REPRESENTATIONS OF POSITIVE PROJECTIONS ON LIPSCHITZ VECTOR MEASURE GAMES

OMER EDHAN

Abstract. Among the single-valued solution concepts studied in cooperative game theory and economics, those which are also positive projections play an important role. The value (e.g., [1],[6],[13]), semivalues (e.g., [2],[7],[8],[23],[26]), and quasivalues (e.g., [1, Chapter12], [14]-[16], [27]) of a cooperative game are several examples of solution concepts which are positive projections. These solution concepts are known to have many important applications in economics. In many applications the specific positive projection discussed is represented as an expectation of marginal contributions of agents to "random" coalitions. Usually these representations are used to characterize positive projections obeying certain additional axioms. It is thus of interest to study the representation theory of positive projections and its relation with some common axioms. We study positive projections defined over certain spaces of nonatomic Lipschitz vector measure games. To this end, we develop a general notion of "calculus" for such games, which in a manner extends the notion of the Radon-Nykodim derivative for measures. We prove several representation results for positive projections, which essentially state that the image of a game under the action of a positive projection can be represented as an averaging of its derivative w.r.t. some vector measure. We then introduce a specific calculus for the space $\mathcal{CON}$ generated by concave, monotonically nondecreasing, and Lipschitz continuous functions of finitely many nonatomic probability measures. We study in detail the properties of the resulting representations of positive projections on $\mathcal{CON}$ and especially those of values on $\mathcal{CON}$. The latter results are of great importance in various applications in economics.

1. Introduction

The study of payoffs in systems of interacting players is one of the most basic issues and interests of economic theory. In many applications it is frequently necessary to study payoffs in games that involve a large number of individually insignificant players. This setting is usually modeled by assuming that the players form a nonatomic continuum, as first considered by Aumann and Shapley [1]. This model is usually referred to as *nonatomic games*.

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Studying payoffs in the setting of nonatomic games has a long and rich history. Usually, the payoff is required to fulfill certain properties, or *axioms*. Among the different kinds of payoffs studied in the setting of nonatomic games we may find the value (e.g., [1],[6],[13]), semivalues (e.g., [2],[7],[8],[23],[26]), and quasivalues (e.g., [1, Chapter12], [14]-[16], [27]).

The payoffs mentioned above have a certain common property - they are positive projections, namely they obey the *linearity*, *positivity* and *projection* axioms: linearity that the payoff map is linear; positivity means that the payoff map of a monotonic game is monotonic; and projection means that the payoff map is an idempotent.

There has been a tremendous advancement in the study of payoffs in "differentiable" games. However, the advancement almost stopped once the differentiability assumption was removed. The reason for that halt was the lack of a general representation theory for positive projections; in every case mentioned above the payoff could be represented as an aggregation of the game's derivative - an adaptation of the marginal contribution to the nonatomic setting. In fact, devising such a representation for a payoff is one of the basic steps (and sometimes, goals) of its study.

The idea of finding such a representation has also proved to be useful in some examples of spaces of "nondifferentiable" games (e.g., [12, 13]). Thus, it seems productive to initiate the study of representations of positive projections in general. In this paper we make the first steps in this direction. Namely, we first construct a theory of "differential calculus" for certain spaces of games which consist of the linear combinations of Lipschitz continuous vector measure games. This "calculus" may be viewed as an extension of the well–known integral and Radon–Nikodym derivative in measure theory, and it is quite different from the traditional notion of the derivative of a game which is found in the literature. We obtain various representation results for positive projections on spaces which admit such a calculus. That is, we prove that any positive projections on such a space which admits a calculus may be written as the expectation (w.r.t. some vector measure) of the game's "derivative". We then construct a calculus for the space $\mathcal{CON}$, generated by concave, monotonically nondecreasing, and Lipschitz continuous functions of finitely many nonatomic probability measures, and study the properties of the resulting representations of positive projections on this space, and those of values especially. The latter results have already played an important role in settling the age–old problem of characterizing the value on spaces of market games with a finite–dimensional core (see [9]-[10]).

2. Definitions and Main Results

2.1. **Basic definitions.** Let $(I,\mathcal{C})$ be a standard[1] measurable space. I is the set of *players*, and $\mathcal{C}$ is the σ-algebra of *coalitions*. A *game* is a real valued function $v:\mathcal{C}\to\mathbb{R}$ with $v(\emptyset)=0$. A game v is:

(1) *finitely additive* if $v(S\cup T)=v(S)+v(T)$ whenever $S,T\in\mathcal{C}$ are mutually disjoint;
(2) *monotonic* if $v(S)\le v(T)$ whenever $S\subseteq T$; and,
(3) of *bounded variation* if it is the difference of two monotonic games.

If Q is a space of games we denote by Q^+ its subset of monotonic games, and $Q^1=\{v\in Q^+ \,:\, v(I)=1\}$. We denote the space of all games of bounded variation by BV. The *variation* of a game $v\in BV$ is the supremum of the variation of v over all increasing chains $S_0\subseteq S_1\subseteq ...\subseteq S_m$ in $\mathcal{C}$, or equivalently

$$\|v\|_{BV}=\inf\left\{u(I)+w(I):u,w\in BV^+,\ v=u-w\right\}.$$

$\|\cdot\|_{BV}$ is a norm on BV (see [1]). Denote by FA the subspace of BV of finitely additive games, and by NA its subspace of all non-atomic and countably additive measures. A game v is *Lipschitz continuous* iff there are $K>0$ and $\lambda\in NA^1$ s.t. $|v(S)-v(T)|\le K\lambda(S\triangle T)$ for every $S,T\in\mathcal{C}$. In this case we denote $v\asymp\lambda$. Denote by LIP the space of all Lipschitz continuous games. Obviously $LIP\subseteq BV$.

For $x\in\mathbb{R}^k$ let $\overline{x}=\frac{1}{k}\sum_{i=1}^{k}x_i$. For any $k\ge1$ let

$$\mathcal{Z}_\lambda^k=\left\{\mu\in\left(NA^1\right)^k:\overline{\mu}\ll\lambda,\ \frac{d\overline{\mu}}{d\lambda}\in L^\infty(\lambda)\right\}.$$

Given a space of vector measure games[2] Q, denote by Q_λ its subspace consisting of games of the form $f\circ\mu$ with $\mu\in\mathcal{Z}_\lambda^k$ for some $k\ge1$.

Denote by Θ the group of measurable automorphisms of $(I,\mathcal{C})$. Each $\theta\in\Theta$ induces a linear mapping θ^* of BV onto itself by $(\theta^*v)(S)=v(\theta S)$. A set of games $Q\subseteq BV$ is *symmetric* if $\theta^*Q=Q$ for each $\theta\in\Theta$. For $\lambda\in NA^1$ denote by $\Theta(\lambda)\le\Theta$ the group of λ-preserving automorphisms. Denote by $B(I,\mathcal{C})$ the space of real valued bounded measurable functions on $(I,\mathcal{C})$, and by $B_+^1(I,\mathcal{C})$ its subset consisting of $\chi\in B(I,\mathcal{C})$ with $0\le\chi\le1$.

Given a linear space of games Q, a *projection* $\Psi:Q\to FA$ is a linear map satisfying the *projection axiom*, namely, $\Psi(\mu)=\mu$ whenever[3] $\mu\in FA\cap Q$. If Q is symmetric then[4] Ψ is a *value* iff it is linear and satisfies the following list of axioms:

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[1]Namely, $(I,\mathcal{C})$ is isomorphic to $([0,1],\mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra on $[0,1]$.
[2]Namely, each $v\in Q$ may be represented as $v=f\circ\mu$ with $f:\mathbb{R}_+^k\to\mathbb{R}$ and $\mu\in\left(NA^1\right)^k$, for some $k\ge1$
[3]See [17].
[4]Following Aumann and Shapley [1].

(1) *efficiency*- $\Psi(v)(I) = v(I)$ for every $v \in Q$;
(2) *symmetry*- $\theta^*\Psi(v) = \Psi(\theta^* v)$ for every $\theta \in \Theta,\ v \in Q$; and,
(3) *positivity*- $v \in Q^+ \Rightarrow \Psi(v) \in FA^+$.

2.2. **Calculus on Spaces of Vector Measure Games.** From now on we shall limit ourselves to *massive* spaces of *vector measure games* $Q \subseteq BV$, namely we assume that $NA \subseteq Q$ and that for every $v \in Q$ there is a vector measure $\left(NA^1\right)^k$ and a function $f : \mathbb{R}^k_+ \longrightarrow \mathbb{R}$ with $v = f \circ \mu$, respectively. A set $\widehat{Q} = \left\{\widehat{Q}_\lambda\right\}_{\lambda \in NA^1}$ is a *game data* of Q if each $\widehat{Q}_\lambda$ is a linear space generated by formal linear combinations (over $\mathbb{R}$) of pairs (f, μ) with $\mu \in \mathcal{Z}^k_\lambda$ and $f : \mathbb{R}^k_+ \longrightarrow \mathbb{R}$ for some $k \geq 2$ (these are the *generators* of $\widehat{Q}_\lambda$), and the linear map $\widehat{Q}_\lambda \overset{\sigma_\lambda}{\mapsto} Q_\lambda$ given by $\sum_{i=1}^n a_i(f_i, \mu^i) \overset{\sigma_\lambda}{\mapsto} \sum_{i=1}^n a_i f_i \circ \mu^i$ is onto for each $\lambda \in NA^1$. The map σ_λ induces a partial order relation on $\widehat{Q}_\lambda$ by $h \leq h' \Leftrightarrow \sigma_\lambda(h) \leq \sigma_\lambda(h')$.

For every $k \geq 2$ denote by Δ^k the $(k-1)$-dimensional simplex in $\mathbb{R}^k$, by $e_i \in \mathbb{R}^k$ the $1 \leq i \leq k$ unit vector, and let $\mathbf{1}_k = \sum_{i=1}^k e_i$. The *diagonal* of $\mathbb{R}^k$ is given by $D^k = \{t\mathbf{1}_k \ : \ t \in \mathbb{R}\}$, and its *perpendicular sphere* is given by[5] $\mathbb{S}^k_\perp = \left\{\frac{x}{\|x\|_2} \ : \ x \in \mathbb{R}^k,\ \overline{x} = 0\right\}$. For each $\mu \in \left(NA^1\right)^k$ let $AF(\mu)$ denote the affine space generated by the range of μ, $\mathcal{R}(\mu)$. Denote by Λ_μ the set $\left(\mathcal{R}(\mu) \setminus D^k\right) \sqcup \left([0,1]\mathbf{1}_k + \mathbb{S}^k_\perp \cap AF(\mu)\right)$ endowed with a topology $\mathcal{T}_\mu$ that makes it homeomorphic to $[0,1]\mathbf{1}_k + \left\{x \in (\mathbf{1}_k)^\perp : \|x\|_2 \in [1,2] \cup \{0\}\right\} \cap AF(\mu)$ via the homeomorphism ϱ_μ that satisfies

$$\varrho_\mu(x) = \overline{x}\mathbf{1}_k + \left(1 + \frac{\|x - \overline{x}\mathbf{1}_k\|_2}{d_2(\partial\mathcal{R}(\mu), \overline{x}\mathbf{1}_k)}\right)\frac{x - \overline{x}\mathbf{1}_k}{\|x - \overline{x}\mathbf{1}_k\|_2} \tag{2.1}$$

for $x \in \mathcal{R}(\mu) \setminus D^k$, and $\varrho_\mu\big|_{[0,1]\mathbf{1}_k + \mathbb{S}^k_\perp \cap AF(\mu)} = id$.

A *generalized direction space with perspective* $\lambda \in NA^1$ is a compact Hasudorff space Ω_λ s.t. there is an injective map $B^1_+(I, \mathcal{C}) \overset{i_\lambda}{\hookrightarrow} \Omega_\lambda$, and for every $\mu \in \mathcal{Z}^k_\lambda$ there is a mapping $\pi_\mu : \Omega_\lambda \to \Lambda_\mu$ s.t. the following diagram is commutative:

$$\begin{tikzcd} B^1_+(I,\mathcal{C}) \arrow[rr, "i_\lambda"] \arrow[dr, "\mu"'] & & \Omega_\lambda \arrow[dl, "\pi_\mu"] \\ & \Lambda_\mu & \end{tikzcd}$$

where $\mu(y) = \int_I y(s)d\mu(s)$.

A *Radon-Nikodym calculus* (a *calculus* for short) for Q w.r.t. a data set $\widehat{Q}$ is a set of 4-tuples $\mathfrak{C} = \left\{\mathfrak{C}_\lambda = \langle \Omega_\lambda, \partial Q_\lambda, \partial_\lambda, \textstyle\int_\lambda \rangle\right\}_{\lambda \in NA^1}$ s.t. Ω_λ is a generalized direction space with perspective λ, ∂Q_λ is a linear

[5]The convention $\frac{0_k}{0} = 0_k$ is used.

subspace[6] of $B(\Omega_\lambda, L^\infty(\lambda))$ containing the constant functions, $\partial_\lambda : \widehat{Q}_\lambda \to \partial Q_\lambda$ is a linear map and $\int_\lambda : \partial Q_\lambda \to Q_\lambda$ is surjective linear map s.t. the following conditions hold for each $\lambda \in NA^1$:

1. $\int_\lambda$ is order preserving;
2. for every constant function g, $\left(\int_\lambda(g)\right)(S) = \int_S g(x)(s)d\lambda(s)$ for every $x \in \Omega_\lambda$;
3. $\partial_\lambda((f,\mu))(x) = \frac{d(f\circ\mu)}{d\lambda}$ for every $x \in \Omega_\lambda$ whenever $f \circ \mu \in NA$; and,
4. the following diagram is commutative:

$$\begin{tikzcd}
\widehat{Q}_\lambda \arrow[rr, "\partial_\lambda"] \arrow[dr, "\sigma_\lambda"'] & & \partial Q_\lambda \arrow[dl, "\int_\lambda"] \\
& Q_\lambda &
\end{tikzcd}$$

The space ∂Q_λ is a *space of derivatives w.r.t* λ. Every game datum $h \in \widehat{Q}_\lambda$ is attached to a *Radon-Nikodym derivative* by the operator ∂_λ s.t. if $h, h' \in \widehat{Q}_\lambda$ satisfy $\sigma_\lambda(h) = \sigma_\lambda(h') = v$ then $\left(\int_\lambda \circ \partial_\lambda\right)(h) = \left(\int_\lambda \circ \partial_\lambda\right)(h') = v$.

