QUORIC MANIFOLDS

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Davis and Januszkiewicz introduced in 1981 a family of compact real manifolds, the *Quasi-Toric Manifolds*, with a group action by a torus, a direct product of circle (T) groups. Their manifolds have an orbit space which is a simple polytope with a distinct isotropy subgroup associated to each face of the polytope, subject to some consistency conditions. They defined a characteristic function which captured the properties of the isotropy subgroups, and showed that their manifolds can be classified by the polytope and characteristic function. They further showed that the cohomology ring of the manifold can be written down directly from properties derived from the polytope and the characteristic function.

This work considers the question of how far the circle group T can be replaced by the group of unit quaternions Q in the construction and description of quasi-toric manifolds. Unlike T, the group Q is not commutative, so the actions of Q^n on the product $\mathbb{H}^n$ of the set of quaternions using quaternionic multiplication are studied in detail. Then, in direct analogy to the quasi-toric manifolds, a family of compact real manifolds, the *quoric manifolds*, is introduced which have an action by Q^n, and whose orbit space is a polytope. A characteristic functor is defined on the faces of the polytope which captures the properties of the isotropy classes of the orbits of the action. It is shown that quoric manifolds can be classified in a manner similar to the quasi-toric manifolds, by the polytope and characteristic functor. A restricted family, the *global quoric manifolds*, which satisfy an additional condition are defined. It is shown that an infinite number of polytopes exist in any dimension over which a global quoric manifold can be defined. It is shown that any *global quoric manifold* can be described as a quotient space of a *moment angle complex* over the polytope, and that its integral cohomology ring can be calculated, taking a form analagous to that in the quasi-toric case.

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I would also like to acknowledge the contribution of my wife through her constant encouragement and support.

$\cong$	group or ring isomorphism	
$\cong$	homeomorphism	
$\simeq$	homotopy equivalence	
$\sim$	conjugate group equivalence	
$\sim$	equivalence relation	
$\sim_\ell$	equivalence relation from ℓ	66
E	exponent matrix, corner (2.2.9)	29
L	isotropy matrix, corner (2.2.16)	36
Λ	characteristic matrix, polytope (4.2.3)	82
Γ	facet graph, corner (3.1.7)	50
Σ	characteristic graph, polytope (4.2.3)	82
ℓ	isotropy functor, corner (3.1.18)	57
λ	characteristic functor, polytope (4.2.3)	82
$\widehat{\ell}(\tau)$	canonical group in class $\ell(\tau)$	66
$\mathbb{H}$	space of quaternions	14
Q	group of unit quaternions (2.0.2)	14
$\mathcal{D}$	disc of quaternions, $\lvert h\rvert \leqslant 1$	115
T	group of unit complex numbers	99
$\mathbb{R}_{\geqslant}$	orbit space, $\mathbb{H}/\mathsf{Q}$	27
$[n]$	set $\{1, \ldots, n\}$	17
P^n	simple polytope, dimension n (§4.1)	71
M^{4n}	manifold with Q^n action, over P^n (§4.2)	80
$\mathcal{H}(\ell)$	derived corner, from ℓ (3.2.1)	66
$\mathcal{M}(\lambda)$	derived manifold from (P,λ) (4.2.8)	86
$\mathcal{Z}_{P,\ell}$	moment angle complex, over P^n (§5.2)	100
$B_{\mathsf{Q}}M$	Borel construction of M^{4n} (6.1.1)	115
$B_{\mathsf{Q}}\mathcal{Z}$	Borel construction of $\mathcal{Z}_{P,\ell}$ (6.1.1)	115
$DJ_{\mathsf{Q}}(K_P)$	Quaternionic DJ space, over P^n (6.1.3)	115

Figure 1: Table of Symbols

Chapter 1

Introduction

In their ground-breaking paper some 20 years ago, Davis and Januszkiewicz [8] (abbreviated to *DJ* throughout this work) introduced and analysed a class of topological manifolds with a group action by a torus, a direct product of circle (T) groups.

Their manifolds, now known as quasi-toric manifolds, have an orbit space which is a simple polytope with a distinct isotropy subgroup associated to each face of the polytope, satisfying some consistency conditions. They defined a characteristic function which captured the properties of the pattern of isotropy subgroups, and showed that their manifolds can be classified by the polytope and the characteristic function. They further showed that the cohomology of the manifold can be written down directly from properties derived from the polytope and the characteristic function.

In their paper (DJ [8]), they built on earlier work on toric varieties constructed from cones and fans, but recast everything in terms of topological spaces with a toric action, and constructed proofs appropriate to topological spaces. They showed that the cohomology of these new spaces could be expressed using essentially the same formula as in the toric case. The book of Buchstaber and Panov [4] (abbreviated to *BP* throughout this work) contains an excellent discussion of this and much related material.

Scott [17] (1995) considered the construction of spaces using quaternions instead of complex numbers to define them, but did not maintain an action of the n-fold group of quaternions (Q^n), merely having the conjugate action of a diagonal Q. He used some general conditions for their definition derived from properties of possible isotropy subgroups on the faces. He did, however, show that the cohomology of these spaces was given by the expected formula.

The current work considers the problem of replacing the circle group in the manifolds of Davis and Januszkiewicz by the group of unit quaternions (Q), describing manifolds which have a Q^n action and which have an orbit space of an n dimensional simple polytope with distinct isotropy subgroups associated to each face. It considers the extent to which the treatment and results of Davis and Januszkiewicz can be reproduced in this case, including the classification and cohomology of these spaces. There are significant differences between these two cases, which derive from the fact that, unlike the circle group, the quaternion group is not commutative. Thus, for example, when a polytope supports a toric characteristic function there are in general an infinite number of incongruent such functions, but there can only ever be a finite number of quaternionic functors. Nevertheless, the manifolds can be similarly classified in the quaternionic case by a polytope and a characteristic functor. By taking appropriate sections, any such quaternionic manifold has a quasi-toric submanifold with the same orbit space polytope and closely related characteristic function, while the converse is not true. It turns out that we need to distinguish a subclass of characteristic functors (which we call *global* characteristic functors). For a *global* quaternionic characteristic functor over a polytope we obtain a formula for the cohomology of the manifold, which is given by a similar construction to that of the toric case.

It is worth noting here the differences as well as the similarities between the quaternionic actions described below and the toric actions.

The most obvious difference is that the quaternions do not commute, whereas the complex numbers do. This has immediate effects in the nature of the collection of subgroups that are available to define the combinatorial data. These are always finite in number, where the toric ones are countable in number. It also results in there being a number of incongruent (up to equivariant homeomorphism) *regular corners*, where in the toric case all corners are congruent to the standard corner. This difference also has more serious effects. Since we allow the group to act both on the left and on the right of the space of quaternions $\mathbb{H}$, the action on $\mathbb{H}$ cannot preserve the quaternionic vector space structure, but only its real vector space structure, so we are not looking at any kind of symplectic spaces.

The similarities come from the fact that the unit quaternions act on the set of all quaternions, by left or right multiplication, such that the orbit space is the real half line, and the isotropy subgroup is trivial except at the origin. The unit complex numbers act on the complex plane with these same characteristics. This allows the construction of manifolds in the form $\mathsf{T}^n \times P^n/\!\sim$, for a suitable simple polytope P^n with combinatorial data defining the equivalence relation via the isotropy subgroups on the faces of P^n, to be generalised directly to the form $\mathsf{Q}^n \times P^n/\!\sim$. The set of suitable simple polytopes is smaller and the combinatorial data is more restricted than for the toric case, but the similarities are still striking. It also turns out that a large class of these spaces are a quotient space of a moment angle complex, even though the details are significantly different. This result allows the calculation of their cohomology in terms of a formula which is formally identical to the toric case.

Chapter 2 discusses those subgroups of the n-fold product Q^n that we are interested in, and the classification of their conjugacy classes.

Chapter 3 describes the various actions of Q^n on the n-fold product space, $\mathbb{H}^n$, and summarises these actions by an *exponent matrix*. The possible patterns of isotropy subgroups are classified, and related to the exponent matrix. Unlike the toric case, there are a number of incongruent actions of Q^n on $\mathbb{H}^n$, and these are classified. An alternative description of the manifold is presented, which is useful particularly for the construction of manifolds over a polytope given a characteristic functor.

Chapter 4 introduces quoric manifolds, and discusses various descriptions and their important properties. It is shown that they can be classified by an orbit polytope and a characteristic functor.

Chapter 5 introduces a restricted class of characteristic functors, the *global* ones. The construction of the moment angle complex for a polytope with a *global* characteristic functor is described, and the various possible Q^m actions described. It is shown that there are plenty of polytopes which support a quoric manifold, including an infinite number in each dimension supporting a *global* characteristic functor.

Chapter 6 obtains the formula for the cohomology of *global* quoric manifolds, by considering the cohomology of the Borel construction of the moment angle complex and its relation to that of the manifold.

Chapter 2

Q^n Conjugacy Classes and Actions

The group of unit quaternions Q acts naturally on the space of quaternions by multiplication, but there are two possible actions corresponding to multiplication on the left or right.

We begin by analysing those subgroups of the direct (or Cartesian) product of n copies of the quaternion group that we are interested in, and obtain a characterisation of the conjugacy classes of these subgroups. We then discuss (section 2.2) the various actions on a product space of quaternions.

Let $\mathbb{H}$ denote the space of quaternions, which is a skew field, or division ring. The set of unit quaternions forms a compact non-commutative Lie group,

$$\mathsf{Q} := \{h \in \mathbb{H} \mid |h| = 1\} \subset \mathbb{H} \tag{2.0.2}$$

using the multiplication in $\mathbb{H}$. Our main focus of interest is the direct product Q^n of n copies of Q, and in particular those subgroups of Q^n which are isomorphic to Q^k, for some $k \leqslant n$. Our goal is to obtain a description of these subgroups as a means of describing the conjugacy classes of these subgroups and their subclasses, which we need in the next section.

We note the well-known basic properties of the simple Lie group Q, that it is the double cover of $SO(3)$, and that the automorphisms of Q are all inner automorphisms. The space of Q is homeomorphic to the 3-sphere S^3, and the space of $SO(3)$ to the projective space $\mathbb{R}P^3$, and further $SO(3)$ is not a subgroup of Q since $\mathbb{R}P^3$ cannot be embedded in S^3. Adams [1], Bourbaki [3] or tom Dieck [9], for example, give a general introduction to Lie groups, and the introductory chapter of Bredon [6] provides a concise summary. Conway and Smith [7] include an excellent discussion of the quaternions and various subgroups of quaternions.

We begin by considering the endomorphisms of Q. The only proper normal subgroup of Q is its centre, $Z = \{1, -1\}$, since Q is the double cover of a simple group. Furthermore, the quotient group $SO(3)$ is not isomorphic to any subgroup of Q, since $\mathbb{R}P^3$ cannot be embedded in S^3. Thus the kernel of any endomorphism is either the trivial subgroup $\mathbf{1}$ or Q itself, so any endomorphism is trivial or an inner automorphism. Thus we have the result:

Proposition 2.0.3 *The endomorphisms of* Q, $\psi_w\colon \mathsf{Q} \to \mathsf{Q}$ *are*

$$\begin{aligned} w = 0 : &\qquad \psi_0\colon \mathsf{Q} \to \mathsf{Q}, \qquad s \mapsto 1, \\ w = u : &\qquad \psi_u\colon \mathsf{Q} \to \mathsf{Q}, \qquad s \mapsto usu^{-1}, \end{aligned}$$

where u is a unit quaternion. ❑

Note that u and $-u$ define the same endomorphism.

2.1 Conjugacy Classes of Q^n

Two subgroups H, H' of any group G are said to be conjugate if there is some element $g \in G$ such that $H' = gHg^{-1}$. Conjugation defines an equivalence relation, and we define the *conjugacy class* (H) of any subgroup H as

$$(H) = \{H' < G \mid \exists g \in G,\ H' = gHg^{-1}\}. \tag{2.1.1}$$

Subgroups Isomorphic to Q. Apart from the coordinate subgroups Q_i of Q^n, there are also various diagonal subgroups isomorphic to Q, and subgroups related to them. We note immediately that, for example, the subgroups $\{(q,q) \mid q \in \mathsf{Q}\}$ and $\{(q, uqu^{-1}) \mid q \in \mathsf{Q}\}$ of Q^2 are different subgroups (for $u \neq \pm 1$), although they are conjugate. We next identify all subgroups isomorphic to Q.

Proposition 2.1.2 *A subgroup of Q^n is isomorphic to Q if and only if it is of the form:*

$$\mathsf{Q}(u) = \{(s_1, \ldots, s_n) \in \mathsf{Q}^n \mid s_i = \psi_{u_i}(t),\ t \in \mathsf{Q}\},$$

where $\psi_u(t) = utu^{-1}$, for some non-zero $(u_1, \ldots, u_n) \in \mathbb{H}^n$, with each u_i equal to 0 or a unit quaternion.

Proof. To say a subgroup $H < \mathsf{Q}^n$ is isomorphic to Q means that there is a monomorphism $\psi\colon \mathsf{Q} \to \mathsf{Q}^n$ with H equal to the image of ψ. Let pr_i be the canonical projection $\mathrm{pr}_i\colon \mathsf{Q}^n \to \mathsf{Q}$, $q = (q_1, \ldots, q_n) \mapsto q_i$. Now compose ψ with this projection, $\mathrm{pr}_i \circ \psi\colon \mathsf{Q} \to \mathsf{Q}^n \to \mathsf{Q}$. For each i, this composition is an endomorphism, so labelled by 0 or a unit quaternion (Proposition 2.0.3). If u_i is 0 for all i, then the image of ψ is the identity of Q^n, so ψ would not be injective. Thus any monomorphism is labelled by a non-zero $u = (u_1, \ldots, u_n)$, and the subgroup H of Q^n is the image of ψ so is of the required form.

Conversely, given a non-zero $u \in \mathbb{H}^n$ with each component equal to zero or a unit quaternion, the map $t \mapsto (s_1, \ldots, s_n)$ defined as in the proposition is clearly a homomorphism. For all $u_i \neq 0$, $u_i t u_i^{-1} = u_i t' u_i^{-1}$ implies $t = t'$, and there is a $u_i \neq 0$ for some i by definition, so the map is a monomorphism. ❏

Of course, not all distinct u define different subgroups; if w is any unit quaternion and $u'_i = u_i w$ for each i, the subgroups $\operatorname{im}\psi_{u'}$ and $\operatorname{im}\psi_u$ are clearly equal (set $t = w\,t' w^{-1}$). It is sufficient here that all subgroups isomorphic to Q are of this form.

If only one element of $u = (u_1, \ldots, u_n)$ is non-zero, then the subgroup is just a coordinate subgroup, but if more than one element of u is non-zero then we will sometimes refer to the group $\mathsf{Q}(u)$ as a *generalised diagonal subgroup* of Q^n. If all elements of u are equal and non-zero, then the subgroup is the usual diagonal subgroup of Q^n.

We next consider the conjugacy classes of subgroups $\mathsf{Q}(u)$. Two subgroups H, H' of Q^n are conjugate if there is some element $q = (q_1, \ldots, q_n) \in \mathsf{Q}^n$ such that $H' = qHq^{-1}$. If $H = \mathsf{Q}(u)$ is of the form of Proposition 2.1.2, then a conjugate subgroup is of the same form with $u'_i = q_i u_i$ for each i. We denote the set containing the first n natural numbers by $[n]$.

Definition 2.1.3 For a subgroup $\mathsf{Q}(u)$ of Q^n, as in Proposition 2.1.2, define its *characteristic set* as the non-empty set

$$\gamma(u) = \{i \in [n] \mid u_i \neq 0\} \ \subset [n].$$

Proposition 2.1.4 *Two subgroups $\mathsf{Q}(u)$ and $\mathsf{Q}(u')$, as in Proposition 2.1.2, are conjugate if and only their characteristic sets $\gamma(u)$, $\gamma(u')$ are equal.*

Proof. Suppose two subgroups $\mathsf{Q}(u)$ and $\mathsf{Q}(u')$ are conjugate. If for any i, $u'_i = 0$ and $u_i \neq 0$ say, then $1 = q_i u_i t u_i^{-1} q_i^{-1}$ for all $t \in \mathsf{Q}$, which is a contradiction. Thus, $u_i \neq 0 \Leftrightarrow u'_i \neq 0$ for all i, and hence define the same set $\gamma(u) = \gamma(u')$.

Conversely, suppose for two such subgroups $\gamma(u) = \gamma(u')$. If for any i, $u_i \neq 0$ then $u'_i \neq 0$, and define $q_i = u'_i\, u_i{}^{-1}$ so $q_i u_i t u_i^{-1} q_i^{-1} = u'_i t u'^{-1}_i$. Otherwise, $u_i = 0$ and $u'_i = 0$, so set $q_i = 1$ and $1 = 1$. Hence q_i is defined for all i and $\mathsf{Q}(u') = q\mathsf{Q}(u)q^{-1}$, that is the subgroups are conjugate. ❏

Corollary 2.1.5 *The conjugacy classes of subgroups of Q^n isomorphic to Q are classified by their characteristic sets $\gamma \subset [n]$.*

Write the class $(\mathsf{Q}(u))$ as $\mathsf{Q}_{\gamma(u)}$.

Proof. Given any subgroup $\mathsf{Q}(u)$ defined by $u \in \mathbb{H}^n$ as in Proposition 2.1.2, then all subgroups conjugate to it define the same set $\gamma(u)$. Thus, each conjugacy class defines a characteristic set γ.

Conversely, for any non-empty set γ define u by $u_i = 1$ if $i \in \gamma$, and $u_i = 0$ otherwise. This defines a subgroup $\mathsf{Q}(u)$ as in Proposition 2.1.2, hence γ defines a conjugacy class, $(\mathsf{Q}(u))$. ❏

The coordinate subgroups Q_i are normal subgroups of the direct product Q^n, as are their products. In contrast, a generalised diagonal subgroup $\mathsf{Q}(u)$ in the class

Q_γ for any γ containing more than one element is never a normal subgroup, since in general the conjugate subgroup $q\mathsf{Q}(u)q^{-1}$ is not equal to $\mathsf{Q}(u)$. Thus, there is an essential difference between the generalised diagonal subgroups and the coordinate subgroups.

Example 2.1.6 Let $n = 4$, and consider the set $\gamma = \{1, 3, 4\} \subset [4]$.

This set γ specifies the class $\mathsf{Q}_{\{1,3,4\}}$, which contains the following subgroups of Q^4:

$$\begin{aligned} H_1 &= \{(iti^{-1}, 1, jtj^{-1}, ktk^{-1}) \mid t \in \mathsf{Q}\}, && u = (i, 0, j, k) \\ H_2 &= \{(wtw^{-1}, 1, wtw^{-1}, wtw^{-1}) \mid t \in \mathsf{Q}\}, && u = (w, 0, w, w) \\ H_3 &= \{(t, 1, t, t) \mid t \in \mathsf{Q}\}, && u = (1, 0, 1, 1) \end{aligned}$$

where i, j, k are the standard pure unit quaternions in $\mathbb{H}$, and w is any unit quaternion. Clearly, $H_2 = H_3$ and can be thought of as the diagonal subgroup of $\mathsf{Q} \times \{1\} \times \mathsf{Q} \times \mathsf{Q} < \mathsf{Q}^4$.

H_1 is a different subgroup, but belongs to the same conjugacy class since $H_3 = qH_1q^{-1}$ for $q = (-i, 1, -j, -k)$.

Remark. In the toric case, in contrast, the group T^n is commutative so each conjugacy class has a unique element, and all subgroups are normal.

Subgroups Isomorphic to Q^k. Having identified the conjugacy classes of subgroups of Q^n isomorphic to Q, we now extend the analysis to identify the conjugacy classes of subgroups isomorphic to Q^k for any $k \leqslant n$. Then we examine the properties of subclasses of conjugacy classes.

Proposition 2.1.7 *A subgroup of Q^n is isomorphic to Q^k, for $0 \leqslant k \leqslant n$, if and only if it is of the form*

$$\mathsf{Q}(u^1, \ldots, u^k) = \mathsf{Q}(u^1) \cdots \mathsf{Q}(u^k) \; < \mathsf{Q}^n$$

for some set of k non-zero elements $u^1, \ldots, u^k$ of $\mathbb{H}^n$, such that each component u_i^j is zero or a unit quaternion, and for each $i = 1, \ldots, n$, there is at most one j such that u_i^j is non-zero.

Proof. The degenerate case, $k = 0$, corresponds to the trivial subgroup $\mathbf{1} < \mathsf{Q}^n$, and the result is immediate. For $k = 1$, $\mathsf{Q}(u^1)$ is of the form given by Proposition 2.1.2, and the result follows.

For $k > 1$, let H be a subgroup of Q^n isomorphic to Q^k. For each $j = 1, \ldots, k$ there is a subgroup $\mathsf{Q}(u^j) < \mathsf{Q}^n$ equal to to the coordinate subgroup Q_j of Q^k, where $u^j \in \mathbb{H}^n$ as in Proposition 2.1.2, and H is the product of these subgroups in Q^n. By using the projection pr_i from Q^n to Q, we see that at most one $\mathsf{Q}_j < \mathsf{Q}^k$ can have a non-trivial image in Q, since distinct coordinate subgroups of Q^k commute. That is, for each i there is at most one non-zero u_i^j, and H is of the form given.

Conversely, if a subgroup is of the form given in the proposition, then the condition that for each i there is at most one u^j such that u_i^j is non-zero ensures that each coordinate subgroup of Q^n has a non-trivial intersection with at most one $\mathsf{Q}(u^j)$ so the product is well defined, and is isomorphic to Q^k. ❑

Proposition 2.1.8 *The conjugacy classes of subgroups of Q^n isomorphic to Q^k for $k \leqslant n$ are classified by sets of k disjoint characteristic sets $\gamma_j \subset [n]$, $j = 1, \ldots, k$.*

Proof. Suppose two subgroups $\mathsf{Q}(u^1, .., u^k)$ and $\mathsf{Q}(v^1, .., v^k)$ isomorphic to Q^k are conjugate. Then the images of the coordinate subgroup Q_j of Q^k are $\mathsf{Q}(u^j)$ and $\mathsf{Q}(v^j)$ and are conjugate, so the characteristic sets $\gamma(u^j)$ and $\gamma(v^j)$ are equal by Proposition 2.1.4. Hence the conjugacy class $(\mathsf{Q}(u^1, .., u^k))$ defines the set of characteristic sets $\gamma(u^1), \ldots, \gamma(u^k)$.

Conversely, given characteristic sets $\gamma_1, \ldots, \gamma_k$, define the set $u^1, \ldots, u^k \in \mathbb{H}^n$ by $u_i^j = 1$ if $i \in \gamma_j$ and 0 otherwise, as in Proposition 2.1.2. Then each $\mathsf{Q}(u^j)$ is well defined, and for each i there is at most one j such that u_i^j is non-zero since the γ_j are disjoint. Hence, the subgroup $\mathsf{Q}(u^1, \ldots, u^k)$ is well defined, with characteristic sets $\gamma_1, \ldots, \gamma_k$. ❑

Definition 2.1.9 A conjugacy class classified by the set of k disjoint sets $\gamma_1, \ldots, \gamma_k$ is written $\mathsf{Q}_{\gamma_1, .., \gamma_k}$, and is said to be of *rank* k. A rank 1 class Q_γ is said to be of *degree* $|\gamma|$, where $|\gamma|$ is the cardinality of the set γ.

Example 2.1.10 Consider the disjoint sets $\gamma_1 = \{1,3,4\}$, $\gamma_2 = \{2\} \subset [4]$ (extending Example 2.1.6).

The corresponding rank 2 class is $\mathsf{Q}_{\{1,3,4\},\{2\}}$, which contains, for example, the following subgroups of Q^4 isomorphic to Q^2

$$\begin{aligned} H_4 &= \{(it_1i^{-1}, wt_2w^{-1}, jt_1j^{-1}, kt_1k^{-1}) \mid t_1, t_2 \in \mathsf{Q}\}, \\ &\qquad u^1 = (i,0,j,k), u^2 = (0,w,0,0) \\ H_5 &= \{(t_1,t_2,t_1,t_1) \mid t_1,t_2 \in \mathsf{Q}\}, \\ &\qquad u^1 = (1,0,1,1), u^2 = (0,1,0,0) \end{aligned}$$

Clearly, H_4 and H_5 are not the same subgroup, but they are conjugate with $H_4 = qH_5q^{-1}$ for any $q = (\pm i, \pm w, \pm j, \pm k)$.

Subclasses of Conjugacy Classes. We now examine the properties of subclasses of conjugacy classes, by examining the corresponding properties of subgroups. Classes are classified by the characteristic sets, and we need to be able to recognise when one class is a subclass of another, in terms of properties of the characteristic sets.

Given subgroups H, K of a group G, the conjugacy class (H) is said to be a *subclass* of (K) if H is subconjugate to K, that is, if there exists $g \in G$ such that H is a subgroup of gKg^{-1}. Then every subgroup $g'Hg'^{-1} \in (H)$ is a subgroup of $(g'g)K(g'g)^{-1} \in (K)$. (See for example, tom Dieck [9]).

Proposition 2.1.11 *If subgroups $\mathsf{Q}(u) \in \mathsf{Q}_{\gamma(u)}$ and $\mathsf{Q}(v) \in \mathsf{Q}_{\gamma(v)}$ of Q^n are elements of different rank 1 conjugacy classes, the intersection $\mathsf{Q}(u) \cap \mathsf{Q}(v)$ is the trivial subgroup **1**, so neither is a subgroup of the other.*

Proof. The intersection of the subgroups is the set of elements defined by $u_isu_i^{-1} = v_is'v_i^{-1}$ for each $i = 1, \ldots, n$, where $u_isu_i^{-1}$ is replaced by 1 for each $u_i = 0$. Since the subgroups belong to different rank 1 classes, $\gamma(u) \neq \gamma(v)$, and so there is some i such that either $u_i = 0$ and $v_i \neq 0$, or vice versa. Hence, either $1 = v_is'v_i^{-1}$ and $s' = 1$, or $u_isu_i^{-1} = 1$ and $s = 1$. In either case $s = s' = 1$, so the only common element is the identity. Thus, since both are elements of a rank 1 class, neither is a subgroup of the other. ❑

Corollary 2.1.12 *The only subclasses of the rank 1 class Q_γ are Q_γ itself and the trivial rank 0 class $(\mathbf{1}) = \mathsf{Q}_\emptyset$.* ❑

Proposition 2.1.13 *The only proper subclasses of the rank 2 class $\mathsf{Q}_{\gamma_1,\gamma_2}$ are the rank 1 classes Q_{γ_1}, Q_{γ_2} and $\mathsf{Q}_{\gamma_1\cup\gamma_2}$.*

Proof. Suppose subgroups $\mathsf{Q}(u) \in \mathsf{Q}_{\gamma_1}$ and $\mathsf{Q}(v) \in \mathsf{Q}_{\gamma_2}$, then the subgroup $\mathsf{Q}(u,v) = \mathsf{Q}(u)\mathsf{Q}(v)$ is an element of the class $\mathsf{Q}_{\gamma_1,\gamma_2}$, where γ_1, γ_2 are disjoint by definition. Clearly, the rank 1 classes Q_{γ_1} and Q_{γ_2} are subclasses of $\mathsf{Q}_{\gamma_1,\gamma_2}$, since both $\mathsf{Q}(u)$ and $\mathsf{Q}(v)$ are subgroups of $\mathsf{Q}(u,v)$. Since for each i at most one of u_i and v_i is non-zero (Proposition 2.1.7), then each component of $u+v \in \mathbb{H}^n$ is zero or a unit quaternion, and $\mathsf{Q}(u+v)$ is the diagonal subgroup of $\mathsf{Q}(u,v)$. Hence, the rank 1 class $\mathsf{Q}_{\gamma_1\cup\gamma_2}$ is a subclass of $\mathsf{Q}_{\gamma_1,\gamma_2}$.

Let $\mathsf{Q}(w)$, in the class Q_ϵ, be a subgroup of some $\mathsf{Q}(u,v)$ in the class $\mathsf{Q}_{\gamma_1,\gamma_2}$. Then $\mathsf{Q}(w) \cap \mathsf{Q}(u)$ is a subgroup of $\mathsf{Q}(u)$, and so is either $\mathsf{Q}(u)$ or $\mathbf{1}$ (Corollary 2.1.12), hence $\epsilon \cap \gamma_1 = \gamma_1$ or $\emptyset$ using Proposition 2.1.4. Similarly, $\mathsf{Q}(w) \cap \mathsf{Q}(v)$ is a subgroup of $\mathsf{Q}(v)$, so is either $\mathsf{Q}(v)$ or $\mathbf{1}$, so $\epsilon \cap \gamma_2 = \gamma_2$ or $\emptyset$. Hence, $\mathsf{Q}(w)$ must be $\mathsf{Q}(u)$, $\mathsf{Q}(v)$, or $\mathsf{Q}(u+v)$, and ϵ is γ_1, γ_2 or $\gamma_1 \cup \gamma_2$. ❑

We want later to build up collections of classes which form a poset using the subclass relationship, so we need to examine in more detail the relationship between classes and subclasses. We want to be able not only to identify the subclasses of any given rank k class, but also to determine the rank k class which has a given set of k rank 1 classes as subclasses. It turns out to be convenient to express these relationships directly in terms of the characteristic sets of the classes (Proposition 2.1.8). We introduce the following definition.

Definition 2.1.14 Any two sets $\gamma_1, \gamma_2 \subset [n]$ will be called *compatible* if they are disjoint or if one is a subset of the other.

Two conjugacy classes Q_{γ_1}, Q_{γ_2} will be called *compatible* if their characteristic sets γ_1, γ_2 are compatible.

Two subgroups will be called *compatible* if they are elements of compatible classes.

This definition allows a set to be compatible with itself. It is clearly not an equivalence relation, since the set $\{1\}$ is compatible with both $\{1,2\}$ and $\{1,3\}$, which are not compatible with each other.

Any collection of sets $\gamma_1, \ldots, \gamma_k \subset [n]$ will be called *compatible* if γ_i and γ_j are compatible for all $i, j = 1, \ldots, k$.

Proposition 2.1.15 *Given two distinct rank 1 classes Q_{γ_1}, Q_{γ_2} there is a rank 2 class which contains both as subclasses if and only if they are compatible.*

In the case that Q_{γ_1}, Q_{γ_2} are compatible, the rank 2 class is given by

$$\begin{aligned} \mathsf{Q}_{\gamma_1,\gamma_2} &\quad \text{if } \gamma_1 \cap \gamma_2 = \emptyset, \\ \mathsf{Q}_{\gamma_1,\gamma_2\setminus\gamma_1} &\quad \text{if } \gamma_1 \subset \gamma_2, \\ \mathsf{Q}_{\gamma_1\setminus\gamma_2,\gamma_2} &\quad \text{if } \gamma_2 \subset \gamma_1. \end{aligned}$$

and is unique.

Proof. This result follows from Proposition 2.1.13. For any rank 2 class $\mathsf{Q}_{\beta_1.\beta_2}$ the sets β_1, β_2 are disjoint by definition, and only has rank 1 subclasses Q_{β_1}, Q_{β_2} and $\mathsf{Q}_{\beta_1\cup\beta_2}$ which are are pairwise compatible. So no rank 2 class can contain subclasses which are not compatible.

Suppose γ_1 and γ_2 are compatible, then since the characteristic sets of the rank 2 class must be disjoint, the rank 2 class is as given, and is unique. ❏

Example 2.1.16 The sets $\gamma_1 = \{1,3,4\}$, $\gamma_2 = \{1,4\} \subset [4]$ are compatible since $\gamma_2 \subset \gamma_1$ (extending Example 2.1.6). Consider the rank 2 class $\mathsf{Q}_{\{1,4\},\{3\}}$, with characteristic sets γ_2 and $\gamma_1\setminus\gamma_2$. By Proposition 2.1.13, this class contains the classes $\mathsf{Q}_{\{1,3,4\}}$ and $\mathsf{Q}_{\{1,4\}}$ as rank 1 subclasses (as well as $\mathsf{Q}_{\{3\}}$).

These classes contain the following subgroups of Q^4:

$$H_1 = \{(iti^{-1}, 1, jtj^{-1}, ktk^{-1}) \mid t \in \mathsf{Q}\} \ \in \mathsf{Q}_{\{1,3,4\}}$$
$$H_6 = \{(wtw^{-1}, 1, 1, wtw^{-1}) \mid t \in \mathsf{Q}\} \ \in \mathsf{Q}_{\{1,4\}}$$
$$H_7 = \{(t_1, 1, t_2, jt_1j^{-1}) \mid t_1, t_2 \in \mathsf{Q}\} \ \in \mathsf{Q}_{\{1,4\},\{3\}}$$
$$H_8 = \{(it_1i^{-1}, 1, it_2i^{-1}, it_1i^{-1}) \mid t_1, t_2 \in \mathsf{Q}\} \ \in \mathsf{Q}_{\{1,4\},\{3\}}$$

Then H_7 and H_8 are subgroups in the same rank 2 class. Also, $H_1 < H_7$ (set $t_1 = iti^{-1}$, $t_2 = jtj^{-1}$) and $H_6 < H_8$ (set $t_1 = iwtw^{-1}i^{-1}$, $t_2 = 1$).

Proposition 2.1.17 *A rank 1 class Q_β is a subclass of $\mathsf{Q}_{\gamma_1,\ldots,\gamma_k}$ if and only if β is the union $\beta = \gamma_{j_1} \cup \ldots \cup \gamma_{j_t}$ of some non-empty collection of the γ_j.*

Proof. For each $j = 1, \ldots, k$, define $u_i^j = 1$ if $i \in \gamma_j$ and 0 otherwise.

If β is the union $\gamma_{j_1} \cup \ldots \cup \gamma_{j_t}$ for some non-empty collection of the γ_j, define $w = u^{j_1} + \ldots + u^{j_t}$. Since the γ_j are disjoint by definition, the components of w are each equal to zero or a unit quaternion, so the subgroup $\mathsf{Q}(w)$ is an element of the class Q_β. The subgroup $\mathsf{Q}(w)$ is also the diagonal subgroup of $\mathsf{Q}(u^{j_1}, \cdots, u^{j_t})$. Hence, Q_β is a subclass of $\mathsf{Q}_{\gamma_1,\ldots,\gamma_k}$

Conversely, if Q_β is a subclass of $\mathsf{Q}_{\gamma_1,\ldots,\gamma_k}$ then any subgroup $\mathsf{Q}(w) \in \mathsf{Q}_\beta$ is a subgroup of some $\mathsf{Q}(u^1, \ldots, u^k)$. So $\mathsf{Q}(w) \cap \mathsf{Q}(u^j)$ is a subgroup of $\mathsf{Q}(u^j)$, so is equal to $\mathsf{Q}(u^j)$ or $\mathbf{1}$. Thus, the characteristic set β of w satisfies $\beta \cap \gamma_j$ is equal to γ_j or $\emptyset$. Hence, β is the union of some non-empty collection of the γ_j. ❑

Corollary 2.1.18 *A class $\mathsf{Q}_{\beta_1,\ldots,\beta_t}$ is a subclass of $\mathsf{Q}_{\gamma_1,\ldots,\gamma_k}$ if and only if each β_i is the union $\beta_i = \gamma_{j_1} \cup \ldots \cup \gamma_{j_t}$ of some non-empty collection of the γ_j.*

Proof. If β_i, for $i = 1, \ldots, t$, is the union of some non-empty collection of the γ_j, then Q_{β_i} is a subclass of $\mathsf{Q}_{\gamma_1,\ldots,\gamma_k}$, so there is a group $\mathsf{Q}(w^i)$ which is a subgroup of some group in the class $\mathsf{Q}_{\gamma_1,\ldots,\gamma_k}$. The β_i are disjoint so the group $\mathsf{Q}(w^1, \ldots, w^b)$ is also a subgroup of some group in the class $\mathsf{Q}_{\gamma_1,\ldots,\gamma_k}$. Hence, $\mathsf{Q}_{\beta_1,\ldots,\beta_t}$ is a subclass of $\mathsf{Q}_{\gamma_1,\ldots,\gamma_k}$.

Conversely, if $\mathsf{Q}_{\beta_1,\ldots,\beta_t}$ is a subclass of $\mathsf{Q}_{\gamma_1,\ldots,\gamma_k}$, then each Q_{β_i} is a subclass of $\mathsf{Q}_{\gamma_1,\ldots,\gamma_k}$, since the group $\mathsf{Q}(w^i)$ is a subgroup some group in the class $\mathsf{Q}_{\gamma_1,\ldots,\gamma_k}$. Thus, β_i is a union of a non-empty collection of the γ_j. ❑

2.1.19 Category of Conjugacy Classes. For each $n \geqslant 0$, the set of all conjugacy classes of subgroups of Q^n isomorphic to Q^k (some $0 \leqslant k \leqslant n$) has a natural partial order derived from the subclass relation, so has the structure of a poset. Thus, we can define a category with these conjugacy classes as objects and subclass inclusion defining the morphisms, which we denote by $\textsc{conj}(\mathsf{Q}^n)$.

Examples 2.1.20 Subclasses.

1. The sets $\gamma_1 = \{1,3,4\}$, $\gamma_2 = \{1,4\}$, $\gamma_3 = \{3\}$ are compatible. The classes Q_{γ_1}, Q_{γ_2} and Q_{γ_3} are all subclasses of $\mathsf{Q}_{\{3\},\{1,4\}}$.

2. For $n = 3$, the objects of $\textsc{conj}(\mathsf{Q}^3)$ are the conjugacy classes:

rank 0: $\mathsf{Q}_\emptyset$

rank 1: $\mathsf{Q}_{\{1\}}$, $\mathsf{Q}_{\{2\}}$, $\mathsf{Q}_{\{3\}}$, $\mathsf{Q}_{\{1,2\}}$, $\mathsf{Q}_{\{1,3\}}$, $\mathsf{Q}_{\{2,3\}}$, $\mathsf{Q}_{\{1,2,3\}}$.

rank 2: $\mathsf{Q}_{\{1\},\{2\}}$, $\mathsf{Q}_{\{1\},\{3\}}$, $\mathsf{Q}_{\{2\},\{3\}}$, $\mathsf{Q}_{\{1\},\{2,3\}}$, $\mathsf{Q}_{\{2\},\{1,3\}}$, $\mathsf{Q}_{\{3\},\{1,2\}}$.

rank 3: $\mathsf{Q}_{\{1\},\{2\},\{3\}}$.

The class $\mathsf{Q}_{\{1,2\}}$ is a subclass of itself and $\mathsf{Q}_{\{1\},\{2\}}$, $\mathsf{Q}_{\{1,2\},\{3\}}$, $\mathsf{Q}_{\{1\},\{2\},\{3\}}$.

Remark. These notions of compatible conjugacy classes (Q_{γ_1},Q_{γ_2}) and compatible sets (γ_1,γ_2) are important in everything that follows. It underlies all the combinatorics in the definition of *regular corners*, *characteristic functors*, and in the classification of manifolds (see below).

Remark. In the toric case, a subgroup of T^n which is an embedding of T is specified by a vector of integers $(k_1, \ldots, k_n)$ which is a basis of some rank 1 direct summand of $\mathbb{Z}^n$, and each conjugacy class consists of a single subgroup so is similarly labelled. The analogue of a rank k class is a subgroup isomorphic to T^k, and is labelled by a set of k such vectors that span a rank k direct summand of $\mathbb{Z}^n$.

Automorphisms of Q^n. The only endomorphisms of Q are ψ_0 and ψ_u, for u a unit quaternion (Proposition 2.0.3), and those with non-zero u are automorphisms (since clearly bijective). The automorphisms of Q^n are obtained by combining these endomorphisms.

Proposition 2.1.21 *Any automorphism of Q^n is of the form*

$$\theta_{u_i,\pi}\colon \mathsf{Q}^n \to \mathsf{Q}^n, \qquad s_i \mapsto t_{\pi i} = u_i s_i u_i^{-1}, \qquad i = 1, \dots, n.$$

where each u_i $(i = 1, \dots, n)$ is a unit quaternion, and π is a permutation of $[n]$.

Proof. For notational convenience, we write $\pi(i)$ as πi. Restricting θ to the i^{th} coordinate subgroup of Q^n and composing with the canonical projection pr_j, gives $\mathrm{pr}_j \circ \theta|_{\mathsf{Q}_i}\colon \mathsf{Q}_i \to \mathsf{Q}^n \to \mathsf{Q}$, which is an endomorphism $\psi_{v_{i,j}}$ of Q for some $v_{i,j}$ equal to zero or a unit quaternion (for $i, j = 1, \dots, n$). But for each j, there can be at most one i for which $v_{i,j}$ is non-zero, since the coordinate subgroups commute with each other, so at most n elements $v_{i,j}$ which are non-zero. Also, for each i there must be some j for which $v_{i,j}$ is non-zero, since θ is an automorphism so injective, so at least n elements $v_{i,j}$ are non-zero. Together these conditions imply that the $n \times n$ matrix with elements $v_{i,j}$ has exactly n non-zero elements, with one non-zero element in each row and one in each column. Define the permutation π by $j = \pi i$ if $v_{i,j}$ is non-zero, and set $u_i = v_{i,\pi i}$. Hence $\theta\colon \mathsf{Q}^n \to \mathsf{Q}^n$, where $\theta(s_i) = t_{\pi i} = u_i s_i u_i^{-1}$. ❑

In particular, it follows that a subgroup in the rank 1 class Q_γ maps to a subgroup in the class $\mathsf{Q}_{\pi(\gamma)}$, where $\pi(\gamma) = \{\pi i \mid i \in \gamma\}$. Since π is a permutation, γ and $\pi(\gamma)$ have the same cardinality, which defines the degree of the corresponding class (see Definition 2.1.9), so we have proved:

Corollary 2.1.22 *Any rank 1 class Q_γ and its image $\mathsf{Q}_{\pi(\gamma)}$ under an automorphism $\theta_{u_i,\pi}$ have the same degree.* ❑

Remark. In the toric case, an automorphism of T^n is a transformation $T^n \to T^n$, defined by $s \mapsto t$, with $t_i = s_1^{e_{i,1}} s_2^{e_{i,2}} \cdots s_n^{e_{i,n}}$ for each i, where the matrix of elements $e_{i,j}$ has a determinant equal to ± 1. There does not exist any notion corresponding to the degree of a class.

2.2 Actions on $\mathbb{H}^n$

Given a topological group G, a topological space X is called a (left) G-space if there is a continuous map (left action) $G \times X \to X$, $(g, x) \mapsto gx$ which satisfies

$$g_1(g_2 x) = (g_1 g_2)x, \quad 1_G x = x, \qquad \forall g_1, g_2 \in G, x \in X.$$

A right G-space may be similarly defined with action on the right, but there is a bijective correspondence between left and right G-spaces so only left spaces will be considered here. For any point $x \in X$, the *orbit* of x is the set $[x]_G := Gx \subset X$, and the orbit under the action of a (not necessarily proper) subgroup $K < G$ is denoted by $[x]_K$. The *isotropy subgroup* is the subgroup $G_x = \{g \in G \mid gx = x\} < G$. The orbits are disjoint subsets of X, and the set of orbits is the *orbit space* X/G (with the quotient topology, so the projection $X \to X/G$ is open and continuous). Different elements of an orbit will in general have different isotropy subgroups, but they are related by

$$G_{gx} = gG_x g^{-1}.$$

As g runs over elements of G, gx runs over all points of the orbit of x, and G_{gx} runs over all subgroups conjugate to G_x. Thus, although G_x is associated with the point x, it is the conjugacy class $(G_x) = \{G_{gx} \mid g \in G\}$ that is naturally associated with the orbit $[x]_G \in X/G$. We will usually refer to this class as an *isotropy class*.

Definition 2.2.1 Two G-spaces X and X' are said to be *equivariantly homeomorphic* if there exists a homeomorphism $f\colon X \to X'$ and an automorphism θ of G such that $f(gx) = \theta(g)f(x)$, for all $g \in G, x \in X$.

Note that this definition corresponds to the usual *weak equivariant homeomorphism*, but we are only interested in this weaker property so will abreviate the phrase.

It is a very difficult problem to consider how patterns of isotropy classes can occur for a given orbit space of some group G, see for example the work of Bredon [6] or tom Dieck [9]. Here, following the example of quasi-toric manifolds, we only consider manifolds with an action that locally matches Q^n on $\mathbb{H}^n$, and whose orbit spaces are simple polytopes with faces defined by the isotropy classes. We next consider how to characterise actions of Q^n on $\mathbb{H}^n$.

