On the topology of sums in powers of an algebraic number

by

NIKITA SIDOROV (Manchester) and BORIS SOLOMYAK (Seattle, WA)

1. Introduction and auxiliary results. Let \( q \in (1, 2) \) and put

\[
\Lambda_n(q) = \left\{ \sum_{k=0}^{n} a_k q^k \mid a_k \in \{-1, 0, 1\} \right\},
\]

and \( \Lambda(q) = \bigcup_{n \geq 1} \Lambda_n(q) \). (It is obvious that the sets \( \Lambda_n(q) \) are nested.) The question we want to address is the topological structure of \( \Lambda(q) \). Is it dense? discrete? mixed?

The first important result has been obtained by A. Garsia [12]: if \( q \) is a Pisot number (an algebraic integer greater than 1 whose conjugates are less than 1 in modulus), then \( \Lambda(q) \) is uniformly discrete. On the other hand, if \( q \) does not satisfy an algebraic equation with coefficients 0, \( \pm 1 \), then it is a simple consequence of the pigeonhole principle that 0 is a limit point of \( \Lambda(q) \), and thus it is dense—see below.

Surprisingly little is known about the case when \( q \) is a root of a polynomial with coefficients 0, \( \pm 1 \). The most notable result is [11, Theorem I] in which the authors prove in particular that if \( q < (1 + \sqrt{5})/2 \) and \( q \) is not Pisot, then \( \Lambda(q) \) has a finite accumulation point.

In this paper we study this case and give two sufficient conditions for \( \Lambda(q) \) to be dense. These conditions are rather general and cover a substantial subset of such \( q \)'s—see Theorems 2.1 and 2.4.

Put

\[
Y_n(q) = \left\{ \sum_{k=0}^{n} a_k q^k \mid a_k \in \{0, 1\} \right\}
\]

and \( Y(q) = \bigcup_{n \geq 1} Y_n(q) \). The set \( Y(q) \) is discrete and we can write its elements in ascending order:

\[
Y(q) = \{0 = y_0(q) < y_1(q) < y_2(q) < \cdots \}.
\]

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Following [11], we define

\[ l(q) = \lim_{n \to \infty} (y_{n+1}(q) - y_n(q)). \]

**Theorem 1.1** ([8]). If 0 is a limit point of \( \Lambda(q) \), then \( \Lambda(q) \) is dense in \( \mathbb{R} \).

It is obvious that 0 is a limit point of \( \Lambda(q) \) if and only if \( l(q) = 0 \). This yields

**Corollary 1.2.** The set \( \Lambda(q) \) is dense in \( \mathbb{R} \) if and only if \( l(q) = 0 \).

The purpose of this paper is to find some wide classes of algebraic \( q \) for which \( l(q) = 0 \).

Put for any \( \beta \in \mathbb{C} \),

\[ Y_n(\beta) = \left\{ \sum_{k=0}^{n} a_k \beta^k \mid a_k \in \{0, 1\}, 0 \leq k \leq n \right\} \]

and \( z_n(\beta) := \#Y_n(\beta) \). It is obvious that \( z_n(\beta) \leq 2^{n+1} \).

In order to estimate \( z_n(\beta) \) for \( |\beta| > 1 \), it is useful to consider the set

\[ A_\lambda := \left\{ \sum_{k=0}^{\infty} a_k \lambda^k \mid a_k \in \{0, 1\}, k \geq 0 \right\}, \quad \text{where } \lambda = \beta^{-1}. \]

We have \( |\lambda| < 1 \), so the series converges for any choice of the coefficients \( a_k \in \{0, 1\} \). It is easy to see that the set \( A_\lambda \) is compact, being the image of the infinite product space \( \{0, 1\}^\infty \) under a continuous mapping. It satisfies the set equation

\[ A_\lambda = \lambda A_\lambda \cup (1 + \lambda A_\lambda), \]

and can be characterized as the unique compact set with this property [14]. It is thus the attractor of the iterated function system \( \{z \mapsto \lambda z, z \mapsto \lambda z + 1\} \) in the complex plane; see [14] for details.