If Q and R are massive spaces with data sets $\widehat{Q}$ and $\widehat{R}$ respectively, then $\widehat{Q}$ is a *subdata set* of $\widehat{R}$, and denote $\widehat{Q} \preceq \widehat{R}$, iff $\widehat{Q}_\lambda \subseteq \widehat{R}_\lambda$ for each $\lambda \in NA^1$ (which also implies $Q \subseteq R$). Given massive spaces Q and R with data sets $\widehat{Q} \preceq \widehat{R}$ respectively, and a calculus $\mathfrak{C}$ of R w.r.t. $\widehat{R}$, denote $\partial R_\lambda^{\widehat{Q}} = \left\{h \in \partial R_\lambda : \int_\lambda(h) \in Q_\lambda\right\}$, and let $\int_\lambda^{\widehat{Q}} = \int_\lambda\big|_{\partial R_\lambda^{\widehat{Q}}}$, and $\partial_\lambda^{\widehat{Q}} = \partial_\lambda\big|_{\widehat{Q}_\lambda}$. Denote $\mathfrak{C}_\lambda^{\widehat{Q}} = \langle \Omega_\lambda, \partial R_\lambda^{\widehat{Q}}, \partial_\lambda^{\widehat{Q}}, \int_\lambda^{\widehat{Q}} \rangle$, and $\mathfrak{C}^{\widehat{Q}} = \left\{\mathfrak{C}_\lambda^{\widehat{Q}}\right\}_{\lambda \in NA^1}$. The calculus $\mathfrak{C}$ of R w.r.t. $\widehat{R}$ is *inductive* iff $\mathfrak{C}^{\widehat{Q}}$ is a calculus of Q w.r.t. $\widehat{Q}$ whenever[7] $\widehat{Q} \preceq \widehat{R}$.

In many applications it is necessary to consider positive projections which are symmetric w.r.t. a nontrivial subgroup of Θ. For this end we now introduce a notion of symmetry to our definitions. Suppose that for some $\lambda \in NA^1$ we have a group action of a subgroup $H_\lambda \leq \Theta(\lambda)$ on Ω_λ. For each $x \in \Omega_\lambda$ and $\theta \in H_\lambda$ denote θx for the action of θ on x. This group action induces group action of H_λ on $B(\Omega_\lambda, L^\infty(\lambda))$ by the linear transformations $A_\theta(g)(x) = g(\theta x) \circ \theta$ for each $\theta \in H_\lambda$, $g \in B(\Omega_\lambda, L^\infty(\lambda))$, and $x \in \Omega_\lambda$. Let $\mathfrak{C}$ be a calculus of Q w.r.t. a data set $\widehat{Q}$. Then $\theta^*\left(\sum_{i=1}^n a_i(f_i, \mu^i)\right) = \sum_{i=1}^n a_i(f_i, \theta^*\mu^i)$ defines a linear group action of H_λ on $\widehat{Q}_\lambda$. We say that $\mathfrak{C}_\lambda$ is *symmetric w.r.t.* H_λ iff for every $\theta \in H_\lambda$ we have $\pi_{\theta^*\mu} = \pi_\mu \circ \theta$, $\partial_\lambda \circ \theta^* = A_\theta \circ \partial_\lambda$, and $\int_\lambda \circ A_\theta = \theta^* \circ \int_\lambda$. The calculus $\mathfrak{C}$ is *symmetric w.r.t.* $\{H_\lambda \leq \Theta(\lambda)\}_{\lambda \in NA^1}$ iff $\mathfrak{C}_\lambda$ is symmetric w.r.t. H_λ for each $\lambda \in NA^1$. The calculus $\mathfrak{C}$ is *symmetric* iff it is symmetric w.r.t. $\{\Theta(\lambda)\}_{\lambda \in NA^1}$.

2.3. **Representations of Positive Projections on Massive Spaces of Vector Measure Games.** A massive space of vector measure games Q is *Radon-Nikodym differentiable* (*differentiable* for short) iff it attains a calculus $\mathfrak{C}$ w.r.t. some data set $\widehat{Q}$. A positive projection $\Psi : Q \to FA$ on a differentiable space

[6] $B(X,Y)$ stands for the space of bounded measurable functions from X to Y.
[7] This is equivalent to the property that the range of $\partial_\lambda^{\widehat{Q}}$ is $\partial R_\lambda^{\widehat{Q}}$ for every massive subspace $Q \subseteq R$ with $\widehat{Q} \preceq \widehat{R}$.

Q with a calculus $\mathfrak{C}$ w.r.t. a data set $\widehat{Q}$ is *representable w.r.t.* $\mathfrak{C}$ iff there is a set of finitely additive vector measures $\{P_\lambda\}_{\lambda\in NA^1}$, the *representing measures of* Ψ *w.r.t.* $\mathfrak{C}$, s.t. for every $\lambda \in NA^1$ the vector measure P_λ is a Borel measure on Ω_λ with values in $\mathcal{L}\left(L^\infty(\lambda), L^2(\lambda)\right)$ of bounded semi-variation[8], for every coalition $S \in \mathcal{C}$ the vector measure $P^S_\lambda = \langle P_\lambda, \chi_S\rangle$ is positive, regular, and countably additive of bounded variation, and for every $g \in C(X_\lambda, L^\infty(\lambda)) \cap \partial Q_\lambda$ we have for every $S \in \mathcal{C}$

$$\Psi\left(\int_\lambda (g)\right)(S) = \int_S \left(\int_{\Omega_\lambda} g(x)dP_\lambda(x)\right)(s)d\lambda(s) = \int_{\Omega_\lambda} g(x)dP^S_\lambda(x). \tag{2.2}$$

If no confusion regarding the calculus $\mathfrak{C}$ may result we shall refer to each P_λ as the *representing measure of* Ψ *w.r.t.* λ.

Given representing measures $\{P_\lambda\}_{\lambda\in NA^1}$ of Ψ w.r.t. the calculus $\mathfrak{C}$, and $S \in \mathcal{C}$, define the *cover of the representation w.r.t.* S (or *the cover w.r.t.* S for short) as the set of linear functionals $\widehat{\Gamma}^S = \left\{\widehat{\Gamma}^S_\lambda : B(\Omega_\lambda, L^\infty(\lambda)) \longrightarrow \mathbb{R}\right\}_{\lambda\in NA^1}$ s.t. for each $\lambda \in NA^1$

$$\widehat{\Gamma}^S_\lambda(g) = \int_{\Omega_\lambda} g(x)dP^S_\lambda(x). \tag{2.3}$$

The cover w.r.t. S will prove useful in the study of symmetric positive projections.

2.4. **Main Results.**

We first establish a connection between the existence of a calculus and representations of positive projections on massive spaces of Lipschitz vector measure games.

Theorem 2.1. *Let Q be a differentiable massive space of Lipschitz vector measure games. If $\mathfrak{C}$ is a calculus for Q w.r.t. a data set $\widehat{Q}$ of Q, then every positive projection on Q is representable w.r.t. $\mathfrak{C}$.*

We shall also prove a result which strengthens Theorem 2.1 and Equation (2.2):

Theorem 2.2. *Let Q be a differentiable massive space of Lipschitz vector measure games, and let $\mathfrak{C}$ be a calculus for Q w.r.t. a data set $\widehat{Q}$ of Q. If $\{P_\eta\}_{\eta\in NA}$ are representing measures of a positive projection $\Psi : Q \longrightarrow FA$ w.r.t. $\mathfrak{C}$ and $g \in \partial Q_\lambda$ is $\langle 1, P^I_\lambda\rangle$-a.e. continuous on Ω_λ then for every $S \in \mathcal{C}$*

$$\Psi(\int_\lambda (g))(S) = \int_{\Omega_\lambda} g(x)dP^S_\lambda(x). \tag{2.4}$$

Our third theorem is a symmetric version of the previous theorems:

Theorem 2.3. *If Ψ in Theorem 2.1 is symmetric w.r.t. an Abelian subgroup $G \leq \Theta(\lambda)$ and $\mathfrak{C}$ is a calculus s.t. $\mathfrak{C}_\lambda$ is symmetric w.r.t. G, then the representing vector measure P_λ may be chosen to satisfy*

$$\widehat{\Gamma}^{\tau S}_\lambda = \widehat{\Gamma}^S_\lambda \circ A_\tau \tag{2.5}$$

[8] i.e. $|P_\lambda|(X_\lambda) < \infty$. See Appendix A for details.

for every $\tau \in G$.

While these results establish a connection between the existence of a calculus and representability of positive projections, they do not prove, however, the existence of representable positive projections. In section 4 we turn to proving the existence of an inductive symmetric calculus for the space $\mathcal{CON}$, whose subspaces are of great importance in many economic applications.

For $k \geq 2$ let CON_+^k be the positive cone of Lipschitz continuous, monotonically nondecreasing, and concave functions $f : \mathbb{R}_+^k \to \mathbb{R}$ with $f(0_k) = 0$, and let CON^k be the linear space of differences of members of CON_+^k. Let $\mathcal{CON}$ be the linear space of games of the form $f \circ \mu$ with $f \in CON^k$ and $\mu \in \left(NA^1\right)^k$ for some $k \geq 2$. We choose canonically $\widehat{\mathcal{CON}}_\lambda$ to be the linear space generated by formal linear combinations of pairs (f, μ) with $f \in CON_+^k$ and $\mu \in \mathcal{Z}_\lambda^k$ for some $k \geq 2$. The definition of $\widehat{\mathcal{CON}}$ follows. For $k \geq 2$ denote by HM_+^k the positive cone generated by the functions $f_C(x) = \min\limits_{c \in C} c \cdot x$ with $C \subseteq \Delta^k$ compact and strictly convex, and by HM^k the space of differences of members of HM_+^k. The space $\mathcal{HM}$ and its data set $\widehat{\mathcal{HM}}$ will be defined in a similar manner to $\mathcal{CON}$ and $\widehat{\mathcal{CON}}$, respectively.

Theorem 2.4. $\widehat{\mathcal{CON}}$ *admits an inductive symmetric calculus*

$$\mathfrak{D} = \{\mathfrak{D}_\lambda = \langle X_\lambda, \partial\mathcal{CON}_\lambda, \partial_\lambda, \Phi_\lambda \rangle\}_{\lambda \in NA}.$$

Combining that with Theorems 2.1-2.2 we obtain

Corollary 2.5. *If* $Q \subseteq \mathcal{CON}$ *is a massive space with a data set* $\widehat{Q} \preceq \widehat{\mathcal{CON}}$*, and* $\Psi : Q \to FA$ *is a positive projection then* Ψ *is representable w.r.t.* $\mathfrak{D}^{\widehat{Q}}$*. Furthermore, if* $\{P_\eta\}_{\eta \in NA^1}$ *is a set of representing measures of* Ψ *w.r.t.* $\widehat{Q}$ *and* $g \in \partial\mathcal{CON}_\lambda^{\widehat{Q}}$ *is* $\langle 1, P_\lambda^I \rangle$*-a.e. continuous on* X_λ *then for every* $S \in \mathcal{C}$

$$\Psi(\Phi_\lambda(g))(S) = \int_{X_\lambda} g(x) dP_\lambda^S(x). \tag{2.6}$$

We shall finally turn our efforts to study representations of positive projections, especially values[9], on symmetric massive subspaces $Q \subseteq \mathcal{CON}$ w.r.t. $\mathfrak{D}^{\widehat{Q}}$, where $\widehat{Q} \preceq \widehat{\mathcal{CON}}$. This is done in section 5. We shall prove various results concerning the geometric and measure theoretic symmetries of such representations. Our main result in this section is:

Theorem 2.6. *Suppose* Ψ *is a value on a symmetric massive space* Q *with data set* $\widehat{Q}$ *satisfying* $\widehat{\mathcal{HM}} \preceq \widehat{Q} \preceq \widehat{\mathcal{CON}}$*. Then there are representing measures* $\{P_\eta\}_{\eta \in NA^1}$ *of* Ψ *w.r.t.* $\mathfrak{D}^{\widehat{Q}}$ *s.t. for every coalition* $S \in \mathcal{C}$*,*

[9] I.e. linear, efficient, symmetric, and positive maps.

and every $g \in \partial\mathcal{CON}_{\lambda}^{\widehat{Q}}$

$$\Psi(\Phi_\lambda(g))(S) = \int_{X_\lambda} g(x) dP_\lambda^S(x). \tag{2.7}$$

3. Representations of Positive Projections

Throughout this section we assume that $Q \subseteq BV$ is a massive[10] space of Lipschitz vector measure games, and that $\Psi : Q \to FA$ is a positive projection.

3.1. **Proof of Theorem 2.1.**

Remark 3.1. If $f \in B(\Omega_\lambda, L^\infty(\lambda))$ then for every $x \in \Omega_\lambda$ we have $f(x) \in L^\infty(\lambda)$ and thus we write $\|f(x)\|_\infty$ for the $L^\infty(\lambda)$ norm of $f(x)$. We further denote $\|f\|_\infty = \sup\limits_{x\in\Omega_\lambda} \|f(x)\|_\infty$. By our assumption $\|f\|_\infty < \infty$ and this norm induces the *uniform convergence topology* on $B(\Omega_\lambda, L^\infty(\lambda))$.

Lemma 3.2. *For every* $v \in Q$ *with* $v \asymp \lambda$ *we have* $\Psi(v) \ll \lambda$ *and* $\frac{d\Psi(v)}{d\lambda} \in L^2(\lambda)$.

Proof. As $v + a\lambda \in Q^+$ for any large enough $a > 0$ and as Ψ is a projection, it is sufficient to prove the lemma for the case $v \in Q^+$. Choose $K_v > 0$ s.t. $w = K_v\lambda - v \in Q^+$. Therefore, as v is monotonic and Ψ is a positive projection we obtain (in BV)

$$0 \le \Psi(v) \le K_v\lambda. \tag{3.1}$$

Therefore $\Psi(v) \ll \lambda$ and

$$0 \le \frac{d\Psi(v)}{d\lambda} \le K_v, \tag{3.2}$$

where the inequalities above hold in $L^1(\lambda)$. □

The operator $\gamma_\lambda : \partial Q_\lambda \to L^2(\lambda)$ given by

$$\gamma_\lambda(g) = \frac{d\Psi(\int_\lambda(g))}{d\lambda}, \tag{3.3}$$

is well defined. As the maps $\int_\lambda$ and Ψ are linear and positive then so is γ_λ. By definition the constant functions are contained in ∂Q_λ, thus ∂Q_λ is, by definition, a massive[11] subspace of $B(X_\lambda, L^\infty(\lambda))$ as every $g \in \partial Q_\lambda$ is bounded by $\|g\|_\infty$. Therefore, by Kantorovich's theorem (Theorem A.1, in the Appendix) γ_λ can be extended to a positive linear operator $\Gamma_\lambda : B(\Omega_\lambda, L^\infty(\lambda)) \to L^2(\lambda)$.