Standard Corner. There are two distinct left actions of Q on $\mathbb{H}$ which can be defined using the non-commutative multiplication of quaternions,

$$\mathsf{Q} \times \mathbb{H} \to \mathbb{H}, \qquad \mu\colon (s,h) \mapsto sh, \qquad \text{and} \quad \bar{\mu}\colon (s,h) \mapsto hs^{-1}. \tag{2.2.2}$$

These two actions are equivariantly homeomorphic if $\mathbb{H}$ is regarded as a real manifold, by

$$\mathbb{H} \to \mathbb{H},\ h \mapsto \bar{h} \qquad \theta = id_{\mathsf{Q}}, \qquad\qquad sh \mapsto \overline{sh} = \bar{h}s^{-1}, \tag{2.2.3}$$

where $\bar{h}$ represents the quaternionic conjugate of h. In both cases, providing $h \neq 0$, the orbit of h is the 3-sphere of radius $|h|$ with centre at the origin, since $|sh| = |h|$ and any other h' with $|h'| = |h|$ can be obtained by acting with $s = h'h^{-1}$ on h. If $h = 0$, the orbit is the origin. The orbit space is thus $\mathbb{R}_{\geqslant}$, the set of non-negative reals, which is the set of possible radii $|h|$. The isotropy subgroup is clearly $\mathbf{1}$ for any $h \neq 0$, or Q for $h = 0$, so the isotropy classes for the orbits of these points are $\mathsf{Q}_\emptyset$ and $\mathsf{Q}_{\{1\}}$ respectively.

Taking the n-fold product of the action μ, above, we have the *standard action*

$$\mathsf{Q}^n \times \mathbb{H}^n \to \mathbb{H}^n, \quad ((s_1,\dots,s_n),(h_1,\dots,h_n)) \mapsto (s_1h_1,\dots,s_nh_n). \tag{2.2.4}$$

We will refer to $\mathbb{H}^n$ with this Q^n action as the *standard n-corner.*

For any point $h \in \mathbb{H}^n$, if all the components h_i are non-zero then the orbit is a product of n 3-spheres, and the isotropy class is the rank 0 class $\mathsf{Q}_\emptyset$. If, however, some of the h_i are zero, writing $\eta = \{i \mid h_i = 0\}$, then the orbit is a product of $(n - |\eta|)$ 3-spheres, sometimes written as $(S^3)^{[n]\setminus\eta}$. Since the action is a direct product, the isotropy subgroup of any point is the product of the coordinate subgroups Q_i for each coordinate $h_i = 0$. The isotropy class for the orbit of this point is the rank $|\eta|$ class $\mathsf{Q}_{\{i_1\},\dots\{i_{|\eta|}\}}$ for $i_j \in \eta$. The orbit space is clearly the direct product $\mathbb{R}^n_{\geqslant}$.

Note that the set η records the coordinates of $h \in \mathbb{H}^n$ which are zero, whereas the characteristic set $\gamma(u)$ of a subgroup $\mathsf{Q}(u)$ of Q^n records the elements of u which are non-zero. These turn out to be the useful definitions, and reflect the fact that for the actions of Q on $\mathbb{H}$ defined by Equation (2.2.2), the isotropy subgroup of the origin is Q, whereas the isotropy subgroup of a point with $h \neq 0$ is $\mathbf{1}$.

2.2.5 Faces of $\mathbb{R}^n_{\geqslant}$. A *face* of the orbit space $\mathbb{R}^n_{\geqslant}$ is a subset of points for which some subset of the coordinates are zero. A *facet* is a face of codimension 1

$$F_i = \{(r_1, \ldots, r_n) \in \mathbb{R}^n_{\geqslant} \mid r_i = 0\}.$$

Any face can be specified by the facets that contain it,

$$F_\sigma = \bigcap_{i \in \sigma} F_i = \{(r_1, \ldots, r_n) \in \mathbb{R}^n_{\geqslant} \mid r_i = 0 \text{ if } i \in \sigma\} \tag{2.2.6}$$

which defines the set $\sigma \subset [n]$.

Let $\mathcal{F}(\mathbb{R}^n_{\geqslant})$ denote the set of faces of $\mathbb{R}^n_{\geqslant}$ (including $\mathbb{R}^n_{\geqslant} = F_\emptyset$). There is a natural partial order on $\mathcal{F}(\mathbb{R}^n_{\geqslant})$ from set inclusion, so $\mathcal{F}(\mathbb{R}^n_{\geqslant})$ is a poset. Hence, we can define a category with the faces of $\mathbb{R}^n$ as objects and the inclusions as morphisms, which we denote by $\text{FACE}(\mathbb{R}^n_{\geqslant})$. The set of all $\sigma \subset [n]$ is partially ordered by inclusion, so defines a category, which we denote by $\text{CAT}([n])$. For any $\sigma \subset [n]$, there is a face F_σ, and if $\sigma \supset \tau$ then $F_\sigma \subset F_\tau$, and conversely. Hence, there is an isomorphism of categories

$$\text{CAT}([n]) \leftrightarrow \text{FACE}(\mathbb{R}^n_{\geqslant})^{\text{OP}}.$$

For the standard action on $\mathbb{H}^n$ (Equation 2.2.4), the isotropy class for any interior point of a face F_σ is $\mathsf{Q}_{\{i_1\},\ldots,\{i_{|\sigma|}\}}$, for $\sigma = \{i_1, \ldots, i_{|\sigma|}\}$. If F_τ is a subface of F_σ then $\sigma \subset \tau$, and $\mathsf{Q}_{\{i_1\},\ldots,\{i_{|\sigma|}\}}$ is clearly a subclass of $\mathsf{Q}_{\{i_1\},\ldots,\{i_{|\tau|}\}}$. That is, the isotropy class of a face F_σ is a rank $|\sigma|$ class, and the isotropy class of a face is a proper subclass of that of any proper subface.

2.2.7 General Actions. We now introduce a more general action of Q^n on $\mathbb{H}^n$ than the standard action (Equation 2.2.4), and in this section discuss its properties. We use the quaternionic multiplication in $\mathbb{H}$ to define the action, but make use of the two possible left actions on $\mathbb{H}$ defined by μ and $\bar{\mu}$ of Equation (2.2.2). We are eventually interested in those actions which satisfy the general properties that we have identified in the standard action, namely

(A1) the orbits are products of 3-spheres,

(A2) the orbit space is the product space $\mathbb{R}^n_{\geqslant}$, and

(A3) the isotropy class of an orbit in a face F_σ is a rank $|\sigma|$ class of subgroups in Q^n.

In this section, we show how to characterise actions and their isotropy classes, and discuss how they are related to the properties (A1–A3). In the next Chapter (3) we restrict attention to those actions which satisfy these three properties.

Let Q^n act on $\mathbb{H}^n$ by $\mathsf{Q}^n \times \mathbb{H}^n \to \mathbb{H}^n$,

$$((s_1, \ldots, s_n), (h_1, \ldots, h_n)) \mapsto (s_{l_1} h_1 s_{r_1}^{-1}, \ldots, s_{l_n} h_n s_{r_n}^{-1}) \tag{2.2.8}$$

where each subscript is the index of some coordinate subgroup Q_k of Q^n or is the empty set, with $s_\emptyset := 1$. This clearly defines a Q^n action on $\mathbb{H}^n$, which leaves each coordinate subspace $\mathbb{H}_i$ invariant, but need not satisfy the conditions (A1–A3). If for each $i = 1, \ldots, n$ the subscripts satisfy $l_i \neq r_i$, then we can collect the exponents of s_{l_i} and $s_{r_i}^{-1}$ into an *exponent matrix.*

Definition 2.2.9 Let the action of Q^n on $\mathbb{H}^n$ be given by Equation (2.2.8). The $n \times n$ *exponent matrix* E is defined for this action by its rows e_i, corresponding to each coordinate h_i,

$$E_{i,j} = (e_i)_j = \delta_{j,l_i} - \delta_{j,r_i}, \qquad i, j = 1, \ldots, n$$

where $\delta_{a,b} = 1$ if $a = b$ and $= 0$ otherwise.

An exponent matrix E then defines an action of Q^n on $\mathbb{H}^n$ in the form given by equation 2.2.8.

Two immediate consequences of condition (A3) are that the isotropy subgroup of an interior point of $\mathbb{H}^n$ (i.e. $h_i \neq 0$, each i) is the trivial subgroup, and that the origin is the only fixed point of $\mathbb{H}^n$. We can then give a geometrical reason for requiring the indices l_i, r_i to be different for each i.

Proposition 2.2.10 *Let the action of Q^n on $\mathbb{H}^n$ be given by Equation* (2.2.8). *If*

(1) *the isotropy subgroup of any interior point $h \in \mathbb{H}^n$ ($h_i \neq 0$, each i) is the trivial subgroup* **1**, *and*

(2) *the origin is the only fixed point,*

then $l_i \neq r_i$ for each i, and $[n] \subset \{l_1, r_1, \ldots, l_n, r_n\}$.

Proof. For any i, the indices l_i, r_i cannot both be empty since then the action would leave each point $(0, \ldots, h_i, \ldots, 0)$ of the subspace $\mathbb{H}_i$ fixed, contradicting (2).

Suppose the non-empty indices l_i and r_i are equal. Then the isotropy subgroup of any interior point contains those elements s_{l_i} of Q_{l_i} such that $s_{l_i} h_i s_{l_i}^{-1} = h_i$, that is, those unit quaternions which commute with h_i. Providing h_i is not real ($h_i \neq \bar{h}_i$), the circle group defined by $\{\cos\alpha + \sin\alpha\ \eta\ \in \mathsf{Q}_{l_i} \mid \alpha \in [0, 2\pi)\}$, where $\eta = (h_i - \bar{h}_i)/|h_i - \bar{h}_i|$, commutes with h_i. If h_i is real, any unit quaternion commutes with it, so the isotropy subgroup is Q_{l_i}. In both cases the isotropy subgroup is not the trivial group, contradicting (1), so $l_i \neq r_i$.

If the set $\{l_1, r_1, \ldots, l_n, r_n\}$ omitted any non-empty index k then the isotropy subgroup of every point, including any interior point, would contain Q_k as a subgroup, contradicting (1). ❑

We sometimes make the choice that each l_i does not take an empty value.

The result of this proposition can be expressed directly in terms of the exponent matrix.

Corollary 2.2.11 *Under the conditions of the proposition, the exponent matrix E satisfies: (1) each row of E is non-zero, (2) each row of E contains at most one positive element and at most one negative element, (3) each column of E is non-zero.*

Proof. (1) Any zero row e_i corresponds to $l_i = r_i = \emptyset$. (2) For each row e_i, there is only one l_i which if non-empty corresponds to a positive $(e_i)_{l_i}$, and only one r_i

which if non-empty corresponds to a negative $(e_i)_{r_i}$. (3) No column is zero, since each non-empty index $k \in [n]$ occurs in the set $\{l_i, r_i \mid i = 1, \ldots, n\}$. ❑

Remark 2.2.13 For any i, interchanging the indices l_i and r_i changes the sign of the i^{th} row of E, and corresponds to taking the quaternionic conjugate of the i^{th} coordinate subspace $\mathbb{H}_i$ of $\mathbb{H}^n$. That is, given an exponent matrix E which specifies an action $\mu\colon \mathsf{Q}^n \times \mathbb{H}^n \to \mathbb{H}^n$ of Q^n on $\mathbb{H}^n$, and an action $\mu'\colon \mathsf{Q}^n \times \mathbb{H}'^n \to \mathbb{H}'^n$ specified by the matrix E' which differs from E only by changing the sign of the row e_i, then there is an equivariant map $\phi\colon \mathbb{H}^n \to \mathbb{H}'^n$, defined by $h = (h_1, \ldots, h_n) \mapsto h' = (h'_1, \ldots, h'_n)$ where $h'_i = \bar{h}_i$ and $h'_j = h_j$ for $j \neq i$, for which $\mu' \circ (id, \phi) = \phi \circ \mu$. Two spaces with actions related by such a change are equivariantly homeomorphic as real manifolds, as in Equation (2.2.3).

Example 2.2.14 Consider an action $\mathsf{Q}^n \times \mathbb{H}^n \to \mathbb{H}^n$ defined by

$$((s_1, \ldots, s_n), (h_1, \ldots, h_n)) \mapsto (s_1 h_1 s_n^{-1}, \ldots, s_{n-1} h_{n-1} s_n^{-1}, s_n h_n).$$

Then, $l_i = i$ for all i, and $r_i = \emptyset$ if $i = n$, and $r_i = n$ otherwise. The exponent matrix of this action is given by,

$$E = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Isotropy Classes. Given an action of Q^n on $\mathbb{H}^n$ as in Equation (2.2.8), the isotropy subgroup of $h = (h_1, \ldots, h_n) \in \mathbb{H}^n$, is the set of $s \in \mathsf{Q}^n$ such that $s_{l_i} h_i s_{r_i}^{-1} = h_i$, for each i. Thus, for $h \in \mathbb{H}^n$, with components $h_i \neq 0$ all i, we obtain n equations $s_{l_i} h_i = h_i s_{r_i}$ in the n unknown unit quaternions s_i. If the isotropy subgroup of an interior point h of $\mathbb{H}^n$ is trivial, these equations must have the unique solution $s_i = 1$, for each i.

For a point h in the subspace of $\mathbb{H}^n$ defined for some j by $h_j = 0$ and $h_i \neq 0$ for $i \neq j$, the equation in h_j is $0 = 0$, leaving $(n-1)$ equations in the n unknowns s_i. If the isotropy subgroup of h is $\mathsf{Q}(u)$ for some non-zero $u \in \mathbb{H}^n$, then these equations must have a solution of the form $s_i = u_i t u_i^{-1}$ or $s_i = 1$ each i, with $t \in \mathsf{Q}$. For a point h in the subspace of $\mathbb{H}^n$ defined by $h_{j_1} = \ldots = h_{j_k} = 0$, with the remaining $(n-k)$ components $h_i \neq 0$ for some $k \leqslant n$, the equation in each h_{j_i} becomes $0 = 0$, leaving $(n-k)$ equations in the n unknowns s_i.

Examples 2.2.15 We consider 4 examples. Define the subspaces of $\mathbb{H}^n$

$$\mathbb{I}_{\widehat{\jmath}} = \{h \in \mathbb{H}^n \mid h_j = 0,\ h_i \neq 0,\ i \neq j\}, \qquad j \in [n]$$
$$\mathbb{I}_{\widehat{\jmath},\widehat{\jmath}'} = \{h \in \mathbb{H}^n \mid h_j = h_{j'} = 0,\ h_i \neq 0,\ i \neq j, j'\} \qquad j \neq j' \in [n].$$

1. Consider the standard action of Q^n on $\mathbb{H}^n$ defined by Equation (2.2.4). Clearly, $l_i = i$ and $r_i = \emptyset$ for each $i = 1, \ldots, n$. The exponent matrix is the $n \times n$ identity matrix.

For an interior point $h \in \mathbb{H}^n$ ($h_i \neq 0$ each i), any $s \in \mathsf{Q}^n$ in the isotropy subgroup of h, satisfies $s_i h_i = h_i$, each i. Thus $s_i = 1$ for each i, and the isotropy subgroup is trivial. For any h in the subspace $\mathbb{I}_{\widehat{\jmath}}$, any $s \in \mathsf{Q}^n$ in the isotropy subgroup of h, satisfies $s_i h_i = h_i$ for $i \neq j$, and the equation involving h_j and s_j is $0 = 0$. Hence the isotropy subgroup of h is Q_j, which is in the class $\mathsf{Q}_{\{j\}}$.

For an h in the subspace $\mathbb{I}_{\widehat{\jmath},\widehat{k}}$, any s in the isotropy subgroup of h, satisfies $s_i h_i = h_i$ for $i \neq j, k$, and the equations involving h_j and h_k are $0 = 0$. Hence the isotropy subgroup of h is $\mathsf{Q}_j \times \mathsf{Q}_k$, which is in the class $\mathsf{Q}_{\{j\},\{k\}}$.

2. Fix $k \in [n]$, and let the action of Q^n on $\mathbb{H}^n$ be given by

$$((s_1, \ldots, s_n), (h_1, \ldots, h_n)) \mapsto (s_1 h_1 s_k^{-1}, \ldots, h_k s_k^{-1}, \ldots, s_n h_n s_k^{-1}).$$

The exponent matrix is given by $E_{i,j} = 1$ (for $i = j \neq k$), $E_{i,j} = -1$ (for $j = k$), and 0 otherwise.

For an interior point h of $\mathbb{H}^n$, any $s \in \mathsf{Q}^n$ in the isotropy subgroup of h, satisfies $h_k s_k^{-1} = h_k$, so $s_k = 1$, and $s_i h_i s_k^{-1} = h_i$, so $s_i = 1$ (for $i \neq k$) and the isotropy subgroup is trivial.

Consider a point h in the subspace $\mathbb{I}_{\widehat{\jmath}}$ for $j \neq k$. Then any s in the isotropy subgroup of h is determined by $h_k s_k^{-1} = h_k$ so $s_k = 1$, and for $i \neq j,k$ by $s_i h_i s_k^{-1} = h_i$ so $s_i = h_i s_k h_i^{-1} = 1$. For $i = j$ there is no condition $(0 = 0)$ so s_j is unrestricted. Thus the isotropy subgroup of $h \in \mathbb{I}_{\widehat{\jmath}}$ is Q_j, and the isotropy class of its orbit is $\mathsf{Q}_{\{j\}}$.

For $h \in \mathbb{I}_{\widehat{k}}$, any s in the isotropy subgroup of h is determined by $s_i h_i s_k^{-1} = h_i$, so $s_i = h_i s_k h_i^{-1}$ for each $i \neq k$, and the condition involving h_k is $0 = 0$. Thus the isotropy subgroup can be written as $\mathsf{Q}(u)$ with the components of u given by $u_i = h_i/|h_i|$ for $i \neq k$ and $u_k = 1$. The isotropy class of its orbit is $\mathsf{Q}_{\{1,\ldots,n\}}$.

For $h \in \mathbb{I}_{\widehat{\jmath},\widehat{\jmath'}}$, for $j, j' \neq k$, any s in the isotropy subgroup of h is determined by $h_k s_k^{-1} = h_k$ so $s_k = 1$, and for $i \neq j, j'$ by $s_i h_i s_k^{-1} = h_i$ so $s_i = h_i s_k h_i^{-1} = 1$. For $i = j, j'$ the conditions are $0 = 0$, so s_j and $s_{j'}$ are unrestricted. Thus the isotropy subgroup of h is $\mathsf{Q}_j \times \mathsf{Q}_{j'}$, and the isotropy class of its orbit is $\mathsf{Q}_{\{j\},\{j'\}}$.

For $h \in \mathbb{I}_{\widehat{\jmath},\widehat{k}}$, any s in the isotropy subgroup of h is determined by $s_i h_i s_k^{-1} = h_i$, so $s_i = h_i s_k h_i^{-1}$ for each $i \neq j,k$, and the conditions involving h_j and h_k are $0 = 0$. Thus the isotropy subgroup can be written as $\mathsf{Q}(u) \times \mathsf{Q}_j$ with the components of u given by $u_i = h_i/|h_i|$ for $i \neq j,k$, $u_j = 0$ and $u_k = 1$. Thus, the isotropy class of the orbit of h is $\mathsf{Q}_{\{j\},\{1,..,\widehat{\jmath},..,n\}}$, where $\widehat{\jmath}$ indicates that the index j is omitted.

3. Let $n = 3$, and consider the action of Q^3 on $\mathbb{H}^3$ specified by

$$((s_1, s_2, s_3), (h_1, h_2, h_3)) \mapsto (s_3 h_1 s_2^{-1}, s_1 h_2, s_1 h_3 s_2^{-1}).$$

For an interior point h of $\mathbb{H}^n$, any $s \in \mathsf{Q}^n$ in the isotropy subgroup of h satisfies $s_1 h_2 = h_2$, so $s_1 = 1$; $s_1 h_3 s_2^{-1} = h_3$, so $s_2 = 1$; and $s_3 h_1 s_2^{-1} = h_1$, so $s_3 = 1$; so the isotropy subgroup is trivial.

For $h \in \mathbb{I}_{\widehat{1}}$, any s in the isotropy subgroup of h satisfies $s_1 = 1$, $s_2 = 1$, and s_3 is unrestricted. Thus the isotropy subgroup of $h \in \mathbb{I}_{\widehat{1}}$ is Q_3, and the isotropy class of its orbit is $\mathsf{Q}_{\{3\}}$.

For $h \in \mathbb{I}_{\widehat{2}}$, any s in the isotropy subgroup of h satisfies $s_2 = h_3^{-1} s_1 h_3$, $s_3 =$

$h_1 s_2 h_1^{-1}$, and s_1 is unrestricted. Thus the isotropy subgroup of $h \in \mathbb{I}_{\hat{2}}$ is $\mathsf{Q}(u)$ for $u = (1, h_3^{-1}|h_3|, h_1 h_3^{-1}/|h_1 h_3^{-1}|)$, and the isotropy class of its orbit is $\mathsf{Q}_{\{1,2,3\}}$.

For $h \in \mathbb{I}_{\hat{3}}$, any s in the isotropy subgroup of h satisfies $s_1 = 1$, $s_3 = h_1 s_2 h_1^{-1}$, and s_2 is unrestricted. Thus the isotropy subgroup of $h \in \mathbb{I}_{\hat{3}}$ is $\mathsf{Q}(u)$ for $u = (0, 1, h_1/|h_1|)$, and the isotropy class of its orbit is $\mathsf{Q}_{\{2,3\}}$.

For $h \in \mathbb{I}_{\hat{1},\hat{2}}$, any s in the isotropy subgroup of h satisfies $s_1 = h_3 s_2 h_3^{-1}$, and s_2, s_3 are unrestricted. Thus the isotropy subgroup of $h \in \mathbb{I}_{\hat{2}}$ is $\mathsf{Q}_3 \times \mathsf{Q}(u)$ for $u = (h_3/|h_3|, 1, 0)$, and the isotropy class of its orbit is $\mathsf{Q}_{\{1,2\},\{3\}}$.

For $h \in \mathbb{I}_{\hat{1},\hat{3}}$, any s in the isotropy subgroup of h satisfies $s_1 = 1$ and s_2, s_3 are unrestricted. Thus the isotropy subgroup of h is $\mathsf{Q}_2 \times \mathsf{Q}_3$, and the isotropy class of its orbit is $\mathsf{Q}_{\{2\},\{3\}}$.

For $h \in \mathbb{I}_{\hat{2},\hat{3}}$, any s in the isotropy subgroup of h satisfies $s_3 = h_1 s_2 h_1^{-1}$, and s_1, s_2 are unrestricted. Thus the isotropy subgroup of h is $\mathsf{Q}_1 \times \mathsf{Q}(u)$ for $u = (0, 1, h_1/|h_1|)$, and the isotropy class of its orbit is $\mathsf{Q}_{\{1\},\{2,3\}}$.

The subspace $\mathbb{I}_{\hat{1},\hat{2},\hat{3}}$ contains only the origin, whose isotropy subgroup is Q^3, and the isotropy class of its orbit is $\mathsf{Q}_{\{1\},\{2\},\{3\}}$.

4. Let $n = 3$, and consider the action of Q^3 on $\mathbb{H}^3$ specified by

$$((s_1, s_2, s_3), (h_1, h_2, h_3)) \mapsto (s_1 h_1 s_2^{-1}, s_2 h_2 s_1^{-1}, s_3 h_3).$$

For an interior point h of $\mathbb{H}^n$, any $s \in \mathsf{Q}^n$ in the isotropy subgroup of h satisfies $s_3 h_3 = h_3$, so $s_3 = 1$; $s_2 h_2 s_1^{-1} = h_2$, so $s_2 = h_2 s_1 h_2^{-1}$; and $s_1 h_1 s_2^{-1} = h_1$, so $s_1 = h_1 s_2 h_1^{-1}$. Hence we have the relation $s_1 = h_1 h_2 s_1 h_2^{-1} h_1^{-1}$. That is, s_1 commutes with $h_1 h_2$, so providing that $h_1 h_2$ is not real s_1 is an element of the circle group $\{\cos\alpha + \sin\alpha\,\eta \mid \alpha \in [0, 2\pi)\}$, for $\eta = (h_1 h_2 - \bar{h}_2 \bar{h}_1)/|h_1 h_2 - \bar{h}_2 \bar{h}_1|$. Thus the isotropy subgroup may be written as $\mathsf{T}(u) = \{(u_1 t u_1^{-1}, u_2 t u_2^{-1}, 1) \in \mathsf{Q}^3 \mid h_1 h_2 t = t h_1 h_2\}$ for $u = (1, h_2/|h_2|, 0)$. If $h_1 h_2$ is real then any s_1 commutes with $h_1 h_2$, so the isotropy subgroup is $\mathsf{Q}(u)$ for $u = (1, h_2/|h_2|, 0)$.

For $h \in \mathbb{I}_{\hat{3}}$, any s in the isotropy subgroup of h satisfies $s_2 = h_2 s_1 h_2^{-1}$, $s_1 = h_1 s_2 h_1^{-1}$, and s_3 is unrestricted. Thus the isotropy subgroup of $h \in \mathbb{I}_{\hat{3}}$ is the product of Q_3 with either $\mathsf{Q}(u)$ or the circle group $\mathsf{T}(u)$ defined above, for h_1h_2 real or not respectively.

For $h \in \mathbb{I}_{\widehat{2}}$, any s in the isotropy subgroup of h satisfies $s_3 = 1$, $s_1 = h_1s_2h_1^{-1}$, and s_2 is unrestricted. Thus the isotropy subgroup of $h \in \mathbb{I}_{\widehat{2}}$ is $\mathsf{Q}(u)$ for $u = (h_1/|h_1|, 1, 0)$, and the isotropy class of its orbit is $\mathsf{Q}_{\{1,2\}}$.

For $h \in \mathbb{I}_{\widehat{1}}$, any s in the isotropy subgroup of h satisfies $s_3 = 1$, $s_2 = h_2s_1h_2^{-1}$, and s_1 is unrestricted. Thus the isotropy subgroup of $h \in \mathbb{I}_{\widehat{1}}$ is $\mathsf{Q}(u)$ for $u = (1, h_2/|h_2|, 0)$, and the isotropy class of its orbit is $\mathsf{Q}_{\{1,2\}}$.

Thus, we have three examples (1-3 above) of actions of Q^n on $\mathbb{H}^n$ of the form Equation (2.2.8), which have the trivial rank 0 isotropy class for any interior point of $\mathbb{H}^n$, rank 1 isotropy classes for orbits of points in the subspaces $\mathbb{I}_{\widehat{\jmath}}$ for each $j \in [n]$, and rank 2 isotropy classes for orbits of points in the subspaces $\mathbb{I}_{\widehat{\jmath},\widehat{\jmath'}}$ for each $j \neq j' \in [n]$. For each of the subspaces, the isotropy class is the same for the orbit of any point in the subspace, so we may think of the class as being attached to the subspace itself. Furthermore, for all $j \neq j' \in [n]$ the isotropy class of $\mathbb{I}_{\widehat{\jmath}}$ is a subclass of $\mathbb{I}_{\widehat{\jmath},\widehat{\jmath'}}$. Also, in each example the rank 1 isotropy classes of the subspaces $\mathbb{I}_{\widehat{\jmath}}$ and $\mathbb{I}_{\widehat{\jmath'}}$ (for $j \neq j' \in [n]$) are compatible (Definition 2.1.14).

The fourth example above shows that we need further conditions even to ensure that the isotropy class is trivial for any interior point of $\mathbb{H}^n$. In other words, the converse of Proposition 2.2.10 does not hold. In this example (4) the isotropy subgroup of an interior point of $\mathbb{H}^3$ is either $\mathsf{Q}(u)$ or the circle group $\mathsf{T}(u)$ for h_1h_2 real or not respectively, where $u = (1, h_2/|h_2|, 0)$. The isotropy subgroup of a point in $\mathbb{I}_{\widehat{3}}$ is either $\mathsf{Q}_3 \times \mathsf{Q}(u)$ or $\mathsf{Q}_3 \times \mathsf{T}(u)$ for h_1h_2 real or not respectively. Furthermore, the isotropy subgroup of points in the subspaces $\mathbb{I}_{\widehat{1}}$ and $\mathbb{I}_{\widehat{2}}$ are in the same class $\mathsf{Q}_{\{1,2\}}$, so the classes do not distinguish the faces. In the next Chapter (3) we will exclude such actions.

For those cases where the isotropy class of each subspace $\mathbb{I}_{\widehat{\imath}}$ of $\mathbb{H}^n$ is a rank 1 class, we introduce the following definition.

Definition 2.2.16 If an action of Q^n on $\mathbb{H}^n$ of the form of Equation (2.2.8) defines for each subspace $\mathbb{I}_{\widehat{i}}$ of $\mathbb{H}^n$ a rank 1 isotropy class Q_{γ_i} the $n \times n$ *isotropy matrix* L is defined by its columns y_i,

$$L_{j,i} = (y_i)_j \quad = 1 \quad j \in \gamma_i, \qquad 0 \quad \text{otherwise}, \qquad\qquad i, j = 1, \ldots, n.$$

We say that two columns y_i, y_k of L are compatible if the γ_i, γ_k that define them are compatible. Also, we write $y_i \prec y_k$ if $\gamma_i \subset \gamma_k$.

Proposition 2.2.17 *If an action of Q^n on $\mathbb{H}^n$ is specified by an $n \times n$ exponent matrix E which satisfies (1a) each row is non-zero, (1b) each row has at most one positive and one negative element, (1c) each column is non-zero, (see Corollary 2.2.11), and has an $n \times n$ isotropy matrix L (see Definition 2.2.16), then $EL = \mathcal{E}$, where $\mathcal{E}$ is a diagonal matrix with diagonal elements equal to $0, \pm 1$.*

Furthermore, if all diagonal elements of $\mathcal{E}$ are non-zero then E and L each have determinant ± 1.

Proof. Consider the subspace $\mathbb{I}_{\widehat{k}}$ (as in Example 2.2.15), and the column y_k of L representing the isotropy class Q_{γ_k} of $\mathbb{I}_{\widehat{k}}$. For any $h \in \mathbb{I}_{\widehat{k}}$, let the isotropy subgroup of h be $\mathsf{Q}(u) < \mathsf{Q}^n$, for some non-zero $u \in \mathbb{H}^n$ for which γ_k is the characteristic set. Then any $s = (s_1, \ldots, s_n) \in \mathsf{Q}(u)$ must satisfy the conditions (C_i:) $s_{l_i} h_i s_{r_i}^{-1} = h_i$, for $i \neq k$.

If l_i and r_i are both non-empty, then condition (C_i) can be written as $s_{l_i} = (h_i/|h_i|)\, s_{r_i} (h_i/|h_i|)^{-1}$, and the components s_{l_i} and s_{r_i} are conjugate, since $h_i \neq 0$ so $h_i/|h_i|$ is a unit quaternion. Hence, the components u_{l_i} and u_{r_i} of u are both 0 or both non-zero. Hence, the indices l_i and r_i are both elements of γ_k or neither are. Hence, the components $(y_k)_{l_i}$ and $(y_k)_{r_i}$ of the column y_k of L, are are both 0 or both 1. The i^{th} row of E contains the non-zero elements $e_{i,l_i} = 1$ and $e_{i,r_i} = -1$, so we may express the relation between the components of the column y_k in the form $e_{i,l_i}(y_k)_{l_i} + e_{i,r_i}(y_k)_{r_i} = 0$. Since the other elements of the i^{th} row of E are zero, and (y_k) is the k^{th} column of L, we may rewrite this relation as $\sum_j E_{i,j} L_{j,k} = 0$, for $i \neq k$.

If only one of l_i, r_i is non-empty, say l_i, then the condition (C_i) can be written as $s_{l_i} h_i = h_i$, so $s_{l_i} = 1$ and $u_{l_i} = 0$. Hence, $l_i \notin \gamma_k$ and $(y_k)_{l_i} = 0$, so $e_{i,l_i}(y_k)_{l_i} = 0$.

Since e_{i,l_i} is the only non-zero element of the i^{th} row of E, we have $\sum_j E_{i,j}L_{j,k} = 0$, for $i \neq k$. A similar argument applies if r_i is non-empty and $l_i = \emptyset$.

If $i = k$ then $h_k = 0$, and there is no further condition on the group components, so $\sum_j E_{k,j}L_{j,k}$ is undetermined. There can be at most two non-zero elements of E, of opposite sign, in the sum, and all elements of L are 0 or 1, so the sum can only be 0 or ± 1. Thus, $\mathcal{E} = EL$ is diagonal, and can only have non-zero elements of ± 1.

If all the diagonal elements of $\mathcal{E}$ are non-zero, it has determinant ± 1. Since E and L are integer matrices their determinants are integral, so each determinant must be ± 1. ❑

The sign ambiguity of the rows of E, from interchanging the subscripts l_i, r_i noted above (Remark 2.2.13), is apparent here. A change of sign of any row, corresponding to changing a coordinate subspace from $\mathbb{H}$ to $\overline{\mathbb{H}}$, changes the sign of the corresponding element of $\mathcal{E}$. In the case where no diagonal elements of $\mathcal{E}$ are zero we have $(\mathcal{E}E)L = \mathbf{1}$, and taking $E' = \mathcal{E}E$ is the canonical choice for the exponent matrix.

Examples 2.2.18 Continuing Examples 2.2.15.

1. For the standard action of Q^n on $\mathbb{H}^n$ (see Example 2.2.15(1)), the exponent matrix is the $n \times n$ identity matrix. The isotropy class of the orbit of any point in the subspace $\mathbb{I}_{\hat{\imath}}$ is $\mathsf{Q}_{\{i\}}$, so the isotropy matrix L is also the identity matrix, as is $\mathcal{E} = EL$.

2. For the action of Q^3 on $\mathbb{H}^3$ (taking $k = n = 3$ in Example 2.2.15(2)) defined by,

$$((s_1, s_2, s_3), (h_1, h_2, h_3)) \mapsto (s_1 h_1 s_3^{-1}, s_2 h_2 s_3^{-1}, h_3 s_3^{-1}) :$$

$$E = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \; L = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{E} = \operatorname{diag}(1, 1, -1).$$

3. For the action defined by (2.2.15(3)),

$$((s_1, s_2, s_3), (h_1, h_2, h_3)) \mapsto (s_3 h_1 s_2^{-1}, s_1 h_2, s_1 h_3 s_2^{-1}) :$$

$$E = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \; L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathcal{E} = \operatorname{diag}(1, 1, -1).$$

4. But, for the action defined by (2.2.15(4)),

$$((s_1, s_2, s_3), (h_1, h_2, h_3)) \mapsto (s_1 h_1 s_2^{-1}, s_2 h_2 s_1^{-1}, s_3 h_3) :$$

$$E = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

No isotropy matrix can be defined since the isotropy group of any point in the subspace $\mathbb{I}_{\hat{3}}$ is not isomorphic to Q. Also, note that the determinant of E is zero.

We now show that having determinant ± 1 is the crucial extra condition on the exponent matrix to ensure that the isotropy class at interior points of $\mathbb{H}^n$ is trivial, and also to obtain orbits of the expected form.

Proposition 2.2.19 *For an action of Q^n on $\mathbb{H}^n$ specified by an $n \times n$ exponent matrix E, the isotropy subgroup of every interior point of $\mathbb{H}^n$ is trivial if and only if* $\det E = \pm 1$.

We first prove the following lemma.

Lemma 2.2.20 *If a square matrix E with non-zero elements equal to ± 1, satisfies (1) every row has at most one positive element and at most one negative element, and (2) its determinant is non-zero, then there exists a row of E with only one non-zero element.*

Proof. Suppose otherwise, then every row has either two non-zero elements or none. But if any row has no non-zero elements, the determinant of E is zero. If every row has two non-zero elements they are $+1$ and -1, so the sum of the elements of every row is zero, so the sum of the columns is zero, hence the determinant is zero. Thus, there is some row with only one non-zero element. ❏

Proof of 2.2.19. We show three things: (1) the determinant of E can only take values 0 or ± 1; (2) if the determinant is ± 1 then the isotropy subgroup of any interior point is trivial; and (3) if the determinant is zero then some interior points have a non-trivial isotropy subgroup.

(1) $\det E$.

Suppose $\det E \neq 0$. There is some row with only one non-zero element (Lemma 2.2.20). Then inductively, for a $p \times p$ submatrix E_p of E, starting with $p = n$ and $E_n = E$, delete the row and column containing this non-zero element to obtain E_{p-1}. Then E_{p-1} has the same determinant up to a factor of ± 1, and each row still has at most one positive and one negative element. Hence again E_{p-1} has some row with a single non-zero element. The induction ends with E_1, with element equal to ± 1. Hence the determinant of E is ± 1, if it is non-zero.

(2) Case $\det E = \pm 1$.

Suppose $h \in \mathbb{H}^n$ is an interior point, $h_i \neq 0$ all i, and let $s \in \mathsf{Q}^n$ be in the isotropy subgroup of h, so $s_{l_i} h_i s_{r_i}^{-1} = h_i$, for each i. There exists some row of E that contains only one non-zero element, say $e_{i,j}$, (Lemma 2.2.20). Hence either $s_j h_i = h_i$ or $h_i s_j^{-1} = h_i$, for $e_{i,j} = 1$ or -1 respectively, and so in either case $s_j = 1$. Now consider the submatrix E_{p-1} of E obtained by removing the i^{th} row and j^{th} column from E_p, which corresponds to the subgroup action on the subspace obtained by omitting Q_j and $\mathbb{H}_i$. Then E_{p-1} has determinant ± 1, since the removed row had only one non-zero element, and each row still has at most one positive or negative element. Hence inductively, every $s_j = 1$, so the isotropy subgroup is trivial.

(3) Case $\det E = 0$.

Since the determinant is zero, there exists a non-zero column vector of integers x such that $Ex = 0$. The i^{th} row of the equation $Ex = 0$ gives $x_{l_i} - x_{r_i} = 0$ if l_i, r_i are both non-empty, or $x_{l_i} = 0$ or $-x_{r_i} = 0$ if one index is empty, so any non-zero elements of x that are related must be equal. Thus, we can take every element of x to be 0 or 1, since x is only defined up to a normalisation factor. We now define a subgroup of Q^n, isomorphic to Q, as follows. For any $t \in \mathsf{Q}$, define the element $s_j \in \mathsf{Q}_j$ by $s_j = t^{x_j}$, so s_j is equal to t or 1 for each j. Then, for real $h_i = |h_i|$, we have $s_{l_i}|h_i|s_{r_i}^{-1} = \prod_j (s_j)^{e_{i,j}}|h_i| = \prod_j (t^{x_j})^{e_{i,j}}|h_i| = (t^{\sum_j e_{i,j} x_j})|h_i| = (t^0)|h_i| = |h_i|$. Thus, the subgroup $\{(t^{x_1}, \ldots, t^{x_n})\}$ is isomorphic to Q, and leaves the points $h = (|h_1|, \ldots, |h_n|) \in \mathbb{H}^n$ fixed, so the isotropy subgroup is not trivial for all interior points. ❏

Corollary 2.2.21 *For an action of Q^n on $\mathbb{H}^n$ specified by an $n \times n$ exponent matrix E, if the isotropy subgroup of an interior point of $\mathbb{H}^n$ is trivial, then the isotropy subgroup of a point in any subspace $\mathbb{H}_{\widehat{k}} = \{h \in \mathbb{H}^n \mid h_i = 0 \iff i = k\}$ for any $k \in [n]$, is in a rank 1 class.*

Furthermore, the isotropy subgroups of all points in a subspace $\mathbb{H}_{\widehat{k}}$ are in the same isotropy class.

Proof. From the proposition, if the isotropy subgroup of an interior point $h \in \mathbb{H}^n$ is trivial then $\det E$ is ± 1. Suppose $h \in \mathbb{H}_{\widehat{k}}$, and $s \in \mathsf{Q}^n$ is an element of the isotropy subgroup of h, then we repeat the proof (2) of the proposition. The induction proceeds as before until for some p the relation is $s_j h_k = h_k$ or $h_k s_j^{-1} = h_k$. Such a p exists since each of the n components of h occurs in one of the n steps of the induction. But now $h_k = 0$, so instead of forcing $s_j = 1$, we have s_j unrestricted. Continuing the induction, the subsequent relations take the form $s_{l_i} h_i s_{r_i}^{-1} = h_i$, so $s_{l_i} = (h_i/|h_i|)\, s_{r_i} (h_i/|h_i|)^{-1}$. Since every s_i occurs in some relation (no zero column since $\det E = \pm 1$), each s_i is either 1 or conjugate to s_j, Hence the isotropy subgroup is of the form $\mathsf{Q}(u)$ for some non-zero $u \in \mathbb{H}^n$, (since the component u_j of u is equal to 1), so an element of a rank 1 class.

Furthermore, for any other $h' \in \mathbb{H}_{\widehat{k}}$ the relation $s_{l_j} = (h'_j/|h'_j|)\, s_{r_j} (h'_j/|h'_j|)^{-1}$ holds precisely when $s_{l_j} = (h_j/|h_j|)\, s_{r_j} (h_j/|h_j|)^{-1}$ so s_{l_j} is conjugate to s_{r_j}, since $h'_j \neq 0$ if and only if $h_j \neq 0$. Thus the isotropy subgroups $\mathsf{Q}(u)$ and $\mathsf{Q}(u')$ of h and h' respectively have the same characteristic sets, so their orbits have the same isotropy class. ❑

Proposition 2.2.22 *For an action of Q^n on $\mathbb{H}^n$ specified by an exponent matrix E, every point $h = (h_1, \ldots, h_n) \in \mathbb{H}^n$ is in the orbit of $(|h_1|, \ldots, |h_n|)$ if and only if* $\det E \neq 0$.

Proof. From the proof of the previous proposition $\det E$ is equal to 0 or ± 1.

(1) Case $\det E = \pm 1$.

Suppose that for any point $h \in \mathbb{H}^n$ there is some $s \in \mathsf{Q}^n$ such that $h_i = s_{l_i} |h_i| s_{r_i}^{-1}$ for each i. We can express these relations as

$$h_i = \prod_j s_j^{e_{i,j}} |h_i|, \qquad i = 1, \dots, n \tag{2.2.23}$$

since each $|h_i|$ is real and commutes with the s_j, providing we are careful about the order in which the factors are multiplied, ensuring that for each i any positive exponent factor occurs to the left of any negative exponent factor. We can regard Equation (2.2.23) as a system of n equations in the n unknowns s_j specified by the matrix E, for which we want to show there always exists a solution, whether or not the solution is unique. The proof is by induction on p, for a $p \times p$ submatrix E_p of E, starting with $p = n$ and $E_n = E$. Suppose inductively that there are p equations specified by E_p in the p remaining unknowns among the components s_j of s. Since E_p has determinant ± 1 and at most one positive and one negative element in each row, there exists some row of E_p with only one non-zero element, say $e_{k,l}$ (Lemma 2.2.20). Then either $h_k = s_l^{e_{k,l}} |h_k|$, or $h_k = s_l u^{-1} |h_k|$, or $h_k = u\, s_l^{-1} |h_k|$, where u represents the value of some s_j already found and is a product of some of the $(h_j/|h_j|)^{\pm 1}$ in some particular order derived from the inductive procedure. So s_l is equal to $(h_k/|h_k|)^{e_{k,l}}$ or $(h_k/|h_k|)\, u$ or $u\,(h_k/|h_k|)^{-1}$, respectively. In each case s_l is a product of some $(h_j/|h_j|)^{\pm 1}$. If $|h_k| = 0$ then any value for s_l can be used. Now substitute this value for s_l throughout, and delete the equation containing h_k from the set of equations. Then we have a new system of $(p-1)$ equations in the $(p-1)$ remaining unknowns among the s_j. These equations correspond to the matrix E_{p-1}, derived from E_p by deleting the row and column containing $e_{k,l}$. Then E_{p-1} also has at most one positive and one negative element in each row, and its determinant is also ± 1 since the deleted row had only one non-zero element, equal to ± 1. The induction ends after $p = 1$, when all the s_j have been assigned some value and there is no further submatrix. Thus, there is always a solution for $s = (s_1, \dots, s_n)$, so every point (h_i) is in the orbit of $(|h_i|)$.

