Some results and problems on complex germs with definable Mittag-Leffler stars

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Abstract

Working in an o-minimal expansion of the real field, we investigate when a germ (around 0 say) of a complex analytic function has a definable analytic continuation to its Mittag-Leffler star.

As an application we show that any algebro-logarithmic function that is complex analytic in a neighbourhood of the origin in $\mathbb{C}$ has an analytic continuation to all but finitely many points in $\mathbb{C}$.

It is my pleasure to dedicate this paper to Anand Pillay on the occasion of his 60th birthday.

1 Introduction

This paper is motivated by the conjecture of Zilber stating that every $\mathbb{C}_{exp}$-definable subset of $\mathbb{C}$ is either countable or co-countable. Here, $\mathbb{C}_{exp}$ is the expansion of the ring of complex numbers by the complex exponential function. As far as I know, even sets of the form

$$\{z \in \mathbb{C} : \exists w \in \mathbb{C}\ F(z, w) = 0\} \qquad (*)$$

where $F(z,w)$ is a (two variable) term of the language $\mathcal{L}(\mathbb{C}_{exp})$ have not been shown to satisfy Zilber's conjecture.

Our approach to this particular case is as follows. Let us suppose that

$$F(0,0) = 0 \neq \frac{\partial F}{\partial w}(0,0).$$

Then by the implicit function theorem there exists $\epsilon > 0$ and a complex analytic function $\phi : \Delta(0;\epsilon) \to \mathbb{C}$ (where, in general, $\Delta(a;r)$ denotes the disk centred at $a \in \mathbb{C}$ and having radius r) such that for all $z \in \Delta(0;\epsilon)$, we have $F(z,\phi(z)) = 0$. We must show that the set $(*)$ is co-countable and it seems reasonable to conjecture that the function element ϕ has an analytic continuation (which necessarily preserves the equation $F(z,\phi(z)) = 0$) to all but countably many points in the complex plane. Indeed, one can fairly easily show that if one proves a suitably generalized version of this *analytic continuation conjecture* (in which w is allowed to be an n-tuple of variables and F an n-tuple of terms in the $1+n$ variables z, w, and where the countably many exceptional points have a certain specific form) then Zilber's conjecture (even for subsets of $\mathbb{C}$ defined by formulas of the language $\mathcal{L}_{\omega_1,\omega}(\mathbb{C}_{exp})$) would follow.

Let us now consider issues of definability. The approach to Zilber's conjecture suggested above transcends $\mathcal{L}(\mathbb{C}_{exp})$-definablility (at least, if Zilber's conjecture is true!): one cannot define *restricted* functions $\phi : \Delta(0;\epsilon) \to \mathbb{C}$ without the resource of the real line and the usual metric. So we follow the Peterzil-Starchenko idea of doing complex analysis definably in a suitable o-minimal structure via the usual identifications $\mathbb{C} \sim \mathbb{R} \oplus i\mathbb{R} \sim \mathbb{R} \times \mathbb{R}$. Actually, we will only be considering a fixed o-minimal expansion $\widetilde{\mathbb{R}}$ of the ordered field of real numbers $\overline{\mathbb{R}}$, so many of the subtleties of [4] will not be required here. But the uniform finiteness of the winding number for definable functions will be, and this was inspired by the Peterzil-Starchenko approach.

My aim in this paper, then, is to consider definable analytic continuation relative to an o-minimal expansion $\widetilde{\mathbb{R}}$ of $\overline{\mathbb{R}}$. I shall only consider continuations along straight line paths emanating from the origin in $\mathbb{C}$, so let me discuss this now. The mathematical theory (i.e. without definability considerations) may be found in [1] and [3], but in very few modern texts as far as I can see.