The sets \( A_\lambda \) with \( |\lambda| < 1 \) have been extensively studied in the “fractal” literature; see e.g. [2, 4, 15, 21] and the book [3, Chapter 8.2]. Note that some of these sources are concerned with the sets

\[ \tilde{A}_\lambda := \left\{ \sum_{k=0}^{\infty} a_k \lambda^k \mid a_k \in \{-1, 1\}, k \geq 0 \right\}, \]

however, it is clear that \( A_\lambda = T(\tilde{A}_\lambda) \), where \( T(z) = \frac{1}{2}(z + (1 - \lambda)^{-1}) \), so all the results immediately transfer.

**Lemma 1.3.**

(i) If \( \lambda \in \mathbb{C} \) with \( |\lambda| \in (1/2, 1) \), then \( z_n(\lambda) = \#Y_n(\lambda) \geq |\lambda|^{-n-1} \) for all \( n \).
(ii) If \( \lambda \in \mathbb{C} \) with \( 2^{-1/2} \leq |\lambda| < 1 \), and \( |\text{Re} \lambda| \leq |\lambda|^2 - 1/2 \), then
\[
z_n(\lambda) \geq |\lambda|^{-2(n+1)}
\]
for all \( n \geq 0 \).

Proof. By the definition of the set \( A_\lambda \), we have, for all \( n \geq 0 \),
\[
A_\lambda = \bigcup_{z \in Y_n(\lambda)} (z + \lambda^{n+1}A_\lambda).
\]

(i) Suppose that the set \( A_\lambda \) is connected, and let \( u, v \in A_\lambda \) be such that
\[ |u - v| = \text{diam}(A_\lambda). \]
Then there exists a “chain” of distinct subsets \( A_j := z_j + \lambda^nA_\lambda \subseteq A_\lambda, j = 1, \ldots, m \), with \( z_j \in Y_n(\lambda) \), such that \( u \in A_1, v \in A_m \) and \( A_j \cap A_{j+1} \neq \emptyset \) for all \( j \leq m - 1 \). Therefore,
\[
diam(A_\lambda) \leq \sum_{j=1}^{m} diam(A_j) = m diam(\lambda^{n+1}A_\lambda)
\]
and the claim follows. If, on the other hand, \( A_\lambda \) is disconnected, then
\[ \lambda A_\lambda \cap (\lambda A_\lambda + 1) = \emptyset. \]
This is a general principle for attractors of iterated function systems with two contracting maps (see [13, 4] or [3, Chapter 8.2]). Therefore, in this case \( \lambda \) is not a zero of a power series with coefficients \( \{-1, 0, 1\} \), much less of a polynomial, hence \( z_n(\lambda) = 2^{n+1} > |\lambda|^{-n-1} \) for all \( n \).

(ii) By [21, Prop. 2.6(i)], in view of the above remark concerning \( \tilde{A}_\lambda \), we know that \( A_\lambda \) has nonempty interior for all \( \lambda \) in the open unit disc such that
\[ 0 \leq |\text{Re} \lambda| \leq |\lambda|^2 - 0.5. \]
Then from (1.1), for the Lebesgue measure \( L^2 \), we have
\[
L^2(A_\lambda) \leq \#Y_n(\lambda)L^2(\lambda^{n+1}A_\lambda) = z_n(\lambda)|\lambda|^{2(n+1)}L^2(A_\lambda),
\]
as desired. 

Note that the proof of Lemma 1.3 does not use the fact that \( \lambda \) is nonreal. Hence we obtain the following result as a direct corollary:

**Lemma 1.4.** If \( q \in (1, 2) \), then \( z_n(\pm q) \geq Cq^n \) for some \( C > 0 \).

**Remarks 1.5.**

(i) Lemma 1.4 for \( + q \) was proved in [11], using the fact that
\[ y_{n+1}(q) - y_n(q) \leq 1 \]
for all \( n \) and any \( q \in (1, 2) \).

(ii) With a bit more work one can show that in the setting of Lemma 1.3(i) we have \( z_n(\lambda) \geq C_n|\lambda|^{-n} \) for some \( C_n \uparrow \infty \), assuming that \( \lambda \) is nonreal. However, this is not needed in this paper.