(2) Case $\det E = 0$.

For any interior point $h \in \mathbb{H}^n$ ($h_i \neq 0$, all i) with isotropy subgroup H_h the orbit O_h is homeomorphic to Q^n/H_h (since the action of the Lie group Q^n is continuous). For real interior points, $h_i = |h_i| > 0$ all i, the isotropy subgroup H_h contains a copy of Q, from the proof of the previous proposition for the case $\det E = 0$, so the real dimension of the orbit is less than that of Q^n, so not all points (h'_i) with $|h'_i| = |h_i|$ can be in the orbit of h. ❑

Corollary 2.2.24 *Under the conditions of the proposition, if* $\det E \neq 0$ *the element* $s \in \mathsf{Q}^n$ *such that* $h = s|h|$ *then each* s_i *is a product of some of the* $(h_j/|h_j|)^{\pm 1}$.

Proof. The inductive procedure in the proof (1) determines this form, and the order in which the products are multiplied. ❑

Corollary 2.2.25 *Under the conditions of the proposition, if* $\det E \neq 0$ *the orbit space is* $\mathbb{R}^n_{\geqslant}$*, and the orbits are all products of 3-spheres, one for each* $h_i \neq 0$.

Proof. Any point $(h'_1, \ldots, h'_n) \in \mathbb{H}^n$ is in the orbit of $(|h_1|, \ldots, |h_n|)$ if $|h'_i| = |h_i|$ for each i by the proposition; and h' is in the orbit of $(|h_1|, \ldots, |h_n|)$ only if $|h'_i| = |h_i|$ each i since the group action only involves multiplication by unit quaternions ($|s_{l_i} h_i s_{r_i}^{-1}| = |h_i|$). Thus each orbit is specified by the set of possible radii, $(|h_1|, \ldots, |h_n|)$, and the orbit space is $\mathbb{R}^n_{\geqslant}$.

For each $i = 1, \ldots, n$ the set of points $|h_i| = r \geqslant 0$ is a 3-sphere if $r > 0$, or is the origin if $r = 0$. Hence, the orbit of $(h_1, \ldots, h_n)$ is the product of 3-spheres, one for each $|h_i| \neq 0$. ❑

Remark 2.2.26 The corresponding results of the last two propositions and corollaries are also true in the toric case, using similar arguments, even though the determinant of the analogue of the exponent matrix can take any integer value.

A general T^n action on $\mathbb{C}^n$ can be specified by a matrix with integer elements $E = (e_{i,j})$. For $t \in \mathsf{T}^n$ and $z, z' \in \mathbb{C}^n$, define $\mathsf{T}^n \times \mathbb{C}^n \to \mathbb{C}^n$ by $(t, z) \mapsto z'$ where each component $z'_i = \prod_j t_j^{e_{i,j}} z_i$. Then (1) the isotropy subgroup of any interior point of $\mathbb{C}^n$ is trivial if and only if $\det E = \pm 1$, and (2) every point $(z_1, \ldots, z_n)$ lies in the orbit of $(|z_1|, \ldots, |z_n|)$ if and only if $\det E \neq 0$. We omit the proof here.

Chapter 3

Regular Corners

In this section we now restrict our attention to actions of Q^n on $\mathbb{H}^n$ corresponding to exponent and isotropy matrices which have a non-zero determinant, which we call *acceptable*. We also introduce two other related objects. The properties of these objects are determined, and the relationship between them is made precise (Corollary 3.1.21). Any $n \times n$ *acceptable exponent matrix* is an exponent matrix so defines an action of Q^n on $\mathbb{H}^n$, hence we define a class of *regular actions* of Q^n on $\mathbb{H}^n$, which we call *regular corners* (Definition 3.1.22). In particular, these actions have the properties that the orbits are products of 3-spheres, the orbit space is $\mathbb{R}^n_{\geqslant}$, and the isotropy class of any interior point of $\mathbb{H}^n$ is trivial.

It also turns out that the regular actions on $\mathbb{H}^n$ are not all equivariantly homeomorphic, in contrast to the toric case, and we show how to classify the regular actions for each $\mathbb{H}^n$ up to equivariant homeomorphism.

In section 3.2 we give an alternative way of writing a regular corner, directly in terms of an *acceptable isotropy functor*, which can be derived directly from an acceptable isotropy matrix, and show that it is equivariantly homeomorphic to regular corner defined by the exponent matrix which is the inverse of the isotropy matrix.

3.1 Properties of a Regular Corner

We begin with the definition of acceptable exponent and isotropy matrices.

Definition 3.1.1 An $n \times n$ exponent matrix E is called an *acceptable exponent matrix* if its determinant is ± 1.
Two acceptable exponent matrices are defined to be *congruent* if one can be obtained from the other by a permutation of the rows, a permutation of the columns, and a change of sign of some rows.

If two acceptable exponent matrices are congruent, the row permutation corresponds to a permutation of the coordinate subspaces of $\mathbb{H}^n$, and the column permutation to a permutation of the coordinate subgroups of Q^n. A change of sign of a row corresponds to the conjugation of some coordinate subspace $\mathbb{H} \to \bar{\mathbb{H}}$ (Remark 2.2.13). Hence a congruence between E matrices induces an equivariant homeomorphism between the corresponding actions of Q^n on $\mathbb{H}^n$, regarded as real manifolds.

The unit matrix is clearly an acceptable exponent matrix, and corresponds to the standard action (Equation 2.2.4, Example 2.2.18(1)).

For an action of Q^n on $\mathbb{H}^n$ specified by an acceptable exponent matrix E the orbit space of the action is $\mathbb{R}^n_{\geqslant}$ (Corollary 2.2.25), so we can define the orbit projection

$$p\colon \mathbb{H}^n \to \mathbb{R}^n_{\geqslant}, \qquad h = (h_1, \ldots, h_n) \mapsto (|h_1|, \ldots, |h_n|). \tag{3.1.2}$$

The subspaces $\mathbb{H}_{\widehat{k}}$ of $\mathbb{H}^n$ defined by $h_i = 0$ if and only if $i = k \in [n]$ are the interiors of the inverse images $p^{-1}(F_k)$ of the facets F_k of the orbit space $\mathbb{R}^n_{\geqslant}$. (See Paragraph 2.2.5 for faces and facets of $\mathbb{R}^n$.) The isotropy subgroup of an interior point of $\mathbb{H}^n$ is the trivial subgroup (Proposition 2.2.19). The isotropy subgroup of each interior point of $p^{-1}(F_k)$ is an element of the same rank 1 class (Corollary 2.2.21), so the isotropy class can be associated with the facet F_k. Thus, the isotropy matrix L is defined (Definition 2.2.16), and $(\mathcal{E}E)$ and L are inverses (Proposition 2.2.17).

Definition 3.1.3 An $n \times n$ isotropy matrix L is called an *acceptable isotropy matrix* if its inverse $L^{-1} = (\mathcal{E}E)$ is an acceptable exponent matrix.
Two acceptable isotropy matrices are defined to be *congruent* if one can be obtained from the other by a permutation of the rows and a permutation of the columns.

If two acceptable isotropy matrices are congruent, the column permutation corresponds to a row permutation of E or a permutation of the coordinate subspaces of $\mathbb{H}^n$, and the row permutation to a permutation of the coordinate subgroups of Q^n.

The unit matrix is clearly an acceptable isotropy matrix, and corresponds to the standard action (Equation 2.2.4, Example 2.2.18(1)).

Proposition 3.1.4 *For an action of Q^n on $\mathbb{H}^n$ specified by an exponent matrix E, then $L = (\mathcal{E}E)^{-1}$ is an acceptable isotropy matrix, if and only if*

(1) $\det L = \pm 1$,

(2) the columns of L are compatible.

Proof. If L is an acceptable isotropy matrix, then by definition $E' = \mathcal{E}E = L^{-1}$ is an acceptable exponent matrix, so each row of E' has at most one positive and one negative element, and det $= \pm 1$. Hence, $\det L = \pm 1$.

Claim. An acceptable exponent matrix $E' = L^{-1}$ and L can be transformed into upper triangular matrices E_n and L_n respectively, by simultaneous permutations of the rows of E_n and columns of L_n, and simultaneous permutations of the columns of E_n and rows of L_n, and possibly changing the sign of some rows of E_n. Each diagonal element of E_n and L_n is 1, and $E_n L_n = \mathbf{1}$.

Justification. Set $L_n = L$ and $E_n = E'$. For any row of E_n that has only one non-zero element ensure the element is positive, by changing the sign of the row if necessary. The proof is by induction, permuting rows and columns of E_n and L_n, so that successively the bottom rows of E_n and L_n are zero to the left of the main diagonal. Let E_p be the $p \times p$ top left submatrix of E_n, and L_P be the $p \times p$ top left submatrix of L_n, starting with $p = n$. We know (inductive hypothesis) that (a) E_p is an acceptable exponent matrix, and (b) $E_n L_n = \mathbf{1}$. Select a row of E_p which has only 1 non-zero element $(e_{r,c})$ (Lemma 2.2.20). Then permute this element to the bottom row of E_p by a row permutation $\pi_r(p) = (p, p-1, \ldots, r)$ of E_n, and right-most column of E_p by a column permutation $\pi_c(p) = (p, p-1, \ldots, c)$ of E_n. Simultaneously, permute the columns of L_n with the permutation $\pi_r(p)$, and the rows of L_n with the permutation $\pi_c(p)$. That is, $(E_n)_{i,j}$ is replaced by $(E_n)_{\pi_r i, \pi_c j}$ and $(L_n)_{i,j}$ is replaced by $(L_n)_{\pi_c i, \pi_r j}$. Then (b) $E_n L_n$ is still the unit matrix, since $\sum_j (E_n)_{\pi_r i, \pi_c j} (L_n)_{\pi_c j, \pi_r i'}$

$= \mathbf{1}_{\pi_r i, \pi_r i'}$. For any row with non-zero element in the new right-most column of E_p, ensure this element has the value -1 by changing the sign of the row if necessary, so the other non-zero element in this row is 1 (which must exist otherwise this row would be identical up to sign to the bottom row of E_p giving a 0 determinant). Also, this row of E_p cannot have had its sign changed in an earlier step in the induction, since it still has two non-zero elements. Define the new submatrix E_{p-1} as the upper left $(p-1) \times (p-1)$ submatrix of E_n. The bottom row of E_p contained only one non-zero element (with value 1), so $\det E_{p-1} = \det E_p = \pm 1$, and any row of E_{p-1} will contain either two non-zero elements, or one non-zero element with value 1. Hence, (a) E_{p-1} is an acceptable exponent matrix. The induction ends with the $p = 1$ step, for $E_1 = (1)$ and $L_1 = (1)$, with E_n being upper triangular with diagonal elements equal to 1, and $E_n L_n = \mathbf{1}$. Hence, L_n is also upper triangular with diagonal elements equal to 1. ❑

Claim. The inverse of an upper triangular acceptable exponent matrix has compatible columns.

Justification. Suppose E_n is an acceptable exponent matrix in upper triangular form with diagonal elements equal to 1, and L_n is a matrix such that $E_n L_n = \mathbf{1}$, as above, then L_n is in upper triangular form with diagonal elements equal to 1. Then inductively, for the $p \times p$ bottom right submatrix E_p of E_n and the $p \times p$ bottom right submatrix L_p of L_n, starting with $p = 1$, the columns of L_p are compatible. There is only one column of L_1 so compatibility is trivial. Now, E_{p+1} is E_p bordered with a top row of either $(1, 0 \ldots,)$ or $(1, 0, \ldots, -1_c, 0, \ldots)$, and completed on the left with zeros. Since $E_{p+1} L_{p+1} = \mathbf{1}$ and both are upper triangular, the leftmost column of L_{p+1} must be $(1, 0, \ldots)^T$, so that $(0, e_j)(1, 0, \ldots)^T = 0$, for each row e_j of E_p, and $(1, e_j)(1, 0, \ldots)^T = 1$, where $(1, e_j)$ is the top row of E_{p+1}, Now, the top row of E_{p+1} is either $(1, 0, \ldots)$ or $(1, 0, \ldots, -1_j, 0, \ldots)$, then the top row of L_{p+1} must be either $(1, 0, \ldots)$ or $(1, r_j)$ respectively, where r_j is the j^{th} row of L_p. Since then, $(1, 0, \ldots)(0, L_{\ldots,c})^T = 0$ or $(1, 0, \ldots, -1_j, 0, \ldots)((r_j)_c, L_{\ldots,c}) = (r_j)_c - (r_j)_c = 0$ (respectively) for each column $L_{\ldots,c}$ of L_p; and for any other row of E_{p+1}, $(0, E_{r,\ldots})((r_j)_c, L_{\ldots,c})= (E_p L_p)_{r,c} = \mathbf{1}_{r,c}$. where $E_{r,\ldots}$ is a row of E_p and $L_{\ldots,c}$ is a column of L_p. The column $(1, 0, \ldots)^T$ of L_{p+1} is compatible with any column, and if columns $(x, c_r, y)^T$ and $(x', c_r', y')^T$ of L_p are compatible then so are $(c_r, x, c_r, y)^T$ and $(c_r', x', c_r', y')^T$ of L_{p+1}. Thus the columns of L_{p+1} are compatible. The induction ends with the $p = n$ step, with L_n determined and its columns compatible. ❑

After the induction, L_n is related to L by a permutation of its rows and a permutation of its columns. Any two columns of L_n are compatible, so trivially they are compatible after any permutation of the columns. A permutation of the rows L_n will also leave them compatible, since the permutation affect both columns equally. That is, the columns of L are compatible.

Conversely, suppose an isotropy matrix L has $\det L = \pm 1$, and compatible columns. Immediately, $\det E = \pm 1$, for $E = L^{-1}$.

Claim. An isotropy matrix L with $\det L = \pm 1$, and compatible columns can be transformed into an upper triangular matrix L_n, by a permutation of the rows and columns.

Justification. Inductively, for a $p \times p$ submatrix L_p of L_n, starting with $p = n$ and $L_n = L$, select a column which has only 1 non-zero element $(l_{r,c})$. Such a column must exist, since otherwise for a column (y_c) with a minimal number of non-zero elements, all columns would contain or be disjoint from y_c, so those rows which contain the non-zero elements of y_c would be equal, giving zero determinant. Then permute this element to the top row with row permutation $\pi_r(p) = (1, 2, \ldots, r)$ of L_n, and to the left-most column with column permutation $\pi_c = (1, 2, \ldots, c)$ of L_n. Then define L_{p-1} as the bottom right submatrix of L_n. If the columns y_i and y_j of L_n are compatible, then they still are when restricted to their bottom p elements, and since the leftmost column of L_p after permutation contains only one non-zero element (of value 1), $\det L_{p-1} = \det L_p$. Thus, L_{p-1} has determinant ± 1, and compatible columns. The induction ends at the $p = 1$ step, when L_n is in upper triangular form. ❑

Claim. The inverse of an upper triangular matrix with determinant ± 1 and compatible columns is an acceptable exponent matrix.

Justification. Suppose the upper triangular matrix L_n has determinant ± 1 and compatible columns. Then inductively, for the $p \times p$ bottom right submatrix L_p of L_n and a $p \times p$ matrix E_p, starting with $p = 1$ and $E_1 = L_1 = 1$, so $E_1 L_1 = 1$. Then

L_{p+1} is L_p bordered at the top with $(1, 0, \ldots, 1_k, r_k)$ or $(1, 0, \ldots)$ and completed on the left with zeros, where r_k represents the remainder of the row. If k is defined, then the k^{th} row of L_p is $(0, \ldots, 1_k, r_k)$, since for $k < s$ if (1) $(r_k)_s = 0$ the columns y_k and y_s must have no elements in common, so the new element of the column y_s must also be 0, and if (2) $(r_k)_s = 1$ then $(y_k)_r \leqslant (y_s)_r$ (since $(y_k)_s = 0$, $(y_s)_s = 1$, from the upper triangular form of L) so the new element of the column y_s must also be 1. Hence, the top row of E_{p+1} is $(1, 0, \ldots, -1_k, 0, \ldots)$ or $(1, 0, \ldots)$. Then, $E_{p+1}L_{p+1} = \mathbf{1}$, since $E_pL_p = \mathbf{1}$ and a non-diagonal element of the top row is $1_k - 1_k$ or is an element of $r_k - r_k$, and clearly $\det E_{p+1} = \det E_p$. Thus, E_{p+1} is an acceptable exponent matrix. Hence, E_n has determinant ± 1, and each row has one positive and at most one negative element. ❑

Suppose π_r is the product of the row permutations of the above process, π_c the product of the column permutations. Applying the permutations π_r^{-1} (respectively, π_c^{-1}) to the rows (respectively, columns) of L_n and to the columns (respectively, rows) of E_n, gives matrices L and E. Then E an acceptable exponent matrix, since permutations of rows and columns does not alter the number of positive and negative elements in any row, and can only change the determinant by ± 1. Also, $\sum_j E_{\pi_r i, \pi_c j} L_{\pi_c j, \pi_r i'} = \mathbf{1}_{\pi_r i, \pi_r i'}$, so $EL = \mathbf{1}$. ❑

Remark. Note that the conditions on a matrix L of $\det L = \pm 1$ and of compatible columns are independent. For example, the matrix $L = ((1,1,0)^T, (0,1,1)^T, (0,1,0)^T)$ has determinant 1, but the first two columns are not compatible; and the matrix $L = ((1,0,0)^T, (0,1,1)^T, (1,1,1)^T)$ has compatible columns but determinant 0. This is in contrast to the toric case where a determinant of ± 1 is sufficient to guarantee that any subset of columns spans a subspace which is a direct summand of $\mathbb{Z}^n$.

To motivate the next structures that we define, we continue with the Examples 2.2.15(1-3), 2.2.18(1-3).

Examples 3.1.5 Continuing Examples 2.2.15.

1. For the standard action of Q^n on $\mathbb{H}^n$ the isotropy class associated with an interior point of the orbit space $\mathbb{R}^n_{\geqslant}$ is $\mathsf{Q}_\emptyset$, and that of any interior point the facet F_k

of $\mathbb{R}^n_\geqslant$ is $\mathsf{Q}_{\{k\}}$. It is straightforward to check that the isotropy class for any face $F_\sigma = F_{i_1} \cap \ldots \cap F_{i_k}$ is $\mathsf{Q}(F_\sigma) = \mathsf{Q}_{\gamma_1,\ldots,\gamma_k}$, where $\gamma_j = \{i_j\}$, for $\sigma = \{i_1, \ldots, i_k\}$. Then trivially, $\mathsf{Q}(F_\tau)$ is a subclass of $\mathsf{Q}(F_\sigma)$ for any $\tau \subset \sigma$.

Now consider the collection of sets γ_i for the rank 1 classes associated with the facets F_i, for $i = 1, \ldots, n$. These are just the singleton sets $\gamma_i = \{i\}$, for each i, and are clearly compatible.

2. For the action of Q^3 on $\mathbb{H}^3$ (taking $k = n = 3$ in Example 2.2.15(2)) defined by,

$$((s_1, s_2, s_3), (h_1, h_2, h_3)) \mapsto (s_1 h_1 s_3^{-1}, s_2 h_2 s_3^{-1}, h_3 s_3^{-1}),$$

the isotropy classes associated with the faces of $\mathbb{R}^n_\geqslant$ are:

$$\begin{array}{rlrlrl}
F_\emptyset : & \mathsf{Q}_\emptyset & & & & \\
F_{\{1\}} : & \mathsf{Q}_{\{1\}} & F_{\{2\}} : & \mathsf{Q}_{\{2\}} & F_{\{3\}} : & \mathsf{Q}_{\{1,2,3\}} \\
F_{\{1,2\}} : & \mathsf{Q}_{\{1\},\{2\}} & F_{\{1,3\}} : & \mathsf{Q}_{\{1\},\{2,3\}} & F_{\{2,3\}} : & \mathsf{Q}_{\{2\},\{1,3\}} \\
F_{\{1,2,3\}} : & \mathsf{Q}_{\{1\},\{2\},\{3\}} & & & &
\end{array}$$

A straightforward check confirms that $\mathsf{Q}(F_\tau)$ is a proper subclass of $\mathsf{Q}(F_\sigma)$ for any τ a proper subset of σ.

The γ_i for the rank 1 classes of the facets are $\{1\}, \{2\}, \{1, 2, 3\}$, which are compatible.

3. For the action of Q^3 on $\mathbb{H}^3$ (Example 2.2.15(3)) defined by,

$$((s_1, s_2, s_3), (h_1, h_2, h_3)) \mapsto (s_3 h_1 s_2^{-1}, s_1 h_2, s_1 h_3 s_2^{-1}),$$

the isotropy classes associated with the faces of $\mathbb{R}^n_\geqslant$ are:

$$\begin{array}{rlrlrl}
F_\emptyset : & \mathsf{Q}_\emptyset & & & & \\
F_{\{1\}} : & \mathsf{Q}_{\{3\}} & F_{\{2\}} : & \mathsf{Q}_{\{1,2,3\}} & F_{\{3\}} : & \mathsf{Q}_{\{2,3\}} \\
F_{\{1,2\}} : & \mathsf{Q}_{\{1,2\},\{3\}} & F_{\{1,3\}} : & \mathsf{Q}_{\{2\},\{3\}} & F_{\{2,3\}} : & \mathsf{Q}_{\{1\},\{2,3\}} \\
F_{\{1,2,3\}} : & \mathsf{Q}_{\{1\},\{2\},\{3\}} & & & &
\end{array}$$

Again, $\mathsf{Q}(F_\tau)$ is a proper subclass of $\mathsf{Q}(F_\sigma)$ for any proper subset τ of σ.

The collection of the γ_i for the rank 1 classes corresponding to the facets are $\{3\}, \{2, 3\}, \{1, 2, 3\}$, which are compatible.

The features common to these three examples that we highlight are (1) the determinant of the exponent matrix is ± 1, (2) the association of faces of $\mathbb{R}^n_{\geqslant}$ to isotropy classes is functorial, preserving the inclusion relationships. We formalise these properties in the next set of definitions.

3.1.6 Facet Graphs. A directed graph Γ is a pair (N, A), where N is a set of nodes $n_1, \ldots$ and A is a set of edges, where an edge is an ordered pair of nodes (n_1, n_2), that is, directed from n_1 to n_2. A path is an ordered sequence of nodes which has an edge from each node to the next, $(n_0, n_1, \ldots, n_k)$ for which each (n_i, n_{i+1}) is an edge in A, and is said to be of length k. A morphism of directed graphs $\Gamma_a \to \Gamma_b$ is a map of the nodes which preserves the directed edges, $\phi\colon N_a \to N_b$ such that if $(n_1, n_2) \in A_a$ then $(\phi(n_1), \phi(n_2)) \in A_b$. An isomorphism is then a morphism of directed graphs which has an inverse.

Our interest here is in graphs where the nodes are the characteristic sets of the rank 1 isotropy classes of an n corner, and paths correspond to subset inclusion.

Definition 3.1.7 A *facet graph* Γ is a finite collection N of non-empty sets

$$\gamma_1, \ldots, \gamma_p \subset [n],$$

and a collection A of directed edges, where (γ_i, γ_j) is an edge if (1) γ_i is a proper subset of γ_j, and (2) there is no γ_k in N such that γ_i is a proper subset of γ_k and γ_k is a proper subset of γ_j.

Note that if any γ_a is a proper subset of γ_b, there is a path from γ_a to γ_b, since if there is no proper intermediate subset, then (γ_a, γ_b) is an edge, otherwise inductively there is a sequence of edges which generate a path. In general the path need not be unique. If two nodes are equal, there is no path between them, although there will be paths from any proper subset to both.

A facet graph has nodes γ_i, for $i = 1, \ldots, p$, so it is a labelled graph. A permutation π_l of the labels permutes the nodes γ_i to $\gamma_{\pi_l i}$, and preserves the edge structure, so generates a graph isomorphism. The number p of sets in the collection need have no relation to n. Below, we restrict to cases where $p = n$, although for subgraphs $p < n$. In later sections we also consider facet graphs where $p > n$.

Definition 3.1.8 Given a facet graph Γ of subsets of $[n]$, for any node γ an element k is said to be a *distinguished element* of γ if (1) $k \in \gamma$ and (2) $k \notin \beta$ for all nodes β of Γ that are proper subsets of γ.

In an arbitrary facet graph a node can have one or many distinguished elements or none, but we need to restrict the possibilities to obtain a correspondence with the acceptable matrices already defined.

Definition 3.1.9 A facet graph Γ of subsets of $[n]$ will be called an *acceptable facet graph* if:

(1) it contains n nodes;

(2) the nodes are compatible;

(3) each node has a different distinguished element.

Two acceptable facet graphs are defined to be *congruent* if there is an isomorphism between them.

Note that condition (3) says that each node has a distinguished element, and that different nodes have different distinguished elements. Since there are n nodes, each a subset of $[n]$, each node can only have one distinguished element. Also, each element $k \in [n]$ is the distinguished element of some node.

A leaf of an acceptable facet graph is a singleton set, the set of its distinguished element, since for any set with more than one element to have a single distinguished element it must have a proper subset which is a node.

Proposition 3.1.10 *If Γ is an acceptable facet graph of subsets of $[n]$, then Γ consists of a collection of rooted trees, with all edges directed towards a local root.*

Proof. There is at most one edge directed away from any node, since otherwise there are edges (γ, β) and (γ, ϵ), for some distinct nodes γ, β and ϵ. Then $\gamma \subset \beta$ and $\gamma \subset \epsilon$, so β and ϵ are not disjoint, so one is a subset of the other by compatibility, which contradicts (γ, β) and (γ, ϵ) both being edges.

There are no cycles, since γ can never be a proper subset of itself. Hence, following the edges from any node generates a unique path, and this path must terminate at some maximal node γ_t. Then, for any node which is a subset of γ_t, there is a path to γ_t, and these nodes form a tree with γ_t as the root. Different maximal nodes are disjoint, since otherwise one would be a subset of the other and so not maximal. Thus the graph Γ is a collection of rooted trees, with each maximal node as a local root. ❑

For an acceptable facet graph the distinguished elements of the nodes can be regarded as providing a second labelling of the graph (see Examples 3.1.13 below). A congruence between two acceptable graphs corresponds to some relabelling of the γ_i and some relabelling of their distinguished elements.

The map between these two labellings turns out to be useful later.

Definition 3.1.11 For an acceptable facet graph Γ of subsets of $[n]$, define the *facet subgroup map* by

$$\alpha\colon [n] \to [n], \qquad i \mapsto \operatorname{dist}(\gamma_i),$$

where $\operatorname{dist}(\gamma)$ is the unique distinguished element of the node γ.

Where the γ_i are the characteristic sets of the rank 1 isotropy classes of an action of Q^n on $\mathbb{H}^n$ and generate an acceptable facet graph, then α picks out a particular coordinate subgroup $\mathsf{Q}_{\alpha(i)}$ of Q^n for each facet F_i of $\mathbb{R}^n_{\geqslant}$.

Proposition 3.1.12 *If Γ is an acceptable facet graph of subsets $\gamma_1, \ldots, \gamma_n$ of $[n]$, then*

(1) the facet subgroup map $\alpha : i \mapsto \operatorname{dist}(\gamma_i)$ is a permutation of $[n]$,

(2) α can be extended to subsets of $[n]$,

$$\alpha\colon \{i_1, \ldots, i_k\} \mapsto \{\operatorname{dist}(\gamma_{i_1}), \ldots, \operatorname{dist}(\gamma_{i_k})\}$$

such that for subsets β, γ of $[n]$,

$$\beta \subset \gamma \iff \alpha(\beta) \subset \alpha(\gamma), \qquad |\alpha(\gamma)| = |\gamma|.$$

Proof. (1) Each node has one distinguished element, so the map α is well defined, each has a different distinguished element, so α is injective, and for each $k \in [n]$ there is some node with k as its distinguished element, so α is surjective. Thus, α is a permutation.

(2) Since α is bijective on $[n]$, it is bijective on subsets of $[n]$. Clearly, for $\beta \subset \gamma \subset [n]$, we have $\alpha(\beta) \subset \alpha(\gamma)$. Since α is bijective, if $\alpha(\beta) \subset \alpha(\gamma)$ then $\beta \subset \gamma$, and $|\alpha(\gamma)| = |\gamma|$. ❑

Examples 3.1.13 Continuing Examples 2.2.15(1-3). For convenience the E and L matrices (Example 2.2.18) are repeated here.

For each node γ_i of the facet graph, where Q_{γ_i} is the isotropy class of the facet F_i and corresponds to the i^{th} column of L, with distinguished element $k = \text{dist}(\gamma_i) = \alpha(i)$, the node is displayed as $\bullet_i^{(k)}$.

1. For the standard action of Q^n on $\mathbb{H}^n$ the isotropy class associated with each facet F_i is $\mathsf{Q}_{\{i\}}$. Thus, the facet graph consists of the n nodes $\gamma_i = \{i\}$, with no inclusion relations, hence is the graph with n components each having a single node. The distinguished element of $\{i\}$ is clearly i. The E and L matrices are both equal to the unit matrix, and the facet graph is

$$\Gamma = \left(\bullet_1^{(1)} \ \bullet_2^{(2)} \ \cdots \ \bullet_n^{(n)} \right).$$

2. For the action of Q^3 on $\mathbb{H}^3$ (taking $k = n = 3$ in Example 2.2.15(2)) defined by,

$$((s_1, s_2, s_3), (h_1, h_2, h_3)) \mapsto (s_1 h_1 s_3^{-1}, s_2 h_2 s_3^{-1}, h_3 s_3^{-1}),$$

the isotropy classes associated with the facets F_1, F_2, F_3 of $\mathbb{R}^n_{\geqslant}$ are $\mathsf{Q}_{\{1\}}$, $\mathsf{Q}_{\{2\}}$ and $\mathsf{Q}_{\{1,2,3\}}$ respectively. Then

$$E = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Gamma = \left(\begin{array}{cc} & \bullet_3^{(3)} \\ \nearrow & \uparrow \\ \bullet_1^{(1)} & \bullet_2^{(2)} \end{array} \right)$$

3. For the action of Q^3 on $\mathbb{H}^3$ (Example 2.2.15(2)) defined by,

$$((s_1, s_2, s_3), (h_1, h_2, h_3)) \mapsto (s_3 h_1 s_2^{-1}, s_1 h_2, s_1 h_3 s_2^{-1}),$$

the isotropy classes associated with the facets F_1, F_2, F_3 of $\mathbb{R}^n_{\geqslant}$ are $\mathsf{Q}_{\{3\}}$, $\mathsf{Q}_{\{1,2,3\}}$ and $\mathsf{Q}_{\{2,3\}}$ respectively, corresponding to the columns of L. Then

$$E = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \; L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \Gamma = \left(\bullet_1^{(3)} \to \bullet_3^{(2)} \to \bullet_2^{(1)} \right)$$

Proposition 3.1.14 *Given an action of Q^n on $\mathbb{H}^n$ with an isotropy class Q_{γ_i} for each facet F_i of the orbit space $\mathbb{R}^n_{\geqslant}$, then the $n \times n$ isotropy matrix L is acceptable if and only if the facet graph Γ of nodes γ_i is acceptable.*

Furthermore, two acceptable facet graphs are congruent if and only if the corresponding acceptable isotropy matrices are congruent.

Proof. Suppose L is an acceptable isotropy matrix.

Claim. An acceptable isotropy matrix L defines a facet graph Γ with n compatible nodes each a subset of $[n]$.

Justification. Each column y_i of L corresponds by definition to the set $\gamma_i = \{t \in [n] \mid (y_i)_t = 1\} \subset [n]$. The columns y_i are compatible by Proposition 3.1.4, so the sets γ_i are also. The γ_i can be partially ordered by set inclusion, so define a facet graph Γ. There are n nodes since n columns in L. ❑

Claim. Each node in the directed graph Γ has a different distinguished element.

Justification. The rows and columns of L can be permuted to bring it into upper triangular form L_n (as in the proof of 3.1.4(2) above). Then for each column $y_{\pi_c i}$ of L_n there is one non-zero element $(y_{\pi_c i})_{\pi_c i}$ such that all lower elements in this column are zero, and all elements to the left in the same row are zero. Hence, any column y_j to the left of $y_{\pi_c i}$ is disjoint or $y_j \prec y_{\pi_c i}$, and any column y_j to the right of $y_{\pi_c i}$ is disjoint or $y_{\pi_c i} \prec y_j$. So $\gamma_{\pi_c i}$ has a distinguished element $\pi_c i$, and it is different from that of any column to the left. Hence, reversing the column permutations, each γ_i has a different distinguished element. The row permutations only permute the elements of all the γ_i together, so do not change the subset relations between the γ_i, so leave each γ_i with a different distinguished element. ❑

Thus, Γ is an acceptable facet graph.

Two congruent acceptable isotropy matrices are related by permutations π_r of the rows and π_c of the columns. Hence the nodes, which are defined by the columns, are permuted by π_c, and the elements of every node, which derive from the row labellings, are permuted by π_r. The permutations π_r and π_c define an isomorphism between the facet graphs, so the two facet graphs are congruent.

Conversely, suppose Γ is an acceptable facet graph.

Claim. An acceptable facet graph Γ defines an isotropy matrix L with compatible columns.

Justification. Each node γ_i of Γ defines by definition a column y_i by $(y_i)_t = 1$ if $t \in \gamma_i$, and 0 otherwise. The nodes γ_i are compatible by definition, so the columns y_i are also. ❑

Claim. The determinant of L is ± 1.

Justification. Each $k \in [n]$ is the distinguished element of some γ_i, since there are n nodes and each has a different distinguished element in $[n]$.

The *parent* of any node γ is the node β for which an edge (γ, β) exists, and is unique if it exists. Two sub-trees are said to be *adjacent* if they have the same parent node. Now define a traverse of the graph Γ recursively as follows: (1) start at any leaf node; (2) then traverse any adjacent sub-tree not yet traversed; (3) then move to the parent node, unless already at the root node. (4) then repeat for any other connected component not yet traversed.

Use the ordering of this traverse to label the nodes $\gamma_{i_1}, \gamma_{i_2}, \ldots$ with their distinguished elements $k_1, k_2, \ldots$. Now re-order the rows of L to be are in the order $k_1, k_2, \ldots$, and the columns to be in the order $i_1, i_2, \ldots$. Then the element k_j appears first in γ_{i_j}, so the first non-zero entry in any row is the diagonal element. Thus, L has been put into upper triangular form with diagonal elements equal to 1 by a permutation of the rows and columns, hence $\det L$ is ± 1. ❑

Two congruent acceptable facet graphs are related by an isomorphism, which defines a permutation π_e of the elements of the nodes, and a permutation π_l of the node labels. These induce a permutation π_l of the columns of L, and a permutation

π_e of the rows of L. Thus, the two L matrices are congruent. ❑

Definition 3.1.15 For an acceptable facet graph Γ on subsets of $[n]$, and a non-empty subset $\sigma \subset [n]$, let $N(\sigma)$ denote the collection of nodes γ_i for all $i \in \sigma$. Define the *derived sets*

$$\gamma_i^\sigma = \gamma_i \backslash \bigcup_{\beta \subsetneq \gamma_i} \beta \qquad\qquad i \in \sigma,$$

where the union is taken over nodes in $N(\sigma)$ that are proper subsets of γ_i.

Proposition 3.1.16 *For an acceptable facet graph Γ on subsets of $[n]$, and a non-empty subset $\sigma \subset [n]$, the derived sets γ_i^σ for $i \in \sigma$ are non-empty and disjoint.*

Furthermore, for any $\sigma \subset \tau \subset [n]$, the rank $|\sigma|$ class $\mathsf{Q}_{\gamma_{i_1}^\sigma,\ldots,\gamma_{i_{|\sigma|}}^\sigma}$ is a subclass of the rank $|\tau|$ class $\mathsf{Q}_{\gamma_{i_1}^\tau,\ldots,\gamma_{i_{|\tau|}}^\tau}$.

Proof. By definition, for an acceptable facet graph Γ, the distinguished element $\text{dist}(\gamma_i)$ is an element of γ_i and not of any node of Γ that is proper subset. Hence $\text{dist}(\gamma_i)$ is an element of γ_i^σ, and in particular γ_i^σ is not empty.

For any $\beta \subset \gamma_i$, the expression $\gamma_i \backslash \beta$ removes all elements of β, so no element of β^σ (which is a subset of β) can be in γ_i^σ. Hence, the sets γ_i^σ are disjoint.

For any γ_i for which there is no proper subset $\beta \in N(\sigma)$ we have $\gamma_i^\sigma = \gamma_i$. Any node γ_i is the union of γ_i^σ and those nodes $\beta \in N(\sigma)$ which are subsets of γ_i. Hence inductively, γ_i is the union of γ_i^σ and those sets β^σ for which $\beta \subset \gamma_i$. That is, each γ_i is a non-empty union of some of the γ_j^σ, and we can write $\gamma_i = \bigcup_{\beta \subseteq \gamma_i} \beta^\sigma$.

Suppose $\sigma \subset \tau \subset [n]$, and $\gamma_i \in N(\sigma)$. We want to show that for any γ_j^τ, the intersection $\gamma_i^\sigma \cap \gamma_j^\tau$ is equal to γ_j^τ or empty. If $\gamma_i \cap \gamma_j = \emptyset$, then immediately $\gamma_i^\sigma \cap \gamma_j^\tau = \emptyset$. If $\gamma_j \supset \gamma_i$, then $\gamma_i^\sigma \cap \gamma_j^\tau = \emptyset$, since $\beta = \gamma_i$ occurs in the definition of γ_j^τ. So let us suppose that $\gamma_j \subset \gamma_i$. We can rearrange the expression for γ_i^σ, as $\bigcap_{\beta \subsetneq \gamma_i \in N(\sigma)} (\gamma_i \backslash \beta)$, then

$$\gamma_i^\sigma \cap \gamma_j^\tau \quad = \quad \bigcap_{\beta \subsetneq \gamma_i \in N(\sigma)} (\gamma_i \backslash \beta) \quad \cap \bigcap_{\beta \subsetneq \gamma_j \in N(\tau)} (\gamma_j \backslash \beta).$$

Since $\gamma_j \subset \gamma_i$, we have $\gamma_j \cap \gamma_i = \gamma_j$, so we can replace γ_i by γ_j throughout the expression. If any β occurs in the first intersection but not in the second, then either

β contains γ_j or β and γ_j are disjoint, by compatibility. In the first case $\gamma_j \backslash \beta$ is empty, so $\gamma_i^\sigma \cap \gamma_j^\tau$ is empty. In the second case $\gamma_j \backslash \beta = \gamma_j$, and can be omitted from the intersection. Then, since $N(\sigma) \subset N(\tau)$, every remaining term in the first intersection is equal to one in the second, which is just the definition of γ_j^τ. Thus, $\gamma_i^\sigma \cap \gamma_j^\tau$ is equal to γ_j^τ or empty. Then, since γ_i is equal to the union of some of the γ_j^τ, each γ_i^σ is the union of some of the γ_j^τ. Hence, by Proposition 2.1.18, the class $\mathsf{Q}_{\gamma_{i_1}^\sigma,\ldots,\gamma_{i_{|\sigma|}}^\sigma}$ is a subclass of $\mathsf{Q}_{\gamma_{i_1}^\tau,\ldots,\gamma_{i_{|\tau|}}^\tau}$. ❑

In particular, each rank 1 class Q_{γ_i} is a subclass of $\mathsf{Q}_{\gamma_1^\sigma,\ldots,\gamma_{|\sigma|}^\sigma}$.

3.1.17 Isotropy Functor. We have seen in some of the examples (2.2.15) that for an acceptable action of Q^n on $\mathbb{H}^n$ we have an isotropy class corresponding to every face of the orbit space $\mathbb{R}^n_\geqslant$, with the class of a face a subclass of any subface. To complete the collection of objects which we use to characterise these actions, we now introduce a function which associates an isotropy class with every face. The most convenient language here is that of categories and functors, to emphasise that the inclusion relations are preserved, using categories $\textsc{cat}([n])$ and $\textsc{conj}(\mathsf{Q}^n)$ already defined (Paragraphs 2.2.5, 2.1.19)

Definition 3.1.18 A functor $\ell\colon \textsc{cat}([n]) \to \textsc{conj}(\mathsf{Q}^n)$ will be called an *acceptable isotropy functor* if ℓ is injective on the objects of $\textsc{cat}([n])$.
Two acceptable isotropy functors ℓ_1, ℓ_2 are defined to be *congruent* if there is an automorphism θ^* of $\textsc{conj}(\mathsf{Q}^n)$ and an automorphism π^* of $\textsc{cat}([n])$ such that $\ell_2 = \theta^* \circ \ell_1 \circ \pi^*$.

For any ordering of the facets, $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$ of $\mathbb{R}^n_\geqslant$, there is a corresponding maximal sequence of nested faces,

$$\mathbb{R}^n_\geqslant \supset F_{\{i_1\}} \supset F_{\{i_1,i_2\}} \supset \ldots \supset F_{\{i_1,i_2,\ldots,i_n\}} = \{0\}$$

Hence, for any acceptable isotropy functor, there is a sequence of nested isotropy classes

$$\mathbf{1} < \ell(\{i_1\}) < \ell(\{i_1,i_2\}) < \cdots < \ell(\{i_1,i_2,\ldots,i_n\})$$

Since (1) the rank of a proper subclass is strictly less than that of the class (from the injective condition on ℓ), (2) the rank of any non-trivial class is positive and no greater than n (working within Q^n), and (3) the nested chain has n steps, it follows that the rank of the isotropy class is equal to the number of facets in the intersection. Any face of $\mathbb{R}^n_{\geqslant}$ can be written as an intersection of facets, so can be contained in such a sequence, thus we have shown:

Proposition 3.1.19 *If $\ell\colon$ CAT$([n]) \to$ CONJ(Q^n) is an acceptable isotropy functor, then for a face F_σ of $\mathbb{R}^n_{\geqslant}$ (of codimension $k = |\sigma|$) the isotropy class $\ell(\sigma)$ has rank k.*

❑

We now relate acceptable isotropy functors and facet graphs.

Proposition 3.1.20 *Given a function ℓ which assigns to each facet F_i of $\mathbb{R}^n_{\geqslant}$ an isotropy class $\ell(\{i\}) = \mathsf{Q}_{\gamma_i}$, the facet graph Γ with nodes $\gamma_1, \ldots, \gamma_n$ is an acceptable facet graph if and only if ℓ can be extended to an acceptable isotropy functor $\ell\colon$ CAT$([n]) \to$ CONJ(Q^n).*

Furthermore, two isotropy functors are congruent if and only if the corresponding facet graphs are congruent.

Proof. Suppose the graph Γ of the γ_i is an acceptable facet graph.

Claim. ℓ can be extended to a functor CAT$([n]) \to$ CONJ(Q^n).

Justification. For each facet F_i we have $\ell(\{i\}) = \mathsf{Q}_{\gamma_i}$. The set of γ_i are compatible by definition, so for any face $F_\sigma = F_{\{i_1,\ldots,i_k\}}$ define $\ell(\sigma) = \mathsf{Q}_{\gamma^\sigma_{i_1},\ldots,\gamma^\sigma_{i_k}}$, where the γ^σ_i are given by Definition 3.1.15. For any $\sigma \subset \tau$, then $F_\tau \subset F_\sigma$, so $\ell(\sigma)$ is a subclass of $\ell(\tau)$. Hence ℓ extends to a functor. ❑

Claim. ℓ is injective on the faces.

Justification. Since the nodes (γ_i) are compatible and each has a different distinguished element (by definition), when we combine any number of them the distinguished element of each γ_i is in γ^σ_i, by Proposition 3.1.16, so is non-empty. Hence the subclass associated with face of co-dimension k has rank k, so faces of different dimension have subgroups of different rank, which are different. If F_τ is not a sub face of F_σ, then there is some facet F_k which contains one but not the other, so there is some γ_k for which Q_{γ_k} is a subclass of one of $\ell(\tau)$ or $\ell(\sigma)$ but not the other. Hence, different faces generate different classes. ❑

Two congruent acceptable facet graphs are related by an automorphism, which defines a permutation π_e of the elements of the nodes, and a permutation π_l of the node labels. These induce a permutation π_l of the facets of $\mathbb{R}^n_\geqslant$ so an automorphism of CAT($[n]$), and induce a permutation π_e of the coordinate subclasses of (Q^n) so an automorphism of CONJ(Q^n). Thus, the two l functors are congruent.

