OPTIMAL STOCKPILES UNDER STOCHASTIC UNCERTAINTY

A thesis submitted to the University of Manchester for the degree of Doctor of Philosophy in the Faculty of Engineering and Physical Sciences

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We study stockpiling problems under uncertain economic and physical factors, and investigate the valuation and optimisation of storage systems where the availability and spot price of the underlying are both subject to stochasticity. Following a Real Options valuation approach, we first study financial derivatives linked to Asian options. A comprehensive set of boundary conditions is compiled, and an alternative (and novel) similarity reduction for fixed-strike Asian options is derived. Hybrid semi-Lagrangian methods for numerically solving the related partial differential equations (PDEs) are implemented, and we assess the accuracy of the valuations thus obtained with respect to results from classical finite-difference valuation methods and with respect to high precision calculations for valuing Asian options with spectral expansion theory techniques.

Next we derive a PDE model for valuing the storage of electricity from a wind farm, with an attached back-up battery, that operates by trading electricity in a volatile market in order to meet a contracted fixed rate of energy generation; this system comprises two diffusive-type (stochastic) variables, namely the energy production and the electricity spot price, and two time-like (deterministic) variables, specifically the battery state and time itself. An efficient and novel semi-Lagrangian alternating-direction implicit (SLADI) methodology for numerically solving advection-diffusion problems is developed: here a semi-Lagrangian approach for hyperbolic problems of advection is combined with an alternating-direction implicit method for parabolic problems involving diffusion. Efficiency is obtained by solving (just) tridiagonal systems of equations at every time step. The results are compared to more standard semi-Lagrangian Crank-Nicolson (SLCN) and semi-Lagrangian fully implicit (SLFI) methods.

Once we have established our PDE model for a storage-upgraded wind farm, a system that depends heavily on the highly stochastic nature of wind and the volatile market where electricity is sold, we derive a Hamilton-Jacobi-Bellman (HJB) equation for optimally controlling charging and discharging rates of the battery in time, and we assess a series of operation regimes. The solution of the related PDE models is approached numerically using our SLADI methodology to efficiently treat this mixed advection and diffusion problem in four dimensions. Extensive numerical experimentation confirms our SLADI methodology to be robust and yields highly accurate solutions and efficient computations, we also explore effects from correlation between stochastic electricity generation and random prices of electricity as well as effects from a seasonal electricity spot price. Ultimately, the objective of approximating optimal storage policies for a system under uncertain economic and physical factors is accomplished.

Finally we examine the steady-state solution of a stochastic storage problem under uncertain electricity market prices and fixed demand. We use a HJB formulation for optimally controlling charging and discharging rates of the storage device with respect to the electricity spot price. A projected successive over-relaxation coupled with the semi-Lagrangian method is implemented, and we explore the use of boundary-fitted coordinates techniques.
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Chapter 1

Introduction

1.1 Overview

Storage systems have received much attention as they serve many applications for processing or utilising resources in systems, particularly those systems subject to variation on the availability or consumption of the underlying. For example physical factors such as temperature, rain fall, sun light, ice or snow can affect the availability or the requirements of a given commodity, also economical factors such as interest rates, exchange rates, commodity prices or seasonal trends can have effects on the production and demand of the related commodity.

Then, for a production system with variable output, stockpiles serve as a buffer that can be used to shift the availability of units from times when the underlying production is in surplus, to times when the underlying production is in deficit with respect to a particular level of demand. For a commodity that is mostly sold during a particular season in the year but requiring to be produced during the full year, and assuming higher prices during the peak demand, stockpiles can be used to shift units from low price (not in season) to high price (in season) in the market. Another situation can be a system with unpredictable demand and a stable level of production, in this case stockpiles can be used to shift units from times with low demand to times with high demand in the market. There can be a production system that processes raw materials coming from a distant country, with variable currency exchange rates then there can be a large effect on the cost faced by the producer, and so stockpiles can be used to keep reserves for times with adverse exchange rates.
Uncertainty will cause times when the system has a gain but also times when the system has a loss, and so storage systems can help with reducing negative effects. Basically stockpiles prove their usefulness by ensuring the availability of resources when the underlying is more valuable, but in order to ensure this deliverability there is a stockpiling requirement such that the underlying resource is stored when it is more conveniently obtained.

Given the potential practical benefits of storage by helping to guarantee the availability of resources we identify that there is some value for stockpiling, it is then important to determine the value of storage for systems subject to uncertainty, as well as for appraising investment projects with these kinds of facilities. Here we investigate the valuation and optimisation of storage systems subject to uncertain economic and physical factors.

A particular application of interest is that of energy generation utilising renewable resources. As it cannot be fully predicted the amount of a resource that will be available at a given moment, then it is not possible to deliver specific amounts of energy. A storage device can be used to shift energy from times when there is a surplus of the resource (by storing units of energy) to times when there is a deficit of the resource (by depleting stored units of energy). In this thesis, the principal usage that we give to storage is for smoothing fluctuations of the output, and we implement this strategy on a wind-powered energy generator.

In addition to the practical benefits of valuing storage systems, the models require sophisticated algorithms to efficiently obtain accurate solutions, as the equations are often difficult to solve. Given the assumption of uncertainty, then we estimate the value of stochastic storage with a model relating a small variation in the value of the system with a small fluctuation of the input variables. The model includes predictable movements and unpredictable variations of the related factors. As stated above these could be physical (temperature, rain fall, wind speed, etc.) or economical (commodity spot price, interest rates, etc.), then the model accounts for any small income or outcome related to the small fluctuations of the related input variables to find the value of the storage system. Additionally, by determining these variations the model can be extended to determine the convenience of moving units into storage, to keep units in storage or to remove units from storage, these by comparing the magnitude of
CHAPTER 1. INTRODUCTION

a potential change in value of the system with respect to a small change in the input variables.

The financial model for Asian options captures the essence of stochastic storage in physical systems. Both problems feature an advection-diffusion process, where deterministic (advection) effects are mixed with stochastic (diffusive) effects. The following example relates to both problems. Consider the salt stockpiling problem for regions with snow during winter season; salt is used to melt snow and keep the road network transitable. Salt must be purchased continuously or discretely at the price per ton on the market and is stockpiled for a period of time before it is required. Assume the presence of snow, then the salt is sold at a predefined price per ton or at the prevailing price per ton on the market. The total cost of the stockpile can be obtained from multiplying the average cost by the total amount of salt, and the income from sales is obtained from multiplying the amount of salt times the selling price. Then the maximum loss that can be incurred per ton is the difference between the average cost and the selling price, and a profit is obtained when the selling price is larger than the average cost. A fixed-strike Asian call option or a floating-strike Asian put option can be bought to cover that possible loss, but the cost for issuing the corresponding Asian option contract must be enough to raise an amount of money equal to the possible loss from the issue of the contract.

Asian option models are financial representations that resemble the advection-diffusion features of a stockpiling problem, thus we use Asian option models to implement the Real Options approach. We conduct a literature survey that is mostly based on the partial differential equations (PDEs) approach, but also includes results from other areas like probability and spectral expansions theory for comparison purposes. The resulting PDE is then embedded into the Black-Scholes framework, and a comprehensive set of appropriate boundary conditions for different variants of the problem are given.

