) < \pi$.

The integrand in (6.5.1) is an analytic function in the restricted region and so by Cauchy's theorem the loop integral can be deformed to the paths starting at $-\infty - i0$ looping around the origin and ending up at $-\infty + i0$.

The integral around the loop is

$$\frac{1}{2\pi i}\int_{-\pi}^{\pi} i\epsilon e^{i\theta}e^{\epsilon e^{i\theta}}e^{-z(\log\epsilon + i\theta)}\,d\theta \to 0$$

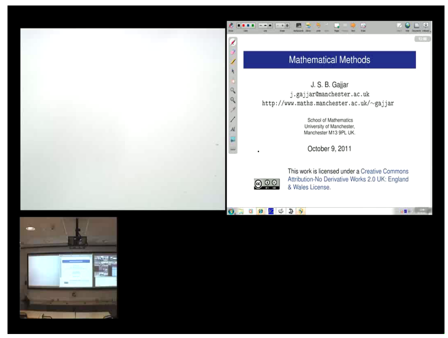
as we take the radius of the loop $\epsilon \to 0$. On the lower part of $\mathcal{D}$ put $t = \tau e^{-i\pi}$ and on the upper part $t = \tau e^{i\pi}$ to get

$$I(z) = -\frac{1}{2\pi i}\int_{\infty}^{0} e^{-\tau}\tau^{-z}e^{i\pi z}\,d\tau - \frac{1}{2\pi i}\int_{0}^{\infty} e^{-\tau}\tau^{-z}e^{-i\pi z}\,d\tau$$

$$= \frac{1}{\pi}\sin(\pi z)\int_0^\infty e^{-\tau}\tau^{-z}\,d\tau. = \frac{1}{\pi}\sin(\pi z)\Gamma(1-z).$$

Using the identity $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, we obtain

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\mathcal{D}} t^{-z}e^t\,dt.$$




Video clip showing Hankel's integral representation of $\Gamma(z)$. Click here to open video clip in external player.

## 6.6 Stirling's formula for $\Gamma(z)$ for large $z$.

We will also need the asymptotic form for $\Gamma(z)$ for $|z|$ large and this is given by Stirling's formula:

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z\left[1 + O(\frac{1}{z})\right], \quad |z| \to \infty, |\arg(z)| < \pi.$$

This can be derived from the integral representation above, see later in the course.

# Bibliography

Olver, F. W. J. 1974 *Introduction to Asymptotics and Special Functions*. Dover.

Whittaker, E. T. & Watson, G. N. 1927 *A course of Modern Analysis*. Cambridge University Press.

# Chapter 7

# Matched expansions, Boundary Layer Theory, WKB method.

## 7.1 Boundary layer theory - regular and singular perturbation problems.

In this section we will consider boundary layer and WKB theory for obtaining asymptotic solutions to differential equations whose highest derivatives are multiplied by a small parameter $\epsilon$. We will find that the solutions change rapidly in thin regions as $\epsilon \to 0$. A **singular perturbation** problem is characterised by the fact that the $\epsilon = 0$ problem has quite different solution properties as compared to the $0 < \epsilon << 1$ problem. In a **regular perturbation** problem as $\epsilon \to 0$ the solution tends to the solution for $\epsilon = 0$. This is best illustrated by looking at some simple examples.

**Example**

Consider

$$y'' + 2\epsilon y' - y = 0, \quad y(0) = 0, \quad y(1) = 1 \tag{7.1.1}$$

and $0 < \epsilon << 1$. The general solution is

$$y(x, \epsilon) = \frac{e^{m_1 x} - e^{m_2 x}}{e^{m_1} - e^{m_2}},$$

where

$$m_1 = -\epsilon + \sqrt{1 + \epsilon^2}, \quad m_2 = -\epsilon - \sqrt{1 + \epsilon^2}.$$

As $\epsilon \to 0$ we have

$$y(x) \to \frac{\sinh(x)}{\sinh(1)},$$

and everything seems ok.

We can also obtain a solution as follows: Write

$$y = Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \ldots.$$

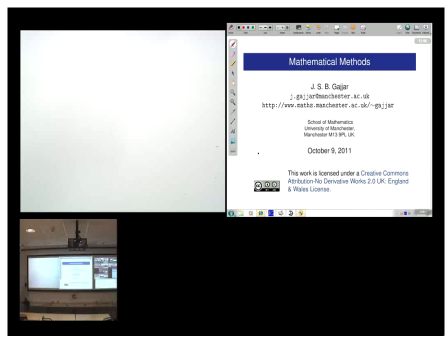
Substitution into the equation (7.1.1) and equating coefficents of like powers of $\epsilon$ to zero gives

$$\begin{aligned} Y_0'' - Y_0 &= 0, \quad Y_0(0) = 0, \quad Y_0(1) = 1 \\ Y_1'' - Y_1 &= -2Y_0', \quad Y_1(0) = 0, Y_1(1) = 0. \end{aligned} \tag{7.1.2}$$

Solving (7.1.2) gives

$$Y_0 = \frac{\sinh(x)}{\sinh(1)}, \quad Y_1 = (1-x)\frac{\sinh(x)}{\sinh(1)}.$$

Again there are no problems - we have a regular perturbation problem.




Video clip for regular perturbation problem example. Click here to open video clip in external player.

**Example**

Consider

$$\epsilon y'' + 2y' - y = 0, \quad y(0) = 0, \quad y(1) = 1, \tag{7.1.3}$$

for $0 < \epsilon << 1$. The solution is as before

$$y(x, \epsilon) = \frac{e^{m_1 x} - e^{m_2 x}}{e^{m_1} - e^{m_2}},$$

where now

$$m_1 = \frac{1}{\epsilon}(-1 + \sqrt{1+\epsilon}), \quad m_2 = \frac{1}{\epsilon}(-1 - \sqrt{1+\epsilon}).$$

As $\epsilon \to 0$ we have

$$m_1 \to \frac{1}{2}, \quad m_2 \sim -\frac{2}{\epsilon}.$$

Note that as $\epsilon \to 0$

$$y \sim \frac{1}{(e^{\frac{1}{2}} - e^{-\frac{2}{\epsilon}})}(e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}}) \sim e^{-\frac{1}{2}}(e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}}).$$

Clearly there are two distinct regions:

- $\frac{x}{\epsilon} = O(1)$, and then
$$y \sim e^{-\frac{1}{2}}(e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}}).$$
- $x >> \epsilon$ and then
$$y \sim e^{-\frac{1}{2}}e^{\frac{x}{2}}.$$

The analytic solution for different values of $\epsilon$ is shown in Fig. 7.1. Note that

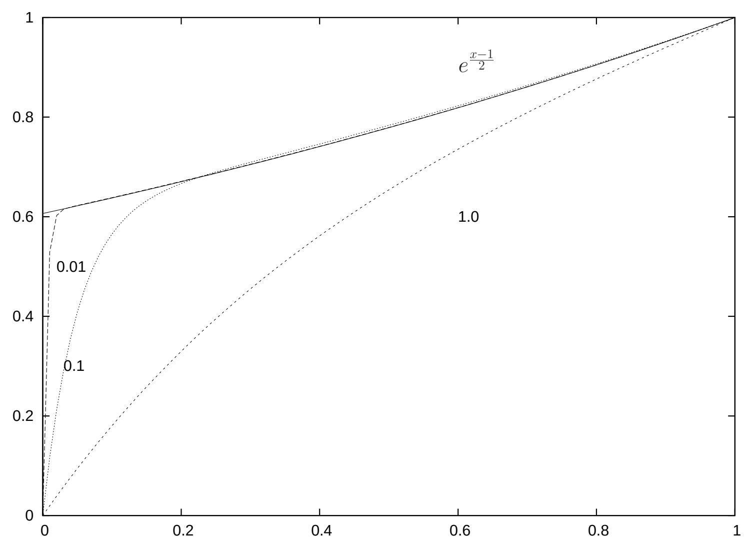



Figure 7.1: Solution $y(x, \epsilon)$ for different values of $\epsilon$.

the solution changes rapidly in the region $x = O(\epsilon)$. We have an example of a singular limit as $\epsilon \to 0$. The region $x = O(\epsilon)$ is called a *boundary layer.*

Suppose we try solving the equation as before. Put

$$y = Y_0 + \epsilon Y_1 + \dots .$$

This gives after substitution into (7.1.3)

$$2Y_0' - Y_0 = 0, \quad 2Y_1' - Y_1 = Y_0'', \tag{7.1.4}$$

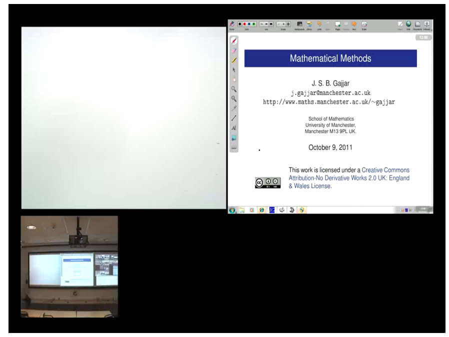
and boundary conditions

$$Y_0(0) = 0, \quad Y_0(1) = 1,$$

$$Y_1(0) = 0, \quad Y_1(1) = 0,$$

etc. Now there is a problem! The order of the equations (7.1.4) is reduced, ie we now have first order equations for the $Y_i$. Consequently which boundary conditions do we choose? The exact solution suggests we can satisfy the condition at $x = 1$. Let us continue with the boundary condition at $x = 1$.

Video clip showing working for singular perturbation example.Click here to open video clip in external player.




Solution of first order problem

$$2Y_0' - Y_0 = 0, \quad Y_0(1) = 1,$$

gives

$$Y_0 = e^{\frac{x-1}{2}}.$$

Clearly this solution is not valid for all $x$ since the condition at $x = 0$ is not satisfied. When $x$ is small the solution fails and we need to examine this region in more detail. The $Y_0$ solution is the leading order *outer solution*. Now when $x$ is small we have

$$Y_0 \sim e^{-\frac{1}{2}}(1 + \frac{x}{2}) = O(1).$$

Put $x = \epsilon^n X$ say where $n > 0$ is to be found. The variable $X$ is called the inner variable and is $O(1)$ in the *inner region* of thickness $O(\epsilon^n)$. The differential equation (7.1.3) in terms of $X$ is

$$\epsilon^{1-2n}\frac{d^2y}{dX^2} + 2\epsilon^{-n}\frac{dy}{dX} - y = 0. \tag{7.1.5}$$

For $n > 0$ the dominant terms are the first two terms and these balance if

$$1 - 2n = -n \quad \Longrightarrow n = 1.$$

A quick consistency check shows that this is ok, (other choices for $n$ eg $n = 1/2$ are not ). In the inner region if we put

$$y = y_0(X) + \epsilon^\alpha y_1(X) + \dots$$

with $\alpha > 0$ and substitute into (7.1.5) (with $n = 1$) we find that the leading order problem is

$$\frac{d^2y_0}{dX^2} + 2\frac{dy_0}{dX} = 0,$$

and one boundary condition is $y_0(X = 0) = 0$.

The other condition must come from *matching* with the outer solution taking $X$ large. Solving yields

$$y_0(X) = A + Be^{-2X}$$

and $y_0(0) = 0$ implies that $A = -B$. Thus

$$y_0(X) = A(1 - e^{-X}).$$

To obtain the constant $A$ we match the inner solution just derived with the outer solution.

$$y_0(X) \sim A \quad \text{for} \quad X >> 1,$$

and

$$Y_0(x) \sim e^{-\frac{1}{2}} \quad \text{for} \quad x \to 0.$$

This gives $A = e^{-\frac{1}{2}}$ and

$$y_0(X) = e^{-\frac{1}{2}}(1 - e^{-2X}).$$

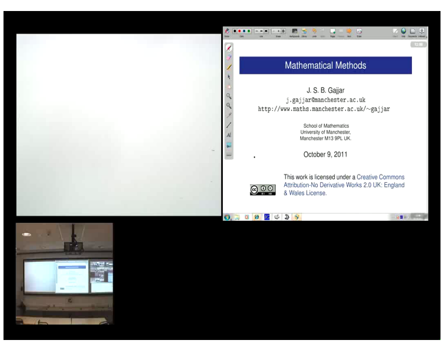
**Summary so far:** 1 term inner and 1 term outer expansions.

**outer:**

$$x = O(1), \quad y = Y_0(x) + \epsilon Y_1(x) + \ldots,$$
$$Y_0(x) = e^{-\frac{1}{2}} e^{\frac{x}{2}}.$$

**inner:**

$$x = \epsilon X, \quad y = y_0(X) + \epsilon^{\alpha} y_1(X) + \ldots,$$
$$y_0(X) = e^{-\frac{1}{2}}(1 - e^{-2X}).$$




Video clip for discussion boundary layer solution. Click here to open video clip in external player.

These are the basics of boundary layer theory and matched asymptotic expansions. The solution can be continued to higher order. Notice that the outer solution expanded for small $x$ gives

$$y \sim e^{-\frac{1}{2}}(1 + \frac{x}{2} + \ldots) + \epsilon Y_1(x) + \ldots.$$

When written in terms of $x = \epsilon X$ this suggests that the inner solution should proceed as

$$y = y_0 + \epsilon y_1 + \ldots.$$

We had assumed that the outer expansion proceeded in powers of $\epsilon$ but this does not have to be the case. One needs to proceed on a term by term basis matching the inner and outer solutions systematically and this will inform how the addtional terms behave. We will continue to the next order for both the inner and outer solutions. Now for the outer solution

$$y = Y_0 + \epsilon Y_1 + \dots,$$

and the problem for $Y_1$ is

$$2Y_1' - Y_1 = Y_0'' = \frac{1}{4}e^{\frac{x-1}{2}}, \quad Y_1(1) = 0.$$

Solving gives

$$Y_1 = \frac{(x-1)}{8}e^{\frac{x-1}{2}}.$$

For the inner problem, we have $x = \epsilon X$ and

$$y = y_0(X) + \epsilon y_1(X) + \dots.$$

The problem for $y_1$ is

$$\frac{d^2y_1}{dX^2} + 2\frac{dy_1}{dX} = y_0 = e^{-\frac{1}{2}}(1 - e^{-2X}), \quad y_1(X=0) = 0. \tag{7.1.6}$$

The solution of (7.1.6) gives

$$y_1 = A(1 - e^{-2X}) + \frac{1}{2}X(1 + e^{-2X})e^{-\frac{1}{2}},$$

where we have incorporated the boundary condition and $A$ is an arbitrary constant to be determined from matching with the outer solution. The outer solution expanded for small $x$ gives

$$y_{outer} = e^{\frac{x-1}{2}} + \epsilon\frac{1}{8}(x-1)e^{\frac{x-1}{2}} + \dots,$$

$$\sim e^{-\frac{1}{2}}(1 + \frac{x}{2} + \dots) + \epsilon e^{-\frac{1}{2}}(\frac{(x-1)}{8}(1 + \frac{x}{2} + \dots)).$$

Written in terms of inner variables this is

$$y_{outer} \sim e^{-\frac{1}{2}} + \epsilon e^{-\frac{1}{2}}(\frac{X}{2} - \frac{1}{8}) + \dots.$$

The two term inner solution is

$$y_{inn} = e^{-\frac{1}{2}}(1 - e^{-2X}) + \epsilon[A(1 - e^{-2X}) + \frac{1}{2}X(e^{-2X} + 1)e^{-\frac{1}{2}}] + \dots,$$

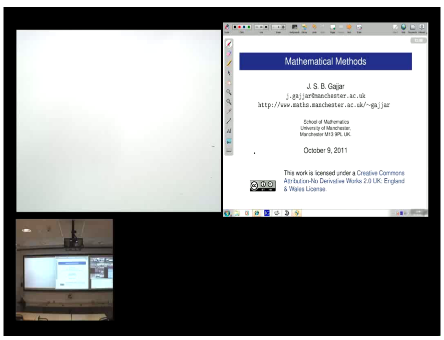
$$\sim e^{-\frac{1}{2}} + \epsilon(A + \frac{1}{2}Xe^{-\frac{1}{2}}) + \ldots, \tag{7.1.7}$$

as $X \to \infty$. This has to match with the two term outer solution written in terms of inner variables, i.e.,

$$y_{outer} \sim e^{-\frac{1}{2}} + \epsilon e^{-\frac{1}{2}}(\frac{X}{2} - \frac{1}{8}) + \ldots. \tag{7.1.8}$$

A match is only possible if $A = -\frac{1}{8}e^{-\frac{1}{2}}$. Thus

$$y_1 = -\frac{1}{8}e^{-\frac{1}{2}}(1 - e^{-2X}) + \frac{X}{2}e^{-\frac{1}{2}}(1 + e^{-2X}).$$




Video clip showing example of higher-order matching.Click here to open video clip in external player.

## 7.2 Uniform approximations

A uniform approximation to the solution valid in the whole region is defined by

$$y_{unif} = Y_{outer} + y_{inn} - y_{match}$$

where $y_{match}$ is the approximation to $y(x)$ in the matching region.

For the above problem we had

$$Y_{outer} = e^{-\frac{1}{2}}e^{\frac{x}{2}} + \epsilon\frac{e^{-\frac{1}{2}}}{8}(x-1)e^{-\frac{x}{2}} + O(\epsilon^2).$$

$$y_{inn} = e^{-\frac{1}{2}}(1 - e^{-\frac{2x}{\epsilon}}) + \epsilon[-\frac{1}{8}e^{-\frac{1}{2}}(1 - e^{-\frac{2x}{\epsilon}}) + \frac{e^{-\frac{1}{2}}}{2}\frac{x}{\epsilon}(1 + e^{-\frac{2x}{\epsilon}})] + \ldots.$$

The matching region is $X(= x/\epsilon) >> 1$ and $x << 1$, ie,

$$\epsilon << x << 1.$$

Thus a one-term uniform approximation is

$$y_{unif} = e^{-\frac{1}{2}}e^{\frac{x}{2}} + e^{-\frac{1}{2}}(1 - e^{-\frac{2x}{\epsilon}}) - e^{-\frac{1}{2}}$$

ie

$$y_{unif} = e^{-\frac{1}{2}}[e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}}].$$

A two term uniform approximation is

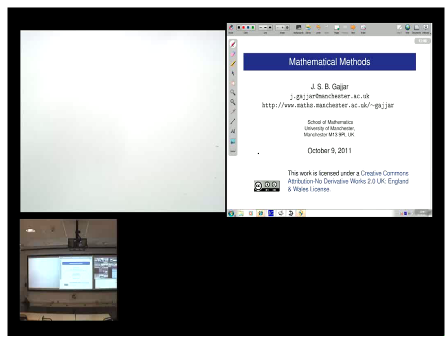
$$y_{unif} = e^{-\frac{1}{2}}e^{\frac{x}{2}} + \epsilon\frac{e^{-\frac{1}{2}}}{8}(x-1)e^{-\frac{x}{2}}$$

$$+e^{-\frac{1}{2}}(1-e^{-\frac{2x}{\epsilon}}) + \epsilon[-\frac{1}{8}e^{-\frac{1}{2}}(1-e^{-\frac{2x}{\epsilon}}) + \frac{e^{-\frac{1}{2}}}{2}\frac{x}{\epsilon}(1+e^{-\frac{2x}{\epsilon}})]$$

$$-[e^{-\frac{1}{2}} + \epsilon(-\frac{1}{8} + \frac{x}{2\epsilon})e^{-\frac{1}{2}}].$$

ie

$$y_{unif} = e^{-\frac{1}{2}}(e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}} + \frac{x}{2}e^{-\frac{2x}{\epsilon}}) + \epsilon(\frac{e^{-\frac{1}{2}}}{8}(x-1)e^{-\frac{x}{2}} + \frac{e^{-\frac{1}{2}}}{8}e^{-\frac{2x}{\epsilon}}).$$




Video clip for uniform approximations. Click here to open video clip in external player.

## 7.3 More on matching and intermediate variables

In the previous example we constructed an outer solution with $x$ fixed and $\epsilon$ tending to zero, and an inner expansion with $X = x/\epsilon$ fixed and $\epsilon$ going to zero. Grapically the process may be represented as in fig. 7.2 with the region A representing the outer solution and region B the inner solution. The figure also shows an overlap region where the two solutions agree. However closer examination of the figure might suggest that there is a possibility of a region C not accessible by the inner or outer solutions. In reality the actual domains of validity of the two solutions may be larger than the above limiting process allows. The difficulty here is arises from the way the matching is done.

A different way to match the two solutions is to introduce an intermediate variable, say $x = \epsilon^{\alpha}\xi$ with (in the above example) $0 < \alpha < 1$. We have $X = x/\epsilon = \epsilon^{-1+\alpha}\xi$ and so as $\epsilon \to 0$ with $\xi$ fixed gives $X \to \infty$ and $\epsilon \to 0$ with $\xi$ fixed also gives $x \to 0$. Thus $\xi$ is an *intermediate* variable and it is in this variable that we attempt to match the inner and outer solutions. The region defined by

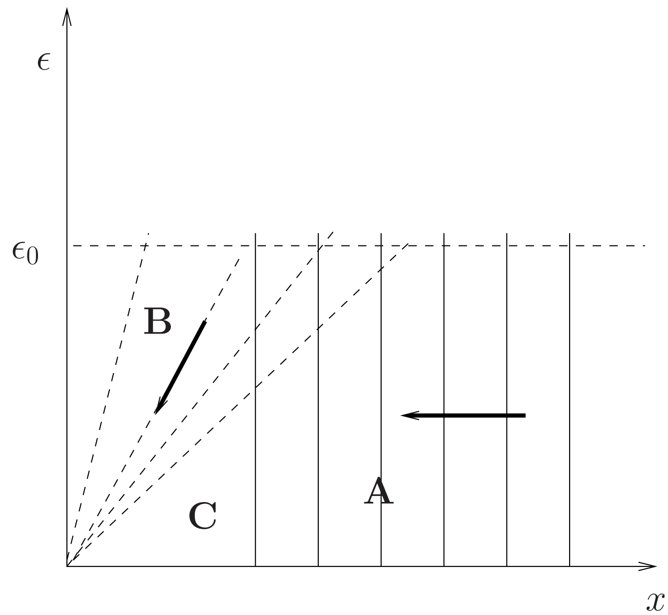



Figure 7.2: Outer solution represented by region **A** with $\epsilon \to 0$ $x$ fixed, and inner solution by **B** with $\epsilon \to 0$ with $X = x/\epsilon$ fixed.

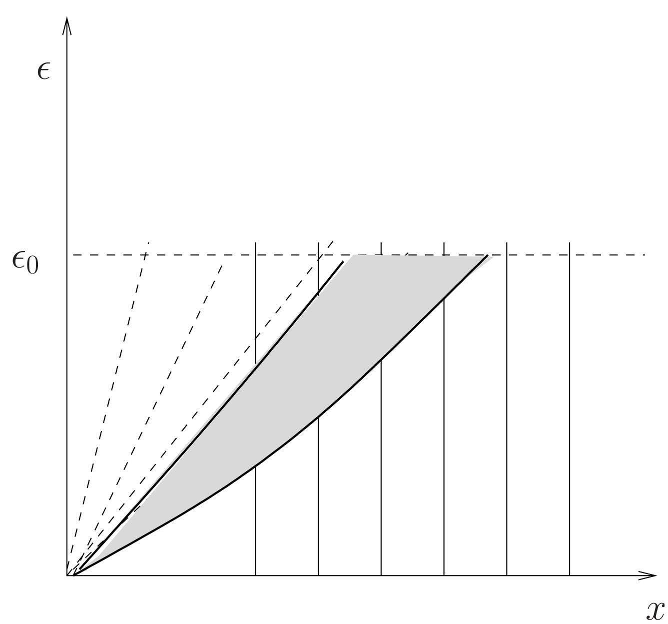
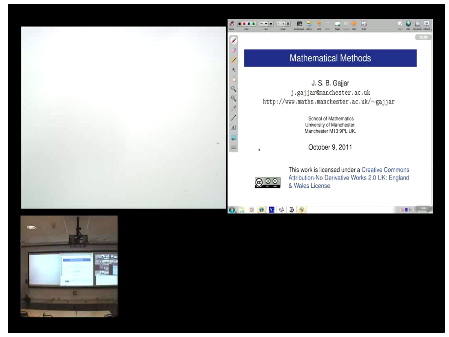



Figure 7.3: Overlap region (shaded) working in terms of intermediate variables $x = \epsilon^\alpha \xi$ and $X = \epsilon^{-1+\alpha}\xi$ with $0 < \alpha < 1$, and $\epsilon \to 0+$.

Video clip showing use of intermediate variables for previous example.Click here to open video clip in external player.




$\xi = O(1)$ is *an overlap region* for the two solutions, as shown schematically in fig. 7.3.

We will show how this works with another example in which the differential equation is nonlinear.

**Example** Consider

$$\epsilon y'' + y' + y^2 = 0, \quad y(0) = 0, y(1) = 1/2. \tag{7.3.1}$$

Suppose we look for an outer solution of the form

$$y = y_0 + \epsilon y_1 + \dots .$$

Then from (7.3.1) we obtain

$$y_0' + y_0^2 = 0, \quad y_0'' + y_1' + 2y_0 y_1 = 0. \tag{7.3.2}$$

The solution of the outer problem shows that

$$-\frac{y_0'}{y_0^2} = 1, \quad \frac{1}{y_0} = x + k,$$

and so

$$y_0 = \frac{1}{x+k}.$$

The boundary layer occurs at $x = 0$ (why?) and so we need to use the condition $y_0(1) = 1/2$ giving $k = 1$, and so

$$y_0 = \frac{1}{x+1}.$$

At next order

$$y_1' + 2y_0 y_1 + y_0'' = 0, \quad y_1(1) = 0.$$

Substituting for $y_0 = 1/(x+1)$ gives

$$y_1' + \frac{2}{x+1} y_1 = \frac{-2}{(1+x)^3}.$$

Hence

$$((1+x)^2 y_1)' = -\frac{2}{1+x},$$

$$(1+x)^2 y_1 + k_1 = -2\log(x+1).$$

Applying the condition $y_1(1) = 0$ gives $k_1 = -2\log 2$ and thus

$$y_1 = \frac{2\log(\frac{2}{1+x})}{(1+x)^2}.$$

For the inner solution we need to seek a solution in terms of an inner variable say $x = \epsilon^n X$ and substitution in (7.3.1) shows that $n = 1$ for a distinguished limit. The inner solution may be expanded as

$$y = Y_0(X) + \epsilon Y_1(X) + \dots.$$

After substitution into (7.3.1) and using $x = \epsilon X$ we obtain

$$Y_0'' + Y_0' = 0, \quad Y_1'' + Y_1' + Y_0^2 = 0.$$

The boundary conditions are

$$Y_0(0) = 0, \quad Y_1(0) = 0.$$

Solving for $Y_0$ yields

$$Y_0 = A_0 + B_0 e^{-X}, \quad \text{and} \quad A_0 + B_0 = 0.$$

Thus

$$Y_0 = A_0(1 - e^{-X}).$$

To find $A_0$ we match with intermediate variables and put $x = \epsilon^\alpha \xi, X = \epsilon^{-1+\alpha}\xi$, and $0 < \alpha < 1$ with $\xi = O(1)$. The one term outer solution written in terms of $\xi$ is

$$y = y_0(x) + \dots \sim \frac{1}{1+\epsilon^\alpha \xi} \sim 1 - \epsilon^\alpha \xi + \dots. \tag{7.3.3}$$

Similarly the outer solution in terms of $\xi$ is

$$y = Y_0(X) + \dots \sim A_0(1 - e^{-\epsilon^{-1+\alpha}\xi}) \sim A_0.$$

Thus matching with (7.3.3) shows that $A_0 = 1$ with error $O(\epsilon^\alpha)$.

Before we match to second order we need to find $Y_1$ which satisfies

$$Y_1'' + Y_1' + Y_0^2 = 0, \quad Y_1(0) = 0.$$

Thus

$$Y_1'' + Y_1' = -(1 - e^{-X})^2.$$

Solving and applying the condition on $X = 0$ gives (check)

$$Y_1(X) = A_1(1 - e^{-X}) + \frac{1}{2}(1 - e^{-2X}) - X(1 + 2e^{-X}).$$

Next we write the outer and inner expansions in terms of the intermediate variables and do the matching. The outer expansion written in terms of $\xi$ is

$$\begin{aligned} y_{out} &= \frac{1}{1+x} + \epsilon\frac{1}{(1+x)^2} 2\log(\frac{2}{1+x}) + \dots, \\ &= \frac{1}{1+\epsilon^\alpha\xi} + \epsilon\frac{1}{(1+\epsilon^\alpha\xi)^2} 2\log(\frac{2}{1+\epsilon^\alpha\xi}) + \dots, \\ &\sim 1 - \epsilon^\alpha\xi + \epsilon^{2\alpha}\xi^2 + \dots + 2\log 2(\epsilon - 2\epsilon^{\alpha+1}\xi + O(\epsilon^{2\alpha})) - 2\epsilon(1 - 2\epsilon^\alpha\xi)(\epsilon^\alpha\xi - O(\epsilon^{2\alpha})). \end{aligned} \tag{7.3.4}$$

Next the inner solution written in terms of $\xi$ is

$$\begin{aligned} y_{inn} &= (1 - e^{-X}) + \epsilon(A_1(1 - e^{-X}) + \frac{1}{2}(1 - e^{-2X}) - X(1 + 2e^{-X})) + \dots, \\ &= (1 - e^{-\epsilon^{\alpha-1}\xi}) + \epsilon\left[A_1(1 - e^{-\epsilon^{\alpha-1}\xi}) + \frac{1}{2}(1 - e^{-2\epsilon^{\alpha-1}\xi}) - \epsilon^{\alpha-1}\xi(1 + 2e^{-\epsilon^{\alpha-1}\xi})\right] + \dots, \\ &\sim 1 + \epsilon A_1 + \frac{\epsilon}{2} - \epsilon^\alpha\xi + \dots \end{aligned} \tag{7.3.5}$$

In (7.3.4) if we keep terms to order $\epsilon$ and assuming that the terms $O(\epsilon^{2\alpha})$ are smaller than terms of $O(\epsilon)$ we require $0 < \alpha < 1/2$. This gives

$$y_{out} \sim 1 - \epsilon^\alpha\xi + \epsilon 2\log 2 + O(\epsilon^2, \epsilon^{1+\alpha}, \epsilon^{2\alpha}). \tag{7.3.6}$$

Comparing (7.3.6) and (7.3.5) we see that the terms of $O(\epsilon^\alpha)$ match automatically and to match the $O(\epsilon)$ terms we require

$$\epsilon A_1 + \frac{\epsilon}{2} = 2\epsilon\log 2,$$

giving

$$A_1 = -\frac{1}{2} + 2\log 2.$$

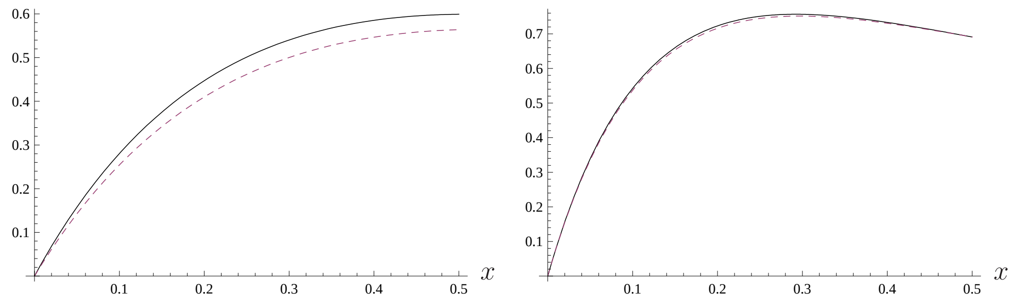
At the next order of matching the terms of $O(\epsilon^{2\alpha})$ match automatically.

The composite solution to $O(\epsilon^2)$ is

$$y_{comp} = y_{out} + y_{inn} - y_{match}.$$

In the above example we find

$$y_{comp} = \frac{1}{x+1} + \frac{2\epsilon}{(x+1)^2}\log\frac{2}{x+1} + (1 - e^{-\frac{x}{\epsilon}}) +$$



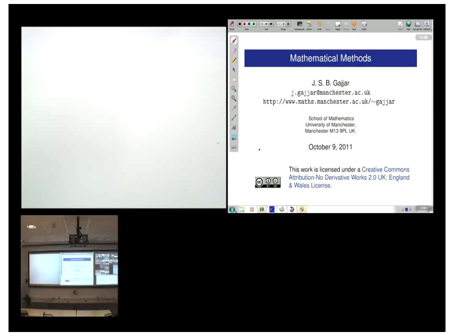

(a) (b)

Figure 7.4: A comparison of the numerical solution of (7.3.1) (solid lines) with the composite solution given by (7.3.7) (dashed line) taking (a) $\epsilon = 0.2$, and (b) $\epsilon = 0.1$

$$\epsilon \left[ (-\frac{1}{2} + 2\log 2)(1 - e^{-\frac{x}{\epsilon}}) + \frac{1}{2}(1 - e^{-\frac{2x}{\epsilon}}) - \frac{x}{\epsilon}(1 + 2e^{-\frac{x}{\epsilon}}) \right]$$
$$- (1 + \epsilon(-\frac{1}{2} + 2\log 2 + \frac{1}{2} - \frac{x}{\epsilon})). \tag{7.3.7}$$

A comparison of the numerical solution of (7.3.1) with the composite solution is shown in Fig. (7.4) and shows excellent agreement for $\epsilon$ small.

## 7.4 Interior boundary layers




Video clip covering an example of hidden boundary layer. Click here to open video clip in external player.

Consider

$$\epsilon y'' + a(x)y' + b(x)y = 0, \quad y(0) = A, y(1) = B. \tag{7.4.1}$$

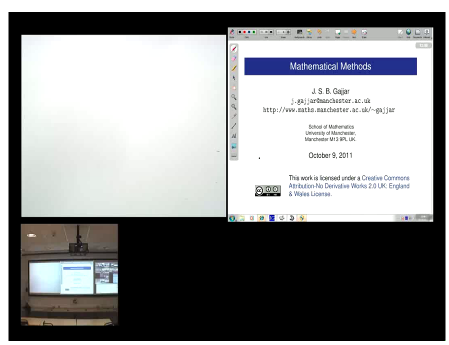
The outer problem (set $\epsilon = 0$) is just

$$a(x)y' + b(x)y = 0.$$

Take $a(x) > 0$ and then

$$y' = -\frac{b(x)}{a(x)}y, \quad y = Ce^{-\int_{x_0}^{x} \frac{b(s)}{a(s)}\, ds}.$$

Again there are two boundary conditions to satisfy and so there must be a boundary layer, but where is the boundary located?




Video clip for example of boundary layer not at $x = 0$. Click here to open video clip in external player.

Suppose that we have a boundary layer at $x = \bar{x}$ of thickness $\gamma(\epsilon)$. We write

$$x = \bar{x} + \gamma(\epsilon)X, \quad \text{where} \quad X = O(1).$$

Then substituting into (7.4.1) with $y = Y$ gives

$$\frac{\epsilon}{\gamma^2}\frac{d^2Y}{dX^2} + \frac{a(\bar{x} + \gamma X)}{\gamma}\frac{dY}{dX} + b(\bar{x} + \gamma X)Y = 0.$$

Now expand $a, b$ as

$$a(\bar{x} + \gamma X) = a(\bar{x}) + \gamma X a'(\bar{x}) + \ldots, \quad b(\bar{x} + \gamma X) = b(\bar{x}) + \gamma X b'(\bar{x}) + \ldots,$$

to get

$$\frac{\epsilon}{\gamma^2}\frac{d^2Y}{dX^2} + \frac{a(\bar{x})}{\gamma}\frac{dY}{dX} + b(\bar{x})Y + \cdots = 0. \tag{7.4.2}$$

For $|\gamma| << 1$ the dominant terms are

$$\frac{\epsilon}{\gamma^2}\frac{d^2Y}{dX^2}, \quad \frac{a}{\gamma}\frac{dY}{dX}.$$

For a balance we require

$$\frac{\epsilon}{\gamma^2} \sim \frac{1}{\gamma} \quad \Longrightarrow \gamma = O(\epsilon).$$

Hence set $\gamma = \epsilon$ ie $x = \bar{x} + \epsilon X$. From (7.4.2) the reduced inner equation is

$$\epsilon^{-1}\left[\frac{d^2Y}{dX^2} + a(\bar{x})\frac{dY}{dX}\right] + b(\bar{x})Y + \cdots = 0. \tag{7.4.3}$$

The leading order inner problem is

$$\frac{d^2Y}{dX^2} + a(\bar{x})\frac{dY}{dX} = 0,$$

giving

$$Y = C_0 + C_1 e^{-a(\bar{x})X}.$$

Now we have assumed that $a(\bar{x}) > 0$. If $\bar{x} > 0$ we need to match as we go out of the boundary layer, ie we need limits $X \to \pm\infty$.

As $X \to \infty$ everything is ok, but as $X \to -\infty$ it suggests that $C_1$ must be zero to avoid exponential growth.

But $C_1 = 0$ implies no boundary layer. Hence $\bar{x} = 0$ and the boundary layer is at $x = 0$ if $a(x) > 0$.

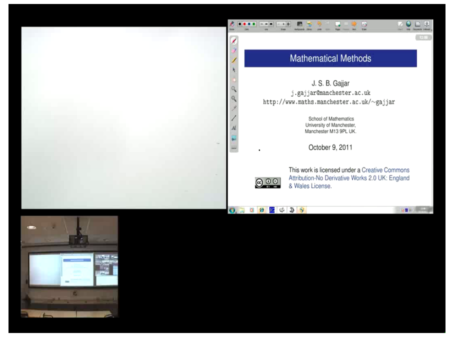
Similarly if $a(x) < 0$ then we have a bounday layer at $x = 1$. If $a(x) = 0$ inside the region we have an internal boundary layer. The above analysis also breaks down.