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[10] Namely, $NA \subseteq Q$.
[11] See Appendix A for the definition.

If we restrict our attention to the subspace $C(X_\lambda, L^\infty(\lambda))$ of $B(X_\lambda, L^\infty(\lambda))$, then by the Dinculeanu-Singer theorem (Theorem A.4 in the Appendix) we obtain[12] that there exists a unique positive, finitely additive vector measure P_λ of bounded semi-variation defined on the Borel sets of Ω_λ with values in $\mathcal{L}(L^\infty(\lambda), L^2(\lambda))$, s.t. for every $g \in C(\Omega_\lambda, L^\infty(\lambda))$

$$\Gamma_\lambda(g) = \int_{\Omega_\lambda} g(x) dP_\lambda(x). \tag{3.4}$$

By Remark A.5 (in the Appendix) the positivity of the operator Γ_λ yields the positivity of the vector measure P_λ. By property (i) of the Dinculeanu-Singer theorem (Theorem A.4 in the Appendix) for every $S \in \mathcal{C}$ the vector measure $P_\lambda^S = \langle P_\lambda, \chi_S \rangle$ is a positive, regular, and countably additive vector measure on the Borel subsets of Ω_λ with values in $\mathcal{L}(L^\infty(\lambda), \mathbb{R}) \cong (L^\infty(\lambda))^*$, and by definition it has a bounded variation. Now, if $g \in C(X_\lambda, L^\infty(\lambda)) \cap \partial Q_\lambda$ and $S \in \mathcal{C}$ then

$$\Psi\left(\int_\lambda (g)\right)(S) = \int_S \gamma_\lambda(g)(s) d\lambda(s) = \tag{3.5}$$

$$\int_S \left(\int_{\Omega_\lambda} g(x) dP_\lambda(x)\right)(s) d\lambda(s) = \int_{\Omega_\lambda} g(x) dP_\lambda^S(x), \tag{3.6}$$

where the first equality in line (3.5) follows from the definition of γ_λ, the second equality in that line follows from Equation (3.4), and the equality in line (3.6) follows from property (iii) of the Dinculeanu-Singer theorem (Theorem A.4 in the Appendix). This proves Equation (2.2) and completes the proof of Theorem 2.1.

3.2. **Proof of Theorem 2.2.** Suppose $g \in \partial Q_\lambda$ is $\langle 1, P_\lambda^I \rangle$-a.e. continuous, i.e., there is $A \subseteq \Omega_\lambda$ with $\langle 1, P_\lambda^I \rangle(A) = 0$ s.t. g is continuous on $\Omega_\lambda \setminus A$. Notice first that the positivity of the vector measure P_λ entails that $\langle 1, P_\lambda^S \rangle(A) = 0$ for every $S \in \mathcal{C}$. There is a l.s.c. function g_- and an u.s.c. function g_+ on Ω_λ with $g_- \leq g \leq g_+$, and the inequalities hold as equalities on $\Omega_\lambda \setminus A$. By Proposition A.9 (in the Appendix) there are bounded sequences $\left(g_-^n\right)_{n=1}^\infty, \left(g_+^n\right)_{n=1}^\infty \subseteq C(\Omega_\lambda, L^\infty(\lambda))$, s.t. $g_-^n \leq g_-$, and $g_+ \leq g_+^n$ for every $n \geq 1$, and $g_-^n, g_+^n \underset{n\to\infty}{\longrightarrow} g$ pointwise on $\Omega_\lambda \setminus A$ (w.r.t. the $L^\infty(\lambda)$ norm). By the positivity of Γ_λ and P_λ we have

$$\Gamma_\lambda(g_-^n) \leq \Gamma_\lambda(g) \leq \Gamma_\lambda(g_+^n) \Rightarrow$$

$$\forall S \in \mathcal{C}, \quad \int_{\Omega_\lambda} g_-^n(x) dP_\lambda^S \leq \int_S \Gamma_\lambda(g)(s) d\lambda(s) \leq \int_{\Omega_\lambda} g_+^n(x) dP_\lambda^S. \tag{3.7}$$

[12]Notice that in this case every positive linear operator $A : C(\Omega_\lambda, L^\infty(\lambda)) \to L^2(\lambda)$ is bounded; indeed, for every $f \in C(\Omega_\lambda, L^\infty(\lambda))$ and every $x \in \Omega_\lambda$ we have $-\|f\|_\infty \leq f(x) \leq \|f\|_\infty$, thus $-\|f\|_\infty \leq f \leq \|f\|_\infty$. Now $A(f)$ is a member of the Banach lattice $L^2(\lambda)$. By the positivity of A we obtain $|A(f)| \leq \|f\|_\infty |A(1)|$ in the Banach lattice $L^2(\lambda)$ and therefore $\|A(f)\|_2 \leq \|A(1)\|_2 \|f\|_\infty$. Hence A is bounded.

As $\int_S \Gamma_\lambda(g)(s)d\lambda(s) = \Psi\left(\int_\lambda(g)\right)(S)$ we obtain

$$\Psi\left(\int_\lambda(g)\right)(S) = \int_{\Omega_\lambda} g(x)dP_\lambda^S(x), \tag{3.8}$$

where the equality in line (3.8) follows by applying the bounded convergence theorem (Theorem A.3 in the Appendix) to the inequality in line (3.7), and Theorem 2.2 follows.

Remark 3.3. Notice that we have in fact also proved that if $g \in \partial Q_\lambda$ is lower semi-continuous then for every $S \in \mathcal{C}$

$$\Psi\left(\int_\lambda(g)\right)(S) \geq \int_{\Omega_\lambda} g(x)dP_\lambda^S(x). \tag{3.9}$$

If g is upper semi-continuous the inverse inequality holds.

3.3. **Proof of Theorem 2.3.** Denote by $\mathcal{F}$ the set consisting of the extensions of the operator γ_λ to a positive linear operator from $B(\Omega_\lambda, L^\infty(\lambda))$ to $L^2(\lambda)$. Notice that every $\phi \in \mathcal{F}$ is bounded with norm[13] 1, thus $\mathcal{F}$ is norm bounded. It is also a closed subset, in the operator weak* topology[14], of the space $\mathcal{O}_\lambda$ of bounded linear operators from $B(\Omega_\lambda, L^\infty(\lambda))$ to $L^2(\lambda)$. Hence, by Theorem A.6 (in the Appendix) we deduce that $\mathcal{F}$ is also compact in this topology. Furthermore $\mathcal{F}$ is convex.

Define a group action of G on $\mathcal{O}_\lambda$ by

$$\forall \tau \in G, \phi \in \mathcal{O}_\lambda, h \in B(\Omega_\lambda, L^\infty(\lambda)), \quad (\tau, \phi)(h) = \phi(A_\tau(h)) \circ \tau^{-1} \tag{3.10}$$

This group action maps $\mathcal{F}$ to itself. Indeed for every $\tau \in G$ and $\phi \in \mathcal{F}$, (τ, ϕ) is a positive linear operator from $B(\Omega_\lambda, L^\infty(\lambda))$ to $L^2(\lambda)$, and for every $g \in \partial Q_\lambda$ we have

$$(\tau, \phi)(g) = \phi(A_\tau(g)) \circ \tau^{-1} = \gamma_\lambda(A_\tau(g)) \circ \tau^{-1} = \frac{d\Psi\left(\int_\lambda(A_\tau(g))\right)}{d\lambda} \circ \tau^{-1} = \tag{3.11}$$

$$\frac{d\Psi\left(\tau^* \int_\lambda(g)\right)}{d\lambda} \circ \tau^{-1} = \frac{d\Psi\left(\int_\lambda(g)\right)}{d\lambda} = \gamma_\lambda(g), \tag{3.12}$$

where the last equality in line (3.11) follows as, by assumption, $\int_\lambda(A_\tau(g)) = \tau^* \int_\lambda(g)$ and the first equality in line (3.12) follows from the symmetry axiom. Hence $G(\mathcal{F}) \subseteq \mathcal{F}$. Notice now that for $\tau \in G$ the map $\phi \mapsto (\tau, \phi)$ defined on $\mathcal{O}_\lambda$ is continuous. Indeed, if $\phi_\beta \xrightarrow[\beta \in B]{} \phi$ is a net in $\mathcal{O}_\lambda$ converging to $\phi \in \mathcal{O}_\lambda$ in the weak* operator topology, then for every $h \in B(\Omega_\lambda, L^\infty(\lambda))$ and $\chi \in L^2(\lambda)$ we have $\langle(\phi_\beta - \phi)(h), \chi\rangle \xrightarrow[\beta \in B]{} 0$,

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[13] Indeed, as in footnote 11 on page 12 we have $\|\phi(g)\|_2 \leq \|\phi(1)\|_2 \|g\|_\infty$. Since $\phi(1) = \gamma_\lambda(1) = 1$ then $\|\phi\| = 1$.

[14] Namely, the weakest topology on $\mathcal{O}_\lambda$ s.t. the maps

$$\phi \overset{g}{\mapsto} \phi(g), \quad \phi \in \mathcal{O}_\lambda,\ g \in B(\Omega_\lambda, L^\infty(\lambda))$$

are continuous for every $g \in B(\Omega_\lambda, L^\infty(\lambda))$.

hence

$$\langle(((\tau,\phi)-(\tau,\phi_\beta))(h),\chi\rangle = \langle(\phi-\phi_\beta)(A_\tau(h))\circ\tau^{-1},\chi\rangle = \tag{3.13}$$
$$\langle(\phi-\phi_\beta)(A_\tau(h)),\chi\circ\tau\rangle \underset{\beta\in B}{\longrightarrow} 0.$$

Now, the action of G induces a commuting family of continuous linear mappings on $\mathcal{O}_\lambda$ which maps its compact and convex subset $\mathcal{F}$ to itself. Hence by Markov-Kakutani fixed point theorem (Theorem A.8 in the Appendix) there is some $\phi_0 \in \mathcal{F}$ with $(\tau,\phi_0)=\phi_0$ for every $\tau\in G$. Take $\Gamma_\lambda=\phi_0$ in the proof of Theorem 2.1 and let P_λ be the representing measure of the restriction of this operator to $C(\Omega_\lambda, L^\infty(\lambda))$ (given by Equation (3.4)). For every $g\in C(\Omega_\lambda, L^\infty(\lambda))$, every $S\in\mathcal{C}$, and every $\tau\in G$ we thus have

$$\int_{\Omega_\lambda} g(x)dP_\lambda^{\tau S}(x) = \int_{\Omega_\lambda} A_\tau(g)(x)dP_\lambda^S(x). \tag{3.14}$$

For $h\in B(\Omega_\lambda, L^\infty(\lambda))$ take a uniformly bounded sequence $(g_m)_{m=1}^\infty \subseteq C(\Omega_\lambda, L^\infty(\lambda))$ converging $\langle 1, P_\lambda^I\rangle$-a.e. to h. By applying the bounded convergence theorem (Theorem A.3 in the Appendix) to Equation (3.14) we obtain for every $\tau\in G$ and $S\in\mathcal{C}$ as $m\to\infty$

$$\int_{\Omega_\lambda} h(x)dP_\lambda^{\tau S}(x) = \int_{\Omega_\lambda} A_\tau(h)(x)dP_\lambda^S(x).$$

Hence

$$\widehat{\Gamma}_\lambda^{\tau S} = \widehat{\Gamma}_\lambda^S \circ A_\tau,$$

which proves the theorem.

4. An Inductive and Symmetric Calculus for $\mathcal{CON}$

4.1. Superdifferentials of Lipschitz Continuous Concave Functions. Given a function $f\in CON_+^k$, a point $x\in\mathbb{R}_{++}^k$, and $y\in\mathbb{R}^k$, the directional derivative $df(x,y)$ of f at x in the direction y is given by

$$df(x,y) = \lim_{\varepsilon\searrow 0}\frac{f(x+\varepsilon y)-f(x)}{\varepsilon}. \tag{4.1}$$

The limit exists as f is concave. The limit in line (4.1) thus also exists for every $f\in CON^k$.

For $f\in CON_+^k$ and $x\in\mathbb{R}_{++}^k$, $p\in\mathbb{R}^k$ is a *supergradient* of f at x iff

$$\forall y\in\mathbb{R}_+^k,\quad f(y)-f(x)\le p\cdot(y-x). \tag{4.2}$$

The set of all supergradients of f at x is denoted by $\partial f(x)$. It is well known (e.g. [22, Theorem 23.4]) that for every $x \in \mathbb{R}^k_{++}$ and $y \in \mathbb{R}^k$

$$df(x, y) = \min_{p \in \partial f(x)} p \cdot y. \tag{4.3}$$

For every $f \in CON^k_+$ and $x \in \mathbb{R}^k_{++}$ the function $df(x, \cdot)$ is concave. The set of supergradients of $df(x, \cdot)$ at $y \in \mathbb{R}^k$ is

$$[\partial f(x)]_y = \left\{ p' \in \partial f(x) \ : \ p' \cdot y = \min_{p \in \partial f(x)} p \cdot y \right\}. \tag{4.4}$$

Thus the directional derivative of $df(x, \cdot)$ at $y \in \mathbb{R}^k$ in the direction $z \in \mathbb{R}^k$ given by

$$df(x, y, z) = \lim_{\varepsilon \searrow 0} \frac{df(x, y + \varepsilon z) - df(x, y)}{\varepsilon}, \tag{4.5}$$

exists, and

$$df(x, y, z) = \min_{p \in [\partial f(x)]_y} p \cdot z. \tag{4.6}$$

Remark 4.1. Let $\chi : I \to \mathbb{R}^k$. For every $x \in \mathbb{R}^k_{++}$ and $y \in \mathbb{R}^k$ denote by $df(x, y, \chi)$ the function from I to $\mathbb{R}$ given by

$$\forall s \in I, \ df(x, y, \chi)(s) = df(x, y, \chi(s)). \tag{4.7}$$

Remark 4.2. Let W be a subspace of $\mathbb{R}^k$ with $W \cap \mathbb{R}^k_{++} \neq \emptyset$. For every function $f \in CON^k$ denote $f_W = f|_{\mathbb{R}^k_+ \cap W}$. Then f_W is Lipschitz continuous. By Rademacher's theorem f_W is Fréchet-differentiable a.e. in $\mathbb{R}^k_+ \cap W$ w.r.t. the Lebesgue measure on W, i.e. for a.e. $x \in \mathbb{R}^k_+ \cap W$ there is (a unique) $\nabla f_W(x)$ with $df_W(x, y) = \nabla f_W(x) \cdot y$ for every direction $y \in W$. We denote the set of differentiability points of f_W in $\mathbb{R}^k_+ \cap W$ by D_{f_W}. Furthermore, if $x \in \mathbb{R}^k_+ \cap W$ and $L_x(v)$ is the half line through x in direction[15] $v \in \mathbb{S}^{k-1} \cap W$ in W (i.e. $L_x(v) = \{x + tv : t \in \mathbb{R}_+\}$) then for a.e. direction $v \in \mathbb{S}^{k-1} \cap W$ (w.r.t. the Haar measure on the sphere $\mathbb{S}^{k-1} \cap W$) the set $\{t \in \mathbb{R}_+ : x + tv \in (D_{f_W})^c\}$ is of Lebesgue measure 0 (in $\mathbb{R}$). This follows immediately from the fact that $W \cap \mathbb{R}^k_+ = \bigcup_{v \in \mathbb{S}^{k-1} \cap W} \left(L_x(v) \cap R^k_+\right)$ and $L_x(v) \cap L_x(v') = \{x\}$ for $v \neq v' \in \mathbb{S}^{k-1}$, as the set $\left(\mathbb{R}^k_+ \cap W\right) \setminus D_{f_W}$ is of Lebesgue measure 0 (in W).