We present formulations of numerical methods that can be used to value such options. An overview of approximation methods using finite-differences is presented; the fully implicit method, the Crank-Nicolson method, and the semi-Lagrangian schemes are defined. Computational effort and accuracy are studied in detail and the results are compared to benchmark examples.
Next we develop a relevant model incorporating both economical and physical stochastic factors. We study the storage of electricity from a wind farm upgraded with a back-up battery; this system comprises two diffusive-type (stochastic) variables, namely the energy production and the electricity spot price, and two time-like variables, specifically the battery state and time itself. Using a Feynman-Kac framework we then obtain an advection-diffusion PDE-type problem whose solution is approached numerically.

A hybrid semi-Lagrangian alternating-direction implicit (SLADI) method is developed and implemented to value the derived PDE for the wind farm model. The accuracy and stability are tested and compared with those observed for more conventional semi-Lagrangian Crank-Nicolson and semi-Lagrangian fully implicit methodologies. We obtain a good comparison with the added benefit of efficiency as the alternating-direction methodology is incorporated into every time-step.

With efficient methodology at hand, we proceed to numerically determine the optimal storage operation regime for the wind farm system, by selecting the best rate for charging and discharging the battery at each point in state space. We use our SLADI methodology to numerically approximate the solution of the HJB equation for maximising the expected cash flows for managing electricity from the storage device alongside energy generation.

We also study steady-state solutions of related electricity storage systems. Here our objective is to define optimal storage operation regimes under uncertain electricity market prices and fixed demand. Here we implement a projected successive over-relaxation method coupled with the semi-Lagrangian technique to determine solutions of the stochastic storage problem.

1.1.1 Asian options

An Asian option is a contract written at time zero giving the holder the right to exercise it in the future at a predefined time, the expiry time. The payoff of an Asian option depends on some kind of price average, over a time period, of the underlying asset to be traded, and the payoff may depend on the final asset price or it may depend on a predefined strike price.

Common examples of these financial derivatives are the ‘fixed-strike’ and the
‘floating-strike’ Asian options. The holder of a fixed-strike Asian call option is entitled to receive (at expiry) the difference between the price average and the strike price, and the holder of a fixed-strike Asian put option is entitled to receive the difference between the strike price and the price average. In the case of a floating-strike Asian call option, the holder is entitled to receive (at expiry) the difference between the spot price and the price average. For a floating-strike Asian put option, the holder is entitled to receive (at expiry) the difference between the price average and the spot price. Asian options with fixed strike are also called average rate or average value options, whereas Asian options with floating strike are also called average strike options.

The problem of the valuation of Asian options has been studied from several perspectives e.g., analytical approximations, Monte Carlo simulations, numerical PDE methods, and numerical Laplace transform inverse algorithms. Since their introduction by Boyle and Emanuel (Boyle, 1993)\footnote{Boyle (1993) makes a note that the original paper by Boyle and Emanuel was not accepted for publication.}, Asian options have attracted the interest of many investigators and there is still work to be done on this research area.

Kemna and Vorst (1990) used Monte Carlo simulation for pricing options based on average asset values; they stated that it is impossible to obtain a formula to explicitly calculate the value of this kind of option, and thus the need to use numerical methods, as their simulations; finite-difference methods were discarded at that time because of the number of variables involved. Kemna and Vorst derived a PDE model to price the average value Asian option under the Black-Scholes hedging arguments for a stock with random price.

Rogers and Shi (1995) exploited a scaling property on the problem to reduce the dimensions from three to two dimensions, for both the fixed-strike and the floating-strike Asian options. They followed a martingale approach to derive their equation; thus the problem is stated in terms of expectations. In addition to the PDE formulation, lower and upper bounds for the value of the option were introduced, with a really accurate lower bound.

Zvan et al. (1998) developed robust numerical methods to solve Asian options PDEs. They enhanced flux-limiting techniques to treat convection-dominated PDEs, a regime that occurs when the velocity term is larger than the diffusion term, in other
words when the magnitude of the coefficient of a first-order term is large compared to the magnitude of the coefficient of the related second-order term, a condition that is prevalent with the absence of diffusion in one dimension (characteristic of Asian options). Their many examples, produced oscillatory solutions, showing how erroneous diffusion may be introduced by using standard methods (even for more routine European options), thus suggesting the use of more sophisticated techniques.

Večer (2001) observed that the Asian option is a special case of an option on a traded account and characterised the problem by the corresponding Hamilton-Jacobi-Bellman equation, which is a problem of stochastic optimal control, with resulting PDEs similar to forms considered in this thesis.

D’Halluin et al. (2005) followed a partial integro-differential equation (PIDE) approach to value fixed-strike Asian options of both European and American style; a hybrid semi-Lagrangian method was used in conjunction with fully implicit (FI), Crank-Nicolson (CN) and backward differencing (BDF) time stepping to value the price of fixed-strike Asian options with jump diffusion processes.

Hugger (2006) addressed the well-posedness of the boundary-value formulation for the fixed-strike Asian option; a complete set of boundary value conditions were obtained with financial information. Also elements of existence, uniqueness and smoothness were presented.

In an alternative approach to PDEs, Thompson (1999) improved the upper bound for the value of fixed-strike and floating-strike Asian options from Rogers and Shi (1995). Simplifications for the lower bounds formulae were obtained, requiring less computational effort, and the results confirmed the accuracy of the lower bounds obtained by Rogers and Shi.

Linetsky (2004) derived two analytical formulae for the value of continuously sampled arithmetic Asian options; the first with an infinite series of terms involving Whittaker functions $M$ and $W$, and the second is defined as a single real integral of an expression involving the Whittaker function $W$ and the addition of a finite number of terms from incomplete gamma functions and Laguerre polynomials. Linetsky used this spectral expansions to provide high precision results (ten digits accurate) to a series of benchmark examples for valuing Asian options, which include low volatility cases.
Dewynne and Shaw (2008) analysed the problem with asymptotic expansions and presented simplified PDEs and a simple derivation of the exact Laplace transform. The simplified equations were derived using the autonomous property in the ‘weighted running sum’ used to calculate Asian options.

Geman and Yor (1993) used Bessel functions to produce a formula for the Laplace transform of the value of Asian options. The Laplace transform formula was then implemented numerically by Shaw (1998) and efficiency and accuracy of these results were verified by Shaw (2000). Lewis (2002) proposed simplifications for the calculation of integral terms in the analytical inversion of the Laplace transformation for Asian options.

The valuation of Asian options has also been approached using a Monte Carlo (MC) approach by Fu et al. (1998), obtaining accurate estimates with using control variates. Zhang (2001) used a semi-analytical method to find the value of Asian options and derived an analytical approximation formula. Turnbull and Wakeman (1991) valued Asian options with an approximation for the probability distribution of the average, an approximation obtained with an Edgeworth series expansion (Abramowitz and Stegun, 1964).

In fact, Asian options do have an analytical expression albeit as an infinite sum, as a Laplace inversion (Geman and Yor, 1993) or spectral expansions Linetsky (2004), and the fixed and floating strike arithmetically-averaged options can be solved using a similarity reduction (Dewynne and Shaw, 2008). There is a rich availability of analytical approximations, Monte Carlo simulations, numerical PDE methods, and numerical Laplace transform inverse algorithms, and also very stringent bounds derived from expectations arguments (Thompson, 1999), and high precision values obtained with spectral expansions calculations (Linetsky, 2004). This previous work makes Asian options a standard (and challenging) benchmark for demonstrating the effectiveness and efficiency of new developments, which we use to assess our models and algorithms.