(iii) It follows from the results of [7, 17] that for any \( \varphi \neq 0, \pi \), the set \( A_\lambda \) has nonempty interior for \( \lambda = re^{i\varphi} \) with \( r \) sufficiently close to 1, but it seems difficult to apply them in the absence of quantitative estimates.
Lemma 1.6. If $\beta \in \mathbb{C} \setminus \{0\}$, then $z_n(\beta) = z_n(1/\beta)$.

Proof. Define $\phi : Y_n(\beta) \rightarrow Y_n(1/\beta)$ as follows:

$$\phi\left(\sum_{k=0}^{n} a_k \beta^k\right) = \sum_{k=0}^{n} a_{n-k}(1/\beta)^k.$$ 

A relation $\sum_{k=0}^{n} a_k \beta^k = \sum_{k=0}^{n} b_k \beta^k$ is equivalent to $\sum_{k=0}^{n} a_k \beta^{k-n} = \sum_{k=0}^{n} b_k \beta^{k-n}$, which in turn is equivalent to $\phi(\sum_{k=0}^{n} a_k \beta^k) = \phi(\sum_{k=0}^{n} b_k \beta^k)$. Thus, $\phi$ is a bijection. ■

Lemma 1.7. Let $q \in (1, 2)$. If $z_n(q) \gg q^n$ (i.e., $\lim_{n \rightarrow \infty} q^{-n} z_n(q) = +\infty$), then $l(q) = 0$.

Proof. Since $\sum_{k=0}^{n} a_k q^k < q^{n+1}/(q - 1)$, the result follows immediately from the pigeonhole principle. ■

Consequently, if $q$ is not a root of a polynomial with coefficients $0, \pm 1$, then $z_n(q) = 2^{n+1}$, and $l(q) = 0$ (which is well known, of course—see, e.g., [8]). If $q$ is such a root, it is obvious that $z_n(q) \ll 2^n$, and the problem becomes nontrivial. It is generally believed that $l(q) = 0$ unless $q$ is Pisot, but this is probably a very tough conjecture.

2. Main results. We need some preliminaries. Put

$$L(q) = \lim_{n \rightarrow \infty} (y_{n+1}(q) - y_n(q)).$$

Note that $L(q) = 0$ is equivalent to $y_{n+1}(q) - y_n(q) \rightarrow 0$ as $n \rightarrow \infty$. This condition was studied in the seminal paper [11]; in particular, it was shown that if $q < 2^{1/4} \approx 1.18921$ and $q$ is not equal to the square root of the second Pisot number $\approx 1.17485$, then $L(q) = 0$ [1]. It was also shown in the same paper that $L(\sqrt{2}) = 0$.

It is worth noting that the two conditions $l(q) = 0$ and $L(q) = 0$ are, generally speaking, very different in nature; for instance, $l(q) = 0$ for all transcendental $q$, whereas $L(q) = 1$ for all $q \geq (1 + \sqrt{5})/2$ (see, e.g., [10]) and no $q \in (\sqrt{2}, (1 + \sqrt{5})/2)$ with $L(q) = 0$ is known.

Throughout this section we assume that $q \in (1, 2)$ is a root of a polynomial with coefficients $0, \pm 1$. It is easy to show that in this case any conjugate of $q$ is less than 2 in modulus.

Finally, recall that an algebraic integer $q > 1$ is called a Perron number if each of its conjugates is less than $q$ in modulus.

Theorem 2.1. If $q \in (1, 2)$ is not a Perron number, then $l(q) = 0$. If, in addition, $q < \sqrt{2}$ and $-q$ is not a conjugate of $q$, then $L(q) = 0$.

(1) V. Komornik has recently shown [16] that the second condition can be removed, so $L(q) = 0$ if $q < 2^{1/4}$. 
Proof. We first prove \( l(q) = 0 \). We have three cases.