2 The Mittag-Leffler star

So consider any complex analytic function $\phi : \Delta(0;r) \to \mathbb{C}$. The *Mittag-Leffler star* of ϕ (henceforth just the *star* of ϕ), denoted S_ϕ, is defined to be the set of all $z \in \mathbb{C}$ such that there exists an open set $U_z \subseteq \mathbb{C}$ with $\Delta(0;r) \cup [0,z] \subseteq U_z$ and a complex analytic function $\psi : U_z \to \mathbb{C}$ with $\psi \restriction \Delta(0;r) = \phi$. (Here, $[0,z]$ denotes the straight line segment in $\mathbb{C}$ from 0 to z, i.e. $[0,z] := \{tz : 0 \leq t \leq 1\}$.)

It can be shown (see [3] Volume 3) that S_ϕ is an open, connected and simply connected set containing $\Delta(0;r)$, and that there exists a unique complex analytic function $\star\phi : S_\phi \to \mathbb{C}$ with $\star\phi \restriction \Delta(0;r) = \phi$. I call $\star\phi$ the *star function* of ϕ. Also, a point z such that $z \notin S_\phi$ but satisfying $[0,w] \subseteq S_\phi$ for all $w \in [0,z] \setminus \{z\}$ will be called a *singular point* of ϕ.

In general one can say very little else about the nature of the set S_ϕ (it could, for example, be bounded). However, we have the following result in the definable situation.

Theorem 1. *Let $\widetilde{\mathbb{R}}$ be an o-minimal expansion of the ordered field of real numbers $\overline{\mathbb{R}}$ and assume that $\widetilde{\mathbb{R}}$ has analytic cell decomposition. Let $\phi : \Delta(0;r) \to \mathbb{C}$ be a complex analytic function and suppose that its star function $\star\phi : S_\phi \to \mathbb{C}$ (and hence its star S_ϕ) is definable (in $\widetilde{\mathbb{R}}$). Then ϕ has only finitely many singular points.*

Proof. Let $\phi : \Delta(0;r) \to \mathbb{C}$ and $\star\phi : S_\phi \to \mathbb{C}$ be as in the hypotheses of the theorem and suppose that ϕ has infinitely many singular points. Then by analytic cell decomposition there would exist (possibly after rotating $\mathbb{C}$ about 0) a 2-cell C of the form

$$C = \{x + iy : a < x < b, f(x) < y < g(x)\},$$

where $f, g : (a,b) \to \mathbb{R}$ are definable real analytic functions, such that $C \subseteq S_\phi$ and such that $\star\phi \restriction C$ has no analytic continuation to any open set in $\mathbb{C}$ containing a point of $graph(g)$. We may further assume (by refining the original cell decomposition) that either for all $z \in C$, $|\star\phi(z)| < 1$ or for all $z \in C$, $|\star\phi(z)| > 1$.

Let us consider the first case. By o-minimality there is a finite set $s \subseteq graph(g)$ such that $\star\phi$ has a definable continuous extension (which we also denote by $\star\phi$) to $(C \cup (graph(g))) \setminus s$. By analytic cell decomposition again, there exists a', b' with $a < a' < b' < b$ such that $\star\phi \circ g \restriction (a', b')$ is a definable real analytic function. We now obtain a contradiction by a using a classical argument (as described in, for example, [2], Chapter IX). Namely, fix $x_0 \in (a', b')$ and let $\epsilon > 0$ be chosen small enough so that $a' < x_0 - \epsilon < x_0 < x_0 + \epsilon < b'$ and also so that the (real) Taylor series of both g and $\star\phi \circ g$ extend (via the same power series) to (not necessarily definable) complex analytic functions $G : \Delta(x_0; \epsilon) \to \mathbb{C}$ and $\Phi : \Delta(x_0; \epsilon) \to \mathbb{C}$ respectively.

Define the complex analytic function $H : \Delta(x_0; \epsilon) \to \mathbb{C}$ by $H(z) := z + iG(z)$. Since the Taylor coefficients of G are real it follows that $H'(x_0) \neq 0$ and hence (by reducing ϵ if necessary) that H is a holomorphic homeomorphism from $\Delta(x_0; \epsilon)$ onto an open set, U say. Further, H maps the interval $(x_0 - \epsilon, x_0 + \epsilon)$ onto $graph(g \restriction (x_0 - \epsilon, x_0 + \epsilon))$.