Conversely, suppose ℓ is an acceptable isotropy functor.

Claim. Sets $\gamma_1, \ldots, \gamma_n$ define a directed graph with compatible nodes.

Justification. For each facet F_i, the class $\ell(\{i\}) = \mathsf{Q}_{\gamma_i}$ defines a node γ_i, and the edges are defined from the inclusion relations between them (Definition 3.1.7). If any distinct γ_i, γ_j are not compatible, the class of common sub-face $\ell(\{i,j\})$ does not include $\ell(\{i\})$ and $\ell(\{j\})$ as subclasses, by Proposition 2.1.15, contradicting the functorial property of ℓ. ❑

Claim. Every node γ_i has a different distinguished element.

Justification. If any γ_j has no distinguished element it is the union $\gamma_{i_1} \cup \ldots \cup \gamma_{i_e}$. Then for any set σ such that $\{j, i_1, \ldots, i_e\} \subset \sigma$ the derived set γ_j^σ is empty (Definition 3.1.15). Then the isotropy class $\ell(\sigma) = Q_{\gamma(\sigma)}$, where $\gamma(\sigma)$ stands for $(\gamma^\sigma_{s_1}, \ldots, \gamma^\sigma_{s_{|\sigma|}})$, has rank less than $|\sigma|$ which contradicts ℓ being an acceptable isotropy functor. No two different γ_i, γ_j can have the same distinguished element, since then they would not be disjoint so one would be a subset of the other by compatibility.

Thus, each γ_i has a different distinguished element. ❑

Two congruent acceptable isotropy functors are related by an automorphism of CONJ(Q^n) and an automorphism of CAT($[n]$). These automorphisms induce a permutation π_q on the classes of the coordinate subgroups of Q^n, and a permutation π_f on the facets of $\mathbb{R}^n_\geqslant$. These in turn induce a permutation π_f of the labels of the nodes, and a permutation π_q of the elements of each node. Thus, the two facet graphs are congruent. ❑

Collecting together Propositions 3.1.4, 3.1.14, 3.1.20, we obtain a summary of the relations between the acceptable objects.

Corollary 3.1.21 *Let E be an exponent matrix defining an action of Q^n on $\mathbb{H}^n$, such that the isotropy class is Q_{γ_i} for each facet F_i of the orbit space $\mathbb{R}^n_{\geqslant}$. Let ℓ be the function which assigns to each F_i the class Q_{γ_i}, let Γ be the facet graph of subsets of $[n]$ defined by the nodes γ_i, and let L be the $n \times n$ matrix of columns defined by the γ_i. Then the following are equivalent:*

(C1) ℓ can be extended to an acceptable isotropy functor $\ell\colon \textsc{cat}([n]) \to \textsc{conj}(\mathsf{Q}^n)$;

(C2) the directed graph Γ is an acceptable facet graph;

(C3) the matrix L is an acceptable isotropy matrix;

(C4) the matrix $E' = L^{-1}$ is an acceptable exponent matrix.

Furthermore, the notions of congruence are the same for each of these characterisations of the action. ❑

Definition 3.1.22 A space $\mathbb{H}^n$ with a Q^n action which satisfies any of the equivalent conditions $(C1)\dots(C4)$ will be called a *regular corner*.

Proposition 3.1.23 *Any regular corner satisfies the requirements (A1–A3) laid out in Paragraph 2.2.7.*

Proof. Any regular n corner is specified by an acceptable exponent matrix, which has determinant ±1, and determines an acceptable isotropy functor ℓ from the facets of $\mathbb{R}^n$ to the conjugacy classes of Q^n. By corollary 2.2.25, the orbits of the Q^n action are products of 3-spheres (A1), and the orbit space is $\mathbb{R}^n_{\geqslant}$ (A2). The isotropy class of any face F_σ of the orbit space $\mathbb{R}^n_{\geqslant}$ is a rank $|\sigma|$ class (A3), by Proposition 3.1.19. ❑

Remark. One important point to note here is that for a regular corner, an acceptable isotropy functor (C1) describes relations between the isotropy classes over all faces of $\mathbb{R}^n_{\geqslant}$. In contrast, the other conditions (C2,C3,C4) define restrictions on objects which are defined only on the facets of $\mathbb{R}^n_{\geqslant}$, hence are much simpler to work with.

Proposition 3.1.24 *Two regular corners are equivariantly diffeomorphic if and only if their exponent matrices are congruent.*

Proof. Consider two regular n corners $\mathbb{H}^n$ with Q^n actions specified by $E1$, $E2$ respectively, which we label $\mathbb{H}1^n$ and $\mathbb{H}2^n$ respectively.

If $E1$ and $E2$ are congruent, then there is some permutation ρ of $[n]$, some permutation μ of $[n]$, and some diagonal matrix $\mathcal{E}$ with diagonal elements $\mathcal{E}_i = \pm 1$, such that $(E2)_{\rho i, \mu j} = \mathcal{E}_i\,(E1)_{i,j}$. Hence, expanding each exponent matrix $E_{i,j} = \delta_{j,l_i} - \delta_{j,r_i}$, we obtain $l2_{\rho i} = \mu\, l1_i$ and $r2_{\rho i} = \mu\, r1_i$ if $\mathcal{E}_i = 1$, or $l2_{\rho i} = \mu\, r1_i$ and $r2_{\rho i} = \mu\, l1_i$ if $\mathcal{E}_i = -1$. Define an automorphism $\theta_{1,\mu}\colon (s_1, \ldots, s_n) \mapsto (s'_1, \ldots, s'_n)$ by $s'_{\mu i} = s_i$, and a diffeomorphism $f\colon (h1_1, \ldots, h1_n) \mapsto (h2_1, \ldots, h2_n)$ by

$$h2_{\rho i} = h1_i \quad (\mathcal{E}_i = 1), \qquad h2_{\rho i} = \overline{h1}_i \quad (\mathcal{E}_i = -1), \qquad\qquad i = 1, \ldots, n,$$

remembering that we are regarding $\mathbb{H}1^n$ and $\mathbb{H}2^n$ as real spaces.

Then, the component of $f(s\,h1)$ in $\mathbb{H}2_{\rho i}$ is $s_{l1_i} h1_i s_{r1_i}^{-1}$ (or its quaternionic conjugate if $\mathcal{E}_i = -1$). The component of $\theta(s)f(h)$ in $\mathbb{H}2_{\rho i}$ is $s'_{l2_{\rho i}} f(h1_i) s'^{-1}_{r2_{\rho i}}$, which equals $s_{l1_i} f(h1_i) s_{r2_i}^{-1}$ (or its conjugate if $\mathcal{E}_i = -1$), since $s'_{l2_{\rho i}} = s'_{\mu\, l1_i} = s_{l1_i}$ and $s'_{r2_{\rho i}} = s'_{\mu\, r1_i} = s_{r1_i}$ (or vice versa if $\mathcal{E}_i = -1$). Thus $f(s\,h) = \theta(s)f(h)$, so θ and f define an equivariant diffeomorphism.

Conversely, suppose $\mathbb{H}1$ and $\mathbb{H}2$ are equivariantly diffeomorphic, so there is some diffeomorphism $f\colon \mathbb{H}1^n \to \mathbb{H}2^n$, and some automorphism $\theta_{u,\mu}\colon \mathsf{Q}^n \to \mathsf{Q}^n$ (Proposition 2.1.21), such that $f(sh) = \theta(s)f(h)$ for all $s \in \mathsf{Q}^n$ and $h \in \mathbb{H}1$, where $u \in \mathsf{Q}^n$ and μ is a permutation of $[n]$. Each coordinate subspace $\mathbb{H}1_i$ and $\mathbb{H}2_i$ is an invariant subspace of their respective actions, so suppose f maps each $\mathbb{H}1_i$ to $\mathbb{H}2_{\rho i}$, then ρ is a permutation of $[n]$, since f is invertible. The action Q^n restricted to $\mathbb{H}1_i$ is given by $h1_i \mapsto s_{l_i} h1_i s_{r_i}^{-1}$. Equivariance implies that the component in $\mathbb{H}2_{\rho i}$ is $s'_{l2_{\rho i}} f(h_1) s'^{-1}_{r2_{\rho i}}$. But $s'_{\mu i} = s_i$, so $l2_{\rho i} = \mu l1_i$, and $r2_{\rho i} = \mu r1_i$ (and set $\mathcal{E}_i = 1$), or vice versa (set $\mathcal{E}_i = -1$). This step is justified since $E2$ is an acceptable exponent matrix ($\mathbb{H}2^n$ is a regular corner), so there is one row (j) of $E2$ with a single element (Lemma 2.2.20) which identifies $l2_j$ and $r2_j$ (with one empty), and inductively there is some row with a single element corresponding to some $l2_k$ or $r2_k$ not yet identified. Then, $(E2)_{\rho i, \mu j}$

$= \delta_{\mu j,l2_{\rho i}} - \delta_{\mu j,r2_{\rho i}} = \mathcal{E}_i\left(\delta_{\mu j,\mu l1_i} - \delta_{\mu j,\mu r1_i}\right) = \mathcal{E}_i\left(\delta_{j,l1_i} - \delta_{j,r1_i}\right) = \mathcal{E}_i\left(E1\right)_{i,j}$, so $E1$ and $E2$ are congruent. ❏

We now turn to the interesting question of how many congruence classes there are for a regular n corner.

Proposition 3.1.25 *There is a bijection between congruence classes of regular n-corners and unlabelled rooted trees with $(n+1)$ nodes.*

Proof. Any regular n corner can be characterised by an acceptable facet graph Γ of subsets of $[n]$, which consists of a number of trees with all edges directed towards a local root node. Adding one new root node, and a new edge from the root node of each component to the new new root defines a connected unlabelled rooted tree with $(n+1)$ nodes. Congruent n corners correspond to congruent facet graphs, which differ only their labellings, that is they have the same underlying unlabelled graph, so create the same unlabelled rooted tree with $(n+1)$ nodes.

Conversely, for any connected unlabelled rooted tree with $(n+1)$ nodes, construct an acceptable facet graph as follows: (1) Direct each edge towards the root node, this is well defined for any tree. (2) Remove the root node together with those edges which include it, leaving a collection of rooted trees with n nodes in total. (3) Assign any permutation of $[n]$ to the nodes, as the labels of the γ_i. (4) Assign any permutation of $[n]$ to the nodes, as the distinguished elements of the γ_i. (5) Define the set γ_i for any node to be the union of the distinguished elements of γ_i and all nodes with a path to γ_i.

If there is a path from γ_j to γ_i then γ_i contains all elements of γ_j by construction, so $\gamma_j \subset \gamma_i$, and otherwise they belong to different trees or subtrees, so have no element in common so they are disjoint. In either case they are compatible. By construction, each γ_i has a distinguished element and they are all different. Thus, the graph has n nodes, each a subset of $[n]$, which are compatible, and each has a different distinguished element, that is, is an acceptable facet graph. ❏

The usual symbol for the number of unlabelled rooted trees with $(n+1)$ nodes is T_{n+1}, and can be calculated from a recurrence relation, see Riordan [16] or Sloane [19], [18](A000081) for more details and references. The number T_{n+1} of congruence classes of n-corners for $n = 1, \dots$ is $1, 2, 4, 9, 20, 48, 115, 286, \dots$, and increases exponentially.

Examples 3.1.26 Incongruent acceptable exponent and isotropy matrices.

Recall that each node γ_i in the facet graph, corresponding to the facet F_i and the i^{th} column of L, with distinguished element $k = \operatorname{dist}(\gamma_i) = \alpha(i)$, is displayed as $\bullet_i^{(k)}$.

1. For $n = 2$, $T_3 = 2$, and examples from the congruence classes of regular 2-corners are defined by

a: $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\qquad \Gamma : \quad \left(\bullet_1^{(1)} \ \bullet_2^{(2)} \right)$

b: $E = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $\qquad \Gamma : \quad \left(\bullet_2^{(1)} \rightarrow \bullet_1^{(2)} \right)$

2. For $n = 3$, $T_4 = 4$, and examples of two regular 3-corners have been given in Example 3.1.13(2,3). Examples from the other two congruence classes are:

a: $E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\qquad \Gamma : \quad \left(\bullet_1^{(3)} \ \bullet_2^{(2)} \ \bullet_3^{(1)} \right)$.

b: $E = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$, $L = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $\qquad \Gamma : \quad \left(\begin{matrix} & \bullet_1^{(3)} \\ & \uparrow \\ \bullet_2^{(1)} & \bullet_3^{(2)} \end{matrix} \right)$

Adding another node $\gamma_0 = \{0, 1, 2, 3\}$ to each of the above graphs together with an edge from each local root to γ_0 clearly results in a directed graph which is one of the 4 possible connected rooted unlabelled trees with 4 nodes.

3.1.27 Sub Corners. For any regular corner we next consider the subspace corresponding to a face of the orbit space, obtained by setting some of the coordinates to zero. We expect this subspace to be a regular corner, with an inherited action, but it is useful to characterise this behaviour in terms of the isotropy matrix.

For any $i \in [n]$, let $\mathbb{H}_{\hat{\imath}}$ be the subspace $\{h \in \mathbb{H}^n \mid h_i = 0\}$ of $\mathbb{H}^n$. (Then $\mathbb{H}_{\hat{\imath}}$ is the closure of the space $\mathbb{I}_{\hat{\imath}}$ defined earlier, Example 2.2.15). Let $\mathsf{Q}_{\hat{\imath}}$ be the subgroup $\{q \in \mathsf{Q}^n \mid q_i = 1\}$ of Q^n. Let us assume (as in the proof of Proposition 3.1.4) that the coordinate subspaces of $\mathbb{H}^n$ and Q^n have been permuted so that the exponent matrix E is in upper triangular form. The action of $\mathsf{Q}_{\widehat{\imath}}$ on $\mathbb{H}_{\widehat{\imath}}$ is then specified by the submatrix $E_{\widehat{\imath}}$ of E obtained by omitting the i^{th} row and column. The submatrix $E_{\widehat{\imath}}$ still has at most one positive and one negative element in each row, and has determinant ± 1, so is an acceptable exponent matrix. Thus, the action of $\mathsf{Q}_{\widehat{\imath}}$ on $\mathbb{H}_{\widehat{\imath}}$ is that of a regular $(n-1)$-corner. We can extend this argument to the subspace of $\mathbb{H}^n$ over any face of $\mathbb{R}^n_{\geqslant}$.

Recall that given an acceptable graph Γ and a set $\tau \subset [n]$, the $|\tau|$ compatible rank 1 classes Q_{γ_i} for $i \in \tau$ are each a subclass of the rank $|\tau|$ class $\mathsf{Q}_{\gamma(\tau)}$ where $\gamma(\tau)$ stands for $(\gamma^\tau_{i_1}, \ldots, \gamma^\tau_{i_{|\tau|}})$ for $i_j \in \tau$, as in Definition 3.1.15.

Proposition 3.1.28 *Given a regular n corner $\mathbb{H}^n$, with an action specified by exponent matrix E, let F_τ be a face of $\mathbb{R}^n_{\geqslant}$ of codimension $k = |\tau|$, with isotropy class $\ell(\tau) = \mathsf{Q}_{\gamma(\tau)}$, and let $\beta = \alpha(\tau)$. Then the subspace $\mathbb{H}_{\widehat{\tau}}$ of $\mathbb{H}^n$ is a regular $(n-k)$ corner, with an inherited action of $\mathsf{Q}_{\widehat{\beta}}$ specified by $E_{\widehat{\tau},\widehat{\beta}}$, obtained from E by deleting the rows $i \in \tau$ and the columns $j \in \beta$.*

Proof. Permuting the rows and columns to put E into upper triangular form sends the element $E_{i,\alpha(i)}$ to a non-zero diagonal element, since the corresponding permutations of the isotropy matrix L also transforms it to upper triangular form, and then each column L has its diagonal element as the first non-zero element from the left. Hence, deleting the row and column containing the element $E_{i,\alpha(i)}$ only changes the determinant by ± 1. So deleting the rows $i \in \tau$ and columns $j \in \alpha(\tau)$ of E, we have $\det E_{\widehat{\tau},\widehat{\beta}} = \pm 1 \det E = \pm 1$. Deleting rows and columns of E leaves each row of $E_{\widehat{\tau},\widehat{\beta}}$ with at most one positive and one negative element. Hence, the submatrix $E_{\widehat{\tau},\widehat{\beta}}$ of E is an acceptable exponent matrix (Definition 3.1.1). ❑

For any particular face F_τ of $\mathbb{R}^n$ there may be sets β other than $\alpha(\tau)$ for which the subgroup $\mathsf{Q}_{\widehat{\beta}}$ has an inherited action on $\mathbb{H}_{\widehat{\tau}}$ specified by the acceptable exponent matrix $E_{\widehat{\tau},\widehat{\beta}}$ (see Example 3.1.29, below). Where they exist, different sets β for which $E_{\widehat{\tau},\widehat{\beta}}$ is an acceptable exponent matrix need not give rise to corners which are congruent to each other.

Example 3.1.29 Action on a face.

Consider the Q^3 action on $\mathbb{H}^3$ defined by

$$((s_1, s_2, s_3), (h_1, h_2, h_3)) \mapsto (s_2 h_1, s_1 h_2 s_3^{-1}, s_3 h_3 s_2^{-1}),$$

then

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

with $\gamma_1 = \{1, 2, 3\}$, $\gamma_2 = \{1\}$, $\gamma_3 = \{1, 3\}$. Consider the submanifold $\mathbb{H}_{\widehat{1}}$ of $\mathbb{H}^3$ so $\tau = \{1\}$, and we can take the set β to be $\{1\}$, $\{2\}$ or $\{3\}$.

(1) Take $\beta = \{1\}$ so $s_1 = 1$.
Then $\mathsf{Q}_{\widehat{\beta}} = \mathsf{Q}_2 \times \mathsf{Q}_3$ acts on $\mathbb{H}_{\widehat{\tau}} = \mathbb{H}_2 \times \mathbb{H}_3$ by $((s_2, s_3), (h_2, h_3)) \mapsto (h_2 s_3^{-1}, s_3 h_3 s_2^{-1})$. The isotropy subgroups on the facets of this face are given by $\gamma_2 = \{2, 3\}$, and $\gamma_3 = \{2\}$.

$$\text{Hence} \quad E_{\widehat{\tau},\widehat{\beta}} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, L_{\widehat{\beta},\widehat{\tau}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_{\widehat{\tau},\widehat{\beta}} L_{\widehat{\beta},\widehat{\tau}} = \operatorname{diag}(-1, -1).$$

(2) Take $\beta = \{2\}$ so $s_2 = 1$.
Then $\mathsf{Q}_{\widehat{\beta}} = \mathsf{Q}_1 \times \mathsf{Q}_3$ acts on $\mathbb{H}_{\widehat{\tau}} = \mathbb{H}_2 \times \mathbb{H}_3$ by $((s_1, s_3), (h_2, h_3)) \mapsto (s_1 h_2 s_3^{-1}, s_3 h_3)$. The isotropy subgroups on the facets of this face are given by $\gamma_2 = \{1\}$, and $\gamma_3 = \{1, 3\}$.

$$\text{Hence} \quad E_{\widehat{\tau},\widehat{\beta}} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, L_{\widehat{\beta},\widehat{\tau}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad E_{\widehat{\tau},\widehat{\beta}} L_{\widehat{\beta},\widehat{\tau}} = \operatorname{diag}(1, 1).$$

(3) Take $\beta = \{3\}$ so $s_3 = 1$.
Then $\mathsf{Q}_{\widehat{\beta}} = \mathsf{Q}_1 \times \mathsf{Q}_2$ acts on $\mathbb{H}_{\widehat{\tau}} = \mathbb{H}_2 \times \mathbb{H}_3$ by $((s_1, s_2), (h_2, h_3)) \mapsto (s_1 h_2, h_3 s_2^{-1})$. The isotropy subgroups on the facets of this face are given by $\gamma_2 = \{\{1\}, \{2, 3\}\}|_{3 \to \emptyset} = \{1\}$, and $\gamma_3 = \{2\}$.

$$\text{Hence} \quad E_{\widehat{\tau},\widehat{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, L_{\widehat{\beta},\widehat{\tau}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_{\widehat{\tau},\widehat{\beta}} L_{\widehat{\beta},\widehat{\tau}} = \operatorname{diag}(1, -1).$$

The third case cannot be equivariant to either of the others, since its isotropy matrix is not congruent to the others.

3.2 Reconstruction of $\mathbb{H}^n$ from L

A regular action of Q^n on $\mathbb{H}^n$ is specified by an action $((s_1,\ldots,s_n),(h_1,\ldots,h_n)) \mapsto (s_{l_1}h_1s_{r_1}^{-1},\ldots,s_{l_n}h_ns_{r_n}^{-1})$ for an acceptable exponent matrix. From this action we can obtain the isotropy matrix L where $L=(\mathcal{E}E)^{-1}$.

If instead we are given a set γ_i for each facet F_i of $\mathbb{R}^n_{\geqslant}$ which define an acceptable facet graph or an acceptable isotropy matrix, then we can reconstruct a Q^n action on $\mathbb{H}^n$ as follows. For a point $x\in\mathbb{R}^n_{\geqslant}$ let $F_{\tau(x)}$ denote the smallest face of $\mathbb{R}^n_{\geqslant}$ containing x (or the face which contains x in its relative interior, allowing the point x to be in the interior of $\{x\}$).

For each isotropy class $\ell(\tau)=\mathsf{Q}_{\gamma(\tau)}$, where $\gamma(\tau)$ stands for $(\gamma_1^\tau,\ldots,\gamma_{|\tau|}^\tau)$, define the *canonical subgroup* $\widehat{\ell}(\tau)=\mathsf{Q}(u^1,\ldots,u^{|\tau|})$ where each vector $u^j\in\mathbb{H}^n$ for $j=1,\ldots,|\tau|$ has components u_t^j equal to 1 if $t\in\gamma_j^\tau$, and 0 otherwise. We note that (1) each subgroup $\widehat{\ell}(\tau)$ is in the isotropy class $\ell(\tau)$, and (2) if $F_\tau\supset F_\sigma$ then $\widehat{\ell}(\tau)$ is a subgroup of $\widehat{\ell}(\sigma)$. Given an acceptable isotropy matrix L, we have an acceptable isotropy functor ℓ, (Corollary 3.1.21), and so an isotropy class and canonical subgroup is defined for every face of $\mathbb{R}^n_{\geqslant}$.

Construction 3.2.1 For an acceptable isotropy matrix L, define

$$\mathcal{H}(\ell)=\mathsf{Q}^n\times\mathbb{R}^n_{\geqslant}/\!\sim_\ell$$

where the equivalence is defined by

$$(t;x)\sim_\ell(t';x') \quad\Longleftrightarrow\quad x=x',\quad t^{-1}t'\in\widehat{\ell}(\tau(x)),$$

for $t=(t_1,\ldots,t_n)\in\mathsf{Q}^n$, $x=(x_1,\ldots,x_n)\in\mathbb{R}^n_{\geqslant}$, with $\widehat{\ell}(\tau(x))$ the canonical subgroup in the class $\ell(\tau(x))$, for $F_{\tau(x)}$ the smallest face of $\mathbb{R}^n_{\geqslant}$ containing x.

Let Q^n act coordinate-wise on the first factor of $\mathcal{H}(\ell)$, that is, define the Q^n action on $\mathcal{H}(\ell)$ by

$$((s_1,\ldots,s_n),[t_1,\ldots,t_n;x_1,\ldots,x_n])\mapsto[s_1t_1,\ldots,s_nt_n;x_1,\ldots,x_n],$$

where $[t;x]$ denotes an equivalence class.

The relation $\sim_\ell$ is clearly an equivalence relation, since $\widehat{\ell}(\tau(x))$ is a group. The action of Q^n on $\mathcal{H}(\ell)$ is well defined, since if $(t;x) \sim_\ell (t';x)$ then $t^{-1}t' \in \widehat{\ell}(\tau(x))$ so $(st)^{-1}(st') = t^{-1}t' \in \widehat{\ell}(\tau(x))$ and $(st;x) \sim_\ell (st';x)$ for any $s \in \mathsf{Q}^n$. The isotropy class of $[t,x]$ is $\ell(\tau(x))$, since if $s[t;x] = [st;x] = [t;x]$ then $(st)^{-1}t \in \widehat{\ell}(\tau(x))$ so $s \in t\widehat{\ell}(\tau(x))t^{-1}$, and $t\widehat{\ell}(\tau(x))t^{-1}$ is in the class $\ell(\tau(x))$.

For any acceptable exponent matrix E, let $E_{i,j} = \delta_{j,l_i} - \delta_{j,r_i}$ define the l_i, r_i, for each $i = 1, \ldots, n$, and let the Q^n action on $\mathbb{H}^n$ be specified by E. Any point $h = (h_1, \ldots, h_n) \in \mathbb{H}^n$ is in the orbit of $|h| = (|h_1|, \ldots, |h_n|)$ (Proposition 2.2.22). Let the isotropy subgroup of $|h|$ be $\mathsf{Q}(u^1, \ldots, u^k)$ where $u^1, \ldots, u^k \in \mathbb{H}^n$. The components of any $s \in \mathsf{Q}(u^1, \ldots, u^k)$ are related by $s_{l_i}|h_i|s_{r_i}^{-1} = |h_i|$, for each i. For any $|h_i| > 0$, if neither l_i nor r_i is empty then $s_{l_i} = s_{r_i}$ or if l_1 or r_i is empty then $s_{r_i} = 1$ or $s_{l_i} = 1$. So all components of the u^j can be taken to be 0 or 1, and the canonical group $\widehat{\ell}(\tau(|h|))$ is just $\mathsf{Q}(u^1, \ldots, u^k)$, where $F_{\tau(|h|)}$ is the smallest face containing $|h|$. Let $EL = \mathcal{E}$, and define

$$(3.2.2) \qquad \psi\colon \mathcal{H}(\ell) \to \mathbb{H}^n$$
$$[t_1, \ldots, t_n; x_1, \ldots, x_n] \mapsto (h_1, \ldots, h_n) = (x_1 t_{l_1} t_{r_1}^{-1}, \ldots, x_n t_{l_n} t_{r_n}^{-1}).$$

The action of Q^n on $\mathbb{H}^n$ is specified by E, so $(x_1 t_{l_1} t_{r_1}^{-1}, \ldots, x_n t_{l_n} t_{r_n}^{-1})$ can be written as $(t_1, \ldots, t_n)(x_1, \ldots, x_n) = tx$, where x now represents a point in $\mathbb{H}^n$.

Proposition 3.2.3 *Let L be an acceptable isotropy matrix and let E specify a Q^n action on $\mathbb{H}^n$ such that $EL = \mathcal{E}$. Then the map $\psi\colon \mathcal{H}(\ell) \to \mathbb{H}^n$ is an equivariant homeomorphism.*

Proof. We check that ψ is well defined and equivariant, then that it is a homeomorphism.

(1) Well Defined:

If two points $(t;x), (t';x) \in \mathsf{Q}^n \times \mathbb{R}^n_{\geqslant}$ belong to the same equivalence class then $t^{-1}t' \in \widehat{\ell}(\tau(x))$. Now, $\psi([t,x]) = tx$ and $\psi([t',x]) = t'x$, but $\widehat{\ell}(\tau(x))$ is the isotropy subgroup of $x = (x_1, \ldots, x_n) \in \mathbb{H}^n$, so $t^{-1}t'x = x$, that is $t'x = tx$. Thus, all points in the equivalence class $[t,x]$ map to the same element of $\mathbb{H}^n$.

(2) Equivariant:

Now, $\psi([t;x]) = tx$, so for any $s \in \mathsf{Q}^n$, $\psi(s[t;x]) = \psi([st;x]) = (st)x = s(tx)$. That is, $\psi(s[t;x]) = s\psi([t;x])$.

(3) Bijective:

If $\psi([t;x]) = \psi([t';x'])$ then $tx = t'x'$, so x' and x belong to the same orbit in $\mathbb{H}^n$. But x is uniquely defined for any orbit, so $x' = x$ and $t^{-1}t'x = x$. Thus, $(t^{-1}t')$ is an element of $\widehat{\ell}(\tau(x))$, the isotropy subgroup of x, and $[t;x] = [t';x]$, so ψ is injective.

Any point $h = (h_1, \ldots, h_n) \in \mathbb{H}^n$ is in the orbit of $|h| = (|h_1|, \ldots, |h_n|)$ (Proposition 2.2.22), so there is some $t_h \in \mathsf{Q}^n$ such that $h = t_h|h|$. Then, $\psi([t_h;|h|]) = t_h|h|$, so ψ is surjective.

(4) Continuous:

The quotient topology is assumed for the space $\mathcal{H}(\ell)$, so the quotient map $\mathsf{Q}^n \times \mathbb{R}^n_{\geqslant} \to \mathcal{H}(\ell)$ is open and continuous. The functions $x_i t_{l_i} t_{r_i}^{-1}$ are elementary and well defined, hence define a map from $\mathsf{Q}^n \times \mathbb{R}^n_{\geqslant}$ to $\mathbb{H}^n$ that is also continuous and open. Thus, since the composite map ψ is bijective, ψ and ψ^{-1} are both continuous. ❏

The map $\psi\colon \mathcal{H}(\ell) \to \mathbb{H}^n$ can be regarded a single chart which defines a smooth structure on $\mathcal{H}(\ell)$. Any other regular n-corner with isotropy matrix L' congruent to L defines a smoothly equivalent structure, by Proposition 3.1.24.

Proposition 3.2.4 *The inverse map* $\psi^{-1}\colon \mathbb{H}^n \to \mathcal{H}(\ell)$

$$(h_1, \ldots, h_n) = (x_1 t_{l_1} t_{r_1}^{-1}, \ldots, x_n t_{l_n} t_{r_n}^{-1}) \mapsto [t_1, \ldots, t_n; x_1, \ldots, x_n]$$

is given by $x_i = |h_i|$ *and each* t_i *is a product of some of the* $(h_j/|h_j|)^{\pm 1}$. *If* $|h_j| = 0$ *then any component of* t *which contains a factor* $(h_j/|h_j|)^{\pm 1}$ *can take any value in* Q, *and is an element of the isotropy subgroup of* $|h|$.

Proof. Each component h_i is equal to $x_i t_{l_i} t_{r_i}^{-1}$, so $x_i = |h_i|$.

For any $h \in \mathbb{H}^n$, we have $h = t_h|h|$ for some $t_h \in \mathsf{Q}^n$ (since any h is in the orbit of $|h| = (|h_1|, \ldots, |h_n|)$, Proposition 2.2.22), so $\psi^{-1}(h) = [t_h;|h|]$. Each component $(t_h)_i$ is the product of some of the $(h_j/|h_j|)^{\pm 1}$ (Corollary 2.2.24). If any components $|h_j|$ are zero, then some components of t_h are not defined, and t_h can take any value in the isotropy subgroup $\widehat{\ell}(\tau(|h|))$ of $|h|$. ❏

In particular cases, it is not difficult to write the inverse map ψ^{-1} explicitly. Because of the non-commutativity of the quaternionic multiplication the order in which products $(h_j/|h_j|)^{\pm 1}$ are written is crucial and is obtained from the details of the inductive procedure in the proof of Proposition 2.2.22(1). We only include details in the example below.

Example 3.2.5 Inverse map, ψ^{-1}. Suppose a Q^3 action on $\mathbb{H}^3$ is specified by

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \text{ and } L = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Define permutations $\pi_r = (3,2,1)$, and $\pi_c = (2,3)$. Then permuting the rows of E and columns of L by π_r and permuting the columns of E and rows of L by π_c transforms both matrices into upper triangular form. Write $h_i = \widehat{h}_i|h_i|$ for each component of h, and any value in Q can be used for $\widehat{h}_i$ when $|h_i| = 0$. The maps $\psi\colon \mathcal{H}(\ell) \to \mathbb{H}^3$ and ψ^{-1} are given by

$$\begin{aligned} \psi\colon \ [t;x] &\mapsto h = (x_1 t_2,\ x_2 t_1 t_3^{-1},\ x_3 t_2 t_3^{-1}) \\ \psi^{-1}\colon \quad h \ &\mapsto [t;x] = [\widehat{h}_2\widehat{h}_3^{-1}\widehat{h}_1,\ \widehat{h}_1,\ \widehat{h}_3^{-1}\widehat{h}_1;\ |h_1|,\ |h_2|,\ |h_3|\,] \end{aligned}$$

The compositions then give,

$$\begin{aligned} [t;x] &\overset{\psi}{\mapsto} (x_1 t_2,\ x_2 t_1 t_3^{-1},\ x_3 t_2 t_3^{-1}) \\ &\overset{\psi^{-1}}{\mapsto} [t_1 t_3^{-1}\cdot t_3 t_2^{-1}\cdot t_2,\ t_2,\ t_3 t_2^{-1}\cdot t_2;\ x_1,\ x_2,\ x_3] \\ &\quad = [t_1,\ t_2,\ t_3;\ x_1,\ x_2,\ x_3], \\ h &\overset{\psi^{-1}}{\mapsto} [\widehat{h}_2\widehat{h}_3^{-1}\widehat{h}_1,\ \widehat{h}_1,\ \widehat{h}_3^{-1}\widehat{h}_1;\ |h_1|,\ |h_2|,\ |h_2|] \\ &\overset{\psi}{\mapsto} (|h_1|\widehat{h}_1,\ |h_2|(\widehat{h}_2\widehat{h}_3^{-1}\widehat{h}_1)(\widehat{h}_3^{-1}\widehat{h}_1)^{-1},\ |h_3|(\widehat{h}_1)(\widehat{h}_3^{-1}\widehat{h}_1)^{-1}) \\ &\quad = (h_1,\ h_2,\ h_3). \end{aligned}$$

Chapter 4

Quoric Manifolds

Davis and Januszkiewicz (DJ [8]) introduced a class of manifolds, toric manifolds, now generally referred to as quasi-toric manifolds, which are T^n-manifolds with an orbit space homeomorphic to a simple polytope whose faces are defined as the connected subspaces of orbits with the same isotropy subgroups. In this section we show that we can similarly define Q^n-manifolds, of dimension $4n$, each with an orbit space homeomorphic to a simple polytope whose faces are defined by the isotropy classes, which we call *quoric manifolds*. Despite the significant differences between the groups T^n and Q^n, much of the development follows a similar path to that of Davis and Januszkiewicz. We show that quoric manifolds can be classified up to equivariant diffeomorphism by a polytope and an isotropy functor λ (Theorem 4.3.3), where the isotropy functor now defines an isotropy class for each face of the polytope instead of an isotropy subgroup. In the toric case, a polytope P^n has in general an infinite number of T^n-manifolds over it, since there are an infinite number of vectors of integers which span a direct summand of $\mathbb{Z}^n$ (see DJ [8], for the example of $P^2 = I^2$). For the quoric manifolds introduced here, a polytope can only support a finite number of congruence classes of isotropy functors, since there are only a finite number of subsets of $[n]$ available at each facet of P^n.

4.1 Polytopes

Let us recall a number of descriptions and decompositions used for a polytope. We give a brief summary of the definitions and results that we need subsequently. Excellent descriptions with much more detail can be found in BP ([4] Ch 1, 4) or Ziegler [22]. A *convex polyhedron* P^n is a subset of $\mathbb{R}^n$ defined as the intersection of m half spaces H_a,

$$P^n = \bigcap_{a=1}^{m} H_a, \qquad H_a = \{x \in \mathbb{R}^n \mid y_a = \langle l_a, x\rangle + c_a \geqslant 0\} \tag{4.1.1}$$

where $l_a \in \mathbb{R}^n$, $c_a \in \mathbb{R}$, for $a = 1, \ldots, m$, and $\langle \cdot, \cdot \rangle$ denotes the natural scalar product in $\mathbb{R}^n$. The orbit spaces $\mathbb{R}^n_{\geqslant}$ introduced earlier (Paragraph 2.2.5) are simple examples. A *polytope* P^n is a bounded convex polyhedron.

If c_a is strictly positive then H_a contains the origin of $\mathbb{R}^n$ in its interior, and we will assume this to be the case for each a, so the origin is in the interior of P^n. We will also assume that this set of half spaces is a minimal set, that is, omitting any one of them defines a different polyhedron. The boundary ∂H_a of H_a is the affine hyperplane corresponding to the set of points with $y_a(x) = 0$. The orbit spaces $\mathbb{R}^n_{\geqslant}$ considered in the previous section (Equation 3.1.2) are polyhedra defined by n half spaces which each select the non-negative values of one coordinate subspace of $\mathbb{R}^n$.

For half spaces in general position, that is the combinatorial properties of the set of faces are not changed by any small perturbation of the half spaces, there are precisely n facets meeting in any vertex. A polytope P^n which has n facets meeting at each vertex is called *simple*. A polytope P^n for which every proper face is a simplex is called *simplicial*.

For a polytope P^n, define its *polar* P^* (sometimes *dual*) by

$$P^* = \{x' \in \mathbb{R}^n \mid \langle x', x\rangle \geqslant -1, \forall x \in P^n\}.$$

The polar of the polar $(P^*)^*$ of any polytope P is equal to P, providing P contains the origin as we are assuming here. If P is simplicial then P^* is simple, and vice versa.

For example, an n simplex Δ^n is both simple and simplicial. An n dimensional cross polytope (for $n > 2$) is not simple, with $2(n-1)$ facets meeting at each vertex. It is however simplicial, and its polar is an n cube which is simple.

We can use the description of P^n in terms of half spaces (Equation 4.1.1) to define a map

$$A_P\colon \mathbb{R}^n \to \mathbb{R}^m, \quad x \mapsto y = (y_1, \dots, y_m), \qquad y_a = \langle l_a, x\rangle + c_a \geqslant 0. \tag{4.1.3}$$

This map is affine, so the image of $\mathbb{R}^n$ is an n-dimensional affine hyperplane in $\mathbb{R}^m$. Let the perpendicular subspace have an outward-pointing basis $\{w_b \mid b = n+1, \dots, m\}$. Define $w_b^*(y) = \langle w_b, y\rangle$, so any point y in the image of $\mathbb{R}^n$ satisfies $w_b^*(y) = w_b^*(c)$ (> 0), for $c = (c_1, \dots, c_m)$, and the image of P^n is

$$A_P(P^n) = \{y \in \mathbb{R}^m_\geqslant \mid w_b^*(y) = w_b^*(c)\},$$

that is, the intersection of the n-dimensional affine hyperplane $A_P(\mathbb{R}^n)$, and the semi-positive cone $\mathbb{R}^m_\geqslant$ of $\mathbb{R}^m$.

4.1.4 Face Vectors and Dehn-Sommerville Relations. For any polytope P^n let f_i denote the number of faces of dimension i (for $i = 0, \dots, n$), including $f_n = 1$ for P^n itself. The vector of these integers $(f_0, \dots, f_n)$ will be called the f-vector. Let $f(x) = \sum_{i=0}^n f_i x^i$, and define $h(t) = f(t-1)$ where $h(t) = \sum_{j=0}^n h_j t^j$. Since any polytope is contractible, its Euler characteristic χ is 1, so $\chi = f(-1) = h(0) = 1$. The components of these vectors are related by

$$h_j = \sum_{i=j}^{n} (-)^{j-i} \binom{i}{j} f_i \qquad\qquad f_i = \sum_{j=i}^{n} \binom{j}{i} h_j. \tag{4.1.5}$$

Now let P^n be a simple polytope.

We can always choose a line L in $\mathbb{R}^n$ such that the perpendicular projection of the vertices of P^n onto L are distinct. Let the direction vector of L be $l \in \mathbb{R}^n$, then $\langle l, v\rangle \neq \langle l, v'\rangle$ for all pairs of vertices $v \neq v'$ of P^n. Thus L is not perpendicular to any face of P^n, since then all vertices of that face would project to the same point. The line L defines a function $L\colon \mathbb{R}^n \to \mathbb{R}$, by $x \mapsto L(x) = \langle l, x\rangle$.

Each edge of P^n can be given a direction, by orienting the edge so that $L(x)$ increases along it. Thus, the 1-skeleton of P^n becomes a directed graph, with unique maximal and minimal vertices. For each vertex v of P^n, define its *index* $ind(v)$ as the number of edges incident at v directed towards it. Let $I_L(i)$ denote the number of vertices of index i. Each face F^k of dimension k has a unique maximal vertex v_F (with respect to L). Since P^n is simple so is every face, so there are k inward pointing edges incident at v_F within F^k, hence, $ind(v_F) \geqslant k$. Every vertex of index q is the maximal vertex of $\binom{q}{k}$ faces of dimension $k \leqslant q$. Hence the number of faces of dimension k is given by $f_k = \sum_{q \geqslant k} \binom{q}{k} I_L(q)$. Comparing with Equation (4.1.5), we deduce that $I_L(q) = h_q$. In particular, $I_L(q)$ does not depend on L. Moreover, since P^n is simple, we have $ind_L(v) + ind_{-L}(v) = n$ for any vertex v, that is $I_L(q) = I_{-L}(n-q)$. Thus, for a simple polytope, we obtain the Dehn-Sommerville relations,

$$h_{n-q} = h_q. \tag{4.1.6}$$

4.1.7 Simplicial and Cubical Complexes. Given a finite set $\mathcal{S}$, a *simplicial complex* on $\mathcal{S}$ is a collection K of subsets of $\mathcal{S}$, such that for each $\sigma \in K$ every subset of σ (including $\emptyset$) is also in K. Each $\sigma \in K$ is a *simplex* of K. One-element sets are the *vertices* of K. The dimension of a simplex $\sigma \in K$ is one less than its cardinality, $\dim \sigma = |\sigma| - 1$. The dimension of a simplicial complex is the largest dimension of its simplices. A subcomplex of K is a subcollection $K' \subset K$ which is also a simplicial complex (see BP [4] Ch 2, or Dwyer [11] §3 for more details).

The standard unit cube $I^q = [0,1]^q$ in $\mathbb{R}^q$, together with all its faces, is an example of a q-dimensional cubical complex. Combinatorially, a q-dimensional cube I^q is characterised by its 2^q vertices, and for each face F of dimension k there are 2^{q-k} faces of I^q which contain F. An *cubical complex* on a finite set $\mathcal{S}$ is a collection K of subsets of $\mathcal{S}$ such that for each $\sigma \in K$, its cardinality $|\sigma|$ is 2^k for some positive integer k, and there are k pairs of disjoint subsets τ, $\sigma \backslash \tau$ of σ of cardinality 2^{k-1} which are in K. Each $\sigma \in K$ is a *cube* of K, of dimension k for $|\sigma| = 2^k$. One-element sets are the *vertices* of K. The dimension of a cubical complex is the largest dimension of its cubes. A subcomplex of K is a subcollection $K' \subset K$ which is also a cubical complex (see BP [4] Ch 4 for more details).

Faces, Posets and Categories. In Paragraph 2.2.5 we discussed the faces F_τ of $\mathbb{R}^n_\geqslant$ and the categories $\text{FACE}(\mathbb{R}^n_\geqslant)$ and $\text{CAT}([n])$. We now extend that discussion to the faces of a simple polytope. Any face F is the intersection of the facets which contain it, $F_\sigma = \bigcap_{a\in\sigma} F_a$. For a simple polytope each vertex is contained in n facets, so each face of codimension k is characterised by a set σ of cardinality $k = |\sigma|$, and this description is unique. The poset $\mathcal{F}(P^n)$ is defined as the set of faces partially ordered by set inclusion, including $F_\emptyset = P^n$ as a maximal element. The category corresponding to this poset, with the faces as objects and the inclusion relations defining the morphisms, is denoted $\text{FACE}(P^n)$. The set of all σ for which F_σ is a face of P^n is partially ordered by set inclusion so is a poset, conventionally denoted by K_P. The category derived from this poset, with the sets of K_P as objects and the inclusion relations defining the morphisms, is denoted by $\text{CAT}(K_P)$. Since $F_\mu \subset F_\sigma$ whenever $\sigma \subset \mu$, and conversely, we clearly have an isomorphism

$$\text{FACE}(P^n)^{\text{OP}} \leftrightarrow \text{CAT}(K_P). \tag{4.1.8}$$

Corresponding to the poset K_P is the simplicial complex of all sets $\sigma \in K_P$ on $[m]$, which we will also call K_P. Any vertex v of P^n is the intersection of n facets $F_{a_1}, \dots, F_{a_n}$, and corresponds to the vertex $\sigma_v = (a_1, \dots, a_n)$ of K_P. Then any subset of σ_v corresponds to a face of P^n so is an element of K_P, and every face F_σ contains some vertex so σ is a subset of some σ_v with n elements.