Case 1: \( q \) has a real conjugate \( p \) and \( q < |p| \). Since \( p \) is an algebraic conjugate of \( q \), it follows from Galois theory that the map \( \psi : Y_n(q) \rightarrow Y_n(p) \) given by \( \psi(\sum_{i=0}^{n} a_i q^i) = \sum_{i=0}^{n} a_i p^i \) is a bijection. Hence \( z_n(q) = z_n(p) \geq C|p|^n \) by Lemma 1.4 and \( z_n(q) \gg q^n \). Now the claim follows from Lemma 1.7.

Case 2: \( q \) has a complex nonreal conjugate \( p \) and \( q < |p| \). This case is similar to Case 1: \( z_n(q) = z_n(p) \geq C|p|^n \) by Lemma 1.3(i) and \( z_n(q) \gg q^n \).

Case 3: \( q \) has a conjugate \( p \) and \( q = |p| \). Let \( f \) denote the minimal polynomial for \( q \). Then \( f(x) = g(x^m) \) for some \( m \geq 2 \) by [6]. Put \( \beta = q^m \). We have

\[
Y_{mk}(q) = \{ a_0 + a_1 \beta^{1/m} + a_2 \beta^{2/m} + \cdots + a_{mk} \beta^m \mid a_i \in \{0, 1\} \}
= \{ A_1 + \beta^{1/m} A_2 + \beta^{2/m} A_3 + \cdots + \beta^{(m-1)/m} A_m : A_1 \in Y_k(\beta), A_i \in Y_{k-1}(\beta), 2 \leq i \leq m \}.
\]

Observe that any relation of the form

\[
A_1 + \beta^{1/m} A_2 + \cdots + \beta^{(m-1)/m} A_m = A'_1 + \beta^{1/m} A'_2 + \cdots + \beta^{(m-1)/m} A'_m
\]

implies \( A_1 = A'_1, \ldots, A_m = A'_m \). Indeed, if \( q \) satisfies an equation \( B_1 + qB_2 + \cdots + q^{m-1} B_m = 0 \) with \( B_i \in \mathbb{Z}[q^m] \), then \( q e^{2\pi ij/m} \) satisfies the same equation for \( j = 1, \ldots, m - 1, \) hence \( B_i = 0 \) for all \( i \). Thus, \( z_{mk}(\beta^{1/m}) = z_k(\beta)(z_{k-1}(\beta))^{m-1} \).

Now, if \( q \geq 2^{1/m} \), then \( \beta \geq 2 \), so \( z_k(\beta) = 2^{k+1} \), and we see from the above argument that for \( n = mk \) we have \( z_n(q) \geq C 2^n \gg q^n \). Otherwise \( z_n(q) \geq z_n(\beta) \geq C \beta^n \gg q^n \). Hence by Lemma 1.7, \( l(q) = 0 \).

Let us now prove the second part of the theorem. Suppose \( q < \sqrt{2} \) is not Perron and \( -q \) is not its conjugate; then \( q \) has a conjugate \( \alpha \neq -q \) with \( |\alpha| \geq q \). Thus, \( q^2 \) has a conjugate \( \alpha^2 \), and \( |\alpha|^2 \geq q^2 \) with \( \alpha^2 \neq q^2 \).

If \( |\alpha| > \sqrt{2} \), then \( \alpha^2 \) (and hence \( q^2 \)) is not a root of a \(-1, 0, 1\) polynomial. Otherwise, we can apply the first part of this theorem to \( q^2 \). In either case, \( l(q^2) = 0 \), whence by [10] Theorem 5, \( L(q) = 0 \).

Remark 2.2. Stankov [22] has proved a similar result for the set

\[
(2.1) \quad A(q) = \left\{ \sum_{k=0}^{n} a_k q^k \mid a_k \in \{-1, 1\}, n \geq 1 \right\}.
\]

More precisely, he has shown that if \( A(q) \) is discrete, then all real conjugates of \( q \) are of modulus strictly less than \( q \).

Corollary 2.3. If \( q \in (1, 2) \) is the square root of a Pisot number and not itself Pisot, then \( l(q) = 0 \).
Proof. If \( q = \sqrt{\beta} \) and \( \beta \) is Pisot, then either \( -q \) is a conjugate of \( q \), or \( q \) is Pisot. ■

**Theorem 2.4.**

(i) Suppose \( q \in (1, 2) \) has a conjugate \( \alpha \) such that \( |\alpha|q < 1 \). Then \( l(q) = 0 \), and consequently \( \Lambda(q) \) is dense in \( \mathbb{R} \).