Now consider the function

$$\Psi := \star\phi - \Phi \circ H^{-1} : (C \cup graph(g \restriction (x_0 - \epsilon, x_0 + \epsilon))) \cap U \to \mathbb{C}.$$

By our construction Ψ is continuous, holomorphic on $C \cap U$, and identically zero on the analytic curve $graph(g \restriction (x_0 - \epsilon, x_0 + \epsilon))$ which forms (a nontrivial) part of the boundary of $C \cap U$. This implies (see [2], page 303, exercise 6) that Ψ is identically zero throughout $(C \cup graph(g \restriction (x_0 - \epsilon, x_0 + \epsilon))) \cap U$. It follows that $\Phi \circ H^{-1}$ provides an analytic continuation of $\star\phi$ to the open set $C \cup U$. But this is a contradiction since U contains the point $x_0 + ig(x_0)$ of $graph(g)$.

Remark: I am grateful to Chris Miller for pointing out to me that the statement in the exercise from Lang's book cited above may well be false in full generality. (The analytic arc mentioned there might not be accessible from within the open set.) However, it is true, and easy to prove via Cauchy's Theorem, for open sets whose boundary consists of finitely many analytic arcs, which is the case here.

The case that $|\star\phi(z)| > 1$ for all $z \in C$ is dealt with by applying the above argument to the function $\frac{1}{\star\phi}$ and then inverting the analytic continuation. (The proof actually shows that $\star\phi$ is necessarily locally bounded at all but finitely many points of $graph(g)$.)

This completes the proof of Theorem 1. $\square$

Later I shall show that the collection of all those complex analytic germs having a definable Mittag-Leffler star has a reasonably rich structure, at least if $\widetilde{\mathbb{R}}$ does. This is in contrast to those germs having definable entire, or definable meromorphic, extensions which, as one can easily show, are (for any o-minimal $\widetilde{\mathbb{R}}$) just polynomials or rational functions respectively.

The proof of Theorem 1 shows that any definable complex analytic function whose domain is an open cell C in $\mathbb{C}$ has an analytic continuation (though not necessarily a definable one) across the boundary of C at all but finitely many points. I leave the reader to combine this remark with Theorem 1 itself to give a proof of the following result.

Theorem 2. *Let $\widetilde{\mathbb{R}}$ be an o-minimal expansion of the ordered field of real numbers $\overline{\mathbb{R}}$ and assume that $\widetilde{\mathbb{R}}$ has analytic cell decomposition. Let $\phi : \Delta(0;r) \to \mathbb{C}$ be a complex analytic function and suppose that its star function $\star\phi : S_\phi \to \mathbb{C}$ (and hence its star S_ϕ) is definable (in $\widetilde{\mathbb{R}}$). Then for all but finitely many points $z \in \mathbb{C}$ there exists a continuous, piecewise linear path begining at 0 and terminating at z (and, in fact, consisting of at most two line segments) along which ϕ has an analytic continuation.*

However, I am unable to settle the following question.

Open Problem 1

Let ϕ and $\star\phi$ be as in the hypotheses of Theorems 1 and 2. Does there exist a finite set $s \subseteq \mathbb{C} \setminus \{0\}$ such that ϕ has an analytic continuation along all continuous, definable paths that begin at 0 and avoid s?

Before finishing this section I should mention the Mittag-Leffler Star Theorem. This provides a remarkable series expansion for $\star\phi$ which converges uniformly to $\star\phi$ on compact subsets of S_ϕ. It is completely analogous to the Taylor expansion on the disk of convergence of ϕ in the sense that the only dependence of the series on the germ ϕ is a fixed (i.e. independent of ϕ) linear one on the numbers $\phi(0), \phi'(0), \ldots, \phi^{(n)}(0), \ldots$. I will not need this result and so I will not expand on this remark. The interested reader may consult [3] for further information and proofs.

3 The ring of definable star functions

I now fix an o-minimal expasion $\widetilde{\mathbb{R}}$ of $\overline{\mathbb{R}}$ and I assume that $\widetilde{\mathbb{R}}$ has analytic cell decomposition. Notions of definability are relative to $\widetilde{\mathbb{R}}$ and are *without* parameters.