### 7.4.1 Further Examples including interior layers

Consider

$$\epsilon y'' + xy' - (\epsilon^2 x^3 + 1)y = 0, \quad y(-1) = 1, y(1) = 2$$

and $-1 \leq x \leq 1, \quad 0 < \epsilon << 1$. The above discussion suggests an interior layer at $x = 0$.




Video clip for above section. Click here to open video clip in external player.

For the outer solution put

$$y = y_0 + \epsilon y_1 + \dots,$$

to get

$$xy_0' - y_0 = 0.$$

Thus

$$y_0 = Ax.$$

Here we have a new difficulty. Which boundary condition do we choose? We can show that there are no boundary layers near $x = \pm 1$. We write

$$y = A_{\pm} x$$

where the $+$ stands for $x > 0$ and $-$ for $x < 0$.

From the boundary conditions it suggests that

$$A_+ = 2, \quad A_- = -1.$$

When $x$ is small the $\epsilon y''$ term is not negligble, and hence we look for an interior layer at $x = 0$ and write

$$x = \gamma(\epsilon) X, \quad \gamma(\epsilon) << 1.$$

This gives with $y = Y$

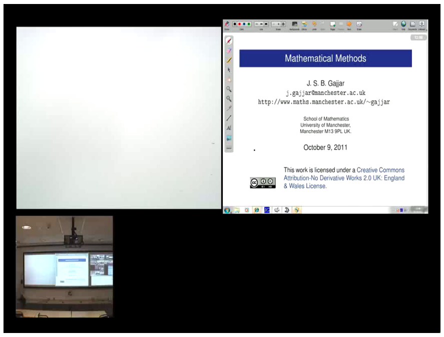
$$\frac{\epsilon}{\gamma^2}\frac{d^2Y}{dX^2} + \frac{\gamma X}{\gamma}\frac{dY}{dX} - Y + \cdots = 0.$$

For a dominant balance this suggests that

$$\frac{\epsilon}{\gamma^2} \sim O(1) \quad \implies \gamma = O(\epsilon^{\frac{1}{2}}).$$

Hence set $x = \epsilon^{\frac{1}{2}} X$ and from the outer solution it suggests that we expand the inner solution as

$$y = \epsilon^{\frac{1}{2}} Y_0 + \epsilon Y_1 + \dots.$$




Video clip for interior layer problem, outer solution for above example. Click here to open video clip in external player.

Substituting into the equation gives

$$\epsilon\epsilon^{-1}\left(\epsilon^{\frac{1}{2}}\frac{d^2Y_0}{dX^2}+\dots\right)+\epsilon^{\frac{1}{2}}X\epsilon^{-\frac{1}{2}}\left(\epsilon^{\frac{1}{2}}\frac{dY_0}{dX}+\dots\right)-\epsilon^{\frac{1}{2}}Y_0+\dots=0.$$

Hence the leading order problem is

$$\frac{d^2Y_0}{dX^2}+X\frac{dY_0}{dX}-Y_0=0. \tag{7.4.4}$$

The boundary conditions suggest that we must match with the outer solution as $X\to\pm\infty$. This suggests that

$$Y_0\sim A_\pm X \quad \text{as} \quad X\to\pm\infty. \tag{7.4.5}$$

The equation (7.4.4) can be solved in terms of parabolic cylinder functions. If we put

$$Y_0=e^{-\frac{X^2}{4}}W_0$$

then $W_0$ satisfies

$$W_0''+(\frac{1}{2}-2-\frac{X^2}{4})W_0=0.$$

Note that two linearly independent solutions of the equation

$$W''+(\frac{1}{2}+\nu-\frac{X^2}{4})W=0,$$

are the parabolic cylinder functions $W=D_\nu(X)$ and $D_{-\nu-1}(iX)$.

In order to do the matching we require the behaviours of $D_\nu(x)$ for $|x|$ large. The properties of $D_\nu(z)$ are summarized below (see for example Abramovitz & Stegun[1]:

$$D_\nu(z)\sim z^\nu e^{-\frac{z^2}{4}}\sum_{n=0}^{\infty}(-1)^n a_n z^{-2n} \quad \text{as} \quad z\to\infty, \quad |\arg(z)|<\frac{3\pi}{4}. \tag{7.4.6}$$

[1] M. Abramovitz and I. A. Stegun *Handbook of Mathematical Function*, Dover. [web version also available]

$$D_\nu(z) \sim z^\nu e^{-\frac{z^2}{4}} \sum_{n=0}^{\infty} (-1)^n a_n z^{-2n} - \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)} e^{i\pi\nu} z^{-\nu-1} e^{\frac{z^2}{4}} \sum_{n=0}^{\infty} b_n z^{-2n}$$

$$\text{as} \quad z \to \infty, \quad \frac{\pi}{4} < \arg(z) < \frac{5\pi}{4}. \tag{7.4.7}$$

Here $a_0 = b_0 = 1$, and

$$a_n = \frac{\nu(\nu-1)\dots(\nu-2n+1)}{2^n n!} \quad b_n = \frac{(\nu+1)(\nu+2)\dots(\nu+n)}{2^n n!}.$$

Hence we can write the solution of (7.4.4) as

$$Y_0 = e^{-\frac{X^2}{4}} (CD_{-2}(X) + ED_1(iX)).$$

Now using

$$D_{-2}(X) \sim X^{-2} e^{-\frac{X^2}{4}}, \quad D_1(iX) \sim (iX) e^{\frac{X^2}{4}} \quad \text{as} \quad X \to \infty$$

and

$$D_{-2}(X) \sim X^{-2} e^{-\frac{X^2}{4}} - \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(2)} e^{2\pi i} X e^{\frac{X^2}{4}}, \quad \text{as} \quad X \to -\infty$$

$$D_1(iX) \sim (iX) e^{\frac{X^2}{4}} \quad \text{as} \quad X \to -\infty,$$

we find that

$$Y_0 \sim e^{-\frac{X^2}{4}} \left[ \frac{C}{X^2} e^{-\frac{X^2}{4}} + E(iX) e^{\frac{X^2}{4}} \right] \quad X \to \infty,$$

ie

$$Y_0 \sim EiX \quad \text{as} \quad X \to \infty.$$

Hence

$$Ei = A_+.$$

Similarly

$$Y_0 \sim e^{-\frac{X^2}{4}} \left[ -C\sqrt{(2\pi)} X e^{\frac{X^2}{4}} + E(iX) e^{\frac{X^2}{4}} \right] \quad X \to -\infty.$$

Hence

$$Y_0 \sim (-\sqrt{(2\pi)}C + iE)X + O(1) \quad X \to -\infty,$$

giving

$$-\sqrt{(2\pi)}C + iE = A_-.$$

Using the given values for $A_\pm$ leads to

$$C = \frac{3}{\sqrt{2\pi}}, \quad E = -2i,$$

and the inner solution as

$$Y_0 = \left( \frac{3}{\sqrt{2\pi}} D_{-2}(X) - 2iD_1(iX) \right) e^{-\frac{X^2}{4}}.$$

A uniform approximation can be calculated to give

$$y_{unif} = \epsilon^{\frac{1}{2}} \left( \frac{3}{\sqrt{2\pi}} D_{-2}(\frac{x}{\sqrt{\epsilon}}) - 2iD_1(\frac{ix}{\sqrt{\epsilon}}) \right) e^{-\frac{x^2}{4\epsilon}}.$$

A comparison of the uniform approxmation

$$y_{unif} = \epsilon^{\frac{1}{2}} \left( \frac{3}{\sqrt{2\pi}} D_{-2}(\frac{x}{\sqrt{\epsilon}}) - 2iD_1(\frac{ix}{\sqrt{\epsilon}}) \right) e^{-\frac{x^2}{4\epsilon}}.$$

with a numerical solution of the differential equation

$$\epsilon y'' + xy' - (\epsilon^2 x^3 + 1)y = 0, \quad y(-1) = 1, y(1) = 2$$

and $-1 \le x \le 1, \quad 0 < \epsilon << 1$, for $\epsilon = 0.05$ is shown in Fig. 7.5 below.

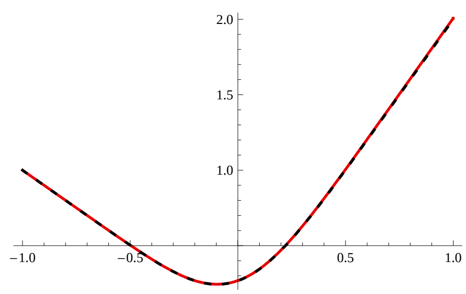
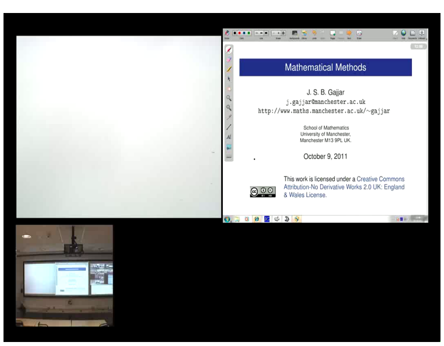



Figure 7.5: A comparison of the (exact) numerical solution to the full equation as compared with the uniform approximation (dashed line) for $\epsilon = 0.05$.




Video clip for interior layer problem, inner solution for above example. Click here to open video clip in external player.

## 7.5 The LG approximation, WKBJ Method

Boundary layer theory fails when we have a rapid variation in the solution throughout the region rather than locally at some location.

**Example** Consider

$$\epsilon y'' + by = 0, \quad y(0) = 0, \quad y(1) = 1,$$

where $b > 0$ and $0 < \epsilon << 1$. Note that the general solution is

$$y = \frac{\sin(x\sqrt{\frac{b}{\epsilon}})}{\sin(\sqrt{\frac{b}{\epsilon}})}.$$

The outer solution is just $y = 0$. For the inner solution, suppose we set

$$x = \bar{x} + \gamma(\epsilon)X, \quad \gamma << 1.$$

Then the equations gives

$$\frac{\epsilon}{\gamma^2}\frac{d^2y}{dX^2} + by = 0.$$

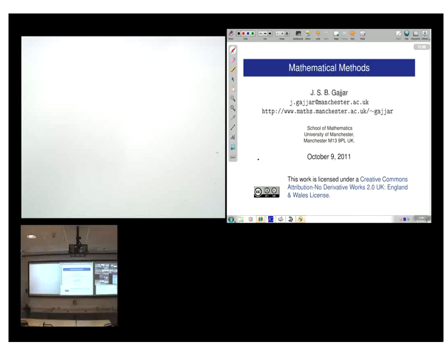
A dominant balance gives $\gamma = \epsilon^{\frac{1}{2}}$ and the resulting inner problem is

$$\frac{d^2y}{dX^2} + by = 0.$$

The solution gives

$$y = A\sin(\sqrt{b}X) + B\cos(\sqrt{b}X).$$

We can choose any $\bar{x}$ but note that for any choice of $\bar{x}$ the solution is not of boundary layer form and cannot be matched to the outer solution as $X \to \pm\infty$ because the inner solution oscillates.




Video clip for above example. Click here to open video clip in external player.

Boundary layer theory fails for these types of singular perturbation problems in which we have wavelike behaviour (as opposed to dissipative or dispersive behaviour). The LG approximation or WKBJ theory is ideal for these classes of problems. The technique we describe below leads to an approximation which was obtained by Liouville (1837) and Green (1837). In fact as noted earlier, Carlini (1817) had also used the same ideas.

The method is more commonly known as the WKBJ method after Wentzel (1926), Kramers (1926), Brillouin (1926), and Jeffreys (1924). (Theoretical physicists call it the WKB method). However it is more correct it to call it the LG approximation which was used by Jeffreys, Wentzel, Kramers and Brillouin, to derive the connection formula in the presence of turning points (see later).

Consider

$$\epsilon y'' = Q(x)y, \quad Q(x) \neq 0. \tag{7.5.1}$$

The basic idea of the theory is that for $\epsilon \to 0$ we look for a solution to (7.5.1) of the form

$$y \sim A(x,\delta)e^{\frac{s(x,\delta)}{\delta}}, \quad \delta(\epsilon) \to 0$$

where $A(x,\delta), s(x,\delta)$ are slowly varying functions of $x$, but note the rapid variation of the solution because of the exponential factor. We can absorb the $A$ into the exponential by writing

$$y = e^{\frac{S(x,\delta)}{\delta}}. \tag{7.5.2}$$

Substitution (7.5.2) into the equation (7.5.1) gives

$$\epsilon\left[\frac{S'^2}{\delta^2} + \frac{S''}{\delta}\right] - Q(x) = 0,$$

where primes denote differentiation with respect to $x$.

For a dominant balance we have $\delta = \epsilon^{\frac{1}{2}}$, and the equation for $S$ reduces to

$$S'^2 - Q(x) = -\epsilon^{\frac{1}{2}}S''. \tag{7.5.3}$$

This suggests that we write

$$S = \sum_{n=0}^{\infty} \epsilon^{\frac{n}{2}} S_n, \quad \epsilon \to 0.$$

Substitution into (7.5.3) gives

$$(S_0' + \epsilon^{\frac{1}{2}}S_1' + \dots)^2 - Q(x) = -\epsilon^{\frac{1}{2}}(S_0'' + \epsilon^{\frac{1}{2}}S_1'' + \dots). \tag{7.5.4a}$$

Equating like powers of $\epsilon$ in (7.5.4a) to zero gives

$$(S_0')^2 = Q(x), \tag{7.5.4b}$$
$$2S_0'S_1' = -S_0'', \tag{7.5.4c}$$
$$2S_0'S_n' + \sum_{j=1}^{n-1} S_j'S_{n-j}' = -S_{n-1}'', \quad n \geq 2. \tag{7.5.4d}$$

We can solve (7.5.4b) to obtain

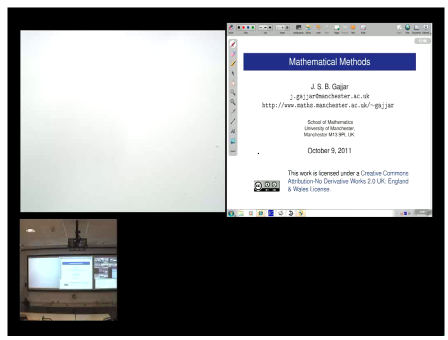
$$S_0 = \pm \int^x Q^{\frac{1}{2}}\, dx,$$

$$S_1' = -\frac{S_0''}{2S_0'} \qquad \Longrightarrow S_1 = -\frac{1}{4}\log|Q|.$$

Hence the leading order behaviour of the solution can be written down as

$$y \sim |Q|^{-\frac{1}{4}}\left[C_1 \exp\left(\frac{1}{\epsilon^{\frac{1}{2}}}\int_a^x Q^{\frac{1}{2}}\right) + C_2 \exp\left(-\frac{1}{\epsilon^{\frac{1}{2}}}\int_a^x Q^{\frac{1}{2}}\right)\right], \tag{7.5.5}$$

where $C_1, C_2, a$ are determined from the boundary conditions. This is the LG approximation to the solution. The approximation with just the leading order term $S_0$ gives what the physicists like to call the *geometrical optics* approximation. The approximation (7.5.5) is also referred to as the *physical optics* approximation.




Video clip for WKB method- general theory. Click here to open video clip in external player.

**Example** Consider again

$$\epsilon y'' + by = 0, \quad y(0) = 0, y(1) = 1,$$

and $b > 0$. Here $Q(x) = -b$, and so

$$S_0 = \pm i\sqrt{b}x.$$

Hence using (7.5.5)

$$y \sim b^{-\frac{1}{4}}(C_1 e^{ix\sqrt{\frac{b}{\epsilon}}} + C_2 e^{-ix\sqrt{\frac{b}{\epsilon}}}),$$

or

$$y \sim A_1 \sin(\sqrt{\frac{b}{\epsilon}}x) + A_2 \cos(\sqrt{\frac{b}{\epsilon}}x).$$

Applying the boundary conditions gives the exact solution

$$y = \frac{\sin(\sqrt{\frac{b}{\epsilon}}x)}{\sin(\sqrt{\frac{b}{\epsilon}})}.$$

**Example** Consider

$$\epsilon y'' - (1+x^2)^2 y = 0, \quad y(0) = 0, y'(0) = 1.$$

If we look for a solution

$$y \sim \exp\left(\frac{1}{\delta}\sum_0^\infty \delta^n S_n\right)$$

then again with $\delta = \epsilon^{\frac{1}{2}}$ we obtain

$$S_0'^2 = (1+x^2)^2, \quad S_0 = \pm\left(\frac{x^3}{3} + x\right).$$

Next

$$S_1 = -\frac{1}{4}\log(1+x^2)^2 = -\frac{1}{2}\log(1+x^2).$$

Thus

$$y \sim (1+x^2)^{-\frac{1}{2}}\left(C_1 \exp\left(\frac{\frac{x^3}{3}+x}{\epsilon^{\frac{1}{2}}}\right) + C_2\exp\left(-\frac{\frac{x^3}{3}+x}{\epsilon^{\frac{1}{2}}}\right)\right). \tag{7.5.6}$$

To find the constants we need to apply the boundary conditions. The condition $y(0) = 0$ leads to (after substituting $x = 0$ in (7.5.6)

$$0 = C_1 + C_2.$$

Now assuming that differentition of (7.5.6) is valid, we find that

$$y'(x) \sim \frac{(1+x^2)^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}}}\left(C_1\exp\left(\frac{(\frac{x^3}{3}+x)}{\epsilon^{\frac{1}{2}}}\right) - C_2\exp\left(\frac{-(\frac{x^3}{3}+x)}{\epsilon^{\frac{1}{2}}}\right)\right)$$

$$-x(1+x^2)^{-\frac{3}{2}}\left(C_1\exp\left(\frac{(\frac{x^3}{3}+x)}{\epsilon^{\frac{1}{2}}}\right) + C_2\exp\left(\frac{-(\frac{x^3}{3}+x)}{\epsilon^{\frac{1}{2}}}\right)\right).$$

Hence applying $y'(0) = 1$ gives

$$1 = \epsilon^{-\frac{1}{2}}(C_1 - C_2).$$

Solving for $C_1, C_2$ gives

$$C_1 = \frac{1}{2}\epsilon^{\frac{1}{2}}, \quad C_2 = -\frac{1}{2}\epsilon^{\frac{1}{2}}.$$

Hence a WKB approximation to the solution is

$$y \sim \epsilon^{\frac{1}{2}}(1+x^2)^{-\frac{1}{2}}\sinh\left(\frac{\frac{x^3}{3}+x}{\epsilon^{\frac{1}{2}}}\right), \quad \epsilon \to 0.$$

The WKB method can also be used for certain eigenvalue problems.

**Example** Consider

$$y'' + \lambda p(x)y = 0, \quad y(0) = 0, \quad y(\pi) = 0. \tag{7.5.7}$$

This equation has nontrivial solutions only for certain discrete values of $\lambda$ say $(\lambda_1, \lambda_2, \dots)$. We can obtain an approximation to the eigenvalues and eigenfunction for large $\lambda$.

Look for an asymptotic solution in WKB form as

$$y \sim \exp(\lambda^{\frac{1}{2}} \sum_{n=0}^{\infty} \lambda^{-n/2} S_n(x)).$$

Substitution into the equation (7.5.7) gives

$$\lambda(S_0' + \lambda^{-\frac{1}{2}} S_1' + \dots)^2 + \lambda^{\frac{1}{2}}(S_0'' + \lambda^{-\frac{1}{2}} S_1'' + \dots) + \lambda p(x) = 0.$$

Solving for $S_0, S_1$ gives

$$S_0 = \pm i \int^x (p(x))^{\frac{1}{2}}\, dx, \quad S_1 = -\frac{1}{4} \log |p(x)|.$$

Hence

$$y \sim |p|^{-\frac{1}{4}} \left[ C_1 \sin(\lambda^{\frac{1}{2}} \int_0^x (p(x))^{\frac{1}{2}}\, dx) + C_2 \cos(\lambda^{\frac{1}{2}} \int_0^x (p(x))^{\frac{1}{2}}\, dx) \right].$$

The boundary conditions in (7.5.7)imply

$$C_2 = 0,$$

and

$$\sin(\lambda^{\frac{1}{2}} \int_0^\pi (p(x))^{\frac{1}{2}}\, dx) = 0.$$

Hence

$$\lambda^{\frac{1}{2}} = \frac{\pm n\pi}{\int_0^\pi (p(x))^{\frac{1}{4}}\, dx} \quad .$$

Thus

$$\lambda \sim \lambda_n = \frac{n^2\pi^2}{[\int_0^\pi (p(x))^{\frac{1}{4}}\, dx]^2}, \quad n >> 1,$$

and approximate solution to (7.5.7) is

$$y \sim |p|^{-\frac{1}{4}} C_n \sin(\lambda_n^{\frac{1}{2}} \int_0^x (p(x))^{\frac{1}{2}}\, dx).$$

### 7.5.1 Additional notes

Implicit in the use of the WKB (LG) method ie

$$y \sim \exp(\sum_{n=0}^{\infty} \delta^{n-1} S_n(x))$$

is that the series

$$\sum_{n=0}^{\infty} \delta^{n-1} S_n(x), \quad \text{as} \quad \delta \to 0$$

is an asymptotic series, uniformly valid for all $x$ throughout the interval. This requires that

$$\delta^n S_{n+1}(x) = o(\delta^{n-1} S_n(x)), \quad n = 1, 2, \ldots,$$

holds uniformly in $x$.

Since we take the exponential of the above series, for the WKB (LG) approximation to be a good approximation, if we truncate the series at $n = M - 1$ say, then we should have

$$\delta^M S_{M+1}(x) = o(1) \quad \delta \to 0$$

since

$$\exp(\delta^M S_{M+1}(x)) = 1 + O(\delta^M S_{M+1}(x)), \quad \text{as} \quad \delta \to 0.$$

### 7.5.2 Turning points and connection formulae

So far in

$$\epsilon y'' - Q(x, \epsilon) y = 0$$

we have taken $Q(x, \epsilon) > 0$ in the interval.

We will now consider

$$\epsilon y'' - Q(x) y = 0, \quad a < x < b, \quad Q(x_0) = 0, \quad Q'(x_0) > 0, \quad a < x_0 < b. \tag{7.5.8}$$

We will assume that there is only one zero in the $a < x < b$. A WKB approximation to the equation (7.5.8) is

$$y \sim C|Q(x)|^{-\frac{1}{4}} \exp\left(\pm \frac{1}{\epsilon^{\frac{1}{2}}} \int^x (Q(s))^{\frac{1}{2}}\, ds.\right).$$

Thus for $x > x_0$ we write

$$y \sim |Q(x)|^{-\frac{1}{4}} \left[A_1 \exp\left(\frac{1}{\epsilon^{\frac{1}{2}}} \int^x (Q(s))^{\frac{1}{2}}\, ds\right) + A_2 \exp\left(-\frac{1}{\epsilon^{\frac{1}{2}}} \int^x (Q(s))^{\frac{1}{2}}\, ds.\right)\right], \tag{7.5.9}$$

and for $x < x_0$ we have

$$y \sim |Q|^{-\frac{1}{4}} \left[B_1 \cos(\frac{1}{\epsilon^{\frac{1}{2}}} \int^x |Q(s)|^{\frac{1}{2}}\, ds) + B_2 \sin(\frac{1}{\epsilon^{\frac{1}{2}}} \int^x |Q(s)|^{\frac{1}{2}}\, ds)\right]. \tag{7.5.10}$$

The above approximation fails near $x = x_0$, where we have

$$Q(x) \sim (x - x_0)Q'(x_0) + \dots. \tag{7.5.11}$$

If we put $x = x_0 + \epsilon^\gamma X$ and substitute into the differential equation (7.5.8) and use (7.5.11) we obtain

$$\epsilon\epsilon^{-2\gamma}\frac{d^2y}{dX^2} - (\epsilon^\gamma X Q'(x_0)y + \dots) = 0.$$

For a dominant balance we require

$$\epsilon^{1-2\gamma} \sim \epsilon^\gamma, \qquad \Longrightarrow \gamma = \frac{1}{3}.$$

The dominant equation in this region reduces to Airy's equation

$$\frac{d^2y}{dX^2} - Xcy = 0, \quad c = Q'(x_0) > 0.$$

This has the solution

$$y_{inn} = D_1\mathrm{Ai}(c^{\frac{1}{3}}X) + D_2\mathrm{Bi}(c^{\frac{1}{3}}X), \tag{7.5.12}$$

which is the inner solution. We need to match this with the outer solution (7.5.9.7.5.10) as $X \to \pm\infty$ or $x \to x_0 \pm$. Now

$$\mathrm{Ai}(X) \sim \frac{1}{2\sqrt{\pi}}X^{-\frac{1}{4}}e^{-\frac{2}{3}X^{\frac{3}{2}}}, \quad X \to \infty,$$

$$\mathrm{Bi}(X) \sim \frac{1}{\sqrt{\pi}}X^{-\frac{1}{4}}e^{\frac{2}{3}X^{\frac{3}{2}}}, \quad X \to \infty.$$

Thus for $X \to +\infty$ from (7.5.12)

$$y_{inn} \sim \frac{1}{\sqrt{\pi}}X^{-\frac{1}{4}}c^{-\frac{1}{12}}\left(\frac{D_1}{2}e^{-\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}} + D_2e^{\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}}\right).$$

Also

$$\mathrm{Ai}(X) \sim \frac{1}{\sqrt{\pi}}(-X)^{-\frac{1}{4}}\sin(\frac{2}{3}(-X)^{\frac{3}{2}} + \frac{\pi}{4}) \quad X \to -\infty,$$
$$\mathrm{Bi}(X) \sim \frac{1}{\sqrt{\pi}}(-X)^{-\frac{1}{4}}\cos(\frac{2}{3}(-X)^{\frac{3}{2}} + \frac{\pi}{4}) \quad X \to -\infty.$$

Hence from (7.5.12) for $X \to -\infty$

$$y_{inn} \sim \frac{1}{\sqrt{\pi}}(-X)^{-\frac{1}{4}}c^{-\frac{1}{12}}\left[D_1\sin(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}} + \frac{\pi}{4}) + D_2\cos(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}} + \frac{\pi}{4})\right]$$

Now if we take the lower limit in (7.5.9) to be equal to $x_0$ (this is not necessary but it simplifies the expressions) then

$$\int_{x_0}^{x} (Q(s))^{\frac{1}{2}}\, ds = \int_0^{x-x_0} [Q(x_0+T)]^{\frac{1}{2}}\, dT,$$

$$\sim \int_0^{x-x_0} \left[cT + \frac{Q''(x_0)}{2}T^2 + \dots\right]^{\frac{1}{2}} dT,$$

$$\sim \int_0^{x-x_0} c^{\frac{1}{2}}T^{\frac{1}{2}} \left[1 + \frac{Q''(x_0)}{4c}T + \dots\right] dT.$$

Hence

$$\int_{x_0}^{x} (Q(s))^{\frac{1}{2}}\, ds \sim \frac{2}{3}c^{\frac{1}{2}}(x-x_0)^{\frac{3}{2}}$$

Hence as $x \to x_0+$ the outer solution behaves as

$$y_{out}^{+} \sim [c(x-x_0)]^{-\frac{1}{4}} \left[A_1 \exp(\frac{1}{\epsilon^{\frac{1}{2}}}\frac{2}{3}c^{\frac{1}{2}}(x-x_0)^{\frac{3}{2}}) + A_2 \exp(-\frac{1}{\epsilon^{\frac{1}{2}}}\frac{2}{3}c^{\frac{1}{2}}(x-x_0)^{\frac{3}{2}})\right],$$

ie

$$y_{out}^{+} \sim c^{-\frac{1}{4}}\epsilon^{\frac{-1}{12}}X^{-\frac{1}{4}} \left[A_1 \exp(\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}) + A_2 \exp(-\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}})\right].$$

Also

$$y_{inn} \sim \frac{1}{\sqrt{\pi}}X^{-\frac{1}{4}}c^{-\frac{1}{12}} \left(\frac{D_1}{2}e^{-\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}} + D_2 e^{\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}}\right). \qquad (7.5.13)$$

To match with the inner solution (7.5.13) as $X >> 1$ we must have

$$\frac{D_1}{2\sqrt{\pi}}c^{-\frac{1}{12}} = A_2 c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}, \quad \frac{D_2}{\sqrt{\pi}}c^{-\frac{1}{12}} = A_1 c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}.$$

Similarly as $x \to 0-$ we have

$$\int_{x_0}^{x} (-Q(s))^{\frac{1}{2}}\, ds \sim -\frac{2}{3}c^{\frac{1}{2}}(x_0-x)^{\frac{3}{2}}.$$

Thus

$$y_{out}^{-} \sim c^{-\frac{1}{4}}(x_0-x)^{-\frac{1}{4}} \left[B_1 \cos(\frac{2}{3}c^{\frac{1}{2}}(x_0-x)^{\frac{3}{2}}) - B_2 \sin(\frac{2}{3}c^{\frac{1}{2}}(x_0-x)^{\frac{3}{2}})\right],$$

$$y_{out}^{-} \sim c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}|X|^{-\frac{1}{4}} \left[B_1 \cos(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}}) - B_2 \sin(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}})\right]. \qquad (7.5.14)$$

And

$$y_{inn} \sim \frac{1}{\sqrt{\pi}}(-X)^{-\frac{1}{4}}c^{-\frac{1}{12}} \left[D_1 \sin(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}} + \frac{\pi}{4}) + D_2 \cos(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}} + \frac{\pi}{4})\right] \qquad (7.5.15)$$

To match (7.5.14, 7.5.15) we must have

$$c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}B_1 = \frac{c^{-\frac{1}{12}}}{\sqrt{2\pi}}(D_1+D_2) = (\frac{A_1}{\sqrt{2}} + A_2\sqrt{2})c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}},$$

$$-c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}B_2 = \frac{c^{-\frac{1}{12}}}{\sqrt{2\pi}}(D_1-D_2) = -(\frac{A_1}{\sqrt{2}} - A_2\sqrt{2})c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}.$$

Hence solving for $B_1, B_2$ gives

$$B_1 = \frac{A_1}{\sqrt{2}} + A_2\sqrt{2},$$

$$B_2 = \frac{A_1}{\sqrt{2}} - A_2\sqrt{2}.$$

**Summary:** For $x > x_0$

$$y \sim |Q(x)|^{-\frac{1}{4}}\left[A_1\exp\left(\frac{1}{\epsilon^{\frac{1}{2}}}\int_{x_0}^{x}(Q(s))^{\frac{1}{2}}\,ds\right)\right.$$

$$\left. + A_2\exp\left(-\frac{1}{\epsilon^{\frac{1}{2}}}\int_{x_0}^{x}(Q(s))^{\frac{1}{2}}\,ds.\right)\right], \tag{7.5.16a}$$

and for $x < x_0$ we have

$$y \sim |Q|^{-\frac{1}{4}}\left[A_1\sin(\frac{1}{\epsilon^{\frac{1}{2}}}\int_{x_0}^{x}|Q(s)|^{\frac{1}{2}}\,ds + \frac{\pi}{4})\right.$$

$$\left. + 2A_2\cos(\frac{1}{\epsilon^{\frac{1}{2}}}\int_{x_0}^{x}|Q(s)|^{\frac{1}{2}}\,ds + \frac{\pi}{4})\right]. \tag{7.5.16b}$$

For $(x - x_0) << 1$

$$y \sim \sqrt{\pi}c^{-\frac{1}{6}}\epsilon^{-\frac{1}{12}}\left[2A_2\mathrm{Ai}(c^{\frac{1}{3}}\epsilon^{-\frac{1}{3}}(x-x_0)) + A_1\mathrm{Bi}(c^{\frac{1}{3}}\epsilon^{-\frac{1}{3}}(x-x_0))\right].$$

The formulae (7.5.16) are known as the connection formulae. The constants $A_1, A_2$ are determined by the boundary conditions. ** CHECK video**

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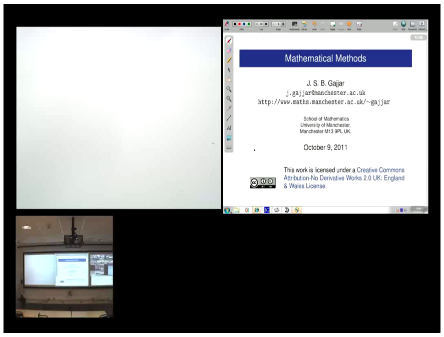
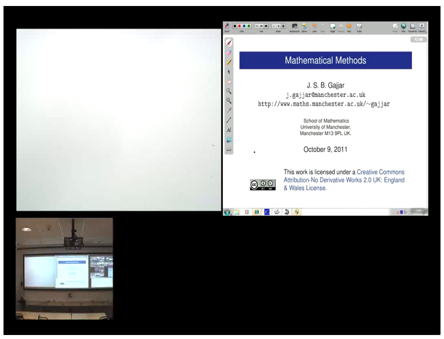



Video clip discussing the theory for a single turning point. Click here to open video clip in external player.

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# Chapter 8

# Introduction to generalised Functions

## 8.1 Introduction

Consider the the following function.

$$\delta_\epsilon(x) = \begin{cases} 0, & x < 0, \\ \epsilon^{-1}, & 0 < x < \epsilon, \\ 0, & x > \epsilon. \end{cases}$$

If $f(x)$ is continuous in an interval which includes the origin and $(0, \epsilon)$ then

$$\int_{-\infty}^{\infty} \delta_\epsilon(x) f(x)\, dx = \epsilon^{-1} \int_0^\epsilon f(x)\, dx.$$

From the mean value theorem

$$\int_0^\epsilon f(x)\, dx = \epsilon f(\epsilon\xi), \quad 0 \leq \xi \leq 1,$$

and therefore

$$\int_{-\infty}^{\infty} \delta_\epsilon(x) f(x)\, dx = f(\epsilon\xi), \quad 0 \leq \xi \leq 1.$$

If we let $\epsilon \to 0$ we obtain

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0),$$

for all functions $f(x)$ continuous in a neighbourhood including the origin. The function $\delta(x)$ is the limit of the $\delta_\epsilon(x)$ as $\epsilon \to 0$ is called the *delta function*, and is an example of a *generalised function*. Note that

$$\delta(x) = \begin{cases} 0, & x < 0 \\ 0, & x > 0. \end{cases},$$

and is undefined at $x = 0$. It is not an ordinary function and also not integrable in the usual sense.

The concept of a delta function dates back to the time of Kirchoff (1882) and Heaviside (1893). The physicist Paul Dirac, after whom the function is named, in the 1920's popularised the concept of the delta function in quantum mechanics, see Dirac (1947), but from a mathematical viewpoint there were many shortcomings. The theory of generalised functions dates back to the work of Sobolev (1936) and Schwartz (1950), Schwartz (1951). A popular and readable text (*An introduction to Fourier Analysis and generalised functions*) was produced by Lighthill (1980) based on the theory of Mikunsinski (1948) and Temple (1953), Temple (1955). The brief introduction below follows Lighthill's book, but see also Jones (1982) (*Generalised Functions*) which does things in a more formal setting. If you want a very formal treatment with linear functionals and measure theory, the book by Vladimirov (2002) *Methods of the theory of generalised functions* is highly recommended.

We first need to introduce the idea of what Lighthill calls *good functions* and *fairly good functions.*

**Definition** *We say that* $f(x) \in \mathcal{C}^m(a,b)$ *if* $f(x)$ *and its first* $m$ *derivatives are continuous in the interval* $(a,b)$.

$f(x) \in \mathcal{C}^\infty(R)$ *is the class of infinitely smooth function in* $R$.

**Example** The function $e^{-x^2} \in \mathcal{C}^\infty(R)$.

**Definition** *A function is said to belong to* $\mathcal{G}$ *if* $f(x) \in \mathcal{C}^\infty(R)$ *and*

$$\lim_{|x|\to\infty} |x^m \frac{d^k}{dx^k} f(x)| = 0$$

*for every* $k$ *and for every integer* $m \geq 0$.

*The space* $\mathcal{G}$ *is the space of* good functions *in the sense of Lighthill. The space* $\mathcal{G}$ *is also called the* Schwartz space*.*

**Example** $e^{-x^2} \in \mathcal{G}$.

**Definition** *A function is said to belong to* $\mathcal{N}$ *if* $f(x) \in \mathcal{C}^\infty(R)$ *and if there exists some* $N$ *such that*

$$\lim_{|x|\to\infty} |x^{-N} \frac{d^k}{dx^k} f(x)| = 0$$

*for every* $k \geq 0$.

*The space* $\mathcal{N}$ *is the space of* fairly good functions *in the sense of Lighthill.*

**Example** $x^p \in \mathcal{N}$. Any polynomial expression belongs to $\mathcal{N}$.

The following properties are straightforward to demonstrate.

- $f(x) \in \mathcal{G} \implies f'(x) \in \mathcal{G}$.
- $f(x), g(x) \in \mathcal{G} \implies f(x) + g(x) \in \mathcal{G}$.
- $f(x) \in \mathcal{G}, g(x) \in \mathcal{N} \implies f(x)g(x) \in \mathcal{G}$.