4.2. **The Direction Space X_λ and its Properties.** Let $NA^* = \bigcup_{k=1}^{\infty} \left(NA^1\right)^k$ and for $\lambda \in NA^1$ let $\mathcal{Z}^*_\lambda = \bigcup_{k=1}^{\infty} \mathcal{Z}^k_\lambda$. For every $\mu \in NA^*$ with $\mu \in \left(NA^1\right)^m$ denote $k(\mu) = m$. Recall that for every $\mu \in NA^*$ we denoted by Λ_μ the set $\left(\mathcal{R}(\mu) \setminus D^{k(\mu)}\right) \sqcup \left([0,1]\mathbf{1}_{k(\mu)} + \mathbb{S}^{k(\mu)}_\perp \cap (AF(\mu)\right)$ endowed with a topology $\mathcal{T}_\mu$ which

[15]Where $\mathbb{S}^{k-1}$ is the Euclidean unit sphere in $\mathbb{R}^k$.

makes it homeomorphic to[16]

$$[0,1]\mathbf{1}_{k(\mu)} + \left\{x \in (\mathbf{1}_{k(\mu)})^{\perp} : \|x\|_2 \in [1,2] \cup \{0\}\right\} \cap AF(\mu)$$

with its Euclidean topology. Let

$$Z_\lambda = \prod_{\mu \in \mathcal{Z}_\lambda^*} \Lambda_\mu \tag{4.8}$$

be endowed with the product topology. Every $z \in Z_\lambda$ has the form

$$z = (z(\mu))_{\mu \in \mathcal{Z}_\lambda^*},$$

where for every $\mu \in \mathcal{Z}_\lambda^*$ we have $z(\mu) \in \Lambda_\mu$.

Let Y_λ be the topological space with the underlying space $B_+^1(I, \mathcal{C})$ and the weakest topology s.t. the map $T : Y_\lambda \to Z_\lambda$, given by $T(y) = (\mu(y)))_{\mu \in \mathcal{Z}_\lambda^*}$, is continuous. For matters of consistency with future notation, we let $y(\mu) = \mu(y)$.

Choose the closure X_λ of $T(Y_\lambda)$ in Z_λ to be the direction space with perspective λ. Every vector $x \in X_\lambda$ is therefore of the form $(x(\mu))_{\mu \in \mathcal{Z}_\lambda^*}$ with $x(\mu) \in \Lambda_\mu$ for every $\mu \in \mathcal{Z}_\lambda^*$.

Denote

$$X_\lambda^\perp = \left\{x \in X_\lambda : \forall \mu \in \mathcal{Z}_\lambda^*,\;\; x(\mu) \in [0,1] \times \mathbb{S}_\perp^{k(\mu)}\right\}. \tag{4.9}$$

As $T(t) \in X_\lambda^\perp$ for every $t \in [0,1]$ we have $X_\lambda^\perp \neq \emptyset$. In fact, this set is much more vast; for $m \geq 1$ and $x \in \mathbb{R}^m$ denote[17] $\Upsilon^m(x) = \frac{x - \overline{x}\mathbf{1}_m}{\|x - \overline{x}\mathbf{1}_m\|_2}$. If $(y_\beta)_{\beta \in B} \subseteq Y_\lambda$ is a net with $T(y_\beta) \underset{\beta \in B}{\longrightarrow} x \in X_\lambda$ then for any $t \in [0,1)$ we can construct a net $(z_{\beta,\tau}^t = t + \tau y_\beta)_{(\beta,\tau) \in B \times (0,1-t)}$ s.t. for every $\mu \in \mathcal{Z}_\lambda^*$, $T(z_{(\beta,\tau)})(\mu)$ converges to $t\mathbf{1}_{k(\mu)} + \Upsilon^{k(\mu)}(x(\mu)) \in [0,1]\mathbf{1}_{k(\mu)} + \mathbb{S}_\perp^{k(\mu)}$. Hence $T(z_{(\beta,\tau)})$ converges to a nontrivial element in $X_\lambda^\perp$ whenever $x(\mu) \notin D^{k(\mu)}$ for some $\mu \in \mathcal{Z}_\lambda^*$. Obviously the spaces X_λ and $X_\lambda^\perp$ are compact and Hausdorff.

4.3. **Calculus for $\mathcal{CON}$.** For every $f \in CON_+^k$, $\mu \in \mathcal{Z}_\lambda^k$, and $\xi \in L^\infty(\lambda)$ define $g(f,\mu,\xi) : X_\lambda \to L^\infty(\lambda)$ as follows: If $dim(AF(\mu)) = 1$, let

$$g_\lambda(f,\mu,\xi)(x) = df\left(x(\mu) + (1 - sign\left(\overline{x(\mu)}\right))\mathbf{1}_k, \frac{d\mu}{d\lambda}\right) \tag{4.10}$$

[16]Consult p. 4 for further details.

[17]The convention $\frac{0_m}{0} = 0_m$ is used.

for every $x \in X_\lambda$. Otherwise, let

$$g_\lambda(f,\mu,\xi)(x) = \begin{cases} \xi, & x(\mu) \in D^k \\ df\left(x(\mu), \frac{d\mu}{d\lambda}\right), & x(\mu) \in \mathcal{R}(\mu) \setminus D^k, \\ df\left(\overline{x(\mu)}\mathbf{1}_k, \Upsilon^k(x(\mu)) + (1 - sign\left(\overline{x(\mu)}\right))\mathbf{1}_k, \frac{d\mu}{d\lambda}\right), & otherwise. \end{cases} \tag{4.11}$$

Obviously $g_\lambda(f,\mu,\xi)$ is well defined.

Remark 4.3. Notice that $g_\lambda(f,\mu,\xi)(x) \in L^\infty(\lambda)$ for every $x \in X_\lambda$. Indeed, let $K \subseteq \mathbb{R}^k$ be a compact set s.t. $\frac{d\mu}{d\lambda}(s) \in K$ λ-a.e., and let M be the Lipschitz constant of f (w.r.t. the ℓ_1^k norm). Then for every $x \in X_\lambda$ we have

$$\|g_\lambda(f,\mu,\xi)(x)\|_\infty \leq \max\left\{M\sum_{i=1}^{k}\left\|\frac{d\mu_i}{d\lambda}\right\|_\infty, \|\xi\|_\infty\right\}.$$

Remark 4.4. Notice that if $f \in CON_+^k$, $\mu \in \mathcal{Z}_\lambda^k$, and $\xi \in L^\infty(\lambda)$ then:

(1) If $dim(AF(\mu)) = 1$ then for every $y \in Y$ with $\overline{\mu(y)} > 0$

$$g_\lambda\left(f,\mu,\xi\right)\left(T(y)\right) = df\left(\mu(y), \frac{d\mu}{d\lambda}\right). \tag{4.12}$$

(2) If $dim(AF(\mu)) \geq 2$ then for every $y \in Y$ with $\mu(y) \notin D^k$

$$g_\lambda\left(f,\mu,\xi\right)\left(T(y)\right) = df\left(\mu(y), \frac{d\mu}{d\lambda}\right). \tag{4.13}$$

Let $\partial\mathcal{CON}_\lambda$ be the linear space spanned by the functions $g_\lambda(f,\mu,\xi)$ with $f \in CON_+^k$, $\mu \in \mathcal{Z}_\lambda^k$, and $\xi \in L^\infty(\lambda)$. For every $\sum_{i=1}^{n} a_i g_\lambda(f_i,\mu^i,\xi^i) \in \mathcal{CON}_\lambda$ define

$$\Phi_\lambda\left(\sum_{i=1}^{n} a_i g_\lambda(f_i,\mu^i,\xi^i)\right) = \sum_{i=1}^{n} a_i f_i \circ \mu^i. \tag{4.14}$$

Remark 4.5. We shall denote $g_\lambda(f,\mu) = g_\lambda\left(f,\mu,df\left(\mathbf{1}_k, \frac{d\mu}{d\lambda}\right)\right)$. As one may assume, we shall choose $\partial\mathcal{CON}_\lambda$ as our space of "derivatives" w.r.t. λ. One may wonder why not consider the space generated be the functions $g_\lambda(f,\mu)$? It will turn out in section 5 that our choice yields several desirable properties for the representing measures $\{P_\lambda\}_{\lambda \in NA^1}$.

In the following, for every $f_1 : \mathbb{R}^k \to \mathbb{R}$ and $f_2 : \mathbb{R}^\ell \to \mathbb{R}$ define $f_1 \oplus f_2 : \mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}$ by $(f_1 \oplus f_2)(x,y) = f_1(x) + f_2(y)$.

Lemma 4.6. *The map Φ_λ is well defined and linear.*

Proof. It is sufficient to prove that the map is well defined. Let $h = \sum_{i=1}^{n} a_i g_\lambda(f_i, \mu^i, \xi^i) \in \partial\mathcal{CON}_\lambda$, with $f_i \in CON_+^{k_i}$, $\mu^i \in \mathcal{Z}_\lambda^{k_i}$, $\xi^i \in L^\infty(\lambda)$, and $a_i \in \mathbb{R}$ for every $1 \leq i \leq n$. Let $k = \sum_{i=1}^{n} k_i$, $F = \bigoplus_{i=1}^{n} a_i f_i$, and $\mu = (\mu^1, ..., \mu^n)$. Then $F \in CON^k$, $\mu \in \mathcal{Z}_\lambda^k$, and hence $F \circ \mu \in \mathcal{CON}$. It is sufficient to prove that $F \circ \mu$ is determined by the values of h alone on $T(Y_\lambda)$. Recall that $F_{AF(\mu)}$ denotes the restriction of F to $AF(\mu) \cap \mathbb{R}_+^k$, and that $D_{F_{AF(\mu)}}$ denotes the set of $x \in \mathbb{R}_+^k \cap AF(\mu)$ where $F_{AF(\mu)}$ is Fréchet differentiable.

Suppose $y \in Y_\lambda$ satisfies $\mu(y) \in D_{F_{AF(\mu)}}$ and $\mu^i(y) \notin D^{k_i}$ whenever $dim(AF(\mu^i)) \geq 2$. Then

$$F(\mu(y)) = F_{AF(\mu)}(\mu(y)) = \int_0^1 dF_{AF(\mu)}(s\mu(y), \mu(y))ds = \tag{4.15}$$

$$\int_0^1 \nabla F_{AF(\mu)}(s\mu(y)) \cdot \mu(y)ds = \int_0^1 \left(\int_I \nabla F_{AF(\mu)}(s\mu(y)) \cdot \frac{d\mu}{d\lambda}(t)y(t)d\lambda(t) \right) ds = \tag{4.16}$$

$$\int_0^1 \left(\int_I dF_{AF(\mu)} \left(s\mu(y), \frac{d\mu}{d\lambda}(t) \right) y(t)d\lambda(t) \right) ds =$$

$$\int_0^1 \int_I \left(\sum_{i=1}^{n} df_i \left(s\mu^i(y), \frac{d\mu^i}{d\lambda}(t) \right) \right) y(t)d\lambda(t)ds = \tag{4.17}$$

$$\int_I \left(\sum_{i=1}^{n} \int_0^1 g_\lambda(f_i, \mu^i, \xi^i)(T(sy))(t)ds \right) y(t)d\lambda(t) = \tag{4.18}$$

$$\int_I \left(\int_{\left\{ s\in[0,1]: s\mu(y) \in D_{F_{AF(\mu)}} \right\}} h(T(sy))ds \right) (t)y(t)d\lambda(t),$$

where the last equalities in lines (4.15) and (4.16) above follow as $\mu(y) \in D_{F_{AF(\mu)}}$, the equality in line (4.17) follows by combining Fubini's theorem, the definition of $g_\lambda(f_i, \mu^i, \xi^i)$, Remark 4.4, and the choice of $y \in Y_\lambda$, and the equality in line (4.18) follows as the set $\left\{ s \in [0,1] : s\mu(y) \in D_{F_{AF(\mu)}} \right\}$ is, by the choice of y, of measure 1. We have thus proved that h determines the values of the $F \circ \mu$ on the set

$$E(\mu, F) =$$
$$\left\{ y \in Y_\lambda : \mu(y) \in D_{F_{AF(\mu)}} \right\} \cap \left\{ y \in Y_\lambda : \forall 1 \leq i \leq n,\ dim(AF(\mu^i)) \geq 2 \to \mu^i(y) \notin D^{k_i} \right\}.$$

The set $E(\mu, F)$ is dense[18] in Y_λ (w.r.t. the norm topology) and $F \circ \mu$ is continuous on Y_λ (w.r.t. the norm topology). Thus $F \circ \mu$ is determined on Y_λ by the values of h and the choices[19] of μ and F. Notice that for different choices of F, F', μ, μ' the set $E = E(\mu, F) \cap E(\mu', F')$ is dense[20] in Y_λ (w.r.t. the norm topology)

[18]By Remark 4.2 the set $\left\{ x \in \mathcal{R}(\mu) : x \in D_{F_{AF(\mu)}} \right\}$ is dense in $\mathcal{R}(\mu)$ and therefore $\left\{ y \in Y_\lambda : \mu(y) \in D_{F_{AF(\mu)}} \right\}$ is dense in Y_λ (w.r.t. the norm topology). Indeed, the set $\{y \in Y_\lambda : \forall 1 \leq i \leq k,\ dim(AF(\mu^i)) \geq 2 \to \mu^i(y) \notin D^{k_i}\}$ is the intersection of Y_λ with the complement of a union of finitely many proper subspaces (as $dim(AF(\mu^i) \geq 2$ for at least one $1 \leq i \leq n$) of $B(I, \mathcal{C})$, hence it is an open dense set in Y_λ (w.r.t. the norm topology), which proves that $E(\mu, F)$ is dense.