I denote by $\widetilde{\mathcal{G}}$ the collection of all definable, complex analytic germs at 0, i.e. the collection of definable, complex analytic functions $f : U \to \mathbb{C}$ (where U is a (definable) open neighbourhood of 0), where two such functions are identified if there is some open neighbourhood of 0 on which they agree. I will, however, not distinguish notationally betwen functions and their germs.

It is clear that $\widetilde{\mathcal{G}}$ is an integral domain (under pointwise operations) and a differential ring (under the usual derivative $\frac{d}{dz}$). We are interested in its subset consisting of those $\phi \in \widetilde{\mathcal{G}}$ having a definable star function $\star\phi : S_\phi \to \mathbb{C}$.

It is not immediately obvious this is a subring of $\widetilde{\mathcal{G}}$: it could be the case, for example, that $\star\phi : S_\phi \to \mathbb{C}$ and $\star\psi : S_\psi \to \mathbb{C}$ are definable but that the domain of $\star(\phi + \psi)$ (i.e. $S_{(\phi+\psi)}$) is strictly larger than $S_\phi \cap S_\psi$. So we would need to show that the extension of the (obviously definable) function $\star\phi + \star\psi : S_\phi \cap S_\psi \to \mathbb{C}$ to $S_{(\phi+\psi)}$ is definable.

To resolve this rather annoying difficulty, we first let $\widetilde{\mathcal{S}}$ be the collection of all definable, open subsets of $\mathbb{C}$ of the form $\mathbb{C} \setminus \bigcup_{j=1}^{n}[a_j, \infty)$, where $a_1, \ldots, a_n$ are (necessarily definable) nonzero complex numbers (and where, for $a \in \mathbb{C}$, $[a, \infty) := \{ta : 1 \leq t\}$).

Now let

$$\widetilde{\mathcal{M}} := \{\phi \in \widetilde{\mathcal{G}} : \phi \,\text{has a (definable) representative}\ \bar{\phi} : U \to \mathbb{C} \text{ for some } U \in \widetilde{\mathcal{S}}\}.$$

Now it is certainly clear that $\widetilde{\mathcal{M}}$ is a subring of $\widetilde{\mathcal{G}}$ since $\widetilde{\mathcal{S}}$ is closed under intersection. We would like to show that $\widetilde{\mathcal{M}}$ may be identified with the collection of those $\phi \in \widetilde{\mathcal{G}}$ having a definable star function. Such a result is in the spirit of those in section 2.7 of [4] but does not seem to follow directly from them. So instead we argue as follows.

Consider, more generally, any definable complex analytic function $F : U \to \mathbb{C}$ where U is a (definable) open subset of $\mathbb{C}$ of co-dimension at most 1 (in the sense of the o-minimal structure $\widetilde{\mathbb{R}}$). Let $\mathcal{E}(F)$ denote the collection of all (not necessarily definable) complex analytic functions $G : V \to \mathbb{C}$ with $U \subseteq V \subseteq \mathbb{C}$, V open, and $G \restriction U = F$. Now if $G_i : V_i \to \mathbb{C}$ are in $\mathcal{E}(F)$ for $i = 1, 2$, and $z \in V_1 \cap V_2$ then $G_1(z) = G_2(z)$. This is because for some $\epsilon > 0$, $\Delta(z; \epsilon) \subseteq V_1 \cap V_2$ and $\Delta(z; \epsilon) \cap U$ is a nonempty open set (as U has codimension 1) on which G_1 and G_2 agree (with F). It follows that G_1 and G_2, being complex analytic, agree throughout $\Delta(z; \epsilon)$ and hence in particular that $G_1(z) = G_2(z)$.