**Definition** *A sequence* $\{\phi_n(x)\}_{n=1}^{\infty}$, *and* $\phi_n(x) \in \mathcal{G}$ *is called a* regular sequence in $\mathcal{G}$ *if for any* $f(x) \in \mathcal{G}$ *the limit*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x) f(x)\, dx$$

*exists.*

*Two regular sequences* $\{\phi_n(x)\}_{n=1}^{\infty}, \{\psi_n(x)\}_{n=1}^{\infty}$,*are* equivalent sequences in $\mathcal{G}$ *if*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x) f(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \psi_n(x) f(x)\, dx.$$

**Example** $e^{-x^2/n^2}, e^{-x^4/n^2}$ are equivalent sequences in $\mathcal{G}$.

**Definition** *Each equivalent class of regular sequences in* $\mathcal{G}$ *defines a* generalised function.

**Definition** *The sequence* $\delta_n(x) = \sqrt{\frac{n}{\pi}} e^{-nx^2}$ *defines the function* $\delta(x)$ *such that*

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$$

*for* $f(x) \in \mathcal{G}$.

**Proof** We have to prove that the limit of the sequence

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\, dx = f(0).$$

Now

$$\int_{-\infty}^{\infty} \delta_n(x)\, dx = \int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}} e^{-nx^2}\, dx = 1.$$

Also

$$|f(x) - f(0)| = |\int_0^x f'(s)\, ds| \le M|x|,$$

since $f(x) \in \mathcal{G}$ and is bounded. Thus

$$\left| \int_{-\infty}^{\infty} \delta_n(x) f(x)\, dx - f(0) \right| = \left| \int_{-\infty}^{\infty} \delta_n(x)(f(x) - f(0))\, dx \right|$$

$$\leq M \int_{-\infty}^{\infty} |x|\delta_n(x)\,dx = M \int_{-\infty}^{\infty} |x|\sqrt{\frac{n}{\pi}} e^{-nx^2}\,dx$$

$$\leq 2M\sqrt{\frac{n}{\pi}} \int_{0}^{\infty} x e^{-nx^2}\,dx = \frac{M}{\sqrt{n\pi}}[-e^{nx^2}]_0^{\infty} = \frac{M}{\sqrt{n\pi}}.$$

Hence taking the limit as $n \to \infty$ proves the result.

## 8.2 Derivatives of generalised functions

Suppose that $\{\phi_n(x)\}_{n=0}^{\infty}$ is a regular sequence in $\mathcal{G}$ then since $\phi_n'(x) \in \mathcal{G}$ we have after integrating by parts

$$\int_{-\infty}^{\infty} \phi_n'(x) f(x)\,dx = -\int_{-\infty}^{\infty} \phi_n(x) f'(x)\,dx,$$

for every $f(x) \in \mathcal{G}$. Letting $n \to \infty$ we see that $\{\phi_n'(x)\}_{n=0}^{\infty}$ is also a regular sequence. We denote the generalised function defined by this sequence as $\phi'(x)$ and we see that

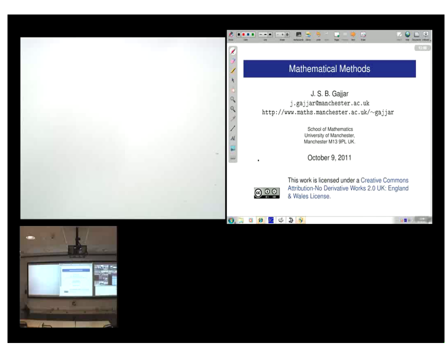
$$\int_{-\infty}^{\infty} \phi'(x) f(x)\,dx = -\int_{-\infty}^{\infty} \phi(x) f'(x)\,dx,$$

We can continue in this way and we see that generalised functions possess derivatives to all orders and in an obvious notation

$$\int_{-\infty}^{\infty} \frac{d^k\phi(x)}{dx^k} f(x)\,dx = (-1)^k \int_{-\infty}^{\infty} \phi(x) \frac{d^k}{dx^k} f(x)\,dx.$$

**Example**

$$\int_{-\infty}^{\infty} \frac{d^k\delta(x)}{dx^k} f(x)\,dx = (-1)^k f^{(k)}(0).$$




Video clip on introduction to generalised functions. Click here to open video clip in external player.

Suppose $\{\phi_n(x)\}_{n=0}^{\infty}$ is a regular sequence in $\mathcal{G}$. Then

$$\int_{-\infty}^{\infty} \phi_n(ax+b)F(x)\,dx = |a|^{-1}\int_{-\infty}^{\infty} \phi_n(x)F(\frac{x-b}{a})\,dx.$$

Hence for $F(x) \in \mathcal{G}$

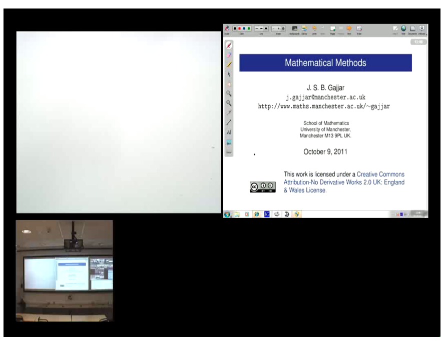
$$\int_{-\infty}^{\infty} \phi(ax+b)F(x)\,dx = |a|^{-1}\int_{-\infty}^{\infty} \phi(x)F(\frac{x-b}{a})\,dx, \quad a \neq 0.$$

**Example**

$$\int_{-\infty}^{\infty} \delta(ax-b)F(x)\,dx = |a|^{-1}\int_{-\infty}^{\infty} \delta(x)F(\frac{x+b}{a})\,dx = |a|^{-1}F(\frac{b}{a}), \quad a \neq 0.$$

**Definition** *We say that $f(x) \in L_p(R)$ if $\int_{-\infty}^{\infty} |f(x)|^p\,dx$ exists.*

Thus for example $L_1(R)$ is the space of absolutely integrable functions.




Video clip discussing properties of generalised functions. Click here to open video clip in external player.

**Definition** *The function $f(x) \in K_p(R)$ if for some $N \geq 0$*

$$\int_{-\infty}^{\infty} \frac{|f(x)|^p}{(1+x^2)^N}\,dx < \infty.$$

**Example** Consider the Heaviside function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

Clearly $H(x)$ is not absolutely integrable but $H(x) \in K_1(R)$, with $N = 1$.

**Example** $f(x) = H(x)x^3 \in K_1(R)$ with $N = 3$.

**Theorem** *Suppose $f(x) \in K_1(R)$. Then it is possible to construct a regular sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ in $\mathcal{G}$ which defines a generalised function $\phi(x)$ such that*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x)F(x)\,dx = \int_{-\infty}^{\infty} \phi(x)F(x)\,dx = \int_{-\infty}^{\infty} f(x)F(x)\,dx, \tag{8.2.1}$$

*for any $F(x) \in \mathcal{G}$.*

**Proof** For a proof see Jones, section 3.2.

The main point of this theorem is that it allows one to define a whole range of generalised functions for which the integral on the right hand side of (8.2.1) exists. The integral exists in the normal sense.

**Example** Consider the Heaviside function introduced earlier. This satisfies the condition of the theorem. Hence we can consider $H(x)$ as a generalised function and using an earlier result for the derivatives of generalised functions, we have

$$\int_{-\infty}^{\infty} H'(x)F(x)\,dx = -\int_{-\infty}^{\infty} H(x)F'(x)\,dx$$

for any $F(x) \in \mathcal{G}$. Now

$$\int_{-\infty}^{\infty} H(x)F'(x)\,dx = \int_{0}^{\infty} F'(x)\,dx = [F(x)]_0^{\infty} = -F(0).$$

Hence we see that

$$\int_{-\infty}^{\infty} H'(x)F(x)\,dx = F(0)$$

and thus

$$H'(x) = \delta(x).$$

**Example** Consider the function sgn(x) defined by

$$\mathrm{sgn(x)} = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$$

This satisfies the condition of the theorem with $N = 1$. Hence

$$\int_{-\infty}^{\infty} \frac{d\mathrm{sgn(x)}}{\mathrm{dx}} F(x)\,dx = -\int_{-\infty}^{\infty} \mathrm{sgn(x)F'(x)\,dx},$$

$$= \int_{-\infty}^{0} F'(x)\,dx - \int_{0}^{\infty} F'(x)\,dx = 2F(0).$$

Hence we have

$$\frac{d\mathrm{sgn(x)}}{\mathrm{dx}} = 2\delta(x).$$

Consider the function $|x|^\alpha$. Now

$$\int_{-\infty}^{\infty} (1+x^2)^{-N}|x|^\alpha \, dx$$

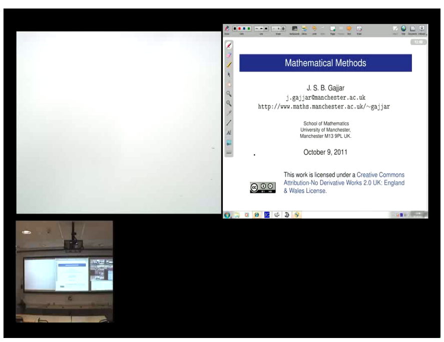
is convergent only if $\alpha > -1$ and if we take $2N > 1 + \alpha$. Thus we can define a generalised function $|x|^\alpha$ if $\alpha > -1$. Now

$$\frac{d}{dx} \log|x| = |x|^{-1}\mathrm{sgn(x)},$$

and so

$$\frac{d}{dx}|x|^\alpha = \alpha|x|^{\alpha-1}\mathrm{sgn(x)},$$

provided also $\alpha > 0$.




Video clip discussing extensions to include functions in $K_p(R)$. Click here to open video clip in external player.

We make use of a result which states that if $f(x)$ is an ordinary function and both $f'(x)$ and $f(x)$ belong to $\mathcal{K}_1(R)$ then the derivative of the generalised function formed by $f(x)$ is the generalised function formed by $f'(x)$.

This can be used to define generalised functions such as $|x|^\alpha$ for non-integral $\alpha < 0$. For all $\alpha$ and $\alpha$ not equal to a negative integer, we can define the generalised functions

$$|x|^\alpha = \frac{1}{(\alpha+1)(\alpha+2)\dots(\alpha+n)} \frac{d^n}{dx^n}[|x|^{\alpha+n}(\mathrm{sgn}(x))^n],$$

$$|x|^\alpha \mathrm{sgn}(x) = \frac{1}{(\alpha+1)(\alpha+2)\dots(\alpha+n)} \frac{d^n}{dx^n}[|x|^{\alpha+n}(\mathrm{sgn}(x))^{n+1}],$$

$$|x|^\alpha H(x) = \frac{1}{(\alpha+1)(\alpha+2)\dots(\alpha+n)} \frac{d^n}{dx^n}[x^{\alpha+n}H(x))],$$

where $n$ is a positive integer such that $n + \Re(\alpha) > -1$. For completeness, the generalised function $x^{-1}$ is defined by

$$x^{-1} = \frac{d}{dx}[\log|x|],$$

and if $m$ is a positive integer

$$x^{-m} = \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{dx^{m-1}}(\log|x|)^{m-1}.$$

## 8.3 Application to singular integrals

We will show one use of the above results for handling singular integrals. But first we need the following result.

*Suppose $f(x)$ is a continuous function with a derivative $f'(x)$ both belonging to $\mathcal{K}_1$. Then*

$$\frac{d}{dx}[f(x)H(x-a)] = \frac{df}{dx}H(x-a) + f(a)\delta(x-a).$$

**Proof**

With the given conditions $f(x)H(x-a)$ and the derivative $\frac{d}{dx}[f(x)H(x-a)]$ define generalised functions. Hence for any good function $\phi(x)$ we have

$$\int_{-\infty}^{\infty}\frac{d}{dx}[f(x)H(x-a)]\phi(x)\,dx = -\int_{-\infty}^{\infty} f(x)H(x-a)\phi'(x)\,dx$$

$$= -\int_a^{\infty} f(x)\phi'(x)\,dx.$$

Integrating by parts we obtain

$$-\int_{-\infty}^{\infty} f(x)H(x-a)\phi'(x)\,dx = -[f(x)\phi(x)]_a^{\infty} + \int_a^{\infty} f'(x)\phi(x)\,dx$$

$$= \int_{-\infty}^{\infty}[f'(x)H(x-a) + \delta(x-a)f(a)]\phi(x)\,dx.$$

Hence

$$\frac{d}{dx}[f(x)H(x-a)] = \frac{df}{dx}H(x-a) + f(a)\delta(x-a).$$

Consider the integral

$$\int_a^b \frac{1}{x}\phi(x)\,dx$$

where $\phi(x)$ is a continuous differentiable function of $x$ and $\phi(0) \neq 0$ and $a < 0 < b$. In the normal sense the integral does not exist since

$$\left(\lim_{\epsilon_1\to 0-}\int_a^{\epsilon_1} + \lim_{\epsilon_2\to 0+}\int_{\epsilon_2}^b\right)\frac{\phi(x)}{x}\,dx$$

does not exist if $\epsilon_1, \epsilon_2$ go to zero independently.

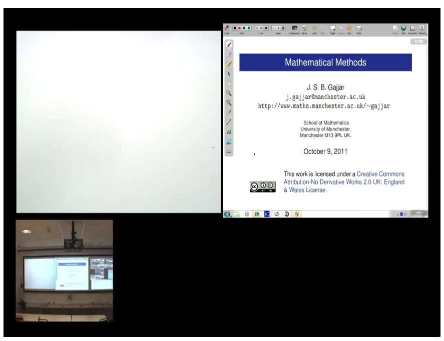
However the *Cauchy principal value* of the integral is defined by

$$\fint_a^b \frac{\phi(x)}{x}\,dx = \lim_{\epsilon\to 0}\left(\int_a^{-\epsilon} + \int_\epsilon^b\right)\frac{\phi(x)}{x}\,dx$$

$$= \phi(b)\log|b| - \phi(a)\log|a| - \int_a^b \phi'(x)\log|x|\,dx.$$

**Example**

$$\fint_{-1}^2 \frac{1}{x}\,dx = \log(2).$$




Video clip discussing Cauchy principal value integral. Click here to open video clip in external player.

Let us see how we can tackle the integral using generalised functions. Consider

$$\int_a^b \frac{\phi(x)}{x}\,dx = \int_{-\infty}^{\infty}[H(x-a)x^{-1} - H(x-b)x^{-1}]\phi(x)\,dx.$$

Now

$$[(H(x-a) - H(x-b))\log|x|]' =$$
$$[H(x-a)\log(|x|/|a|) - H(x-b)\log(|x|/|b|)$$
$$+H(x-a)\log|a| - H(x-b)\log|b|]'$$
$$= \{x^{-1}H(x-a) - x^{-1}H(x-b)\} + \delta(x-a)\log|a| - \delta(x-b)\log|b|.$$

Hence

$$\int_{-\infty}^{\infty}[H(x-a)x^{-1} - H(x-b)x^{-1}]\phi(x)\,dx =$$
$$\int_{-\infty}^{\infty}\{(H(x-a) - H(x-b))\log|x|]'\phi(x)\,dx$$
$$+\int_{-\infty}^{\infty}(\delta(x-b)\log|b| - \delta(x-a)\log|a|)\phi(x)\,dx$$

$$= \phi(b)\log|b| - \phi(a)\log|a| - \int_{-\infty}^{\infty}\{(H(x-a)-H(x-b))\log|x|\}\phi'(x)\,dx$$

which is the same as the Cauchy principal value interpretation.

Consider next singular integrals of the form

$$\int_0^b x^\beta \phi(x)\,dx$$

where $0 < b$ and $\beta$ is not a negative integer. We can write the integral as

$$\int_{-\infty}^{\infty}(x^\beta H(x) - x^\beta H(x-b))\phi(x)\,dx.$$

Next note that

$$[x^{\beta+n}H(x-b)]' = [(x^{\beta+n} - b^{\beta+n})H(x-b) + b^{\beta+n}H(x-b)]'.$$
$$= (\beta+n)x^{\beta+n-1}H(x-b) + b^{\beta+n}\delta(x-b).$$

Continue differentiating like this to obtain

$$[x^{\beta+n}H(x-b)]^{(n)} =$$
$$(\beta+n)(\beta+n-1)...(\beta+1)x^\beta H(x-b) + b^{\beta+n}\delta^{(n-1)}(x-b)+$$
$$+(\beta+n)b^{\beta+n-1}\delta^{(n-2)}(x-b) + \cdots + (\beta+n)...(\beta+2)b^{\beta+1}\delta(x-b).$$

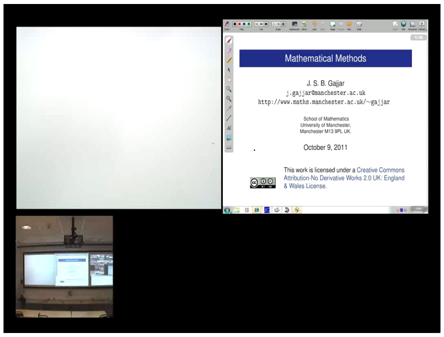
Hence the integral can be written as

$$\int_{-\infty}^{\infty}\{x^\beta H(x) - x^\beta H(x-b)\}\phi(x)\,dx =$$
$$\int_{-\infty}^{\infty}\left[\frac{(x^{\beta+n}H(x) - x^{\beta+n}H(x-b))^{(n)}}{(\beta+n)(\beta+n-1)...(\beta+1)}\right.$$
$$\left.+\frac{b^{\beta+n}\delta^{(n-1)}(x-b)}{(\beta+n)...(\beta+1)} + \cdots + \frac{b^{\beta+1}\delta(x-b)}{\beta+1}\right]\phi(x)\,dx.$$

Finally we obtain

$$\int_0^b x^\beta\phi(x)\,dx =$$
$$\frac{(-1)^n}{(\beta+1)(\beta+2)...(\beta+n)}\int_{-\infty}^{\infty}x^{\beta+1}(H(x)-H(x-b))\phi^{(n)}(x)\,dx$$
$$+\frac{b^{\beta+1}}{(\beta+1)}\phi(b) - \frac{b^{\beta+2}\phi'(b)}{(\beta+1)(\beta+2)} + \cdots + \frac{(-1)^{n-1}b^{\beta+n}\phi^{(n-1)}(b)}{(\beta+1)(\beta+2)...(\beta+n)}.$$

The above intepretation agrees with the *Hadamard finite part* of the integral $\int_0^b x^\beta\phi(x)\,dx$.




Video clip discussing Hadamard finite part integral. Click here to open video clip in external player.

**Example** Consider

$$\int_0^{*b} x^{-3/2} f(x)\, dx = \lim_{\epsilon \to 0} \int_\epsilon^b x^{-3/2} f(x)\, dx.$$

[The notation $\int^*$ is sometimes used to denote a singular integral and that we need to interpret in the Hadamard finite part sense]. Now

$$\int_\epsilon^b x^{-3/2} f(x)\, dx. = [-2x^{-1/2} f(x)]_\epsilon^b + \int_\epsilon^b 2x^{-1/2} f(x)\, dx$$

$$= [-2b^{-1/2} f(b)] + 2\epsilon^{-1/2} f(\epsilon) + \int_\epsilon^b 2x^{-1/2} f'(x)\, dx$$

The Hadamard finite part of the integral is defined by ignoring the $\epsilon^{-1/2} f(\epsilon)$ term and taking the limit as $\epsilon \to 0$ giving

$$\int_0^{*b} x^{-3/2} f(x)\, dx = [-2b^{-1/2} f(b)] + \int_\epsilon^b 2x^{-1/2} f'(x)\, dx.$$

**Example** Consider

$$\int_0^1 \frac{x^{-5/2}}{1+x}\, dx$$

We can write

$$x^{-\frac{5}{2}}(H(x) - H(x-1)) =$$
$$\frac{1}{(-\frac{5}{2}+1)(-\frac{5}{2}+2)} \frac{d^2}{dx^2}\left[x^{-\frac{1}{2}}(H(x) - H(x-1))\right]$$
$$+ \frac{\delta(x-1)}{(-\frac{5}{2}+1)} + \frac{\delta'(x-1)}{(-\frac{5}{2}+1)(-\frac{5}{2}+2)}.$$

Hence with $\phi(x) = 1/(x+1)$ we have

$$\int_0^1 \frac{x^{-\frac{5}{2}}}{1+x}\, dx = \int_{-\infty}^{\infty} \frac{4}{3} \frac{d^2}{dx^2}\left[x^{-\frac{1}{2}}(H(x) - H(x-1))\right] \phi(x)\, dx$$

$$+\int_{-\infty}^{\infty}(-\frac{2}{3}\delta(x-1)+\frac{4}{3}\delta'(x-1))\phi(x)\,dx,$$

$$=\frac{4}{3}\int_{-\infty}^{\infty}x^{-\frac{1}{2}}(H(x)-H(x-1))\phi''(x)\,dx-\frac{2}{3}\phi(1)-\frac{4}{3}\phi'(1)$$

$$=\frac{4}{3}\int_{0}^{1}\frac{2}{x^{\frac{1}{2}}(x+1)^{3}}\,dx=\frac{\pi}{2}+\frac{4}{3}$$

We will investigate Fourier transforms properties of generalised functions after we have discussed Fourier transforms.

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# Chapter 9

# Integral Transforms

## 9.1 Fourier Transform

**Definition** *We say that*

$$f(x) \in L_p(\Omega) \quad \textit{if} \quad \int_\Omega |f(x)|^p \, dx < \infty.$$

Note that $L_1(R)$ is the space of absolutely integrable functions in $R$.

**Theorem** *Suppose $f(t)$ and its derivative is continuous on $R$ except at a finite number of points for which $f$ has integrable bounded discontinuities, and $f(t) \in L^1(R)$. Then Fourier's Integral Theorem states that*

$$\frac{1}{2}[f(x+) + f(x-)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dk \int_{-\infty}^{\infty} e^{-ikt} f(t) \, dt \qquad (9.1.1)$$

Note that if $f(t)$ is continuous at $t = x$ the left hand side of (9.1.1) reduces to $f(x)$. For a proof of (9.1.1) see, for example, the book by Sneddon *The use of integral transforms*.

The theorem can also be proved under less restrictive conditions, see Titchmarsh.

**Definition** *The Fourier transform $F(k)$ of the function $f(x)$ is defined by*

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx, \qquad (9.1.2)$$

*and the inverse Fourier transform by*

$$\frac{1}{2}[f(x+) + f(x-)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} \, dk. \qquad (9.1.3)$$

Note that different authors may use definitions of a FT different to that given above.

We will sometimes write $\mathcal{F}(f(x);k)$ to denote the FT when we want to show the function explictly. Likewise $\mathcal{F}^{-1}(F(k))$ will denote the inverse Fourier transform.

### 9.1.1 Basic properties of FT

Some of the following properties are easy to prove. Suppose $F(k)$ is the FT of $f(x)$. Then

1. $\mathcal{F}(f(x-a);k) = e^{-ika}\mathcal{F}(f(x);k)$.
2. $\mathcal{F}(f(ax);k) = \frac{1}{|a|}F(\frac{k}{a})$.
3. $\mathcal{F}(e^{iax}f(x);k) = F(k-a)$.
4. $\mathcal{F}(\overline{f(-x)};k) = \overline{\mathcal{F}(f(x);k)}$.
5. $\mathcal{F}(F(x);k) = f(-k)$.

In the above the overbar denotes the complex conjugate.

**Theorem** *If $f(x)$ satisfies the conditions of the Fourier Integral Theorem, then*

1. *$F(k)$ is bounded for $-\infty < k < \infty$.*
2. *$F(k)$ is continuous for $-\infty < k < \infty$.*

**Proof of (1)**

$$|F(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ikx}||f(x)|\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\,dx < M$$

for some constant $M$ since $f$ is absolutely integrable. Hence $F(k)$ is bounded for real $k$.

**Proof of (2)** Consider

$$|F(k+h) - F(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ikh} - 1||f(x)|\,dx \leq \sqrt{\frac{2}{\pi}}M.$$

Also

$$\lim_{h\to 0} |e^{-ihx} - 1| = 0$$

for all $x \in R$. Hence

$$\lim_{h\to 0} |F(k+h) - F(k)| \le \lim_{h\to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ihx} - 1||f(x)|\, dx = 0.$$

Thus $F(k)$ is continuous.

*If $f(x)$ is continuously differentiable and $f(x) \to 0$ as $|x| \to \infty$ then*

$$\mathcal{F}(f'(x); k) = ikF(k).$$

The proof follows easily from integration by parts.

*If $f(x)$ has a jump discontinuity at $x = x_0$ then*

$$\begin{aligned}
\mathcal{F}(f'(x); k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-ikx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{x_0} + \int_{x_0}^{\infty} [f'(x) e^{-ikx}] \right) dx \\
&= ikF(k) + e^{-ikx_0}[f(x_0+) - f(x_0-)].
\end{aligned}$$

**Example** Let $a, b > 0$ and consider

$$\begin{aligned}
f(x) &= e^{ax} \quad x < 0 \\
&= e^{-bx} \quad x > 0.
\end{aligned}$$

Then

$$\begin{aligned}
F(k) &= \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{0} e^{ax} e^{-ikx}\, dx + \int_{0}^{\infty} e^{-bx} e^{-ikx}\, dx \right), \\
&= \frac{1}{\sqrt{2\pi}} \left( \frac{1}{a - ik} + \frac{1}{b + ik} \right).
\end{aligned}$$

We can invert the transform obtained in the last example using contour integration. Now

$$I_1 = \int_{-\infty}^{\infty} e^{ikx} \frac{1}{a - ik}\, dk = \int_{-\infty}^{\infty} e^{ikx} \frac{i}{k + ia}\, dk.$$

Consider

$$\int_C i \frac{e^{ixz}}{z + ia}\, dz$$

where the contour $C$ is chosen appropriately.

For $x < 0$ we choose $C$ to be the real axis and the semicircular arc $C_1$ in the lower half plane, see figure 9.1.

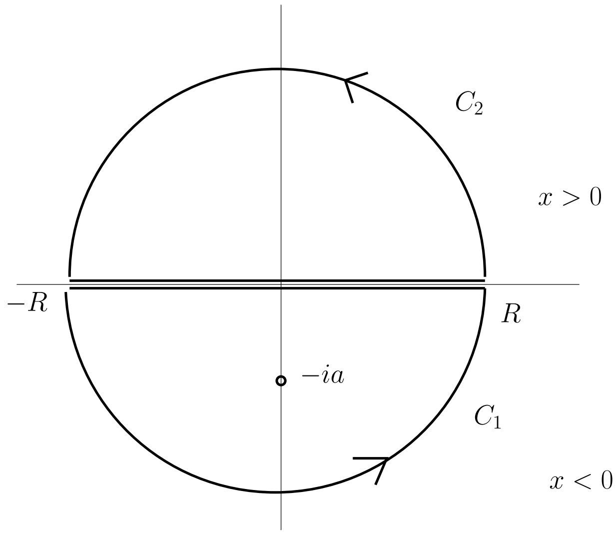



Figure 9.1: Contours for example.

The integrand has a simple pole at $z = -ia$ and so using Cauchy's theorem

$$-\int_{-R}^{R} i\frac{e^{ikx}}{k+ia}\,dk + \int_{C_1} i\frac{e^{izx}}{z+ia}\,dk = 2\pi i Res[\frac{ie^{izx}}{z+ia}]_{z=-ia} = -2\pi e^{xa}.$$

From Jordan's Lemma $\int_{C_1} \to 0$ as the radius of the circle $R$ increases. Hence

$$I_1 = 2\pi e^{xa}, \quad x < 0$$

Similarly for $x > 0$ if we deform in the upper-half plane see figure 9.1, and applying Cauchy's Theorem gives

$$\int_{-R}^{R} i\frac{e^{ikx}}{k+ia}\,dk + \int_{C_2} i\frac{e^{izx}}{z+ia}\,dz = 0$$

This gives $I_1 = 0$ for $x > 0$. For $x = 0$ using Cauchy's theorem

$$\int_{-R}^{R} \frac{i}{k+ia}\,dk + \int_{C_2} \frac{i}{z+ia}\,dz = 0$$

Thus

$$\int_{-R}^{R} \frac{i}{k+ia}\,dk = -i\int_{0}^{\pi} \frac{iRe^{i\theta}}{Re^{i\theta}+ia}\,d\theta,$$

$$= \frac{1}{i}[\log(Re^{i\theta}+ia)]_0^{\pi} = \frac{1}{i}\log\frac{[Re^{i\pi}+ia]}{[R+ia]}.$$

Hence taking the limit $R \to \infty$ we find

$$\int_{-\infty}^{\infty} \frac{i}{k+ia}\,dk = \pi.$$

Putting the results together we find that

$$\mathcal{F}^{-1}[\frac{1}{\sqrt{2\pi}}\frac{1}{a-ik}] = \begin{cases} e^{ax} & x<0, \\ \frac{1}{2} & x=0, \\ 0 & x>0 \end{cases} .$$

In the same way we can invert $\frac{1}{\sqrt{2\pi}(b+ik)}$ using the contours as shown in figure 9.2 to obtain

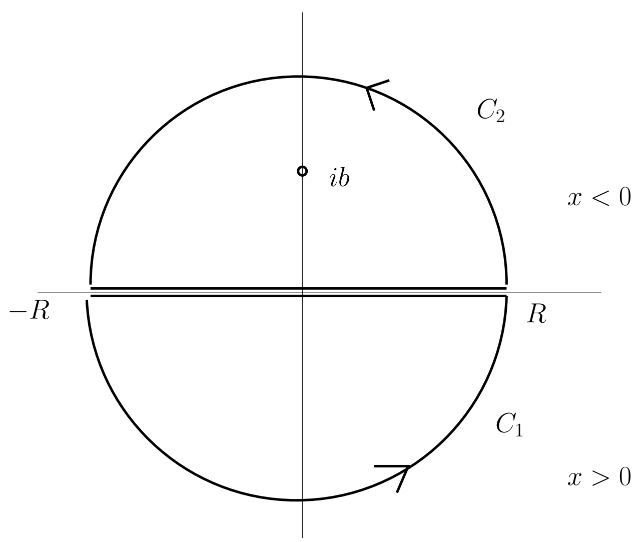



Figure 9.2: Contours for example.

$$\mathcal{F}^{-1}[\frac{1}{\sqrt{2\pi}}\frac{1}{(b+ik)}] = \begin{cases} 0 & x<0, \\ \frac{1}{2} & x=0, \\ e^{-bx} & x>0. \end{cases} .$$

Putting it all together we find

$$\mathcal{F}^{-1}(F(k)) = \begin{cases} e^{ax} & x<0, \\ 1 & x=0, \\ e^{-bx}. & x>0 \end{cases}$$

### 9.1.2 Convolution Theorem

**Definition** *We define the convolution of two integrable functions $f(x)$ and $g(x)$ as the operation $f*g$ given by*

$$f(x)*g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi.$$

**Convolution Theorem** *Suppose that $F(k)$ and $G(k)$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively. Then*

$$\mathcal{F}\{f(x) * g(x)\} = F(k)G(k),$$

*and*

$$f(x) * g(x) = \mathcal{F}^{-1}(F(k)G(k)),$$

*ie*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\, d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)G(k)e^{ikx}\, d\xi.$$

**Proof**

$$\begin{aligned}\mathcal{F}\{f(x) * g(x)\} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx}\, dx \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\, d\xi, \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik\xi} g(\xi)\, d\xi \int_{-\infty}^{\infty} e^{-ik(x-\xi)} f(x-\xi)\, dx, \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik\xi} g(\xi)\, d\xi \int_{-\infty}^{\infty} e^{-ik\eta} f(\eta)\, d\eta, \\ &= G(k)F(k).\end{aligned}$$

Hence proof.

### 9.1.3 Parseval's Relation

Suppose we put $x = 0$ in the convolution theorem

$$\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\, d\xi = \int_{-\infty}^{\infty} e^{ikx} F(k)G(k)\, dk.$$

This gives

$$\int_{-\infty}^{\infty} f(\xi)g(-\xi)\, d\xi = \int_{-\infty}^{\infty} F(k)G(k)\, dk.$$

Next substitute $g(x) = \overline{f(-x)}$ and note that (see section 9.1.1)

$$G(k) = \mathcal{F}\{(g(x)\} = \mathcal{F}\{\overline{f(-x)}\} = \overline{\mathcal{F}\{f(x)\}} = \overline{F(k)}.$$

Hence

$$\int_{-\infty}^{\infty} f(x)\overline{f(x)}\, dx = \int_{-\infty}^{\infty} F(k)\overline{F(k)}\, dk,$$

or

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(k)|^2\, dk. \tag{9.1.4}$$

This result is known as Parseval's relation. It is important in many signal processing applications. The integral $\int_{-\infty}^{\infty} |f(x)|^2\, dx$ can be thought of as the energy of a signal and $\int_{-\infty}^{\infty} |F(k)|^2\, dk$ is what is really measured and known as the power spectrum.

### 9.1.4 Fourier transform of generalised functions

Consider the generalised function $\delta(x)$ introduced earlier. If we proceed naively the Fourier Transform of $\delta(x)$ is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}},$$

and by the Fourier inversion theorem

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\,dx.$$

The results are correct but recall the 'sifting properties' were only established for the class of good functions $f(x) \in \mathcal{G}$ and $e^{ikx}$ does not belong to this class. We have to first establish some properties for Fourier Transforms of generalised functions.

**Theorem** *If $\{\phi_n(x)\} \in \mathcal{G}$ then its Fourier Transform $\Gamma_n(k)$ also belongs to $\mathcal{G}$.*

**Proof** For any $p \geq 0$

$$\left|\int_{-\infty}^{\infty} x^p\phi_n(x)e^{-ikx}\,dx\right| \leq \int_{-\infty}^{\infty}|x|^p|\phi_n(x)|\,dx < \infty$$

since $\phi_n(x)$ is a good function.

Note that if $f(x) \in \mathcal{G}$ then its Fourier Transform $F(k)$ exists and also

$$\frac{dF}{dk} = \int_{-\infty}^{\infty} -ixf(x)e^{-ikx}\,dx$$

exists. In fact all the derivatives of $F(k)$ exist and

$$\frac{d^qF}{dk^q} = \int_{-\infty}^{\infty}(-ix)^q f(x)e^{-ikx}\,dx$$

We also need to prove that for any $r > 0$

$$\lim_{|k|\to\infty}|k^r\frac{d^qF(k)}{dk^q}| = 0$$

for all $q$.

Now using

$$\frac{d^qF}{dk^q} = \int_{-\infty}^{\infty}(-ix)^q f(x)e^{-ikx}\,dx$$

integration by parts gives

$$= \frac{1}{ik}\int_{-\infty}^{\infty}e^{-ikx}\frac{dg}{dx}\,dx$$

with $g(x) = (-ix)^q f(x)$.

Thus

$$\frac{d^qF}{dk^q} = \frac{1}{(ik)^m}\int_{-\infty}^{\infty} e^{-ikx}\frac{d^mg}{dx^m}\,dx$$

Hence one can choose $m$ such that for any $r > 0$

$$\lim_{|k|\to\infty} |k^r \frac{d^qF(k)}{dk^q}| = 0$$

Thus $F(k)$ is also a good function belonging to $\mathcal{G}$.

*Suppose $\{\phi_n(x)\}$ is a regular sequence and $f(x) \in \mathcal{G}$. If $\Phi_n(k)$ is the Fourier Transform of $\phi_n(x)$ and $F(k)$ the Fourier transform of $f(x)$, then from Parseval's theorem*

$$\int_{-\infty}^{\infty} \Phi_n(x)F(x)\,dx = \int_{-\infty}^{\infty} \phi_n(x)f(-x)\,dx.$$

Hence if the limit

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} \phi_n(x)f(-x)\,dx$$

exists for every arbitrary member of $\mathcal{G}$ then the limit

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} \Phi_n(x)F(x)\,dx$$

also exists for every $F \in \mathcal{G}$. If $\phi(x)$ is the generalised function defined by the regular sequence $\{\phi_n(x)\}$ then we define the generalised function $\Phi(x)$ by the regular sequence $\{\Phi_n(k)\}$ and we call $\Phi(k)$ the Fourier transform of $\phi(x)$.

**Example** Consider

$$\delta_n(x) = \sqrt{\frac{n}{\pi}}e^{-x^2n}.$$

This defines the delta function $\delta(x)$ and note that

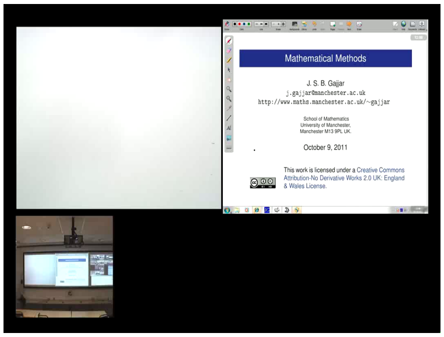
$$\Delta_n(k) = \int_{-\infty}^{\infty}\sqrt{\frac{n}{\pi}}e^{-nx^2-ikx}\,dx$$

$$= \sqrt{\frac{n}{\pi}}e^{\frac{-k^2}{4n}}\int_{-\infty}^{\infty} e^{-n(x+\frac{ik}{2n})^2}\,dx,$$

$$= e^{\frac{-k^2}{4n}}.$$

Taking the limit as $n \to \infty$ shows that $\Delta_n(k)$ defines the generalised function 1. Hence the Fourier Transform of $\delta(x)$ is $1/\sqrt{2\pi}$, ie

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}.$$




Video clip on Fourier Transforms of generalised functions. Click here to open video clip in external player.

The inverse transform transform can be worked out in a similar way by taking the sequence $\Delta_n(k) = e^{-k^2/(4n)}$ and shows that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \Delta_n(k) e^{ikx}\, dk = 2\pi.$$

Hence

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} 1 e^{ikx}\, dk = \delta(x),$$

which is a restatement of the Fourier inversion for $\delta(x)$.