[19]Namely, the choices of $\mu^1, ..., \mu^n$, $f_1, ..., f_n$, and $a_1, ..., a_n$.

[20]That follows using Remark 4.2 and footnote 18 above.

and for every $y \in E$ we have $F \circ \mu(y) = F' \circ \mu'(y)$ by combining Equations (4.15)-(4.18). Therefore $F \circ \mu$ is determined by the values of h alone. □

Lemma 4.7. *The map Φ_λ is order preserving.*

Proof. Let $h = \sum_{i=1}^{n} a_i g_\lambda(f_i, \mu^i, \xi^i)$ with $f_i \in CON^{k_i}$, $\mu^i \in \mathcal{Z}_\lambda^{k_i}$, $\xi^i \in L^\infty(\lambda)$, and $a_i \in \mathbb{R}$ for every $1 \le i \le n$. Let $k = \sum_{i=1}^{n} k_i$, $F = \bigoplus_{i=1}^{n} f_i$, and $\mu = (\mu^1, ..., \mu^n)$. Then $F \in CON^k$ and $\mu \in \mathcal{Z}_\lambda^k$.

Recall that $F_{AF(\mu)}$ denotes the restriction of F to $AF(\mu) \cap \mathbb{R}_+^k$ and that $D_{F_{AF(\mu)}}$ denotes the set of $x \in \mathbb{R}_+^k \cap AF(\mu)$ where $F_{AF(\mu)}$ is Fréchet differentiable. By Remark 4.2 if $x \in AF(\mu) \cap \mathbb{R}_+^k$ then for a.e. $x' \in AF(\mu) \cap \mathbb{R}_+^k$ with $x' \neq x$ we have $z \in D_{F_{AF(\mu)}}$ for a.e. $z \in [x, x']$. Thus, for a.e. $x' \in AF(\mu) \cap \mathbb{R}_+^k$ it holds that for a.e. $z \in [x, x']$ we have for every $y \in AF(\mu)$.

$$dF_{AF(\mu)}(z, y) = \nabla F_{AF(\mu)}(z) \cdot y. \tag{4.19}$$

Choose $y, y' \in Y_\lambda$ with $y \le y'$ and $\|\mu(y' - y)\|_2 > 0$. For every $\ell \ge 1$ there is[21] some $y'_\ell \in Y_\lambda$ s.t. the following properties hold:

i. $\mu(y) \neq \mu(y'_\ell)$, and $y'_\ell - y \ge -\frac{1}{\ell}$;
ii. $\|\mu(y'_\ell) - \mu(y')\|_2 \le \frac{1}{\ell}$;
iii. Equation (4.19) holds for a.e. $z \in [\mu(y), \mu(y'_\ell)]$; and
iv. $\mu^i(y'_\ell) \notin D^{k_i}$ whenever $dim(AF(\mu^i)) \ge 2$.

Thus

$$F(\mu(y'_\ell)) - F(\mu(y)) = F_{AF(\mu)}(\mu(y'_\ell)) - F_{AF(\mu)}(\mu(y)) = \tag{4.20}$$

$$\int_0^1 dF_{AF(\mu)}(\mu(y) + t\mu(y'_\ell - y), \mu(y'_\ell - y))dt = \tag{4.21}$$

$$\int_0^1 \nabla F_{AF(\mu)}(\mu(y) + t\mu(y'_\ell - y)) \cdot \mu(y'_\ell - y)dt =$$

$$\int_0^1 \left(\int_I \nabla F_{AF(\mu)}(\mu(y + t(y'_\ell - y))) \cdot \frac{d\mu}{d\lambda}(s)(y'_\ell - y)(s)d\lambda(s) \right) dt = \tag{4.22}$$

$$\int_0^1 \left(\int_I dF_{AF(\mu)} \left(\mu(y + t(y'_\ell - y)), \frac{d\mu}{d\lambda}(s) \right) (y'_\ell - y)(s)d\lambda(s) \right) dt =$$

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[21] Choose some $0 < \epsilon_\ell < \min\left\{ \frac{1}{2\sqrt{k}\ell}, \frac{1}{4\sqrt{k}} \|\mu(y') - \mu(y)\|_2 \right\}$ and consider the set $E_\ell = \mu(y') + \epsilon_\ell \mathcal{R}(\mu) \subseteq AF(\mu)$. Let $x = \mu(y)$ and let $A(x)$ be the set of points $w \in AF(\mu)$ s.t. 1. Equation (4.19) hold for a.e. $z \in [x, w]$, and 2. $w^i \notin D^{k_i}$ whenever $dim(AF(\mu^i)) \ge 2$ (where $w = (w^1, ..., w^n) \in \prod_{i=1}^{n} \mathbb{R}^{k_i}$). Then $A(x)$ is dense in $AF(\mu)$. As E_ℓ has a nonempty relative interior in $AF(\mu)$, there is a point $x' = \mu(y') + \epsilon_\ell \mu(z_\ell) \in E_\ell \cap A(x)$ with $z_\ell \in Y_\lambda$ and $\mu(z_\ell) \neq 0_k$. Set $y'_\ell = \frac{y' + \epsilon_\ell z_\ell}{1 + \epsilon_\ell}$. By the choice of ϵ_ℓ and y'_ℓ we have $0 \le y'_\ell \le 1$, $y'_\ell - y \ge -2\epsilon_\ell \ge -\frac{1}{\ell}$, $\|\mu(y'_\ell) - \mu(y'))\|_2 \le \frac{1}{\ell}$, and $\mu(y) \neq \mu(y'_\ell)$. As $(1 + \epsilon_\ell)\mu(y'_\ell) = x' \in A(x)$ then $\mu^i(y'_\ell) \notin D^{k_i}$ whenever $dim(AF(\mu^i)) \ge 2$ and Equation (4.19) holds for a.e. $z \in [\mu(y), \mu(y'_\ell)]$, and we are done.

$$\int_0^1 \int_I \left(\sum_{i=1}^n df_i \left(\mu^i(y + t(y'_\ell - y)), \frac{d\mu^i}{d\lambda}(s) \right) (y'_\ell - y)(s) \right) d\lambda(s)dt = \tag{4.23}$$

$$\int_0^1 \int_I \left(\sum_{i=1}^n g_\lambda(f_i, \mu^i, \xi^i)(T((1-t)y + ty'_\ell))(s)(y'_\ell - y)(s) \right) d\lambda(s)dt =$$

$$\int_0^1 \int_I \left(h(T((1-t)y + ty'_\ell))(s)(y'_\ell - y)(s) \right) d\lambda(s)dt \geq \tag{4.24}$$

$$-\frac{1}{\ell} \int_0^1 \int_I h(T((1-t)y + ty'_\ell))(s) d\lambda(s)dt, \tag{4.25}$$

where the equalities in lines (4.21)-(4.22) follow as Equation (4.19) holds for a.e. $z \in [\mu(y), \mu(y'_\ell)]$, the equality in line (4.23) follows by combining the definition of $g_\lambda(f, \mu^i, \xi^i)$ with Remark 4.4 and the fact that the interval $[\mu^i(y), \mu(y'_\ell)]$ intersects D^{k_i} in at most one point whenever $dim(AF(\mu^i)) \geq 2$, and the inequality in line (4.24) follows by combining the fact that $h \geq 0$ with property (i) of y'_ℓ above. As F is continuous and the sequence $\left(\int_0^1 \int_I h(T(y + t(y'_\ell - y)))(s) d\lambda(s)dt \right)_{\ell=1}^\infty$ is bounded, the lemma follows by taking the limit $\ell \to \infty$ in Equations (4.20)-(4.25). $\square$

Remark 4.8. Notice that for every $f \in HM_+^k$, every $\mu \in \mathcal{Z}_\lambda^k$, and every $\xi \in L^\infty(\lambda)$ we have $g_\lambda(f, \mu, \xi) \in C(X_\lambda, L^\infty(\lambda))$, the space continuous functions from X_λ to $L^\infty(\lambda)$ with its strong topology (see Appendix A for details). If $dim(AF(\mu)) = 1$ then this immediately follows. If $dim(AF(\mu)) \geq 2$, then $g_\lambda(f, \mu, \xi) = \begin{cases} df\left(\mathbf{1}_k, x(\mu), \frac{d\mu}{d\lambda}\right), & x(\mu) \notin D^k \\ \xi, & x(\mu) \in D^k. \end{cases}$ As $(y \mapsto df(\mathbf{1}_k, y, z))_{z \in K}$ is a family of equicontinuous function on $\Lambda_\mu \backslash D^k$ for every compact $K \subseteq R_+^k$, $D^k \cap \Lambda_\mu$ is a connected component of Λ_μ, and $x \mapsto x(\mu)$ is continuous on X_λ it follows that $g_\lambda(f, \mu, \xi) \in C(X_\lambda, L^\infty(\lambda))$.

4.4. **Proof of Theorem 2.4.** For each $\lambda \in NA^1$, X_λ is a compact Hausdorff space. By its construction, for every $\mu \in \mathcal{Z}_\lambda^k$ the diagram

$$\begin{tikzcd} B_+^1(I, \mathcal{C}) \arrow[rr, "T"] \arrow[dr, "\mu"'] & & X_\lambda \arrow[dl, "\pi_\mu"] \\ & \Lambda_\mu & \end{tikzcd}$$

is commutative, hence X_λ is a generalized direction space with perspective λ. Notice that for $\theta \in \Theta(\lambda)$, $(\theta x)(\mu) = x(\theta^* \mu)$ defines a group action of $\Theta(\lambda)$ on X_λ s.t. $\pi_{\theta^* \mu} = \pi_\mu \circ \theta$.

By construction Φ_λ is surjective and by Lemma 4.7 it is order preseving. Define $\partial_\lambda : \widehat{Q}_\lambda \to \partial\mathcal{CON}_\lambda$ by $\partial_\lambda(\sum_{i=1}^{n} a_i(f_i,\mu^i)) = \sum_{i=1}^{n} a_i g_\lambda(f_i,\mu^i)$. This linear map is well defined and as

$$\Phi_\lambda \circ \partial_\lambda \left(\sum_{i=1}^{n} a_i(f_i,\mu^i) \right) = \sum_{i=1}^{n} a_i f_i \circ \mu^i, \tag{4.26}$$

it follows that $\Phi_\lambda \circ \partial_\lambda = \sigma_\lambda$. If $\theta \in \Theta(\lambda)$, $f \in CON_+^k$, $\mu \in \mathcal{Z}_\lambda^k$, and $\xi \in L^\infty(\lambda)$ then $A_\theta(g_\lambda(f,\mu,\xi)) = g_\lambda(f,\theta^*\mu,\theta^*\xi)$, hence

$$\Phi_\lambda(A_\theta(g_\lambda(f,\mu,\xi))) = \theta^*(f\circ\mu), \tag{4.27}$$

and

$$\partial_\lambda(\theta^*(f,\mu)) = A_\theta(g_\lambda(f,\mu)). \tag{4.28}$$

As $\partial_\lambda(f,\mu) = \frac{d(f\circ\mu)}{d\lambda}$ whenever $f\circ\mu \in NA$, and $\Phi_\lambda(\xi)(S) = \int_S \xi(s)d\lambda(s)$ for every $\xi \in L^\infty(\lambda)$ and $S \in \mathcal{C}$ we have thus proved that $\mathfrak{D}$ is a symmetric calculus for $\mathcal{CON}$ w.r.t. $\widehat{\mathcal{CON}}$.

Suppose that $Q \subseteq \mathcal{CON}$ is a massive subspace with data set $\widehat{Q} \preceq \widehat{\mathcal{CON}}$. In order to prove that $\mathfrak{D}^{\widehat{Q}}$ is a calculus for Q w.r.t. $\widehat{Q}$ it is sufficient to verify that the range of $\partial_\lambda^{\widehat{Q}}$ is contained in $\partial\mathcal{CON}_\lambda^{\widehat{Q}}$ for each $\lambda \in NA^1$ (see footnote 7 on p. 6). Indeed, if $(f,\mu) \in \widehat{Q}_\lambda$ then $\Phi_\lambda(g_\lambda(f,\mu)) = f\circ\mu \in Q_\lambda$, and hence $\partial_\lambda((f,\mu)) \in \partial\mathcal{CON}_\lambda^{\widehat{Q}}$. Thus $\mathfrak{D}$ is also inductive and the theorem is proved.

5. Properties of the Representing Measures for $\mathcal{CON}$

Throughout this section we assume that $Q \subseteq \mathcal{CON}$ is a massive subspace with $\widehat{Q} \preceq \widehat{\mathcal{CON}}$.