It now follows that all functions in $\mathcal{E}(F)$ have a common extension, $H : W \to \mathbb{C}$ say, which also lies in $\mathcal{E}(F)$. Further, H is definable. To see this let $A \subseteq \mathbb{C} \times \mathbb{C}$ be the closure of the graph of $F : U \to \mathbb{C}$. Then one easily shows that for all $z, w \in \mathbb{C}$, $\quad H(z) = w$ if and only if $\langle z, w \rangle \in A$ and for some $\epsilon > 0$, $\quad A \cap (\Delta(z; \epsilon) \times \mathbb{C})$ is the graph of a continuously (complex) differentiable function with domain $\Delta(z; \epsilon)$, and this is a definable condition.

Now suppose that $\phi \in \widetilde{\mathcal{M}}$, represented by $\bar{\phi} : U \to \mathbb{C}$ with $U \in \widetilde{\mathcal{S}}$. Since sets in $\widetilde{\mathcal{S}}$ obviously have co-dimension at most 1, we may apply the argument above to $F = \bar{\phi}$ and let $H : W \to \mathbb{C}$ be the resulting maximal extension. Then as the function $\star\phi : S_\phi \to \mathbb{C}$ lies in $\mathcal{E}(\bar{\phi})$ it follows that $S_\phi \subseteq W$ and $H \restriction S_\phi = \star\phi$. Now it may be the case that the inclusion here is proper (e.g. if $\phi(z) = (1 - z)^{-1}$, then $S_\phi = \mathbb{C} \setminus [1, \infty)$ whereas $W = \mathbb{C} \setminus \{1\}$), but, given W it is very easy to define the singular points of ϕ, and hence also the set S_ϕ. Since $\star\phi$ is just the restriction of H to S_ϕ its definabilty also follows, as required.

The rest of this paper is devoted to proving that $\widetilde{\mathcal{M}}$ is closed under various operations. Our first observation is now clear.

Theorem 3. *$\widetilde{\mathcal{M}}$ is a subring (in fact, a differential subring) of $\widetilde{\mathcal{G}}$.*

We also have the following result where I regard one domain, R_0 say, as being algebraically closed in another domain, R say, if every zero in R of a (not necessarily monic) polynomial with coefficients in R_0, actually lies in R_0.

Theorem 4. *$\widetilde{\mathcal{M}}$ is algebraically closed in $\widetilde{\mathcal{G}}$.*

Proof. Firstly, if $f \in \widetilde{\mathcal{M}}$ is invertible in $\widetilde{\mathcal{G}}$ (i.e. if $f(0) \neq 0$) then it is invertible in $\widetilde{\mathcal{M}}$. For if $domain(f) = U \in \widetilde{\mathcal{S}}$, let $Z_f := \{a \in U : f(a) = 0\}$. Since Z_f is a discrete set, it is finite (by o-minimality). So if we set $\quad V := U \setminus \bigcup_{a \in Z_f} [a, \infty)$ then $V \in \widetilde{\mathcal{S}}$ and $\frac{1}{f} : V \to \mathbb{C}$ is a definable, complex analytic function. Hence $\frac{1}{f} \in \widetilde{\mathcal{M}}$.

So to prove the theorem it is now sufficient to consider a monic polynomial

$$P(w) = w^n + f_1 \cdot w^{n-1} + \cdots + f_n$$

where $f_1, \ldots, f_n \in \widetilde{\mathcal{M}}$, which has a root, ϕ say, in $\widetilde{\mathcal{G}}$. We may assume that P is irreducible over (the field of fractions of) $\widetilde{\mathcal{M}}$ and, in particular, that its discriminant, D say, is a nonzero element of $\widetilde{\mathcal{M}}$. It follows, as above, that we can find a set $U \in \widetilde{\mathcal{S}}$ such that both $U \subseteq \bigcap_{j=1}^{n} domain(f_j)$ and $D(z) \neq 0$ for all $z \in U \setminus \{0\}$.