(ii) Suppose \( q \in (1, 2) \) has a nonreal conjugate \( \alpha \) such that \( |\alpha|q = 1 \). Then \( l(q) = 0 \).

If, in addition, \( q < \sqrt{2} \) in either case, then \( L(q) = 0 \).

**Proof.** (i) As above, we have \( z_n(q) = z_n(\alpha) \). On the other hand, by Lemma 1.6 \( z_n(\alpha) = z_n(1/\alpha) \), and by Lemmas 1.4 and 1.3 \( z_n(1/\alpha) \geq C(|1/\alpha|)^n \). Hence \( z_n(q) \geq C(|1/\alpha|)^n \gg q^n \), in view of \( |\alpha q| < 1 \). Hence by Lemma 1.7 \( l(q) = 0 \).

If \( q < \sqrt{2} \), then \( q^2 \) has a conjugate \( \alpha^2 \), and \( q^2|\alpha|^2 < 1 \). Hence \( l(q^2) = 0 \), whence \( L(q) = 0 \).

(ii) Denote \( \alpha_1 = q \), \( \alpha_2 = \alpha \), and \( \alpha_3 = \overline{\alpha} \). Since \( |\alpha|q = 1 \) and \( \alpha \) is nonreal, we have three conjugates satisfying \( \alpha_1^2\alpha_2\alpha_3 = 1 \). Smyth [20, Lemma 1] characterizes such situations, but it is easier for us to proceed directly. The Galois group of the minimal polynomial for \( q \) is transitive, so there is an automorphism of the Galois group mapping \( \alpha_1 \) to \( \alpha_2 \). We deduce that \( \alpha_2^2\alpha_i\alpha_j = 1 \) for some distinct conjugates \( \alpha_i \) and \( \alpha_j \) of \( \alpha_1 \). But this implies \( \max\{|\alpha_i|, |\alpha_j|\} \geq \alpha_1 = q \), hence \( q \) is not a Perron number, and \( l(q) = 0 \) by Theorem 2.1.

If \( q < \sqrt{2} \), then \( q^2|\alpha|^2 = 1 \), and (ii) applies to \( q^2 \), unless \( \alpha^2 \in \mathbb{R} \). In the latter case \( \alpha = \pm i/q \), whence the minimal polynomial for \( q \) contains only powers divisible by 4. Hence the minimal polynomial for \( q^2 \) contains only even powers, which implies that \( -q^2 \) is conjugate to \( q^2 \), whence \( q^2 \) is not Perron, and \( l(q^2) = 0 \). ■

**Remark 2.5.** If \( |\alpha|q = 1 \) and \( \alpha \) is real, we do not know if \( l(q) = 0 \). In fact, this includes the interesting (and probably, difficult) case of Salem numbers (2).

**Definition 2.6.** We say that an algebraic integer \( q > 1 \) is anti-Pisot if it has only one conjugate less than 1 in modulus and at least one conjugate greater than 1 in modulus other than \( q \) itself.

**Corollary 2.7.** If \( q \in (1, 2) \) is anti-Pisot and also a root of a polynomial with coefficients in \( \{-1, 0, 1\} \), then \( l(q) = 0 \).

(2) Recall that an algebraic number \( q > 1 \) is called a Salem number if all its conjugates have absolute value no greater than 1, and at least one has absolute value exactly 1.
Proof. Let \( \alpha = \alpha_1, \alpha_2, \ldots, \alpha_{k-1}, q \) be all the conjugates of \( q \). We have
\[
|\prod_{j=1}^{k-1} \alpha_j| q = 1,
\]
because \( q \) satisfies an algebraic equation with coefficients 0, \( \pm 1 \), whence its minimal polynomial must have a constant term \( \pm 1 \).