1.1.2 Storage systems

Thompson et al. (2009)\textsuperscript{2} introduced a partial integro-differential equation (PIDE)
model for the valuation and optimisation of natural gas storage facilities. The PIDE
model accounts for operational constraints of storage capacity, rates of injection and
extraction, as well as the possibility of alternating between injecting and extracting, the
objective being to maximise the cash flow from buying and selling gas with stochastic
spot price under the assumption of leakage. With the use of a Bellman-type equation
the value of the storage facility is then optimised by numerical solution.

Later, Thompson et al. (2004) derived models for the valuation and optimal opera-
tion of electricity power plants; their models feature the stochastic nature of electricity
spot prices and were defined to incorporate a second source of stochasticity from the
resources used to run hydroelectric and thermal generators. The nature of storage is
present in the hydroelectric generator model, as water is released from a reservoir to
generate electricity and water is pumped back to the reservoir to accumulate potential
energy. Even though the formulation was made with respect to two sources of ran-
doneness, Thompson et al. (2004) solved the problem assuming a constant value for
the resource’s random variable, thus solving only for the case of random prices of elec-
tricity. For the numerical solution, explicit finite-difference schemes with second-order
slope limiters were utilised, although those techniques are time-step restricted due to
stability issues (in our case time-steps can be decoupled from the space resolution,
such that a fully four-dimensional model can be efficiently solved).

Semi-Lagrangian schemes have been used throughout the gas storage valuation
research to solve the derived PDE and PIDE models, for example Chen and Forsyth
(2007) valued and optimised the operation of gas storage facilities based on the model
of Thompson et al. (2009); seasonality was additionally introduced into the stochastic
process.

Chen and Forsyth (2008) then priced hydroelectric power plants subject to op-
erational restrictions with a stochastic control approach. Later, Chen and Forsyth
(2009) presented implications of a regime-switching model on natural gas storage val-
uation and optimal operation, the objective being to reproduce natural gas spot prices
with more accuracy. The underlying stochastic process is allowed to jump between
two regimes (different combinations of mean-reverting and general Brownian motion
regimes).

A methodology for accurately valuing stochastic optimal control problems has been
developed recently. Ware (2013) has proposed a semi-Lagrangian time-stepping algorithm with a Fourier-cosine discretisation; as compared to other methodologies, this algorithm has demonstrated second-order convergence.

Ahn et al. (2002) used a flexible multi-factor model of forward curve dynamics to solve the associated stochastic control problem and conclude that the value of gas storage is highly impacted by the volatility term structure of gas futures. Results account for physical storage with limited injection and withdrawal rates, and also for virtual storage without restrictions (injection/withdrawal) as this is merely a financial instrument.

Manoliu (2004) approached the valuation of storage using a multilevel tree methodology by modelling a stochastic one-factor and two-factor diffusion spot price for the underlying, and also accounting for storage operational constraints. In the case of the two-factor diffusion commodity price, a quasi-analytical solution is presented when the storage operational constraints can be neglected.

For the valuation of natural gas storage facilities, Parsons (2013) compared results from trading simulations of a two-factor tree approach with respect to historical price data. The results from the test highlighted the expected optionality of storage facilities under mean-reverting spot prices, for both fast and low-cycle contracts.

The Monte Carlo method has also been used to tackle the valuation problem to lease gas storage facilities. For example Boogert and De Jong (2008) generalised the Least Squares Monte Carlo (LSMC) approach for American options for valuing gas storage facilities, Carmona and Ludkovski (2010) extended the Monte Carlo approach for valuing energy storage facilities and considered the gas storage problem and the hydroelectric pumped storage problem, whilst Lai et al. (2010) coupled an approximate dynamic method with the Monte Carlo approach for valuing the natural gas storage.

Howell et al. (2011) developed a partial differential equation system for modelling stochastic storage in physical systems. Long-term (perpetual) valuations for storage facilities were obtained for an application involving a random output (from a wind power generator) and the storage is used for smoothing out fluctuations. The model is related with the aforementioned gas storage problem and the hydroelectric pumped storage problem, all of these settings include dynamics for replenishing and consuming reserves in the storage, and account for the profit flow obtained after the storage
operation. The accuracy and speed of PDE numerical methods is highlighted, in comparison to simulation methods where computations tend to be slow and errors can be difficult to estimate.

1.1.3 Numerical PDE methods

Essentially Asian options and the storage related PDEs share the mixed advection and diffusion nature, which often poses difficulties for their numerical solution; erroneous diffusion may be introduced, producing oscillatory solutions when the magnitude of the advection coefficients is large compared to the magnitude of the diffusion coefficients. Valuing Asian options is well known to be prone to these numerical sensitivities (see for example, Zvan et al., 1998; Rogers and Shi, 1995; D’Halluin et al., 2005). Other problems, including many from the physical sciences, also possess advection-diffusion features and the algorithms used to solve these numerically must also address similar difficulties (see Spiegelman and Katz, 2006).

Recent developments are taking advantage of more suitable (and sophisticated) computational techniques to treat mixed advection and diffusion problems. Here we present a semi-Lagrangian alternating-direction implicit (SLADI) methodology, whose stability compares well to that observed for a semi-Lagrangian Crank-Nicolson (SLCN) and semi-Lagrangian fully implicit (SLFI) methods, the latter being unconditionally stable (D’Halluin et al., 2005). A literature review indicates that a semi-Lagrangian and alternating-direction implicit method has been applied to a physical problem involving the integration of the shallow water equations in spherical coordinates (Bates, 1984). However, we are unaware of any previous work in finance research or storage valuation problems in particular, where these techniques have been used together. This hybrid approach is used in this thesis to value a four-dimensional storage option involving three space variables and time; the space variables have advective terms, of which two model/include a diffusive (stochastic) feature.

Alternating-direction implicit (ADI) methods work efficiently by splitting an operator in an $n$-dimensional diffusion space into $n$ parts of a full time step (Smith, 1965). In each fraction of time, only terms from one of the diffusion dimensions are implicitly calculated, while the terms from the other diffusion dimensions are treated explicitly, requiring the solution of (only) simple systems of equations. Otherwise a
full-dimensional system of equations may be required to be solved directly (which can raise computational memory or time issues) or iteratively, at each time step.

Semi-Lagrangian schemes are valuable tools for calculating the solution of advection differential equations by following (virtual) particles in a flow arriving at convenient positions (mesh nodes of a uniform grid) – see Staniforth and Coté (1991). If single advective first-order derivative terms appear in an equation, semi-Lagrangian schemes generate systems of equations independent of the corresponding dimension, allowing model reduction, in effect, by following flow characteristics. Here we use our semi-Lagrangian alternating-direction implicit methodology to approximate the solution of a model with two sources of uncertainty.

Four-dimensional storage systems have already been defined, but it may be possible that because previous techniques are time prohibitive they have not been fully solved numerically. Here we extend the knowledge on hybrid semi-Lagrangian and finite-difference algorithms to efficiently value and optimise systems with this dimensionality.

1.2 Objectives of the Research

The main aims of this thesis are:

1. To demonstrate the effectiveness and efficiency of hybrid finite-difference methodologies coupled with the semi-Lagrangian technique in solving the PDE for determining the price of a financial Asian option, which was selected since its mathematical complexity precludes analytical solutions and challenges numerical solution methods.

2. To extend this knowledge for determining the value of a wind-powered electricity generator subject to price and volume stochastic behavior and to a fixed price fixed volume output contract, with a storage opportunity.