*One can show that if $G(k)$ is the Fourier transform of a generalised function $g(x)$ then*

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(k) e^{ikx} dk.$$

More generally, suppose $\{\frac{d^p\phi_n}{dx^p}(x)\}$ defines the generalised function $\frac{d^p\phi}{dx^p}(x)$ then the Fourier transform of $\{\frac{d^p\phi_n}{dx^p}(x)\}$ is given by

$$\int_{-\infty}^{\infty} \{\frac{d^p\phi_n}{dx^p}(x)\} e^{-ikx}\, dx = (ik)^p \Phi(k),$$

where $\Phi(k)$ is the FT of $\phi(x)$.

**Example**

$$\mathcal{F}(\delta^{(p)}(x); k) = (ik)^p \frac{1}{\sqrt{2\pi}},$$

and using the inversion formula

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (ik)^p e^{ikx}\, dk = \delta^{(p)}(x).$$

Other properties of generalised functions also carry over to Fourier transforms, thus for example

$$\int_{-\infty}^{\infty} \delta(ax-b) e^{-ikx}\, dx = \frac{e^{-ibk/a}}{|a|}.$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-ibk}\,dk = \delta(x-b).$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (ik)^p e^{ikx-ibk}\,dk = \delta^{(p)}(x-b).$$

**Example** Consider the Fourier transform of the function

$$f(x) = e^{-ax}H(x) - e^{ax}H(-x)$$

where $a > 0$ and $H(x)$ is the Heaviside function. Now

$$\sqrt{2\pi}F(k) = \int_{-\infty}^{\infty} (e^{-ax}H(x) - e^{-ax}H(-x))e^{-ikx}\,dx,$$

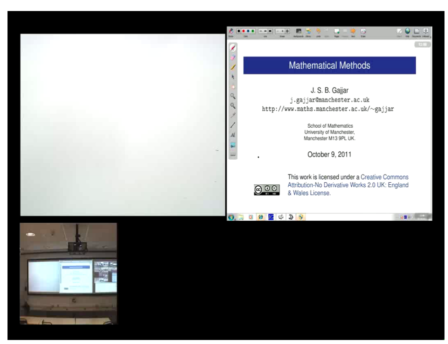
$$= \int_{0}^{\infty} e^{-(a+ik)x}\,dx - \int_{-\infty}^{0} e^{(a-ik)x}\,dx,$$

$$= \frac{1}{(a+ik)} - \frac{1}{a-ik}.$$

Now take the limit as $a \to 0$. Then we see that $f(x)$ approaches the generalised function sgn(x). Thus the Fourier transform of sgn(x) is $\frac{2}{ik}\frac{1}{\sqrt{2\pi}}$. Next note that $H(x) = \frac{1}{2}(1 + \text{sgn(x)})$. Thus the Fourier transform of $H(x)$ is

$$\mathcal{F}(H(x);k) = (\frac{1}{ik} + \sqrt{\frac{\pi}{2}}\delta(-k))\frac{1}{\sqrt{2\pi}}.$$

Video clip on Fourier Transforms of generalised functions- examples. Click here to open video clip in external player.




### 9.1.5 Solutions of PDE's using FT and Green's functions

Consider the solution of Laplace's equation in the half-plane with Dirichlet boundary conditions, ie

$$\phi_{xx} + \phi_{yy} = 0, \quad -\infty < x < \infty, \quad y \geq 0 \tag{9.1.5}$$

with boundary conditions

$$\phi(x,0) = f(x), \quad -\infty < x < \infty,$$

$$\phi(x,y) \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty.$$

Let $\Phi(k,y)$ be the FT of $\phi(x,y)$ ie

$$\Phi(k,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x,y) e^{-ikx}\, dx.$$

The FT of the equation (9.1.5) and boundary conditions gives

$$\frac{d^2\Phi}{dy^2} - k^2\Phi = 0, \tag{9.1.6}$$

$$\Phi(k,0) = F(k), \quad \phi(k,y) \to 0 \quad \text{as } y \to \infty,$$

where $F(k)$ is the FT of $f(x)$. The solution of (9.1.6) gives

$$\Phi(k,y) = F(k) e^{-|k|y}. \tag{9.1.7}$$

Inverting (9.1.7) and using the convolution theorem yields

$$\phi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi,$$

where $g(x)$ is the inverse Fourier transform of $e^{-|k|y}$, and

$$\mathcal{F}^{-1}(e^{-|k|y}) = \sqrt{\frac{2}{\pi}} \frac{y}{x^2+y^2}.$$

Hence

$$\phi(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \frac{y}{((x-\xi)^2 + y^2)}\, d\xi, \quad y > 0.$$

This is Poisson's integral formula for the half-plane. From

$$\phi(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \frac{y}{((x-\xi)^2 + y^2)}\, d\xi, \quad y > 0$$

taking the limit as $y \to 0+$ shows that

$$\phi(x,0) = f(x) = \frac{1}{\pi} \lim_{y \to 0+} \int_{-\infty}^{\infty} f(\xi) \frac{y}{((x-\xi)^2 + y^2)}\, d\xi, \quad y > 0.$$

Thus another representation of the delta function is

$$\delta(x-\xi) = \lim_{y \to 0} \frac{1}{\pi} \frac{y}{[(x-\xi)^2 + y^2]}.$$

Consider the diffusion equation

$$\phi_t = \kappa\phi_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$

subject to

$$\phi(x,0) = f(x), \quad -\infty < x < \infty,$$

and $\kappa > 0$ is a constant. Define the FT

$$\Phi(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x,t) e^{-ikx}\, dx.$$

Then the transform of the equation becomes

$$\Phi_t = -\kappa k^2 \Phi, \quad t > 0,$$

and

$$\Phi(k, t = 0) = F(k)$$

where $F(k)$ is the FT of $f(x)$. Solving gives

$$\Phi(k,t) = F(k)e^{-\kappa k^2 t}.$$

Inverting gives

$$\begin{aligned}\phi(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} e^{-\kappa k^2 t}\, dk, \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi)\, d\xi,\end{aligned}$$

after using the convolution theorem and where $g(x)$ has the Fourier transform $e^{-\kappa k^2 t}$. Thus

$$\begin{aligned}g(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\kappa k^2 t} e^{ikx}\, dk, \\ &= \frac{1}{\sqrt{2\kappa t}} e^{-\frac{x^2}{4\kappa t}}.\end{aligned}$$

Hence

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} f(\xi) \exp[-\frac{(x-\xi)^2}{4\kappa t}]\, d\xi.$$

We can write

$$\phi(x,t) = \int_{-\infty}^{\infty} f(\xi) G(x-\xi, t)\, d\xi,$$

where the function

$$G(x-\xi, t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp[-\frac{(x-\xi)^2}{4\kappa t}]$$

is the Green's function of the diffusion equation for the infinite interval. Note that we have another representation of the Delta function by taking the limit $t \to 0+$ ie

$$\delta(x-\xi) = \lim_{t\to 0+} \frac{1}{\sqrt{4\pi\kappa t}} \exp[-\frac{(x-\xi)^2}{4\kappa t}].$$

Consider the linearised K-dV equation

$$u_t + cu_x + u_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

where $c > 0$ is a constant. The initial conditions are

$$u(x, t = 0) = f(x).$$

Define a FT

$$U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ikx}\, dx$$

and taking a FT of the equation shows that

$$U(k,t) = F(k) e^{-(ikc - ik^3)t}.$$

Hence

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ik(x-ct)} e^{ik^3 t}\, dk.$$

Suppose we take $f(x) = \delta(x)$, and $F(k) = \frac{1}{\sqrt{2\pi}}$ Then

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-ct)} e^{ik^3 t}\, dk$$

$$= \frac{1}{\pi} \int_{o}^{\infty} \cos(k(x-ct) + k^3 t)\, dk$$

$$= \frac{1}{\pi 3 t^{\frac{1}{3}}} \int_{0}^{\infty} \cos(\frac{k}{3t^{\frac{1}{3}}}(x-ct) + \frac{k^3}{3})\, dk.$$

Hence

$$u(x,t) = \frac{1}{3t^{\frac{1}{3}}} \mathrm{Ai}(\frac{(x-ct)}{3t^{\frac{1}{3}}}),$$

where we have used the integral representation of the Airy function

$$\mathrm{Ai}(z) = \frac{1}{\pi} \int_{0}^{\infty} \cos(zk + \frac{k^3}{3})\, dk.$$

## 9.2 Laplace Transform

From the Fourier integral formula (9.1.1), namely

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\int_{-\infty}^{\infty} e^{-ikx} g(t)\,dt\,dk. \tag{9.2.1}$$

Suppose we let

$$g(x) = e^{-cx}H(x)f(x), \quad c > 0$$

so that $g(x) = 0$ for $x < 0$. Then the above formula (9.2.1) becomes

$$f(x) = \frac{e^{cx}}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\int_{0}^{\infty} e^{-t(c+ik)} f(t)\,dt\,dk.$$

Next let $s = c + ik$ to obtain

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx}\int_{0}^{\infty} e^{-st} f(t)\,dt\,ds.$$

The Laplace transform of $f(x)$ is defined as

$$\mathcal{L}(f(x); s) = F(s) = \int_{0}^{\infty} e^{-sx} f(x)\,dx, \quad \Re(s) > 0.$$

The inversion formula is given by

$$\mathcal{F}^{-1}(F(s)) = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx}F(s)\,ds, \quad c > 0.$$

**Theorem**

*Suppose (i) $f(x)$ is integrable over every finite interval $[a, b], 0 < a < b$ and (ii) there exists a real number $c$ such that for arbitrary $b > 0$ the integral $\int_b^T e^{ct} f(t)\,dt$ tends to a finite limit as $T \to \infty$, and for arbitrary $a > 0$ the integral $\int_\epsilon^a |f(t)|\,dt$ tends to a finite limit as $\epsilon \to 0+$, then the Laplace transform of $f(x)$ exists for $\Re(s) \geq c$.*

**Proof** If $a, \epsilon$ are arbitrary and $(\epsilon < a)$, and suppose $c > 0$, then

$$\left|\int_\epsilon^a e^{-st} f(t)\,dt\right| \leq \int_\epsilon^a e^{-ct}|f(t)|\,dt \leq \int_\epsilon^a |f(t)|\,dt.$$

If $c < 0$ then

$$\left|\int_\epsilon^a e^{-pt} f(t)\,dt\right| \leq \int_\epsilon^a e^{-ct}|f(t)|\,dt \leq e^{-ca}\int_\epsilon^a |f(t)|\,dt.$$

Thus $\int_0^a e^{-st} f(t)\, dt$ exists for arbitrary $a$. Note that

$$\int_0^T e^{-pt} f(t)\, dt = \left( \int_0^a + \int_a^b + \int_b^T \right) e^{-st} f(t)\, dt$$

and letting $T \to \infty$ and using the given conditions proves the result. One can further prove that if, in addition to the above mentioned conditions, $f(t) = O(e^{ct})$ as $t \to \infty$ then the Laplace transform converges absolutely for $\Re(s) > c$. If $f(t) = O(t^c)$ as $t \to \infty$ then the Laplace transform converges absolutely for $\Re(s) > 0$.

One other result worth noting is that with the conditions stated above the Laplace transform $\mathcal{L}(f(x); s) = F(s)$ is an analytic function of $s$ in the half-plane $\Re(s) > c$. [For a proof see, Sneddon.]

The following basic results are easy to establish. Suppose $F(s)$ is the Laplace transform of $f(t)$. Then

- $\mathcal{L}(e^{-at} ft) = F(s + a)$.
- $\mathcal{L}(f^{(n)}(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$ where $f^{(n)}(t) = d^n f/dt^n$.
- $\mathcal{L}(t^n e^{-at}) = \frac{n!}{(s+a)^{n+1}}, \quad a > 0$.

**Example**

$$\mathcal{L}(x^\nu; s) = \int_0^\infty x^\nu e^{-sx}\, dx = s^{-\nu-1} \int_0^\infty t^\nu e^{-t}\, dt = s^{-\nu-1} \Gamma(\nu + 1),$$

where $\Gamma(\nu + 1) = \int_0^\infty t^\nu e^{-t}\, dt$ is the Gamma function and $\Re(\nu) > -1$.

**Example** Consider

$$\int_0^\infty e^{-st} e^{iat}\, dt = \frac{s + ia}{s^2 + a^2}, \quad \Re(s) > 0,$$

and $a$ is a real. Taking real and imaginary parts shows that

$$\mathcal{L}(\cos(at); s) = \frac{s}{s^2 + a^2},$$

$$\mathcal{L}(\sin(at); s) = \frac{a}{s^2 + a^2}.$$

**Example** Consider the Laplace Transform of the Bessel function $J_0(at), \quad a > 0$. We have

$$J_0(ax) = \frac{2}{\pi} \int_0^{\pi/2} \cos(at \cos\theta)\, d\theta.$$

So

$$\mathcal{L}(J_0(at);s) = \frac{2}{\pi}\int_0^\infty e^{-st}\int_0^{\pi/2}\cos(at\cos\theta)\,d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\int_0^\infty e^{-st}\cos(at\cos\theta)\,dt\,d\theta,$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\frac{sd\theta}{s^2+a^2\cos^2\theta} = \frac{2s}{\pi}\int_0^\infty \frac{du}{s^2+a^2+s^2u^2}.$$

Hence

$$\mathcal{L}(J_0(at);s) = \frac{1}{(s^2+a^2)^{\frac{1}{2}}}.$$

### 9.2.1 Convolution theorems

**Definition** *Suppose $\mathcal{L}(f(t)) = F(s)$ and $\mathcal{L}(g(t)) = G(s)$. The convolution of $f(t)$ and $g(t)$ is defined by*

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

**Theorem**

$$\mathcal{L}(f(t) * g(t)) = F(s)G(s),$$

or

$$f(t) * g(t) = \mathcal{L}^{-1}(F(s)G(s)).$$

**Proof**

Now

$$\mathcal{L}(f(t) * g(t)) = \int_0^\infty e^{-st}\int_0^t f(t-\tau)g(\tau)\,d\tau\,dt.$$

$$= \int_0^\infty g(\tau)\int_\tau^\infty e^{-st}f(t-\tau)\,dt\,d\tau =$$

$$\int_0^\infty e^{-s\tau}g(\tau)\,d\tau\int_0^\infty e^{-st}f(t)\,dt = F(s)G(s),$$

after reversing the order of integration, see figure 9.3.

### 9.2.2 Tauberian and Abelian theorems

Consider

$$f(t) = \sum_{k=0}^n a_k\frac{t^k}{k!}.$$

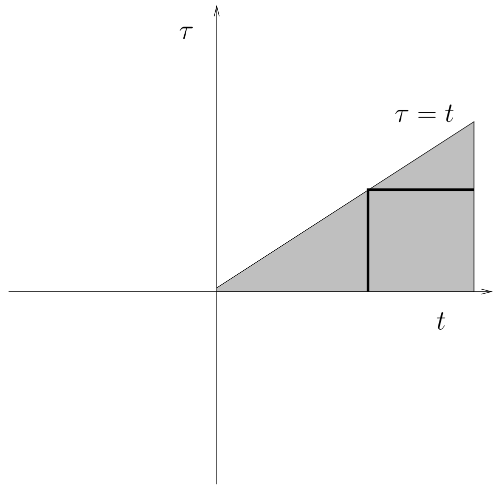



Figure 9.3: Shaded region in the convolution integral

The Laplace transform of $f(t)$ gives

$$F(s) = \int_0^\infty e^{-st} \sum_{k=0}^{n} a_k \frac{t^k}{k!}\, dt$$

$$= \sum_{k=0}^{n} a_k s^{-k-1}.$$

We deduce that

$$\lim_{s\to\infty} (sF(s)) = \lim_{t\to 0+} (f(t)) = a_0.$$

and

$$\lim_{s\to\infty} (s^{n+1}F(s)) = \lim_{t\to 0+} (f^{(n)}(t)) = a_n.$$

Thus the behaviour of the function near $t = 0+$ is reflected in the behaviour of the LT for large $s$. The above result holds more generally and are known as the Tauberian and Abelian theorems.

Suppose the Laplace transform $F(s)$ of a function $f(t)$ exists and in addition $f(t)$ and its derivatives exist as $t \to 0+$. Then

1. $\lim_{s\to\infty} F(s) = 0$,

2. $\lim_{s\to\infty} s^{n+1}F(s) - s^n f(0) - \cdots - sf^{(n-1)}(0) = f^{(n)}(0)$.

3. Suppose $f(t)$ is bounded for all $t > 0$ and the limit $\lim_{t\to\infty} f(t) = f(\infty)$ exists, then
$$\lim_{s\to 0+} sF(s) = f(\infty).$$

**Example** Consider

$$f(t) = 1 - e^{-at}, (a > 0), \quad F(s) = \frac{a}{s(s+a)}$$

$$\lim_{s \to 0+} (sF(s)) = 1 = f(\infty).$$

$$\lim_{s \to \infty} (s^2 F(s) - sf(0)) = \lim_{s \to \infty} (\frac{as}{s+a}) = a = f'(0).$$

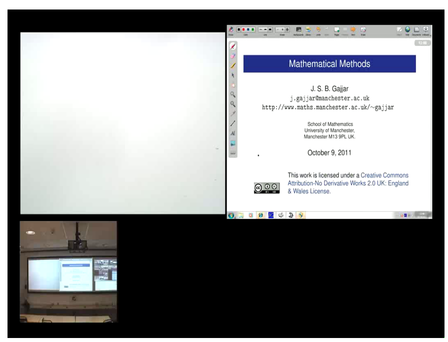
### 9.2.3 Watson's Lemma

The use of Laplace transforms leads to integrals of the form

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt. \tag{9.2.2}$$

One is often interested in trying to estimate this integral for $s$ large. This is where Watson's lemma becomes extremely useful. Observe that for well behaved functions $f(t)$ the dominant value of the integral (9.2.2) will occur in the vicinity of $t = 0$. This suggests that we should be able to estimate the integral by replacing the $f(t)$ by its local expansion for $t = 0$. The more formal result is summarised in Watson's lemma.

Video clip showing proof of $\Gamma(z)\Gamma(1-z) = \pi \sin \pi z$. Click here to open video clip in external player.




**Theorem: Watson's Lemma** *Suppose $f(t) = O(e^{at})$ as $t \to \infty$ and in some neighbourhood of $t = 0$, $f(t)$ can be expanded as*

$$f(t) = t^\alpha \left[ \sum_{k=0}^{n} a_k t^k + R_{n+1}(t) \right], \quad 0 < t < \tau, \quad \alpha > -1,$$

*where $|R_{n+1}(t)| < At^{n+1}$ for $0 < t < \tau$. Then*

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

*has the asymptotic expansion*

$$F(s) \sim \sum_{k=0}^{n} a_k \frac{\Gamma(\alpha+k+1)}{s^{\alpha+k+1}} + O(\frac{1}{s^{\alpha+n+2}}), \quad s \to \infty.$$

**Proof**
Now

$$F(s) = \int_0^\tau e^{-st} f(t)\,dt + \int_\tau^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\tau e^{-st} t^\alpha \sum_{k=0}^{n} a_k t^k\,dt + \int_0^\tau e^{-st} t^\alpha R_{n+1}(t)\,dt + \int_\tau^\infty e^{-st} f(t)\,dt. \tag{9.2.3}$$

Also

$$\int_0^\tau e^{-st} a_k t^{\alpha+k}\,dt = \int_0^\infty e^{-st} a_k t^{\alpha+k}\,dt - \int_\tau^\infty e^{-st} a_k t^{\alpha+k}\,dt$$

$$= a_k \frac{\Gamma(\alpha+k+1)}{s^{\alpha+k+1}} + O(e^{-s\tau}).$$

Here we have used

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\,dt, \quad \Re(\alpha) > 0.$$

Using the given behaviour for $R_{n+1}(t)$ the second term in (9.2.3) can be estimated as follows:

$$\left| \int_0^\tau e^{-st} t^\alpha R_{n+1}(t)\,dt \right| \leq A \int_0^\tau e^{-st} t^{\alpha+n+1} = O(\frac{1}{s^{\alpha+n+1}}).$$

Finally for the last term in (9.2.3) we have

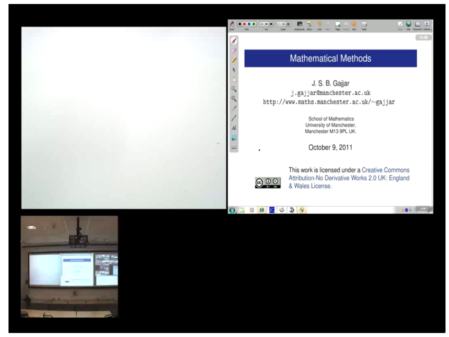
$$\left| \int_\tau^\infty e^{-st} f(t)\,dt \right| \leq B \int_\tau^\infty e^{-(s-a)t}\,dt = Be^{-(s-a)\tau}$$

which tends to zero exponentially for $s \to \infty$. Combining the estimates leads to the required result that

$$F(s) \sim \sum_{k=0}^{n} a_k \frac{\Gamma(\alpha+k+1)}{s^{\alpha+k+1}} + O(\frac{1}{s^{\alpha+n+2}}), \quad s \to \infty.$$

Watson's lemma is extremely powerful and can be used to generate asymptotic expansions from the knowledge of the local behaviour of the integrand in Laplace type integrals.

**Example**




Video clip showing proof of Watson's Lemma. Click here to open video clip in external player.

Consider for example the parabolic cylinder function $D_\nu(z)$. An integral representation of $D_\nu(z)$ is given by

$$D_\nu(z) = \frac{e^{-\frac{z^2}{4}}}{\Gamma(-\nu)} \int_0^\infty e^{-zt} e^{-\frac{t^2}{2}} \frac{dt}{t^{\nu+1}}$$

which is valid for $\Re(\nu) < 0$.

We apply Watson's lemma to the function

$$f(t) = e^{-\frac{t^2}{2}} t^{-\nu-1} = \sum_{n=0}^\infty (-1)^n \frac{t^{2n-\nu-1}}{2^n n!}.$$

Using Watson's lemma gives

$$D_\nu(z) \sim \frac{e^{-\frac{z^2}{4}}}{\Gamma(-\nu)} \sum_{n=0}^\infty \frac{(-1)^n}{2^n n!} \frac{\Gamma(2n-\nu)}{z^{2n-\nu}}.$$

The result is also valid for $\Re(\nu) \geq 0$.

The LT is very useful for solving a number of ODE, PDE and other problems. We will just look at one or two (ununusal) examples.

**Example** Suppose

$$F(n) = \int_0^\infty e^{-nx} f(x)\, dx$$

for $n$ integer, and

$$S = \sum_{n=0}^\infty F(n) = \sum_{n=0}^\infty \int_0^\infty f(x) e^{-nx}\, dx.$$

Assuming that we can interchange the summation and integration we find

$$S = \int_0^\infty f(x) h(x)\, dx,$$

where

$$h(x) = \sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}}.$$

Suppose we take $a \geq 0, p > 0$ and

$$f(x) = \frac{x^{p-1}e^{-ax}}{\Gamma(p)},$$

so that

$$\begin{aligned} F(n) &= \int_0^\infty \frac{e^{-nx}x^{p-1}e^{-ax}}{\Gamma(p)}\,dx \\ &= \frac{1}{\Gamma(p)}\int_0^\infty x^{p-1}e^{-(n+a)x}\,dx = \frac{1}{(n+a)^p}. \end{aligned}$$

Hence

$$S = \sum_{n=0}^{\infty} \frac{1}{(n+a)^p} = \int_0^\infty \frac{1}{\Gamma(p)} x^{p-1} \frac{e^{-ax}}{1-e^{-x}}\,dx.$$

Take $a = 1$ and we see that

$$\int_0^\infty \frac{x^{p-1}e^{-x}}{1-e^{-x}}\,dx = \Gamma(p)\zeta(p)$$

where $\zeta(p)$ is the Riemann zeta function.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \zeta(p).$$

Eg.,

$$\zeta(2) = \sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The function

$$\zeta(p,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^p}$$

is called the generalised Riemann zeta function. Note that $\zeta(p,1) = \zeta(p)$.

One can express the zeta function in terms of a Hankel type loop integral, see figure 9.4.

$$\zeta(p,a) = -\frac{\Gamma(1-p)}{2\pi i}\int_C \frac{(-z)^{p-1}e^{-az}}{1-e^{-z}}\,dz$$

It can be further shown using these representations that $\zeta(p,a)$ is an analytic function for all $p$ except $p = 1$ where $\zeta(p,a)$ has a simple pole with residue 1.

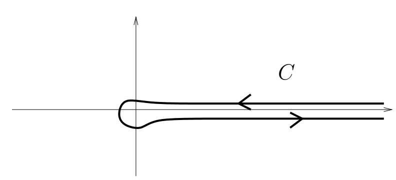



Figure 9.4: Loop contour for Hankel's integral representation of $\zeta(z)$.

## 9.3 Mellin Transform

The Mellin transform is extremely useful for certain applications including solving Laplace's equation in polar coordinates, as well as for estimating integrals.

We will first consider the generalised Laplace transform.

### 9.3.1 Generalised Laplace transform

Suppose that for finite $T$ the function $g(t)$ is absolutely integrable on $(0, T)$, ie

$$\int_0^T |g(t)|\, dt < \infty$$

and $g(t) = O(e^{\alpha t})$ as $t \to \infty$ for some real constant $\alpha$. Then the one-sided Laplace transform

$$\mathcal{L}^+[g; s] = \int_0^\infty e^{-st} g(t)\, dt$$

converges absolutely and is holomorphic (analytic) in $\Re(s) > \alpha$.

Similarly suppose

$$\int_0^T |g(-t)|\, dt < \infty$$

and $g(t) = O(e^{\beta t})$ as $t \to -\infty$ for some real constant $\beta$. Then the one-sided Laplace transform

$$\mathcal{L}^-[g; s] = \int_{-\infty}^0 e^{-st} g(t)\, dt = \int_0^\infty e^{st} g(-t)\, dt$$

converges absolutely and is holomrophic in the left half-plane $\Re(s) < \beta$. Under the same conditions on $g(t)$ and if $\beta > \alpha$ then the two-sided Laplace transform

$$\mathcal{L}[g; s] = \int_{-\infty}^\infty e^{-st} g(t)\, dt$$

converges absolutely and is holomorphic in the vertical strip $\alpha < \Re(s) < \beta$ see figure 9.5. If $\beta < \alpha$ then the generalised Laplace transform does not exist for any $s$.

Suppose $\beta > \alpha$ and let $t = -\log(x)$, and $g(-\log x) = f(x)$. Then note that

$$e^{-st} = e^{s \log(x)} = x^s.$$

Hence

$$\begin{aligned} \mathcal{L}[g, s] &= \int_0^\infty x^{s-1} f(x)\, dx. \\ &= \mathcal{M}[f; s] \end{aligned}$$

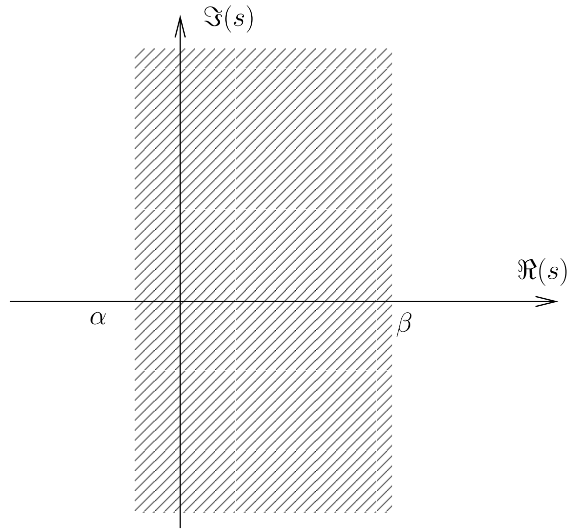



Figure 9.5: Common strip of analyticity of $g(s)$

where $\mathcal{M}$ is the Mellin transform of $f(x)$.

So the Mellin transform of $f(x)$ is the two-sided Laplace transform of $g(t)$ where $t = -\log(x)$ and it converges absolutely and is holomorphic in the strip $\alpha < \Re(s) < \beta$.

**Example** Consider $g(t) = e^{-a|t|}$. Then

$$G^+(s) = \int_0^\infty e^{-st}e^{-at}\,dt$$

$$= \left[-\frac{e^{-(s+a)t}}{s+a}\right]_0^\infty = \frac{1}{s+a}$$

provided $\Re(s) > -a$.

Similarly

$$G^-(t) = \int_{-\infty}^0 e^{-st}e^{at}\,dt = \int_{-\infty}^0 e^{-(s-a)t}\,dt$$

$$= \left[-\frac{e^{-(s-a)t}}{s-a}\right]_{-\infty}^0 = \frac{1}{a-s},$$

for $\Re(s) < a$. Thus $\mathcal{M}[e^{-a|t|}; s]$ converges and is analytic in the strip $-a < \Re(s) < a$.

**Example** Consider $g(t) = e^{-at}$. Then

$$G^+(s) = \frac{1}{s+a}$$

provided $\Re(s) > -a$. Similarly

$$G^{-}(t) = \int_{-\infty}^{0} e^{-st} e^{-at}\, dt = \int_{-\infty}^{0} e^{-(s-a)t}\, dt$$

$$= \left[ -\frac{e^{-(s+a)t}}{s+a} \right]_{-\infty}^{0} = -\frac{1}{a+s},$$

for $\Re(s) < -a$. Thus $\mathcal{M}[e^{-at}; s]$ where $t = \log(x)$ does not exist.

Now since

$$g(t) = O(e^{\alpha t}), \quad \text{as} \quad t \to \infty$$

we obtain

$$f(x) = O(x^{-\alpha}), \quad \text{as} \quad x \to 0+ .$$

Also

$$g(t) = O(e^{\beta t}), \quad \text{as} \quad t \to -\infty,$$

and so

$$f(x) = O(x^{-\beta}), \quad \text{as} \quad x \to \infty.$$

Thus when the Mellin transform of $f(x)$ exists its strip of analyticity is determined by the behaviour of $f(x)$ as $x \to 0+$ and $x \to \infty$. The inversion formula for the Mellin transform follows from the inversion formula for the two-sided Laplace transform of $g(t)$. Now

$$g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \mathcal{L}[g; s]\, ds,$$

where we require that $\alpha < c < \beta$.

Putting $x = -\log(t)$ we find

$$f(x) = g(-\log(t)) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \mathcal{M}[f; s]\, ds.$$

This inversion formula is valid at all points $x > 0$ where $f(x)$ is continuous.

**Theorem**

*Suppose $F(s)$ is a function of the complex variable $s = \sigma + i\tau$ which is regular in the infinite strip $a < \sigma < b$ and that for any arbitrary small positive $\epsilon$, $F(s)$ tends to zero uniformly as $\tau \to \infty$ in the strip $a + \epsilon \leq \sigma \leq b + \epsilon$. Then if the integral*

$$\int_{-\infty}^{\infty} F(\sigma + i\tau)\, d\tau$$

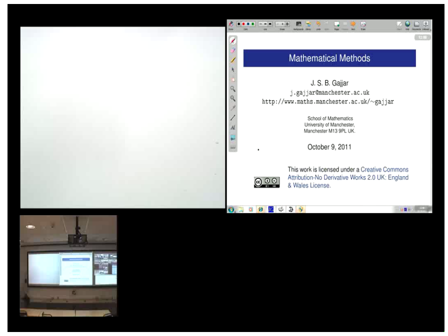
*converges absolutely for each value of $\sigma \in (a, b)$ and, if for positive real values of $x$ and a fixed $c \in (a, b)$ we define*

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s)\, ds$$

*then in the strip $a < \sigma < b$*

$$F(s) = \int_0^\infty x^{s-1} f(x)\, dx.$$

For a proof, see Sneddon.




Video clip on a discussion of the Mellin Transform. Click here to open video clip in external player.

**Example** Consider $f(x) = e^{-ax}$ where $a > 0$. Then

$$\mathcal{M}[e^{-ax}; s] = \int_0^\infty x^{s-1} e^{-ax}\, dx$$

$$= \frac{1}{a^s} \int_0^\infty x^{s-1} e^{-x}\, dx.$$

$$= \frac{\Gamma(s)}{a^s}.$$

**Example** Consider $f(x) = 1/(e^x - 1)$. Then Mellin transform of $f$ is

$$\mathcal{M}[f; s] = \int_0^\infty x^{s-1} \frac{1}{e^x - 1}\, dx$$

$$= \int_0^\infty x^{s-1} \sum_{n=1}^\infty e^{-nx}\, dx,$$

$$= \sum_{n=1}^\infty \int_0^\infty x^{s-1} e^{-nx}\, dx$$

$$\sum_{n=1}^\infty \frac{\Gamma(s)}{n^s} = \Gamma(s)\zeta(s),$$

where $\zeta(s)$ is the Riemann zeta function. We require that $\Re(s) > 1$ for convergence.

**Example** Consider $f(x) = 1/(1+x)$. The Mellin transform of $f$ is

$$\mathcal{M}[f;s] = \int_0^\infty x^{s-1}\frac{1}{1+x}\,dx.$$

In the section on basic complex analysis we saw how to evaluate integrals like this and

$$\int_0^\infty \frac{x^{s-1}}{1+x}\,dx =$$

$$-\frac{\pi e^{-\pi s i}}{\sin(s\pi)}\sum\{res[z^{s-1}\frac{1}{1+z}] \quad \text{at} \quad z=-1\},$$

$$= \frac{\pi}{\sin(\pi s)}.$$

Note that we can also evaluate the Mellin transform of $f(x) = 1/(1+x)$ as

$$\mathcal{M}[f;s] = \int_0^\infty x^{s-1}\frac{1}{1+x}\,dx$$

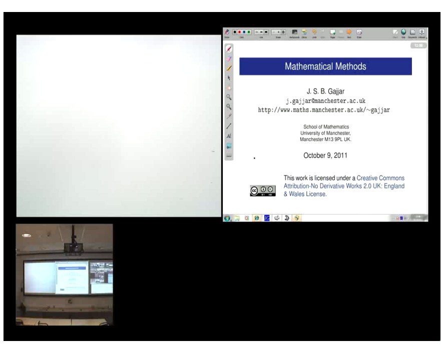
by letting $x = t/(1-t)$. Thus

$$\mathcal{M}[f;s] = \int_0^1 \frac{t^{s-1}}{(1-t)^{s-1}}\frac{(1-t)}{(1-t)^2}\,dt = \int_0^1 t^{s-1}(1-t)^{-s}\,dt$$

$$= \int_0^1 t^{s-1}(1-t)^{1-s-1}\,dt = B(s,1-s) = \Gamma(s)\Gamma(1-s),$$

where $B(p,q)$ is the Beta function. Hence

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(s\pi)}.$$

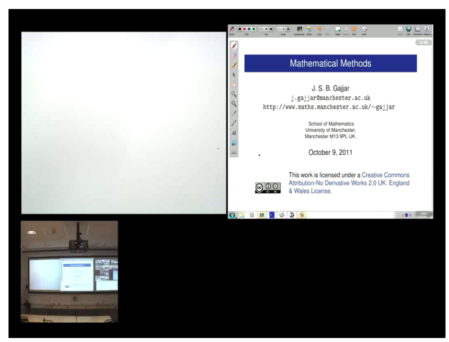



Video clip on a discussion of the Mellin Transform of $1/(1+x)$. Click here to open video clip in external player.

**Example** Suppose $f(x) = 1/(1+x)^n$ then by the same technique we obtain

$$\mathcal{M}[f;s] = \int_0^1 t^{s-1}(1-t)^{n-s-1}\,dt = B(s,n-s) = \frac{\Gamma(p)\Gamma(1-p)}{\Gamma(n)}.$$

Video clip discussing examples. Click here to open video clip in external player.



### 9.3.2 Mellin transform- basic properties

Suppose $F(s)$ is the Mellin transform of $f(x)$.

- $\mathcal{M}[f(ax); s] = a^{-s}F(s), \quad a > 0.$
- $\mathcal{M}[x^a f(x); s] = F(s+a).$
- $$\mathcal{M}[f'(x); s] = \int_0^\infty x^{s-1} f'(x)\, dx$$
  $$= [x^{s-1}f(x)]_0^\infty - (s-1)\int_0^\infty x^{s-2} f(x)\, dx$$
  $$= -(s-1)F(s-1),$$
  provided that $x^{s-1}f(x)$ goes to zero at $x = 0+$ and $x \to \infty$.
- $\mathcal{M}[xf'(x); s] = \int_0^\infty x^s f'(s)\, ds = -sF(p).$
- $\mathcal{M}[x^n f^{(n)}(x); s] = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} F(s).$

  Both require that $x^{n+s-q} f^{(n-q)}(x)$ is zero as $x \to 0+$ and $x \to \infty$ for $q = 1, \ldots n$.

### 9.3.3 Mellin transform- Parseval's formula

*Suppose $F(s)$ is the Mellin transform of $f(x)$ and $G(s)$ is the Mellin transform of $g(x)$. Then*

$$\mathcal{M}[f(x)g(x); s] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)G(s-z)\, dz.$$

*The result for $s = 1$ gives the Parseval formula for Mellin transforms ie*

$$\int_0^\infty f(x)g(x)\, dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)G(1-z)\, dz.$$

**Proof**

$$\mathcal{M}[f(x)g(x); s] = \int_0^\infty x^{s-1} f(x)g(x)\, dx$$

$$= \frac{1}{2\pi i}\int_0^\infty x^{s-1} g(x)\, dx \int_{c-i\infty}^{c+i\infty} x^{-z} F(z)\, dz,$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)\,dz \int_0^\infty x^{s-z-1} g(x)\,dx$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)G(s-z)\,dz.$$

In the above, the interchange of the order of integration requires some suitable conditions on $f(x)$ and $g(x)$ .