It now follows from classical theory that ϕ has an analytic continuation to all points of U. (The continuation is single valued since U is simply connected.) In particular, $U \subseteq S_\phi$ and this continuation is necessarily equal to $\star\phi \restriction U$ (as both functions agree on an open neighbourhood of zero) and, further, $P(\star\phi) = 0$ (in the ring $\widetilde{\mathcal{M}}$). It remains to show that $\star\phi \restriction U$ is definable. However, this follows easily by considering a cell decomposition of $\mathbb{R}^3$ compatible with the (definable) set consisting of all $\langle x, y, t \rangle \in \mathbb{R}^3$ such that $x + iy \in U$ and for some $u \in \mathbb{R}$,

$$(t + iu)^n + f_1(x + iy) \cdot (t + iu)^{n-1} + \cdots + f_0(x + iy) = 0.$$

Then the graph of the real part of $\star\phi \restriction U$ is given by piecing together certain 0, 1 and 2 cells of this decomposition. The imaginary part of $\star\phi \restriction U$ is dealt with similarly and this completes the proof of the theorem. □

It follows from Theorem 4 that if $\widetilde{\mathbb{R}} = \overline{\mathbb{R}}$ then $\widetilde{\mathcal{M}} = \widetilde{\mathcal{G}}$.

Open Problem 2

Does there exist an o-minimal expansion $\widetilde{\mathbb{R}}$ of $\overline{\mathbb{R}}$ in which some nonalgebraic, complex analytic germ $f : \Delta(0; r) \to \mathbb{C}$ is definable, but is such that $\widetilde{\mathcal{M}} = \widetilde{\mathcal{G}}$?

Certainly the complex exponential function restricted to a disk $\Delta(0; r)$ could not be definable in such an $\widetilde{\mathbb{R}}$ since its star function is the entire exponential function which is not definable in any o-minimal structure.

From now on we assume that, for any $R > 0$, $\exp \restriction \{x + iy : -R < y < R\}$ is definable in $\widetilde{\mathbb{R}}$. This is equivalent to both the real exponential function $\exp \restriction \mathbb{R}$ and the restricted sine function $\sin : [0, 2\pi) \to \mathbb{R}$ being definable in $\widetilde{\mathbb{R}}$. The structure $\mathbb{R}_{an,\exp}$ is an example.

Theorem 5. *Let $f \in \widetilde{\mathcal{M}}$ and assume that $f(0) \neq 0$. Then any branch of* $\log f$ *(restricted to some set in $\widetilde{S}$) is in $\widetilde{\mathcal{M}}$.*

Proof. Clearly we may choose $r > 0$ small enough so that all determinations of $\log f \restriction \Delta(0; r)$ are in $\widetilde{\mathcal{G}}$. Let $L_f : \Delta(0; r) \to \mathbb{C}$ be such a determination. Let $U = domain(f)$, so that $U \in \widetilde{\mathcal{M}}$. Now arguing as before we may assume that $f(z) \neq 0$ for all $z \in U$. Thus, since U is simply connected, L_f extends to a single valued logarithm of f on all of U via the usual formula

$$L_f(z) = L_f(0) + \int_0^z \frac{f'(w)}{f(w)} dw \quad (z \in U)$$

where the integration is along the straight line segment $[0, z] \subseteq U$.

To see that $L_f : U \longrightarrow \mathbb{C}$ is definable let us first note that the functions $|f| : U \longrightarrow \mathbb{R}_{>0}$ and $\frac{f}{|f|} : U \longrightarrow \{w \in \mathbb{C} : |w| = 1\}$ are definable, continuous functions. Since the real logarithm function from $\mathbb{R}_{>0}$ to $\mathbb{R}$ is definable, we obtain immediately that the real part of L_f $(= \log |f|)$ is definable.

To deal with the imaginary part we note that as L_f is definable in some neighbourhood of 0, the number $L_f(0)$ is definable and hence so is its imaginary part, θ_0 say. Then for $z \in U$, the imaginary part of $L_f(z)$ is given by $\theta_z(z)$, where $\theta_z : [0, z] \longrightarrow \mathbb{R}$ is the unique continuous function satisfying (a) $\theta_z(0) = \theta_0$, and (b) $\frac{f}{|f|}(w) = e^{i\theta_z(w)}$ for $w \in [0, z]$.

So we must show that $\theta_z(z)$ is a definable function.