3. To extend this approach to ascertain the optimal storage rate.

The objectives to develop in this thesis are:

- Provide a Real Options framework for valuing storage systems under uncertain economic (commodity price, interest rates, etc) and physical (temperature, rainfall, snow/ice, etc) factors.
• Define and develop numerical methods for efficiently solving financial (Asian-style) and storage related PDE-type models.

• Develop a model for storage systems under uncertain economic and physical factors, including (deterministic) storage dynamics.

• Test the robustness of the proposed numerical methods and benchmark with existing methodologies.

• Calculate the value of storage systems subject to uncertainty in both economic and physical factors.

• Determine optimal operation regimes for stochastic storage systems for the transient state and also for the steady state.

1.3 Layout of Thesis

Chapter 2 “Mathematical background” presents the concepts and frameworks to derive models for determining the price of financial Asian options, and for determining the value of systems under uncertain economic and physical factors. Random walks are used to model the stochastic nature of asset prices, the Black-Scholes model is then used to describe the value of a contract depending on the random price of the related underlying asset; Itô’s Lemma and a modified Feynman-Kac equation provide the support to link the effects from small changes in the price of the underlying with small changes in the value of a contract related to the price of the underlying. Then we present the Real Options valuation approach to appraise the value of a real system or real asset with intrinsic stochastic dynamics due to trading a related underlying asset. The resulting mathematical models are derived as partial differential equations (PDEs). In our interest to study the numerical valuation and optimisation of stochastic storage systems, we next conduct a literature review on Asian options as this financial derivative is found to resemble the essence of stochastic storage systems, the valuation of both problems is described by advection-diffusion PDEs with no diffusion in one space dimension. We then find that the mathematical complexity of the fixed-strike Asian option precludes analytical (closed-form) solutions and
CHAPTER 1. INTRODUCTION

Chapter 3 “Numerical methods for financial and storage-related PDEs” presents classical approaches for numerically solving advection problems and also for solving diffusion problems. Since the valuation of Asian options and the valuation of the stochastic storage problem present properties from both advection and diffusion problems, then we present recent hybrid approaches for numerically solving these kinds of models, namely the semi-Lagrangian fully implicit method (SLFI) and the semi-Lagrangian Crank-Nicolson (SLCN) method; in this hybrid approaches the method of characteristics for advection problems is combined with finite-difference methods for diffusion. Here we introduce an efficient and novel contribution in this field, the semi-Lagrangian alternating-direction implicit (SLADI) method for numerically treating mixed advection and diffusion problems with two sources of uncertainty and two time-like variables. Next we demonstrate the effectiveness and efficiency of hybrid semi-Lagrangian fully implicit methods and semi-Lagrangian Crank-Nicolson methods with respect to numerical approximations for the analytical formulae of fixed-strike Asian options. Here we also assess our similarity reduction described in Chapter 2.

In Chapter 4 “Storage valuation, an application to wind power generation” we introduce a model for determining the value of a wind-powered electricity generator subject to price and volume stochastic behavior and to a fixed-price fixed-volume output contract, with a storage opportunity. We utilise a Feynman-Kac framework to support the model development and thus define the corresponding boundary value problem. Once the wind farm valuation model has been defined, we proceed to apply our knowledge to effective and efficient valuation, then we present detailed implementations of four variants of our semi-Lagrangian alternating-direction implicit (SLADI) methodology as well of detailed implementations of more conventional semi-Lagrangian Crank-Nicolson (SLCN) and semi-Lagrangian fully implicit (SLFI) methods in order to solve numerically the wind farm valuation problem. We benchmark precision, convergence and efficiency of the results from these different methodologies and include
additional convergence results for models with smoother coefficients. We then analyse valuation results by presenting the sources of value with respect to energy systems with and without storage (assuming a fixed operation of the storage device based on electricity spot prices and the rate of the output), and also with and without a revenue guarantee; detailed results are analysed for the system with storage and a revenue guarantee. Finally we present results for introducing correlation and seasonality effects into the model. In the appendix at the end of the chapter we explore the behavior on the domain extrema of the wind farm valuation model and justify the linear solutions imposed on these boundaries of the domain.

Next, in Chapter 5 “Storage optimisation, an application to wind power generation” we remove the constraint imposed in the previous chapter of a fixed rule for operating the battery, and then we extend our valuation approach to ascertain the best way to operate the battery by determining optimal rates of energy injection and extraction for every system state. We verify that our PDE boundary value formulation associated with the Hamilton-Jacobi-Bellman equation is consistent with the assumptions from the Strong Comparison Result; which is a principle to establish that the value function is continuous and the unique viscosity solution of the associated control problem. Next we define conditions for numerically approximating optimal rates in the battery for injection and extraction of energy, and verify numerically the convergence of the solution. The optimal (numerical) results are then analysed and compared with various regimes for operating the storage device; a seasonal electricity price and correlation effects are introduced. We also explore a solution towards the steady state to additionally verify stability of our SLADI methodology. In the appendix at the end of the chapter we explore the time invariant version for a particular stochastic storage system subject to fixed demand and a stochastic electricity price, solutions on fixed grids are obtained and results are investigated for boundary-fitted coordinates.

Chapter 6 “Conclusions and Further Work” highlights advantages and drawbacks of the results of the research; valuable outputs are obtained but difficulties are also identified. On the concluding remarks for future work, we address the applicability and limitations of the research, indicate areas that can be approached and problems identified, explore ideas for continuing the development of our techniques and present challenges in the research area.
Chapter 2

Mathematical background

In this chapter we present concepts and frameworks to derive and support models representing systems under uncertain economical and physical factors.

The dynamics of the variables studied here present simultaneously a predictable and an unpredictable behavior, incorporating effects of a mixed deterministic and stochastic nature. The core idea for approaching these problems is by characterising how a small change in the effect is obtained as a result of a small change in the input variables.

Spot prices of stocks and currency exchange rates are examples of variables of stochastic behavior in the financial world, and temperature, rain fall or windspeed are examples of stochastic variables in the physical world. As such, these factors produce effects in contracts related to trading a related underlying.

We present the ideas for valuing financial derivatives linked to stock prices (option pricing) and how these ideas relate to valuing real assets (or real world investments) under the Real Options approach.

After relating the valuation of financial derivatives with the valuation of real world systems, we then introduce concepts for optimally controlling a system to achieve the best performance; concepts that we will apply (in Chapter 5) for determining the best operation of storage and consequently the best valuation of the system.

The first instrument that we study is a financial Asian option under uncertain price of the underlying. A detailed literature review is provided to identify the direction and gaps in this research area, we identify the complexity of the problem and the difficulties (and challenges) that arise in determining numerical solutions; selected
references are obtained as a benchmark. We derive a (novel) similarity reduction for the fixed-strike Asian option with European style and compile a comprehensive set of boundary conditions. We then find that the mathematical structure for the partial differential equation model representing this financial derivative is related to a mathematical model for the stochastic storage problem, which is also introduced.

2.1 Concepts and frameworks

2.1.1 Random walks

Before defining the Black-Scholes framework, it is necessary to describe the model that will be used to represent the behavior of the asset price; here geometric Brownian motion will be considered. Brownian motion can be interpreted as the trajectory of a particle inside a dynamic fluid; the particle has a direction, but it will be modified as there exist continuous collisions with other particles in every other direction. In the finance analogy, the spot price of an asset inside a dynamic market has a movement in its value over time, corresponding to its tendency or drift and the randomness or volatility of the environment. The above interpretation of motion implicitly considers an elapsed period of time. The variables are the actual position and how this position changes over time.