**Example** Consider the exponential integral

$$\mathrm{Ei}(x) = \int_x^\infty \frac{e^{-u}}{u}\,du$$
$$= \int_1^\infty \frac{e^{-qx}}{q}\,dq.$$

The Mellin transform of $\mathrm{Ei}(x)$ is given by

$$F(s) = \int_0^\infty x^{s-1}\,dx \int_1^\infty \frac{e^{-qx}}{q}\,dq,$$
$$= \int_1^\infty \frac{dq}{q}\int_0^\infty x^{s-1}e^{-qx}\,dx,$$
$$= \frac{\Gamma(s)}{s}, \quad \Re(s) > 0.$$

**Example** It is a useful exercise to invert the previous transform. So consider

$$I_1 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)x^{-s}}{s}\,ds$$

where $c > 0$. We will evaluate

$$I = \frac{1}{2\pi i}\oint_{L+L_1+C+L_2} \frac{\Gamma(z)x^{-z}}{z}\,dz,$$

where the contour is as shown. The integrand $\Gamma(z)x^{-z}/z$ has a double pole at $z = 0$ and simple poles at $z = -1, -2, .., $ , see figure 9.6. From the properties of the integrand, the integrals along $L_1, L_2$ tend to zero as $R \to \infty$ where we parameterize $L_1$ by $z = y + iR_n, \quad 0 < y < c$ and similarly for $L_2$.

On the contour $C$ where $z = R_n e^{i\theta}, \quad \pi/2 < \theta < \pi$ and $-\pi < \theta < -\pi/2$ and $R_n$ lies between zeros of the Gamma function on the negative real axis. The values of $\theta$ are chosen so that we may use the asymptotic form for the Gamma function given by Stirling's formula.

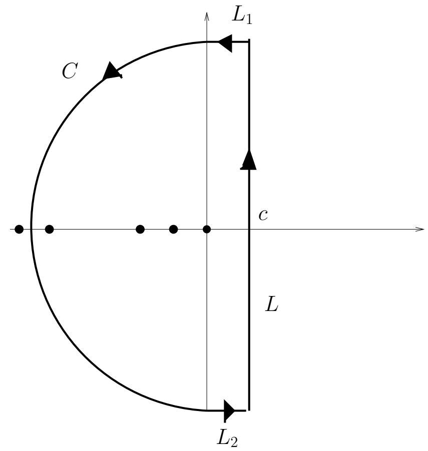



Figure 9.6: Singularities of $\Gamma(z)/z$ lie on the negative real axis at $z = 0, -1, -2, ..$

If $R_n$ is large we may use the asymptotic form for $\Gamma(z)$ to obtain the estimate

$$\left| x^{-z} \frac{\Gamma(z)}{z} \right|$$

$$\sim \left| e^{-R_n \log(x)(\cos\theta + i\sin\theta)} \sqrt{\frac{2\pi}{R_n^3}} e^{\frac{-3i\theta}{2}} e^{R_n(\cos\theta + i\sin\theta)(\log\frac{R_n}{e} + i\theta)} \right|$$

$$= \left| e^{-R_n \log(x)\cos\theta} \sqrt{\frac{2\pi}{R_n^3}} e^{R_n \log\frac{R_n}{e}\cos\theta} e^{-R_n\theta\sin\theta} \right|.$$

If we choose $R_n > ex$ and note that for $-\pi < \theta < -\frac{\pi}{2}$ and $\frac{\pi}{2} < \theta < \pi$ we have $\cos\theta < 0$ and $-\theta\sin\theta < 0$, we see that the integrand is exponentially small, and thus the integral around $\mathcal{C}$ goes to zero as $R_n \to \infty$. Using Cauchy's theorem we obtain

$$\frac{1}{2\pi i}\int_L x^{-s}\frac{\Gamma(s)}{s}\,ds = \sum\{\text{Residues of } z^{-s}\frac{\Gamma(z)}{z}, z \le 0\}.$$

We can write

$$\Gamma(1+z) = 1 + zg(z)$$

in the neighbourhood of $z = 0$ where $g(z)$ is analytic near $z = 0$. Hence

$$\Gamma(z) = \frac{1}{z}\Gamma(1+z) = \frac{1}{z} + g(z),$$

and so $z = 0$ is a simple pole of $\Gamma(z)$ with residue 1. In the same way we obtain

$$\Gamma(z) = (z-1)(z-2)\ldots(z-n)\Gamma(z-n)$$

and so

$$\Gamma(z-n) = \frac{1+g(z)}{z(z-1)\ldots(z-n)} = \frac{(-1)^n}{zn!}(1+O(z)),$$

showing that $z = -n$ is a simple pole with residue $(-1)^n/n!$ for $n \geq 1$. Thus for the function

$$x^{-z}\frac{\Gamma(z)}{z}$$

the residue contribution from $z = -n, \quad n \geq 1$ is

$$\frac{x^n}{-n}\frac{(-1)^n}{n!}.$$

Now near $z = 0$ we can write

$$\Gamma(z) = \frac{1}{z} - \gamma + O(z)$$

and so

$$x^{-z}\frac{\Gamma(z)}{z} = e^{-z\log(x)}\frac{\frac{1}{z} - \gamma + O(z)}{z} = \frac{1}{z}(1 - z\log(x) + O(z^2))(\frac{1}{z} - \gamma + O(z))$$

$$= \frac{1}{z^2} - \frac{(\gamma + \log x)}{z} + O(1).$$

Hence the residue contribution from the double pole at $z = 0$ is $-\gamma - \log(x)$. The inversion gives

$$\mathrm{Ei(x) = -\log(x) - \gamma - \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{nn!}}.$$

**Example**

Suppose that $F(s)$ is the Mellin transform of a real-valued function $f(r)$. Then consider

$$\mathcal{M}[f(re^{i\theta}); s] = \int_0^\infty f(re^{i\theta})r^{s-1}\,dr.$$

We will assume that $f(z)$ with $z = re^{i\theta}$ is the analytic continuation of a the function $f(r)$ defined in some sector $-\alpha < \arg(z) < \alpha$.

Thus

$$\int_0^R f(r)r^{s-1}\,dr - e^{is\theta}\int_0^R f(re^{i\theta})r^{s-1}\,dr + \int_C f(z)z^{s-1}\,dz = 0,$$

where the contour $C$ is as shown in figure 9.7.

Provided the integral along $C$ goes to zero as $R \to \infty$ we obtain

$$\int_0^\infty f(re^{i\theta})r^{s-1}\,dr = e^{-is\theta}\int_0^R f(r)r^{s-1}\,dr = e^{-is\theta}F(s).$$

A sufficient condition for this is that

$$z^s f(z) \to 0, \quad |z| \to \infty, \quad |\arg(z)| < \alpha.$$

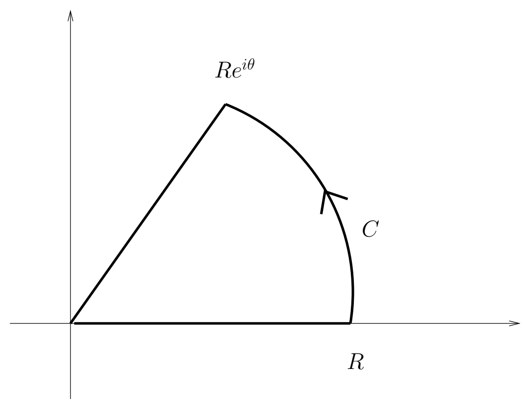



Figure 9.7

Since the Mellin transform of $f(r)$ exists we have

$$r^s f(r) \to 0, \quad r \to \infty,$$

in the strip of analyticity and so the above condition is also valid provided $\alpha$ is suitably chosen.

From

$$\int_0^\infty f(re^{i\theta}) r^{s-1}\, dr = e^{-is\theta} F(s).$$

taking real and imaginary parts gives

$$F(s)\cos s\theta = \mathcal{M}[\Re\{f(re^{i\theta})\}; s],$$

$$F(s)\sin s\theta = \mathcal{M}[-\Im\{f(re^{i\theta})\}; s],$$

These results are due to Harrington (1967).

### 9.3.4 Mellin transform- Applications to solution of pde's

Consider the solution of

$$r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0$$

in the wedge $0 < r < \infty, \quad 0 < \theta < \alpha$ together with the boundary conditions

$$\phi_\theta(r, 0) = 0, \quad \phi(r, \alpha) = f(r), \quad 0 < r < \infty,$$

$$\phi(r, \theta) \to 0 \quad \text{as} \quad r \to \infty, 0 < \theta < \alpha.$$

Now let us take the Mellin transform of the equation and suppose that $\Phi(s, \theta)$ is the MT of $\phi(r, \theta)$. This gives

$$\int_0^\infty r^{s-1}(r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta})\, dr$$

This becomes

$$[r^{s+1}\phi_r - sr^s\phi]_0^\infty + s^2\Phi(s,\theta) + \Phi_{\theta\theta} = 0.$$

For the integrated term to vanish we require that

$$[r^{s+1}\phi_r - sr^s\phi]_0^\infty = 0.$$

The transformed equation is then

$$\frac{d^2\Phi(s,\theta)}{d\theta^2} + s^2\Phi = 0.$$

Solving and using the transformed boundary conditions that

$$\Phi_\theta(s,0) = 0, \quad \Phi(s,\alpha) = F(s),$$

we find that

$$\Phi(s,\theta) = F(s)\frac{\cos(s\theta)}{\cos(s\alpha)}.$$

In the previous section we discussed how to invert Mellin transforms of the type

$$F(s)\sin(s\theta), \quad F(s)\cos(s\theta).$$

The next example shows how to do this for a specific case.

**Example**

In the problem we met earlier

$$\nabla^2\phi = 0, \quad 0 < r < \infty, \quad 0 \le \theta \le \alpha$$

$$\phi_\theta(r,0) = 0, \quad \phi(r,\alpha) = f(r),$$

we found that the Mellin transform of $\phi$ was given by

$$\Phi(s,\theta) = \frac{F(s)\cos(s\theta)}{\cos(s\alpha)}.$$

Suppose we take

$$f(r) = \begin{cases} 1 & 0 < r < 1, \\ 0 & 1 < r \end{cases}.$$

Then for $0 < \Re(s)$,

$$F(s) = \int_0^1 r^{s-1}\,dr = \frac{1}{s}.$$

Hence

$$\Phi(s,\theta) = \frac{\cos(s\theta)}{s\cos(s\alpha)}.$$

This can be inverted using residues, but we can make use of the above results to obtain the solution in a different manner.

We see that

$$\phi(r,\theta) = \mathcal{M}[\Re\{g(re^{i\theta})\}; s]$$

where the Mellin transform of $g(z)$ is $\frac{1}{s\cos(s\alpha)}$

To find $g(z)$ first note that if $h(r) = \frac{\pi}{2} - \tan^{-1} r$ then

$$M(s) = \int_0^\infty r^{s-1}(\frac{\pi}{2} - \tan^{-1} r)\, dr$$

$$= [\frac{1}{s}r^s(\frac{\pi}{2} - \tan^{-1} r)])_0^\infty + \int_0^\infty \frac{1}{s}r^s \frac{1}{1+r^2}\, dr.$$

The integrated term vanishes provided that $0 < \Re(s) < 1$. We are left with

$$M(s) = \frac{1}{s}\int_0^\infty \frac{r^s}{1+r^2}\, dr.$$

The integral is of the form

$$\int_0^\infty r^{a-1} f(r)\, dr = -\frac{\pi e^{-\pi a i}}{\sin(a\pi)} \sum Res[z^{a-1}f(z)]$$

with $a = s+1$ which we have met earlier and we find that

$$M(s) = \frac{\pi}{2s\cos(\frac{s\pi}{2})}.$$

Next using the property that if $H(s)$ is the Mellin transform of $h(r)$

$$M[h(r^k); s] = k^{-1}H(\frac{s}{k}),$$

With $k = \frac{\pi}{2\alpha}$ the Mellin transform of

$$1 - \frac{2}{\pi}\tan^{-1}(r^k)$$

is therefore

$$\frac{1}{s\cos(\alpha s)}$$

provided $0 < \Re(s) < k$. Finally pulling all the results together we see that

$$\mathcal{M}^{-1}[\frac{\cos(s\theta)}{s\cos(s\alpha)}] = \Re[g(re^{i\theta})]$$

where

$$g[z] = 1 - \frac{2}{\pi}\tan^{-1}(z^k),$$

where $k = \frac{\pi}{2\alpha}$.

Note that for $|\arg(z)| < \frac{\pi}{2}$

$$\frac{\pi}{2}h(z) = \frac{\pi}{2} - \tan^{-1} z = \frac{\pi}{2} - \int_0^z \frac{1}{1+t^2}\,dt$$

$$= -\frac{1}{2i}[\log(z-i) - \log(z+i)].$$

Now

$$\Re[g(z)] = \Re[h(z^k)] = \Re[h(r^k e^{ik\theta})].$$

Putting $z^k = -i + r_1 e^{i\theta_1}$, and $z^k = i + r_2 e^{i\theta_2}$ we find that

$$\frac{\pi}{2}\Re[g(z)] = -\frac{1}{2}(\theta_2 - \theta_1).$$

Now

$$\tan\theta_1 = \frac{r^k\sin(k\theta)+1}{r^k\cos(k\theta)}, \quad \tan\theta_2 = \frac{r^k\sin(k\theta)-1}{r^k\cos(k\theta)},$$

Hence finally we obtain the solution

$$\phi(r,\theta) = \Re[g(z)] = \frac{(\theta_1-\theta_2)}{\pi}$$

$$\phi(r,\theta) = \frac{1}{\pi}\begin{cases} \pi - \tan^{-1}\left[\frac{2r^k\cos(k\theta)}{1-r^{2k}}\right] & 0 \le r < 1 \\ \tan^{-1}\left[\frac{2r^k\cos(k\theta)}{r^{2k}-1}\right] & 1 < r \end{cases}$$

where $k = \frac{\pi}{2\alpha}$.

## 9.4 Riemann-Lebesgue Lemma, and analytic continuation of Mellin transforms.

In the later examples we make use of the **Riemann-Lebesgue lemma** stated below.

*Suppose $f(t)$ is sectionally continuous in a compact interval $[a,b]$. Then (i)*

$$\int_a^b e^{i\lambda t} f(t)\,dt = o(1), \quad \text{as} \quad \lambda \to \infty.$$

*(ii) If $a$ is finite or $-\infty$, $b$ finite or $\infty$ and $f(t)$ is continuous in $(a,b)$ and*

$$I = \int_a^b e^{i\lambda t} f(t)\, dt$$

*converges uniformly at $a, b$ for sufficently large $\lambda$ then*

$$I = o(1), \quad as \quad \lambda \to \infty.$$

**Proof of Riemann-Lebesgue Lemma** We will only prove part (i) for the case when $f(t)$ is continuous in $(a,b)$. If $f(t)$ is sectionally continuous, we split the interval into portions in which is it continuous and use the result repeatedly.

Given $\epsilon{>}0$ we can partition the interval $(a,b)$ into points $a = t_0 < t_1 < \cdots < t_n = b$ so that

$$|f(t) - f(t_k)| \le \frac{\epsilon}{2(b-a)}.$$

Then

$$\int_a^b e^{i\lambda t} f(t)\, dt = \sum_{k=1}^n f(t_k) \int_{t_{k-1}}^{t_k} e^{i\lambda t}\, dt + \sum_{k=1}^n \int_{t_{k-1}}^{t_k} e^{i\lambda t}(f(t) - f(t_k))\, dt.$$

Suppose $M = \max|f(t)|$ for $t \in [a,b]$. Then for $\lambda > 0$

$$\left| \int_{t_{k-1}}^{t_k} e^{i\lambda t}\, dt \right| = \left| \frac{e^{i\lambda t_k} - e^{i\lambda t_{k-1}}}{i\lambda} \right| \le \frac{2}{\lambda}.$$

Hence

$$\left| \int_a^b e^{i\lambda t} f(t)\, dt \right| \le \frac{2Mn}{\lambda} + \sum_{k=1}^n (t_k - t_{k-1}) \frac{\epsilon}{2(b-a)}$$
$$= \frac{2Mn}{\lambda} + \frac{\epsilon}{2} < \epsilon$$

provided $\lambda > 4Mn/\epsilon$.

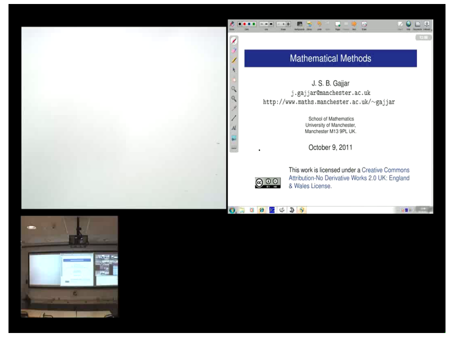
Let $s = \sigma + i\lambda$ and consider

$$\mathcal{M}[f(x); s] = \int_0^\infty x^{s-1} f(x)\, dx = \int_0^\infty x^\sigma e^{i\lambda \log(x)} f(x)\, dx.$$

If we make use of the Riemann-Lebesgue Lemma then for any $\sigma$ which lies in the strip of analyticity of the MT of $f(x)$

$$\lim_{\lambda\to\infty} |\mathcal{M}[f(x); \sigma + i\lambda] = 0,$$

ie the MT of $f(x)$ tends to zero along vertical line in the strip of analyticity.




Video clip on a proof of the Riemann Lebesgue Lemma. Click here to open video clip in external player.

## 9.5 Analytic continuation of Mellin transforms

Now

$$\mathcal{M}[e^{-x}; s] = \int_0^\infty x^{s-1} e^{-x}\, dx = \Gamma(s)$$

is analytic in the right-hand plane $0 < \Re(s)$.

Consider next the MT of the function $e^{-it}$. This is given by

$$\mathcal{M}[e^{-ix}; s] = \int_0^\infty x^{s-1} e^{-ix}\, dx.$$

Note that the integral does not converge absolutely, only conditionally. We can evaluate the integral by considering

$$\oint_{\mathcal{C}} z^{s-1} e^{-iz}\, dz$$

where the contour $\mathcal{C}$ is as shown in figure 10.1. From Cauchy's theorem

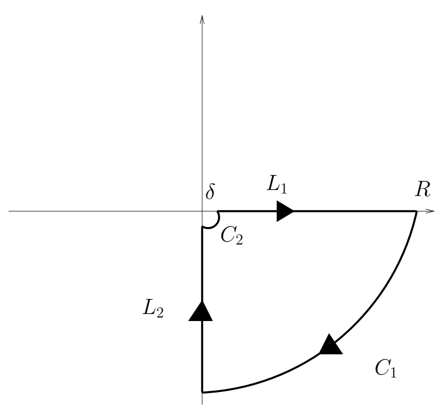



Figure 9.8: Contour $C = L_1 + C_1 + L_2 + C_2$

$$\oint_{\mathcal{C}} z^{s-1} e^{-iz}\, dz = 0$$

since the integrand is analytic inside $\mathcal{C}$. Now

$$|I_{C_2}| = \left| \int_{-\frac{\pi}{2}}^{0} i\delta e^{i\theta} (\delta e^{i\theta})^{s-1} e^{-i\delta(\cos\theta + i\sin\theta)}\, d\theta \right| \to 0,$$

as $\delta \to 0$ provided $\Re(s) > 0$. Next with $s = \sigma + i\lambda$

$$|\int_{C_1}| = \left| -\int_{-\frac{\pi}{2}}^{0} (Re^{i\theta})^{s-1} iRe^{i\theta} e^{-iR\cos\theta + R\sin\theta}\, d\theta \right|.$$

So

$$\begin{aligned} |\int_{C_1}| &\le \left| e^{|\lambda|\pi/2} \int_0^{\frac{\pi}{2}} e^{\lambda\theta} R^{\sigma} e^{-R\sin\theta}\, d\theta \right| \\ &\le e^{|\lambda|\pi/2} R^{\sigma} e^{|\lambda|\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} e^{-\frac{2R\theta}{\pi}}\, d\theta \\ &< \pi e^{|\lambda|\pi/2} e^{|\lambda|\frac{\pi}{2}} R^{\sigma-1}. \end{aligned}$$

Thus

$$|\int_{C_1}| \to 0 \quad \text{as} \quad R \to \infty$$

provided $\Re(s) < 1$. Hence taking the limit as $\delta \to 0$ and $R \to \infty$ we obtain

$$\int_{L_1} + \int_{L_2} = 0$$

giving

$$\begin{aligned} \int_0^{\infty} x^{s-1} e^{-ix}\, dx &= \int_0^{\infty} (e^{-i\frac{\pi}{2}} y)^{s-1} e^{-i\frac{\pi}{2}} e^{-y}\, dy \\ &= e^{-is\frac{\pi}{2}} \int_0^{\infty} y^{s-1} e^{-y} dy = e^{-is\frac{\pi}{2}} \Gamma(s) \end{aligned}$$

Hence the Mellin transform of $e^{ix}$ is given by

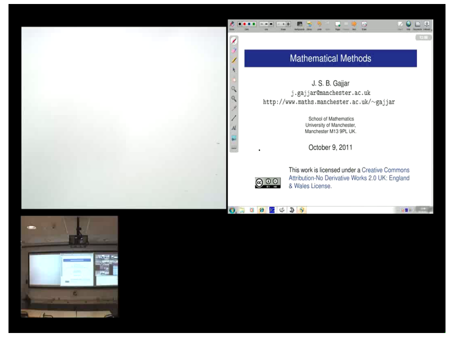
$$\mathcal{M}[e^{-ix}; s] = \Gamma(s) e^{-is\frac{\pi}{2}}, \tag{9.5.1}$$

provided $0 < \Re(s) < 1$.

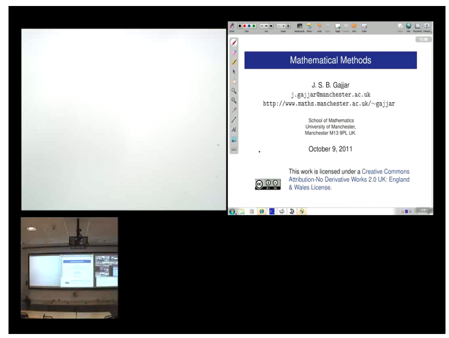
We can analytically continue $\mathcal{M}[e^{-ix}; s]$ into the right half-plane $\Re(s) \ge 1$ using the formula (9.5.1).

Similarly we showed earlier that the function $f(x) = 1/(1+x)$ has the Mellin transform

$$\mathcal{M}[\frac{1}{1+x}; s] = \frac{\pi}{\sin(\pi s)}, \tag{9.5.2}$$




Video clip on a discussion of the Mellin Transform of $e^{-it}$. Click here to open video clip in external player.




Video clip on a discussion of the analytic continuation of Mellin Transform. Click here to open video clip in external player.

again provided $0 < \Re(s) < 1$. We can use (9.5.2) to analytically continue $\mathcal{M}[1/(1+x); s]$ into the right half-plane $\Re(s) \geq 1$. The analytic continuation gives a function which has simple poles at the positive integers.

Let $F(s) = \mathcal{M}[f(x); s]$ be the Mellin transform of $f(x)$ which is analytic in the strip $\alpha < \Re(s) < \beta$. Suppose that as $x \to \infty$ we have

$$f(x) \sim e^{-dx^{\nu}} \sum_{m=0}^{\infty} x^{-r_m} \sum_{n=0}^{N(m)} c_{mn} (\log x)^n,$$

where $\Re(d) \geq 0, \nu > 0$ and $\Re(r_m)$ is an increasing sequence in $m$, and $N(m)$ are non-negative integers. Then $F(s)$ can be continued analytically into the right-half plane $\beta < \Re(s)$. The analytic continuation is such that

1. If $\Re(d) > 0$, then $F(s)$ is analytic in the right-half plane.

2. If $d = -i\omega, \quad \omega \neq 0$ then $F(z)$ is analytic in the right half-plane.

3. If $d = 0$ then $F(s)$ is analytic in the right-half plane except for poles.

The proofs of the results are given in detail in (Bleistein & Handelsman, 1975, chap. 4). Outline proofs are as follows:

**Proof of (1)** This follows from the properties of the Mellin transform given earlier. **Proof of (2)** Write

$$F(s) = \int_0^{\infty} x^{s-1} f(x)\, dx = \int_0^1 x^{s-1} f(x)\, dx + \int_1^{\infty} x^{s-1} f(x),\, dx,$$

and express the second integral as

$$\int_1^\infty (f(x) - h_k(x))x^{s-1}\,dx + \int_1^\infty h_k(x)x^{s-1}\,dx,$$

where

$$h_k(x) = e^{i\omega x^\nu} \sum_{\Re(r_m)<k} x^{-r_m} \sum_{n=0}^{N(m)} c_{mn}(\log x)^n.$$

Note that $f(x) - h_k(x) = O(e^{i\omega x^\nu} x^{-\Re(r_j)}(\log x)^{N(j)})$ where $j$ is such that $\Re(r_j) \geq k$.

So the first term is analytic in the extended strip $\alpha < \Re(s) < k$.

We need to prove that the term

$$\int_1^\infty h_k(x)x^{s-1}\,dx,$$

can be analytically continued in the extended strip.

Let $h_k(x) = e^{i\omega x^\nu} x^{-r_0} H_k(x)$, where $H_k(x) = O((\log x)^{N(0)})$ as $x \to \infty$.

Now

$$\begin{aligned}\int_1^\infty h_k(x)x^{s-1}\,dx, &= \int_1^\infty e^{i\omega x^\nu} x^{s-r_0-1} H_k(x)\,dx \\ &= \int_1^\infty e^{i\omega x^\nu} x^{\nu-1} x^{s-r_0-\nu} H_k(x)\,dx \\ &= \left[\frac{x^{s-r_0-\nu} e^{i\omega x^\nu} H_k(x)}{i\omega\nu}\right]_1^\infty \\ &- \int_1^\infty e^{i\omega x^\nu} x^{s-r_0-\nu-1} \left[\frac{(s-r_0-\nu)xH_k(x) + x\frac{dH_k(x)}{dx}}{i\omega\nu}\right] dx\end{aligned}$$

The last term is of the form

$$\int_1^\infty x^{s-r_0-\nu-1} e^{i\omega x^\nu} \tilde{H}_k(x)\,dx,$$

and is analytic in the extended region $\Re(z) < \Re(r_0) + \nu$.

By integrating by parts $q$ times the integral

$$\int_1^\infty h_k(x)x^{s-1}\,dx,$$

can be analytically continued into the extended region $\alpha < \Re(s) < \Re(r_0) + q\nu$.

Putting it all together this shows that the Mellin transform of $f(x)$ can be analytically continued into the right-half plane $\alpha < \Re(s)$, as an analytic function.

**Proof of (3)** Introduce

$$h_k(x) = \sum_{\Re(r_m)<k} x^{-r_m} \sum_{n=0}^{N(m)} c_{mn} (\log x)^n.$$

and note that the Mellin transform of $f(x)$ is

$$\int_0^\infty x^{s-1} f(x)\, dx$$

$$= \int_0^1 f(x) x^{s-1}\, dx + \int_1^\infty (f(x) - h_k(x))\, dx + \int_1^\infty x^{s-1} h_k(x)\, dx.$$

The first term is analytic in $\alpha < \Re(s)$.

As before the term

$$I_2 = \int_1^\infty x^{s-1}(f(x) - h_k(x))\, dx$$

is analytic in the strip $\alpha < \Re(s) < k$.

The last term can be computed directly noting that

$$\int_1^\infty x^{s-r_m-1} (\log x)^n\, dx = \frac{n!(-1)^{n+1}}{(s-r_m)^{n+1}}.$$

Hence

$$\mathcal{M}[f;s] = \sum_{\substack{m \\ \Re(r_m) < k}} \sum_{n=0}^{N(m)} \frac{(-1)^{n+1} c_{mn} n!}{(s-r_m)^{n+1}} + I_2.$$

The right hand is analytic except for poles at $s = r_m$ in the extended strip $\alpha < \Re(s) < k$, and since $k$ is arbitrary this proves the result.

The analytic continuation of $\mathcal{M}[f;s]$ into the left-half plane $\Re(s) < \alpha$ can be obtained and depends on the properties of $f(x)$ as $x \to 0+$

Suppose that as $x \to 0+$ we have

$$f(x) \sim e^{-qx^\mu} \sum_{m=0}^{\infty} x^{a_m} \sum_{n=0}^{K(m)} b_{mn} (\log x)^n,$$

where $\Re(q) \geq 0, \mu < 0$ and $\Re(a_m)$ is an increasing sequence in $m$, and $K(m)$ are non-negative integers. Then $F(s)$ can be continued analytically into the left-half plane $\Re(s) < \alpha$. The analytic continuation is such that

1. If $\Re(q) > 0$, then $F(s)$ is analytic in the left-half plane $\Re(s) < \beta$.

2. If $q = -i\omega, \quad \omega \neq 0$ then $F(z)$ is analytic in the left half-plane $\Re(s) < \alpha$.
3. If $q = 0$ then $F(s)$ is analytic in the left-half plane except for poles at $s = -a_m$ with Laurent expansion
$$\sum_{n=0}^{K(m)} \frac{b_{mn}(-1)^m n!}{(s+a_m)^{n+1}}.$$

In summary, the above results show that the Mellin transform of a function $f(x)$ , which has a particular type of asymptotic behaviour for small or large $x$, can be analytically continued into the whole complex plane with at worst pole singularities at points which can be predicted by the asymptotic behaviour of $f(x)$ for $x \to 0+$ and $x \to \infty$.

**Example** Consider
$$f(x) = e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}.$$
Now
$$\mathcal{M}[e^{ix}; s] = \Gamma(s) e^{is\frac{\pi}{2}}, \tag{9.5.3}$$
which is analytic in the strip $0 < \Re(s) < 1$. The analytic continuation of the transform into the left-half plane using (9.5.3) shows that the function is analytic in $\Re(s) < 0$ except for simple poles at $s = 0, 1, -2, \cdots - n, ...$, as predicted by the theorem.

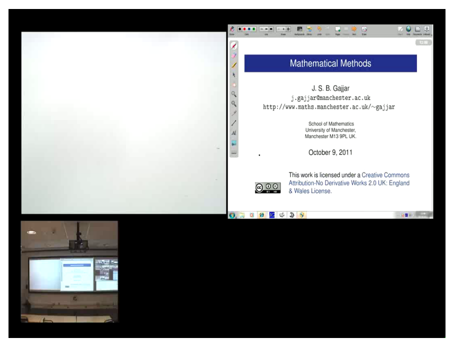
**Example**

Consider
$$\begin{aligned} \mathrm{Ei}(x) &= \int_x^{\infty} \frac{e^{-u}}{u}\, du \\ &= -\log(x) - \gamma - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n n!}. \end{aligned}$$
The Mellin transform of $\mathrm{Ei}(x)$ is given by
$$\mathcal{M}[\mathrm{Ei}(x); s] = \frac{\Gamma(s)}{s}, \quad \Re(s) > 0. \tag{9.5.4}$$
We can analytically continue (9.5.4) into the left-half plane and the analytic continuation gives a function which has simple poles at the negative integers and a double pole at $s = 0$.

# Bibliography

Bleistein, N. & Handelsman, R. A. 1975 *Asymptotic Expansions of Integrals*. McGraw-Hill.

Harrington, W. T. 1967 A property of mellin transforms. *SIAM Review* **9**, 542–547.




Video clip on a proof of the Riemann Lebesgue Lemma. Click here to open video clip in external player.

# Chapter 10

# Asymptotic expansion of integrals

In this section we will look at techniques for finding estimates for certain types of integrals containing a large (or small) parameter. Chapters 4-7 of Bleistein & Handelsman are essential reading for those who need to use these ideas for their research.

We will study

- Use of Mellin transforms.
- Laplace's method
- Steepest descent method.

## 10.1 Asymptotic expansion of integrals using the Mellin transform technique

Consider

$$H[f(x);\lambda] = \int_0^\infty h(\lambda x) f(x)\, dx, \tag{10.1.1}$$

where we will assume that $\lambda$ is real and we will investigate the behaviour of $H[f(x);\lambda]$ limit as $\lambda \to \infty$.

**Example**

If for example, $h(x) = e^{-x}$ then (10.1.1) is the Laplace transform of $f(x)$.

Suppose that $f(x)$ and $h(x)$ have Mellin transforms which are analytic in the strip $\alpha < \Re(s) < \beta$ and $\gamma < \Re(s) < \delta$. If we make use of Parseval's formula, see section 9.3.3, we obtain

$$\int_0^\infty h(\lambda x) f(x)\, dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \lambda^{-s} M[h;s] M[f;1-s]\, ds, \tag{10.1.2}$$

since

$$M[h(\lambda x);s] = \int_0^\infty x^{s-1} h(\lambda x)\,dx$$
$$= \lambda^{-s} \int_0^\infty x^{s-1} h(x)\,dx = \lambda^{-s} M[h(x);s].$$

Here we assume that there is a common strip of analyticity of $M[h;s], M[f,1-s]$ and the integral in (10.1.2) is taken along a line in this strip of analyticity. Next apply Cauchy's theorem and consider

$$\oint_{\mathcal{C}} \lambda^{-z} G(z)\,dz$$

where $G(z) = M[h;z]M[f;1-z]$, where $\mathcal{C} = L_1 + L_2 + L_3 + L_4$ as shown in figure 10.1.

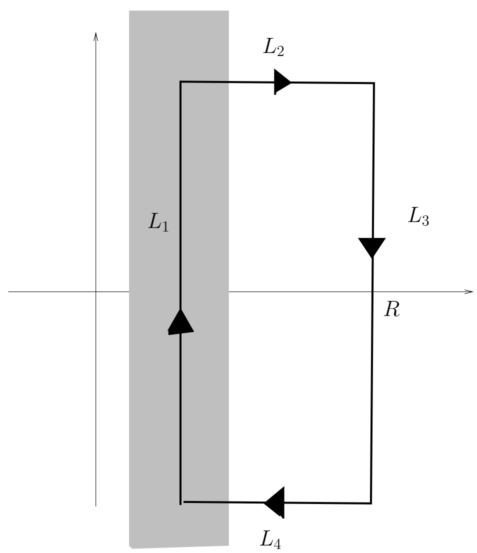



Figure 10.1: Contour $\mathcal{C}$ for application of Cauchy's theorem

Let

$$I_j = \int_{L_j} \lambda^{-z} G(z)\,dz.$$

We can write

$$I_2 = \int_c^R \lambda^{-(x+iY_1)} G(x+iY_1)\,dx,$$

and

$$I_3 = -\int_c^R \lambda^{-(x-iY_2)} G(x-iY_2)\,dx,$$

where $Y_1, Y_2$ are large and positive. If $G(z)$ is such that $|G(x+iy)| \to 0$ as $|y| \to \infty$ in $c \leq x \leq R$ then the integrals $I_2, I_4$ tend to zero as $Y_1 \to \infty$ and $Y_2 \to \infty$.

Cauchy's theorem together with the limit $Y_1, Y_2 \to \infty$ gives

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \lambda^{-z} G(z)\,dz = \frac{1}{2\pi i}\int_{R-i\infty}^{(R+i\infty)} \lambda^{-z} G(z)\,dz - \sum_{c<\Re(s)<R} \text{residues}[\lambda^{-s}G(s)].$$

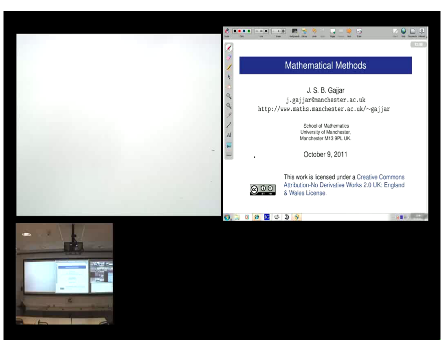
If we assume that $G(R+iy)$ is absolutely integrable then

$$\left|\int_{R-i\infty}^{R+i\infty} \lambda^{-z} G(z)\,dz\right| = \left|\int_{-\infty}^{\infty} \lambda^{-(R+iy)} G(R+iy)\,dy\right|$$
$$\leq |\lambda|^{-R}\int_{-\infty}^{\infty} |G(R+iy)|\,dy = O(\lambda^{-R}).$$

Thus

$$H[f(x);\lambda] =$$
$$\int_0^\infty h(\lambda x) f(x)\,dx = -\sum_{c<\Re(s)<R} \text{residues}[\lambda^{-s}G(s)] + O(\lambda^{-R}). \tag{10.1.3}$$

The sum of residues gives a finite asymptotic expansion for $H[f,\lambda]$. Note that $G(s) = \mathcal{M}[h(x);s]\mathcal{M}[f(x);1-s]$ and to be able to use (10.1.3) we need to be able to analytically continue $\mathcal{M}[h(x);s]$ into the right-half plane, and $\mathcal{M}[f(x);s]$ into the left-half plane. In particular this requires knowedge of the behaviour of $f(x)$ as $x \to 0+$, so that we can use the results on analytic continuation of Mellin transforms discussed earlier.




Video clip on a proof of the Riemann Lebesgue Lemma. Click here to open video clip in external player.

**Example** Consider

$$\int_0^\infty e^{-\lambda t} f(t)\,dt$$

so that $h(t) = e^{-t}$.

The Mellin transform of $e^{-t}$ is given by

$$\mathcal{M}[e^{-t};s] = \Gamma(s)$$

with $\Re(s) > 0$. Suppose $F(s)$ the Mellin transform of $f(x)$ is analytic in the region $\alpha < \Re(s) < \beta$ and that as $x \to 0+$ we have

$$f(x) \sim \sum_{m=0}^{\infty} x^{a_m} \sum_{n=0}^{K(m)} b_{mn} (\log x)^n,$$

where $\Re(a_m)$ is an increasing sequence in $m$, and $K(m)$ are non-negative integers. Then $F(s)$ can be continued analytically into the left-half plane $\Re(s) < \alpha$ with at worst pole singularities and local behaviour

$$\sum_{n=0}^{K(m)} \frac{b_{mn}(-1)^n n!}{(s+a_m)^{n+1}}.$$

near $s = -a_m$, see section 9.5.