5.1. **General Properties.** We begin with the following observations:

Remark 5.1. Notice that if $\widehat{\mathcal{HM}} \subseteq \widehat{Q}$ then for every $\mu \in \mathcal{Z}_\lambda^k$ with $dim(AF(\mu)) \geq 2$ we have $\langle 1, P_\lambda\rangle(\{x \in X_\lambda : x(\mu) \in D^k\}) = 0$. Indeed, choose $f \in HM_+^k$, and consider $g_0 = \partial(f,\mu,0)$, and $g_1 = \partial(f,\mu,1)$. Then $g_0, g_1 \in \partial Q_\lambda \cap C(X_\lambda, L^\infty(\lambda))$. As $\Phi_\lambda(g_0) = \Phi_\lambda(g_1)$ we obtain $\Psi_\lambda(g_0) = \Psi_\lambda(g_1)$. As $g_0 = g_1$ outside of the set $\{x \in X_\lambda : x(\mu) \in D^k\}$ we thus obtain from Remark 4.8 and Equation (2.2)

$$\langle 1, P_\lambda\rangle(\left\{x \in X_\lambda : x(\mu) \in D^k\right\}) = \langle 0, P_\lambda\rangle(\left\{x \in X_\lambda : x(\mu) \in D^k\right\}) = 0.$$

Remark 5.2. Notice that for every $E \in \mathcal{B}(X_\lambda)$ and $S \in \mathcal{C}$ we have

$$\langle \chi_S, P_\lambda\rangle(E) = \langle 1, P_\lambda\rangle(E)\chi_S.$$

Indeed, for every $S, T \in \mathcal{C}$ with $\lambda(T \cap S) = 0$ we have

$$0 \leq \langle \chi_S, P_\lambda^T \rangle(E) \leq \langle \chi_S, P_\lambda^T \rangle(X_\lambda) = \int \chi_S(s)\chi_T(s)d\lambda(s) = 0 \Rightarrow \tag{5.1}$$

$$\langle \chi_S, P_\lambda \rangle(E) = \langle \chi_S, P_\lambda \rangle(E)\chi_S,$$

where the second inequality in line (5.1) above follows from the positivity of P_λ and following equality follows as Ψ is a projection. Hence

$$\langle 1, P_\lambda \rangle(E)\chi_S = \langle \chi_S, P_\lambda \rangle(E)\chi_S + \langle \chi_{S^c}, P_\lambda \rangle(E)\chi_S = \langle \chi_S, P_\lambda \rangle(E).$$

Remark 5.3. For every $\phi \in L^\infty(\lambda)$ and $E \in \mathcal{B}(X_\lambda)$ we have

$$\langle \phi, P_\lambda \rangle(E) = \langle 1, P_\lambda \rangle(E)\phi.$$

Indeed, on the one hand the simple functions are dense in $L^\infty(\lambda)$ (e.g. [11, Theorem 6.8]). Thus there is a sequence $(\phi^n)_{n=1}^\infty$ of simple functions with $\lim_{n\to\infty} \|\phi - \phi^n\|_\infty = 0$. By Remark 5.2 we deduce that for every $n \geq 1$ we have $\langle \phi^n, P_\lambda \rangle(E) = \langle 1, P_\lambda \rangle(E)\phi^n$. On the other hand, the vector measure P_λ^T has a bounded variation for every $T \in \mathcal{C}$, thus $\lim_{n\to\infty} \langle \phi^n, P_\lambda^T \rangle(E) = \langle \phi, P_\lambda^T \rangle(E)$. Hence we obtain

$$\forall T \in \mathcal{C}, \quad \langle \phi, P_\lambda^T \rangle(E) = \lim_{n\to\infty} \langle \phi^n, P_\lambda^T \rangle(E) = \lim_{n\to\infty} \int \phi^n(s)\chi_T(s)\langle 1, P_\lambda \rangle(E)(s)d\lambda(s) =$$

$$\int \phi(s)\chi_T(s)\langle 1, P_\lambda \rangle(E)(s)d\lambda(s) \Rightarrow \langle \phi, P_\lambda \rangle(E) = \langle 1, P_\lambda \rangle(E)\phi.$$

5.2. **Properties of P_λ when Ψ is a Value.** Recall that for $S \in \mathcal{C}$, the cover of Ψ w.r.t. S is the set $\left\{\widehat{\Gamma}_\lambda^S : B(X_\lambda, L^\infty(\lambda)) \to \mathbb{R}\right\}_{\lambda \in NA^1}$ of linear functionals given by

$$\widehat{\Gamma}_\lambda^S(g) = \int_{X_\lambda} g(x)dP_\lambda^S(x) \tag{5.2}$$

for every $g \in B(X_\lambda, L^\infty(\lambda))$. Recall that for every $g \in B(X_\lambda, L^\infty(\lambda)$, every $\theta \in \Theta(\lambda)$, and every $x \in X_\lambda$ we defined $A_\theta(g)(x) = g(\theta x) \circ \theta$, where θx is given by $(\theta x)(\mu) = x(\theta^* \mu)$ for every $\mu \in \mathcal{Z}_\lambda^*$. Every A_θ is a bounded linear map from $B(X_\lambda, L^\infty(\lambda))$ to itself.

The following lemma shows that we can choose the representing measures $\{P_\lambda\}_{\lambda \in NA^1}$ in a manner which entails a residue of the efficiency axiom.

Lemma 5.4. *Let $\theta \in \Theta(\lambda)$ be strongly λ-mixing and let P_λ be the representing vector measure of Ψ w.r.t. λ satisfying Equation (2.5) w.r.t. θ. Then for every $g \in \partial\mathcal{CON}_\lambda^{\widehat{Q}}$ we have*

$$\widehat{\Gamma}_\lambda^I(g) = \Phi_\lambda(g)(I). \tag{5.3}$$

Proof. It is sufficient to prove the lemma for $g = g_\lambda(f, \mu, \xi)$ with $f \in CON_+^k$ and $\mu \in \mathcal{Z}_\lambda^k$ for some $k \geq 2$, and $\xi \in L^\infty(\lambda)$. By Remark 3.3 and the efficiency axiom we have

$$f(\mathbf{1}_k) = \Gamma_\lambda^I(g) \geq \widehat{\Gamma}_\lambda^I(g). \tag{5.4}$$

We thus need to prove the inverse inequality. For every Borel set $E \subseteq X_\lambda$ and $\phi \in L^\infty(\lambda)$ with $\int_I \phi(s) d\lambda(s) = 1$ we have

$$\langle \phi\circ\theta^n, P^I_\lambda\rangle(E) = \int_I \phi(\theta^n(s))\langle 1, P_\lambda\rangle(E)(s)d\lambda(s) \underset{n\to\infty}{\longrightarrow} \tag{5.5}$$
$$\left(\int_I \phi(s)d\lambda(s)\right)\left(\int_I \langle 1, P_\lambda\rangle(E)(s)d\lambda(s)\right) = \langle 1, P^I_\lambda\rangle(E),$$

where the equality in the display (5.5) follows by combining Remark 5.3 with the definition of P^I_λ, and the limit follows as θ is strongly mixing. Hence, by Lemma A.7 (in the Appendix), for every $1\le i\le k$ the sequence of measures $\left(\nu^i_n = \langle \frac{d\mu_i}{d\lambda}\circ\theta^n - 1, P^I_\lambda\rangle\right)_{n=1}^{\infty}$ converges to 0 in variation[22]. Also notice that by the concavity and monotonicity of f we may write

$$g(x) \ge f(\mathbf{1}_k) + \sum_{i=1}^k g_i(x)\left(\frac{d\mu_i}{d\lambda} - 1\right) \tag{5.6}$$

for every $x\in X_\lambda$, where $g_i : X_\lambda \to \mathbb{R}_+$ is bounded for every $1\le i\le k$.

Therefore, for every $n\ge 1$

$$\widehat{\Gamma}^I_\lambda(g) = \widehat{\Gamma}^{\theta^n I}_\lambda(g) = \int_{X_\lambda} A_{\theta^n}(g)(x)dP^I_\lambda(x) \ge \tag{5.7}$$
$$f(\mathbf{1}_k) + \sum_{i=1}^k \int_{X_\lambda} g_i(\theta^n x) d\left\langle \left(\frac{d\mu_i}{d\lambda}\circ\theta^n - 1\right), P^I_\lambda\right\rangle(x) =$$
$$f(\mathbf{1}_k) + \sum_{i=1}^k \int_{X_\lambda} g_i(\theta^n x) d\nu^i_n(x) \tag{5.8}$$

where the second equality in line (5.7) follows by Theorem 2.3 and the next inequality follows by combining Equation (5.6) with the positivity of the vector measure P^I_λ. The lemma now follows by taking $n\to\infty$ in Equation (5.8), as for every $1\le i\le k$ the function g_i is bounded on X_λ and $\nu^i_n \underset{n\to\infty}{\longrightarrow} 0$ in variation. □

We refer to any $\{P_\lambda\}_{\lambda\in NA^1}$ which obeys Equation 5.3 as a *canonical representation measures of* Ψ *w.r.t.* $\mathfrak{D}^{\widehat{Q}}$. We have thus proved that for every value Ψ on a massive symmetric space Q with $\widehat{Q}\preceq\widehat{\mathcal{CON}}$ there exists a canonical representation of Ψ w.r.t. $\mathfrak{D}^{\widehat{Q}}$. We are now ready to prove the main theorem of this section.

Proof of Theorem 2.6. Let $\{P_\eta\}_{\eta\in NA^1}$ be canonical representation measures of Ψ w.r.t. $\mathfrak{D}^{\widehat{Q}}$. To prove the theorem, it is sufficient to consider $h = g_\lambda(f,\mu,\xi)\in\partial\mathcal{CON}^{\widehat{Q}}_\lambda$. As $f\in CON^k_+$ for some $k\ge 2$, then by Remark 3.3 we have in this case for every $S\in\mathcal{C}$

$$\Psi(\Phi_\lambda(h))(S) \ge \int_{X_\lambda} h(x)dP^S_\lambda(x). \tag{5.9}$$

[22]As we also have $|\nu^i_n|(E) \le \left(\left\|\frac{d\mu_i}{d\lambda}\right\|_\infty + 1\right)\langle 1, P^I_\lambda\rangle(E)$ for every Borel set $E\subseteq X_\lambda$.

Thus for every $S \in \mathcal{C}$

$$f(\mathbf{1}_k) = \Psi(\Phi_\lambda(h))(I) = \Psi(\Phi_\lambda(h))(S) + \Psi(\Phi_\lambda(h))(S^c) \geq \tag{5.10}$$

$$\int_{X_\lambda} h(x)dP_\lambda^S(x) + \int_{X_\lambda} h(x)dP_\lambda^{S^c}(x) = \int_{X_\lambda} h(x)dP_\lambda^I(x) = f(\mathbf{1}_k), \tag{5.11}$$

where the first equality in line (5.10) follows from the efficiency axiom and the last equality in line (5.10) follows as P_λ is a canonical representation of Ψ w.r.t. λ. By combining Equations (5.9)-(5.11) we obtain

$$\Psi(\Phi_\lambda(h))(S) = \int_{X_\lambda} h(x)dP_\lambda^S(x), \tag{5.12}$$

and the theorem follows. □

A representing measure P_λ is *diagonal* iff it is supported on[23] $X_\lambda^\perp$. Assuming that P_λ is diagonal may seem to be restrictive, but in many important cases this assumption is valid. Our two final results will prove that if the derivative space $\partial\mathcal{CON}_\lambda^{\widehat{Q}}$ is rich enough then, essentially, a canonical and diagonal representing measure is invariant under reflections and assigns a measure zero to hyperplanes. These fundamental symmetries where shown to be important in many applications (e.g. [9],[12]-[13], [19]).

For any $a, b \in \Delta^k$ and $t \in (0,1]$ let $h_{ab}^t \in CON_+^k$ be given by $h_{ab}^t(x) = \min(a \cdot x, b \cdot x, t)$. Denote by $w_{ab} \in \mathbb{S}_\perp^k$ the vector with direction $b - a$.

Lemma 5.5. *Let $\eta \in \mathcal{Z}_\lambda^k$ with $dim(AF(\eta)) \geq 2$. Suppose that $\widehat{\mathcal{HM}} \preceq \widehat{Q}$, that for every $t \in (0,1]$ the set $A_\eta^t = \left\{w_{ab} : a, b \in \Delta^k,\ (h_{ab}^t, \eta) \in \widehat{Q}_\lambda\right\}$ has Haar measure 1 in $\mathbb{S}_\perp^k$, and that $0_k \in A_\eta^t$. Furthermore, suppose that the representing measures $\{P_\lambda\}_{\lambda \in NA^1}$ are canonical and diagonal. Then for every $t \in (0,1]$ it holds that for every $E \in \mathcal{B}(\mathbb{S}_\perp^k)$ we have for λ-a.e. $s \in I$ with $\frac{d\eta}{d\lambda}(s) \notin D^k$*

$$\langle 1, P_\lambda\rangle(\{x \in X_\lambda : x(\eta) \in [0,t]\boldsymbol{1}_k + E\})(s) = \langle 1, P_\lambda\rangle(\{x \in X_\lambda : x(\eta) \in [0,t]\boldsymbol{1}_k - E)\})(s).$$

Proof. Notice first that for $a, b \in \Delta^k$, $t \in (0,1]$, and $z, y \in \mathbb{R}^k$ we have

$$dh_{ab}^t(s\mathbf{1}_k, z + (1 - sign(s))\mathbf{1}_k, y) = \begin{cases} a \cdot y, & w_{ab} \cdot z > 0, \\ b \cdot y, & w_{ab} \cdot z < 0, \\ \min(a \cdot y, b \cdot y), & w_{ab} \cdot z = 0, \end{cases} \tag{5.13}$$

if $s < t$, and

$$dh_{ab}^t(s\mathbf{1}_k, z, y) = 0 \tag{5.14}$$

[23]For the definition of $X_\lambda^\perp$, consult Equation (4.9).

if $s > t$. By Remark 5.1 we have[24] $\langle 1, P_\lambda\rangle(\{x : x(\eta) \in D^k\}) = 0$. It also follows from [17] that $\Psi(h^t_{ab} \circ \eta) = \frac{t}{2}(a+b)\cdot\eta$. Furthermore, denote by F^t_η the subset of A^t_η s.t. $w \in F^t_\eta \Leftrightarrow \langle 1, P^I_\lambda\rangle(\{x : w\cdot x(\eta) = 0\}) = 0$. Then F^t_η is of full Haar measure, and for every $S \in \mathcal{C}$ and for every $w \in F^t_\eta$ it also holds that $\langle 1, P^S_\lambda\rangle(\{x : w\cdot x(\eta) = 0\}) = 0$. By combining Theorem 2.6, Equations (5.13)-(5.14), and the fact that, by assumption, the vector measure P_λ is diagonal, we obtain that if $w_{ab} \in F^t_\eta$ satisfies $w_{ab} \neq 0_k$, then for every $S \in \mathcal{C}$

$$\begin{aligned}\frac{t}{2}(a+b)\cdot\eta(S) = \Psi(h^t_{ab}\circ\eta)(S) = \int_{X_\lambda} \partial(h^t_{ab},\eta)(x)dP^S_\lambda(x) = \\ a\cdot\langle\frac{d\eta}{d\lambda}, P^S_\lambda\rangle\Big(\Big\{x \in X^\perp_\lambda : \overline{x(\eta)} \le t,\ w_{ab}\cdot x(\eta) > 0\Big\}\Big) + \\ b\cdot\langle\frac{d\eta}{d\lambda}, P^S_\lambda\rangle\Big(\Big\{x \in X^\perp_\lambda : \overline{x(\eta)} \le t,\ w_{ab}\cdot x(\eta) < 0\Big\}\Big).\end{aligned} \tag{5.15}$$