Here we basically model the asset price; in a financial world prices are continuously modified by other factors. In order to have a general perception of the order of change, a relative return is defined as the change in the price divided by the former value of the asset; note that an absolute return is impractical, unless the exact value of the asset were known.

Let the price of an asset be \( S \) at a given instant in time \( t \). After a slight change in time \( dt \), the price of the asset becomes \( S + dS \). Brownian motion expresses the return in terms of a small change in time and a small random variation, namely

\[
\frac{dS}{S} = \mu \, dt + \sigma \, dW. \tag{2.1}
\]

Here \( \mu \, dt \) is the contribution to the return that can be calculated with certainty (assuming a zero dividend yield), and states that it is proportional to \( dt \) with a factor of \( \mu \) (the expected rate of growth of the asset price known as the drift). \( \sigma \, dW \) is the
random contribution to the return, and states that it is proportional to $dW$ in a factor of $\sigma$ (the standard deviation of the returns known as the volatility).

The source of random variation is the term $dW$, which is known as a Wiener process, and by definition is a random variable taken from a normal distribution with mean equal to zero and variance equal to $dt$. Let $\phi$ be a random variable taken from a standardised normal distribution with mean equal to zero and variance equal to one, then the Wiener process can be written as

$$dW = \phi \sqrt{dt}.$$  

Brownian motion in the form of equation (2.1) is a special case of a random walk and will be the model used to represent a lognormal random walk of an asset price. Many different random walks exist, for example the parameters $\mu$ and $\sigma$ can be defined as functions of time, the asset price or both, also $dW$ can be modelled as a different random process.

The path followed by a random walk is called a realisation, and it will result in a different one every time the random walk is reinitialised. Figure 2.1 presents an example of a random walk, for an asset with an initial price $S_0 = 100$, a rate of growth $\mu = 0.05$ per year and a volatility of $\sigma = 0.20$ per (year)$^{1/2}$, the asset price is found after a slight change in time $dt$ of one day from a year with 365 days; the Box Muller method (Box and Muller, 1958) is used to obtain the value of a normally distributed random variable $\phi$. This procedure is replicated after every change in the asset price, say for another 364 equally small changes in time, and the price of the asset after one year $S_T = 106.57$ is found for this realisation example.

Given the random nature of the process described, no two trajectories of the asset price are identical, but the final value $S_T$ will be distributed about the original value $S_0$ in a skewed bell-shape curve known as the lognormal distribution (Wilmott, 1995), which has the probability density function,

$$\frac{1}{\sigma S \sqrt{2\pi t}} e^{-\left(\log(S/S_0) - (\mu - \frac{1}{2}\sigma^2)t\right)^2/2\sigma^2 t},$$ (2.2)

for $0 < S < \infty$. Figure 2.2 presents the normalised histogram with 25 intervals when 50000 realisations were calculated, and the dashed line is the theoretical probability density function of $S_T$; equation (2.2) is calculated with parameters $\mu = 0.05, \sigma = 0.20, T = 1$ and $S_0 = 100$. 

CHAPTER 2. MATHEMATICAL BACKGROUND

Figure 2.1: Random walk of an asset price

Figure 2.2: Normalised histogram of the asset price distribution
Financial contracts can be seen as functions defined in terms of the related asset spot price at a given point in time. Next, we will consider how a function of a random variable changes with respect to a slight change in the random variable itself.

2.1.2 Itô’s Lemma

Itô’s Lemma bridges a small change in a random variable with the corresponding small change in a function of such random variable (Øksendal, 2003). The result

\[ dW^2 = dt \] (2.3)

is derived in the formal proof, indicating that \( dW^2 \) behaves as \( dt \), a result that can be interpreted as follows: for a given time interval of length \( t \) that is partitioned into \( n \) pieces, we obtain segments of length \( (t/n) \), then every increment \( \Delta W_j = \left( W\left(\frac{j}{n}\right) - W\left(\frac{j-1}{n}\right) \right) \) is independent and identically distributed and follows a normal distribution \( N(0, t/n) \) with zero mean and variance equal to \( (t/n) \). When all these increments are squared and added up we obtain that \( \int_0^t (dW)^2 = \sum_{j=1}^n (\Delta W_j)^2 \) for \( n \) large, from the variance definition (\( \text{Var}(Y) = E(Y^2) - E(Y)^2 \)) we find that the expected value for each \( (\Delta W_j)^2 \) equals to \( t/n \), the variance. Since we have \( n \) increments, then \( \int_0^t (dW)^2 = t \) for \( n \) large, or \( dW^2 = dt \) (Baxter and Rennie, 1996).

The following is a non-formal interpretation of Itô’s Lemma based on a Taylor series expansion. Let \( f(S) \) be a function in terms of the random variable \( S \), and assuming that such a function is smooth, the variation of \( f \) due to a slightly change \( dS \) on \( S \) as a Taylor series expansion is

\[ df = \frac{df}{dS} dS + \frac{1}{2} \frac{d^2f}{dS^2} dS^2 + \ldots, \] (2.4)

\(...)\) represent the remainder of the series and is smaller compared to the two terms retained. \( dS \) is already defined in (2.1), and at every instant is just a number with random nature, thus it can be squared

\[ dS^2 = (\mu S dt + \sigma S dW)^2 \]
\[ = \mu^2 S^2 dt^2 + 2\mu \sigma S^2 dt dW + \sigma^2 S^2 dW^2. \] (2.5)

The first two terms in (2.5) are smaller than the last one, and so \( dt dW \) and \( dt^2 \) are relatively smaller as \( dt \to 0 \), and only the term of \( dW^2 \) remains since \( dW^2 \to dt \),
then
\[ dS^2 \rightarrow \sigma^2 S^2 \, dt. \] (2.6)

After substitution of (2.1) and (2.6) in (2.4), it is found that
\[ df = \frac{df}{dS}(\mu S \, dt + \sigma S \, dW) + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2} \, dt \]
\[ = \left( \mu S \frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2} \right) \, dt + \sigma S \frac{df}{dS} \, dW \] (2.7)

The last expression is known as Itô’s Lemma. If now the function is in terms of the random variable \( S \) and of time \( t \), \( f(S, t) \). (2.7) can be extended by the expansion of \( f(S + dS, t + dt) \) using again a Taylor series
\[ df = \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dS^2 + \ldots, \]

The expression (2.1) and (2.6) are substituted again, and \( df \) is then
\[ df = \left( \mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) \, dt + \sigma S \frac{\partial f}{\partial S} \, dW. \] (2.8)

Now we present Itô’s lemma for a function in terms of multiple random variables and time, \( f(S_1, S_2, \ldots, S_m, t) \), where \( dS_i = \mu_i S_i \, dt + \sigma_i S_i \, dW_i \), for \( i = 1, 2, \ldots, m \). By using a Taylor series expansion of \( f(S_1 + dS_1, S_2 + dS_2, \ldots, S_m + dS_m, t + dt) \) we obtain
\[ df = \frac{\partial f}{\partial t} dt + \sum_i \frac{\partial f}{\partial S_i} dS_i + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 f}{\partial S_i \partial S_j} dS_i dS_j + \ldots, \]

for \( i = j \) we have \( dS_i^2 \rightarrow \sigma_i^2 S_i^2 \, dt \) as in (2.6), for \( i \neq j \) we are required to find the product of the two numbers of random nature
\[ dS_i dS_j = (\mu_i S_i \, dt + \sigma_i S_i \, dW_i)(\mu_j S_j \, dt + \sigma_j S_j \, dW_j) \]
\[ = \mu_i \mu_j S_i S_j \, dt^2 + \mu_i \sigma_j S_i S_j \, dt \, dW_i + \mu_j \sigma_i S_i S_j \, dt \, dW_j + \sigma_i \sigma_j S_i S_j \, dW_i dW_j \] (2.9)