Hence $F(1-s)$ can be analytically continued into the region $\Re(s) > 1-\alpha$ with pole singularities and the singular part of $F(1-s)$ has the behaviour

$$-\sum_{n=0}^{K(m)} \frac{b_{mn} n!}{(s-a_m-1)^{n+1}}.$$

near $s = a_m + 1$. So the function

$$\lambda^{-z} G(z) = \lambda^{-z} H(z) F(1-z) = \lambda^{-z} \Gamma(z) F(1-z)$$

has only residue contributions from the poles at $z = a_m + 1$ and using the earlier result (assuming that the various properties are satisfied) we have

$$\int_0^\infty e^{-\lambda x} f(x)\, dx = -\sum_{c<\Re(s)<R} \text{residues}[\lambda^{-s} G(s)] + O(\lambda^{-R}). \tag{10.1.4}$$

Let $R \to \infty$ and we obtain

$$\int_0^\infty e^{-\lambda x} f(x)\, dx \sim \sum_{m=0}^{\infty} \sum_{n=0}^{K(m)} b_{mn} \left( \frac{d^n}{dz^n} \left[ \lambda^{-z} \Gamma(z) \right] \right)_{z=1+a_m}.$$

Hence

$$\int_0^\infty e^{-\lambda x} f(x)\, dx \sim$$

$$\sum_{m=0}^{\infty} \lambda^{-1-a_m} \sum_{n=0}^{K(m)} b_{mn} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} (-\log \lambda)^j \left( \frac{d^{n-j}}{dz^{n-j}} \Gamma(z) \right)_{z=1+a_m}.$$

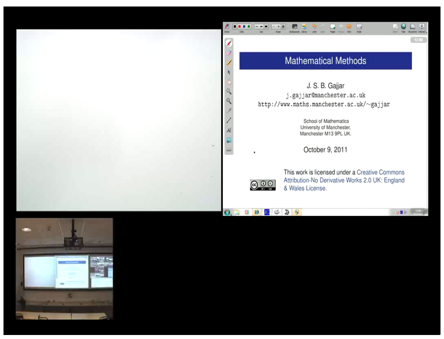
Note that if $b_{mn} = 0$ for $n > 1$ ie

$$f(x) \sim \sum_{m=0}^{\infty} b_{m0} x^{a_m} \quad \text{as} \quad x \to 0+$$

then we obtain Watson's lemma

$$\int_0^\infty e^{-\lambda x} f(x)\, dx \sim \sum_{m=0}^{\infty} \lambda^{-1-a_m} b_{m0} \Gamma(1 + a_m),$$

as $\lambda \to \infty$.




Video clip on application of the Mellin transform technique to estimate $\int_0^\infty e^{-\lambda t} f(t)\, dt$ for large $\lambda$. Click here to open video clip in external player.

**Example** Consider the integral

$$\text{Ei}(\lambda) = \int_\lambda^\infty \frac{e^{-\tau}}{\tau}\, dt.$$

If we put $\tau = \lambda + t$ then

$$\lambda e^\lambda \text{Ei}(\lambda) = \int_0^\infty \frac{e^{-t}}{1 + \lambda^{-1} t}\, dt. \tag{10.1.5}$$

We have met the function $\text{Ei}(x)$ already (in lecture 15) where we showed that

$$\text{Ei}(x) = -\log x - \gamma - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n n!}. \tag{10.1.6}$$

Note that the integral

$$I(q) = q^\nu \int_0^\infty \frac{f(t)}{(1 + qt)^\nu}\, dt$$

is the *Generalised Stieltjes transform* of the function $f(t)$.

Let us see how we can estimate $\text{Ei}(\lambda)$ for $\lambda$ large using the Stieltjes transform given by (10.1.5).

First put $\epsilon = \lambda^{-1}$ and consider

$$I(\epsilon) = \int_0^\infty e^{-t} h(\epsilon t)\, dt$$

where $h(t) = 1/(1+t)$. Now the Mellin transforms of $e^{-t}$ and $1/(1+t)$ are given by

$$\mathcal{M}[e^{-t}; s] = \Gamma(s), \quad 0 < \Re(s),$$
$$\mathcal{M}[\frac{1}{1+t}; s] = \frac{\pi}{\sin \pi s}, \quad 0 < \Re(s) < 1.$$

Hence we have a common strip of analyticity $0 < \Re(s) < 1$ for the function

$$G(s) = \mathcal{M}[e^{-t}; 1-s]\mathcal{M}[\frac{1}{1+t}; s] = \frac{\pi\Gamma(1-s)}{\sin \pi s}.$$

We can analytically continue the function $G(s)$ into the right-half plane $0 < \Re(s)$.

Now $\Gamma(1-s)$ has simple poles at $1-s = -n, \quad n = 0, -1, -2, ...$ and thus the analytic continuation of $G(s)$ has double poles at the positive integers $s = n, \quad n = 1, 2, \ldots$.

Using our earlier results

$$I = \int_0^\infty \frac{e^{-t}}{1+\epsilon t}\, dt = -\sum_{0<\Re(s)} \mathrm{Res}[\epsilon^{-s}\frac{\pi\Gamma(1-s)}{\sin \pi s}].$$

Next using the result

$$\Gamma(z-n) = \frac{\Gamma(z)}{(z-1)\ldots(z-n)} = \frac{\Gamma(z)(-1)^n}{(1-z)\ldots(n-z)},$$
$$= \frac{1}{z}[1-\gamma z+\ldots]\frac{(-1)^n}{n!}(1+z+..)(1+\frac{z}{2}+..)\ldots(1+\frac{z}{n}+\ldots),$$
$$= \frac{(-1)^n}{n!}(\frac{1}{z} - (\gamma - \sum_{j=1}^{n}\frac{1}{j}) + \ldots),$$

for $z$ small.

We can use this to work out the residue at the double poles for our function. Consider

$$g(s) = \epsilon^{-s}\frac{\pi\Gamma(1-s)}{\sin \pi s} = e^{-s\log(\epsilon)}\frac{\pi\Gamma(1-s)}{\sin \pi s},$$

and put $s = n + \delta$ where $n$ is a positive integer and $\delta$ is small. Thus

$$g(s) =$$

$$\epsilon^{-n}(1-\delta\log(\epsilon)+\ldots)\frac{\pi}{(-1)^n\pi\delta}\left[\frac{(-1)^n}{(n-1)!\delta} - \frac{(-1)^{n-1}}{(n-1)!}(\gamma - \sum_{j-1}^{n-1}\frac{1}{j}) + \ldots\right],$$

$$= \epsilon^{-n}(-1)^n \left[ \frac{(-1)^{n+1}}{(n-1)!\delta^2} + \frac{(-1)^{n-1}}{(n-1)!\delta} \left\{ (\gamma - \sum_{j=1}^{n-1} \frac{1}{j}) + \log(\epsilon) \right\} + \dots \right].$$

Hence the residue of the function $g(s)$ at $s = n$ is

$$\frac{(-1)^n(-1)^{n-1}}{(n-1)!} \left[ \gamma - \sum_{j=1}^{n-1} \frac{1}{j} + \log \epsilon \right],$$

with the summation interpreted to be zero if $n = 1$. Hence

$$I \sim -\sum_{n=1}^{\infty} \frac{\epsilon^{-n}}{(n-1)!} \left[ \gamma - \sum_{j=1}^{n-1} \frac{1}{j} + \log \epsilon \right],$$

$$= -(\gamma + \log(\epsilon))\epsilon^{-1} \sum_{n=0}^{\infty} \frac{\epsilon^{-n}}{n!} + \epsilon^{-1} \sum_{n=2}^{\infty} \frac{\epsilon^{-n}}{(n-1)!} \sum_{j=1}^{n-1} \frac{1}{j},$$

$$= -(\gamma + \log(\epsilon))\epsilon^{-1} e^{\epsilon^{-1}} + \epsilon^{-1} \sum_{n=1}^{\infty} \frac{\epsilon^{-n}}{n!} \sum_{j=1}^{n} \frac{1}{j}.$$

If we substitute back for $\epsilon^{-1} = \lambda$ and use (10.1.5) we find that

$$\lambda e^{\lambda} \mathrm{Ei}(\lambda) \sim -\lambda(\gamma + \log(\lambda))e^{\lambda} + \lambda \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \sum_{j=1}^{n} \frac{1}{j} \quad .$$

Hence

$$\mathrm{Ei}(\lambda) \sim -(\gamma + \log \lambda) + e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \sum_{j=1}^{n} \frac{1}{j}.$$

This agrees with the earlier result (10.1.6) provided we use the identity

$$e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \sum_{j=1}^{n} \frac{1}{j} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\lambda^n}{n n!}.$$

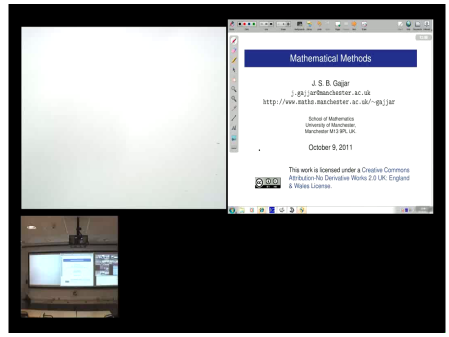
(For a proof of this last identity see the solutions to examples 6).

**Example** Consider

$$I(Y) = \int_0^{\infty} F(t) \frac{e^{iYt} - 1}{it} \, dt, \qquad (10.1.7)$$

where $F(t)$ is a smooth function which decays to zero exponentially as $t \to \infty$. Also

$$F(t) = \sum_{n=0}^{\infty} a_n t^n \quad \text{as} \quad t \to 0+ .$$




Video clip discussing estimation of $Ei(\lambda)$ for $\lambda \to 0$ using Mellin Transforms. Click here to open video clip in external player.

We need to find the behaviour of $I(Y)$ as $Y \to \pm\infty$. Integrals like this occur in hydrodynamic stability theory. Consider

$$H(F;\lambda) = \int_0^\infty F(t)h(\lambda t)\,dt$$

where

$$h(t) = \frac{e^{i\sigma t} - 1}{it},$$

and $\sigma = \pm 1$. Hence

$$I(Y) = YH(F;Y).$$

Now

$$H[F;\lambda] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \lambda^{-s}\mathcal{M}[h(t);s]\mathcal{M}[F(t);1-s]\,ds,$$

and the Mellin transform of $h(t)$ is

$$\mathcal{M}[h[t];s] = \int_0^\infty \frac{(e^{i\sigma t}-1)}{it}t^{s-1}\,dt = -i\int_0^\infty (e^{i\sigma t}-1)t^{s-2}\,ds,$$

$$= -i\left[\frac{t^{s-1}}{(s-1)}(e^{i\sigma t}-1)\right]_0^\infty + i^2\sigma\int_0^\infty \frac{1}{s-1}t^{s-1}e^{i\sigma t}\,dt,$$

$$= \frac{-\sigma}{s-1}\mathcal{M}[e^{i\sigma t};s] = -\frac{\sigma}{s-1}e^{\frac{\pi i\sigma s}{2}}\Gamma(s),$$

provided $0 < Re(s) < 1$.

Note that $\mathcal{M}[h(t);s]$ can be analytically continued into $1 < \Re(s)$ and the analytic continuation has a simple pole at $s = 1$.

Next consider the Mellin transform of $F(t)$. Given the behaviour of the function at $\infty$ this will be analytic in some strip $0 < \Re(s) < \beta$. The behaviour near $t \to 0+$, ie

$$F(t) \sim \sum_{n=0}^\infty a_n t^n$$

means that

$$\mathcal{M}[F(t);1-s] \sim -\frac{a_n}{(s-n-1)}$$

near $s = n+1, n = 0, 1, \ldots$ see lecture 16. Hence

$$\begin{aligned} H[F;s] &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \lambda^{-s}\mathcal{M}[h(t);s]\mathcal{M}[F(t);1-s]\,ds \\ &= -\sum_{0<\Re(s)} \mathrm{Res}(\lambda^{-s}\mathcal{M}(h(t);s)\mathcal{M}[F(t);1-s]). \end{aligned}$$

The function

$$\begin{aligned} G(s) &= -\lambda^{-s}\mathcal{M}(h(t);s)\mathcal{M}[F(t);1-s]) \\ &= \frac{\sigma\lambda^{-s}e^{\frac{i\sigma\pi s}{2}}}{(s-1)}\Gamma(s)\mathcal{M}[F(t);1-s] \end{aligned}$$

has a double pole at $s = 1$ and simple poles at the positive integers $s = n, \quad n > 1$. The residue at $s = 1$ is given by

$$\begin{aligned} &\lim_{s\to 1}\frac{d}{ds}\left((s-1)^2 G(s)\right) \\ &= \lim_{s\to 1}\frac{d}{ds}\left[\sigma(s-1)\lambda^{-s}e^{\frac{\pi i\sigma s}{2}}\Gamma(s)\mathcal{M}[F(t);1-s]\right], \\ &= -a_0\sigma\lambda^{-1}e^{\frac{i\pi\sigma}{2}}\left(-\log(\lambda)+1+\frac{\pi i\sigma}{2}+\Gamma'(1)\right) \\ &+\lambda^{-1}e^{\frac{i\pi\sigma}{2}}\Gamma(1)\lim_{s\to 1}\frac{d}{ds}\left((s-1)\mathcal{M}[F(t);1-s]\right), \end{aligned}$$

where we have used

$$\lim_{s\to 1}\{(s-1)\mathcal{M}[F(t);1-s]\} = -a_0.$$

Hence

$$\begin{aligned} &I(Y) = YH(Y) \\ &\sim -a_0\sigma e^{\frac{\pi i\sigma}{2}}\left[-\log|Y|+1+\Gamma'(1)+\frac{i\pi\sigma}{2}\right]+\sigma e^{\frac{i\pi\sigma}{2}}J+O(\frac{1}{Y}) \quad \text{as}\, Y\to\pm\sigma\infty, \\ &\sim -a_0 i\left[-\log|Y|+1+\Gamma'(1)+\frac{i\pi\sigma}{2}\right]+iJ+O(\frac{1}{Y}) \quad \text{as} \quad Y\to\pm\infty, \end{aligned}$$

where $\sigma = \mathrm{sgn(Y)}$, and

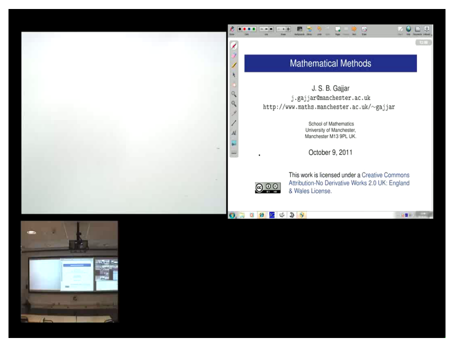
$$J = \lim_{s\to 1}\frac{d}{ds}\left((s-1)\mathcal{M}[F(t);1-s]\right).$$

We see that

$$I(Y\to+\infty) - I(Y\to-\infty) = F(0)\pi$$

since $a_0 = F(0)$.

Video clip on for above example. Click here to open video clip in external player.



## 10.2 Laplace's Method

Laplace's method is useful when trying to estimate integrals of the form

$$I(\lambda) = \int_a^b e^{-\lambda p(t)} q(t)\, dt,$$

where $a, b$ may be finite or infinite.

The following technique dates back to Laplace (1820). Observe that the peak value of the function $e^{-\lambda p(t)}$ occurs at the point $t = t_0$ where $p(t)$ is a minimum. For large $\lambda$ the peak is concentrated in a neighbourhood of $t - t_0$, see for example Fig. 10.2 where a plot of the function $e^{-\lambda(\cosh(t)-1)}$ is shown for varying $\lambda$.

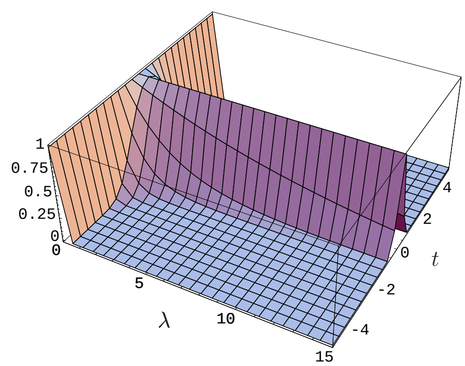



Figure 10.2: Plot of $f(\lambda, t) = e^{-\lambda \cosh[t]} e^{\lambda}$. Observe peak is concentrated near $t = 0$ .

In essence Laplace's method is as follows: Suppose that $t_0 = a$ and $p'(a) > 0, q(a) \neq 0$. In the integral

$$I(\lambda) = \int_a^b e^{-\lambda p(t)} q(t)\, dt,$$

we replace $p(t), q(t)$ by local series expansions near $t = t_0$. Then

$$I(\lambda) \sim \int_a^b e^{-\lambda(p(a)+p'(a)(t-t_0))} q(a)\, dt.$$

We replace the upper-limit by $\infty$ to obtain

$$I(\lambda) \sim q(a) e^{-\lambda p(a)} \int_a^\infty e^{-\lambda(t-a)p'(a)}\, dt.$$

Hence

$$I(\lambda) \sim q(a) \frac{e^{-\lambda p(a)}}{\lambda p'(a)}.$$

If instead $t = t_0$ is an interior point and $p''(t_0) > 0$ then

$$I(\lambda) = \int_a^b e^{-\lambda p(t)} q(t)\, dt \sim \int_a^b e^{-\lambda(p(t_0)+\frac{1}{2}p''(t_0)(t-t_0)^2)} q(t_0)\, dt \qquad (10.2.1)$$

Since the peak is concentrated in the neighbourbood of $t - t_0$ we may replace the upper and lower limits in (10.2.1) by $\pm\infty$ with neglible error. Then using $\int_{-\infty}^{\infty} e^{-at^2}\, dt = \sqrt{\pi/a}$ for $a > 0$ we obtain,

$$I(\lambda) \sim e^{-\lambda p(t_0)} q(t_0) \int_{-\infty}^{\infty} e^{-\lambda \frac{(t-t_0)^2}{2} p''(t_0)}\, dt = e^{-\lambda p(t_0)} q(t_0) \sqrt{\frac{2\pi}{\lambda p''(t_0)}}.$$

These hand waving arguments work remarkably well and are proven more formally below.

**Theorem** *Suppose*

1. *$p(t) > p(a)$ for $t \in (a, b)$ and the minimum of $p(t)$ is only approached at $t = a$.*

2. *$p'(t), q'(t)$ are continuous in a neighbourhood of $t = a$ except possibly at $t = a$.*

3. *As $t \to a+$*

$$p(t) \sim p(a) + \sum_{k=0}^{\infty} p_k (t-a)^{k+\mu}, \quad q(t) \sim \sum_{k=0}^{\infty} q_k (t-a)^{k+\sigma-1},$$

*where $\mu > 0, Re(\sigma) > 0, p_0 \neq 0, q_0 \neq 0$. Also we assume that we can differentiate $p(t)$ to obtain*

$$p'(t) \sim \sum_{k=0}^{\infty} (k+\mu) p_k (t-a)^{k+\mu-1}.$$

4. *$\int_a^b e^{-\lambda p(t)} q(t)\, dt$ converges absolutely for large $\lambda$.*

*Then*

$$I(\lambda) = \int_a^b e^{-\lambda p(t)} q(t)\, dt \sim e^{-\lambda p(a)} \sum_{k=0}^{\infty} \Gamma\left(\frac{k+\sigma}{\mu}\right) \frac{a_k}{\lambda^{\frac{k+\sigma}{\mu}}},$$

*where $v = p(t) - p(a)$ and*

$$f(v) = \frac{q(t)}{p'(t)} \sim \sum_{k=0}^{\infty} a_k v^{\frac{k+\sigma-\mu}{\mu}} \quad as \quad v \to 0+.$$

**Proof**

Let $v = p(t) - p(a)$ then

$$I(\lambda) = \int_a^b e^{-\lambda p(t)} q(t)\, dt$$

$$= e^{-\lambda p(a)} \int_0^{p(b)-p(a)} e^{-\lambda v} f(v)\, dv$$

where $f(v) = q(t)/p'(t)$. Hence

$$I(\lambda) = e^{-\lambda p(a)} \int_0^{\infty} e^{-\lambda v} f(v)\, dv - e^{-\lambda p(a)} \int_{p(b)-p(a)}^{\infty} e^{-\lambda v} f(v)\, dv. \tag{10.2.2}$$

The contribution from the last integral in (10.2.2) can be shown to be negligble. If we use Watson's lemma for the other integral noting that as $t \to a+$, $v \to 0$ and

$$f(v) \sim \sum_{k=0}^{\infty} a_k v^{\frac{k+\sigma-\mu}{\mu}}.$$

This gives

$$I(\lambda) \sim e^{-\lambda p(a)} \int_0^{\infty} e^{-\lambda v} \sum_{k=0}^{\infty} a_k v^{\frac{k+\sigma}{\mu}-1}\, dv$$

$$= e^{-\lambda p(a)} \sum_{k=0}^{\infty} \int_0^{\infty} e^{-\lambda v} a_k v^{\frac{k+\sigma}{\mu}-1}\, dv, .$$

Hence

$$I(\lambda) \sim e^{-\lambda p(a)} \sum_{k=0}^{\infty} a_k \Gamma\left(\frac{k+\sigma}{\mu}\right) \frac{1}{\lambda^{\frac{k+\sigma}{\mu}}}.$$

**Example**

Consider the modified Bessel function of the second kind

$$K_\nu(\lambda) = \int_0^{\infty} e^{-\lambda \cosh t} \cosh(\nu t)\, dt$$

and we need the behaviour for large $\lambda$.

Here $p(t) = \cosh t$ has a minimum value of 1 at $t = 0$. Hence put

$$v = \cosh t - 1$$

For small $t$

$$v = \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots. \tag{10.2.3}$$

We can invert this to find $t$ as a function of $v$ for $v$ small and the leading term is $t = (2v)^{\frac{1}{2}}$. This suggests that for small $v$ we may write,

$$t = (2v)^{\frac{1}{2}} + c_1 v + c_2 v^{\frac{3}{2}} + \dots .$$

Thus substituting into (10.2.3) we find

$$v = \frac{t^2}{2!} + \frac{t^4}{4!} + \dots$$

$$= \frac{1}{2}[(2v)^{\frac{1}{2}} + c_1 v + c_2 v^{\frac{3}{2}} + \dots]^2 + \frac{1}{4!}[(2v)^2 + \dots] + \dots ,$$

$$= v + v^{\frac{3}{2}} c_1 \sqrt{2} + v^2[\sqrt{2} c_2 + \frac{c_1^2}{2} + \frac{1}{6}] + \dots .$$

Comparing like powers of $v$ on both sides implies that

$$c_1 = 0, \quad c_2 = -\frac{1}{6\sqrt{2}}.$$

Hence

$$t = (2v)^{\frac{1}{2}} - \frac{1}{6\sqrt{2}} v^{\frac{3}{2}} + \dots .$$

Hence

$$K_\nu(\lambda) = \int_0^\infty e^{-\lambda \cosh t} \cosh(\nu t)\, dt = e^{-\lambda} \int_0^\infty e^{-\lambda v} \frac{dt}{dv}[1 + \frac{\nu^2}{2} t^2 + \dots]\, dv$$

$$= e^{-\lambda} \int_0^\infty e^{-\lambda v}[\frac{1}{2}\sqrt{2} v^{-\frac{1}{2}} - \frac{1}{4\sqrt{2}} v^{\frac{1}{2}} + \dots][1 + \frac{\nu^2}{2}(2v) + \dots]\, dv,$$

$$= e^{-\lambda} \int_0^\infty e^{-\lambda v}[\frac{\sqrt{2}}{2} v^{-\frac{1}{2}} + v^{\frac{1}{2}}(\frac{\sqrt{2}}{2}\nu^2 - \frac{1}{4\sqrt{2}}) + \dots]\, dv.$$

This gives

$$K_\nu(\lambda) = e^{-\lambda}\sqrt{\frac{\pi}{2\lambda}}\left[1 + \frac{1}{2}(\nu^2 - \frac{1}{4})\frac{1}{\lambda} + \dots\right],$$

as $\lambda \to \infty$.

**Example- Stirling's formula for large $x$.** We will show how Laplace's method can be used to estimate the Gamma function $\Gamma(\lambda)$ for large values of the argument. Consider

$$\Gamma(\lambda+1) = \lambda\Gamma(\lambda) = \int_0^\infty e^{-y} y^\lambda \, dy. \tag{10.2.4}$$

Hence

$$\Gamma(\lambda) = \frac{1}{\lambda}\int_0^\infty e^{-y} y^\lambda \, dy.$$

Now

$$e^{-y} y^\lambda = e^{-y+\lambda \log y},$$

and the function $r(y) = -y + \lambda \log y$ has a minimum at $y = \lambda$. It is better to work with a fixed point rather than one depending on $\lambda$. So put $y = \lambda t$. Then substituting into (10.2.4) gives

$$\Gamma(\lambda) = \frac{1}{\lambda}\int_0^\infty e^{-\lambda t} \lambda^\lambda t^\lambda \lambda \, dt,$$

$$= \lambda^\lambda \int_0^\infty e^{-\lambda(t-\log t)} \, dt.$$

Consider

$$I(\lambda) = \int_0^\infty e^{-\lambda(T-\log T)} \, dT.$$

Now $P(T) = T - \log T$ has a minimum value of 1 at $T = 1$ for $T > 0$. If we are interesed in just the dominant term for $\Gamma(x)$ we can replace $P(T)$ by a local expansion in the vicinity of $T = 1$ and work with that. Below we show how more terms can generated. First we write

$$I(\lambda) = \int_0^1 e^{-\lambda P(T)} \, dT + \int_1^\infty e^{-\lambda P(T)} \, dT, \tag{10.2.5}$$

and estimate the two integrals separately.

Consider

$$I_1 = \int_0^1 e^{-\lambda P(T)} \, dT. \tag{10.2.6}$$

Put $t = 1 - T$ in (10.2.6) so that the minimum occurs at $t = 0$ and then

$$I_1 = \int_0^1 e^{-\lambda(1-t-\log(1-t))} \, dt. \tag{10.2.7}$$

Next let

$$v = 1 - t - \log(1-t) - 1 = -t - \log(1-t).$$

For small $t$ we have

$$v = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \dots.$$

This suggests that for small $v$

$$t = (2v)^{\frac{1}{2}} + c_1 v + c_2 v^{\frac{3}{2}} + \dots .$$

Hence

$$v = \frac{1}{2}[(2v)^{\frac{1}{2}} + c_1 v + c_2 v^{\frac{3}{2}} + \dots]^2 + \frac{1}{3}[(2v)^{\frac{1}{2}} + c_1 v + \dots]^3 + \frac{1}{4}[(2v)^2 + \dots] + \dots,$$

$$= \frac{1}{2}[2v + 2\sqrt{2v}c_1 v + 2\sqrt{2v}c_2 v^{\frac{3}{2}} + c_1^2 v^2 + \dots]$$

$$+\frac{1}{3}[(2v)^{\frac{3}{2}} + 3(2v)(c_1 v + c_2 v^{\frac{3}{2}}) + \dots] + v^2 + \dots,$$

$$= v + v^{\frac{3}{2}}[\sqrt{2}c_1 + \frac{2\sqrt{2}}{3}] + v^2[\sqrt{2}c_2 + \frac{c_1^2}{2} + 2c_1 + 1] + \dots .$$

Equating like powers of $v$ on both sides gives $c_1 = -\frac{2}{3}$ and

$$\sqrt{2}c_2 = -(1 + 2c_1 + \frac{c_1^2}{2}) = -(1 - \frac{4}{3} + \frac{2}{9}) = \frac{1}{9}.$$

Thus $c_2 = \frac{\sqrt{2}}{18}$ and we have

$$t = (2v)^{\frac{1}{2}} - \frac{2}{3}v + \frac{\sqrt{2}}{18}v^{\frac{3}{2}} + \dots .$$

This gives

$$\frac{dt}{dv} = \frac{1}{\sqrt{2}}v^{-\frac{1}{2}} - \frac{2}{3} + \frac{1}{6\sqrt{2}}v^{\frac{1}{2}} + \dots$$

as $v \to 0+$. With the substitution $v = -t - \log(1-t)$ the integral (10.2.5) becomes

$$I_1 = e^{-\lambda} \int_0^\infty e^{-\lambda v} \frac{dt}{dv}\, dv.$$

Using Watson's lemma means replacing $\frac{dt}{dv}$ by the expansion for small $v$ to get

$$I_1(\lambda) \sim e^{-\lambda} \int_0^\infty e^{-\lambda v} \left[\frac{1}{\sqrt{2}}v^{-\frac{1}{2}} - \frac{2}{3} + \frac{1}{6\sqrt{2}}v^{\frac{1}{2}} + \dots\right] dv,$$

$$= e^{-\lambda}\left[\sqrt{\frac{\pi}{2\lambda}} - \frac{2}{3\lambda} + \sqrt{\frac{\pi}{2}}\frac{1}{12\lambda^{\frac{3}{2}}} + \dots\right]. \qquad (10.2.8)$$

We still need to consider the second of the integrals in (10.2.5), ie,

$$I_2 = \int_1^\infty e^{-\lambda(T - \log T)}\, dT = e^{-\lambda} \int_0^\infty e^{-\lambda(t - \log(1+t))}\, dt. \qquad (10.2.9)$$

Here $p(t) = t - \log(1+t)$ has a minimum value of 0 at $t = 0$.Put $v = t - \log(1+t)$. As $t \to 0+$ we have

$$v = \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^4}{4} + \dots .$$

Inverting this for small $v$ suggests that

$$t = (2v)^{\frac{1}{2}} + c_1 v + c_2 v^{\frac{3}{2}} + \dots .$$

Thus

$$v = \frac{1}{2}[(2v)^{\frac{1}{2}} + c_1 v + c_2 v^{\frac{3}{2}} + \dots]^2 - \frac{1}{3}[(2v)^{\frac{1}{2}} + c_1 v + \dots]^3 + \frac{1}{4}[(2v)^2 + \dots] + \dots ,$$

$$= \frac{1}{2}[2v + 2\sqrt{2v}c_1 v + 2\sqrt{2v}c_2 v^{\frac{3}{2}} + c_1^2 v^2 + \dots]$$

$$-\frac{1}{3}[(2v)^{\frac{3}{2}} + 3(2v)(c_1 v + c_2 v^{\frac{3}{2}}) + \dots] + v^2 + \dots ,$$

$$= v + v^{\frac{3}{2}}[\sqrt{2}c_1 - \frac{2\sqrt{2}}{3}] + v^2[\sqrt{2}c_2 + \frac{c_1^2}{2} - 2c_1 + 1] + \dots .$$

Hence $c_1 = \frac{2}{3}$ and

$$\sqrt{2}c_2 = -(1 - 2c_1 + \frac{c_1^2}{2}) = -(1 - \frac{4}{3} + \frac{2}{9}) = \frac{1}{9}.$$

Thus $c_2 = \frac{\sqrt{2}}{18}$ and we have

$$t = (2v)^{\frac{1}{2}} + \frac{2}{3}v + \frac{\sqrt{2}}{18}v^{\frac{3}{2}} + \dots .$$

This gives

$$\frac{dt}{dv} = \frac{1}{\sqrt{2}}v^{-\frac{1}{2}} + \frac{2}{3} + \frac{1}{6\sqrt{2}}v^{\frac{1}{2}} + \dots$$

as $v \to 0+$ . With the substitution $v = t - \log(1+t)$ the integral (10.2.9) for $I_2$ becomes

$$I_2 = e^{-\lambda}\int_0^\infty e^{-\lambda v}\frac{dt}{dv}\,dv.$$

$$I_2(\lambda) \sim e^{-\lambda}\int_0^\infty e^{-\lambda v}\left[\frac{1}{\sqrt{2}}v^{-\frac{1}{2}} + \frac{2}{3} + \frac{1}{6\sqrt{2}}v^{\frac{1}{2}} + \dots\right]dv,$$

$$= e^{-\lambda}\left[\sqrt{\frac{\pi}{2\lambda}} + \frac{2}{3\lambda} + \sqrt{\frac{\pi}{2}}\frac{1}{12\lambda^{\frac{3}{2}}} + \dots\right]. \tag{10.2.10}$$

Combining the two expressions (10.2.8),(10.2.10) for $I_1$ and $I_2$ shows that

$$\Gamma(\lambda) = \lambda^\lambda(I_1(\lambda) + I_2(\lambda)),$$

and using the derived asymptotic expansions for the two integrals gives

$$\Gamma(\lambda) \sim \lambda^{\lambda} e^{-\lambda} \sqrt{\frac{2\pi}{\lambda}} \left[ 1 + \frac{1}{12\lambda} + \dots \right],$$

as $\lambda \to \infty$.

This is Stirling's formula for the Gamma function for large values of the argument.

## 10.3 Method of stationary phase

In place of Laplace type integrals of the form (9.2.2) suppose we consider integrals of the form

$$I(\lambda) = \int_a^b e^{i\lambda p(t)} q(t)\, dt \tag{10.3.1}$$

and we require the behaviour of $I(\lambda)$ for large $\lambda$. A special case of these are Fourier transforms with $a, b$ replaced by $\pm\infty$ and $p(t) = t$. For integrals of the form there is a famous result known as the **Riemann-Lebesgue lemma** which states that $I(\lambda) \to 0$ as $\lambda \to \infty$ provided $|q(t)|$ is integrable in the interval $[a, b]$ and that $p(t)$ is continuously differentiable for $a \le t \le b$ and not constant on any subinterval in $a \le t \le b$.

If $p'(t)$ is non-zero in $a \le t \le b$ then we can use integration by parts and show that $I(\lambda) = O(1/\lambda)$ as $\lambda \to \infty$. The more interesting case is when $p'(t)$ is zero in $a \le t \le b$.

Observe that for large $\lambda$ the integrand in (10.3.1) oscillates and contributions cancel out except near end points and near stationary points of $p(t)$. The behaviour of the integral can be estimated by looking at the local behaviour of the functions $p(t), q(t)$ near end points and near the stationary points of $p(t)$, as we did with Laplace's method. The basic idea of the method of stationary phase is as follows. Suppose that $p(t)$ has a single stationary point for at $t = t_0$ in $a < t < b$ and we can write

$$p(t) = p(t_0) + \frac{1}{2} p''(t_0)(t - t_0)^2 + \dots, \quad q(t) = q(t_0) + \dots.$$

Then we can approximate $I(\lambda)$ as

$$I(\lambda) \sim \int_{-\infty}^{\infty} e^{i\lambda(p(t_0) + \frac{1}{2}(t-t_0)^2 P''(t_0))} q(t_0)\, dt \sim e^{i\lambda p(t_0)} q(t_0) \int_{-\infty}^{\infty} e^{i\lambda \frac{p''(t_0)}{2} T^2}\, dT,$$

and so

$$I(\lambda) \sim \sqrt{\frac{2\pi}{\lambda}} e^{\frac{i\pi}{4}} e^{i\lambda p(t_0)} q(t_0),$$

where we have used

$$\int_{-\infty}^{\infty} e^{i\lambda T^2}\, dT = \sqrt{\frac{\pi}{\lambda}} e^{\frac{i\pi}{4}}.$$

The above can be generalised to deal with other behaviours and to obtain higher order behaviour as follows. Suppose that $p(t)$ has a single stationary point $t = t_0$ in $t \in [a, b]$. We can write

$$I(\lambda) = \int_a^{t_0} e^{i\lambda p(t)} q(t)\, dt + \int_{t_0}^{b} e^{i\lambda p(t)} q(t)\, dt. \tag{10.3.2}$$

Assume that near $t = t_0+$ we have

$$p(t) = p(t_0) + \alpha(t-t_0)^{\nu} + o((t-t_0)^{\nu}), \quad q(t) = \beta(t-t_0)^{\delta-1} + o((t-t_0)^{\nu}), \tag{10.3.3}$$

where $\nu > 0, \delta > 0$, and that the expression for $p(t)$ is differentiable, ie

$$p'(t) \sim \alpha\nu(t-t_0)^{\nu-1} \quad \text{as} \quad t \to t_0+.$$

Consider

$$I_1(\lambda) = \int_{t_0}^{b} e^{i\lambda p(t)} q(t)\, dt.$$

If we make the substitution

$$v = s(p(t) - p(t_0)) \tag{10.3.4}$$

where $s = \mathrm{sgn}(\alpha)$ then

$$I_1(\lambda) = e^{i\lambda p(t_0)} \int_0^{|p(b)-p(t_0)|} e^{is\lambda v} F(v)\, dv \tag{10.3.5}$$

where

$$F(v) = \frac{sq(t)}{p'(t)}.$$

Note that from (10.3.3), (10.3.4) as $t \to t_0+$

$$t - t_0 \sim \left(\frac{v}{|\alpha|}\right)^{\frac{1}{\nu}}.$$

Thus using the behaviour of $q(t)$ given in (10.3.3) we have

$$F(v) \sim \frac{s\beta(t-t_0)^{\delta-1}}{\alpha\nu(t-t_0)^{\nu-1}} \sim \frac{s\beta}{\alpha\nu}\left(\frac{v}{|\alpha|}\right)^{\frac{\delta}{\nu}-1}.$$

If $F(v)$ is well behaved for large $v$ then using the above we can approxmiate $I_1$ by

$$I_1(\lambda) = e^{i\lambda p(t_0)} \int_0^{|p(b)-p(t_0)|} e^{i\lambda sv} F(v)\, dv$$

$$\sim e^{i\lambda p(t_0)} \int_0^\infty e^{i\lambda s v} F(v)\, dv.$$

We can extract the leading order behavior of $I_1$ by replacing $F(v)$ with the local behaviour near $v \to 0+$. Thus

$$I_1(\lambda) \sim s e^{i\lambda p(t_0)} \int_0^\infty e^{i\lambda s v} \frac{\beta}{\alpha\nu} \left(\frac{v}{|\alpha|}\right)^{\frac{\delta}{\nu}-1} dv$$

$$\sim e^{i\lambda p(t_0)} \frac{s\beta}{\alpha\nu} e^{i\frac{\pi}{2}\frac{\delta}{\nu}s} \frac{\Gamma(\frac{\delta}{\nu})}{|\alpha|^{\frac{\delta}{\nu}-1}\lambda^{\frac{\delta}{\nu}}},$$

where we have used the result

$$\int_0^\infty e^{i\lambda\sigma t} t^{s-1}\, dt = \lambda^{-s} e^{i\sigma s\pi/2}\Gamma(s)$$

for $\lambda > 0$ and $\sigma = \pm 1$. Hence

$$I_1(\lambda) \sim e^{i\lambda p(t_0)} \frac{\beta}{\nu} e^{i\frac{\pi}{2}\frac{\delta}{\nu}s} \frac{\Gamma(\frac{\delta}{\nu})}{(|\alpha|\lambda)^{\frac{\delta}{\nu}}}. \tag{10.3.6}$$

Similarly for

$$I_2(\lambda) = \int_a^{t_0} e^{i\lambda p(t)} q(t)\, dt$$

suppose that as $t \to t_0-$

$$p(t) \sim p(t_0) + \gamma(t_0 - t)^\epsilon + o((t-t_0)^\epsilon), \qquad q(t) \sim \rho(t_0 - t)^{\sigma-1} + o((t-t_0)^\sigma),$$

where $\epsilon > 0, \sigma > 0$. Then

$$I_2(\lambda) \sim\sim e^{i\lambda p(t_0)} \frac{\rho}{\epsilon} e^{i\frac{\pi}{2}\frac{\sigma}{\epsilon}S} \frac{\Gamma(\frac{\sigma}{\epsilon})}{(|\gamma|\lambda)^{\frac{\sigma}{\epsilon}}}, \tag{10.3.7}$$

where $S = \mathrm{sgn}(\gamma)$.