Suppose \( |\alpha| < 1 \); then it is clear that \( \alpha \in \mathbb{R} \) (since it is unique). If \( |\alpha_2| > 1 \) and \( |\alpha_j| \geq 1 \) for \( j = 3, \ldots, k-1 \), then it is obvious that \( |\alpha| q \leq |\alpha_2|^{-1} < 1 \), i.e., the condition of Theorem 2.4(i) is satisfied. \( \blacksquare \)

3. Examples

Example 3.1. Let \( q \approx 1.22074 \) be the positive root of \( x^4 = x + 1 \). Then \( q \) has a single conjugate \( \alpha \approx -0.72449 \) inside the open unit disc and no conjugates of modulus 1, whence \( q \) is anti-Pisot, and by Corollary 2.7
\[
l(q) = 0.
\]
Furthermore, \( q < \sqrt{2} \), whence \( L(q) = 0 \) as well.

Note that \( q > 2^{1/4} \), so we cannot derive the latter claim immediately from \([\prod \) Theorem IV].

Example 3.2. An example of \( q \) with a real conjugate \( \alpha \) which is not anti-Pisot but still satisfies the condition of Theorem 2.4(i), is the appropriate root of \( x^5 = x^4 + x^2 + x - 1 \). Here \( q \approx 1.52626 \) and \( \alpha \approx 0.59509 \).

Example 3.3. For the equation \( x^5 = x^4 - x^2 + x + 1 \) we have \( q \approx 1.26278 \) and \( |\alpha| \approx 0.74090 \) so \( |\alpha| q \approx 0.93559 \) (and \( \alpha \notin \mathbb{R} \)). By Theorem 2.4(i), \( L(q) = 0 \).

Example 3.4. For the equation \( x^8 = x^7 + x^6 + x^5 - x^4 - x^3 - x^2 + x - 1 \) we have \( q \approx 1.52501 \). Among its conjugates is \( \alpha \approx 0.3741 + 0.52404i \) with \( |\alpha| \approx 0.64387 < 1/q = 0.65574 \), so again \( l(q) = 0 \) by Theorem 2.4(i). Note that \( q > \sqrt{2} \) so we cannot claim \( L(q) = 0 \).

Example 3.5. The following example illustrates Theorem 2.4(ii). Let \( q \approx 1.19863 \) be the largest root of \( x^{12} = x^9 + x^6 + x^3 - 1 \); then \( \alpha = \zeta q^{-1} \) is a root of this equation as well, where \( \zeta \) is any complex nonreal cubic root of unity. Hence \( q|\alpha| = 1 \), and Theorem 2.4(ii) applies, i.e., \( L(q) = 0 \). Note that \( q = \sqrt[3]{3} \), where \( \beta \) is a quartic Salem number.

Example 3.6. For the equation \( x^{11} = x^{10} + x^9 - x^6 + x^4 - x^2 - 1 \) we have \( q \approx 1.5006 \). Among its conjugates is \( \lambda \approx 0.02625 + 0.7414i \). Theorem 2.4 does not apply, but we can use Lemma 1.3(ii) to obtain
\[
z_n(q) = z_n(\lambda) \geq |\lambda|^{-2(n+1)} \approx 1.81696^{n+1} \gg q^n,
\]
which implies that \( l(q) = 0 \). Note that Lemma 1.3(ii) indeed applies, because \( 0.02625 \approx \text{Re } \lambda < |\lambda|^2 - 1/2 \approx 0.05037 \).

Example 3.7. Consider the equation \( x^{18} = -x^{16} + x^{14} + x^{11} + x^{10} + \cdots + x + 1 \) (no powers missing between \( x^{10} \) and 1). It has a root \( q \approx 1.22289 \), and the conjugates largest in modulus are \( u, \overline{u} \) approximately equal to \( -.03958 \pm 1.3109i \). Then Theorem 2.1 implies \( L(q) = 0 \).
It is worth mentioning that there is another way to obtain this result. Consider $q^2$ and its conjugates $u^2, \overline{u}^2$. We claim that although $|u^2| < 2$, $u^2$, and hence $q^2$, is not a zero of a $-1,0,1$ polynomial (whence $l(q^2) = 0$, which implies $L(q) = 0$).