For $z \in U$, let

$$A_z = \{t \in \mathbb{R} : 0 \leq t \leq 1 \text{ and } \frac{f}{|f|}(tz) = 1\}$$

Then A_z is, uniformly in z, a definable subset of $[0, 1]$. It follows by o-minimality that there exists $N > 0$ such that for all $z \in U$, either A_z contains at most N points, or else A_z contains an open interval. In the former case we clearly have that

$$\theta_0 - 2\pi(N + 1) \leq \theta_z(w) \leq \theta_0 + 2\pi(N + 1)$$

for all $w \in [0, z]$. This holds in the latter case too since then, by analyticity, f is real (and positive) throughout $[0, z]$, and hence θ_z is constant with value θ_0.

We now consider a cell decomposition of $\mathbb{R}^3$ compatible with the set

$\{\langle x, y, \theta\rangle \in \mathbb{R}^3 : x + iy \in U, \theta_0 - 2\pi(N + 1) \leq \theta \leq \theta_0 + 2\pi(N + 1)$
and $\cos\theta + i \sin\theta = \frac{f(x+iy)}{|f(x+iy)|}\}$.

(Notice that this set is definable by our assumptions on $\widetilde{\mathbb{R}}$.)

The graph of the function $z \longmapsto \theta_z(z)$ $(z \in U)$ is now obtained by piecing together certain 0, 1 and 2 cells of this decomposition. $\square$

By a similar argument one can also establish the following result.

Theorem 6. *If $f \in \widetilde{\mathcal{M}}$, $f(0) \neq 0$, and α is an exponent of $\widetilde{\mathbb{R}}$, then (any branch of) f^α also lies in $\widetilde{\mathcal{M}}$.*

I am reasonably confident that Theorems 4 and 5 have a common generalization as suggested by the following problem.

Open Problem 3
Let $f_1, \ldots, f_n \in \widetilde{\mathcal{M}} \setminus \{0\}$ and let $\tau_1(w), \ldots, \tau_n(w)$ be one variable terms of $\mathcal{L}(\mathbb{C}_{exp})$. Assume that $\phi \in \widetilde{\mathcal{G}}$ satisfies

$$\sum_{j=1}^{n} f_j(z) \cdot \tau_j(\phi(z)) = 0$$

for all $z \in domain(\phi)$. Is it the case that $\phi \in \widetilde{\mathcal{M}}$?

As for our original motivation, I conjecture a positive answer to the following.

Open Problem 4
Let $\tau(z, w)$ be a two variable term of $\mathcal{L}(\mathbb{C}_{exp})$ and let $\phi \in \widetilde{\mathcal{G}}$ be such that

$$\tau(z, \phi(z)) = 0$$

for all $z \in domain(\phi)$. Then does the star of ϕ have at most countably many singular points? If so, is each such point definable?

Finally, I state a result that makes no reference to definablity. It follows immediately from Theorems 4, 5, 6 and 2. However, I see no way of proving it without using the o-minimality of, say, $\mathbb{R}_{an,exp}$.

Theorem 7. *Let $\mathcal{O}$ denote the ring of complex analytic germs at the origin in $\mathbb{C}$. Let $\mathcal{H}$ be the smallest subset of $\mathcal{O}$ containing the polynomial ring $\mathbb{C}[z]$ and satisfying the following closure conditions:*
(i) $\mathcal{H}$ is a subring of $\mathcal{O}$ and is algebraically closed in $\mathcal{O}$;
(ii) if $f \in \mathcal{H}$ and $f(0) \neq 0$, then $\log f \in \mathcal{H}$;
(iii) if $f \in \mathcal{H}$, $f(0) \neq 0$ and $\alpha \in \mathbb{R}$, then $f^\alpha \in \mathcal{H}$.
Then for every germ $f \in \mathcal{H}$ there exists a finite set $s_f \subseteq \mathbb{C}$ such that for all $z \in \mathbb{C} \setminus s_f$ there exists a piecewise linear path starting at 0 and terminating at z along which f can be analytically continued.

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