On the other hand, if $w_{ab} = 0_k$ then $a = b$ and $h^t_{ab}(x) = \min(a\cdot x, t)$, hence for every $S \in \mathcal{C}$

$$ta\cdot\eta(S) = a\cdot\langle\frac{d\eta}{d\lambda}, P^S_\lambda\rangle\Big(\Big\{x \in X^\perp_\lambda : x(\mu) \in [0,t]\mathbf{1}_k + \mathbb{S}^k_\perp\Big\}\Big) \tag{5.16}$$

Hence, by passing[25] to the Radon-Nikodym derivatives in Equations (5.15)-(5.16), we obtain (the equalities are in $L^2(\lambda)$)

$$\begin{aligned}\frac{t}{2}(a+b)\cdot\frac{d\eta}{d\lambda} = a\cdot\langle\frac{d\eta}{d\lambda}, P_\lambda\rangle\Big(\Big\{x \in X^\perp_\lambda : \overline{x(\eta)} \le t,\ w_{ab}\cdot x(\eta) \ge 0\Big\}\Big) + \\ b\cdot\langle\frac{d\eta}{d\lambda}, P_\lambda\rangle\Big(\Big\{x \in X^\perp_\lambda : \overline{x(\eta)} \le t,\ w_{ab}\cdot x(\eta) \le 0\Big\}\Big),\end{aligned} \tag{5.17}$$

whenever $w_{ab} \in F^t_\eta$ satisfies $w_{ab} \neq 0_k$, and

$$\begin{aligned}ta\cdot\frac{d\eta}{d\lambda} = a\cdot\langle\frac{d\eta}{d\lambda}, P_\lambda\rangle\Big(\Big\{x \in X^\perp_\lambda : x(\mu) \in [0,t]\mathbf{1}_k + \mathbb{S}^k_\perp\Big\}\Big) \Rightarrow \\ \langle 1, P_\lambda\rangle\Big(\Big\{x \in X^\perp_\lambda : x(\mu) \in [0,t]\mathbf{1}_k + \mathbb{S}^k_\perp\Big\}\Big) = t.\end{aligned} \tag{5.18}$$

By rearranging Equation (5.17) and recalling Equation (5.18) we obtain ,whenever $t \in T$ and $w_{ab} \in F^t_\eta$,

$$\begin{aligned}w_{ab}\cdot\langle\frac{d\eta}{d\lambda}, P_\lambda\rangle\Big(\Big\{x \in X^\perp_\lambda : \overline{x(\eta)} \le t,\ w_{ab}\cdot x(\eta) > 0\Big\}\Big) = \\ w_{ab}\cdot\langle\frac{d\eta}{d\lambda}, P_\lambda\rangle\Big(\Big\{x \in X^\perp_\lambda : \overline{x(\eta)} \le t,\ w_{ab}\cdot x(\eta) < 0\Big\}\Big) \Rightarrow\end{aligned} \tag{5.19}$$

$$\begin{aligned}w_{ab}\cdot\frac{d\eta}{d\lambda}\langle 1, P_\lambda\rangle\Big(\Big\{x \in X^\perp_\lambda : \overline{x(\eta)} \le t,\ w_{ab}\cdot x(\eta) > 0\Big\}\Big) = \\ w_{ab}\cdot\frac{d\eta}{d\lambda}\langle 1, P_\lambda\rangle\Big(\Big\{x \in X^\perp_\lambda : \overline{x(\eta)} \le t,\ w_{ab}\cdot x(\eta) < 0\Big\}\Big),\end{aligned} \tag{5.20}$$

[24] As $dim(AF(\eta)) \ge 2$

[25] Which is possible, as the choice of the full measure set on which Equation (5.13) holds was independent of $S \in \mathcal{C}$

where Equation (5.20) follows from Equation (5.19) by Remark 5.3. Denote

$$S_0 = \left\{ s \in I : \frac{d\eta}{d\lambda}(s) \in D^k \right\}.$$

Notice that for a.e. $w \in \mathbb{S}^k_{\perp} \setminus \{0_k\}$ we have

$$\lambda\left(\left\{ s \in S_0^c : w \cdot \frac{d\eta}{d\lambda} = 0 \right\}\right) = 0.$$

Thus, by Equation (5.20) for a.e. $w \in \mathbb{S}^k_{\perp} \setminus \{0_k\}$ we have for λ-a.e. $s \in S_0^c$

$$\begin{aligned} &\langle 1, P_\lambda\rangle(\left\{ x \in X_\lambda^{\perp} : \overline{x(\eta)} \leq t,\ w \cdot x(\eta) \geq 0 \right\})(s) = \\ &\langle 1, P_\lambda\rangle(\left\{ x \in X_\lambda^{\perp} : \overline{x(\eta)} \leq t,\ w \cdot x(\eta) \leq 0 \right\})(s), \end{aligned}$$

and therefore, for every $S \in \mathcal{C}$ with $S \subseteq S_0^c$

$$\begin{aligned} &\langle 1, P_\lambda^S\rangle(\left\{ x \in X_\lambda^{\perp} : \overline{x(\eta)} \leq t,\ , w \cdot x(\eta) \geq 0 \right\}) = \\ &\langle 1, P_\lambda^S\rangle(\left\{ x \in X_\lambda^{\perp} : \overline{x(\eta)} \leq t,\ w \cdot x(\eta) \leq 0 \right\}). \end{aligned} \tag{5.21}$$

By [24, Lemma 2.3], Equation (5.21) implies that for every $S \in \mathcal{C}$ with $S \subseteq S_0^c$ and every Borel set $E \subseteq \mathbb{S}^k_{\perp}$ we have

$$\langle 1, P_\lambda^S\rangle(\left\{ x \in X_\lambda^{\perp} : x(\eta) \in [0,t]\mathbf{1}_k + E \right\}) = \langle 1, P_\lambda^S\rangle(\left\{ x \in X_\lambda^{\perp} : x(\eta) \in [0,t]\mathbf{1}_k - E \right\}). \tag{5.22}$$

Passing to the Radon-Nikodym derivative in Equation (5.22) proves the lemma. □

Corollary 5.6. *Let $\eta \in \mathcal{Z}^k_\lambda$ with $dim(AF(\eta)) \geq 2$ and $t \in (0,1]$. Suppose that $\widehat{\mathcal{HM}} \preceq \widehat{Q}$, that $A^t_\eta = \left\{ w_{ab} : a, b \in \Delta^k,\ (h^t_{ab}, \eta) \in \widehat{Q}_\lambda \right\}$ has Haar measure 1 in $\mathbb{S}^k_{\perp}$, and that $0_k \in A^t_\eta$. Further suppose that the representing measure $\{P_\lambda\}_{\lambda \in NA^1}$ is canonical and diagonal.*

For every non-empty $J \subseteq \{1, ..., k\}$ with $|J| \geq 2$ let

$$F^t_J(\eta) = \left\{ x \in X_\lambda^{\perp} : \overline{x(\eta)} \leq t, x(\eta) \neq 0_k, \forall i, j \in J\ x(\eta)_i = x(\eta)_j \right\},$$

and

$$S_J(\eta) = \left\{ s \in I : \forall i, j \in J\ \frac{d\mu_i}{d\lambda}(s) = \frac{d\mu_j}{d\lambda}(s) \right\}.$$

Then for λ-a.e. $s \in (S_J(\eta))^c$

$$\langle 1, P_\lambda\rangle(F^t_J(\eta))(s) = 0. \tag{5.23}$$

Proof. For every $i \neq j \in J$ denote $h_{ij}^t = h_{e_ie_j}^t(x)$. As $\{P_\lambda\}_{\lambda\in NA^1}$ is a canonical and diagonal representation of Ψ we have

$$t = h_{ij}^t(\mathbf{1}_k) = \Psi(h_{ij}^t \circ \eta)(I) = \int_{X_\lambda^\perp} dh_{ij}^t\left(\mathbf{1}_k, x(\eta), \frac{d\eta}{d\lambda}\right) dP_\lambda^I(x) =$$
$$\langle \frac{d\eta_i}{d\lambda}, P_\lambda^I\rangle(\left\{x \in X_\lambda^\perp : \overline{x(\mu)} \le t,\ x(\eta)_i < x(\eta)_j\right\})+$$
$$\langle \frac{d\eta_j}{d\lambda}, P_\lambda^I\rangle(\left\{x \in X_\lambda^\perp : \overline{x(\mu)} \le t,\ x(\eta)_i > x(\eta)_j\right\}) + \langle \min\left\{\frac{d\eta_i}{d\lambda}, \frac{d\eta_j}{d\lambda}\right\}, P_\lambda^I\rangle(F_{ij}^t(\eta)).$$

By applying Lemma 5.5 to the set $\left\{x \in X_\lambda^\perp : \overline{x(\mu)} \le t,\ x(\eta)_i < x(\eta)_j\right\}$ we obtain

$$t = \langle \frac{1}{2}\frac{d(\eta_i+\eta_j)}{d\lambda}, P_\lambda^I\rangle(\left\{x \in X_\lambda^\perp : \overline{x(\mu)} \le t,\ x(\eta)_i \neq x(\eta)_j\right\})+$$
$$\langle \min\left\{\frac{d\eta_i}{d\lambda}, \frac{d\eta_j}{d\lambda}\right\}, P_\lambda^I\rangle(F_{ij}^t(\eta)).$$

As $\langle 1, P_\lambda\rangle(\{x : x(\eta) = 0_k\}) = 0$ and by Equation (5.16) $\langle 1, P_\lambda\rangle(\left\{x \in X_\lambda^\perp : [0,t]\mathbf{1}_k + \mathbb{S}_\perp^k\right\}) = t$ we obtain

$$\langle \frac{1}{2}\frac{d(\eta_i+\eta_j)}{d\lambda}, P_\lambda^I\rangle(F_{ij}^t(\eta)) = \langle \min\left\{\frac{d\eta_i}{d\lambda}, \frac{d\eta_j}{d\lambda}\right\}, P_\lambda^I\rangle(F_{ij}^t(\eta)) \Rightarrow$$
$$\langle \left|\frac{d(\eta_i-\eta_j)}{d\lambda}\right|, P_\lambda^I\rangle(F_{ij}^t(\eta)) = 0.$$

By Remark 5.3 we deduce that for λ-a.e. $s \in I$

$$\left|\frac{d(\eta_i-\eta_j)}{d\lambda}(s)\right| \langle 1, P_\lambda\rangle(F_{ij}^t(\eta))(s) = 0,$$

hence for λ-a.e. $s \in (S_{ij}(\eta))^c$ we have

$$\langle 1, P_\lambda\rangle(F_{ij}^t(\eta))(s) = 0 \Rightarrow \tag{5.24}$$

$$\langle \chi_{(S_{ij}(\eta))^c}, P_\lambda\rangle(F_{ij}^t(\eta)) = 0 \tag{5.25}$$

where Equation (5.25) follows from Equation (5.24) by Remark 5.3 and the equality in line (5.25) holds in $L^2(\lambda)$.

Fix $i \in J$. Then (in $L^2(\lambda)$)

$$0 \le \langle \chi_{(S_J(\eta))^c}, P_\lambda\rangle\left(F_J^t(\eta)\right) = \sum_{j\in J\setminus i}\langle \chi_{(S_{ij}(\eta))^c}, P_\lambda\rangle\left(F_J^t(\eta)\right) \le \tag{5.26}$$
$$\sum_{j\in J\setminus i}\langle \chi_{(S_{ij}(\eta))^c}, P_\lambda\rangle\left(F_{ij}^t(\eta)\right) = 0,$$

where the first inequality in line (5.26) follows from the positivity of the vector measure P_λ and the last inequality in that line follows by combining the positivity of the vector measure P_λ with the fact that $F^t_J(\eta) \subseteq F^t_{ij}(\eta)$ for every $i \neq j \in J$. The lemma now immediately follows by Remark 5.3. $\square$

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Appendix A. Rudiments of Functional Analysis

Here we give some functional analysis background which is needed to understand the proof of some of our results. For further reading, one is advised to use the references.

A.1. **Extension of Linear Operators.** A *Banach lattice* Z is a Banach space which is also a lattice, whose lattice structure is commensurable with its Banach space topology, i.e., if $0 \leq x \leq y$ then $\|x\| \leq \|y\|$. A Banach lattice is a *K-space* if it is order complete, i.e., if every nonempty $A \subseteq Z$ which is bounded from above (below) has a least (greatest) upper (lower) bound.

Example: For every $1 < p \leq \infty$, every standard measure space $(I, \mathcal{C})$, and every $\lambda \in NA^1$ the space $L^p(\lambda)$ is a K-space. In fact, if X is a Banach lattice then X^* with its positive cone

$$X^*_+ = \{x^* \in X^* : \forall x \in X_+, x^*(x) \geq 0\} \tag{A.1}$$

is a K-space (see [3, p. 162]), and hence every reflexive Banach lattice is a K-space.