The first three terms in (2.9) are smaller than the last one since \( dt \, dW_i \), \( dt \, dW_j \), \( dt^2 \) are relatively smaller as \( dt \rightarrow 0 \), the last term remains since \( E(dW_i \, dW_j) \rightarrow \rho_{ij} \, dt \) with \( \rho_{ij} \) being the correlation between the two stochastic processes \( dW_i \) and \( dW_j \); which is a result that can be interpreted from the covariance and the correlation definitions (Morgan, 1984)
\[
\rho_{XY} = \text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}
\]

since we have that \(E(W_j) = 0\) and \(\text{Var}(W_j) = dt\), then \(E(dW_i dW_j) \rightarrow \rho_{ij} dt\).

Then we can write \(df\) as

\[
df = \left(\frac{\partial f}{\partial t} + \sum_i \mu_i S_i \frac{\partial f}{\partial S_i} + \frac{1}{2} \sum_i \sigma_i S_i \frac{\partial^2 f}{\partial S_i^2} + \frac{1}{2} \sum_i \sum_j \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 f}{\partial S_i \partial S_j} \right) dt + \sum_i \sigma_i S_i \frac{\partial f}{\partial S_i} dW_i \tag{2.10}
\]

A generalised Itô process \(S(t)\) is a continuous-time stochastic process with

\[
dS = a(S,t) dt + b(S,t) dW, \tag{2.11}
\]

where \(dW\) is a standard Wiener process, and \(a(S,t)\) and \(b(S,t)\) are known functions. Here \(E(\,dS\,) = a(S,t) \, dt\) since \(E(\,dW\,) = 0\), and the variance of \(dS\) is given by \(\text{Var}(\,dS\,) = \text{E}[\,dS^2\,] = \text{E}[\,dS\,] = b^2(S,t) \, dt\), where \(a(S,t)\) and \(b^2(S,t)\) are known as the instantaneous drift rate and the instantaneous variance rate of the Itô process.

If \(a(S,t) = \mu S\), and \(b(S,t) = \sigma S\), with \(\mu\) and \(\sigma\) being the constant drift and volatility, then the equation (2.11) becomes

\[
dS = \mu S \, dt + \sigma S \, dW, \tag{2.12}
\]

which is known as geometric Brownian motion (Dixit and Pindyck, 1994).

For \(a(S,t) = \kappa(\bar{S} - S)\), and \(b(S,t) = \sigma\), with \(\kappa\) (known here as the speed of mean reversion) and \(\sigma\) (the volatility) being constants, and \(\bar{S}\) being the long-run mean value of \(S\), the equation (2.11) becomes

\[
dS = \kappa(\bar{S} - S) \, dt + \sigma \, dW, \tag{2.13}
\]

which is known as the Ornstein-Uhlenbeck (OU) stochastic process. This stochastic process has constant variance which is independent of the drift at any time or any initial condition (Doob, 1942).

If \(a(S,t) = \kappa(\bar{S} - S)\), and \(b(S,t) = \sigma \sqrt{S}\), with \(\kappa\) being a constant speed of mean reversion, \(\sigma\) being a constant volatility, and \(\bar{S}\) being the long-run mean value of \(S\), then the equation (2.11) becomes

\[
dS = \kappa(\bar{S} - S) \, dt + \sigma \sqrt{S} \, dW, \tag{2.14}
\]
which is known as the Cox-Ingersoll-Ross (CIR) stochastic process. In this process, the variable $S$ remains positive if the parameters satisfy the condition $2\kappa \bar{S} \geq \sigma^2$ and it may reach a zero level only if $\sigma^2 > 2\kappa \bar{S}$ (Cox et al., 1985).

These mean-reverting processes can be generalised to the form $a(S, t) = \kappa(\bar{S} - S)$, and $b(S, t) = \sigma S^\nu$, where $\kappa$ is the speed of mean reversion, $\sigma$ is the volatility, $\nu$ is constant, and $\bar{S}$ is the long-run mean value of $S$ (Chan et al., 1992), then the equation (2.11) becomes

$$dS = \kappa(\bar{S} - S)\,dt + \sigma S^\nu\,dW,$$

(2.15)

$v = 0$ gives the OU process, $v = \frac{1}{2}$ gives the CIR process, and $v = 1$ gives a stochastic process of mean-reversion with proportional volatility.

Next, we will present the Black-Scholes model which is derived with the use of a random walk and Itô’s Lemma

### 2.1.3 The Black-Scholes model

The Black-Scholes framework is a backbone for the development of valuation schemes for financial derivatives, when a PDE approach based on hedging arguments is followed. The assumptions in the model (Black and Scholes, 1973) are:

1. The interest rate $r$ on a riskless investment and the volatility of the asset $\sigma$ are deterministic functions of time over the lifespan of the option.

2. The asset price varies continuously in time and follows a lognormal random walk.

3. The asset has no dividend payments.

4. The option can only be exercised at the maturity time (in the case of European options).

5. No transaction cost is associated with buying or selling the asset or the option.

6. There is no chance to make riskless profits.

7. The asset can be traded continuously.

8. Short selling is not penalised and the asset can be bought or sold in any divisible amount.
Let $V(S,t)$ be the value of an European option that depends on the asset price $S$ and time $t$, the asset price follows a geometric Brownian motion

$$dS = (\mu - \delta)S \, dt + \sigma S \, dW.$$  

Here the constants $\mu$ and $\sigma$ are rate of growth and the volatility of the asset, and by removing the third assumption we have an asset continuously paying a dividend yield rate $\delta$.

The value of the option changes due to a small change on the random variable $S$, then using Itô’s Lemma (2.7) the change on the option value is

$$dV = \sigma S \frac{\partial V}{\partial S} \, dW + \left( (\mu - \delta)S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) \, dt \quad (2.16)$$

A portfolio $\mathcal{P}$, with value $\Pi$, is constructed by buying one option $V$ and selling $\Delta$ units of a borrowed underlying asset $S$

$$\Pi = V - \Delta S \quad (2.17)$$

the change in the portfolio with respect to slight changes in the value of the option and the asset price is

$$d\Pi = dV - \Delta \, dS - \delta \Delta S \, dt, \quad (2.18)$$

With the last term subtracted as there is a reduction in the value of the portfolio due to dividends payment.