Subdivisions. Given a connected subspace X of $\mathbb{R}^n$ and some index set A, a *subdivision* of X is a collection of subsets of X

$$K = \{X_a \subset X \mid a \in A\}$$

such that (1) $X = \bigcup_{a\in A} X_a$, and (2) $X_a \cap X_b \in K$. In many common applications, the subsets X_a are restricted to some particular class of sets, such as homeomorphic images of simplices or cubes. Sometimes, we have a subdivision K of X, and look for a finer subdivision K' of X, that is each $X_a \in K$ is the union of some of the elements of K'.

Given any CW complex, homeomorphic to a CW structure defined on a connected subset of $\mathbb{R}^n$, there are a number of useful ways of subdividing the complex. We describe two of them here for later use, the barycentric subdivision and the cubical subdivision (BP [4] Ch 4). The description is adapted to the case of the polytopes that we are considering here.

4.1.9 Barycentric Subdivision of P^n. Let P^n be a simple polytope with m facets, with its face poset $\mathcal{F}(P^n)$ and simplicial complex K_P. Define a *chain* as a set of nested elements of K_P, $C = (\sigma_0 \subset \ldots \subset \sigma_k)$, where the subset inclusions need not be proper. The chain is called *proper* if every subset inclusion is proper, and then its dimension is defined to be k. The empty chain () is regarded as proper, with dimension of -1, and is a subchain of all chains. The *barycentric complex* is defined as the set of all proper chains of K_P,

$$BC(K_P) = \{(\sigma_0 \subset \ldots \subset \sigma_k) \mid \sigma_i \in K_P,\ \sigma_i \neq \sigma_{i+1}\}. \tag{4.1.10}$$

For a chain C of $BC(K_P)$, any subchain of C is clearly also a chain of $BC(K_P)$, and any chain is a subchain of some n-dimensional chain, since every face of P^n contains a vertex. So $BC(K_P)$ is an n-dimensional simplicial complex.

The barycentric complex $BC(K_P)$ may be realised as a subdivision of the polytope P^n. Select a point $A(\sigma)$ in the interior of each face F_σ, including the vertices and $F_\emptyset = P^n$. For each pair of distinct points $A(\sigma_0)$, $A(\sigma_1)$ for which $\sigma_0 \subset \sigma_1$, identify an edge joining them in F_{σ_0} as a homeomorphic image $A(\sigma_0, \sigma_1)$ of a unit interval, corresponding the the 1-dimensional chain $(\sigma_0 \subset \sigma_1)$. Inductively, for any set of $(k+1)$ distinct points $A(\sigma_0), \ldots, A(\sigma_k)$ for which $C = (\sigma_0 \subset \ldots \subset \sigma_k)$ is a chain in $BC(K_P)$, identify the k-dimensional homeomorphic image $A(\sigma_0, \ldots, \sigma_k)$ in F_{σ_0} of a simplex bounded by the images of the simplices of all the subchains of C. Then P^n is the union of the blocks $A(\sigma_0 \subset \ldots \subset \sigma_k)$, and defines B_P the *barycentric subdivision* of P^n,

$$P^n = \bigcup_{(\sigma_0 \subset \ldots \subset \sigma_k) \in BC(K_P)} A(\sigma_0 \subset \ldots \subset \sigma_k).$$

4.1.11 Cubical Subdivision of P^n. Let P^n be a simple polytope with m facets, with its face poset $\mathcal{F}(P^n)$ and simplicial complex K_P. Define the *cubical complex* $CC(K_P)$ as the set of all (not necessarily proper) 2-chains of K_P,

$$CC(K_P) = \{(\sigma_1 \subset \sigma_2) \mid \sigma_1, \sigma_2 \in K_P\}. \tag{4.1.12}$$

The dimension of a 2-chain is defined to be $\dim\,(\sigma_1 \subset \sigma_2) = |\sigma_2| - |\sigma_1|$. A 2-chain $(\sigma_1 \subset \sigma_2)$ is a face of the 2-chain $(\tau_1 \subset \tau_2)$ whenever $(\tau_1 \subset \sigma_1 \subset \sigma_2 \subset \tau_2)$ is a chain (not necessarily proper). Each σ of K_P (including $\sigma = \emptyset$) can be identified with a vertex $(\sigma \subset \sigma)$ of $CC(K_P)$, since this chain has dimension 0, and all vertices of $CC(K_P)$ can be identified in this way. Other chains are proper, and have a positive dimension.

Given a chain $S = (\sigma_1 \subset \sigma_2)$ (of dimension s) which is a face of $T = (\tau_1 \subset \tau_2)$ (of dimension t), so that $(\tau_1 \subset \sigma_1 \subset \sigma_2 \subset \tau_2)$, then there are $(t - s)$ elements of $\tau_2 \backslash \tau_1$ that are not elements of $\sigma_2 \backslash \sigma_1$. Therefore there are 2^{t-s} chains which are faces of T and which contain S as a face (including S and T). Thus, the set of faces of T which contain S forms a cubical complex, and is sufficient to show that $CC(K_P)$ is a cubical complex. Every chain of dimension n is of the form $(\emptyset \subset \sigma_v)$ with $|\sigma_v| = n$, so can be uniquely associated with the vertex $v = F_{\sigma_v}$ of P^n.

Just as for the case of the barycentric subdivision above, we can use the cubical complex $CC(K_P)$ to define a subdivision of P^n. Select a point $B(\emptyset, \emptyset)$ in the interior of P^n (for the chain $(\emptyset \subset \emptyset)$), and points $B(\sigma, \sigma)$ in the interior of each face F_σ (for chains $(\sigma \subset \sigma)$, each $\sigma \in K_P$). Identify an edge $B(\sigma \backslash \{a\}, \sigma)$ between each $B(\sigma, \sigma)$ and $B(\sigma \backslash \{a\}, \sigma \backslash \{a\})$, as a homeomorphic image of a unit interval in F_σ (for chain $(\sigma \backslash \{a\} \subset \sigma)$, for each $a \in \sigma \in K_P$). Inductively, for each chain $S = (\sigma_1 \subset \sigma_2)$ of dimension $k = |\sigma_2| - |\sigma_1|$, identify a block $B(\sigma_1, \sigma_2)$ as the homeomorphic image of a k-cube in P^n bounded by the $3^k - 1$ faces of S of lower dimension.

Since each chain S of dimension n in $CC(K_P)$ can be uniquely associated with a vertex of P^n, there is a single n-dimensional block of the subdivision containing each vertex. Any two blocks can only intersect in a common face, and any block is the face of some n-block, so the union of the n-blocks is just P^n.

Then we can express P^n as the union over all cubes $B(\sigma,\tau)$, including all their faces, so there is a term for every element in $CC(K_P)$, which defines C_P the *cubical subdivision* of P^n

$$P^n = \bigcup_{(\sigma\subset\tau)\in CC(K_P)} B(\sigma,\tau).$$

Since $B(\sigma,\sigma)$ for a non-empty $\sigma \in K_P$ is a point in the interior of the face F_σ of P^n, the block $B(\sigma_1,\sigma_2)$ for non-empty σ_1 is contained in the boundary of P^n, so $\partial P^n = \bigcup_{\emptyset\neq\sigma\subset\tau\in K_P} p(\sigma,\tau)$. In some situations we are only interested in the n-dimensional blocks of C_P and their intersections, so can omit the boundary terms, giving the decomposition

$$P^n = \bigcup_{\sigma\in K_P} B(\sigma), \tag{4.1.13}$$

where $B(\sigma) = B(\emptyset,\sigma) \in C_P$.

Embedding of Cubical Subdivision in I^m. We now use this cubical subdivision of a simple polytope to construct another important embedding of P^n in $\mathbb{R}^m$. In this case the polytope is embedded in a subset of the faces of the m-cube I^m in $\mathbb{R}^m$ rather than in an affine hyperplane of $\mathbb{R}^m$.

For any block $B(\sigma_1,\sigma_2)$ of C_P, corresponding to chain $(\sigma_1 \subset \sigma_2) \in CC(K_P)$, define the $(|\sigma_2| - |\sigma_1|)$-dimensional subspace C_{σ_1,σ_2} of I^m as

$$C_{\sigma_1,\sigma_2} = \{x \in I^m \mid x_a = 0\ (a \in \sigma_1),\quad x_a = 1\ (a \notin \sigma_2)\}$$

It is immediate that if $(\mu_1 \subset \mu_2)$ is a face of $(\sigma_1 \subset \sigma_2)$, then the face C_{μ_1,μ_2} is a subface of C_{σ_1,σ_2}. Thus the complex consisting of the sets C_{σ_1,σ_2} is isomorphic to the cubical complex $CC(K_P)$, and to the cubical division C_P of P^n. Hence, there is an embedding of the subdivision C_P in I^m,

$$P^n \cong \bigcup_{(\tau\subset\sigma)\in K_P} C_{\tau,\sigma}.$$

The union may be restricted to $C_\sigma = C_{\emptyset,\sigma}$ when only the maximal dimension blocks of P^n and their intersections are needed (as in Equation (4.1.13)). We write $C_\sigma =$

$I^\sigma \times 1^{\widehat{\sigma}}$, where $I^\tau = \prod_{a\in\tau} I_a \subset I^m$, and $1^{\widehat{\tau}} = \prod_{a\notin\tau}\{1\}_a \subset I^m$. Then we can write the embedding of the cubical subdivision C_P in I^m as

$$P^n \cong \bigcup_{\sigma\in K_P} I^\sigma \times 1^{\widehat{\sigma}}. \tag{4.1.14}$$

Example 4.1.15 Cubical Decomposition of 2-Simplex.

We illustrate the cubical decomposition of the 2-simplex Δ^2, as the only example which can sensibly be illustrated on a page.

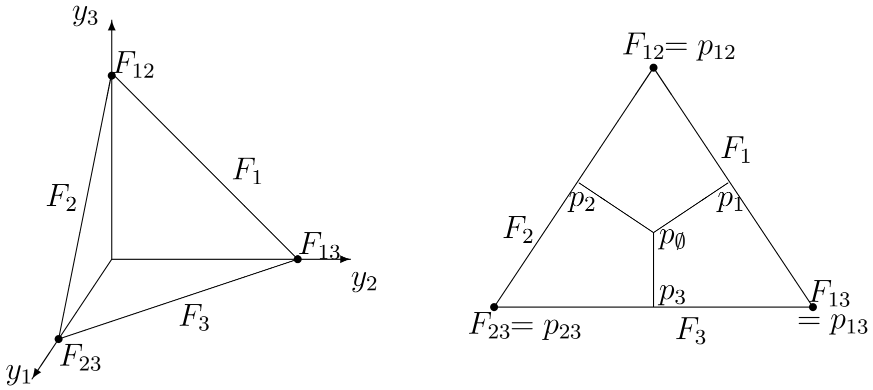


Figure 4.1: (a) Embedding of Δ^2 in $\mathbb{R}^3$, (b) Cubical Decomposition

The facets (edges) of Δ^2 are F_1, F_2, F_3, with vertices as the intersections F_{12}, F_{13}, F_{23}. Then Δ^2 maps to the intersection of the affine hyperplane $y_1 + y_2 + y_3 = 1$ with the positive cone in $\mathbb{R}^3$, see Figure 4.1(a). The points $p_\emptyset = B(\emptyset, \emptyset)$ in P^n and $p_i = B(\{i\}, \{i\})$ in F_i are chosen, then the point p_{12} is determined as the intersection of the plane through $(p_\emptyset, p_1)$ and $(p_\emptyset, p_2)$ with F_{12}, in this case the point F_{12}, and similarly for p_{13} and p_{23}.

The cubical decomposition of Δ^2 is shown (see Figure 4.1(b)) with a 2-dimensional cube (quadrilateral) associated with each vertex. This decomposition maps to the surface of the unit cube (see Figure 4.2, p.79) mapping the points p_i to $I^0_{(i)} = I_{(i),(i)}$ and p_{ij} to $I^0_{(ij)} = I_{(ij),(ij)}$. The edges $(p_\emptyset, p_i)$ map to $I^1_{\emptyset,(i)}$, and (p_i, p_{ij}) map to $I^1_{(i),(ij)}$. The subcube $(p_\emptyset, p_1, p_2, p_{12})$ maps to the face $I^2_{\emptyset,(12)}$, with vertices $I^0_\emptyset$, $I^0_{(1)}$, $I^0_{(2)}$, $I^0_{(12)}$.

4.1.17 Stanley Reisner Ring of a Simple Polytope. The Stanley-Reisner face ring of a simple polytope is an important combinatorial invariant of the polytope.

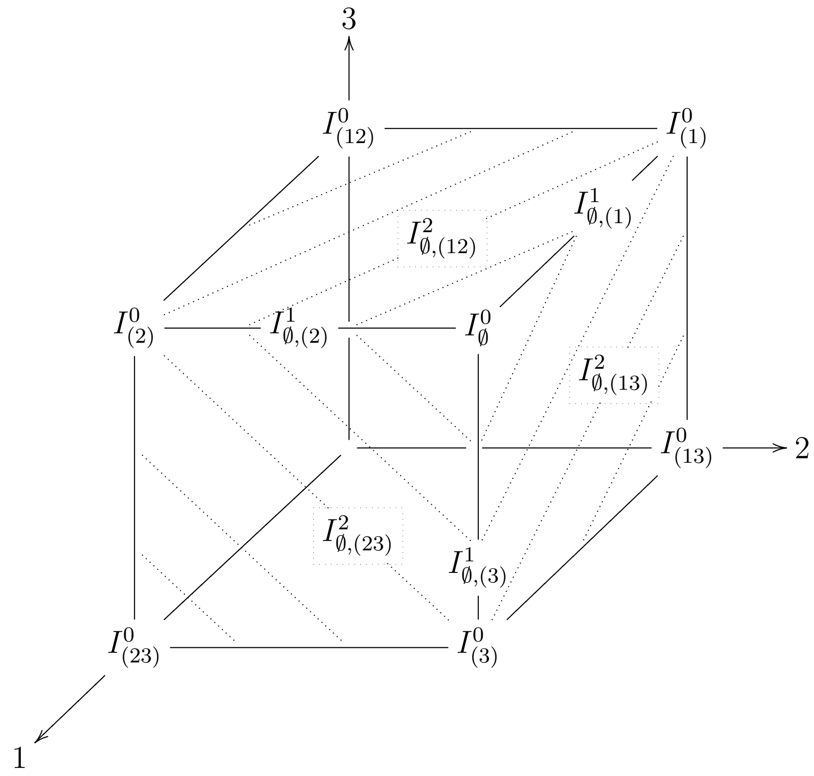


Figure 4.2: Embedding of Δ^2 in I^3

The face ring was initially introduced for simplicial polytopes (see Stanley [20], or BP [4] Ch 3), but since the polar of any simplicial polytope is simple, it is straightforward to adapt the definition of the face ring to simple polytopes.

For any simple polytope P^n, with facets F_a for $a = 1, \dots, m$, let $\mathbb{Z}[v_1, \dots, v_m]$ be the polynomial algebra over $\mathbb{Z}$ on m generators. We make it a graded algebra by defining the degree of each generator $|v_a|$ to be d, for any positive even integer d, conventionally taken to be 2. For any subset $\tau = \{a_1, \dots, a_t\} \subset [m]$ let $v(\tau)$ denote the monomial $v_{a_1} \cdots v_{a_t}$. Let $\mathcal{I}_P$ denote the ideal in $\mathbb{Z}[v_1, \dots, v_m]$ that is generated by all the $v(\tau)$ for which F_τ is *not* a face of P^n. Then the *Stanley Reisner ring* $\mathrm{SR}(P^n)$ is defined as the quotient ring $\mathbb{Z}[v_1, \dots, v_m]/\mathcal{I}_P$. This ring clearly depends only on the combinatorial properties of P^n, and is frequently written to depend on K_P rather than P^n itself.

If F_τ is not a face of P^n, but F_ρ is a face for every proper subset $\rho \subset \tau$, then F_τ is said to be a *missing face* of P^n. For any non-face F_σ there is some missing face

F_τ with $\tau \subset \sigma$, hence the monomial $v(\sigma)$ is a multiple of $v(\tau)$. Thus, the monomials $v(\tau)$ corresponding to the missing faces of P^n form a minimal generating set for the ideal $\mathcal{I}_P$.

4.2 Quoric Manifolds over Simple Polytopes

The defining characteristics that we want to use for our Q^n-manifolds M^{4n} are (1) that every point should belong to an open neighbourhood which is equivariantly diffeomorphic to some regular corner, and (2) that the orbit space is diffeomorphic to a simple polytope whose faces are distinguished by the isotropy classes. Without the second condition, a number of Q^n-manifolds would be included which have properties, such as their cohomology, that cannot be determined in the same way as for the quoric manifolds introduced here. Examples include the spaces $S^{4n} \subset \mathbb{H}^n \times \mathbb{R}$ with Q^n acting coordinatewise on the n quaternionic coordinates which has only two fixed points. These defining characteristics also imply that there is a functor from the category of faces of the orbit space (with subset inclusion defining the morphisms) to the category of isotropy classes $\textsc{conj}(\mathsf{Q}^n)$ such that the restriction to each vertex of the polytope is an acceptable isotropy functor for some regular n-corner. In the next section we will introduce an additional *global* compatibility condition.

Definition 4.2.1 Suppose Q^n acts smoothly on a smooth manifold M^{4n}, a *regular chart* is a pair (U, ϕ), where U is a Q^n-stable open subset of M^{4n}, and ϕ is an equivariant homeomorphism $\phi\colon U \to \mathbb{H}^n$ to a regular n-corner. That is, ϕ satisfies $\phi(t\,y) = t\,\phi(y)$ for all $t \in \mathsf{Q}^n$, $y \in U \subset M^{4n}$, and the action $t\,\phi(y)$ is that of some regular corner (Definition 3.1.22).

A Q^n action on M^{4n} is said to be *locally regular* if M^{4n} has a regular atlas, that is, if every point of M^{4n} lies in a regular chart.

By referring to a smooth manifold M^{4n}, it is implicit in this definition that the if two charts (U, ϕ_U) and (V, ϕ_V) have overlapping domains, $U \cap V \neq \emptyset$, then the transition functions $\phi_U \circ \phi_V^{-1}\colon \phi_V(U \cap V) \to \phi_U(U \cap V)$ are smooth, regarded as real functions between subsets of $\mathbb{R}^{4n}$.

Different charts in the atlas need not refer to regular corners that are congruent. Example 4.2.5 below shows that for a simple case, the quaternionic projective space $\mathbb{H}P^n$ over a simplex Δ^n, two incongruent regular corners occur in the atlas.

Remark. Note that it is not necessary to include an arbitrary automorphism θ_U of Q^n explicitly in this definition, that is to allow $\phi(t\,y) = \theta_U(t)\,\phi(y)$ for each chart, as is done in the toric case. Any automorphism of Q^n defines an equivariant diffeomorphism from one regular n corner to another (Proposition 3.1.24), so a non-trivial automorphism θ_U just determines another equivariant regular corner.

If a Q^n action on M^{4n} is locally regular then any Q^n orbit of M^{4n} that projects to a point in the relative interior of a face of $\mathbb{R}^n_{\geqslant}$ of codimension k has an isotropy class of rank k (for $0 \leqslant k \leqslant n$) (Proposition 3.1.19), and the isotropy classes of all points in the relative interior of a face are equal, since this holds for any regular corner. In particular, the inverse image of the origin of $\mathbb{H}^n$ is an isolated fixed point, with isotropy subgroup equal to Q^n, so M^{4n} has only isolated fixed points. We note (following BP [4] Ch 5, see also Ziegler [22]) that two simple polytopes are diffeomorphic as manifolds with corners if and only if they are combinatorially equivalent, that is they have the same face poset, K_P.

We now define quoric manifolds as the quaternionic analogue of the quasi-toric T^n-manifolds introduced by Davis and Januszkiewicz.

Definition 4.2.2 Given a simple polytope P^n, a *quoric manifold over* P^n is a smooth Q^n-manifold M^{4n} such that:

(1) the action of Q^n on M^{4n} is locally regular,

(2) there is an orbit space projection map $\pi\colon M^{4n} \to P^n$, that is, the fibres of π are the Q^n orbits.

Let $v = F_{\sigma_v}$ be a vertex of a simple polytope P^n, where $\sigma_v = \{a_1, \ldots, a_n\}$ in K_P. Restricting $\textsc{cat}(K_P)$ to the vertex v, that is restricting to the subsets of σ_v (including σ_v and $\emptyset$), defines a full subcategory $\textsc{cat}(K_P|_v)$ of $\textsc{cat}(K_P)$, which is isomorphic to $\textsc{cat}([n]) \cong \textsc{face}(\mathbb{R}^n_{\geqslant})^{\mathrm{OP}}$. That is, $r_v\colon \textsc{cat}([n]) \to \textsc{cat}(K_P|_v)$ defined by $\sigma \mapsto \{a_i \in \sigma_v \subset [m] \mid i \in \sigma\}$ is an isomorphism.

Definition 4.2.3 Given a simple polytope P^n, a *characteristic functor* over P^n is a functor $\lambda\colon \text{CAT}(K_P) \to \text{CONJ}(Q^n)$ such that its restriction to any vertex v of P^n composed with r_v,

$$\ell_v = \lambda|_v \circ r_v\colon \text{CAT}([n]) \xrightarrow{\cong} \text{CAT}(K_P|_v) \to \text{CONJ}(\mathsf{Q}^n),$$

is an acceptable isotropy functor.

For each facet F_a of P^n (for $a = 1, \ldots, m$), let $\lambda(\{a\}) = \mathsf{Q}_{\gamma_a}$, and define a column vector y_a by $(y_a)_t = 1$ if $t \in \gamma_a$, and $= 0$ otherwise (for $t = 1, \ldots, n$). The *characteristic matrix* of λ over P^n is the $n \times m$ matrix Λ defined by the columns y_a. The *characteristic graph* Σ of λ over P^n is the facet graph whose m nodes are the sets $\gamma_a \subset [n]$.

Proposition 4.2.4 *Any quoric manifold M^{4n} over a simple polytope P^n, has a characteristic functor over P^n.*

Proof. This follows directly from the locally regular condition. ❑

Example 4.2.5 The quaternionic projective space $\mathbb{H}P^n$ is conventionally defined as the quotient space $\mathbb{H}^{n+1}_\times/\mathbb{H}_\times$ (for $X_\times = X\backslash\{0\}$). Taking the quotient of both factors by $\mathbb{R}_>$, we can express $\mathbb{H}P^n$ as the set of equivalence classes

$$[h_0, h_1, \ldots, h_n] = [h_0u, h_1u, \ldots, h_nu], \qquad u \in \mathsf{Q}, \quad h \in \mathbb{H}^{n+1}, \quad \sum |h_i|^2 = 1.$$

The Q^n action is defined by

$$\mathsf{Q}^n \times \mathbb{H}P^n \to \mathbb{H}P^n, \quad ((s_1, \ldots, s_n), [h_0, h_1, \ldots, h_n]) \mapsto [h_0, s_1h_1, \ldots, s_nh_n],$$

and we may take the orbit projection map to be

$$\pi\colon [h_0, h_1, \ldots, h_n] \mapsto (|h_0|^2, |h_1|^2, \ldots, |h_n|^2) \qquad \subset \mathbb{R}^{n+1}_\geqslant, \quad \sum |h_i|^2 = 1,$$

so the orbit space is the standard simplex Δ^n.

For each $k = 0, 1, \ldots, n$ define a chart (U_k, ψ_k) by $U_k = \mathbb{H}P^n|_{|h_k|>0}$, and

$$\begin{aligned}\psi_k\colon U_k \to \mathbb{H}^n, \;\; [h_0, h_1, \ldots, h_n] &\mapsto (q_1, \ldots, q_n)\\ &= (h_0h_k^{-1}, h_1h_k^{-1}, \ldots, \widehat{h}_k, \ldots, h_nh_k^{-1}),\end{aligned}$$

where $\widehat{h}_k$ denotes that the coordinate h_k is omitted. Then $\psi_0(U_0)$ is the standard n corner with action $((s_1, \ldots, s_n), (q_1, \ldots, q_n)) \mapsto (s_1 q_1, \ldots, s_n q_n)$, and exponent matrix $E_{(0)}$ equal to the unit matrix. For $k > 0$ the image $\psi_k(U_k)$ is a regular n corner with Q^n action,

$$((s_1, \ldots, s_n), (q_1, \ldots, q_n)) \mapsto (q_1 s_k^{-1}, s_1 q_2 s_k^{-1}, \ldots, s_{k-1} q_k s_k^{-1}, s_{k+1} q_{k+1} s_k^{-1}, \ldots).$$

Thus, the exponent matrix is $(E_{(k)})_{i,j} = -1$ (if $j = k$), $(E_{(k)})_{i,j} = 1$ (if $1 \leqslant i - 1 = j < k$ or $k < i = j \leqslant n$), and $(E_{(k)})_{i,j} = 0$ otherwise (compare Example 2.2.15(2)). For $k > 0$ the $E_{(k)}$ are congruent to each other, but not congruent to $E_{(0)}$.

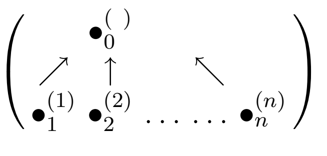
For $k > 0$, the transition functions $\psi_k \circ \psi_0^{-1} \colon \psi_0(U_0 \cap U_k) \to \psi_k(U_0 \cap U_k)$, $q = (q_1, \ldots, q_n) \mapsto q' = (q'_1, \ldots, q'_n)$ are given by

$$\begin{aligned} q'_1 &= q_k^{-1}, \quad & q'_i &= q_{i-1} q_k^{-1} \ (1 < i \leqslant k); \quad & q'_i &= q_i q_k^{-1} \ (k < i \leqslant n); \\ q_k &= {q'_1}^{-1}, \quad & q_i &= q'_{i+1} {q'_1}^{-1} \ (1 \leqslant i < k); \quad & q_i &= q'_i {q'_1}^{-1} \ (k < i \leqslant n); \end{aligned}$$

where $|q_k|, |q'_1| > 0$. These transition functions $\psi_k \circ \psi_0^{-1}$ are clearly diffeomorphisms between the subspaces $\psi_0(U_0 \cap U_k)$ and $\psi_k(U_0 \cap U_k)$ of $\mathbb{H}^n$, and are not diffeomorphisms between the corners $\mathbb{H}^n$, since the corners contain points with components $q_k = 0$ or $q'_1 = 0$. The functions $\psi_k \circ \psi_{k'}^{-1}$ behave similarly.

The isotropy classes for the facets are $\lambda(\{0\}) = \mathsf{Q}_{\gamma_0}$ with $\gamma_0 = \{1, \ldots, n\}$, and $\lambda(\{i\}) = Q_{\gamma_i}$ with $\gamma_i = \{i\}$ $(i = 1, \ldots, n)$. For any face F_σ of Δ^n, the isotropy class is $\lambda(\sigma) = Q_{\gamma^\sigma_{i_1}, \ldots, \gamma^\sigma_{i_{|\sigma|}}}$ where $\gamma^\sigma_i = \{i\}$ for any $0 \neq i \in \sigma$, and $\gamma^\sigma_0 = \{i \in [n] \mid i \notin \sigma\}$ if $0 \in \sigma$. The characteristic matrix and characteristic facet graph are given by

$$\Lambda = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

$\Sigma =$ 

Each $n \times n$ submatrix of Λ is an acceptable isotropy matrix, and each subgraph $\Sigma|_v$ with n nodes is an acceptable facet graph. Of course, Σ itself can never be an acceptable facet graph, since it is a graph of m nodes on subsets of $[n]$, with $m > n$.

Remark 4.2.6 This definition of a characteristic functor is our generalisation of Davis and Januszkiewicz's condition (∗) for quasi-toric manifolds to the case of the non-commutative group Q^n. The isotropy subgroup corresponding to orbits in the interior of a facet of the orbit space of a quasi-toric manifold is specified by a vector of integers $p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$. Since T is a commutative group, all points on any orbit of a T^n-manifold have the same isotropy subgroup, and an isotropy class contains only one subgroup. The isotropy subgroup of orbits in the interior of a face of the orbit space of codimension k is specified by a set of k such vectors of integers. Davis and Januszkiewicz's condition (*) specifies restrictions on the set of vectors for any face, namely that they span a k dimensional subspace which is a direct summand of $\mathbb{Z}^n$. These conditions may be derived from the requirements that the isotropy subgroup of any interior point of the manifold is trivial, and that the orbits of points are products of circles (see Remark 2.2.26). The condition (*) may be regarded as a restriction on actions on a complex T^n corner, together with a requirement that any point in the quasi-toric manifold is in a neighbourhood which is equivariantly diffeomorphic to a complex corner subject to these restrictions. We have taken the view that for our treatment of the quaternionic case, the properties of a corner are better determined separately, since the analysis of a corner is more complicated.

Since Q^n is non-commutative, isotropy subgroups are replaced with isotropy classes as the appropriate object to associate with an orbit. The restrictions on acceptable exponent matrices that define regular corners are derived from essentially the same properties of orbits and isotropy classes (Propositions 2.2.19, 2.2.22). The relationship between the definition of an acceptable isotropy functor and condition (*) is not immediate, even though they ultimately derive from similar requirements. In both cases, the restrictions could expressed in the form that the isotropy class of a face of the orbit space is a proper subclass of any proper subface.

Also, any acceptable isotropy matrix Λ over a polytope P^n satisfies the condition that for any vertex v, the square submatrix corresponding to the facets containing v has determinant ± 1, so defines a quasi-toric manifold over P^n. That is, any quaternionic characteristic matrix Λ over a polytope P^n is also a quasi-toric Λ matrix over P^n.

Proposition 4.2.7 *Given a simple polytope P^n (with m facets) then*
(1) any $n \times m$ matrix Λ with columns labelled by the facets of P^n, is a characteristic isotropy matrix over P^n if and only if the restriction $\Lambda|_v$ to the columns of Λ defined by the facets that contain the vertex v of P^n is an acceptable isotropy matrix;
(2) any facet graph Σ over $[n]$ with m nodes labelled by the facets of P^n, is a characteristic facet graph over P^n if and only if the restriction $\Sigma|_v$ to the nodes corresponding to the facets that contain the vertex v of P^n is an acceptable facet graph.

Proof. If Λ is a characteristic matrix or Σ a characteristic graph, then there is a characteristic functor λ over P^n which defines them (Definition 4.2.3). Then the restriction of λ, to any vertex v of P^n is an acceptable isotropy functor by definition. So the restrictions $\Lambda|_v$ and $\Sigma|_v$ are an acceptable isotropy matrix and an acceptable facet graph respectively, by Propositions 3.1.14, 3.1.20.

Conversely, if the restrictions $\Lambda|_v$ or $\Sigma|_v$ to any vertex v of P^n are an acceptable isotropy matrix or an acceptable facet graph then there is an acceptable isotropy functor $\ell_v\colon \textsc{cat}([n]) \xrightarrow{r_v} \textsc{cat}(K_P|_v) \xrightarrow{\lambda_v} \textsc{conj}(\mathsf{Q}^n)$, such that $(\lambda_v)(\{a\}) = \mathsf{Q}_{\gamma_a}$ for any facet F_a which contains v, where γ_a is the node of Σ, or is the set defined by the column y_a of Λ. Thus, since $\lambda_v(\{a\})$ depends only on the face F_a and not on the vertex v, it is the restriction to v of a characteristic functor $\lambda\colon \textsc{cat}(K_P) \to \textsc{cong}(\mathsf{Q}^n)$ over P^n. ❑

Reconstruction. Any quoric manifold over a polytope is defined by its Q^n action, from which a characteristic functor can be derived (Proposition 4.2.4). It is useful to have available a construction that starts from a characteristic functor λ over a polytope and generates a quoric manifold. Recall the construction of a corner (Construction 3.2.1) $\mathcal{H}(\ell) = \mathsf{Q}^n \times \mathbb{R}^n_{\geqslant}/{\sim_\ell}$, from an acceptable isotropy functor ℓ, which is equivariantly homeomorphic to a regular $\mathbb{H}^n$. We now extend this construction to a manifold over a polytope.

For any point x in P^n, let $F_{\tau(x)}$ denote the smallest face containing x, and let $F_{\tau(x)}$ have codimension k. Let $\widehat{\lambda}(\tau(x)) < \mathsf{Q}^n$ denote the canonical subgroup $\mathsf{Q}(u^1, \ldots, u^k)$

in the class $\lambda(\tau(x)) = \mathsf{Q}_{\gamma_1,\ldots,\gamma_k}$, defined by $u^1, \ldots, u^k \in \mathbb{H}^n$ where $(u^i)_j = 1$ if $j \in \gamma_i$ or 0 otherwise.

Construction 4.2.8 Given a characteristic functor λ over a simple polytope P^n, define the *derived space*,

$$\mathcal{M}(\lambda) = \mathsf{Q}^n \times P^n/{\sim_\lambda},$$

where $(q, x) \sim_\lambda (q', x') \in \mathsf{Q}^n \times P^n$ if $x = x'$ and $q^{-1}q' \in \widehat{\lambda}(\tau(x))$.

The action of Q^n is by left multiplication, $(s, [q, x]) \mapsto [sq, x]$, for any $s \in \mathsf{Q}^n$ and $[q, x] \in \mathcal{M}(\lambda)$, which is well defined, since if $q^{-1}q' \in \widehat{\lambda}(\tau(x))$ then so $(sq)^{-1}(sq) \in \widehat{\lambda}(\tau(x))$.

Proposition 4.2.9 *The derived space $\mathcal{M}(\lambda)$ is a quoric manifold over P^n.*

Proof. We check that $\mathcal{M}(\lambda)$ is a locally regular smooth manifold, and that there is an orbit projection map.

(1) Locally Regular Manifold:
For any $s \in \mathsf{Q}^n$ and $x \in F_{\tau(x)}$, if $(q, x) \sim_\lambda (q', x)$ then $(sq, x) \sim_\lambda (sq', x)$, since $(sq)^{-1}(sq') = q^{-1}q' \in \widehat{\lambda}(\tau(x))$. That is, the Q^n action on $\mathcal{M}(\lambda)$ is well defined.

For any vertex v of P^n, let $P(v)$ denote the subspace of P which contains the interiors of all those faces which contain v, including v and P^n. Then there is a diffeomorphism $f_v \colon P(v) \to \mathbb{R}^n_\geqslant$, since $P(v)$ has a face poset isomorphic to that of $\mathbb{R}^n_\geqslant$. Let ℓ_v be the restriction of the characteristic functor λ to those faces which contain v. By definition, ℓ_v is an acceptable isotropy functor on some regular n-corner, so (using Proposition 3.2.3) $\mathsf{Q}^n \times \mathbb{R}^n_\geqslant/{\sim_{\ell_v}}$ is homeomorphic to a regular corner $\mathbb{H}^n$. Then $U_v = \mathsf{Q}^n \times P(v)/{\sim_{\ell_v}}$ is an open subset of $\mathcal{M}(\lambda)$, and is the pullback of the projection $\mathsf{Q}^n \times \mathbb{R}^n_\geqslant/{\sim_{\ell_v}} \to \mathbb{R}^n_\geqslant$ along f_v, and $\phi_v = (id \times f_v) \colon U_v \to \mathsf{Q}^n \times \mathbb{R}^n_\geqslant/{\sim_{\ell_v}}$ is an equivariant homeomorphism. Thus, (U_v, ϕ_v) provides a regular chart on $\mathcal{M}(\lambda)$.

$$\begin{array}{ccc} U_v = \mathsf{Q}^n \times P(v)/{\sim_{\ell_v}} & \xrightarrow{\phi_v = (id \times f_v)} & \mathsf{Q}^n \times \mathbb{R}^n_\geqslant/{\sim_{\ell_v}} \\ \big\downarrow {\scriptstyle /\mathsf{Q}^n} & & \big\downarrow {\scriptstyle /\mathsf{Q}^n} \\ P(v) & \xrightarrow{\quad f_v \quad} & \mathbb{R}^n_\geqslant \end{array}$$

Furthermore, any transition function $\phi_v \circ \phi_{v'}^{-1} \colon \phi_{v'}(U_v \cap U_{v'}) \to \phi_{v'}(U_v \cap U_{v'})$ is equal to $(id \times f_v) \circ (id \times f_{v'})^{-1}$, and is a diffeomorphism since f_v and $f_{v'}$ are diffeomorphisms. The set $P(v)$ over all vertices of P^n clearly covers P^n, so the set $\{(U(v), \phi_v) \mid v \text{ vertex of } P^n\}$ is a regular atlas, that is $\mathcal{M}(\lambda)$ is smooth and locally regular.

(2) Projection map:
Define the orbit projection map $\pi\colon \mathcal{M}(\lambda) \to P^n$, by $[q, x] \mapsto x$. For any point x of P^n, the fibre of π is $\mathsf{Q}^n \times \{x\}/\sim_\lambda$, which is clearly an orbit since Q^n acts transitively on Q^n. Thus, π is an orbit projection map. ❏

Proposition 4.2.10 *For any quoric manifold M^{4n} over P^n, there is an equivariant diffeomorphism between M^{4n} and $\mathcal{M}(\lambda)$, where λ is the characteristic functor of M^{4n}.*

Proof. Let the orbit projection map be $\pi\colon M^{4n} \to P^n$. For each vertex v of P let (U_v, ϕ_v) be a chart, so the map $\phi_v\colon U_v \to \mathbb{H}^n$ is an equivariant homeomorphism, and let $P(v) = \pi(U_v)$ be the image of U_v in P^n. Let ℓ_v be the restriction of the functor λ to the faces of P^n which contain the vertex v (including v and P^n). Then $\mathbb{H}^n$ can be identified with $\mathsf{Q}^n \times \mathbb{R}^n_{\geqslant}/\sim_{\ell_v}$ (Proposition 3.2.3). Define the map $f_v = \mathrm{pr}_2 \circ \phi_v \circ \pi^{-1}\colon P(v) \to \mathbb{R}^n_{\geqslant}$. Then f_v is well defined and injective, since it maps orbits to orbits since ϕ_v is equivariant, and is a homeomorphism, since ϕ_v is a homeomorphism and the orbit projection maps are open and continuous. The map $(id_{\mathsf{Q}^n} \times f_v)\colon \mathsf{Q}^n \times P(v)/\sim_{\ell_v} \to \mathsf{Q}^n \times \mathbb{R}^n_{\geqslant}/\sim_{\ell_v}$ is an equivariant homeomorphism, and defines a chart on $\mathsf{Q}^n \times P(v)/\sim_{\ell_v}$. Now define the map $\psi_v = (id_{\mathsf{Q}^n} \times f_v)^{-1} \circ \phi_v\colon\ U_v \to \mathsf{Q}^n \times P(v)/\sim_{\ell_v}$. Then ψ_v is equivariant, since ϕ_v is equivariant, and is a homeomorphism, since id_{Q^n}, f_v and ϕ_v are homeomorphisms. Furthermore, the charts (U_v, ϕ_v) and $(\mathsf{Q}^n \times P(v), id_{\mathsf{Q}^n} \times f_v)$ map corresponding points to the same point in $\mathbb{H}^n$ by construction, so ψ_v is smooth.

$$
\begin{array}{ccccc}
U_v(\subset M) & \xrightarrow{\ \phi_v\ } & \mathbb{H}^n \cong \mathsf{Q}^n \times \mathbb{R}^n_{\geqslant}/\sim_{\ell_v} & \xleftarrow{\ (id\times f_v)\ } & \mathsf{Q}^n \times P(v)/\sim_{\ell_v} \\
\Big\downarrow \pi & & \Big\downarrow /\mathsf{Q}^n & & \Big\downarrow /\mathsf{Q}^n \\
P(v) = \pi(U_v) & \xrightarrow{\ f_v\ } & \mathbb{R}^n_{\geqslant} & \xleftarrow{\ f_v\ } & P(v)
\end{array}
$$

❏

4.2.11 Betti Numbers. We are now in a position to determine the Betti numbers of a quoric manifold M^{4n}, although we need more machinery to determine the ring structure of $H^*(M^{4n})$ (Chapter 6 below). The result directly parallels that for quasi-toric manifolds. Following Davis and Januszkiewicz (DJ [8], see also BP [4] Construction 5.15), we define a perfect cellular structure on M^{4n}, that is, a cellular structure for which the boundary operators of the cellular chain complex are all zero, so that there is a generator in the homology of M^{4n} for each cell. Suppose we are given a quoric manifold over a simple polytope, $\pi\colon M^{4n} \to P^n$, and the 1-skeleton of P^n is made into a directed graph relative to some line L in $\mathbb{R}^n$, as in Paragraph 4.1.4. Then, for each vertex v of P^n there are $ind(v)$ incident edges pointing towards v. These inward pointing edges span a face F_v of dimension $ind(v)$. Let $\widehat{F}_v$ denote the subspace of F_v obtained by deleting all its faces which do not contain v. Then the set of $\widehat{F}_v$ over all the vertices are disjoint, and their union is equal to P^n. Each $\widehat{F}_v$ is homeomorphic to an open disc $D^{ind(v)} \subset \mathbb{R}^{ind(v)}_{\geqslant}$. Define the inverse image $\widehat{U}_v := \pi^{-1}\widehat{F}_v$. The cell $\widehat{U}_v$ is homeomorphic to an open disc of dimension $4\,ind(v)$, and its closure U_v is the submanifold $\pi^{-1}F_v \subset M^{4n}$. The boundary of each cell U_v is attached to the union of some cells of lower dimension, namely the inverse images of the faces that were deleted from F_v to obtain $\widehat{F}_v$. Then the set $\{U_v \mid v \text{ vertex of } P^n\}$ over all vertices of P^n defines a cell structure for M^{4n}, for which the $\widehat{U}_v$ are disjoint and their union is equal to M^{4n}. Thus, we have a perfect cell structure, and the Betti numbers of the manifold M^{4n} can be obtained directly.

Theorem 4.2.12 *Given a quoric manifold M^{4n} over a simple polytope P^n, the Betti numbers of M^{4n} are zero except in dimensions which are a multiple of 4, where they are given by*

$$b_{4i}(M^{4n}) = h_i(P^n) \qquad (i = 1, \ldots, n).$$

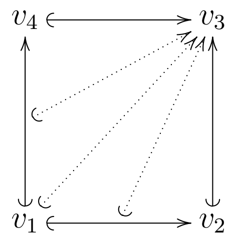
where $h = (h_1, \ldots, h_n)$ is the h-vector of P^n.

Proof. In the perfect cellular structure constructed above the number of cells of dimension $4i$ is $I(i) = h_i$, and there are none in other dimensions. ❑

Remark 4.2.13 We note in particular that this result depends solely on the polytope, and so gives the values of the Betti numbers of any quoric manifold over P^n.

Example 4.2.14 Consider a quoric manifold over a square, $\pi\colon M^8 \to I^2$. Let the vertices of I^2 be v_1, v_2, v_3, v_4, with edges (v_1, v_2), (v_1, v_4), (v_2, v_3), (v_4, v_3), and interior (v_1, v_2, v_3, v_4). With a line L in the direction from v_1 to v_3, the edges are oriented as listed. The subspace $\widehat{F}_{v_1}$ is just v_1. The subspace $\widehat{F}_{v_2}$ contains v_2 and the interior of the edge (v_1, v_2), and the subspace $\widehat{F}_{v_4}$ contains v_4 and the interior of the edge (v_1, v_4). The subspace $\widehat{F}_{v_3}$ contains v_3 and the interiors of the edges (v_2, v_3) and (v_4, v_3) and the interior (v_1, v_2, v_3, v_4).

I^2 :



Then $\widehat{U}_{v_1} = \pi^{-1}\widehat{F}_{v_1}$ is a fixed point of M^8. The subspaces $\widehat{U}_{v_2} = \pi^{-1}\widehat{F}_{v_2}$ and $\widehat{U}_{v_4} = \pi^{-1}\widehat{F}_{v_4}$ are homeomorphic images of an open 4-disc, with their boundaries attached to the point U_{v_1}. The subspace $\widehat{U}_{v_3} = \pi^{-1}\widehat{F}_{v_3}$ is a homeomorphic image of an open 8-disc, with boundary attached to the union of the two 4-discs and the point U_{v_1}. The subspaces $\widehat{U}_{v_i}$ are clearly disjoint, and their union is M^8. Their closures U_{v_i} define a cell structure on M^8, with cells of dimension 0, 4, 4 and 8. Hence the non-zero Betti numbers of M^8 are $b_0 = 1$, $b_4 = 2$, $b_8 = 1$.