A subspace V of a Banach lattice Z is *massive* if for every $z \in Z$ there is a $v \in V$ s.t. $z \leq v$. We will be interested in extending positive linear operators from a subspace $V \leq Z$ into a Banach lattice Y to positive linear operators from Z to Y. The following result solves this problem in the case that V is massive:

Theorem A.1 (Kantorovich)**.** [18, Theorem 3.1.7] *Let Z be a Banach lattice and Y a K-space. Then if V is a massive subspace of Z and $T : V \to Y$ is a positive linear operator then T can be extended to a positive linear operator $\overline{T} : Z \to Y$.*

A.2. **Vector Measures.** A function F from an algebra $\mathfrak{F}$ of subsets of a set Ω to a Banach space Z is a *finitely additive vector measure* or simply a *vector measure* iff whenever $E_1, E_2 \in \mathfrak{F}$ are disjoint then $F(E_1 \cup E_2) = F(E_1) + F(E_2)$. If, in addition, $F\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} F(E_n)$ in the norm topology of Z for all sequences $(E_n)_{n=1}^{\infty}$ of pairwise disjoint members of $\mathfrak{F}$ s.t. $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{F}$ then F is termed a *countably additive vector measure* or simply *countably additive.*

The *strong variation of* F is the function $\|F\| : \mathfrak{F} \to \mathbb{R}$ defined by

$$\|F\|(E) = \sup_{\pi} \sum_{A \in \pi} \|F(A)\|,$$

where the supremum is taken over all finite partitions of E into disjoint members of $\mathfrak{F}$. One may easily check that $\|F\|$ is a monotonic finitely additive measure. A measure F is of *bounded variation* if $\|F\|(\Omega) < \infty$. Furthermore,

Proposition A.2. [4, Proposition I.1.9] *A vector measure of bounded variation is countably additive iff its variation is countably additive.*

A.3. **Integration w.r.t. a Measure with Values in $\mathcal{L}(Y,Z)$.** Let F be a vector measure on an algebra $\mathfrak{F}$ of subsets of Ω with values in the Banach space $\mathcal{L}(Y,Z)$ of bounded linear operators from Y to Z, where Y, Z are Banach lattices. Denote by $\mathcal{S}_{\Omega,\mathfrak{F}}(Y)$ the set of simple functions on Ω w.r.t. $\mathfrak{F}$ taking values in Y, i.e. the set of functions of the form $\sum_{i=1}^{n} a_i\chi_{E_i}$ where $E_i \in \mathfrak{F}$ and $a_i \in Y$ for every $1 \le i \le n$. The (Bartle) integral of such a simple $f = \sum_{i=1}^{n} a_i\chi_{E_i}$ w.r.t. F is given by

$$\int f dF = \sum_{i=1}^{n} F(E_i)(a_i). \tag{A.2}$$

A measurable function $f : \Omega \to Y$ is *strongly F-integrable*, or *integrable* for short, if for every increasing sequence $(f_n)_{n=1}^{\infty}$ of simple functions $f_n : \Omega \to Y$ with $f_n \underset{n\to\infty}{\longrightarrow} f$ pointwise $\|F\|$-a.e. the limit $\nu(E) = \lim\limits_{n\to\infty} \int f_n\chi_E dF$ exists in the strong topology of Z for every $E \in \mathfrak{F}$ and is independent of the choice of $(f_n)_{n=1}^{\infty}$. In that case we denote

$$\int_E f dF = \lim_{n\to\infty} \int_E f_n dF. \tag{A.3}$$

The following theorem is a version of the well-known Bartle bounded convergence theorem:

Theorem A.3 (Bartle bounded convergence theorem)**.** *Let $(f_n)_{n=1}^{\infty}$ be a uniformly bounded sequence of integrable functions $f_n : \Omega \to Y$, and suppose that F above is countably additive of bounded variation. If (f_n) converges $\|F\|$-a.e. to f then f is integrable and*

$$\lim_{n\to\infty} \int f_n dF = \int f dF$$

in the strong topology of Z.

Proof. By Egorof-Lusin's theorem [5, p. 520] for every $\epsilon > 0$ there is a measurable subset $E = E(\epsilon) \subseteq \Omega$ s.t. $\|F(E^c)\| < \epsilon$ and (f_n) converges uniformly to f on E. Let $C > 0$ be s.t. $\sup\limits_{x\in\Omega} \|f_n(x)\| \le C$ for every $n \in \mathbb{N}$. Note that

$$\left\| \int_E f_n dF \right\| \le C \|F\|(E)$$

for every $E \in \mathfrak{F}$, where $\|F\|$ denotes the variation of F. Let $N \in \mathbb{N}$ be s.t. for every $m, n > N$ and every $x \in E$, $\|f_m(x) - f_n(x)\| < \epsilon$. Then for every $m, n > N$ we have

$$\left\| \int f_m dF - \int f_n dF \right\| \leq \left\| \int_E (f_m - f_n) dF \right\| + \left\| \int_{E^c} (f_m - f_n) dF \right\| <$$
$$\epsilon \left\| F \right\| (E) + 2C \left\| F \right\| (E^c).$$

As F is countably additive of finite variation we have $\|F\| (E(\epsilon)^c) \to 0$ as $\epsilon \to 0^+$, hence

$$\lim_{m,n\to\infty} \left\| \int f_m dF - \int f_n dF \right\| = 0, \tag{A.4}$$

proving that the integrals form a Cauchy sequence in Z and hence convergence in its strong topology. As for every sequence of increasing functions $(g_n)_{n=1}^\infty$ converging pointwise to f and $\epsilon > 0$ there is measurable subset E and $N \in \mathbb{N}$ s.t. $|f_n(x) - g_n(x)| < \epsilon$ for every $x \in E$ and $n \geq N$, and as $\|g_n(x)\| \leq \|f(x)\| \leq C$ for every x, we deduce in a similar manner that $\lim_{n\to\infty} \int f_n dF = \lim_{n\to\infty} \int g_n dF$, hence f is integrable, and the rest of the theorem now easily follows. □

A.4. **Representation of Bounded Linear Operators.** Let Z, Y be Banach spaces, Ω a compact and Hausdorff space. If G is a measure on the Borel σ-algebra $\mathcal{B}_\Omega$ of Ω taking values in $\mathcal{L}(Y, Z^{**})$ then for every $z^* \in Z^*$ we define the measure $G_{z^*} : \mathcal{B}_\Omega \to Y^*$ by $G_{z^*}(A)(y) = \langle G(A)(y), z^* \rangle$ where $\langle \cdot, \cdot \rangle$ is the usual pairing. The *semi-variation* $|G|(E)$ of G on $E \in \mathcal{B}_\Omega$ is given by $|G|(E) = \sup\{\|G_{z^*}\| (E) : ||z^*|| \leq 1\}$.

Let $T : C(\Omega, Y) \to Z$ be a bounded linear operator. The following theorem, due to Dinculeanu and Singer, is a fortification of the Riesz representation theorem:

Theorem A.4 (Dinculeanu-Singer)**.** [4, p. 182] *There exists a unique finitely additive measure G of bounded semi-variation (i.e. $|G|(\Omega) < \infty$), defined on $\mathcal{B}_\Omega$ with values in $\mathcal{L}(Y, Z^{**})$ s.t. $T(f) = \int_\Omega f(\omega) dG(\omega)$ and,*

(i) G_{z^*} is a regular and countably additive Borel measure for each $z^* \in Z^*$;

(ii) the mapping $z^* \mapsto G_{z^*}$ of Z^* into[26] $C(\Omega, Y)^*$ is weak* to weak* continuous;

(iii) $\langle T(f), z^* \rangle = \int_\Omega f(\omega) dG_{z^*}(\omega)$, for every $f \in C(\Omega, Y)$ and every $z^* \in Z^*$.

Remark A.5. Notice that if T is positive then its representing measure G is also positive. Indeed, for every $E \in \mathcal{B}_\Omega$ choose a sequence of continuous functions $(f_n)_{n=1}^\infty \subseteq C(\Omega, [0,1])$ with $f_n \xrightarrow[n\to\infty]{} \chi_E$ pointwise. Thus

[26]This space isomorphic to the space of regular countably additive vector measures of bounded variation on $\mathcal{B}_\Omega$ taking values in Y^*.

for every two positive elements $y \in Y$ and $z^* \in Z^*$ we have

$$\langle G(E)(y), z^* \rangle = \lim_{n\to\infty} \int_\Omega (f_n(\omega)y)\, dG_{z^*}(\omega) = \lim_{n\to\infty} \langle T(f_n y), z^* \rangle \geq 0, \tag{A.5}$$

where the first equality in Line (A.5) follows by combining property (i) of Theorem A.4 with the bounded convergence theorem A.3 and the last inequality in that line follows from the positivity of T. Hence $G(E) : Y \to Z^{**}$ is a positive operator for every $E \in \mathcal{B}_\Omega$.

A.5. **Weak* operator topology.** One useful notion in functional analysis is of the weak* topology. Here we introduce the operator weak* topology. Let X, Y be Banach spaces and consider $\mathcal{L}(X, Y^*)$. The *operator weak* topology* on $\mathcal{L}(X, Y^*)$ is the weakest topology in which for every $x \in X$ the map $U \overset{x}{\mapsto} U(x)$ from $\mathcal{L}(X, Y^*)$ to Y^* is continuous w.r.t. to the weak* topology of Y^*. The following is a generalization of the Banach-Alaoglu theorem

Theorem A.6. *The unit ball of $\mathcal{L}(X, Y^*)$ in the strong topology is compact in the operator weak* topology.*

Proof. Denote the unit ball of $\mathcal{L}(X, Y^*)$ by B. Consider the map $\psi : \mathcal{L}(X, Y^*) \to \prod_{x\in X} Y^*$ given by

$$\psi(U) = (U(x))_{x\in X}\,. \tag{A.6}$$

This map is continuous w.r.t. to the operator weak* topology. It is injective as if $\psi(U) = \psi(U')$ we have $U(x) = U'(x)$ for every $x \in X$. Thus B is mapped to a subset W of

$$\prod_{x\in X} \{y^* \in Y^* : \|y^*\| \leq \|x\|\}\,. \tag{A.7}$$

The set given in Equation (A.7) is compact in the product topology, where each Y^* is taken with its weak* topology. Notice that W is also closed, hence compact. Indeed, if $\psi(U_\beta) \underset{\beta\in B}{\longrightarrow} w$ is a converging net in W with $w \in \prod_{x\in X} Y^*$, then the mapping $x \overset{U}{\mapsto} w_x$ from X to Y^*, is linear and is also bounded with $\|U\| \leq 1$, hence $U \in B$, and $\psi(U) = w$, hence W is closed. The inverse mapping from W onto B is also continuous. Indeed, as ψ is injective it is sufficient to prove that $\psi(V)$ is open whenever V is a basic open set of B. By the definition of the operator weak* topology, every basic open subset of B is a finite intersection of B with sets of the form $\psi^{-1}(V')$ where V' is a basic open set of $\prod_{x\in X} Y^*$. But $\psi\left(B \cap \psi^{-1}(V')\right) = V' \cap W$ which is open in the topology induced on W by $\prod_{x\in X} Y^*$, hence ψ^{-1} is continuous. Now $B = \psi^{-1}(W)$ is the continuous image of a compact set, hence compact. $\square$

A.6. **Uniform convergence of measures.**

Lemma A.7. *Let $(I,\mathcal{C})$ be a measure space, $(\mu_n)_{n=1}^{\infty}$ a sequence of signed measures on $(I,\mathcal{C})$, and ν a positive measure on $(I,\mathcal{C})$ s.t. for every $n \geq 1$ and $S \in \mathcal{C}$ $|\mu_n|(S) \leq \nu(S)$, where $|\cdot|$ stands for the variation. Suppose that for every $S \in \mathcal{C}$ we have $\mu_n(S) \underset{n\to\infty}{\longrightarrow} \mu(S)$ for some signed measure μ on $(I,\mathcal{C})$. Then $(\mu_n)_{n=1}^{\infty}$ converges to μ in variation.*

Proof. Notice that $|\mu_n| \ll \nu$, and $|\mu| \ll \nu$. Denote $f_n = \frac{d\mu_n}{d\nu}$, $f = \frac{d\mu}{d\nu}$. As $\mu_n(S) \underset{n\to\infty}{\longrightarrow} \mu(S)$ we deduce that $f_n \underset{n\to\infty}{\longrightarrow} f$ ν-a.e. and as $|f_n - f| \leq 2$ ν-a.e. we have by the dominated convergence theorem $\int_I |f_n(s) - f(s)| d\nu(s) \underset{n\to\infty}{\longrightarrow} 0$, which implies $\mu_n \underset{n\to\infty}{\longrightarrow} \mu$ in variation. □

A.7. **Markov-Kakutani fixed point theorem.** Let X be a locally convex topological vector space. A family $\mathcal{F}$ of linear endomorphisms of X is *commuting* if for every $S, T \in \mathcal{F}$ we have $T \circ S = S \circ T$. The proof of the following well–known theorem may be found in [8, p. 456, Theorem 6]

Theorem A.8. *Let K be a compact convex subset of X, and $\mathcal{F}$ be a commuting family of continuous linear endomorphisms which map K to itself. Then there is $p \in K$ with $T(p) = p$ for every $T \in \mathcal{F}$.*

A.8. **Lower and upper semicontinuous functions with values in a Banach lattice.** Given a compact Hausdorff space Ω and a Banach lattice X, a function $f : \Omega \to X$ is lower-semicontinuous (l.s.c.) iff for every $a \in X$ the set $U_a = \{w \in \Omega : f(w) > a\}$ is open. It is upper-semicontinuous (u.s.c.) iff for every $a \in X$ the set $V_a = \{w \in \Omega : f(w) < a\}$ is open.

Proposition A.9. *Suppose $f : \Omega \to X$ is a bounded l.s.c. (u.s.c.). Then there is a sequence $(f_n : \Omega \to X)_{n=1}^{\infty}$ of continuous functions s.t. $f_n \leq f$ $(f_n \geq f)$ for every $n \geq 1$, and for every $w \in \Omega$, $f_n(w) \underset{n\to\infty}{\longrightarrow} f(w)$ in X.*

Proof. It is sufficient to prove the proposition under the assumptions that f is l.s.c. and $0 \leq f \leq b$ for some $b \in X$. For any $n \geq 1$ consider, for $0 \leq k \leq n-1$, the decreasing open sets $U_{nk} = f^{-1}\left(\left\{x > \frac{kb}{n}\right\}\right)$. Define $g_n = \frac{b}{n}\sum_{k=0}^{n-1} \chi_{U_{nk}}$. Then g_n is l.s.c. and $0 \leq f(w) - g_n(w) \leq \frac{b}{n}$ for each $w \in \Omega$. For given $n, k \geq 1$ choose an increasing sequence of continuous functions h_{nk}^m converging pointwise to $\chi_{U_{nk}}$. Now, for every $w \in \Omega$ there is some $M = M(n,w) \geq 1$ s.t. for every $m \geq M$ we have $0 \leq \chi_{U_{nk}}(w) - h_{nk}^m(w) \leq \frac{1}{n}$ for every $0 \leq k \leq n-1$. Thus for every $m \geq M$

$$0 \leq f(w) - \frac{b}{n}\sum_{k=1}^{n-1} h_{nk}^m(w) \leq \frac{2b}{n}. \tag{A.8}$$

Denote $g_n^m = \frac{b}{n}\sum_{k=1}^{n-1} h_{nk}^m$. Then g_n^m is continuous with $g_n^m \leq g_n \leq f$ for every $n, m \geq 1$. Arrange the family of continuous functions $(g_n^m)_{n,m\geq 1}$ in a sequence, say $(r_n)_{n=1}^{\infty}$. Choose $f_n = \sup(r_1, ..., r_n)$ and we are done. □

School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel

E-mail address: edhan@post.tau.ac.il