The resulting stochastic differential equations for $dS$ (2.1) and $dV$ (2.16) are substituted in the equation for $d\Pi$ (2.18)

$$d\Pi = \sigma S \left( \frac{\partial V}{\partial S} - \Delta \right) \, dW + \left( (\mu - \delta)S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - (\mu - \delta) \Delta S - \delta \Delta S \right) \, dt.$$ 

If

$$\Delta = \frac{\partial V}{\partial S}, \quad (2.19)$$

then, the change in the value of the portfolio is deterministic:

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \delta \Delta S \right) \, dt.$$ 

Claiming that there cannot be any riskless profit, an amount $\Pi$ can only have a growth as if it were deposited on a riskless account, then $d\Pi = r \Pi \, dt$ and

$$r \Pi \, dt = \left( \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \delta \Delta S \right) \, dt. \quad (2.20)$$
Finally, bringing the definitions for $\Delta$ (2.19) and $\Pi$ (2.17) into (2.20)

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta)S \frac{\partial V}{\partial S} - rV = 0.
$$

(2.21)

An European call option $C(S, t)$ gives the holder the right (not the obligation) to buy a stock $S$ for a strike price $K$ at the maturity time $T$. To find the value of this option the boundary conditions required to solve (2.21) are:

At $S = 0$, if ever the stock price takes the value of zero it will remain in that value, then

$$
C(0, t) = 0.
$$

As $S \to \infty$,

$$
C(S, t) \to S.
$$

At expiry,

$$
C(S, T) = \max\{S - K, 0\}.
$$

An European put option $P(S, t)$ gives the holder the right (not the obligation) to sell a stock $S$ for a strike price $K$ at the maturity time $T$. To find the value of this option the boundary conditions required to solve (2.21) are:

At $S = 0$, if ever the stock price takes the value of zero it will remain in that value, then we **discount back**

$$
P(0, t) = Ke^{-r(T-t)}.
$$

As $S \to \infty$, the option is unlikely to be exercised and

$$
P(S, t) \to 0.
$$

At expiry,

$$
P(S, T) = \max\{K - S, 0\}.
$$

### 2.1.4 Feynman-Kac equation

The Feynman-Kac equation is a generalisation to the Kolmogorov backward equation, and when it is used to solve boundary value problems (Oksendal, 2003) considers a class of expectations of the form

$$
\int_{0}^{T} e^{-\int_{0}^{s} \sigma(Z^t) \, ds} \int_{0}^{s} e^{-\int_{0}^{r} \sigma(Z^t) \, ds} g(Z^t) \, dt, \quad (2.22)
$$
where \( q(z) \geq 0 \) is a continuous function on \( \mathbb{R}^n \), the function \( f \) is the value of the solution at the domain boundary \( \partial H \), the function \( g \) is the sum of the corresponding instantaneous source terms. \( Z^t \in \mathbb{R}^n \) is an Itô diffusion, \( \bar{\tau} \) is the first exit time from the solution domain \( H \) of \( Z^t \), and \( E_z \) represents the expected value for \( Z^0 = z \in \mathbb{R}^n \).

Equation (2.22) is then the solution to the boundary value problem

\[
\begin{cases}
Lu(z) - q(z)u(z) = -g(z) & \text{on } H \\
\lim_{z \to w} u(z) = f(w) & \text{for } w \in \partial H,
\end{cases}
\]  

(2.23)

where

\[
Lu = \sum a_{ij} \frac{\partial^2 u}{\partial z_i \partial z_j} + \sum b_i \frac{\partial u}{\partial z_i}.
\]  

(2.24)

With drift coefficients \( b^T = [b_i] \), and diffusion coefficients \( \sigma^T \), and so \( [a_{ij}] = \frac{1}{2} \sigma \sigma^T \).

### 2.1.5 Real Options valuation

A financial option is a contract that gives the holder the right, but not the obligation, to trade an asset at some point in the future. When there is a previously arranged price in the contract and the transaction is for buying the asset then we have a call option, whereas if the transaction is for selling the asset we have a put option. The prearranged price is called the strike price and the time to exercise the option is called the expiry date. When the option can only be exercised at the expiry date then the option is known to have an exercise of European style whereas if the option can be exercised at any time between the issue of the contract and the expiry date is known to have an exercise of American style.

The idea behind the Real Options approach for valuing projects is that they can be seen as potential investments without an obligation of commitment (see for example, Dixit and Pindyck, 1994; Trigeorgis, 1996) as in the case for valuing call options for assets. The investor gets the right (not the obligation) to spend resources in an investment (exercising the option) and obtain the project (a stock share) with stochastic value. Common Real Options include among others: delaying or advancing an investment, abandonment or installment of activities, expansion or contraction of operations, and acquisition or selling of premises.

A storage essentially entitles the owner with the option to sell the stockpile at anytime in the future. This describes a compounded option as first the owner has
the right (not the obligation) to buy and stockpile the underlying asset and then has
the right (without obligation) to sell the underlying asset, the decision can be taken
any time, thus can be referred as a “compounded American option” (Carmona and
Ludkovski, 2010).

Then any technique developed for finding the value of financial options for a par-
ticular asset can be used to find the value of a particular project related to the same
asset, this under the assumptions that the asset can be traded directly in the market
or otherwise it would be possible to trade a different asset that could replicate the
uncertainty of the original asset i.e. both assets being completely correlated.

As described previously, the stochastic dynamics of an asset price can be captured
by Itô’s Lemma and then this can be used to find the value of a financial derivative as
previously presented in the Black-Scholes equation (2.21). If the objective is to find
the potential value of a project involving the same asset, in the appropriate way Itô’s
Lemma can be used for the corresponding valuing purpose, in this case a portfolio with
two components has to be created, one component is the investment opportunity, and
the other is a number of asset units (by selling $\Delta$ units of a borrowed asset) so that
the portfolio is risk-free. The previous basically replicates the value of the portfolio
for the Black-Scholes equation (2.17) but in this case $V$ represents the value of the
investment opportunity.

A contingent claims analysis consists of determining the value of a project through
the future profit flow entitled to the owner, by replicating the associated risk and
return of tradable assets. The valuation is performed with the underlying asset if it
is directly traded in the market; when this is not the case it is required to utilise
tradable assets with the same risk and returns. The key idea is to obtain the value
of the project by equating the rate of return from owning the project with the rate of
return of a portfolio consisting of the investment of one dollar in a riskless asset and
the purchase of $\Delta$ output units from the project (Dixit and Pindyck, 1994). Here $S$,
the output price from the project, is assumed to follow geometric Brownian motion

$$dS = \alpha S \, dt + \sigma S \, dW,$$

where $\alpha$ and $\sigma$ are the rate of growth and the variance of the price for the output, and
d$W$ is a standard Wiener process.
The total expected rate of return is given by $\mu = \alpha + \delta$, where $\delta$ is the dividend yield rate. The interest rate $r$ is considered to be riskless, as commonly present in government bonds, and following the capital asset pricing model (CAPM) the risk-adjusted rate of return is given by

$$
\mu = r + \phi \rho S \sigma
$$

where $\phi$ is the market price of risk and $\rho$ is the coefficient of correlation between the returns of $S$ and the whole market portfolio $m$ (Dixit and Pindyck, 1994). An investor will then ask for this risk-adjusted rate of return in order to own a particular risky project. The risk-free rate of return complements this section, this is interpreted as the rate given by an instrument when its future value is known with certainty: a bond with a riskless rate of return is an example of an instrument with a risk-free rate of return.