4.2.15 Complex and Real Submanifolds. It is always possible to take a subspace of the space of quaternions $\mathbb{H}$ which is a copy of the complex numbers $\mathbb{C}$. Restricting to the unit quaternions gives a copy of the circle group T as a subspace of the group of unit quaternions Q. This observation leads directly to the idea of taking a subspace of any quoric manifold to obtain a quasi-toric manifold, which will have a characteristic function with the same characteristic matrix Λ.

First note that any quoric characteristic matrix Λ over a polytope P^n satisfies the conditions for a quasi-toric Λ matrix, since any non-zero column vector of 0's and

1's is a non-zero element of $\mathbb{Z}^n$, and the restriction to the submatrix corresponding to any vertex has determinant ± 1. Suppose we fix some $\omega \in \mathbb{H}$ such that $\omega^2 = -1$. Then the set $\mathbb{C}_\omega = \{(a + b\omega) \in \mathbb{H} \mid a, b \in \mathbb{R}\}$ is a copy of $\mathbb{C}$ in $\mathbb{H}$, and the set $\mathsf{T}_\omega = \{(a + b\omega) \in \mathbb{H} \mid a, b \in \mathbb{R},\ a^2 + b^2 = 1\}$ is a copy of T in Q. Clearly, the action of Q^n on $\mathbb{H}^n$ restricts to an action of T_ω^n on $\mathbb{C}_\omega^n$. A subgroup Q_{γ_i} defined by column y_i of Λ restricts to a subgroup $\mathsf{T}_{\gamma_i} = \{(t^{(y_i)_1}, \ldots, t^{(y_i)_n}) \in \mathsf{T}^n \mid t \in \mathsf{T}\}$ corresponding to the same column. Thus, the definition of the derived manifold (Construction 4.2.8) $\mathcal{M}(\lambda) = \mathsf{Q}^n \times P^n/\!\sim_\lambda$ over (P^n, λ), restricts to the quasi-toric submanifold with the restriction of the subgroups at the faces of P^n, with the same characteristic matrix Λ.

We cannot expect to derive a quoric characteristic matrix from a general quasi-toric Λ matrix. The elements of an arbitrary quasi-toric Λ matrix are integers, while elements of a quoric characteristic matrix are only 1 or 0. Even taking the elements modulo 2 of a quasi-toric Λ matrix does not in general give a quoric characteristic matrix, since the restriction of Λ to a submatrix corresponding to a vertex must satisfy $\det \Lambda|_v = \pm 1$, not just $\det \Lambda|_v = \pm 1 \pmod 2$. Furthermore, the columns of $\det \Lambda|_v \pmod 2$ need not be compatible.

The space of quaternions $\mathbb{H}$ contains a copy of the reals $\mathbb{R}$, and the group of unit quaternions Q contains the real subgroup $S^0 = \{1, -1\} \cong \mathbb{Z}_2$. Any action of Q^n on $\mathbb{H}^n$ restricts to an action of $\mathbb{Z}_2^n$ on $\mathbb{R}^n$, and a subgroup Q_{γ_i} defined by column y_i of Λ restricts to a subgroup $(\mathbb{Z}_2)_{\gamma_i} = \{(t^{(y_i)_1}, \ldots, t^{(y_i)_n}) \in \mathbb{Z}_2^n \mid t \in \mathbb{Z}_2\}$ corresponding to the same column y_i. For any characteristic functor λ over a simple polytope P^n, restricting the quoric manifold $\mathcal{M}(\lambda) = \mathsf{Q}^n \times P^n/\!\sim_\lambda$ to the real subspace gives a real manifold with $\mathbb{Z}_2^n$ action, which Davis and Januszkiewicz introduced as a *small cover* of P^n. Clearly, restricting $\mathbb{H}^n$ via $\mathbb{C}_\omega^n$ gives the same real subspace.

The space of quaternions $\mathbb{H}$ also contains a copy of the non-negative reals $\mathbb{R}_\geqslant$, as a subspace of $\mathbb{R}$, with the group Q intersecting $\mathbb{R}_\geqslant$ in the single element 1. Hence the quoric manifold $\mathcal{M}(\lambda)$ restricts to a copy of P^n.

4.3 Classification

Two quoric manifolds over a polytope are considered as essentially the same if there is an equivariant diffeomorphism between them. We now consider the question classifying quoric manifolds over a polytope up to equivariant diffeomorphism. We show that the classification of quasi-toric manifolds by the pairs (P^n, λ) (DJ [8], BP [4], [5]) has a direct analogue for quoric manifolds.

An equivariant diffeomorphism of two quoric manifolds $\pi_i\colon M_i^{4n} \to P_i^n$ over P_i^n (for $i = 1, 2$) is an automorphism θ of Q^n and a diffeomorphism $f\colon M_1^{4n} \to M_2^{4n}$ such that $f(s\,x) = \theta(s)\,f(x)$ for all $s \in \mathsf{Q}^n$ and $x \in M_1^{4n}$. Two polytopes, regarded as manifolds with corners, are diffeomorphic if and only if they have the same face structure, that is they have isomorphic face posets. So, if the diffeomorphism is $p\colon P_1^n \to P_2^n$ then the induced map $p^*\colon \textsc{cat}(K_{P_1}) \to \textsc{cat}(K_{P_2})$ is an isomorphism. An automorphism θ of Q^n induces an automorphism $\theta^*\colon \textsc{conj}(\mathsf{Q}^n) \to \textsc{conj}(\mathsf{Q}^n)$.

Definition 4.3.1 Let P_1^n and P_2^n be simple polytopes and let $\lambda_i\colon \textsc{cat}(K_{P_i}) \to \textsc{conj}(\mathsf{Q}^n)$ $(i = 1, 2)$ be characteristic functors over them. An *isomorphism* between pairs (P_1^n, λ_1) and (P_2^n, λ_2) is a pair of maps (p, θ), where p is a diffeomorphism $p\colon P_1^n \to P_2^n$ and θ is an automorphism of Q^n, such that $\theta^* \circ \lambda_1 = \lambda_2 \circ p^*$.

Proposition 4.3.2 *Two quoric manifolds $\pi_i\colon M_i^{4n} \to P_i^n$ with characteristic functors λ_i (for $i = 1, 2$) are equivariantly diffeomorphic if and only if there is an isomorphism of the pairs (P_i^n, λ_i).*

Proof. By Proposition 4.2.10 it is sufficient to consider the derived quoric manifolds $\mathcal{M}_1(\lambda_1)$ and $\mathcal{M}_2(\lambda_2)$.

Given an equivariant diffeomorphism $f\colon \mathcal{M}_1(\lambda_1) \to \mathcal{M}_2(\lambda_2)$, define $p = \pi_2 \circ f \circ \pi_1^{-1}\colon P_1^n \to P_2^n$. Since any chart $\phi_v\colon U_v \to \mathbb{H}^n$ can be written as $\mathsf{Q}^n \times P(v)/{\sim_{\lambda_1}} \to \mathsf{Q}^n \times \mathbb{R}^n_{\geqslant}/{\sim_{\lambda_1}}$, the restriction $\phi_v|_{P(v)}\colon P(v) \to \mathbb{R}^n_{\geqslant}$ provides a chart for P^n as a manifold with corners, and similarly for any chart $\psi_{pv}\colon \mathsf{Q}^n \times P(pv)/{\sim_{\lambda_2}} \to \mathsf{Q}^n \times \mathbb{R}^n_{\geqslant}/{\sim_{\lambda_2}}$. Then, any composition $\phi_{pv} \circ f \circ \psi_v^{-1}$ that shows f is smooth also shows that $\phi_{pv}|_{P(v)} \circ p \circ \psi_v|_{P(fv)}^{-1}$ is smooth. Thus f descends to a diffeomorphism $p\colon P_1^n \to P_2^n$.

Since f is equivariant, there are induced isomorphisms $p^*\colon \text{CAT}(K_{P_1}) \to \text{CAT}(K_{P_2})$ and $\theta^*\colon \text{CONJ}(\mathsf{Q}^n) \to \text{CONJ}(\mathsf{Q}^n)$, such that $\theta^* \circ \lambda_1 = \lambda_2 \circ p^*$. That is, there is an isomorphism of pairs $(P_1^n, \lambda_1) \to (P_2^n, \theta_2)$.

Conversely, suppose there is an isomorphism of pairs, $(P_1^n, \lambda_1) \to (P_2^n, \theta_2)$. Then there is a diffeomorphism $p\colon P_1 \to P_2$, and an automorphism θ of Q^n such that $\theta^* \circ \lambda_1 = \lambda_2 \circ p^*$. Define $f = (\theta \times p)\colon \mathcal{M}_1(\lambda_1) \to \mathcal{M}_2(\lambda_2)$. which maps $[t; x] \mapsto [\theta(t); p(x)]$. Taking the orbit projection recovers $p \circ \pi_1 = \pi_2 \circ f$, and the isotropy class $\lambda_1(\tau)$ of face F_τ of $\mathcal{M}_1(\lambda_1)$ maps to the isotropy class $\theta^*\lambda_1(\tau) = \lambda_2(p^*\tau)$ of face $F_{p^*\tau}$ of $\mathcal{M}_2(\lambda_2)$. Moreover, f is clearly equivariant, since $f(s[t; x]) = f([st; x]) = (\theta(st), p(x)) = \theta(s)(\theta(t), p(x)) = \theta(s)f([t; x])$, and is smooth since p and θ are. ❑

As an immediate corollary, we have:

Theorem 4.3.3 *There is a bijection between quoric manifolds M^{4n} up to equivariant diffeomorphism, and isomorphism classes of characteristic pairs (P^n, λ).* ❑

Examples 4.3.4 Some examples of polytopes in 2 dimensions and their incongruent characteristic functors.

1. **2-simplex $P^2 = \Delta^2$.** The only rank 1 isotropy classes Q_γ have γ equal to $\{1\}$, $\{2\}$ or $\{1, 2\}$, which are compatible, and must be different on edges (facets) which intersect. So λ must assign each of these to one of the three edges, and assign the rank 2 class $\mathsf{Q}_{\{1\},\{2\}}$ to each vertex, and this assignment defines a characteristic functor. Thus, up to rotations and reflections of the simplex and permutations of the coordinate subgroups of Q^2, there is only one characteristic functor on the 2-simplex up to congruence.

   The manifold derived from this characteristic functor over Δ^2 is $\mathbb{H}P^2$, and can be written as $\mathsf{Q}^2 \times \Delta^2/\sim_\lambda$. It is conventionally defined as the quotient space $\mathbb{H}^3_\times/\mathbb{H}_\times$, where for any space $X_\times = X\backslash\{0\}$. After taking the quotient by $\mathbb{R}_>$ of both terms, $\mathbb{H}P^2$ can be written as the set of equivalence classes $[h_1, h_2, h_3] = [h_1 u, h_2 u, h_3 u]$ for $u \in \mathsf{Q}$, $h = (h_1, h_2, h_3) \in \mathbb{H}^3$ with $\sum |h_i|^2 = 1$. The action is given by $((s_1, s_2), [h_1, h_2, h_3]) \mapsto [s_1 h_1, s_2 h_2, h_3]$, and the isotropy subgroups are easily checked.

2. Square $P^2 = I^2$**.** Let the facets F_a be F_1, F_2, F_3, F_4 in sequence around the square. The rank 1 isotropy classes are as in Ex 1. So λ must either assign two of these each to opposite edges, or one to a pair of opposite edges and the other two to the remaining edges. Since an automorphism of Q^2 can interchange the coordinate subgroups (hence $\mathsf{Q}_{\{1\}} \leftrightarrow \mathsf{Q}_{\{2\}}$) while no automorphism can change the degree of the isotropy class ($\mathsf{Q}_{\{1\}}$ is essentially different from $\mathsf{Q}_{\{1,2\}}$), there are in total 4 congruence classes of characteristic functors over I^2.

- $\lambda_{(1)}(\{a\}) = (\mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{2\}})$,
- $\lambda_{(2)}(\{a\}) = (\mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{1,2\}}, \mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{1,2\}})$,
- $\lambda_{(3)}(\{a\}) = (\mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{1,2\}})$,
- $\lambda_{(4)}(\{a\}) = (\mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{1,2\}}, \mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{1,2\}})$.

The manifolds from the first two are each a direct product $(\mathbb{H}P^1)^2 = (\mathbb{H}^2_\times/\mathbb{H}_\times)^2$, but with incongruent actions. Define variables to correspond to the facets, $h = (h_1, h_2, h_3, h_4) \in \mathbb{H}^4$, and with $w_3^* = |h_1|^2 + |h_3|^2 = 1,\ w_4^* = |h_2|^2 + |h_4|^2 = 1$, the points of $M^8_{(1)}$ and $M^8_{(2)}$ are the equivalence classes $[h_1, h_3][h_2, h_4] = [h_1 u, h_3 u][h_2 v, h_4 v],\ u, v \in \mathsf{Q}$. The actions are given by:

- $M^8_{(1)}$: $((s_1, s_2), [h_1, h_3][h_2, h_4]) \mapsto [s_1 h_1, h_3][s_2 h_2, h_4]$
- $M^8_{(2)}$: $((s_1, s_2), [h_1, h_3][h_2, h_4]) \mapsto [s_1 h_1, s_2 h_3][s_2 h_2, h_4]$

for $s = (s_1, s_2) \in \mathsf{Q}^2$. The other two manifolds are not direct products but can be described in a similar way, as we will see later (Example 5.3(2)).

3. Pentagon $P^2 = P^2_{(5)}$**.** The rank 1 isotropy classes are as in Example 1. So λ must assign one of these to a single edge, and the other two each to two edges. Thus, there a total of 2 congruence classes of characteristic functors over P_5, distinguished by whether a degree-1 or degree-2 class occurs once.

- $\lambda_{(1)}(\{a\}) = (\mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{1,2\}})$,
- $\lambda_{(2)}(\{a\}) = (\mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{1,2\}}, \mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{1,2\}}, \mathsf{Q}_{\{2\}})$.

4. Polygons $P^2 = P^2_{(m)}$. Extending the sequence, for polygons with $m = 6, 7, 8$, there are $8, 6, 17$ congruence classes of characteristic functors respectively.

Each of these examples produces an 8-dimensional manifold with a Q^2 action.

Chapter 5

Global Quoric Manifolds

We now place an additional restriction on the characteristic functors over a simple polytope P^n, and obtain the class of *global* quoric manifolds. The properties of this restricted class of manifolds include the existence of a moment angle complex from which the quoric manifold can be obtained as a quotient space, and the calculation of the cohomology of the manifold using techniques derived from those used in the toric case.

The definition of a quoric manifold (Definition 4.2.2) only required the characteristic functor λ to have compatible isotropy classes for the facets around each vertex of the orbit space P^n. We now add the new requirement of compatibility for the isotropy classes of all facets of P^n. The most important effect of this extra condition is that just as a polytope P^n with m facets can be embedded as an affine subspace of $\mathbb{R}^m_\geqslant$ (Equation 4.1.3), a characteristic matrix Λ over P^n can be embedded as a submatrix of an $m \times m$ acceptable isotropy matrix L for some regular m-corner. A moment angle complex over (P^n, λ) can then be defined as a subspace of $\mathbb{H}^m$, with a Q^m action specified by the exponent matrix $E = L^{-1}$, Chapter 5.2 below. The quotient of the moment angle complex by a subgroup Q^{m-n} of Q^m is then equivariantly homeomorphic to the manifold M^{4n} over (P, λ), by analogy with the toric case.

Subsection 5.3 gives some examples, and Subsection 5.4 contains a discussion of the number of polytopes which can support a characteristic functor, or a global characteristic functor. We find there are an infinite number in each dimension, which provides an adequate set of examples!

5.1 Global Characteristic Functors

We introduce some definitions and new objects which allow us to work with these new characteristic functors and manifolds.

Definition 5.1.1 A *global characteristic functor* is a characteristic functor λ over a polytope P^n such that for all facets F_a, F_b of P^n the isotropy classes $\lambda(\{a\})$ and $\lambda(\{b\})$ are compatible.

A characteristic matrix Λ is called *global* if all its columns are compatible, and a characteristic facet graph Σ is called *global* if all its nodes are compatible.

Proposition 5.1.2 *Let λ be a characteristic functor over a simple polytope P^n, and let Λ be the characteristic matrix and Σ the facet graph defined by λ. If any one of λ, Λ, or Σ is global, then so are the others.*

Proof. For any two facets F_a, F_b, the classes $\lambda(\{a\}) = \mathsf{Q}_{\gamma_a}$ and $\lambda(\{b\}) = \mathsf{Q}_{\gamma_b}$ are compatible if and only if the sets γ_a and γ_b are compatible (Definition 2.1.14). The columns y_a and y_b of Λ are compatible if and only if the the sets γ_a and γ_b are compatible (Definition 2.2.16). ❑

A facet graph Σ is said to be *embedded* in a facet graph Σ' if there is an injective map s from the nodes of Σ to the nodes of Σ' such that each node $\gamma \in \Sigma$ is a subset of its image $s(\gamma) \in \Sigma'$. A characteristic matrix Λ is said to be *embedded* in a characteristic matrix Λ' if Λ is a submatrix of Λ'. A characteristic functor λ is said to be *embedded* in a characteristic functor λ' if there is an embedding of the corresponding characteristic graphs. Here we are only interested in embedding a characteristic facet graph of subsets of $[n]$ with m nodes in an acceptable facet graph of subsets of $[m]$, and embedding an $n \times m$ characteristic matrix as a submatrix of some $m \times m$ acceptable isotropy matrix.

Theorem 5.1.3 *Any global characteristic graph Σ over a simple polytope P^n with m facets can be embedded in an acceptable facet graph Γ_m corresponding to some regular m-corner.*

Corollary 5.1.4 *Any global characteristic matrix Λ over a simple polytope P^n with m facets can be embedded as an $n \times m$ submatrix of an $m \times m$ acceptable isotropy matrix L corresponding to some regular m-corner.*

Any global characteristic functor λ over a simple polytope P^n with m facets can be embedded in an acceptable isotropy functor ℓ corresponding to some regular m-corner.

Proof of 5.1.3.

(1) Suppose the nodes $\gamma_a \subset [n]$, for $a = 1, \ldots, m$, of a facet graph Σ over a simple polytope P^n are compatible. The graph Σ may contain some nodes which are equal, some nodes with the same distinguished element and some nodes with no distinguished element, since there are only n values for the distinguished element and m nodes. No node in Σ can have more than one distinguished element since this is true for the nodes restricted to any subgraph corresponding to a vertex, and adding nodes cannot increase the number of distinguished elements of any node. The proof starts with the facet graph Σ, and enlarges some nodes by including additional elements from $n+1, \ldots, m$ $(=[m]\backslash[n])$. The operation used is that of *augmenting* a node γ to $\gamma' = \gamma \cup \{k\}$, by including the next unused element k from $[m]\backslash[n]$.

(2) The proof is by induction on the degree of the nodes. Consider each node γ_a of degree t $(= |\gamma_a|)$. We know (inductive hypothesis) that (a) each node γ_c with $|\gamma_c| < t$ has a single distinguished element k_c, which is different from that of any other node, and (b) all the γ are compatible.

(3) For any γ_a (of degree t) with no distinguished element, augment γ_a and augment with the same element every γ_c which contains γ_a. Then γ_a (and any other that was equal to it) now has degree greater than t, each now has a single distinguished element and is still compatible with every γ. Any γ_c which properly contained γ_a still has the same distinguished element if any, and is still compatible with every γ.

For any identical nodes $\gamma_a = \gamma_c = ..$ (of degree t) with a distinguished element, select one, say γ_a, and augment with the same element all nodes other than γ_a which contain γ_a, including γ_c. Then γ_a has a distinguished element which is different from that of any node. Any γ_c which was equal to γ_a now has degree greater that t, and has the new element as its distinguished element, since γ_a is a proper subset, and is still compatible with all γ. For any γ_d that properly contained γ_a, γ'_d properly contains γ'_c so will have the same distinguished element as before if any, and will still be compatible with all γ

(4) Hence (a) each γ_a such that $|\gamma_a| \leqslant t$ has a single distinguished element k_a, which is different from that of any other node, and (b) all γ are compatible.

(5) The base of the induction is $t = 0$, for which (a) there are no empty nodes $|\gamma| = 0$, and (b) all γ are compatible by hypothesis. Each node can have no more than one distinguished element, and there are the same number m of elements and nodes so there is always an element available for augmentation, and all are used. Hence, by induction, every node γ has a different distinguished element, and all γ are compatible. That is, the graph of all the augmented nodes, Γ_m, is an acceptable facet graph. ❑

Proof of 5.1.4. The embedding of the characteristic facet graph Σ over P^n in the acceptable facet graph Γ_m can be translated directly to an embedding of the characteristic matrix Λ in the acceptable isotropy matrix L_m. For any node γ_a ($\subset [n]$) in Σ, there is a corresponding node γ'_a ($\subset [m]$) in Γ_m such that $\gamma_a \subset \gamma'_a$. Define a column vector y'_a of length m by $(y'_a)_b = 1$ if $b \in \gamma'_a$, and zero otherwise. Then the first n elements of y'_a are exactly the elements of the column y_a of Λ. Thus, defining the matrix L_m by the columns y'_a, we have Λ as the submatrix consisting of the top n rows of L_m, and L_m is an acceptable isotropy matrix since Γ_m is an acceptable facet graph.

Since an embedding of a characteristic functor is defined in terms of the embedding of its characteristic matrix, any global characteristic functor can be embedded in an acceptable isotropy functor. ❑

In general, there are choices to be made in the inductive procedure to construct the facet graph Γ_m from Σ, so there may be different Γ_m, which are not even congruent, in which Σ may be embedded.

Examples 5.1.5 Consider a cube, $P^3 = I^3$. There are many characteristic functors over a cube, not all of which are global. Label the facets $F_1, \ldots, F_6$, with F_i and F_{i+3} opposite each other for $i = 1, 2, 3$.

1. Non-Global: Consider the graph Σ defined by its nodes,

$$\gamma_1 = \{1\}, \quad \gamma_2 = \{2\}, \quad \gamma_3 = \{1,3\},$$
$$\gamma_4 = \{1\}, \quad \gamma_5 = \{2\}, \quad \gamma_6 = \{2,3\}.$$

Then Σ restricted to any vertex v of I^3 corresponds to an acceptable facet graph with nodes either equal to $\{1\}, \{2\}, \{1,3\}$ or equal to $\{1\}, \{2\}, \{2,3\}$. Thus, Σ is a characteristic facet graph, but is not global since the sets $\{1,2\}$ and $\{2,3\}$ are not compatible.

2. Global: Change one node in the previous example, so that $\gamma_6 = \{1,3\}$, then all nodes of Σ are compatible, and Σ is a global characteristic facet graph. We can then augment its nodes to

$$\gamma_1' = \{1\}, \qquad \gamma_2' = \{2\}, \qquad \gamma_3' = \{1,3,4\},$$
$$\gamma_4' = \{1,4\}, \quad \gamma_5' = \{2,5\}, \quad \gamma_6' = \{1,3,4,6\},$$

which is easily checked to be an acceptable facet graph.

Chapter 5.4 below discusses further the existence of polytopes with characteristic functors, including those which are global.

5.2 Moment Angle Complexes

The conventional definition of the toric moment angle complex over a simple polytope P^n begins with a copy of the unit complex numbers $\mathsf{T} \subset \mathbb{C}$ for each facet F_a of P^n (for $a = 1, \ldots, m$). For any face F_τ, let $\mathsf{T}^\tau = \prod_{a \in \tau} \mathsf{T}_a \subset \mathsf{T}^m$, where T_a is a coordinate subgroup of T^m, for each a. Let $F_{\tau(x)}$ denote the smallest face of P^n which contains x, and define the moment angle complex by

$$\mathcal{Z}_P = \mathsf{T}^m \times P^n / \sim_\epsilon \qquad (t, x) \sim_\epsilon (t', x') \iff x = x',\ t^{-1}t' \in \mathsf{T}^{\tau(x)}.$$

The moment angle complex $\mathcal{Z}_P$ is naturally a T^m-space, by defining the coordinatewise action: $((s),[t,x]) \mapsto [st,x]$, for any $s,t \in \mathsf{T}^m$, $x \in P^n$. Alternatively, we can express $\mathcal{Z}_P$ as a pull back of the quotient map $\mathbb{C}^m \to \mathbb{R}^m_{\geqslant}$, $(z_1,\dots,z_m) \mapsto (|z_1|^2,\dots,|z_m|^2)$ along the embedding of P^n in $\mathbb{R}^n_{\geqslant}$, $A_P\colon P^n \to \mathbb{R}^m_{\geqslant}$ (Equation 4.1.3). Then $\mathcal{Z}_P$ is a subspace of the standard complex m-corner $\mathbb{C}^m$, with an inherited T^m action. For a quasi-toric manifold M^{2n} over P^n, there always exists an equivariant map $\rho\colon \mathcal{Z}_P \to M^{2n}$. Furthermore, because of the commutativity of T and the requirement that the isotropy subgroups of M^{2n} are characterised by vectors which span a subspace which is a direct summand in $\mathbb{Z}^m$, the group T^m can be written as a direct product of the kernel of ρ and the quotient $\mathsf{T}^m/\ker\rho$, isomorphic to T^{m-n} and T^n respectively. In the quaternionic case things a more complicated, not only because Q is not commutative, but also because there is more than one congruence class of regular m corners.

A quaternionic moment angle complex $\mathcal{Z}$ can be defined for any polytope P^n with m facets, and any regular corner $\mathbb{H}^m$ with isotropy functor ℓ, as a subspace of $\mathbb{H}^m$. Different m corners (up to congruence) give rise to different to different moment angle complexes. If there is an equivariant projection $\rho\colon \mathcal{Z} \to M^{4n}$, then the kernel of the projection $\ker\rho$ is a normal subgroup of Q^m, so is a product of coordinate subgroups. But not all quoric manifolds can occur as a quotient of some moment angle complex: it turns out that M^{4n} is equivariantly diffeomorphic to a quotient of a moment angle complex only when the characteristic functor of M^{4n} satisfies the global condition (see Proposition 5.2.7).

Definition 5.2.1 Given a simple polytope P^n with m facets, and any regular m-corner with isotropy functor ℓ, then define the *moment angle complex* by

$$\mathcal{Z}_{P,\ell} = \mathsf{Q}^m \times P^n/{\sim_\ell}, \qquad (q,x) \sim_\ell (q',x') \iff x = x',\ q^{-1}q' \in \widehat{\ell}(\tau(x)),$$

where $\widehat{\ell}(\tau(x))$ is the canonical subgroup in the class $\ell(\tau(x))$ for $x = x'$ in the relative interior of the face $F_{\tau(x)}$.

Define a Q^m action on $\mathcal{Z}_{P,\ell}$ by $(s,[q;x]) \mapsto [sq;x]$, for any $s \in \mathsf{Q}^m$ and $[q;x] \in \mathcal{Z}_{P,\ell}$.

We check that $\mathcal{Z}_{P,\ell}$ is a Q^m-manifold, adapting the argument from the toric case (BP [4] Ch 6.2).

Proposition 5.2.2 *A moment angle complex $\mathcal{Z}_{P,\ell}$ over a polytope P^n is a manifold, with a Q^m action.*

Proof. (1) We claim that Q^n embeds in D^{3n+1} hence in S^{3n+1} or in $\mathbb{R}^{3n+1}$, where $D^k \subset \mathbb{R}^k$ is a k dimensional unit disc, and we write $\mathring{D}^k$ to denote its interior. The result is obvious for $n = 1$, since $\mathsf{Q} = S^3 \hookrightarrow D^4$. Then inductively, suppose that $\mathsf{Q}^k \hookrightarrow D^{3k+1}$, so

$$\mathsf{Q}^{k+1} = \mathsf{Q}^k \times \mathsf{Q} \ \hookrightarrow D^{3k+1} \times S^3 \ \cong D^{3k+1} \times \partial D^4 \ \hookrightarrow \partial D^{3k+5} \ \cong S^{3k+4}.$$

The embedding is clearly not surjective, so there is a disc $D^{3k+4} \subset S^{3k+4}$ such that $\mathsf{Q}^{k+1} \hookrightarrow D^{3(k+1)+1}$. That is, Q^n embeds in D^{3n+1}.

(2) For any vertex $v \in P^n$, denote by F_v the subset of P^n obtained by deleting all those faces that do not contain v. Then clearly there is a homeomorphism $\phi_v \colon F_v \to \mathbb{R}^n_{\geqslant}$, so P^n can be regarded as a manifold with corners, with atlas

$$\{(\phi_v, F_v) \mid v \text{ vertex of } P^n\}.$$

Suppose $\pi \colon \mathcal{Z}_{P,\ell} \to P^n$ is the orbit map, then $\pi^{-1}(F_v) \cong \mathsf{Q}^{m-n} \times \mathring{D}^{4n}$. But Q^{m-n} can be embedded in $D^{3(m-n)+1}$ and is orientable, so a small neighbourhood H of $\mathsf{Q}^{m-n} \in D^{3(m-n)+1}$ is homeomorphic to $\mathsf{Q}^{m-n} \times \mathring{D}^1$. Taking the product with $\mathring{D}^{4n-1}$, we obtain an open set in D^{3m+n} homeomorphic to $\mathsf{Q}^{m-n} \times \mathring{D}^{4n}$. Thus, we have an atlas, so $\mathcal{Z}_{P,\ell}$ is a manifold.

(3) The action is well defined, since for any $[q; x] = [q'; x]$ we have $q^{-1}q' \in \widehat{\ell}(\tau(x))$, so $(sq)^{-1}(sq') = q^{-1}q' \in \widehat{\ell}(\tau(x))$, so $[sq; x] = [sq'; x]$. ❑

As in the toric case, the moment angle complex can alternatively be defined as the pullback of the quotient map $\mathbb{H}^m \to \mathbb{R}^m_{\geqslant}$ along the embedding of P^n in $\mathbb{R}^m_{\geqslant}$

(5.2.3)
$$\begin{array}{ccc} \mathcal{Z}_{P,\ell} & \longrightarrow & \mathbb{H}^m_\ell \\ \downarrow & & \downarrow /\mathsf{Q}^m \\ P^n & \xrightarrow{A_P} & \mathbb{R}^m_{\geqslant}, \end{array}$$

where we use the regular corner $\mathbb{H}^m_\ell$ with the action of Q^m specified the exponent matrix corresponding to ℓ in the construction of $\mathcal{Z}_{P,\ell}$.

Proposition 5.2.4 *For any simple polytope P^n with m facets, and an acceptable isotropy functor ℓ*

(1) the moment angle complex $\mathcal{Z}_{P,\ell}$ is equivariantly homeomorphic to the pullback (5.2.3),

(2) the moment angle complex $\mathcal{Z}_{P,\ell}$ is homeomorphic to $\mathcal{Z}_{P,\ell(\mathbf{1})}$.

where $\ell(\mathbf{1})$ is the isotropy functor of the standard m corner.

Proof. (1) the map $A_P\colon P^n \to \mathbb{R}^m_\geqslant$ is injective, and its image is the affine subspace defined by $w^*_{\bar{a}}(y) = w^*_{\bar{a}}(c)$, so we can identify P^n with its parametrisation by $y \in \mathbb{R}^m_\geqslant$ (for $\bar{a} = n+1, \ldots, m$). By Proposition 3.2.3, we can write $\mathbb{H}^m$ as $\mathsf{Q}^m \times \mathbb{R}^m_\geqslant/\!\sim_\ell$, with the quotient map $\mathbb{H}^m \to \mathbb{R}^m_\geqslant$, $h = (h_1, \ldots, h_m) \mapsto (|h_1|^2, \ldots, |h_n|^2)$. Hence a point z in the pullback can be written as

$$([h_1/|h_1|, \ldots, h_m/|h_m|; |h_1|, \ldots, |h_m|], (y_1, \ldots, y_m)) \in (\mathsf{Q}^m \times \mathbb{R}^m_\geqslant/\!\sim_\ell) \times P^n$$

restricted to $|h_a| = y_a$, for $w^*_{\bar{a}}(h) = w^*_{\bar{a}}(y)$. If $|h_a| = 0$ then $h_a/|h_a|$ is not defined but can take any value, since all values belong to the same equivalence class defining $\mathcal{Z}_{P,\ell}$. Thus, we can write $z = [\,h_1/|h_1|, \ldots, h_m/|h_m|;\, y_1, \ldots, y_m\,]$ restricted to $w^*_{\bar{a}}(h) = w^*_{\bar{a}}(y)$, that is $z = [h/|h|; y] \in \mathsf{Q}^m \times P/\!\sim_\ell$, so $\mathcal{Z}_{P,\ell}$ is equivariantly diffeomorphic to the pullback.

(2) This result follows from the diffeomorphism between $\mathbb{H}^m$ and $\mathsf{Q}^m \times \mathbb{R}^m_\geqslant/\!\sim_\ell$ for any acceptable isotropy functor ℓ (Proposition 3.2.3), recalling that the unit matrix is always an acceptable isotropy matrix, since $\mathbb{H}^m$ is diffeomorphic to itself whichever Q^m action is used. ❑

Of course, even though the different moment angle complexes over P^n are diffeomorphic they will not in general be equivariantly diffeomorphic.

Construction 5.2.5 Given a quoric manifold M^{4n} over a simple polytope P^n with global characteristic matrix Λ, construct an acceptable isotropy matrix L for a regular m-corner such that Λ embeds in L (Corollary 5.1.4), and construct the moment angle complex $\mathbb{Z}_{P,\ell}$ using the isotropy matrix L. Then we say the moment angle complex $\mathcal{Z}_{P,\ell}$ *covers* M^{4n}.

Proposition 5.2.6 *Given a moment angle complex $\mathcal{Z}_{P,\ell}$ which covers a global manifold M^{4n} over a simple polytope P^n, then the subgroup $\mathsf{Q}_{n+1} \times \ldots \times \mathsf{Q}_m$ ($\cong \mathsf{Q}^{m-n}$) acts freely on $\mathcal{Z}_{P,\ell}$,*

Essentially, this is because every column of L has a non-zero element among its first n components, by construction from Λ.

Proof. Suppose that $\mathsf{Q}_{n+1} \times \ldots \times \mathsf{Q}_m$ does not act freely on some face F_τ of $\mathbb{R}^m_\geqslant$ of codimension $t = |\tau|$. That is, a subgroup of $\mathsf{Q}_{n+1} \times \ldots \times \mathsf{Q}_m$ is in a subclass of the isotropy class $\ell(\tau) = \mathsf{Q}_{\gamma_{a_1},\ldots,\gamma_{a_t}}$. So, there is some $x \in \tau$ such that $\gamma_x \subset [m]\backslash[n]$. Restrict the action to the first n coordinate subgroups of Q^m to obtain the isotropy class $\lambda(\tau) = \mathsf{Q}_{\gamma'_{a_1},\ldots,\gamma'_{a_t}}$ of the Q^n action on the manifold M^{4n}, where $\gamma'_a = \gamma_a \cap [n]$. But then $\gamma'_x = \gamma_x \cap [n] = \emptyset$, so the rank of the isotropy class $\lambda(\tau)$ of M^{4n} is less than t, which is a contradiction if F_τ is a face of P^n. Thus the image of P^n in $\mathbb{R}^m_\geqslant$ does not meet the face F_τ of $\mathbb{R}^n_\geqslant$, and $\mathcal{Z}_{P,\ell}$ has an empty intersection with the inverse image of F_τ in $\mathbb{H}^n$, so Q_{γ_x} is not part of an isotropy class for any orbit of $\mathcal{Z}_{P,\ell}$. That is, the subgroup $\mathsf{Q}_{m+1} \times \ldots \times \mathsf{Q}_m$ acts freely. ❑

Proposition 5.2.7 *Given a quoric manifold M^{4n} over a simple polytope P^n with characteristic functor λ, there is a moment angle complex $\mathbb{Z}_{P,\ell}$ over P^n with an action of $\mathsf{Q}^m \cong \mathsf{Q}^n \times \mathsf{Q}^{m-n}$ such that the quotient $\mathcal{Z}_{P,\ell}/\mathsf{Q}^{m-n}$ is equivariantly homeomorphic to M^{4n} if and only if λ is global.*

Proof. The Q^n-manifold M^{4n} can be written (Proposition 4.2.10) as $M \cong \mathcal{M}(\lambda) = \mathsf{Q}^n \times P^n/\!\sim_\lambda$, with coordinatewise left multiplication by Q^n. If the isotropy matrix Λ is global then then it can be embedded in an acceptable isotropy matrix L.

The moment angle complex for this isotropy matrix is given by $\mathsf{Q}^m \times P^n/\!\sim_\ell$ with coordinatewise left multiplication by Q^m. The last $(m-n)$ coordinate subgroups act freely, so the quotient map is given by

$$\mathcal{Z}_{P,\ell} = \mathsf{Q}^m \times P^n/\!\sim_\ell \quad \xrightarrow{/\mathsf{Q}^{m-n}} \quad \mathsf{Q}^n \times P^n/\!\sim_\lambda = \mathcal{M}(\lambda),$$

since for any x in the relative interior of $F_{\tau(x)}$, the isotropy subgroup $\widehat{\ell}(\tau(x))$ restricted to Q^n is just $\lambda(\tau(x))$, from the embedding of Λ in L.

Conversely, if the characteristic functor is not global, then Λ cannot be embedded in an acceptable isotropy matrix, since if two columns of Λ are not compatible then the corresponding nodes of the characteristic graph are not compatible, and no procedure for augmenting the nodes can make them compatible. Thus, there does not exist any acceptable isotropy matrix L in which Λ can be embedded. ❏

5.3 Examples

We now present a number of examples of polytopes and characteristic functors and the manifolds and moment angle complexes derived from them to illustrate how these constructions work.

1. n-Simplex $P^n = \Delta^n$. For the polytope $P^n = \Delta^n$ there is only one characteristic functor, up to congruence, as we now show.

Let the vertices of Δ^n be v_a for $a = 1, \dots, n, n+1$, and the opposite facets F_a. Let Σ be the characteristic graph over Δ^n, with a node γ_a for each facet F_a, and let $\Gamma(v_a) = \Sigma|_{v_a}$ be the restriction to the nodes corresponding to the facets which contain v_a, that is excluding the node γ_a. Then $\Gamma(v_a)$ is an acceptable facet graph, and each of its nodes has a unique distinguished element.

There is some node of Σ which contains more than one element, since there are $(n+1)$ nodes of subsets of $[n]$. So let γ_x be any maximal node in Σ which contains more than one element. Then let γ_a be any maximal node of the acceptable graph $\Gamma(v_x)$, with its distinguished element k_a. Now consider the acceptable graph $\Gamma(v_a)$, which differs from $\Gamma(v_x)$ only in containing the node γ_x instead of γ_a. All other nodes γ_c occur in both $\Gamma(v_x)$ and $\Gamma(v_a)$, so have the same distinguished element in both graphs, since γ_c is the union of the distinguished elements of its subtree including itself, and γ_x is maximal so is not a subset of γ_c, Hence, the only remaining value for a distinguished element is that of γ_a in $\Gamma(v_x)$, and so the distinguished element of γ_x in $\Gamma(v_a)$ equals the distinguished element of γ_a in $\Gamma(v_x)$. Since any node contains its distinguished element the nodes γ_a and γ_x are not disjoint, and since γ_x is maximal in Σ, we have $\gamma_a \subset \gamma_x$. But every node is a subset of some maximal node, so the union of the maximal nodes is $[n]$, so $\gamma_x = [n]$. That is, every node in Σ which is maximal and contains more that one element is equal to $[n]$. But the nodes in Σ must all be different, hence the nodes are the n singleton sets and $[n]$.

After possibly permuting the vertices, we can write the characteristic matrix as $\Lambda = (\mathbf{1}, y_1)$, where $\mathbf{1}$ represents an $n \times n$ unit matrix, and y_1 is column vector $(1, \ldots, 1)^T$.

This matrix Λ is global since the columns are compatible, so it can be embedded in the square isotropy matrix $L = \begin{pmatrix} \mathbf{1} & y_1 \\ 0 & 1 \end{pmatrix}$, with $E = \begin{pmatrix} \mathbf{1} & -y_1 \\ 0 & -1 \end{pmatrix}$, providing we take $\mathcal{E} =$ diag$(1, \ldots, 1, -1)$. We can then identify the moment angle complex $\mathcal{Z}_{P,\ell}$ with $\mathbb{H}^{n+1}|_{\sum |h_i|^2 = 1} \cong S^{4n+3}$, and the quotient $\mathcal{Z}_{P,\ell}/\mathsf{Q}_{n+1}$ with $\mathbb{H}P^n$. These have actions (for $i = 1, \ldots, n$),

$$\begin{aligned} \mathsf{Q}^{n+1} \times S^{4n+3} &\to S^{4n+3}, \\ ((s_1, \ldots, s_{n+1}), (h_1, \ldots, h_{n+1})) &\mapsto (s_1 h_1 s_{n+1}^{-1}, \ldots, s_n h_n s_{n+1}^{-1}, h_{n+1} s_{n+1}^{-1}), \\ \mathsf{Q}^n \times \mathbb{H}P^n &\to \mathbb{H}P^n, \\ ((s_1, \ldots, s_n), [h_1, \ldots, h_n, h_{n+1}]) &\mapsto [s_1 h_1, \ldots, s_n h_n, h_{n+1}] \\ &= [s_1 h_1 u, \ldots, s_n h_n u, h_{n+1} u]. \end{aligned}$$

The subgroup Q_{n+1} clearly acts freely on S^{4n+3}, since $0 \notin S^{4n+3}$

2. Square $P^2 = I^2$. There are four congruence classes of characteristic functors on a square $P^2 = I^2$, two of which were considered in Example 4.3.4(2). Here we consider the other two (labelled there as (3) and (4)). We can identify $\mathcal{Z}$ as $\mathbb{H}^4|(w_3^* = w_4^* = 1) \cong S^7 \times S^7$, where $w_3^* = |h_1|^2 + |h_3|^2$ and $w_4^* = |h_2|^2 + |h_4|^2$.

$M^8_{(3)}$**:** $\lambda_{(3)}(\{a\}) = (\mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{1,2\}})$

The matrix $\Lambda_{(3)}$ is global, so can be embedded as the top two rows in

$$L_{(3)} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{with} \quad E_{(3)} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus the moment angle complex $S^7 \times S^7$ has a Q^4 action

$$(s,h) \mapsto (s_1 h_1 s_3^{-1}, s_2 h_2 s_4^{-1}, s_3 h_3 s_4^{-1}, s_4 h_4).$$

Clearly, Q_3 and Q_4 act freely since h_1 and h_3 are not both zero, and h_2 and h_4 are not both zero. Take the quotient $S^7 \times S^7/(\mathsf{Q}_3 \times \mathsf{Q}_4)$ to obtain

$$M^8_{(3)} = \{[h] \in \mathbb{H}^4/\mathsf{Q}^2 \mid w_3^* = w_4^* = 1, [h] = [h_1 u^{-1}, h_2 v^{-1}, u h_3 v^{-1}, v h_4)]\}$$

with Q^2 action $((s_1, s_2), [h_1, h_2, h_3, h_4]) \mapsto [s_1 h_1, s_2 h_2, h_3, h_4]$.

In the neighbourhood of the fixed point $h_3 = h_4 = 0$, that is when $|h_3| < 1$ and $|h_4| < 1$ so $h_1 \neq 0$ and $h_2 \neq 0$, we can use h_3, h_4 as coordinates. Fixing $h_1 u^{-1} = |h_1|$ and $h_2 v^{-1} = |h_2|$, so $u = \widehat{h}_1 = h_1/|h_1|$ and $v = \widehat{h}_2 = h_2/|h_2|$, gives

$$\psi_{3,4} \colon [h] \mapsto (q_3, q_4) = (\widehat{h}_1 h_3 \widehat{h}_2^{-1}, \widehat{h}_2 h_4), \quad (s_1, s_2)(q_3, q_4) \mapsto (s_1 q_3 s_2^{-1}, s_2 q_4)$$

which is clearly a regular corner, with $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This agrees with the top right submatrix of $L_{(3)}$. Neighbourhoods of the other three fixed points of $M^8_{(3)}$ may be analysed similarly, with two neighbourhoods equivariantly homeomorphic to a standard corner and two neighbourhoods equivariantly homeomorphic to the other corner (there are only two 2-corners up to congruence).