Indeed, if it were, then $q^{-2}, u^{-2}, (\overline{u})^{-2}$ would also be zeros of such a polynomial. However, the product of these three numbers is $\approx 0.226024$, so this is impossible, in view of the following

**Claim.** Suppose $z_1, z_2, z_3$ are three different roots of a $-1,0,1$ polynomial. Then $|z_1z_2z_3| \geq 1/2 \cdot (4/3)^{-3/2} = 0.32476\ldots$.

This claim is a slight generalization of [5, Theorem 2]; see [19, Theorem 2.4].

**Example 3.8.** Finally, an example of $q$ for which none of our criteria works is the real root of $x^5 = x^4 + x^3 - x + 1$. Here $q \approx 1.54991$, and the other four conjugates are nonreal, with the moduli $\approx 1.04492$ and $\approx 0.76871$ respectively.

Another example is any Salem number $q \in (1,2)$, for instance $q \approx 1.72208$ which is a root of $x^4 = x^3 + x^2 + x - 1$ (which is of course none other than $\beta$ from Example 3.5).

4. **Final remarks and open problems**

4.1. Our first remark concerns the case $q \in (m, m+1)$ with $m \geq 2$. Here the natural definition for $A(q)$ is

$$A(q) = \left\{ \sum_{k=0}^{n} a_k q^k \mid a_k \in \{-m, -m + 1, \ldots, m - 1, m\}, \; n \geq 1 \right\}.$$  

Theorem 2.4 holds for this case, provided $\alpha \in \mathbb{R}$ (and so does Case 1 of Theorem 2.1)—the proof is essentially the same. The case of nonreal $\alpha$ is less straightforward, since there is no ready-to-apply complex machinery for $m \geq 2$. (Basically, we need that if $\alpha$ is a zero of a polynomial with coefficients in $\{-m, \ldots, m\}$, then the attractor of the iterated function system $\{\alpha z + j\}_{j=0}^{n}$ in the complex plane is connected. This can be verified for $m = 2,3$ but we do not know if this is true in general.) Note also that an analogue of Theorem 1.1 for $m \geq 2$ has been proved in [9].

4.2. We do not know whether the extra condition that $-q$ is not a conjugate of $q$ is really necessary in the second claim of Theorem 2.1. In particular, is it true that $L(\sqrt{\varphi}) = 0$ if $\varphi$ is the golden ratio?

4.3. In [18, Proposition 1.2] it is shown that if $q < \sqrt{2}$ and $q^2$ is not a root of a polynomial with coefficients $0, \pm 1$, then the set $A(q)$ given by (2.1) is dense in $\mathbb{R}$. In fact, what the authors use in their proof is the condition $l(q^2) = 0$. Consequently, Theorems 2.1 and 2.4 provide sufficient conditions
for \( A(q) \) to be dense in the case when \( q^2 \) satisfies an algebraic equation with coefficients 0, \( \pm 1 \).

4.4. Is \( l(q) = 0 \) for \( q \) in Example 3.8 and suchlike?

4.5. All our criteria suggest that \( l(q) = 0 \) implies \( L(q) = 0 \) for \( q < \sqrt{2} \). Is this really the case?

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**Added in proof** (June 2011). In the recent paper by Sh. Akiyama and V. Komornik [1] several results mentioned in the introductory part of the present paper have been significantly improved, namely:

- If \( q \in (1, \sqrt{2}] \) is non-Pisot, then \( l(q) = 0 \) and \( A(q) \) is dense in \( \mathbb{R} \).
- If \( q \in (\sqrt{2}, 2) \) is non-Pisot, then \( A(q) \) has a finite accumulation point.
- For \( q \in (1, 2^{1/3}] \) we have \( L(q) = 0 \).

**References**

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Nikita Sidorov
School of Mathematics
The University of Manchester
Oxford Road
Manchester M13 9PL, United Kingdom
E-mail: sidorov@manchester.ac.uk

Boris Solomyak
Department of Mathematics
University of Washington
Box 354350
Seattle, WA 98195, U.S.A.
E-mail: solomyak@math.washington.edu

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