The dominant contribution to $I$ is given by adding the estimates (10.3.6), (10.3.7) for $I_1$ and $I_2$ to get

$$I(\lambda) \sim e^{i\lambda p(t_0)} \frac{\beta}{\nu} e^{i\frac{\pi}{2}\frac{\delta}{\nu}s} \frac{\Gamma(\frac{\delta}{\nu})}{(|\alpha|\lambda)^{\frac{\delta}{\nu}}} + e^{i\lambda p(t_0)} \frac{\rho}{\epsilon} e^{i\frac{\pi}{2}\frac{\sigma}{\epsilon}S} \frac{\Gamma(\frac{\sigma}{\epsilon})}{(|\gamma|\lambda)^{\frac{\sigma}{\epsilon}}}.$$

Near an end point one can adapt the above analysis as appropriate. The above ideas can be treated more formally, see, for example, chapter 6 of Olver.

**Example** Consider the Bessel function of order $n$ where $n$ is real

$$J_n(\lambda) = \frac{1}{\pi} \int_0^\pi \cos(nt - \lambda \sin t)\, dt.$$

We can write this as

$$J_n(\lambda) = \frac{1}{\pi}\Re\left[\int_0^{\pi} e^{int-i\lambda\sin t}\,dt\right].$$

Here $p(t) = \sin t$ has a single stationary point at $t = \frac{\pi}{2}$ for $t \in [0,\pi]$. First let $t = \frac{\pi}{2} + T$ and then

$$J_n(\lambda) = \int_{-\frac{\pi}{2}}^{0} + \int_0^{\frac{\pi}{2}} \left(e^{in(\frac{\pi}{2}+T)}e^{-i\lambda\cos T}\right)\,dT. \tag{10.3.8}$$

Consider

$$I_1 = \int_{-\frac{\pi}{2}}^{0} e^{in(\frac{\pi}{2}+T)}e^{-i\lambda\cos T}\,dT = e^{in\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} e^{-inT}e^{-i\lambda\cos T}\,dT.$$

Put

$$u = -\cos T + 1 \sim \frac{T^2}{2} + O(T^4) \quad \text{as} \quad T \to 0.$$

Inverting gives

$$T = (2u)^{\frac{1}{2}} + \ldots \quad \text{as} \quad u \to 0+.$$

Thus

$$I_1 \sim e^{in\frac{\pi}{2}}\int_0^{\pi/2} e^{i\lambda(u-1)}(1+\ldots)(2u)^{-\frac{1}{2}}\,du,$$

$$I_1 \sim e^{in\frac{\pi}{2}-i\lambda}\frac{1}{\sqrt{2}}\int_0^{\infty} e^{i\lambda u}u^{-\frac{1}{2}}\,du = e^{in\frac{\pi}{2}-i\lambda}e^{\frac{i\pi}{4}}\sqrt{\frac{\pi}{2\lambda}}. \tag{10.3.9}$$

Next consider

$$I_2 = \int_0^{\frac{\pi}{2}} e^{in(\frac{\pi}{2}+T)}e^{-i\lambda\cos T}\,dT.$$

Put

$$u = -\cos T + 1 \sim \frac{T^2}{2} \quad \text{as} \quad T \to 0+.$$

Thus

$$T = (2u)^{\frac{1}{2}} \quad \text{as} \quad u \to 0+.$$

Hence

$$I_2 \sim e^{in\frac{\pi}{2}}\int_0^{\pi/2}(1+\ldots)e^{i\lambda(u-1)}(2u)^{-\frac{1}{2}}\,du,$$

$$\sim e^{in\frac{\pi}{2}-i\lambda}\int_0^{\infty} e^{i\lambda u}(2u)^{-\frac{1}{2}}\,du.$$

Thus

$$I_2 \sim e^{in\frac{\pi}{2}-i\lambda}e^{\frac{i\pi}{4}}\sqrt{\frac{\pi}{2\lambda}}. \tag{10.3.10}$$

Hence finally using (10.3.8), (10.3.9), (10.3.10) we obtain

$$J_n(\lambda) \sim \frac{1}{\pi}\Re\left[2e^{in\frac{\pi}{2}-i\lambda}e^{\frac{i\pi}{4}}\sqrt{\frac{\pi}{2\lambda}}+\dots\right]$$

$$= \sqrt{\frac{2}{\pi\lambda}}\cos(\frac{\pi}{4}+\frac{n\pi}{2}-\lambda) \quad \text{as} \quad \lambda \to \infty.$$

## 10.4 Method of steepest descent

This technique first developed by Riemann (1892) and is extremely useful for handling integrals of the form

$$I(\lambda) = \int_{\mathcal{C}} e^{\lambda p(z)} q(z)\, dz$$

where $\mathcal{C}$ is a contour in the complex plane and $p(z), q(z)$ are analytic functions, and $\lambda$ is taken to be real.(If $\lambda$ is complex ie $\lambda = |\lambda| e^{i\alpha}$ we can absorb the exponential factor into $p(z)$.) We require the behaviour of $I(\lambda)$ as $\lambda \to \infty$.

The basic idea of the method of steepest descent (or sometimes referred to as the *saddle-point method*), is that we apply Cauchy's theorem to deform the contour $\mathcal{C}$ to contours coinciding with the path of steepest descent. Usually these contours pass through points $z = z_0$ where $p'(z_0) = 0$. As we will see on the steepest descent contours, $\Im(p(z))$ is constant and so we are left with integrals of the type which can be handled using Watson's lemma.

Let $p(z) = u(x, y) + iv(x, y)$ be an analytic function of the complex variable $z = x + iy$ in some domain $\mathcal{D}$. Notice that for any path of integration the exponential function

$$e^{\lambda p(z)} = e^{\lambda u(x,y)} e^{i\lambda v(x,y)}$$

may have a maximum modulus at some point $z = z_0$ on the path. Ideally we would like to choose a path near a point $z = z_0$ such that $u$ attains a peak and decreases away from $z = z_0$. But the imaginary part $v(x, y)$ will in general also change and the exponential factor $e^{i\lambda v}$ will oscillate rapidly near $z_0$.

This suggests that a suitable path is one where $v(x, y)$ is nearly constant as we move away from $z = z_0$. Also by the maximum modulus theorem $u, v$ cannot attain maximum or minimum values in a domain if $p(z)$ is analytic, only on the boundary of the region. Thus the point $z = z_0$ must coincide with a saddle point where $p'(z_0) = 0$.

The method of steepest descent is thus also called the saddle point method. If we consider the surface

$$u(x, y) = u(x_0, y_0)$$

passing through some point $z = z_0$ of $\mathcal{D}$ then note that $\nabla u|_{z_0}$ defines the direction of steepest ascent from the point $z = z_0$ and $-\nabla u|_{z_0}$ the direction of steepest descent.

Now consider the surface

$$v(x, y) = v(x_0, y_0).$$

We have that $\nabla v = (v_x, v_y)$ is in a direction normal to the surface. But using the Cauchy-Riemann equations $v_x = -u_y, v_y = u_x$.

Thus

$$\nabla v = (v_x, v_y) = (-u_y, u_x).$$

Hence a direction tangential to the surface is given by

$$(-v_y, v_x) = (u_x, u_y) = \nabla u.$$

Thus tangents to the surface

$$v(x, y) = v(x_0, y_0)$$

lie in the direction of steepest ascent/descent through $z_0$ and the lines of constant $u$ and constant $v$ intersect at right angles in the regions of analyticity of the functions.

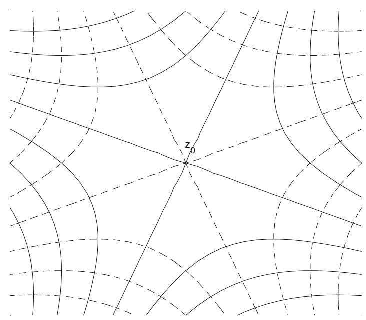



Figure 10.3: Typical steepest ascent/descent curves shown by solid/dashed lines near a simple saddle point $z = z_0$

Observe also that for any change $\delta p$

$$\delta p = \delta u + i\delta v$$

and so

$$|\delta u| \leq |\delta p|$$

and $|\delta u|$ is a maximum at $z = z_0$ only when $\delta v = 0$, ie when $v(x, y) = v(x_0, y_0)$.

**Definition** *We define $z = z_0$ to be* **a saddle point of order** $N - 1$ *if*

$$p'(z_0), \ldots p^{(N-1)}(z_0) = 0, \quad p^{(N)}(z_0) \neq 0.$$

*A saddle point or order 1 is a simple saddle point.*

Near $z = z_0$ we have

$$p(z) = p(z_0) + \frac{(z - z_0)^N}{N!} p^{(N)}(z_0) + o((z - z_0)^N).$$

Putting

$$z = z_0 + \rho e^{i\theta}, \quad p^{(N)}(z_0) = ae^{i\alpha}$$

we have

$$p(z) - p(z_0) \sim \frac{\rho^N a e^{i(N\theta+\alpha)}}{N!}.$$

Thus the curves of steepest ascent/descent through $z = z_0$ are given locally by

$$\Im(p(z) - p(z_0)) = 0 \qquad \Longrightarrow \sin(N\theta + \alpha) = 0,$$

giving $N\theta + \alpha = k\pi$, where $k$ is an integer. In this case

$$p(z) - p(z_0) = u(x, y) - u(x_0, y_0) \sim \frac{\rho^N}{N!} a \cos(N\theta + \alpha).$$

Thus the curves of **steepest descent** are given by

$$\theta = -\frac{\alpha}{N} + (2k+1)\frac{\pi}{N} \quad k = 0, 1, 2, \ldots, N-1,$$

since $\cos(N\theta + \alpha)$ is then negative and $u(x, y) < u(x_0, y_0)$ as we move away from $z = z_0$.

The curves of **steepest ascent** are given by

$$\theta = -\frac{\alpha}{N} + 2k\frac{\pi}{N} \quad k = 0, 1, 2, \ldots, N-1,$$

since $\cos(N\theta + \alpha)$ is then positive and $u(x, y) > u(x_0, y_0)$ as we move away from $z = z_0$.

**Example** Consider $p(z) = z - \frac{z^3}{3}$. Now $p'(z) = 1 - z^2$ and so the *critical points* where $p'(z) = 0$ are given by $z = \pm 1$. Also $p''(z) = -2z$ and thus $p''(1) = -2 = 2e^{i\pi}$, and $p''(-1) = 2$.

Near $z = 1$ the directions of steepest descent are given by the directions $\theta = -\frac{\pi}{2} + (2k+1)\frac{\pi}{2}, \quad k = 0, 1$ ie $\theta = 0, \pi$.

Near $z = -1$ the directions of steepest descent are given by the directions $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Consider the point $z = 1$. The steepest descent/ascent curves satisfy

$$v(x, y) = \Im(z - \frac{z^3}{3}) = y(1 - x^2 + \frac{y^2}{3}) = v(1, 0) = 0.$$

There are two curves of steepest ascent/descent passing through $z_0 = 1$. These are $y = 0$ and $1 - x^2 + \frac{y^2}{3} = 0$. Clearly $y = 0$ is the steepest descent curve, see Fig. 10.4.

Next consider the point $z = -1$. Here the steepest descent/ascent curves satisfy

$$v(x, y) = y(1 - x^2 + \frac{y^2}{3}) = v(-1, 0) = 0.$$

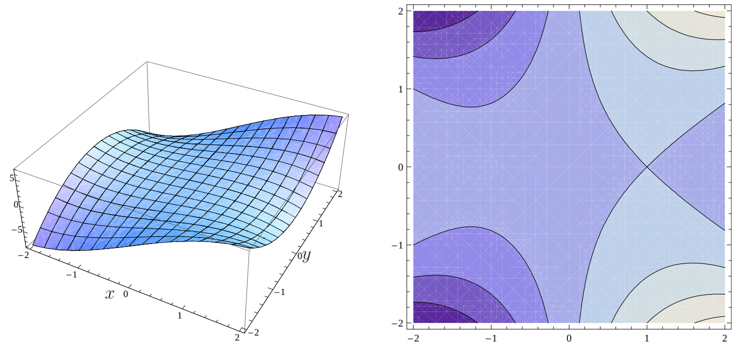
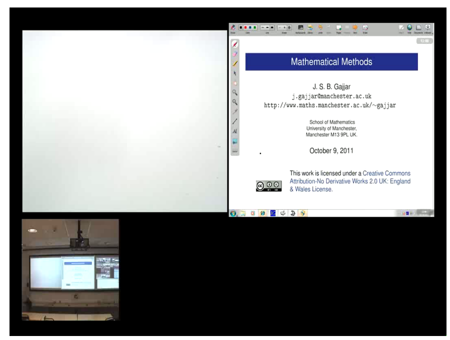


Figure 10.4: Plots of the surface and contour levels for $u(x, y) = Re(z - z^3/3) = u(1, 0)$. The steepest descent path is given by $y = 0$.

Video clip on introduction to the steepest descent method. Click here to open video clip in external player.




This time the curve $1 - x^2 + \frac{y^2}{3} = 0$ is the curve of steepest descent emanating from $x = -1$.

**Example**

Consider $p(z) = \cosh z - \frac{z^2}{2}$. Here

$$p'(z) = \sinh z - z, \quad p''(z) = \cosh z - 1, \quad p'''(z) = \sinh z.$$

Thus $z = 0$ is a saddle point of order 4 and

$$p''''(0) = 1,$$

and the directions of steepest descent are $\theta = (2k+1)\frac{\pi}{4}, \quad k = 0, 1, 2, 3$. A plot of the surface $u(x, y) = \Re(p(z)) - 1$ is shown in Fig. 10.5.

### 10.4.1 Method of steepest descent- key steps

The key steps in using the method of steepest descent are:

- Identify the saddle points, singular points and endpoints likely to contribute to an estimate for the integral.

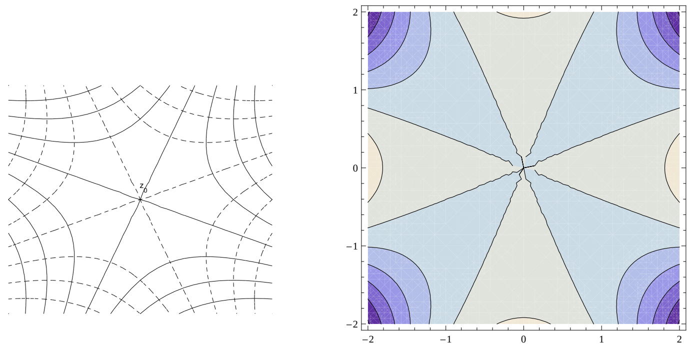



Figure 10.5: A plot of the surface $u(x, y) = \Re(p(z)) - 1$.

- Determine path of steepest descent. It may be the case that there is no continuous path joining the endpoints and one needs two or more steepest descent paths.
- Deform contour making use of Cauchy's theorem.
- Evaluate integral making use of Watson's lemma as appropriate.

The books by (Bender & Orszag, 1999, chap.6), and (Bleistein & Handelsman, 1975, chap. 7) contain many examples which should be studied in detail.

The first example below is taken from Bender and Orszag.

**Example**

Consider

$$I(\lambda) = \int_0^1 e^{i\lambda t} \log t \, dt.$$

Here

$$p(z) = iz = ix - y.$$

There are no saddle points. The steepest descent/ascent paths are given by

$$\Im(p(z)) = x = constant.$$

Since $u(x, y) = \Re(p(z)) = -y$, for $y > 0$ the curves $x = constant$ are steepest descent paths.

Also note that there is no continuous steepest descent path passing through the two endpoints of the integral $x = 0$ and $x = 1$.

This motivates the choice of the contour $C_1 + C_2 + C_3$ that we deform the original path of integration, see Fig. 10.6. We take a branch cut along the negative real axis for $\log(z)$.

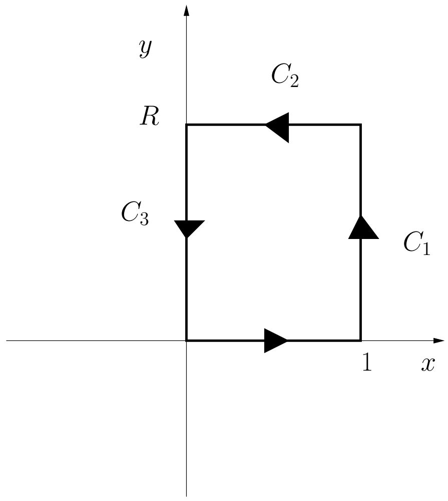



Figure 10.6: Deformed contour for integral $\int_0^1 \log t e^{i\lambda t}\,dt$.

Using Cauchy's theorem we can write

$$\int_0^1 \log(t)e^{i\lambda t}\,dt = -\int_{\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3} e^{i\lambda z}\log z\,dz.$$

For $C_2$ put $z = x + iR$ and then

$$\int_{\mathcal{C}_2} = -\int_0^1 \log(x+iR)e^{i\lambda(x+iR)}\,dx$$

and we see that

$$|\int_{\mathcal{C}_2}| \le e^{-R\lambda}\int_0^1 |\log(x+iR)|\,dx$$

and thus goes to zero as $R \to \infty$.

For $\mathcal{C}_1$ put $z = 1 + iy$ then

$$\int_{\mathcal{C}_1} = i\int_0^R \log(1+iy)e^{i\lambda(1+iy)}\,dy = ie^{i\lambda}\int_0^R \log(1+iy)e^{-y\lambda}\,dy.$$

For $\mathcal{C}_3$ put $z = iy$ then

$$\int_{\mathcal{C}_3} = -i\int_0^R \log(iy)e^{-\lambda y}\,dy.$$

Letting $R \to \infty$ we find that

$$I(\lambda) = i\int_0^\infty e^{-\lambda y}\log(iy)\,dy - ie^{i\lambda}\int_0^\infty \log(1+iy)e^{-\lambda y}\,dy.$$

Now

$$
\begin{aligned}
i\int_0^\infty e^{-\lambda y}\log(iy)\,dy &= i\int_0^\infty e^{-\lambda y}(\log y + \frac{i\pi}{2})\,dy \\
&= -\frac{\pi}{2\lambda} + i\frac{1}{\lambda}\int_0^\infty (\log(y) - \log(\lambda))e^{-y}\,dy, \\
&= -\frac{\pi}{2\lambda} - \frac{i\log\lambda}{\lambda} + \frac{i\gamma}{\lambda}
\end{aligned}
$$

where $\gamma$ is the Euler constant and we have used the result that

$$
-\gamma = \int_0^\infty \log y e^{-y}\,dy.
$$

Next applying Watson's lemma to the integral

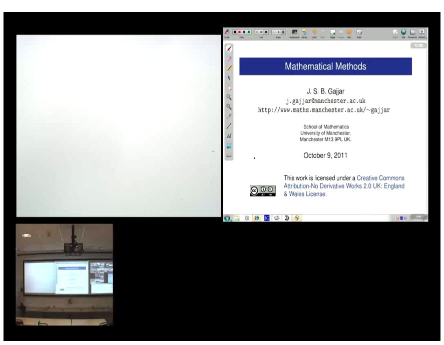
$$
I_1 = ie^{i\lambda}\int_0^\infty \log(1+iy)^{-\lambda y}\,dy
$$

gives

$$
\begin{aligned}
I_1 &\sim ie^{i\lambda}\int_0^\infty \sum_{n=1}^\infty (-1)^{n-1}\frac{(iy)^n}{n}e^{-\lambda y}\,dy, \\
I_1 &\sim e^{i\lambda}\sum_{n=1}^\infty \frac{(-i)^{n+1}\Gamma(n+1)}{n\lambda^{n+1}}.
\end{aligned}
$$

Hence putting it all together

$$
I(\lambda) = \int_0^1 \log(t)e^{i\lambda t}\,dt \sim -\frac{i\log\lambda}{\lambda} - \frac{2i\gamma + \pi}{2\lambda} + e^{i\lambda}\sum_{n=1}^\infty \frac{(-1)^n(i)^{n+1}\Gamma(n)}{\lambda^{n+1}}.
$$




Video clip on use of steepest descent method for $\int_0^1 \log(t)e^{i\lambda t}\,dt$. Click here to open video clip in external player.

**Example**

Consider the integral

$$
I(\lambda) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{\lambda(at - t^{\frac{1}{2}})}}{t}\,dt,
$$

where $a, c$ are positive constants and we take a branch cut along the negative real axis.

We require the behaviour of $I(\lambda)$ as $\lambda \to \infty$. Here

$$p(t) = at - t^{\frac{1}{2}}, \quad p'(t) = a - \frac{1}{2}t^{-\frac{1}{2}}, \quad p''(t) = \frac{1}{4}t^{-\frac{3}{2}}.$$

There is a simple saddle point given by $p'(t) = 0$ ie at $t = t_0 = 1/(4a^2)$.

Note that $p''(t_0) = 2a^3$ and so the steepest descent paths have directions $\theta = \pi/2, 3\pi/2$ emanating from the $t = t_0$.

By Cauchy's theorem we can deform the path of integration to pass through the saddle point as shown in the Fig. 10.7.

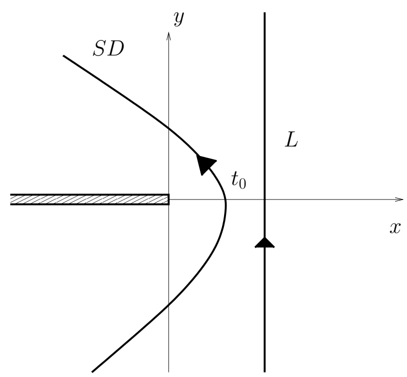



Figure 10.7: Original path $L$ is deformed to the steepest descent path $SD$ passing through the saddle point $t = t_0 = 1/(4a^2)$.

Thus to obtain the leading order estimate for the integral, we can approximate the SD path by a straight line in the direction of steepest descent, ie put $t = \frac{1}{4a^2} + iT$ and note that

$$at - t^{\frac{1}{2}} = -\frac{1}{4a} - \frac{2a^3}{2}T^2 + \ldots,$$

and

$$\frac{1}{t} = 4a^2 + \ldots.$$

Thus the integral becomes

$$I(\lambda) = \frac{1}{2\pi i}\int_{SD} \frac{e^{\lambda(at - t^{\frac{1}{2}})}}{t}\, dt,$$

$$\sim \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{\lambda}{4a}} e^{-a^3 T^2 \lambda} 4a^2 i \, dT.$$

Hence

$$I(\lambda) \sim e^{\frac{-\lambda}{4a}} \frac{2a^2}{\pi} \int_{-\infty}^{\infty} e^{-a^3T^2\lambda}\, dT = 2\sqrt{\frac{a}{\pi\lambda}} e^{-\frac{\lambda}{4a}}.$$

To obtain more terms one needs to work harder.

First note that the steepest descent paths satisfy

$$p(t) = p(t_0) \qquad \Longrightarrow \; at - t^{\frac{1}{2}} = -\frac{1}{4a},$$

and so the imaginary part of $p(t)$ is zero along these paths. Hence if we put

$$-W = at - t^{\frac{1}{2}} + \frac{1}{4a}$$

where $W$ is real and positive, (because we have a SD path) we find that

$$at - t^{\frac{1}{2}} + \frac{1}{4a} + W = 0$$

giving

$$t = \left[\frac{1 \pm [1 - 4a(\frac{1}{4a} + W)]^{\frac{1}{2}}}{2a}\right]^2 = \left(\frac{1 \pm 2a^{\frac{1}{2}} W^{\frac{1}{2}}}{2a}\right)^2.$$

The $\pm$ signs here indicate the two steepest directions emanating from $t = t_0$, see Fig. 10.8 and by expanding for small $W$ we see that $+$ sign corresponds to the $\pi/2$ direction and the $-$ sign the $3\pi/2$ direction. We can write

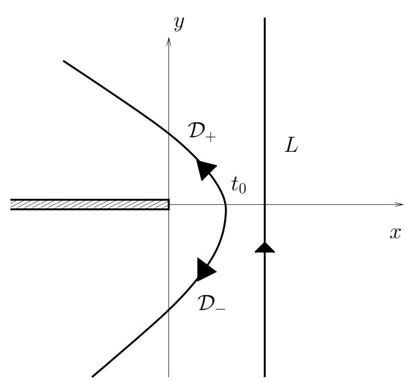



Figure 10.8: The steepest descent paths $\mathcal{D}_+$ and $\mathcal{D}_-$ emanating from $t = 1/4a^2$.

$$I(\lambda) = \frac{1}{2\pi i}\left(\int_{\mathcal{D}_+} - \int_{\mathcal{D}_-}\right) \frac{e^{\lambda(-\frac{1}{4a} - W)}}{t} \frac{dt}{dW}\, dW.$$

Now

$$\frac{dt}{dW} = \sigma(1 + \sigma 2i\sqrt{aW}) \frac{i}{2a^{\frac{3}{2}}} W^{-\frac{1}{2}},$$

where $\sigma = 1$ for $\mathcal{D}_+$ and $\sigma = -1$ for $\mathcal{D}_-$.

Also

$$\frac{1}{t} = 4a^2(1 + 2i\sigma\sqrt{aW})^{-2}.$$

To use Watson's lemma we need the expansion of $\frac{1}{t}\frac{dt}{dW}$ as $W \to 0+$. Using the above expression gives

$$\frac{1}{t}\frac{dt}{dW} = \frac{4a^2 i}{2a^{\frac{3}{2}}}(\sigma W^{-\frac{1}{2}} + 2ia^{\frac{1}{2}})(1 + 2i\sigma a^{\frac{1}{2}} W^{\frac{1}{2}})^{-2},$$

$$\sim 2ia^{\frac{1}{2}}[\sigma W^{-\frac{1}{2}} - 2ia^{\frac{1}{2}} - 4a\sigma W^{\frac{1}{2}} + 8ia^{\frac{3}{2}} W + 16a^2\sigma W^{\frac{3}{2}} + \ldots].$$

Here we have used the fact that $\sigma^2 = 1$. Thus

$$I(\lambda) \sim \frac{1}{2\pi i} 2a^{\frac{1}{2}} i e^{-\frac{\lambda}{4}} \Big[$$

$$\int_0^\infty e^{-\lambda W}(W^{-\frac{1}{2}} - 2ia^{\frac{1}{2}} - 4aW^{\frac{1}{2}} + 8ia^{\frac{3}{2}} W + 16a^2 W^{\frac{3}{2}} + \ldots)\, dW$$

$$+ \int_0^\infty e^{-\lambda W}(W^{-\frac{1}{2}} + 2ia^{\frac{1}{2}} - 4aW^{\frac{1}{2}} - 8ia^{\frac{3}{2}} W + 16a^2 W^{\frac{3}{2}} + \ldots)\, dW \Big],$$

Hence

$$I(\lambda) \sim 2a^{\frac{1}{2}} \frac{e^{-\frac{\lambda}{4}}}{\pi} \left[ \frac{\Gamma(\frac{1}{2})}{\lambda^{\frac{1}{2}}} - 4a \frac{\Gamma(\frac{3}{2})}{\lambda^{\frac{3}{2}}} + 16a^2 \frac{\Gamma(\frac{5}{2})}{\lambda^{\frac{5}{2}}} + \ldots \right],$$

$$\sim 2\sqrt{\frac{a}{\pi\lambda}} e^{-\frac{\lambda}{4a}} \left[ 1 - \frac{2a}{\lambda} + \frac{12a^2}{\lambda^2} + \ldots \right],$$

as $\lambda \to \infty$.

**Example** Consider

$$I(\lambda) = \int_{-\infty}^\infty \frac{e^{i\lambda(t+t^3/3)}}{2t^2+1}\, dt. \tag{10.4.1}$$

Now

$$p(t) = i(t + t^3/3), \quad p'(t) = i(1 + t^2), \quad p''(t) = 2it.$$

Hence we have simple saddle points at $t = \pm i$ and

$$p(\pm i) = \mp 2/3, \quad p''(\pm i) = \mp 2.$$

Thus the directions of steepest descent from $t = i$ are $\theta = 0, \pi$ and the directions of steepest descent from $t = -i$ are $\theta = \pi/2, 3\pi/2$.

Next note that if we set $t = Re^{i\phi}$ then for large $R$ and $\lambda > 0$,

$$e^{i\lambda(t+t^3/3)} \sim O(e^{-\frac{\lambda R^3}{3}\sin(3\phi)})$$

and this decays provided the $\sin(3\phi)$ term is positive. So if we displace the contour in the upper-half plane the contour should begin and end in the sectors

$$2\pi/3 < \phi < \pi, \quad \text{and} \quad 0 < \phi < \pi/3.$$

The steepest descent/ascent paths satisfy

$$\Im(p(t)) = \Im(p(\pm i)) = 0$$

giving with $t = x + iy$,

$$\Im[i(x + iy + \frac{1}{3}(x^3 + 3ix^2y - 3xy^2 - iy^3))] = x(1 + \frac{x^2}{3} - y^2).$$

So the steepest descent paths emanating from $t = i$ are $1 + \frac{x^2}{3} - y^2 = 0$ and from $t = -i$ are $x = 0$.

Note also that if $y^2 = (\frac{x^2}{3} - 1)$ then for large $x$ we have $y \sim \pm\frac{1}{\sqrt{3}}x$. A sketch of the path is given in fig. 10.9. The above analysis suggests that we can deform

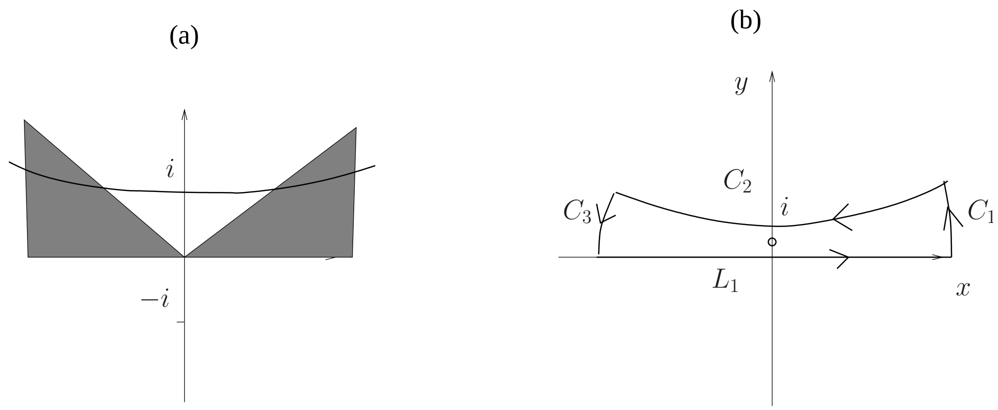



Figure 10.9: (a) Steepest descent path through $t = i$ (b) Contours for application of Cauchy's theorem.

the original contour in (10.4.1) to the upper-half plane on to the steepest decsent path through $t = i$, see fig. 10.9. Applying Cauchy"s theorem we obtain

$$\int_{L_1+C_1+C_2+C_3} \frac{e^{i\lambda(t+t^3/3)}}{2t^2+1}\,dt = 2\pi i Res[t = i/\sqrt{2}],$$

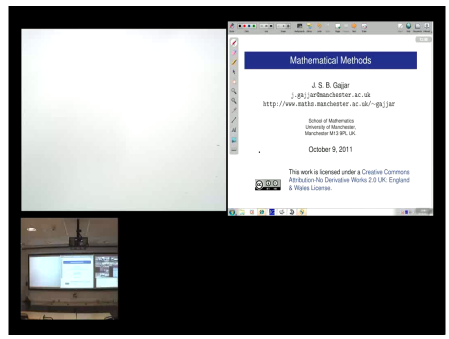
since the integrand has a simple pole at $t = i/\sqrt{2}$. The integrals along $C_1, C_3$ goes to zero for large $R$ and so

$$\int_{L_1} \frac{e^{i\lambda(t+t^3/3)}}{2t^2+1}\,dt = \int_{-C_2} \frac{e^{i\lambda(t+t^3/3)}}{2t^2+1}\,dt + 2\pi i[\frac{e^{i\lambda(\frac{i}{\sqrt{2}} - \frac{i}{6\sqrt{2}})}}{\frac{4i}{\sqrt{2}}}].$$

For the integral along the steepest descent path we can put (for the leading order contribution only) $t = i + T$ to obtain

$$I(\lambda) \sim \int_{-\infty}^{\infty} \frac{e^{\lambda(-\frac{2}{3}-T^2)}}{(-2+1)}\,dT + \frac{\sqrt{2}}{2}\pi e^{-\frac{5\lambda}{6\sqrt{2}}},$$

$$\sim -\sqrt{\frac{\pi}{\lambda}}e^{-2\lambda/3} + \frac{\sqrt{2}}{2}\pi e^{-\frac{5\lambda}{6\sqrt{2}}}.$$




Video clip on use of the steepest descent method for $\int_{-\infty}^{\infty} \frac{e^{i\lambda(t+t^3/3)}}{2t^2+1}\,dt$. Click here to open video clip in external player.

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# Chapter 11

# Introduction to Wiener-Hopf method

## 11.1 Conformal mapping

There are many problems in physical applied mathematics, eg, fluid mechanics, electrostatics, elasticity theory, heat conduction etc, which require the solution of Laplace's equation

$$\nabla^2\phi = 0,$$

in some domain $\mathcal{D}$ with suitable boundary conditions.

Note that if $P(z) = \phi(x, y) + i\psi(\,y)$ is an analytic function of the complex variable $z = x + iy$ then from the Cauchy Riemann equations

$$\phi_x = \psi_y, \quad \phi_y = -\psi_x$$

we obtain

$$\nabla^2\phi = \psi_{xy} - \psi_{yx} = 0$$

and similarly

$$\nabla^2\psi = 0.$$

Thus the problem of solving Laplace's equation can be reduced to finding an analytic function which satisfies certain boundary conditions.

In general if the domain $\mathcal{D}$ is complicated then this might have to done numerically. However by using a suitable mapping function $w = f(z)$ the problem can be simplified if the domain can be transformed to the upper-half plane or the unit disk say. This is where conformal mapping is extremely useful

**Definition***A mapping is conformal if it preserves the angle between two differentiable arcs. A mapping defined by analytic functions is conformal.*

**Proof** Let $C_1, C_2$ be two differentiable arcs which meet at the point $P$ with $z = z_0$ say, and the point $A$ with location $z_1$ is a point along $C_1$ and the point $B$ with location $z_2$ a point along $C_2$, see figure 11.1. We will take the point $A$ and $B$ to be the same distance $r$ from $P$ (this is ok if $r$ is small). Then

$$z_1 - z_0 = re^{i\theta_1}, \quad z_2 - z_0 = re^{i\theta_2},$$

and

$$\frac{z_2 - z_0}{z_1 - z_0} = e^{i(\theta_2 - \theta_1)}.$$

Thus the angle between $PA$ and $PB$ is given by

$$\theta_2 - \theta_1 = \arg \frac{z_2 - z_0}{z_1 - z_0}.$$

The angle $\alpha$ between the two arcs at $P$ is given by the limit

$$\alpha = \lim_{r \to 0} \frac{z_2 - z_0}{z_1 - z_0}.$$

Suppose that $w_0, w_1, w_2$ are the images of the curves when subject to the mapping $w = f(z)$ and

$$w_2 = f(z_2), \quad w_1 = f(z_1), \quad w_0 = f(z_0).$$

The points $P, A, B$ map to $P', A', B'$ respectively. The angle $\beta$ between $P'A'$ and $P'B'$ is given by

$$\begin{aligned} \beta = \arg \frac{w_2 - w_0}{w_1 - w_0} &= \arg \frac{f_2 - f_0}{f_1 - f_0}, \\ &= \arg \left\{ \frac{\frac{f_2 - f_0}{z_2 - z_0}}{\frac{f_1 - f_0}{z_1 - z_0}} . \frac{z_2 - z_0}{z_1 - z_0} \right\}. \end{aligned}$$

Taking the limit as $r \to 0$ and noting that

$$\lim_{r \to 0} \frac{f_2 - f_0}{z_1 - z_0} = \frac{df(z_0)}{dz} = \lim_{r \to 0} \frac{f_1 - f_0}{z_1 - z_0},$$

we find that

$$\beta = \alpha$$

provided $f'(z_0) \neq 0$.