We could take the alternative form for $M^8_{(3)}$, as the space $\mathsf{Q}^2 \times I^2/{\sim_\lambda}$ where $(s_1, s_2; x) \sim_\lambda (s_1', s_2'; x')$ when $x = x'$ and $(s_1, s_2)^{-1}(s_1', s_2') \in \lambda(\tau(x))$. Representing I^2 as $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4_\geqslant \mid w_3^*(x) = w_4^*(x) = 1\}$, it is natural to write $q_1 = s_1 x_1$, $q_2 = s_2 x_2$. Then extend to $q_3 = u x_3$, $q_4 = v x_4$ for $u, v \in \mathsf{Q}$, but take equivalence classes over u, v. The action gives the isotropy classes specified by L, and is just that written above.

$M^8_{(4)}$: $\lambda_{(4)}(\{a\}) = (\mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{1,2\}}, \mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{1,2\}})$

The matrix $\Lambda_{(4)}$ is global, so can be embedded as the top two rows in

$$L_{(4)} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{with} \quad E_{(4)} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus the moment angle complex $S^7 \times S^7$ has a Q^2 action

$$(s,h) \mapsto (s_1 h_1 s_3^{-1}, s_3 h_2 s_4^{-1}, s_2 h_3 s_3^{-1}, s_4 h_4)$$

The subgroups Q_3 and Q_4 clearly act freely. Take the quotient $S^7 \times S^7/(\mathsf{Q}_3 \times \mathsf{Q}_4)$ to obtain

$$M^8_{(4)} = \{[h] \in \mathbb{H}^4/\mathsf{Q}^2 \mid w_3^* = w_4^* = 1, [h] = [h_1u^{-1}, uh_2v^{-1}, h_3u^{-1}, vh_4)]\}$$

with action $((s_1, s_2), [h]) \mapsto [s_1h_1, h_2, s_2h_3, h_4]$.

In the neighbourhood of the fixed point $h_3 = h_4 = 0$, that is when $|h_3| < 1$ and $|h_4| < 1$ so $h_1 \neq 0$ and $h_2 \neq 0$, we can use h_3, h_4 as coordinates. Fixing $h_1u^{-1} = |h_1|$ and $uh_2v^{-1} = |h_2|$, so $u = \widehat{h}_1 = h_1/|h_1|$ and $v = u\widehat{h}_2 = uh_2/|h_2|$, gives

$$\psi_{3,4} \colon [h] \mapsto (q_3, q_4) = (h_3\widehat{h}_1^{-1}, \widehat{h}_1\widehat{h}_2h_4), \quad (s_1, s_2)(q_3, q_4) \mapsto (s_2q_3s_1^{-1}, s_1q_4)$$

which is clearly the regular corner with $L = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. This agrees with the top right submatrix of $L_{(4)}$. Neighbourhoods of the other three fixed points of $M^8_{(4)}$ can be analysed similarly, with all four being equivariantly homeomorphic to the same regular corner.

5.4 Polytopes Supporting Quoric Manifolds

We have seen that a characteristic functor λ can be found over a 2-simplex (Δ^2), and four congruence classes of characteristic functors over a square (I^2) (Example 4.3.4(1, 2)). For an n simplex, $P^n = \Delta^n$, there is always a characteristic functor, by assigning the $(n+1)$ sets $\{1\}, \{2\}, \dots \{n\}$ and $\{1, 2, \dots, n\}$ to the $(n+1)$ facets, and the quoric manifold over Δ^n is $\mathbb{H}P^n$. The n-cube, $P^n = I^n$, also has a number of characteristic functors, for example by assigning the acceptable isotropy functor of any regular n corner to the faces around one vertex, and assigning the same class to each opposite face. *These are all examples of global characteristic functors.* Example 5.1.5 exhibited a non-global as well as a global characteristic functor over a cube. It is of interest to know that there are more than just simplices and cubes (as in these two examples) for which there exists a characteristic functor or even a global characteristic functor. The properties of polytopes covered in this section is classical material, and can be found in BP ([4] Ch 2.2), and references cited there. The properties of the colourings described here are straightforward applications of this material.

Any characteristic matrix Λ over a polytope P^n satisfies the conditions for a quasi-toric Λ matrix (see also Paragraph 4.2.15), so any polytope which supports a quoric manifold also supports a quasi-toric manifold. We do not expect the converse to be true, but have no examples at present (which would need to be of dimension at least 4).

A (convex) simple polytope P^n is said to be k-neighbourly if every set of k facets has a non-empty intersection, for $2 \leqslant k \leqslant [n/2]$. Clearly, k-neighbourly implies $(k-1)$-neighbourly. If P^n is 2-neighbourly, then every pair of facets has a common vertex, so any characteristic matrix Λ over P^n must have all its columns different and compatible. Each column has length n and with elements of 0 or 1, so there can be at most $2^n - 1$ different non-zero columns. As noted by Davis and Januszkiewicz (see also BP [4]), there do exist simple polytopes P^n with $m \geqslant 2^n$ facets, for example, the polars of the (simplicial) cyclic polytopes. The *moment curve* in $\mathbb{R}^n$ is defined by

$$x\colon \mathbb{R} \to \mathbb{R}^n, \qquad t \mapsto x(t) = (t, t^2, \ldots, t^n)$$

For any $m > n$, the *cyclic polytope* $C^n(x_1, \ldots, x_m)$ is defined as the convex hull of the m distinct points $x(t_1), \ldots, x(t_m)$. It can be shown that no $(n+1)$ points on the moment curve belong to a common affine hyperplane, that $C^n(x_1, \ldots, x_m)$ has exactly m vertices and is $[n/2]$ neighbourly, and the combinatorial type of $C^n(t_1, \ldots, t_m)$ is independent of the choice of the points $x(t_1), \ldots, x(t_m)$ (See Ziegler [22], for more details.) The polar $Cy^n(m)$ of $C^n(x_1, \ldots, x_m)$ is a simple n-dimensional polytope with m facets, for any $m > n$. So for $m \geqslant 2^n$ and $n \geqslant 4$ there cannot exist any characteristic functor over $Cy^n(m)$. Thus, we have:

Proposition 5.4.1 *There exist simple polytopes in dimensions $n \geqslant 4$ for which no characteristic functor exists.* ❏

Let $\mathcal{F}_1(P)$ denote the set of facets of the polytope P^n. An n dimensional polytope is said to be *facet k-colourable* if there is a map from its facets to the set $[k]$ of colours, $c\colon \mathcal{F}_1(P) \to [k]$, such that if any two different facets have a non-empty intersection then they are assigned different colours.

Proposition 5.4.2 *If a simple polytope P^n is $(n+1)$-colourable then there exists a global characteristic functor over P^n.*

Proof. Let $c\colon \mathcal{F}_1(P^n) \to [n+1]$ be an $(n+1)$ colouring of the simple polytope P^n. The $(n+1)$ sets $\gamma_1 = \{1\}$, ..., $\gamma_n = \{n\}$ and $\gamma_{n+1} = \{1, \ldots, n\}$ are compatible, so to each colour $i = 1, \ldots, n+1$ assign the set γ_i. Then, for any vertex v of P^n, restricting the sets $\gamma_{c(F_a)}$ to facets containing v gives an acceptable facet graph. That is, the sets $\gamma_{c(F_a)}$ over all the facets of P^n define a global characteristic facet graph, so define a global characteristic functor. ❏

Proposition 5.4.3 *If a simple polytope P^n is n-colourable then there exist T_{n+1} congruence classes of global characteristic functors over P^n, where T_{n+1} is the number of congruence classes of n corners.*

Proof. Let $c\colon \mathcal{F}_1(P^n) \to [n]$ be an n-colouring of the simple polytope P^n. For any regular n corner let the sets $\gamma_1, \ldots, \gamma_n$ be nodes of its facet graph Γ. Assign the set γ_i to each colour $i = 1, \ldots, n$, then the sets $\gamma_{c(F_a)}$, over all the facets of P^n, are the nodes of a global characteristic facet graph Σ, since all nodes are compatible, and the restriction to any vertex is congruent to Γ. ❏

We note that each characteristic facet graph Σ identified in the proof of the proposition satisfies the property that its restriction to any vertex v gives the same Γ. In contrast, for the characteristic functor over a simplex (Example 4.2.5) there was one vertex for which the restriction gave a facet graph which was not congruent to that of any other vertex.

A simplex, Δ^n, is $(n+1)$-colourable, since there are $(n+1)$ facets, and is not n-colourable since every two facets have a non-empty intersection.

A unit hypercube, $I^n \subset \mathbb{R}^n$, is n-colourable by assigning the colour k to each facet perpendicular to the k^{th} axis in $\mathbb{R}^n$.

Proposition 5.4.4 *In dimensions $n = 1$, 2 or 3, all simple polytopes P^n are $(n+1)$-colourable.*

Proof. In dimension 1, there is only one polytope, I, and it has two vertices so is trivially 2-colourable. In dimension 2 any polygon is 3-colourable, by assigning the first two colours to alternate edges around the polygon, and the third for the last edge when there are an odd number of edges. In dimension 3, as noted by Davis and Januszkiewicz (DJ [8]), any polyhedron is 4-colourable by the Four Colour Theorem ([2], or see [21] for more references). Thus, in dimensions $1, 2, 3$, all simple polytopes are $(n+1)$-colourable. ❏

We now consider a number of ways of generating simple $(n+1)$-colourable polytopes.

Given a simple polytope P^n with m facets and a vertex v in P^n, there exists an affine hyperplane defined by $\langle l_{m+1}, x\rangle = c_{m+1}$ for $l_{m+1} \in \mathbb{R}^n$ and $c_{m+1} \in \mathbb{R}$, which splits v from the other vertices; that is, $\langle l_{m+1}, v\rangle < c_{m+1}$ and $\langle l_{m+1}, w\rangle > c_{m+1}$ for any vertex $w \neq v$. The half-space $H_{m+1} = \{x \in \mathbb{R}^n \mid \langle l_{m+1}, x\rangle \geqslant 0\}$ then does not contain the vertex v, and contains the other vertices in its interior. Adding H_{m+1} to the set of half spaces which define P^n then defines a new polytope, $P^n_{\#v}$, with one extra facet and n new vertices replacing v. Note that $P^n_{\#v}$ is combinatorially equivalent to the conventional *connected sum* $P^n \#_v \Delta^n$.

Proposition 5.4.5 *If P^n is a simple $(n+1)$-colourable polytope, then so is $P^n_{\#v}$ for any vertex v, and $P^n_{\#v}$ has one more facet than P^n.*

Proof. The intersection of H_{m+1} with P^n removes the vertex v , and generates one new facet, so $P^n_{\#v}$ has $(m+1)$ facets. The new facet is an $(n-1)$-simplex since P^n is simple, and each of the new vertices is contained in $(n-1)$ of the facets containing v, and the new facet, so $P^n_{\#v}$ is simple. The n facets which contain v define n colours from the set $[n+1]$, so the other colour can be assigned to the new facet, and each new vertex is contained in n facets with different colours assigned. Hence, $P^n_{\#v}$ is also $(n+1)$ colourable. ❏

Given an $(n-1)$-dimensional simplicial complex, $K = \{\sigma_1, \ldots, \sigma_m \subset [n-1]\}$, its barycentric subdivision is a new simplicial complex whose vertices are non-empty chains of simplices of K (see Equation 4.1.10 for more detail). That is, the non-empty set $b = \{\sigma_1, \ldots, \sigma_k\}$ is a vertex of K' if and only if $\sigma_1 \subset \ldots \subset \sigma_k \in K$. If b' is a proper subchain of b then K' has an edge joining them, and in general a simplex of K' is a set of nested chains in K. Assigning the value $c = |b| \in [n]$ to each chain b ensures that each vertex of a simplex in K' has a different value.

Furthermore, the barycentric subdivision of the boundary of a simplicial polytope is the boundary of a simplicial polytope. So, starting from any simple polytope P^n, construct its polar, $(P^n)^*$. The boundary of $(P^n)^*$ defines an $(n-1)$-dimensional simplicial complex, whose barycentric subdivision is the boundary of an n-dimensional simplicial polytope, $(P^n)^{*\prime}$. Now construct the polytope $P^{n\#}$ as the polar of $(P^n)^{*\prime}$.

Proposition 5.4.6 *For any simple polytope P^n the polytope $P^{n\#}$ is simple and n-colourable.*

Proof. Any barycentric subdivision is simplicial, so its polar is simple. Assigning the value $c = |b| \in [n]$ to each chain b of $(P^n)^*$ ensures that each vertex of a simplex in $(P^n)^{*\prime}$ takes a different value. Taking the polar, facets acquire a value (in $[n]$), and any facets that intersect have different values. Thus, $P^{n\#}$ is n-colourable. ❑

Theorem 5.4.7 *In any dimension $n > 1$,*

(1) for any $m > n$ there is a simple $(n+1)$-colourable polytope with m facets,

(2) there are an infinite number of simple polytopes which are n colourable.

Proof. (1) A simplex Δ^n is simple and $(n+1)$-colourable, and has $(n+1)$ facets. Set $P^n_{(n+1)} = \Delta^n$, and for $P^n_{(k)}$ select any vertex v and construct $P^n_{(k+1)} = P^n_{(k)\#v}$. Thus, for any $m > n$ there is a polytope $P^n_{(m)}$ with m facets, and is $(n+1)$-colourable (Proposition 5.4.5).

(2) A hypercube I^n is simple and n colourable. Set $P^n_1 = I^n$, and for $P^n_{(k)}$ construct $P^n_{(k+1)} = P^{n\#}_{(k)}$. Then, for any $p > 0$ there is a polytope $P^n_{(p)}$, which is n-colourable (Proposition 5.4.6). ❑

Proposition 5.4.2 shows that any polytope P^n which is $(n+1)$-colourable supports a global characteristic functor. Example 6.3(3) below shows that for a 3-dimensional prism, Pr^3, there are more global characteristic functors than the one implied by the

Proposition. The prism Pr^3 also shows that there are global characteristic functors which do not correspond to any $(n+1)$-colouring. That is, proposition 5.4.2 exposes only a very small number of possible characteristic functors.

Conjecture 5.4.8

(1) There exist polytopes P^n that are not $(n+1)$ colourable but support a global characteristic functor.

(2) There exist polytopes P^n that support a characteristic functor but not a global characteristic functor.

(3) There exist polytopes P^n that support a quasi-toric characteristic function but not a (quoric) characteristic functor.

In dimensions 1 and 2 all characteristic functors are global, and in dimension 3 all polytopes support a global characteristic functor (Proposition 5.4.2, 5.4.4) as well as some that are not global (Example 5.1.5). In dimension at least 4 there are polytopes that do not support any characteristic functor (Proposition 5.4.1). We might expect the proportion of characteristic functors which are global for any polytope P^n to decrease with dimension, from the general increase in complexity. There is some support for this expectation, from the following results. For an n simplex Δ^n there is only one characteristic functor, and it is global. For a hypercube I^n, Figure 5.1 lists the number of congruence classes of characteristic functors, and the number of them that are global, for $n \leqslant 5$. (These numbers were calculated by a short computer program, written to check every assignment of sets $\gamma \subset [n]$ to facets of I^n, and list one for each congruence class of characteristic functor, indicating whether it was global or not.)

I^n: n	1	2	3	4	5
No. Classes of $\lambda(I^n)$	1	4	19	110	677
No. global	1	4	17	89	482

Figure 5.1: Number of Characteristic Functors over a Hypercube

Chapter 6

Cohomology of Global Quoric Manifolds

The goal of this section is to compute the integral cohomology ring of a quoric manifold M^{4n} over a simple polytope P^n with a characteristic functor over P^n satisfying the global condition, showing that it takes the form familiar from the case of quasitoric manifolds. We approach this calculation in much the same way as Davis and Januszkiewicz, but using the quaternionic group Q instead of the circle group T. We make essential use of the the Borel constructions of the moment angle complex $B_{\mathsf{Q}}\mathcal{Z}$ and of the quoric manifold $B_{\mathsf{Q}}M^{4n}$, together with the quaternionic analogue of the Davis-Januszkiewicz space, $DJ_{\mathsf{Q}}(K_P)$. We show that $B_{\mathsf{Q}}\mathcal{Z}$ is homotopy equivalent to $DJ_{\mathsf{Q}}(K_P)$, and that $B_{\mathsf{Q}}M$ and $B_{\mathsf{Q}}\mathcal{Z}_{P,\ell}$ are homotopy equivalent. The integral cohomology of $DJ_{\mathsf{Q}}(K_P)$ is verified to be isomorphic to the Stanley-Reisner ring SR(P) $= \mathbb{Z}[u_1, \ldots, u_m]/\mathcal{I}$ of the polytope P^n, but with the degree of each generator u_a equal to 4.

We examine the classifying map $B_{\mathsf{Q}}\mathcal{Z}_{P,\ell} \to (B\mathsf{Q})^m$, of the principal bundle $(E\mathsf{Q})^m \times \mathcal{Z}_{P,\ell} \to B_{\mathsf{Q}}\mathcal{Z}_{P,\ell}$, and the classifying map $B_{\mathsf{Q}}M \to (B\mathsf{Q})^n$, of the principal bundle $(E\mathsf{Q})^m \times M^{4n} \to B_{\mathsf{Q}}\mathcal{Z}_{P,\ell}$. We show that these two classifying maps can be related by a commutative square of bundle maps. The main differences from the toric case are that the Q^m action on $\mathcal{Z}_{P,\ell}$ is twisted, (that is, the action is that of a regular rather than a standard m-corner), and that the map $B_{\mathsf{Q}}\mathcal{Z} \to B_{\mathsf{Q}}M$ is simply a projection to the first n factors. Then we use the spectral sequence of the principal bundle $\mathsf{Q}^n \hookrightarrow E\mathsf{Q}^n \times M \to B_{\mathsf{Q}}M$ to show that the expression for the cohomology ring of M^{4n} takes the same form as for the toric case, providing its characteristic functor satisfies the global condition.

6.1 Borel Constructions

Classifying Space. For any topological group G, there is a topological space EG which is contractible and has a free G action. We assume a left action, $(g, e) \mapsto ge$. The orbit space $BG = EG/G$ is a *classifying space* of G, and the quotient map $\mathfrak{p}\colon EG \to BG$ is a principal G bundle with the universal property that any principal G bundle $X \to Y$ is classified up to bundle isomorphism by a homotopy class of classifying maps $c\colon Y \to BG$. The Milnor construction of EG as the infinite join of G is the standard example, see for example, Husemoller [14] Ch 4. Given a classifying space BG of G, we will always use $(BG)^m$ as the classifying space of G^m. That is, the G^m action on (EG) is assumed to be coordinatewise. We always assume a base point $* \in EG$ such that $\mathfrak{p}(*)$ is the basepoint in BG.

The classifying space of Q is well known to be the infinite dimensional quaternionic projective space $\mathbb{H}P^\infty$, with universal bundle, $S^\infty \to \mathbb{H}P^\infty$. Thus, $\mathbb{H}P^\infty$ is the quaternionic analogue of the classifying space $\mathbb{C}P^\infty$ of the circle group T. The projective space $\mathbb{H}P^n$ was introduced in Example 4.2.5, and can be expressed as S^{4n+3}/Q, where S^{4n+3} is the unit sphere in $\mathbb{H}^{n+1}$. There is a natural inclusion of S^{4n+3} in S^{4n+7}, by identifying the first $(n+1)$ coordinates, $(h_0, h_1, \ldots, h_n) \hookrightarrow (h_0, h_1, \ldots, h_n, 0)$, which induces a natural inclusion of $\mathbb{H}P^n$ into $\mathbb{H}P^{n+1}$. Then taking the colimit gives $S^\infty = \bigcup_{n>0} S^{4n+3}$, and $\mathbb{H}P^\infty = \bigcup_{n>0} \mathbb{H}P^n$.

The infinite projective space $\mathbb{H}P^\infty$ has a canonical cell structure, since each $\mathbb{H}P^n$ has, with one cell in each dimension which is a multiple of 4 and none in other dimensions, and with the Hopf map defining the attaching map of any cell to the cell with next smaller dimension. Then, $\mathbb{H}P^\infty$ is a CW complex. Furthermore, the integral cohomology $H^*(\mathbb{H}P^\infty)$ is isomorphic to the polynomial ring $\mathcal{Z}[x]$, where the generator x has degree 4.

Borel Construction. For any G-space X, the Borel construction (or homotopy quotient) of X is the base space

$$B_G X := EG \times_G X = (EG \times X)/G.$$

of a principal G bundle, $G \hookrightarrow EG \times X \to B_G X$, since G acts freely on EG.

Suppose we are given a global quoric manifold M^{4n} over a polytope P^n with a Q^n action corresponding to the global characteristic matrix Λ, together with a moment angle complex $\mathcal{Z}_{P,\ell}$ with a Q^m action specified by an acceptable exponent matrix $E = L^{-1}$ derived from Λ (Corollary 5.1.4). Then the Borel constructions for $\mathcal{Z}_{P,\ell}$ and M^{4n} are defined as (simplifying the notation slightly)

$$(6.1.1) \qquad B_{\mathsf{Q}}\mathcal{Z} = (E\mathsf{Q})^m \times_{\mathsf{Q}^m} \mathcal{Z}_{P,\ell}\ ; \qquad B_{\mathsf{Q}}M = (E\mathsf{Q})^n \times_{\mathsf{Q}^n} M^{4n}.$$

Cubical Decompositions. Recall that we can identify the simple polytope P^n with its image as a subspace of the boundary of the unit cube in $\mathbb{R}^m$ (Equation 4.1.14) $P^n = \bigcup_{\tau \in K_P} I^\tau \times 1^{\widehat{\tau}}$ (where $\widehat{\tau} = [m] \backslash \tau$). This decomposition can be lifted to the moment angle complex $\mathcal{Z}_{P,\ell} = \bigcup_{\tau \in K_P} \mathcal{D}^\tau \times \mathsf{Q}^{\widehat{\tau}}$ (where $\mathcal{D}_a$ is the unit disc in the coordinate subspace $\mathbb{H}_a$ of $\mathbb{H}^m$, and Q_a is naturally identified with the boundary $\partial\mathcal{D}_a$), and can be extended to the homotopy quotient of $\mathcal{Z}_{P,\ell}$

$$(6.1.2) \quad B_{\mathsf{Q}}\mathcal{Z} = \bigcup_{\tau \in K_P} BZ(\tau), \qquad BZ(\tau) = E\mathsf{Q}^m \times_{\mathsf{Q}^m} (\mathcal{D}^\tau \times \mathsf{Q}^{\widehat{\tau}}) \subset (E\mathsf{Q} \times_{\mathsf{Q}} \mathcal{D})^m.$$

The quaternionic analogue of the Davis-Januszkiewicz space is defined as a subspace of $(B\mathsf{Q})^m$,

$$(6.1.3) \qquad DJ_{\mathsf{Q}}(K_P) = \bigcup_{\tau \in K_P} BQ(\tau), \qquad BQ(\tau) = (B\mathsf{Q})^\tau = (E\mathsf{Q}/\mathsf{Q})^\tau \subset (B\mathsf{Q})^m.$$

It is immediate from this definition that $BQ(\mu) \subset BQ(\tau)$ for any $\mu \subset \tau$.

Homotopy Equivalences. We next show that, just as in the toric case, $B_{\mathsf{Q}}\mathcal{Z}$ is homotopy equivalent to $DJ_{\mathsf{Q}}(K_P)$, even though here the action on $\mathcal{Z}_{P,\ell}$ is not diagonal. We will define the maps between the components of the decompositions above which respect the different underlying Q^m actions.

For a G-space X, and any $x \in X$, let $[x]_K$ denote the orbit of x in X under the action of a subgroup $K < G$. We begin with two technical lemmas, in the form which we use them below.

Lemma 6.1.4 *Let G be a topological group, and E, A, B be G-spaces each with a left G action. Let G act transitively on B. Fix a point $b_0 \in B$, and let H be the isotropy subgroup of b_0. Then $E \times_G (A \times B)$ and $E \times_H A$ are homeomorphic.*

Proof. Since G acts transitively on B, for any $b \in B$ there is some $g_b \in G$ such that $g_b b = b_0$. Define the map

$$f\colon E \times_G (A \times B) \to E \times_H A, \qquad [e, a, b]_G \mapsto [g_b e, g_b a]_H.$$

We check the properties of f:

(1) Well defined: (a) For any other g_b' such that $g_b' b = b_0 = g_b b$, it follows that $g_b' g_b^{-1} = h$ for some $h \in H$. Thus $[g_b' e, g_b' a]_H = [hg_b e, hg_b a]_H = [g_b e, g_b a]_H$.
(b) For any $[ge, ga, gb]_G = [e, a, b]_G$ we have $b_0 = g_{gb}(gb) = g_{gb} g g_b^{-1} b_0$, so $g_{gb} = hg_b g^{-1}$ for some $h \in H$. Then, $[ge, ga, gb]_G \mapsto [g_{gb} ge, g_{gb} ga]_H = [hg_b e, hg_b a]_H = [g_b e, g_b a]_H$.

(2) Injective: If $[e, a, b]$ and $[e', a', b']$ map to the same element of $E \times_H A$ then $[g_b e, g_b a]_H = [g_{b'} e', g_{b'} a']_H$, hence $g_{b'} e' = hg_b e$ and $g_{b'} a' = hg_b a$, for some $h \in H$. Setting $g = g_{b'}^{-1} h g_b$ it follows that $e' = ge$ and $a' = ga$, and also that $b' = g_{b'}^{-1} b_0 = gb$. Hence $]e, a, b] = [e', a', b']$.

(3) Surjective: For any $[e, a]_H$, there is $[e, a, b_0]_G$ which maps to it.

(4) Continuous: Since the group action is continuous for a topological group, and the quotient maps are continuous and open for any closed subgroup, f and f^{-1} are continuous. ❏

Lemma 6.1.5 *Let $EG \to BG$ be a universal principal G bundle, and let G act diagonally on the Cartesian product $(EG)^k$ for some $k > 1$. Then the map*

$$p_a\colon (EG)^k/G \to BG_a, \qquad [e_1, \ldots, e_k]_G \mapsto [e_a]_G,$$

is a homotopy equivalence for any $a \in [k]$, ❑

Proof. The space $(EG)^k$ is contractible with a free G action, and is the pull back of $EG \to BG$ along p_a, so p_a is a homotopy equivalence. ❑

We also need to use a result from homotopy theory. A map $f\colon A \to B$ is said to be a cofibration if it satisfies the homotopy extension property with respect to all topological spaces Y. A CW pair (X, A) is a cell complex X and a closed subcomplex A, and any CW pair has the homotopy extension property (see, for example, Hatcher [13] (0.16)). Thus, a map $f\colon A \to B$ between two CW complexes is a cofibration if it is injective and its image is a closed subspace.

We are working in CW, the category of CW complexes. Let $\mathcal{C}$ be a small category, then a $\mathcal{C}$ *diagram* (in CW) is a functor $F\colon \mathcal{C} \to \text{CW}$. Given two $\mathcal{C}$ diagrams with a set of homotopy equivalences between them, we want to know when the colimits of the diagrams are also homotopy equivalent. Model category theory (see for example Dwyer [10], [11]) treats this question in a very abstract form, but the following simplified result is sufficient here.

Lemma 6.1.6 *Let $\mathcal{C}$ be a small category, and let $F_1, F_2\colon \mathcal{C} \to \text{CW}$ be two $\mathcal{C}$ diagrams in the category of CW complexes. If (1) each morphism in $F_1(\mathcal{C})$ and $F_2(\mathcal{C})$ is a cofibration, and (2) there is a natural transformation $t\colon F_1 \to F_2$ such that for every $A \in \mathcal{C}$ the map $t_A\colon F_1(A) \to F_2(A)$ is a homotopy equivalence, then the colimits of F_1 and F_2 are homotopy equivalent.*

We are now ready to show that $B_{\mathsf{Q}}\mathcal{Z}$ is homotopy equivalent to $DJ_{\mathsf{Q}}(K_P)$ and that $B_{\mathsf{Q}}\mathcal{Z}$ is homotopy equivalent to $B_{\mathsf{Q}}M$.

Proposition 6.1.7 *The spaces $B_{\mathsf{Q}}\mathcal{Z}$ and $DJ_{\mathsf{Q}}(K_P)$ are colimits of $\text{CAT}(K_P)$ diagrams.*

Proof. Recall the notation used in the cubical decompositions, $(d_\tau, s_{\widehat{\tau}}) \in \mathcal{D}^\tau \times \mathsf{Q}^{\widehat{\tau}}$ represents $(h_1, \ldots, h_m) \in \mathcal{D}^m$ where $h_a = d_a \in \mathcal{D}_a$ if $a \in \tau$, and $h_a = s_a \in \mathsf{Q}_a = \partial\mathcal{D}_a$ if $a \notin \tau$, for any $\tau \in \text{CAT}(K_P)$. Let $e = (e_1, \ldots, e_m) \in E\mathsf{Q}^m$.

(1) Let $\mathcal{G}_Z\colon \text{CAT}(K_P) \to (E\mathsf{Q})^m \times_{\mathsf{Q}^m} \mathcal{D}^m$, be the functor defined by $\tau \mapsto BZ(\tau)$, (Equation 6.1.2) and $(\tau \subset \mu) \mapsto (i_{\tau,\mu}\colon BZ(\tau) \to BZ(\mu))$, where $i_{\tau,\mu}$ is the inclusion map. The inclusion $\mathsf{Q}^\tau \subset \mathcal{D}^\tau$ is Q^m equivariant, hence for any $\mu \subset \tau$ the inclusions

$$i_{\mu,\tau} : BZ(\mu) \to BZ(\tau),$$
$$[e; d_\mu, s_{\widehat{\mu}}]_{\mathsf{Q}^m} \mapsto [e; d_\mu, s_{\widehat{\mu}\cap\tau}, s_{\widehat{\tau}}]_{\mathsf{Q}^m} = [e; d_\tau, s_{\widehat{\tau}}]_{\mathsf{Q}^m}$$

are well defined, so $BZ(\mu) \subset BZ(\tau)$. Clearly, for nested inclusions $\mu \subset \rho \subset \tau$, we have $i_{\rho,\tau} \circ i_{\mu,\rho} = i_{\mu,\tau}$. That is, $\mathcal{G}_Z$ is a functor, and its colimit is $B_{\mathsf{Q}}\mathcal{Z}$ by construction.

(2) Let $\mathcal{G}_D\colon \text{CAT}(K_P) \to (B\mathsf{Q})^m$, be the functor defined by $\tau \mapsto BQ(\tau)$, (Equation 6.1.3) and $(\tau \subset \mu) \mapsto (i_{\tau,\mu}\colon BQ(\tau) \to BQ(\mu))$, where $i_{\tau,\mu}$ is the inclusion map. By definition, if $\mu \subset \tau$ then $BQ(\mu) \subset BQ(\tau)$, so for any nested inclusion $\mu \subset \rho \subset \tau$ we have $BQ(\mu) \subset BQ(\rho) \subset BQ(\tau)$. That is, $\mathcal{G}_D$ is a functor, and its colimit if $DJ_{\mathsf{Q}}(K_P)$ by definition. ❑

Define maps between the blocks $BZ(\tau)$ and $BQ(\tau)$ by,

(6.1.8)
$$p_\tau\colon BZ(\tau) \to BQ(\alpha(\tau)) \to BQ(\tau),$$
$$[e; d_\tau, s_{\widehat{\tau}}]_{\mathsf{Q}^m} \mapsto (b_1, \ldots, b_m), \qquad b_t = [e_{\alpha(t)}]_{\mathsf{Q}_{\alpha(t)}} \quad (t \in \tau),$$
$$= * \qquad \text{otherwise.}$$

The action of Q^m on $\mathcal{Z}_{P,\ell}$ is specified by the $m \times m$ acceptable isotropy matrix L. Let $\gamma(\tau) = (\gamma_{t_1}, \ldots, \gamma_{t_{|\tau|}})$ label the isotropy class $\mathsf{Q}_{\gamma_{t_1},\ldots,\gamma_{t_{|\tau|}}}$ associated with the face F_τ of P^n, as defined from L. Recall that the γ_t are disjoint, and that the labelling of the γ_t can be chosen so that $\alpha(t) \in \gamma_t$ for each $t \in \tau$ (Proposition 3.1.16). Let $\mathsf{Q}(\gamma(\tau))$ denote the canonical group in the class $\mathsf{Q}_{\gamma(\tau)}$, and let $(E\mathsf{Q})^{\gamma_t}$ denote the product of those coordinate subspaces $(E\mathsf{Q})_a$ for which $a \in \gamma_t$.

Proposition 6.1.9 *The colimits $B_{\mathsf{Q}}\mathcal{Z}$ and $DJ_{\mathsf{Q}}(K_P)$ are homotopy equivalent.*

Proof. We check that p_τ is a homotopy equivalence for each $\tau \in K_P$, and that all squares commute, so the terms define a natural transformation between the diagrams.

(1) Homotopy equivalence: For each $\tau \in K_P$ the block $BZ(\tau) = (E\mathsf{Q})^m \times_{\mathsf{Q}^m} (\mathcal{D}^\tau \times \mathsf{Q}^{\widehat{\tau}})$ is homeomorphic to $(E\mathsf{Q})^m \times_{\mathsf{Q}(\gamma(\tau))} \mathcal{D}^\tau$, by Lemma 6.1.4, since the isotropy subgroup of the element $1^{\widehat{\tau}} \in \mathsf{Q}^{\widehat{\tau}}$ is $\mathsf{Q}(\gamma(\tau))$. Since each $\mathcal{D}_t$ is contractible, $(E\mathsf{Q})^m \times_{\mathsf{Q}(\gamma(\tau))} \mathcal{D}^\tau$ is homotopy equivalent to $(\prod_{t\in\tau}(E\mathsf{Q})^{\gamma_t}) \times_{\mathsf{Q}(\gamma(\tau))} (E\mathsf{Q})^{\widehat{\gamma(\tau)}}$. Each term from the product, $(E\mathsf{Q})^{\gamma_t}/\mathsf{Q}(\gamma_t)$, is homotopy equivalent to $(B\mathsf{Q})^{\alpha(t)}$ (Lemma 6.1.5), and is mapped homeomorphically by p_τ to $(B\mathsf{Q})_t$. The remaining term, $(E\mathsf{Q})^{\widehat{\gamma(\tau)}}$, is a product of some of the $(E\mathsf{Q})_a$, which are all contractible, hence there is a homotopy from $BZ(\tau)$ to $BQ(\tau)$. The projection maps for each $t \in \tau$ (p_a in the proof of Lemma 6.1.5) combine together to give precisely p_τ.

(2) All squares commute:

(6.1.10)
$$\begin{array}{ccc} BZ(\mu) & \xrightarrow{i_Z} & BZ(\tau) \\ \downarrow{\scriptstyle p_\mu} & & \downarrow{\scriptstyle p_\tau} \\ BQ(\mu) & \xrightarrow{i_D} & BQ(\tau) \end{array}$$

For any $\mu \subset \tau$ we have

$$BZ(\mu) \to \cdots \to BQ(\tau)$$

$i_D \circ p_\mu$:

$$[e; d_\mu, s_{\widehat{\mu}}]_{\mathsf{Q}^m} \mapsto [e_{\alpha(\mu)}]_{\mathsf{Q}(\alpha(\mu))} \mapsto [e_\mu]_{\mathsf{Q}(\mu)} \mapsto [e_\mu, *_{\widehat{\mu}\cap\tau}]_{\mathsf{Q}(\tau)}$$

$p_\tau \circ i_Z$:

$$[e; d_\mu, s_{\widehat{\mu}}]_{\mathsf{Q}^m} \mapsto [e; d_\mu, s_{\widehat{\mu}\cap\tau}, s_{\widehat{\tau}}]_{\mathsf{Q}^m} \mapsto [e_{\alpha(\mu)}, *_{\alpha(\widehat{\mu}\cap\tau)}]_{\mathsf{Q}(\alpha(\tau))} \mapsto [e_\mu, *_{\widehat{\mu}\cap\tau}]_{\mathsf{Q}(\tau)},$$

for $e = (e_1, \ldots, e_m) \in E\mathsf{Q}^m$, and $e_\mu = (e'_1, \ldots, e'_m) \in E\mathsf{Q}^m$ where $e'_a = e_a$ if $a \in \mu$, or $= *$ otherwise.

(3) Inclusion maps are cofibrations: Since Q_a is a closed subset of $\mathcal{D}_a$ for $a \in [m]$, and $BZ(\tau)$ is a subspace of the CW complex $(E\mathsf{Q})^m \times_{\mathsf{Q}^m} \mathcal{D}^m$ for all $\tau \in K_P$, each inclusion $i_{\mu,\tau}$ is a cofibration, so the inclusion maps of the blocks $BZ(\tau)$ of $B_{\mathsf{Q}}\mathcal{Z}$ are cofibrations.

Any $BQ(\tau)$ is a closed subspace of the CW complex $(B\mathsf{Q})^m$, so the inclusion maps of the blocks $BQ(\tau)$ of $DJ_{\mathsf{Q}}(K_P)$ are cofibrations.

(4) Hence (Lemma 6.1.6) the colimits $B_{\mathsf{Q}}\mathcal{Z}$ and $DJ_{\mathsf{Q}}(K_P)$ are homotopy equivalent. ❏

Proposition 6.1.11 *The Borel constructions $B_{\mathsf{Q}}M$ and $B_{\mathsf{Q}}\mathcal{Z}$ are homotopy equivalent.*

Proof. The subgroup consisting of the last $(m-n)$ coordinate subgroups of Q^m acts freely on $\mathcal{Z}_{P,\ell}$ (Proposition 5.2.6), so we have

$$\begin{aligned} B_{\mathsf{Q}}\mathcal{Z} &= (E\mathsf{Q})^m \times_{\mathsf{Q}^m} \mathcal{Z}_{P.\ell} \\ &= \big((E\mathsf{Q})^n \times_{\mathsf{Q}^n} (\mathcal{Z}_{P,\ell}/\mathsf{Q}^{m-n})\big) \times (E\mathsf{Q})^{m-n} \\ &\simeq (E\mathsf{Q})^n \times_{\mathsf{Q}^n} M^{4n}. \end{aligned}$$

❏

Combining the last two propositions (6.1.9, 6.1.11) immediately gives:

Corollary 6.1.12 *The Borel construction $B_{\mathsf{Q}}M$ is homotopy equivalent to the quaternionic analogue of the Davis-Januszkiewicz space $DJ_{\mathsf{Q}}(K_P)$.* ❏

6.2 Cohomology of Global Quoric Manifolds

We first give an outline of the steps taken, in the propositions of this section, to calculate the integral cohomology ring of a global quoric manifold M^{4n}.

The cohomology of the DJ-space $DJ_{\mathsf{Q}}(K_P)$ is verified to be isomorphic to the face ring $\mathrm{SR}(P^n) \cong \mathbb{Z}[u_1,\dots,u_m]/\mathcal{I}_P$ with generators u_a $(a = 1,\dots,m)$ of degree 4 (Proposition 6.2.3). The Borel constructions $B_{\mathsf{Q}}\mathcal{Z}$ and $B_{\mathsf{Q}}M$ are both homotopy equivalent to $DJ_{\mathsf{Q}}(K_P)$ (Propositions 6.1.9, 6.1.11), so their cohomology rings are also isomorphic to the face ring.

The Q^m action on the moment angle complex $\mathcal{Z}_{P,\ell}$ generates a principal bundle $\mathsf{Q}^m \hookrightarrow E\mathsf{Q}^m \times \mathcal{Z}_{P,\ell} \to E\mathsf{Q}^m \times_{\mathsf{Q}^m} \mathcal{Z}_{P,\ell} = B_{\mathsf{Q}}\mathcal{Z}$. The classifying map of this bundle $c_\ell\colon B_{\mathsf{Q}}\mathcal{Z} \to B\mathsf{Q}^m$ can be combined with the homotopy equivalence $B_{\mathsf{Q}}\mathcal{Z} \to DJ_{\mathsf{Q}}(K_P) \subset (B\mathsf{Q})^m$ to relate the cohomology $H^*(B_{\mathsf{Q}}\mathcal{Z})$ and $H^*(B\mathsf{Q}^m)$. The generators $x_a \in H^*(B\mathsf{Q}^m)$ map to $\theta_a = c_\ell^*(x_a) \in H^*(B_{\mathsf{Q}}\mathcal{Z})$, and we determine the crucial expressions $\theta_a = \sum_{d=1}^m L_{a,d}u_d$

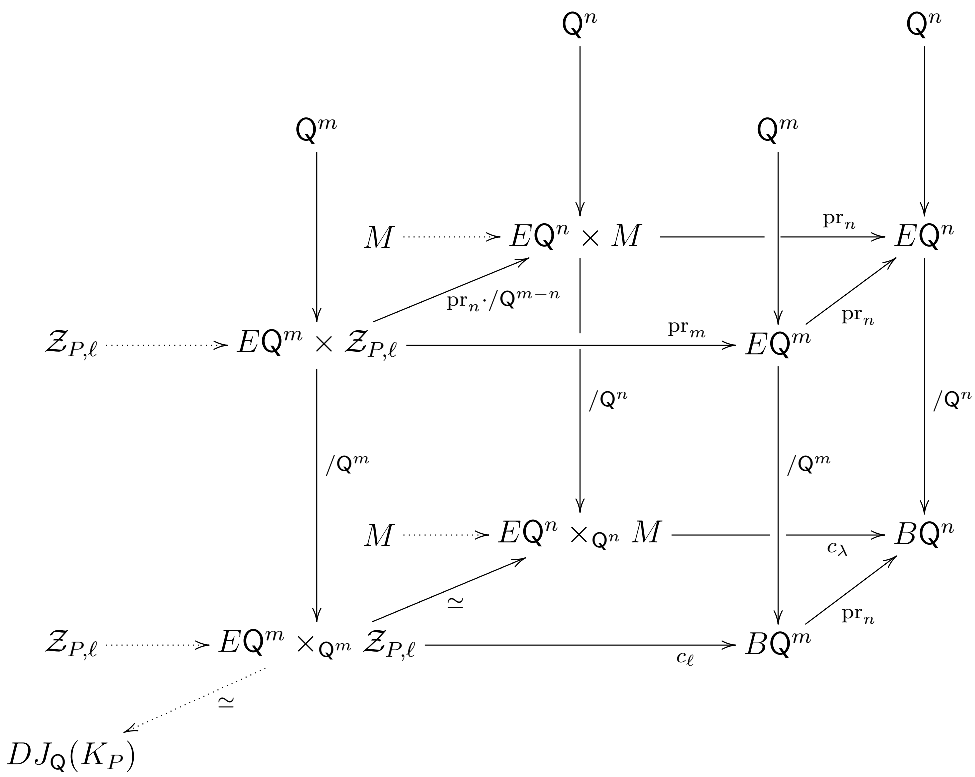


Figure 6.1: Commutative Diagram of Principal Bundle Maps

We have constructed the moment angle complex $\mathcal{Z}_{P,\ell}$ with a Q^m action so that taking the quotient with respect to the last $(m-n)$ coordinate subgroups is equivariantly homeomorphic to the manifold M^{4n} (Prop 5.2.7). This quotient induces a natural quotient map from the principal bundle $\mathsf{Q}^m \hookrightarrow E\mathsf{Q}^m \times \mathcal{Z}_{P,\ell} \to B_{\mathsf{Q}}\mathcal{Z}$ to the principal bundle $\mathsf{Q}^n \hookrightarrow E\mathsf{Q}^n \times M \to B_{\mathsf{Q}}M$. There is a corresponding projection from $\mathsf{Q}^m \hookrightarrow E\mathsf{Q}^m \to B\mathsf{Q}^m$ to $\mathsf{Q}^n \hookrightarrow E\mathsf{Q}^n \to B\mathsf{Q}^n$, so that the pullback square of the classifying map $c_\ell \colon B_{\mathsf{Q}}\mathcal{Z} \to B\mathsf{Q}^m$ maps to the pullback square of the classifying map $c_\lambda \colon B_{\mathsf{Q}}M \to B\mathsf{Q}^n$.

This commutative square of bundle maps allows us to determine the induced map c_λ^* relating the cohomology generators of $H^*(B_{\mathsf{Q}}M)$ to the images of the generators of $H^*(B\mathsf{Q}^n)$. We then use a spectral sequence of the fibration $\mathsf{Q}^n \hookrightarrow E\mathsf{Q}^n \times M \to B_{\mathsf{Q}}M$ to calculate the cohomology $H^*(M^{4n})$ (Theorem 6.2.7). The proof of these results takes the rest of this section.

Figure 6.1 (p.121) summarises the relations between the principal bundles considered here, where any map pr_k indicates a projection onto the first k coordinates, and there are obvious identity and projection maps which are omitted for clarity.

Remark 6.2.2 The same diagram with the corresponding maps may be used in the toric case. In the usual treatment of the toric case (DJ [8], BP [4]), the action of T^n on $\mathcal{Z}$ is coordinate-wise. Then the homotopy map from the space $DJ_{\mathsf{Q}}(K_P)$ to $B_{\mathsf{T}}\mathcal{Z}$ is essentially a projection, but instead the quotient $\mathrm{pr}_{H(l)} \colon \mathcal{Z} \to M^{2n}$, and hence the projection $B_{\mathsf{T}}\mathcal{Z} \to B_{\mathsf{T}}M$, involves the characteristic function. This projection map introduces the twisting action on M. Indeed, it is this map $\mathrm{pr}_{H(l)}$ that induces the familiar linear relations θ_i of the generators u_a of $H^*(B_{\mathsf{Q}}\mathcal{Z})$, which has a direct analogue in the present treatment.

Here, in contrast, the quotient map from $\mathcal{Z}$ to M must be a quotient by a product of coordinate subgroups, since they are the only normal subgroups of Q^m. In this case, it is the classifying map of $B_{\mathsf{Q}}\mathcal{Z}$ which exhibits the twisting.

Of course, this method of using a twisted action on the moment angle complex may be used equally well in the toric case.

Let the generators of the integral cohomology ring $H^*(B\mathsf{Q}^m)$ be x_a, of degree 4, (for $a = 1, \dots, m$), so $H^*(B\mathsf{Q}^m)$ is isomorphic to $\mathbb{Z}[x_1, \dots, x_m]$. We use the projection map $\mathrm{pr}_n \colon B\mathsf{Q}^m \to B\mathsf{Q}^n$, to the first n coordinate subspaces, to identify the generators of $H^*(B\mathsf{Q}^n)$ with the first n generators $(x_1, \dots, x_n)$ of $H^*(B\mathsf{Q}^m)$. Since $DJ_{\mathsf{Q}}(K_P)$, $B_{\mathsf{Q}}\mathcal{Z}$ and $B_{\mathsf{Q}}M$ are homotopy equivalent, we identify the generators $(u_1, \dots, u_m)$ of their cohomology rings.

Proposition 6.2.3 *The integral cohomology ring of $DJ_{\mathsf{Q}}(K_P)$ is isomorphic to the face ring of the polytope P^n, that is $H^*(DJ_{\mathsf{Q}}(K_P)) \cong \mathrm{SR}(P^n) \cong \mathbb{Z}[u_1, \dots, u_m]/\mathcal{I}_P$, where each generator u_a is of degree 4.*

Proof. We write the generators of the cohomology of $DJ_{\mathsf{Q}}(K_P) \subset (B\mathsf{Q})^m$ as u_a, so for now we write the generators of $H^*(B\mathsf{Q}^m)$ as u_a for $a = 1, \dots, m$.

Each block $BQ(\tau)$ of $DJ_{\mathsf{Q}}(K_P)$ is a product of coordinate subspaces of $(B\mathsf{Q})^m$ (Equation 6.1.3). In particular, the canonical inclusion $i_D\colon DJ_{\mathsf{Q}}(K_P) \to (B\mathsf{Q})^m$ restricted to $BQ(\{a\})$ induces the projection $i_D^*|_{BQ(\{a\})}\colon H^*(B\mathsf{Q}^m) \cong \mathcal{Z}[u_1,\ldots,u_m] \to \mathbb{Z}[u_a]$. Since every coordinate subspace $(B\mathsf{Q})_a$ of $(B\mathsf{Q})^m$ occurs in $DJ_{\mathsf{Q}}(K_P)$, the induced map i_D^* is surjective. For any $\rho \notin K_P$, corresponding to a non-face of P^n, the block $BQ(\rho)$ is not a subspace of $DJ_{\mathsf{Q}}(K_P)$, so $\mathcal{Z}[u_{a_1},\ldots,u_{a_{|\rho|}}]$ for $\rho = \{a_1,\ldots,a_{|\rho|}\}$ is not a subring of $H^*(DJ_{\mathsf{Q}}(K_P))$. In particular, the monomial $u(\rho) = u_{a_1}\cdots u_{a_{|\rho|}}$ does not occur in $H^*(DJ_{\mathsf{Q}}(K_P))$, so $i_D^*\colon u(\rho) \mapsto 0$. That is, the kernel of i_D^* contains the square-free monomials $u(\rho)$ of every non-face, so is the ideal generated by them, that is $\mathcal{I}_P$. Thus, $H^*(DJ_{\mathsf{Q}}(K_P))$ is isomorphic to $\mathbb{Z}[u_1,\ldots,u_m]/\mathcal{I}_P$, the face ring $\mathrm{SR}(P^n)$ of P^n. ❑

Let $c_\ell\colon B_{\mathsf{Q}}\mathcal{Z} \to B\mathsf{Q}^m$ be the classifying map of the Borel construction $\mathsf{Q}^m \hookrightarrow (E\mathsf{Q})^m \times \mathcal{Z}_{P,\ell} \to B_{\mathsf{Q}}\mathcal{Z}$, defined by $c_\ell\colon [e;z]_{\mathsf{Q}^m} \mapsto [e]_{\mathsf{Q}^m}$. Define the linear combinations of the generators u_a of $H^*(B_{\mathsf{Q}}\mathcal{Z})$, by $\theta_d = \sum_{a=1}^m L_{d,a}\, u_a$, where L is the isotropy matrix of the Q^m action on $\mathcal{Z}_{P,\ell}$.

Proposition 6.2.4 *The classifying map $c_\ell\colon B_{\mathsf{Q}}\mathcal{Z} \to B\mathsf{Q}^m$, induces the map*

$$c_\ell^*\colon H^*(B\mathsf{Q}^m) \to H^*(B_{\mathsf{Q}}\mathcal{Z}), \quad x_d \mapsto \theta_d$$

on the cohomology generators $x_d \in H^*(B\mathsf{Q}^m)$, $u_a \in H^*(B_{\mathsf{Q}}\mathcal{Z})$.

Proof. For any facet $\tau = \{a\}$, $BZ(\{a\})$ is homotopy equivalent to $BQ(\{a\}) \subset DJ_{\mathsf{Q}}(K_P)$, so identifies the cohomology of $BZ(\{a\})$ as $\mathbb{Z}[u_a]$, where u_a is the generator of $H^*(DJ_{\mathsf{Q}}(K_P)) = \mathbb{Z}[u_1,\ldots,u_m]/\mathcal{I}_P$. The classifying map of $B_{\mathsf{Q}}\mathcal{Z}$ restricted to $BZ(\{a\}) = E\mathsf{Q}^m \times_{\mathsf{Q}^m} (\mathcal{D}^{\{a\}} \times \mathsf{Q}^{\{\widehat{a}\}}) \cong E\mathsf{Q}^m \times_{\mathsf{Q}(\gamma_a)} \mathcal{D}^{\{a\}}$ is

$$\begin{aligned} c_\ell|_{BZ(\{a\})}\colon\ & BZ(\{a\}) \to B\mathsf{Q}^m \\ & [e_1,\ldots,e_m; d_{\{a\}}]_{\mathsf{Q}(\gamma_a)} \mapsto (b_1,\ldots,b_m), \end{aligned}$$

where $b_d = [e_d]_{\mathsf{Q}_d}$ if $d \in \gamma_a$, and $b_d = *$ otherwise. Thus, c_ℓ induces

$$\begin{aligned} c_\ell^*|_{BZ(\{a\})}\colon\ & x_d \mapsto u_a \quad (d \in \gamma_a = \gamma(\{a\})) \\ & x_d \mapsto 0 \qquad (d \notin \gamma(a)) \end{aligned}$$

But the isotropy matrix L (for $\mathcal{Z}_{P,\ell}$) characterises the γ_a, since $L_{b,a} = 1$ if $b \in \gamma_a$ and $= 0$ otherwise (Definition 2.2.16), so we can write the induced map, for each $a, d = 1, \ldots, m$, as $c_\ell^*|_{\{a\}} \colon x_d \mapsto L_{d,a}u_a$. Since c_ℓ^* is a ring homomorphism, we have in $H^*(B_{\mathsf{Q}}\mathcal{Z})$, $c_\ell^* \colon x_d \mapsto \sum_{a=1}^m L_{d,a}u_a = \theta_d$. ❑

Proposition 6.2.5 *The diagram of principal bundle maps (Figure 6.1) is a commutative diagram.*

Proof. (1) We check each face describes a bundle map.

The right-hand face, $(\mathsf{Q}^m \hookrightarrow E\mathsf{Q}^m \to B\mathsf{Q}^m) \to (\mathsf{Q}^n \hookrightarrow E\mathsf{Q}^n \to B\mathsf{Q}^n)$, is simply a projection onto the first n coordinates subspaces, hence clearly a bundle map.

The front and back faces are pull back squares.

The left-hand face $(\mathsf{Q}^m \hookrightarrow E\mathsf{Q}^m \times \mathcal{Z}_{P,\ell} \to E\mathsf{Q}^m \times_{\mathsf{Q}^m} \mathcal{Z}_{P,\ell}) \to (\mathsf{Q}^n \hookrightarrow E\mathsf{Q}^m \times M^{4n} \to E\mathsf{Q}^n \times_{\mathsf{Q}^n} M^{4n})$ is the projection onto the first n coordinate subspaces of $E\mathsf{Q}^n$ and the quotient of $\mathcal{Z}_{P,\ell}$ by Q^{m-n}, hence

$$E\mathsf{Q}^m \times \mathcal{Z}_{P,\ell} \to \cdots \to E\mathsf{Q}^n \times_{\mathsf{Q}^n} M^{4n}$$
$$(e_1, \ldots, e_m; z) \mapsto (e_1, \ldots, e_n; [z]_{\mathsf{Q}^{m-n}}) \mapsto [e_1, \ldots, e_n; [z]_{\mathsf{Q}^{m-n}}]_{\mathsf{Q}^n}$$
$$(e_1, \ldots, e_m; z) \mapsto [e_1, \ldots, e_m; z]_{\mathsf{Q}^m} \mapsto [e_1, \ldots, e_n; [z]_{\mathsf{Q}^{m-n}}]_{\mathsf{Q}^n},$$

so is a bundle map.

(2) Next we check that the diagram commutes,

$$E\mathsf{Q}^m \times \mathcal{Z}_{P,\ell} \to \cdots \to E\mathsf{Q}^n$$
$$(e_1, \ldots, e_m; z) \mapsto (e_1, \ldots, e_n; [z]_{\mathsf{Q}^{m-n}}) \mapsto (e_1, \ldots, e_n)$$
$$(e_1, \ldots, e_m; z) \mapsto (e_1, \ldots, e_m) \mapsto (e_1, \ldots, e_n).$$

Hence, Figure 6.1 is a commutative diagram of principal bundle maps. ❑

We need to use a standard result from homological algebra about a Koszul resolution of a module over a ring. MacLane ([15] VII.2) or Eisenbud ([12] §17) include a standard description of a Koszul resolution. Let K be any commutative ring, and $P = K[x_1, \ldots, x_n]$ be the polynomial algebra over K in n indeterminates of even degree d. Let $\wedge_P[y_1, \dots, y_n]$ be the exterior algebra over P in n indeterminates of odd degree $(d-1)$. Define a differential operator by $\mathrm{d}\colon y_i \mapsto x_i$, $x_i \mapsto 0$, and an augmentation $\varepsilon\colon P \to K$, by $\varepsilon\colon x_i \mapsto 0$. Then (MacLane [15] VII.2.1), the sequence

$$0 \to \wedge_P^n \xrightarrow{\mathrm{d}} \quad \cdots \quad \xrightarrow{\mathrm{d}} \wedge_P^0 = P \xrightarrow{\varepsilon} K \to 0$$

is an exact sequence, or equivalently provides a free resolution of K. Since the ring K is the 0-graded part of P, it is the P module $P/(x_i)$ where (x_i) is the ideal generated by all the x_i. This ideal is generated by the non-zero-graded terms in the image of the differential, which we write as $(\operatorname{im} \mathrm{d}^+)$.

This result can be extended to any commutative ring R and R-module A. Suppose n elements $x_1, \dots, x_n \in R$ are such that each x_i is not a zero-divisor of A/J_{i-1}, where J_k is the ideal $(x_1, \dots, x_k)$. Such a set is called a regular sequence, and suppose that each x_i is of even degree d. If the x_i are a linearly independent subset of generators of R, then they from a regular sequence. Let $\wedge_R[y_1, \dots, y_n]$ be the exterior algebra over R on n indeterminates of odd degree $(d-1)$. Define the differential operator by $\mathrm{d}\colon y_i \mapsto x_i$, $x_i \mapsto 0$, and an augmentation $\varepsilon\colon A \otimes_R \wedge_r^0 = A \otimes R \to A/J_n$ by $\varepsilon\colon (a, r) \mapsto ar + aJ_n$.

Lemma 6.2.6 *(MacLane [15] VII.6.Ex(3)) The sequence*

$$0 \to A \otimes_R \wedge_R^n \to \quad \cdots \quad \to A \otimes_R \wedge_R^0 = A \to A/J_n \to 0$$

is an exact sequence, or equivalently provides a Koszul resolution of A/J_n.

The ideal J_n is again the ideal generated by the non-zero-graded elements in the image of the differential, $(\operatorname{im} \mathrm{d}^+)$.

We are now ready to obtain the main result.

Theorem 6.2.7 *Given a global quoric manifold* M^{4n} *over a simple polytope* P^n, *with* m *facets and characteristic matrix* Λ, *the integral cohomology ring is given by*

$$H^*(M^{4n}) \cong \mathrm{SR}(P^n)/\mathcal{J} \cong \mathbb{Z}[u_1, \dots, u_m]/(\mathcal{I} + \mathcal{J}),$$

where there is a generator u_a (with degree $|u_a| = 4$) for each facet F_a of P^n, where $\mathcal{I}$ is the ideal generated by the monomials in the u_a corresponding to the missing faces of P^n, and $\mathcal{J}$ is the ideal $(\theta_1, \ldots, \theta_n)$ generated by the $\theta_i = \sum_{a=1}^{m} \Lambda_{i,a} u_a$, for $i = 1, \ldots, n$.

Proof. Recall that $E\mathsf{Q}^n \times_{\mathsf{Q}^n} M^{4n}$ is just $B_{\mathsf{Q}}M$ and that $E\mathsf{Q}^n \times M^{4n}$ is homotopy equivalent to M^{4n}, so we can use the Serre spectral sequence of the fibration $\mathsf{Q}^n \hookrightarrow E\mathsf{Q}^n \times M^{4n} \to B_{\mathsf{Q}}M$ to calculate the cohomology of M^{4n}. By Propositions 6.1.12 and 6.2.3, the cohomology $H^*(B_{\mathsf{Q}}M)$ is isomorphic to the face ring $\mathrm{SR}(P)$.

(1) Write the cohomology rings $H^*(\mathsf{Q}^n) \cong \mathbb{Z}[y_1, \ldots, y_n]/(y_i^2)$ with $|y_i| = 3$, and $H^*(B\mathsf{Q}^n) \cong \mathbb{Z}[x_1, \ldots, x_n]$ with $|x_i| = 4$. The spectral sequence of $\mathsf{Q}^n \hookrightarrow E\mathsf{Q}^n \to B\mathsf{Q}^n$ collapses at the E_4 page, since each term in the direct product does, and is a Koszul resolution, with $\mathrm{d}y_i = x_i$.

(2) Let $x_1, \ldots, x_n$ be the generators of $H^*(B\mathsf{Q}^n)$. The composition of maps $\mathrm{pr}_n \circ c_\ell \colon B_{\mathsf{Q}}\mathcal{Z} \to B\mathsf{Q}^m \to B\mathsf{Q}^n$ forms part of the commutative square (Figure 6.1), and induces homomorphisms of the cohomology generators

$$
\begin{aligned}
(x_1, \ldots, x_n) &\mapsto (x_1, \ldots, x_n, 0, \ldots) \quad &\in H^4(B\mathsf{Q}^m)\\
&\mapsto (\theta_1, \ldots, \theta_n, 0, \ldots) \quad &\in H^4(B_{\mathsf{Q}}\mathcal{Z})
\end{aligned}
$$

where $\theta_i = \sum_a^m L_{i,a} u_a$ (for $i = 1, \ldots, n$), and equals $\sum_a^m \Lambda_{i,a} u_a$, since Λ is the submatrix consisting of the top n rows of L. The composition of maps $c_\lambda \circ \simeq \colon B_{\mathsf{Q}}\mathcal{Z} \to B_{\mathsf{Q}}M \to B\mathsf{Q}^n$ forms the other part of the commutative square, and induces homomorphisms of the cohomology generators

$$
\begin{aligned}
(x_1, \ldots, x_n) &\mapsto (w_1, \ldots, w_m) \quad &\in H^*(B_{\mathsf{Q}}M)\\
&\mapsto (w_1, \ldots, w_m) \quad &\in H^*(B_{\mathsf{Q}}\mathcal{Z})\\
&= (\theta_1, \ldots, \theta_n, 0, \ldots) \quad &\in H^*(B_{\mathsf{Q}}\mathcal{Z})
\end{aligned}
$$

where the w_a are the images in $H^4(B_{\mathsf{Q}}M)$ of the generators x_i of $H^*(B\mathsf{Q}^n)$. Since $H^*(B_{\mathsf{Q}}\mathcal{Z})$ and $H^*(B_{\mathsf{Q}}M)$ are each isomorphic to the face ring, we use the images of the generators u_a of $H^*(B_{\mathsf{Q}}\mathcal{Z})$ as the generators of $H^*(B_{\mathsf{Q}}M)$. The commutative square of bundle maps allows us to identify each image $c_\lambda^*(x_i) = w_i$ with $\theta_i = \sum_a^m \Lambda_{i,a} u_a$.

(3) The bundle map from $\mathsf{Q}^n \hookrightarrow E\mathsf{Q}^n \times M \to B_{\mathsf{Q}}M$ to $\mathsf{Q}^n \hookrightarrow E\mathsf{Q}^n \to B\mathsf{Q}^n$ induces a map of spectral sequences. In particular, $H^*(B_{\mathsf{Q}}M) \cong \mathbb{Z}[u_1, \ldots, u_m]/\mathcal{I}$ and the images of the x_i are $\theta_i = \sum_a^m \Lambda_{i,a} u_a$. The images θ_i are linear combinations of the generators u_i, and form a regular sequence since the rows of Λ are linearly independent, since they form a subset of the rows of L, which is invertible. The differentials are given by $\mathrm{d}y_i = \theta_i$ and $\mathrm{d}u_a = 0$.

(4) The spectral sequence of $\mathsf{Q}^n \hookrightarrow E\mathsf{Q}^n \times M \to B_{\mathsf{Q}}M$ contains non-zero terms $E_r^{p,q}$ only for $p = 4k$ and $q = 3l$ (for $k, l \geqslant 0$), from the degrees of the generators of $H^*(B_{\mathsf{Q}}M)$ and $H^*(\mathsf{Q})$. Hence, the differentials must be 0 for $r < 4$. On the E_4 page $E_4^{4k,3l} = A^k \otimes_{\mathbb{Z}} \wedge[y_j]^l$, where A^k represents the ring generated by the monomials of degree k in the generators u_a of $H^*(B_{\mathsf{Q}}M)$, and $\wedge[y_i]^l$ represents the ring generated by the monomials of degree l in the generators y_i of $H^*(\mathsf{Q}^n)$, for $i = 1, \ldots, n$ and $a = 1, \ldots, m$. The differential maps the term $E_4^{4k,3l}$ to the term $E_4^{4(k+1),3(l-1)}$ so gives rise to diagonal sequences, whose direct sum can be written

$$0 \to A \otimes_{\mathbb{Z}} \wedge^n[y_i] \to \cdots \to A \otimes_{\mathbb{Z}} \wedge^0[y_i] = A \to 0,$$

where $A = \oplus_k A^k$. This is clearly a Koszul resolution, where J_n is the ideal $(\operatorname{im} \mathrm{d}^+) = (\theta_1, \ldots, \theta_n)$. Thus the only terms that survive to the next page are on the p-axis ($l = 0$), confirming that the spectral sequence collapses at the E_4 page. Hence, we have $H^*(M^{4n}) = A/J_n \cong \mathbb{Z}[u_1, \ldots, u_m]/\mathcal{I}/\mathcal{J}$ where $\mathcal{J}$ is the ideal $(\theta_1, \ldots, \theta_n)$. ❑

6.3 Examples

1. Square $P^2 = I^2$. Take $\lambda_{(4)}(\{a\})) = (\mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{1,2\}}, \mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{1,2\}})$, (see Example 5.3(2)).

$$\Lambda_{(4)} \text{ embeds in } L = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and } E = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The moment angle complex $\mathcal{Z}_{I^2,\ell_{(4)}}$ is defined by $\mathsf{Q}^4 \times I^2/\sim_{\ell_{(4)}}$, which may be written

$$\mathcal{Z}_{I^2,\ell_{(4)}} \cong \mathbb{H}^4|_{w_3^*=w_4^*=1}, \qquad w_3^* = |h_1|^2 + |h_3|^2,\ w_4^* = |h_2|^2 + |h_4|^2.$$

The Q^4 action on $\mathcal{Z}_{I^2,\ell_{(4)}}$ is then given by

$$((s_1, s_2, s_3, s_4), (h_1, h_2, h_3, h_4)) \mapsto (s_2 h_1 s_3^{-1}, s_3 h_2 s_4^{-1}, s_1 h_3 s_3^{-1}, s_4 h_4).$$

The action on the manifold $M^8_{(4)} = \mathcal{Z}_{P,\ell(4)}/(\mathsf{Q}_3 \times \mathsf{Q}_4)$ is given by

$$((s_1, s_2), [h_1, h_2, h_3, h_4]) \mapsto [s_2 h_1, h_2, s_1 h_3, h_4].$$

where $[h_1, h_2, h_3, h_4] = [h_1 u^{-1}, u\, h_2 v^{-1}, h_3 u^{-1}, v\, h_4]$.

The map from the facet labelling to the coordinate subgroup labelling is $\alpha(1) = 2$, $\alpha(2) = 3$, $\alpha(3) = 1$, $\alpha(4) = 4$. The subgroup $\mathsf{Q}(\gamma(\tau))$ is the canonical subgroup in the class Q_τ. Then the classifying map $c_\ell|_\tau \colon B_{\mathsf{Q}}\mathcal{Z} \to B\mathsf{Q}^4$ restricted to each block is

$$\{a\} \quad c_\ell|_{\{a\}} \colon BZ(\{a\}) = E\mathsf{Q}^4 \times_{\mathsf{Q}(\{a\})} \mathcal{D}_a \to B\mathsf{Q}^4$$

$$\{1\} \quad [e; d_1]_{\mathsf{Q}(\{2\})}$$
$$\mapsto ([e_1]_{\mathbf{1}}, [e_2]_{\mathsf{Q}(\{2\})}, [e_3]_{\mathbf{1}}, [e_4]_{\mathbf{1}}),$$
$$\{2\} \quad [e; d_2]_{\mathsf{Q}(\{1,2,3\})}$$
$$\mapsto ([e_1]_{\mathsf{Q}(\{1,2,3\})}, [e_2]_{\mathsf{Q}(\{1,2,3\})}, [e_3]_{\mathsf{Q}(\{1,2,3\})}, [e_4]_{\mathbf{1}}),$$
$$\{3\} \quad [e; d_3]_{\mathsf{Q}(\{1\})}$$
$$\mapsto ([e_1]_{\mathsf{Q}(\{1\})}, [e_2]_{\mathbf{1}}, [e_3]_{\mathbf{1}}, [e_4]_{\mathbf{1}}),$$
$$\{4\} \quad [e; d_4]_{\mathsf{Q}(\{1,2,3,4\})}$$
$$\mapsto ([e_1]_{\mathsf{Q}(\{1,2,3,4\})}, [e_2]_{\mathsf{Q}(\{1,2,3,4\})}, [e_3]_{\mathsf{Q}(\{1,2,3,4\})}, [e_4]_{\mathsf{Q}(\{1,2,3,4\})}),$$

Taking the cohomology generators $x_a \in H^*(B\mathsf{Q}^4)$ and $u_a \in H^*(B_{\mathsf{Q}}\mathcal{Z}) \cong H^*(DJ_{\mathsf{Q}}(K_P))$, we have the following induced maps on the blocks $BZ(\tau)$, listing only the non-zero maps:

$$\{a\} \quad c^*_\ell|_\tau \colon H^*(B\mathsf{Q}(\{a\})) \to H^*(B_{\mathsf{Q}}\mathcal{Z})$$

$$\{1\} \quad x_2 \mapsto u_1$$
$$\{2\} \quad x_1 \mapsto u_2 \quad x_2 \mapsto u_2 \quad x_3 \mapsto u_2$$
$$\{3\} \quad x_1 \mapsto u_3$$
$$\{4\} \quad x_1 \mapsto u_4 \quad x_2 \mapsto u_4 \quad x_3 \mapsto u_4 \quad x_4 \mapsto u_4$$

$$\theta_1 = u_2 + u_3 + u_4, \quad \theta_2 = u_1 + u_2 + u_4, \quad \theta_3 = u_2 + u_4, \quad \theta_4 = u_4.$$

That is, $x_d \mapsto \theta_d = \sum_a L_{d,a} u_a,$ for $a, d = 1, 2, 3, 4$.

The face ring is given by $\mathrm{SR}(I^2) \cong \mathbb{Z}[u_1, u_2, u_3, u_4] \,/\, (u_1 u_3,\ u_2 u_4)$. The ideal $\mathcal{J}$ is generated by the 2 linear combinations $(\theta_1, \theta_2) = (u_2 + u_3 + u_4,\ u_1 + u_2 + u_4)$. Thus, taking $u = u_1$, $v = u_3$, the cohomology of the manifold $M^8_{(4)}$ over I^2 is

$$H^*(M^8_{(4)}) \cong \ \mathrm{SR}(I^2)/(\theta_1,\, \theta_2) \cong \mathbb{Z}[u, v]/(uv,\ u^2 + v^2).$$

Thus, the Betti numbers for $M^8_{(4)}$ are $b_0 = 1$, $b_4 = 2$, $b_8 = 1$, which illustrates the calculation from the h-vector of a square (Theorem 4.2.12).

2. n-Simplex $P^n = \Delta^n$. The characteristic matrix Λ for an n-simplex was shown (Example 5.3(1)) to be $(\mathbf{1}, y_1)$, where y_1 is the column $(1, \ldots, 1)^T$, and the face ring is given by $\mathrm{SR}(\Delta^n) \cong \mathbb{Z}[u_1, \ldots, u_{n+1}]/(u_1 \cdots u_{n+1})$. The ideal $\mathcal{J}$ is generated by the n elements $(u_i - u_{n+1})$, for $i = 1, \ldots, n$. The manifold over Δ^n is $\mathbb{H}P^n$, so we have

$$H^*(\mathbb{H}P^n) \cong \ \mathrm{SR}(\Delta^n)/\mathcal{J} \cong \mathbb{Z}[u]/(u^{n+1}).$$

3. Prism $P^3 = Pr^3$. A 3-dimensional prism is the Cartesian product of a 2-simplex with an interval (or 1-simplex). It is also the connected sum of two 3-simplices. We label the opposite triangular facets F_1, F_5, and the three square facets F_2, F_3, F_4. The face ring is $\mathrm{SR}(Pr^3) = \mathbb{Z}[u_1, \ldots, u_5]/(u_1 u_5,\ u_2 u_3 u_4)$. The moment angle complex is the subspace of $\mathbb{H}^5$,

$$\mathcal{Z}(Pr^3) = \{(h_1, \ldots, h_5) \in \mathbb{H}^5 \mid w_4^* = w_5^* = 1\}$$

where $w_4^* = |h_1|^2 + |h_5|^2$ and $w_5^* = |h_2|^2 + |h_3|^2 + |h_4|^2$. For a global characteristic functor on Pr^3 the action on $\mathcal{Z}(Pr^3)$ can be determined (Corollary 5.1.4) and the quoric manifold M^{12} obtained as the quotient space $\mathcal{Z}_{Pr^3,\ell}/(\mathsf{Q}_4 \times \mathsf{Q}_5)$. The cohomology ring can be determined directly from the characteristic matrix Λ (Theorem 6.2.7).

There turn out to be seven congruence classes of characteristic functors over a 3-dimensional prism, which we list below, and all are global. The cohomology rings of the quoric manifolds over Pr^3 separate into three isomorphism classes, listed below as A, B, C. Two of the seven quoric manifolds ($M^{12}_{(1)}$,$M^{12}_{(5)}$) are connected sums of quoric manifolds, $\mathbb{H}P^3 \# \mathbb{H}P^3$, and have isomorphic cohomology rings ($\cong A$). Two of the manifolds ($M^{12}_{(4)}$,$M^{12}_{(7)}$) have different isotropy classes on the two end facets of Pr^3 and have isomorphic cohomology rings ($\cong C$). They also have 5 different isotropy classes on the 5 facets, so do not correspond to a 4-colouring. The remaining three manifolds have isomorphic cohomology rings ($\cong B$).

The three rings A, B, C have the same number of generators, showing that the Betti numbers are: $b_0 = 1$, $b_4 = 2$, $b_8 = 2$, $b_{12} = 1$, which agrees with the calculation from the h-vector of the prism (Theorem 4.2.12).

$M^{12}_{(1)}$**:** $\lambda_{(1)}(\{a\}) = (\mathsf{Q}_{\{1,2,3\}}, \mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{3\}}, \mathsf{Q}_{\{1,2,3\}})$.

$$\Lambda_{(1)} \text{ embeds in } L = \begin{pmatrix} 1\,1\,0\,0\,1 \\ 1\,0\,1\,0\,1 \\ 1\,0\,0\,1\,1 \\ 1\,0\,0\,0\,1 \\ 0\,0\,0\,0\,1 \end{pmatrix}, \quad \text{and } E = \begin{pmatrix} 0&0&0&1&-1 \\ 1&0&0&-1&0 \\ 0&1&0&-1&0 \\ 0&0&1&-1&0 \\ 0&0&0&0&1 \end{pmatrix}.$$

Thus, the action on the moment angle complex is given by,

$$((s_i), (h_i)) \mapsto (s_4 h_1 s_5^{-1}, s_1 h_2 s_4^{-1}, s_2 h_3 s_4^{-1}, s_3 h_4 s_4^{-1}, s_5 h_5).$$

The quoric manifold is the quotient

$$\begin{aligned} M^{12}_{(1)} &= \mathcal{Z}_{Pr^3,\ell_{(1)}}/(\mathsf{Q}_4 \times \mathsf{Q}_5) \\ &= \{[u h_1 v^{-1}, h_2 u^{-1}, h_3 u^{-1}, h_4 u^{-1}, v h_5] \in \mathbb{H}^5/\mathsf{Q}^2 \mid w_4^* = w_5^* = 1\} \end{aligned}$$

with the action of $\mathsf{Q}^3 = \mathsf{Q}_1 \times \mathsf{Q}_2 \times \mathsf{Q}_3$ derived from that on $\mathcal{Z}_{Pr^3,\ell_{(1)}}$

$$((s_1, s_2, s_3), [h_1, \ldots, h_5]) \mapsto [h_1, s_1 h_2, s_2 h_3, s_3 h_4, h_5].$$

The cohomology ring $H^*(M^{12}_{(1)})$ is given by $\mathrm{SR}(Pr^3)/\mathcal{J}$, where the ideal $\mathcal{J}$ is generated by the 3 elements $(u_1 + u_2 + u_5)$, $(u_1 + u_3 + u_5)$, and $(u_1 + u_4 + u_5)$. Setting $u = u_1$, $v = -u_5$ we obtain $u_2 = u_3 = u_4 = -u + v$, and

$$H^*(M^{12}_{(1)}) \cong \mathbb{Z}[u, v]/(-uv,\, -(u - v)^3) \cong \mathbb{Z}[u, v]/(uv,\, u^3 - v^3) \qquad \cong A$$

with generators $(\mathbf{1}; u, v; u^2, v^2; u^3 = v^3)$.

$M^{12}_{(2)}$: $\lambda_{(2)}(\{a\}) = (\mathsf{Q}_{\{3\}}, \mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{1,2\}}, \mathsf{Q}_{\{3\}})$.

$$\Lambda_{(2)} \text{ embeds in } L = \begin{pmatrix} 0&1&0&1&0\\ 0&0&1&1&0\\ 1&0&0&0&1\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \quad \text{and } E = \begin{pmatrix} 0&0&1&0&-1\\ 1&0&0&-1&0\\ 0&1&0&-1&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix},$$

with actions on $\mathcal{Z}_{Pr^3,\ell_{(2)}}$ and $M^{12}_{(2)}$ derived as above.

The cohomology ring $H^*(M^{12}_{(2)})$ is given by $\mathrm{SR}(Pr^3)/\mathcal{J}$, where $\mathcal{J}$ is generated by the 3 elements (u_2+u_4), (u_3+u_4), and (u_1+u_5). Setting $u=u_1$, $v=u_4$ we obtain

$$H^*(M^{12}_{(2)}) \cong \mathbb{Z}[u,v]/(-u^2,\, v^3) \cong \mathbb{Z}[u,v]/(u^2,\, v^3) \qquad \cong B$$

with generators $(\mathbf{1}; u, v; uv, v^2; uv^2)$.

$M^{12}_{(3)}$: $\lambda_{(3)}(\{a\}) = (\mathsf{Q}_{\{1,2,3\}}, \mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{1,2\}}, \mathsf{Q}_{\{1,2,3\}})$.

$$\Lambda_{(3)} \text{ embeds in } L = \begin{pmatrix} 1&1&0&1&1\\ 1&0&1&1&1\\ 1&0&0&0&1\\ 1&0&0&1&1\\ 0&0&0&0&1 \end{pmatrix}, \quad \text{and } E = \begin{pmatrix} 0&0&1&0&-1\\ 1&0&0&-1&0\\ 0&1&0&-1&0\\ 0&0&-1&1&0\\ 0&0&0&0&1 \end{pmatrix}.$$

with actions on $\mathcal{Z}_{Pr^3,\ell_{(3)}}$ and $M^{12}_{(3)}$ derived as above.

The cohomology ring $H^*(M^{12}_{(3)})$ is given by $\mathrm{SR}(Pr^3)/\mathcal{J}$, where $\mathcal{J}$ is generated by the 3 elements $(u_1+u_2+u_4+u_5)$, $(u_1+u_3+u_4+u_5)$, and (u_1+u_5). Setting $u=u_1$, $v=u_4$ we obtain

$$H^*(M^{12}_{(3)}) \cong \mathbb{Z}[u,v]/(-u^2,\, v^3) \cong \mathbb{Z}[u,v]/(u^2,\, v^3) \qquad \cong B$$

with generators $(\mathbf{1}; u, v; uv, v^2; uv^2)$.

$M^{12}_{(4)}$: $\lambda_{(4)}(\{a\}) = (\mathsf{Q}_{\{3\}}, \mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{1,2\}}, \mathsf{Q}_{\{1,2,3\}})$.

$$\Lambda_{(4)} \text{ embeds in } L = \begin{pmatrix} 0&1&0&1&1\\ 0&0&1&1&1\\ 1&0&0&0&1\\ 0&0&0&1&1\\ 0&0&0&0&1 \end{pmatrix}, \quad \text{and } E = \begin{pmatrix} 0&0&1&0&-1\\ 1&0&0&-1&0\\ 0&1&0&-1&0\\ 0&0&0&1&-1\\ 0&0&0&0&1 \end{pmatrix}.$$

with actions on $\mathcal{Z}_{Pr^3,\ell_{(4)}}$ and $M^{12}_{(4)}$ derived as above.

The cohomology ring $H^*(M^{12}_{(4)})$ is given by $\mathrm{SR}(Pr^3)/\mathcal{J}$, where $\mathcal{J}$ is generated by the 3 elements $(u_2+u_4+u_5)$, $(u_3+u_4+u_5)$, and (u_1+u_5). Setting $u=u_1$, $v=u_2$ we obtain

$$H^*(M^{12}_{(4)}) \cong \mathbb{Z}[u,v]/(-u^2,\, v^2(u-v)) \cong \mathbb{Z}[u,v]/(u^2,\, (u-v)v^2) \quad \cong C$$

with generators $(\mathbf{1}; u, v; uv, v^2; uv^2 = v^3)$.

$M^{12}_{(5)}$**:** $\lambda_{(5)}(\{a\}) = (\mathsf{Q}_{\{3\}}, \mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{1,2,3\}}, \mathsf{Q}_{\{3\}})$.

$$\Lambda_{(5)} \text{ embeds in } L = \begin{pmatrix} 0&1&0&1&0\\ 0&0&1&1&0\\ 1&0&0&1&1\\ 0&0&0&1&1\\ 0&0&0&1&0 \end{pmatrix}, \quad \text{and } E = \begin{pmatrix} 0&0&1&-1&0\\ 1&0&0&0&-1\\ 0&1&0&0&-1\\ 0&0&0&0&1\\ 0&0&0&1&-1 \end{pmatrix}.$$

with actions on $\mathcal{Z}_{Pr^3,\ell_{(5)}}$ and $M^{12}_{(5)}$ derived as above.

The cohomology ring $H^*(M^{12}_{(5)})$ is given by $\mathrm{SR}(Pr^3)/\mathcal{J}$, where $\mathcal{J}$ is generated by the 3 elements (u_2+u_4), (u_3+u_4), and $(u_1+u_4+u_5)$. Setting $u=u_1$, $v=-u_5$ we obtain

$$H^*(M^{12}_{(5)}) \cong \mathbb{Z}[u,v]/(-uv,\, -(u-v)^3) \cong \mathbb{Z}[u,v]/(uv,\, u^3-v^3) \qquad \cong A$$

with generators $(\mathbf{1}; u, v; u^2, v^2; u^3 = v^3)$.

$M^{12}_{(6)}$**:** $\lambda_{(6)}(\{a\}) = (\mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{2,3\}}, \mathsf{Q}_{\{1,2,3\}}, \mathsf{Q}_{\{2\}})$.

$$\Lambda_{(6)} \text{ embeds in } L = \begin{pmatrix} 0&1&0&1&0\\ 1&0&1&1&1\\ 0&0&1&1&0\\ 0&0&1&1&1\\ 0&0&0&1&0 \end{pmatrix}, \quad \text{and } E = \begin{pmatrix} 0&1&0&-1&0\\ 1&0&0&0&-1\\ 0&0&1&0&-1\\ 0&0&0&0&1\\ 0&0&-1&1&0 \end{pmatrix}.$$

with actions on $\mathcal{Z}_{Pr^3,\ell_{(6)}}$ and $M^{12}_{(6)}$ derived as above.

The cohomology ring $H^*(M^{12}_{(6)})$ is given by $\mathrm{SR}(Pr^3)/\mathcal{J}$, where $\mathcal{J}$ is generated by the 3 elements (u_2+u_4), $(u_1+u_3+u_4+u_5)$, and (u_3+u_4). Setting $u=u_1$, $v=-u_4$ we obtain

$$H^*(M^{12}_{(6)}) \cong \mathbb{Z}[u,v]/(-u^2,\, -v^3) \cong \mathbb{Z}[u,v]/(u^2,\, v^3) \qquad \cong B$$

with generators $(\mathbf{1}; u, v; uv, v^2; uv^2)$.

$M^{12}_{(7)}$**:** $\lambda_{(7)}(\{a\}) = (\mathsf{Q}_{\{2\}}, \mathsf{Q}_{\{1\}}, \mathsf{Q}_{\{2,3\}}, \mathsf{Q}_{\{1,2,3\}}, \mathsf{Q}_{\{3\}})$.

$$\Lambda_{(7)} \text{ embeds in } L = \begin{pmatrix} 0&1&0&1&0 \\ 1&0&1&1&0 \\ 0&0&1&1&1 \\ 0&0&1&1&0 \\ 0&0&0&1&0 \end{pmatrix}, \quad \text{and } E = \begin{pmatrix} 0&1&0&-1&0 \\ 1&0&0&0&-1 \\ 0&0&0&1&-1 \\ 0&0&0&0&1 \\ 0&0&1&-1&0 \end{pmatrix}.$$

with actions on $\mathcal{Z}_{Pr^3,\ell_{(7)}}$ and $M^{12}_{(7)}$ derived as above.

The cohomology ring $H^*(M^{12}_{(7)})$ is given by $\mathrm{SR}(Pr^3)/\mathcal{J}$, where $\mathcal{J}$ is generated by the 3 elements (u_2+u_4), $(u_1+u_3+u_4)$, and $(u_3+u_4+u_5)$. Setting $u = u_1$, $v = -u_4$ we obtain

$$H^*(M^{12}_{(7)}) \cong \mathbb{Z}[u,v]/(-u^2,\ -v^2(v-u)) \cong \mathbb{Z}[u,v]/(u^2,\ (u-v)v^2) \cong C$$

with generators $(\mathbf{1}; u, v; uv, v^2; uv^2 = v^3)$.

Note that none of the three graded rings A, B, C is isomorphic to another.

$B \neq A$: Suppose $h\colon A \to B$ is a ring homomorphism, and suppose that u, v are generators of B with $0 = u^2 = v^3$, and that x, y are generators of A with $0 = xy = x^3 - y^3$. Let $h(x) = au + bv$, $h(y) = cu + dv$, with integer coefficients. Then setting $0 = h(xy) = (au+bv)(cu+dv) = (ad+bc)uv + bdv^2$ gives $ad + bc = bd = 0$, since $u^2 = 0$, and uv and v^2 are distinct quadratic generators. If $b = 0$, then $ad = 0$, so either $a = b = 0$ ($h(x) = 0$) or $b = d = 0$ ($c\,h(x) = a\,h(y) = acu$), or if $d = 0$, then $bc = 0$, so either $b = d = 0$ ($c\,h(x) = a\,h(y) = acu$) or $c = d = 0$ ($h(y) = 0$). In all cases h is not surjective, so cannot be an isomorphism.

$C \neq A$: Suppose $h\colon A \to C$ is a ring homomorphism, and suppose that u, v are generators of B with $0 = u^2 = uv^2 - v^3$, and that x, y are generators of A with $0 = xy = x^3 - y^3$. Let $h(x) = au + bv$, $h(y) = cu + dv$, with integer coefficients. Then setting $0 = h(xy) = (au+bv)(cu+dv) = (ad+bc)uv + bdv^2$ gives $ad + bc = bd = 0$. If $b = 0$, then $ad = 0$, so either $a = b = 0$ ($h(x) = 0$) or $b = d = 0$ ($c\,h(x) = a\,h(y) = acu$), or if $d = 0$, then $bc = 0$, so either $b = d = 0$ ($c\,h(x) = a\,h(y) = acu$) or $c = d = 0$ ($h(y) = 0$). In all cases h is not surjective, so cannot be an isomorphism.

$C \neq B$: Suppose $h\colon C \to B$ is a ring homomorphism, and suppose that u, v are generators of B with $0 = u^2 = v^3$, and that x, y are generators of C with

$0 = x^2 = xy^2 - y^3$. Let $h(x) = au + bv$, $h(y) = cu + dv$, with integer coefficients. Then setting $0 = h(x^2) = (au + bv)^2 = 2abuv + b^2v^2$ gives $b = 0$, and $0 = h(xy^2 - y^3) = ((a - c)u - dv)(2cduv + d^2v^2) = (ad^2 - cd^2 - 2cd^2)uv^2$ so $(a - 3c)d^2 = 0$. Hence, $h\colon \alpha x + \beta y \mapsto c(3\alpha + \beta)u + d\beta v$, and the element u is never in the image of h, so h is not surjective, and cannot be an isomorphism.

The map $h\colon C \to B$ is, however, injective for any non-zero values of c and d, so C is isomorphic to a proper subring of B. A similar analysis for $h\colon B \to C$ shows that B is also isomorphic to a proper subring of C.

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