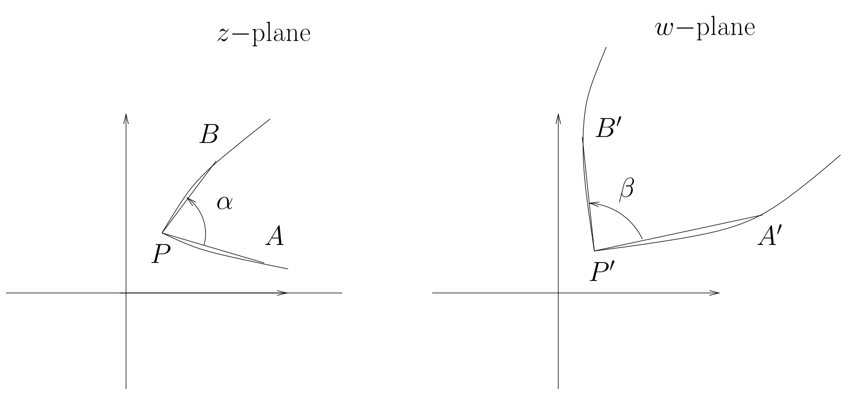


Figure 11.1

### 11.1.1 Conformal mapping - critical points

Suppose next that

$$f'(z_0) = f''(0) = \dots f^{(n-1)}(z_0) = 0, \quad f^{(n)}(z_0) \neq 0,$$

and that

$$f(z) = a_m(z - z_0)^m + \dots \quad \text{as} \quad z \to z_0.$$

Then

$$\beta = \lim_{r\to 0}\{\arg \frac{f(z_2) - f(z_0)}{f(z_1) - f(z_0)}\} = \lim_{r\to 0}\left\{\arg\left(\frac{z_2 - z_0}{z_1 - z_0}\right)^n\right\} = n\alpha.$$

Thus for critical points angles are magnified $n$ times in the mapped plane.

**Example** Consider $w = z^2$. Here if $w = u + iv, \quad z = z + iy$

$$u = x^2 - y^2, \quad v = 2xy.$$

Thus $OA$ where $y = x$ maps to $O'A'$ where $u = 0, v = 2x^2$, see figure 11.2. Similarly $OB$ where $y = -x$ maps to $O'B'$ with $u = 0, v = -2x^2$. Finally on $AB$ we have $y = 1$, and so $u + 1 = v^2/4$. Note that $w'(z) = 2z$ is zero when $z = 0$.

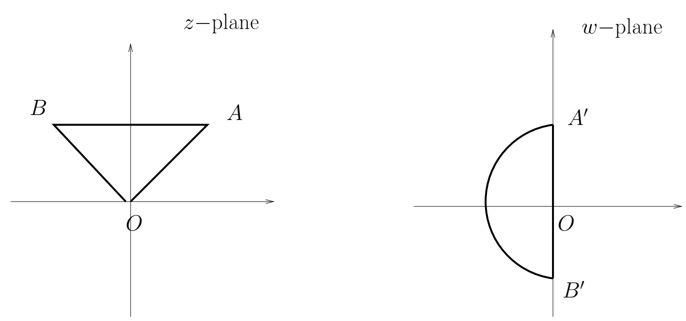


Figure 11.2: The triangle $OAB$ mapped to the $w-$plane with the mapping $w = z^2$.

Maps preserve the connectivity of a domain. Thus the map $w = f(z)$ of a simply connected domain maps to a simply connected domain in the $w$ plane. [Proof not given].

### 11.1.2 Schwartz-Christoffel formula

**Riemann Mapping Theorem** *Let $\mathcal{D}$ be a simply connected domain in the $z$ plane and which is not the entire $z-$plane. There there exists a* univalent *function $f(z)$ such that $w = f(z)$ maps $\mathcal{D}$ onto the disk $|w| < 1$. [A function $f(z)$ is univalent means that $f(z)$ takes no more than one value in $\mathcal{D}$.]*

**Proof** For a proof of the Riemann mapping theorem see the book by **?**. [The proof does not give a recipe for calculating such a function].

One important point which arises when proving this theorem is that for any univalent mappings, only three points can be prescribed arbitrarily on the boundaries of two domain. This formula allows for the mapping of the interior of a polygon onto the upper-half plane. [The exterior of the polygon could also be mapped in the same way.]

Consider first the points $AOC$ mapped to $A'O'C'$ in the $w$ plane, see figure 11.3

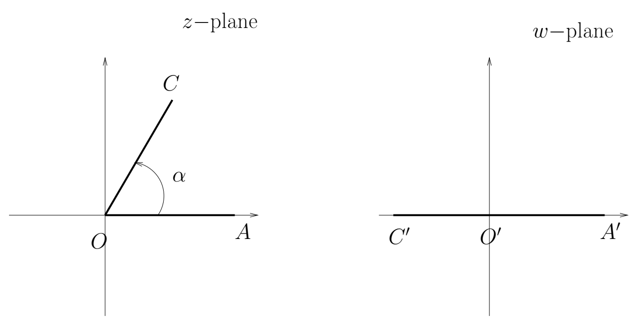


Figure 11.3: $OAC$ mapped to $O'A'C'$ with mapping $w = z^{\pi/\alpha}$,

Write $w = z^p$ and then

$$\arg(w) = p\arg(z).$$

On $OA$ we have $\arg(z) = 0$ and therefore on $O'A', \arg(w) = 0$.

On $OC$, $\quad \arg(z) = \alpha$. On $O'C'$ we have $\arg(w) = p\alpha$. So choose $p$ such that $p\alpha = \pi$. Thus

$$w = z^{\frac{\pi}{\alpha}}$$

straightens out the lines $ABC$ in the $w$ plane.

**Schwartz-Christoffel Transformation** *Let $\mathcal{C}$ be the piecewise linear boundary of a polygon in the $z$ plane having vertices $z_1, z_2, \ldots, z_n$ and let the interior angles of the successive vertices be $\alpha_1\pi, \alpha_2\pi, \ldots \alpha_n\pi$, in anti-clockwise order.*

*The mapping*

$$\frac{dz}{dw} = \gamma\Pi_{k=1}^{n-1}(w - a_k)^{\alpha_k - 1}$$

*maps the polygon in the $z$ plane into a straight line in the $w$ plane.*

*The vertices of the polygon $A_1, A_2, \ldots, A_n$ are mapped to the points $a_1, a_2, \ldots a_n$ on the real axis. Note that $\sum_{i=1}^{n} \alpha_i = n - 2$, and we take branch cuts in the lower half-plane.*

**Proof** A formal proof is not given here and may be found in **?**. We will just make a few observations.

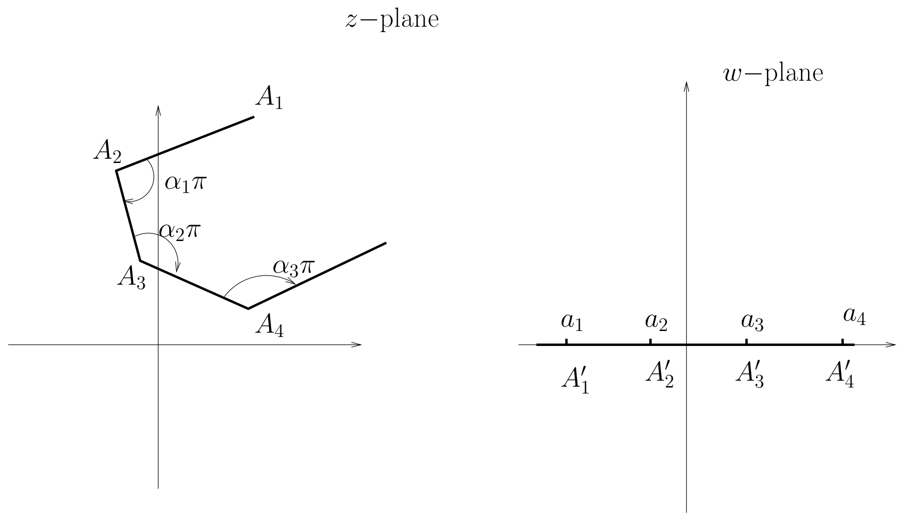


Figure 11.4: Vertices of a polygon$A_1, A_2, ..., A_n$ are mapped to points locations $a_1, a_2, ..., a_n$ on the real axis.

Consider first the line segment and consider the section between $a_{k-1}$ and $a_k$, figure 11.4. Now

$$\arg(\frac{dz}{dw}) = \arg(\gamma) + (\alpha_1 - 1)\arg(w - a_1) + \cdots + (\alpha_n - 1)\arg(w - a_n).$$

As $w$ moves from $a_{k-1}$ towards $a_k$ $\arg(w)$ the change in the argument of $\arg(dz/dw)$ in crossing the point $a_k$ is given by

$$[\arg(\frac{dz}{dw})]_{z_k-}^{z_k+} = (\alpha_k - 1)[\arg(w - a_k)]_{a_k-}^{a_k+} = (\alpha_k - 1)[0 - \pi] = \pi(1 - \alpha_k).$$

This is the same as the turning angle as we move from the segment of the polygon to the left of the vertex at $A_k$ crossing $A_k$.

The SC formula holds also when one of the points maps to a point at infinity. Put

$$w = a_n - \frac{1}{\zeta},$$

which transforms the point $w = a_n$ to $\zeta = \infty$. Now

$$\frac{dz}{dw} = \zeta^2\frac{dz}{d\zeta} = \gamma(a_n - a_1 - \frac{1}{\zeta})^{\alpha_1 - 1}\dots(a_{n-1} - a_n - \frac{1}{\zeta})^{\alpha_{n-1}-1}(-\frac{1}{\zeta})^{\alpha_n - 1}.$$

Thus

$$\frac{dz}{d\zeta} = \tilde{\gamma}(\zeta - \tilde{a}_1)^{\alpha_1 - 1}\dots(\zeta - \tilde{a}_{n-1})^{\alpha_{n-1}-1},$$

where $\tilde{\gamma}$ is a complex constant, $\tilde{a}_k = 1/(a_n - a_k)$ and we have used that $\sum_{k=1}^{n}(\alpha_k - 1) = -2$. This is essentially the same formula as before but with the point $a_n$ which maps to $\zeta = \infty$ removed.

**Example** Consider mapping the semi-infinite strip $-1 \leq \Re(z) \leq 1, \quad \Im(z) \geq 0$ into the upper half-plane.

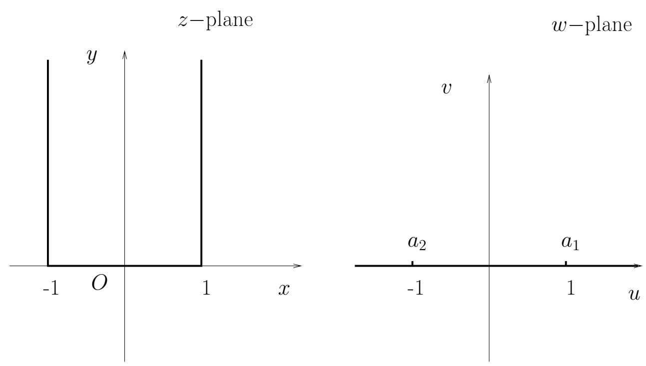


Figure 11.5: Domain in $z-$ and $w-$planes.

The point $A$ with $z = 1$ maps to $A', \quad w = 1$ and $B$ with $a_1 = -1$ to $B'$ with $a_2 = -1$. Here $\alpha_1 = \alpha_2 = 1/2$. Thus

$$\frac{dz}{dw} = \gamma(w+1)^{\frac{1}{2}-1}(w-1)^{\frac{1}{2}-1} = \hat{\gamma}(1-w^2)^{-\frac{1}{2}}.$$

Hence

$$z = \int^w \hat{\gamma}\frac{1}{(1-w^2)^{\frac{1}{2}}}\,dw = \hat{\gamma}\sin^{-1} w + c.$$

Putting in the values $z = 1, w = 1$, and $z = -1, w = -1$ gives

$$1 + \hat{\gamma}\frac{\pi}{2} + c, \quad -1 = -\hat{\gamma}\frac{\pi}{2} + c.$$

Hence

$$c = 0, \quad \hat{\gamma} = \frac{2}{\pi}$$

and

$$z = \frac{2}{\pi}\sin^{-1} w$$

or

$$w = \sin(\frac{\pi z}{2}).$$

**Example** In a heat conduction problem we need to solve

$$\nabla^2 \Phi = 0$$

with the boundary conditions

$$\Phi = T_0 \quad x = -1, \quad \Phi = 2T_0, \quad x = 1,$$

$$\Phi = 0, \quad y = 0, \quad -1 < x < 1.$$

If we use the mapping $w = \sin(\frac{\pi z}{2})$ then in the transformed domain we require an analytic function $\Phi$, say, which takes the value

$$\Phi = 2T_0, \quad \Im(w) = 0, \Re(w) > 1,$$

$$\Phi = 0, \quad \Im(w) = 0, \quad -1 < \Re(w) < 1,$$

$$\Phi = T_0, \quad \Im(w) = 0, \quad \Re(w) < -1.$$

In view of the discontinuities in the boundary conditions, let us try

$$\Psi + i\Phi = A\log(w+1) + B\log(w-1) + iC,$$

and $w = -1 + r_1 e^{i\theta_1}$, and $w = 1 + r_2 e^{i\theta_2}$.

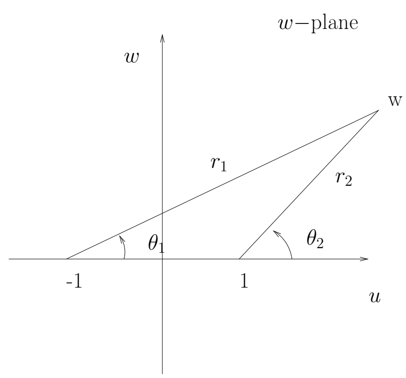


Figure 11.6: $w = -1 + r_1 e^{i\theta_1}$ and $w = 1 + r_2 e^{i\theta_2}$

Then

$$\Psi + i\Phi = A(\log(r_1) + i\theta_1) + B(\log(r_2) + i\theta_2) + iC.$$

Taking $A, B, C$ to be real, we obtain

$$\Phi = A\theta_1 + B\theta_2 + C.$$

Applying the conditions

$$\theta_1 = 0, \theta_2 = 0, \quad \Phi = 2T_0 \implies C = 2T_0.$$

Next

$$\theta_1 = 0, \theta_2 = \pi, \quad \Phi = 0, \implies C + \pi B = 0.$$

Finally

$$\theta_1 = \pi, \theta_2 = \pi, \Phi = T_0 \implies, T_0 = A\pi + B\pi + C.$$

Hence

$$\Phi = \frac{T_0}{\pi}\theta_1 + -\frac{2T_0}{\pi}\theta_2 + 2T_0.$$

Now let $w = u + iv$ and so

$$\theta_1 = \tan^{-1}\frac{v}{w+1}, \quad \theta_2 = \tan^{-1}\frac{v}{u-1}.$$

But $w = \sin(\frac{\pi z}{2})$ and so with $z = x + iy$ we obtain

$$u = \sin(\frac{\pi x}{2})\cosh(\frac{\pi y}{2}), \quad v = \cos(\frac{\pi x}{2})\sinh(\frac{\pi y}{2}),$$

giving

$$\begin{aligned}\Phi(x,y) = 2T_0 + \frac{T_0}{\pi}\tan^{-1}&\left[\frac{\cos(\frac{\pi x}{2})\sinh(\frac{\pi y}{2})}{\sin(\frac{\pi x}{2})\cosh(\frac{\pi y}{2})+1}\right]\\ -\frac{2T_0}{\pi}\tan^{-1}&\left[\frac{\cos(\frac{\pi x}{2})\sinh(\frac{\pi y}{2})}{\sin(\frac{\pi x}{2})\cosh(\frac{\pi y}{2})-1}\right].\end{aligned}$$

### 11.1.3 Bilinear maps- Mobius transform

Consider the mapping function

$$w = f(z) = \frac{az+b}{cz+d}, \quad a,d,c,d \in \mathcal{C}, \quad ad - bc \neq 0.$$

This is somtimes referred to as a bilinear map, or a Mobius transform, or a fractional linear transformation.

**Special cases**

- $c = 0, d = 1$. Linear map (points are translated and strectched.)
- $a = d = 0, c = b = 1$. Thus $f(z) = 1/z$ - reciprocal map.
- If $ad = bc$ for example $f(z) = (6z+4)/(3z+2) = 2$ the map is not conformal.

Every Mobius map is conformal. Note that

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0.$$

The inverse map is given by

$$f^{-1}(w) = \frac{dw-b}{-cw+a}.$$

**Theorem** *Suppose $f(z)$ is a bilinear map. Then the image of a straight line is either a straight line or a circle. The image of a circle is either a straight line or circle.*

**Proof** We can write the map $f(z)$ as a product of four maps

$$f = f_4 f_3 f_2 f_1$$

where

$$f_1 = z + \frac{d}{c}, \quad f_2 = \frac{1}{z}, \quad f_3 = \frac{(bc - az)z}{c^2}, \quad f_4 = z + \frac{a}{c}.$$

If $c = 0$, then $f = \frac{a}{d}z + \frac{b}{d}$. Note that $f_1, f_3, f_4$ map lines to lines and circles to circles. Consider $f_2 = 1/z$. If $z = x + iy \neq 0$ and $w = f_2(z) = u + iv$ then

$$u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2}.$$

Now the equation of a straight line or circle in the $z-$plane must satisfy

$$Ax + By + C(x^2 + y^2) = D$$

for constants $A, B, C$ not all zero. In terms of $u, v$ this is equivalent to

$$Au - Bv - D(u^2 + v^2) = -C,$$

which is also a line or a circle.

**Theorema** *Given any two sets of distinct points* $z_1, z_2, z_3$ *and* $w_1, w_2, w_3$ *with* $(z_1 \neq z_2, \quad z_1 \neq z_3, \quad z_2 \neq z_3)$ *and* $(w_1 \neq w_2, \quad w_1 \neq w_3, \quad w_2 \neq w_3)$ *then there is a unique bilinear map taking* $z_i \to w_i, \quad i = 1, 2, 3$ *given by*

$$\frac{w - w_1}{w - w_2}\frac{w_3 - w_2}{w_3 - w_1} = \frac{z - z_1}{z - z_2}\frac{z_3 - z_2}{z_3 - z_1}.$$

**Proof** For a proof see examples 7.

We can use this to map a circle to a straight line by its effect by choosing where three points go.

A univalent, conformal map of the annulus $\mu < |z| < 1$ onto $\alpha < |w| < 1$ is possible if and only if $\mu = \alpha$. In fact from this results it follows that it is possible to conformally map any doubly connected domain in the $z-$plane onto $\alpha < |w| < 1$. $\alpha$ is known as the Riemann modulus. [For a proof, see **?**].

For further information on conformal mappings one is referred to the book by **?**, *Conformal Mapping.* For computational aspects see also a recent book by **?** *Schwartz-Christoffel Mapping.*

A use collection of well known maps is contained in **?**, *Dictionary of Conformal Representations.*

In the last few years there have significant developments with regard to conformal mappings for multiply connected domains with work by Elcrat and his group, and also Darren Crowdy at Imperial. It is beyond the scope of this course to discuss these new developments, but those interested should look at some of their papers starting with **?**.

## 11.2 Riemann-Hilbert problems and the Wiener-Hopf method

In this last lecture we will discuss the briefly solution of Riemann-Hilbert problems, and also the Wiener-Hopf method which is one particular example of a Riemann-Hilbert problem.

In the **scalar** Riemann-Hilbert problem one is faced with the task of finding two analytic functions $\Phi^+(z), \Phi^-(z)$ defined inside and outside a closed contour $\mathcal{C}$ of the complex plane such that

$$\Phi^+(z) - G(t)\Phi^-(z) = f(z), \quad \text{for} \quad z \quad \text{on} \quad \mathcal{C}. \tag{11.2.1}$$

A standard reference text for problems of this type is the book *Singular Integral Equations* by Muskhelishvili (1953). A short readable account, including recent applications is contained in (Ablowitz & S., 2003, chap. 7). For Wiener-Hopf problems, the book *Wiener-Hopf technique* by Noble (1958) is highly recommened.

Riemann-Hilbert problems arise in numerous applications including wave scattering problems, solution of integral equations, inverse-scattering theory and nonlinear waves, elasticity theory etc. In his thesis Riemann (1851) posed the following problem, find $U(z)$ satisfying

$$\bar{g}(z)U^+(z) + g(z)U^-(z) = f(z) \quad \text{for} \quad z \quad \text{on} \quad \mathcal{D},$$

where $g(z), f(z)$ are analytic on $\mathcal{C}$ and the overbar denotes complex conjugate, and $\mathcal{D}$ is the boundary of a circle. Hilbert (1904) posed the RH problem described earlier. The solution that we will discuss below is one due to Plemelj (1908) and Gakhov (1938). Carleman (1922) had also solved a Wiener-Hopf problem with the same ideas.

**Example** Consider waves on a string with a density discontinuity. The governing equations are

$$\frac{d^2U}{dx^2} + k_1^2 U = 0, \quad x < 0,$$
$$\frac{d^2U}{dx^2} + k_2^2 U = 0, \quad x > 0,$$

with boundary conditions

$$U \sim Ae^{-ik_1x} \quad x \to -\infty,$$

$$U \sim Be^{ik_2x} \quad x \to \infty.$$

We will take $k_1 = K_1 + i\epsilon, k_2 = K_2 + i\epsilon$ where $K_1, K_2, \epsilon$ are positive, and eventually let $\epsilon \to 0$.

At the joint $x = 0$ we require

$$U(x = 0+) - U(x = 0-) = 1, \quad \frac{dU}{dx}(0+) - \frac{dU}{dx}(0-) = ik_1.$$

Let us define the functions

$$U^+(x) = \begin{cases} U(x), & x > 0 \\ 0, & x < 0 \end{cases}, \qquad U^-(x) = \begin{cases} 0, & x > 0 \\ U(x), & x < 0 \end{cases}.$$

Next define Fourier Transforms

$$U^{*+}(s) = \int_0^\infty U^+(x) e^{isx}\, dx, \quad U^{*-}(s) = \int_{-\infty}^0 U^-(x) e^{isx}\, dx.$$

Note that $U^{*+}(s)$ is analytic for $\Im(s) > -\epsilon$ and $U^{*-}(s)$ is analytic for $\Im(s) < \epsilon$. Next the Fourier transform of the differential equations shows that

$$\int_{-\infty}^0 \left( \frac{d^2U^-}{dx^2} + k_1^2 U^- \right) e^{isx}\, dx = 0.$$

$$(k_1^2 - s^2) U^{*-}(s) + \frac{dU}{dx}(0-) - isU(0-) = 0, \quad \Im(s) > -\epsilon. \tag{11.2.2}$$

Similarly,

$$(k_2^2 - s^2) U^{*+}(s) - \frac{dU}{dx}(0+) + isU(0+) = 0, \quad \Im(s) < \epsilon. \tag{11.2.3}$$

Note that both equations hold in a strip of analyticity $-\epsilon < \Im(s) < \epsilon$. Adding both equations and using the given conditions shows that

$$(k_1^2 - s^2) U^{*-} + (k_2^2 - s^2) U^{*+} = -i(s - k_1),$$

or

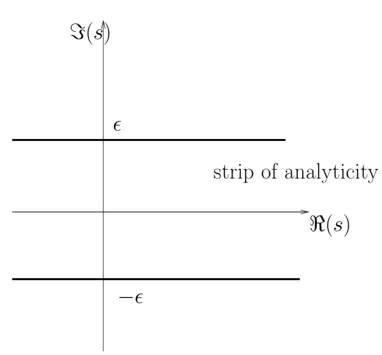


Figure 11.7: Strip of analyticity for the Wiener-Hopf equation.

$$\left(\frac{s^2-k_2^2}{s^2-k_1^2}\right)U^{*+}(s)+U^{*-}(s)=\frac{i}{s+k_1}. \tag{11.2.4}$$

The equation (11.2.4) is referred to as the Wiener-Hopf equation. It holds in the strip of analyticity $-\epsilon < \Im(s) < \epsilon$, see figure 11.7

For the Riemann-Hilbert problem, an equation like (11.2.4) would hold on a closed or open contour $\mathcal{L}$ say, separating the regions where the $+$ and $-$ functions are analytic.

We will first discuss the solution of a more general Wiener Hopf equation. Consider the equation

$$G(s)U^{*+}(s)+U^{*-}(s)=P(s), \quad \alpha < \Im(s) < \beta \tag{11.2.5}$$

where the functions $G(s), P(s)$ are known and have no zeros or poles in the strip $\alpha < \Im(s) < \beta$ and have algebraic behaviour at infinity.

We write

$$G(s)=G_+(s)G_-(s)$$

where $G_-$ is analytic and has no zeros for $\Im(s) < \beta$ and $G_+$ is analytic and has no zeros for $\Im(s) > \alpha$. We will also assume that $G_+, G_-$ and $(1/G_+), (1/G_-)$ are algebraic and bounded by a polynomial at infinity.

Note that the decomposition into the $\pm$ functions is usually one of the hardest steps. There are various techniques which one can use to attempt this. In the example below this is done by inspection, but in general it is not as easy as this. Hence from (11.2.5)

$$G_+(s)U^{*+}(s)+\frac{U^{*-}(s)}{G_-(s)}=\frac{P(s)}{G_-(s)} \quad \text{in} \quad \alpha < \Im(s) < \beta. \tag{11.2.6}$$

The right-hand side of (11.2.6) is known.

For the next step write

$$\frac{P(s)}{G_-(s)}=Q_+(s)+Q_-(s)$$

where $Q_+$ is analytic and of algebraic growth for $\Im(s) > \alpha$ and where $Q_-$ is analytic and of algebraic growth for $\Im(s) < \beta$.

Thus from (11.2.6)

$$G_+(s)U^{*+}(s)-Q_+(s)=-\frac{U^{*-}(s)}{G_-(s)}+Q_-(s), \tag{11.2.7}$$

and (11.2.7) holds in the strip $\alpha < \Im(s) < \beta$.

$$G_+(s)U^{*+}(s)-Q_+(s)=-\frac{U^{*-}(s)}{G_-(s)}+Q_-(s). \tag{11.2.8}$$

The left-hand and right-hand sides of (11.2.8) continue each other analytically to define a function, say $E(s)$, which is analytic in the whole complex plane. We can also use the assumed conditions to infer the behaviour for large $|s|$.

Suppose for example that the left-hand side of (11.2.8) is $O(|s|^{\gamma_1})$ and the right-hand-side $(O(|s|^{\gamma_2})$ as $|s| \to \infty$.

Let $n = \max([\gamma_1], [\gamma_2])$ where [] denotes the integer part. Then by an extended version of Liouville's theorem $E(s)$ must be a polynomial of degree $n$ at most.

This gives us the solution

$$G_+(s)U^{*+}(s) - Q_+(s) = E(s), \quad \frac{U^{*-}(s)}{G_-(s)} - Q_+(s) = E(s),$$

from which $U^{*+}(s)$ and $U^{*-}(s)$ can be found.

**Example** Let us return to our earlier example. The equation we have to solve is

$$\left(\frac{s^2 - k_2^2}{s^2 - k_1^2}\right) U^{*+}(s) + U^{*-}(s) = \frac{i}{s + k_1}. \tag{11.2.9}$$

Here

$$G(s) = \frac{s^2 - k_2^2}{s^2 - k_1^2}, \quad -\epsilon < \Im(s) < \epsilon.$$

Note that

$$G(s) = \frac{(s + k_2)}{(s + k_1)} \frac{(s - k_2)}{(s - k_1)}$$

and we can take

$$G_+(s) = \frac{(s + k_2)}{(s + k_1)}, \quad G_-(s) = \frac{(s - k_2)}{(s - k_1)}.$$

Here we could obtain the $G_\pm$ functions by inspection, but this is not typical, and is usually one of the hardest steps. Thus

$$\frac{(s + k_2)}{(s + k_1)} U^{*+}(s) + \frac{(s - k_1)}{(s - k_2)} U^{*-}(s) = \frac{i(s - k_1)}{(s + k_1)(s - k_2)}.$$

$$= \frac{i(k_2 - k_1)}{(k_2 + k_1)(s - k_2)} + \frac{2ik_1}{(k_2 + k_1)(s + k_1)}.$$

Hence

$$\frac{(s + k_2)}{(s + k_1)} U^{*+}(s) - \frac{2ik_1}{(k_2 + k_1)(s + k_1)} =$$

$$\frac{i(k_2 - k_1)}{(k_2 + k_1)(s - k_2)} - \frac{(s - k_1)}{(s - k_2)} U^{*-}(s). \tag{11.2.10}$$

We have accomplised the decomposition. The left-hand side of (11.2.10) is analytic for $\Im(s) > -\epsilon$ and the right-hand side is analytic for $\Im(s) < \epsilon$. Both sides are analytic in the strip $-\epsilon < \Im(s) < \epsilon$. By analytic continuation both must be equal to an analytic function $E(s)$. We can now use the conditions as $|s| \to \infty$ to determine $E(s)$. Note that since $U^{\pm}$ tends to some constant values as $x \to 0\pm$ from the Abelian, Tauberian theorems, (lecture 12) we must have

$$U^{*+}(s) = O(\frac{1}{s}) \quad \text{as} \quad |s| \to \infty, \quad \Im(s) > -\epsilon,$$

$$U^{*-}(s) = O(\frac{1}{s}) \quad \text{as} \quad |s| \to \infty, \quad \Im(s) < \epsilon.$$

Thus

$$E(s) \to 0 \quad \text{as} \quad |s| \to \infty.$$

By Liouville's theorem we must have $E(s) = 0$. Hence the solution is given by

$$U^{*+}(s) = \frac{2ik_1}{(k_1 + k_2)(s + k_2)}, \quad U^{*-}(s) = -\frac{i(k_1 - k_2)}{(k_1 + k_2)(s - k_1)}.$$

Inverting the transforms gives

$$U^{+}(x) = \frac{1}{2\pi} \frac{2ik_1}{(k_1 + k_2)} \int_{\mathcal{L}} \frac{e^{-isx}}{s + k_2} \, ds$$

where the path of integration runs above the singularity at $s = -k_2$. By deforming the path onto a semi-circular path in the upper/lower half-planes (upper for $x < 0$ and lower for $x > 0$) and using Cauchy's theorem gives

$$U^{+}(x) = \begin{cases} 0, & x < 0 \\ \frac{2k_2}{k_1+k_2} e^{ik_2 x} & x > 0 \end{cases}.$$

Similarly

$$U^{-}(x) = \begin{cases} 0, & x > 0 \\ \frac{(k_1-k_2)}{k_1+k_2} e^{-ik_1 x} & x < 0 \end{cases}.$$

The ideas discussed above are not too dissimilar to those which are used to solve the scalar Riemann-Hilbert problem.

We need a few definitions first.

**Definition** *Consider a simple smooth closed curve $\mathcal{C}$ which divides the complex plane into two regions $D^+$ and $D^-$. A function $\phi(z)$ is called* **sectionally continuous** *if $\phi(z)$ is analytic in each of the regions $D^{\pm}$, except possibly at $z = \infty$ and $\phi(z)$ approaches a definite limiting value as $z \to z_0$ along a path which lies wholly in $D^{\pm}$ where $z_0$ is a point on $\mathcal{C}$. $\phi^{\pm}$ are said to be the boundary values on $\mathcal{C}$.*

**Definition Hölder condition** *A function $\phi(t)$ is said to satisfy the Hölder condition on a smooth curve $\mathcal{L}$ if for any two points $\tau, \tau_1$ on $\mathcal{L}$*

$$|\phi(\tau) - \phi(\tau_1)| \leq A|\tau - \tau_1|^{\lambda}, \quad 0 < A, \quad 0 < \lambda \leq 1.$$

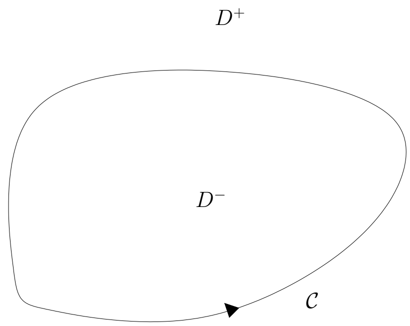


Figure 11.8: some

The Lipschitz condition is given by $\lambda = 1$ in the above. Any differentiable function satisfies the Lipschitz condition.

Consider first the homogeneous version of the Riemann-Hilbert problem. Given a smooth closed curve $\mathcal{C}$, and a function $G(z)$ satisfying the Hölder condition on $\mathcal{C}$, find a sectionally continuous function $\Phi(z)$ such that

$$\Phi^+(z) = G(z)\Phi^-(z) \quad \text{on} \quad \mathcal{C},$$

with $\Phi(z)$ of finite degree $k$ at infinity, (with $k$ integer), ie

$$\Phi(z) \sim c_k z^k + O(z^{k-1}), \quad \text{as} \quad z \to \infty, \quad c_k \neq 0.$$

Recall the **Plemlj formulae** discussed in lecture 10.

Suppose $\mathcal{C}$ is a smooth curve (which may be closed or open) and suppose $\phi(t)$ satisfies the Hölder condition on $\mathcal{C}$. Then if

$$\Phi(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\phi(t)}{t-z}\,dt$$

as $z \to z_0$ along a path lying wholly within $D^\pm$, and where $z_0$ is a point on $\mathcal{C}$ but not an endpoint, then

$$\Phi^\pm(z_0) = \pm\frac{1}{2}\Phi(z_0) + \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\phi(t)}{t-z_0}\,dt. \tag{11.2.11}$$

The proof of this result earlier was for functions $\phi(z)$ which are analytic at $z = z_0$. The proof for the more general case of functions satisfying the Hölder condition is given in Muskhelishvili (1953).

Note from (11.2.11) that

$$\Phi^+(z_0) - \Phi^-(z_0) = \Phi(z_0), \quad \Phi^+(z_0) + \Phi^-(z_0) = \frac{1}{\pi i}\oint_{\mathcal{C}} \frac{\phi(t)}{t-z_0}\,dt. \tag{11.2.12}$$

The problem (11.2.12) is a special case of the more general RH problem with $G(z) = 1$ in the earlier notation, and the solution is given by the Plemlj formula and the Cauchy integral for $\Phi(z)$. If in addition we know that $\Phi(z)$ is of degree $k$ at infinity then the more general solution of (11.2.12) which is sectionally continuous is

$$\Phi(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\phi(t)}{t-z}\,dt + P_k(z),$$

where $P_k(z)$ is an arbitrary polynomial of degree $k$ at most. For the solution of the RH homogeneous problem

$$\Phi^+(z) = G(z)\Phi^-(z)$$

if we take logarithms of both sides we find

$$[\log\Phi(z)]^+ - [\log\Phi(z)]^- = \log G(z). \tag{11.2.13}$$

This is the same form as for the Plemlj formula, except the right-hand side $\log G(z)$ is not necessarily one-valued and continuous. In fact as we make a circuit around $\mathcal{C}$ the function $\log G(z)$ changes by

$$\frac{1}{2\pi i}[\log G(z)]_{\mathcal{C}} = \frac{1}{2\pi}[\arg G(z)]_{\mathcal{C}} = \kappa$$

where $\kappa$ is an integer which may be zero, positive or negative. Assume without loss of generality that the origin is located in the $D^+$ region.

The solution to the above difficulty with the $\log G(z)$ term is obtained by writing

$$\Phi^+(z) = z^{-\kappa}G(z)(z^{\kappa}\Phi^-),$$

which after taking loagithms becomes

$$[\log\Phi(z)]^+ - [\log z^{\kappa}\Phi(z)]^- = \log(z^{-\kappa}G(z)). \tag{11.2.14}$$

The function

$$\log(z^{-\kappa}G(z))$$

is one valued and continuous on $\mathcal{L}$. Hence a particular solution to the homogeneous RH problem is given by $\Phi(z) = S(z)$ where

$$\log[S(z)] = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\log(z^{\kappa}G(z))}{t-z}\,dt.$$

If $\Phi(z)$ is of finite degree $m$ at infinity then the term $\log(z^{\kappa}\Phi^-(z))$ will have degree $\kappa + m$ at infinity and thus the most general solution of the inhomegenous RH problem is given by

$$\Phi(z) = S(z)P_{m+\kappa}(z)$$

where $S(z)$ is our fundamental solution above and $P_{m+\kappa}$ is an arbitrary polynomial of degree $m+\kappa$.

For the non-homogeneous RH problem

$$\Phi^+(z) - G(t)\Phi^-(z) = f(z), \quad \text{for} \quad z$$

the solution is more involved, but the ideas are very similar to those described above. Similar ideas apply also to open contours, see Muskhelishvili for further details.

Vector RH problems are an active research area with many applications in wave-scattering, inverse scattering etc, and beyond the scope of this course.

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