Let $V(S,t)$ be the value of the project and $f(S,t)$ be the profit flow earned on a small time $dt$; as described before, the key idea is to equate, for a small time $dt$, the rate of return from owning the project with the rate of return from the value of the portfolio $\Pi$ i.e.

$$
\frac{dV}{V} = \frac{d\Pi}{\Pi}.
$$

By Itô’s Lemma

$$
dV = \left( \frac{\partial V}{\partial t} + \alpha S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW,
$$

in time $dt$ the project has received the profit flow $f$ and then

$$
\frac{dV}{V} = \frac{1}{V} \left( \left( \frac{\partial V}{\partial t} + \alpha S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f \right) dt + \sigma S \frac{\partial V}{\partial S} dW \right).
$$

The rate of return on the portfolio is

$$
\frac{d\Pi}{\Pi} = \frac{r + \Delta (\alpha + \delta)}{1 + \Delta S} dt + \frac{\sigma \Delta S}{1 + \Delta S} dW,
$$

where the cost of the portfolio is one dollar plus the $\Delta$ output units i.e. $\Pi = 1 + \Delta S$, the dollar invested in the riskless asset returns $r dt$, the investment in the output returns the dividends $\Delta \delta S dt$ plus the random return $\Delta dS = \Delta \alpha S dt + \Delta \sigma S dW$.

The value of the project is replicated with the value of the portfolio when both, the project and the portfolio, are exposed to the same risk, i.e.

$$
\frac{\Delta}{1 + \Delta S} = \frac{1}{V} \frac{\partial V}{\partial S},
$$

(2.25)
and when both earn equal returns, namely
\[
\frac{r + \Delta(\alpha + \delta)S}{1 + \Delta S} = \frac{1}{V} \left( \frac{\partial V}{\partial t} + \alpha S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f \right). \tag{2.26}
\]

From (2.25) we can obtain \( \Delta = \frac{\partial V}{\partial S} / (V - S(\partial V/\partial S)) \), when this last expression is substituted into (2.26) we obtain
\[
\frac{1}{V} \left( \frac{\partial V}{\partial t} + \alpha S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f \right) = \frac{1}{V} \left( rV - rS \frac{\partial V}{\partial S} + (\alpha + \delta)S \frac{\partial V}{\partial S} \right).
\]

Finally by multiplying both sides of the equation by \( V \), we obtain the PDE describing the value \( V \) of the project
\[
\frac{\partial V}{\partial t} + (r - \delta)S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV + f = 0. \tag{2.27}
\]

Alternatively, a Feynman-Kac framework can be used to capture the underlying stochastic dynamics and find the value of an investment project (Evatt et al., 2010). One can identify from the boundary value problem formulation (2.23) that the source terms \( g(Z^t) \) correspond to the (variable) future profit flow in the contingent claims analysis, then \( q(z) \) corresponds to the riskless market rate \( r \), and then correspondingly the Itô diffusion \( Z^t \) behaves with drift rates \( b_i = r - \delta_i \) where \( \delta_i \) is the dividend of the \( i \)-th component, this equivalence is the definition of “risk-neutrality valuation” where future payoffs of the project are discounted at the riskless rate \( r \) with the assumption that the Itô process replicates the behavior \( b_i = r - \delta_i \) (risk-neutral) and the investor would take \( r = \mu = b_i + \delta_i \) as the total expected rate of return; otherwise an investor would select riskless bonds instead of the project. As we have presented, the formulations from using the arguments from the Black-Scholes equation, the contingent claims analysis and the Feynman-Kac equation are consistent and generate equivalent PDE formulations.

### 2.1.6 Dynamic programming

In order to achieve in a future time the best performance (in some regard) of a controllable system, any subsequent decision to control the system’s operation that has to be done from now, has to be optimal regardless of the actual state of the system (rephrasing the Optimality Principle in the Dynamic Programming, Bellman, 1957). As described, the problems of concern in Dynamic Programming are those that are
made in stages, and the objective is to minimise a particular cost, i.e. achieving the best performance by some measure. The stages that are required to solve the problem belong correspondingly to sub-problems that keep the essence of the original problem: assume that it is the last decision to make in the problem, then to determine the best performance from that state, the decision just needs to be optimal from that stage, then the problem of achieving the best performance is transferred to the current stage to ensure optimality. Every sub-problem is defined until the initial conditions are identified, i.e. the original state from which the system starts, and then the optimal sequence of decisions are taken. The two components of the model for the problem are a time dynamic and a cost function (which is additive over time); when a discrete approach in time is considered (Bertsekas, 2005), we have a system

$$x_{k+1} = f_k(x_k, d_k, w_k), \quad \text{for } k = 0, 1, \ldots, N - 1,$$

where $k$ are discrete indexes for time, $x_k$ is the state of the system at time $k$, $d_k$ is the decision taken to control the system at time $k$, $w_k$ is a random parameter, $N$ is the time horizon (total of time stages), and $f_k$ is a function that updates the state of the system to $(x_{k+1})$ given the current state $(x_k)$, the control $(d_k)$ and the disturbance $(w_k)$. The total cost is calculated by the additive property (in time) of the cost function $c$, and $c_k(x_k, d_k, w_k)$ then represents the cost incurred at time $k$. Since the cost is generally a random variable, due to the disturbance $w_k$, the total expected cost to be optimised becomes

$$E \left\{ c_N(x_N) + \sum_{k=0}^{N-1} c_k(x_k, d_k, w_k) \right\},$$

where the expectation is taken over the joint distribution of the random variables involved. Optimisation of the system is attained by the controls $d_0, d_1, \ldots, d_{N-1}$, in turn every control $d_k$ is decided based on information at the current state $x_k$.

The basic problem of Dynamic Programming (Bertsekas, 2005) consists of the following. The state $x_k$ is an element that belongs to the space $S_k$, the control $d_k$ is an element belonging to the space $U_k$, and the random parameter $w_k$ is an element from the space $I_k$. Since the current state of the system $x_k$ imposes conditions restricting the control $d_k$, we have that for all $x_k \in S_k$ and $k$, $d_k \in D_k(x_k)$ where $D_k(x_k)$ is a nonempty set for the constrained values of the control $d_k$, and $D_k(x_k) \subset U_k$. $P(\cdot|x_k, d_k)$ represents the probability distribution of $w_k$ and is independent of previous values.
$w_{k-1}, \ldots, w_0$, but may depend on the values of $x_k$ and $d_k$. A policy or control law $\pi$ consists of a sequence of functions $\{\varphi_0, \ldots, \varphi_{N-1}\}$, and every $\varphi_k$ is a function that maps states $x_k$ into controls $d_k$, i.e. $d_k = \varphi_k(x_k)$. A policy is then called ‘admissible’ if $\varphi(x_k) \in D_k(x_k)$ for all $x_k \in S_k$. With known initial state $x_k$ and admissible policy $\pi = \{\varphi_0, \ldots, \varphi_{N-1}\}$, the random variables for the states ($x_k$) and the disturbances ($w_k$) evolve under distributions governed by the system equation

$$x_{k+1} = f_k(x_k, \varphi_k(x_k), w_k), \quad \text{for } k = 0, 1, \ldots, N - 1.$$ 

Then, for known cost functions $c_k$, $k = 0, 1, \ldots, N$, the expected cost incurred by a policy $\pi$ with initial state $x_0$ is

$$J_\pi(x_0) = E \left\{ c_N(x_N) + \sum_{k=0}^{N-1} c_k(x_k, \varphi_k(x_k), w_k) \right\},$$

where the expectation is with respect to the random variables $x_k$ and $w_k$. Given the set of all admissible policies $\Pi$, $\pi^*$ is an optimal policy if it minimises the cost

$$J_{\pi^*}(x_0) = \min_{\pi \in \Pi} J_\pi(x_0),$$

and the optimal cost function $J^*$ is

$$J^*(x_0) = \min_{\pi \in \Pi} J_\pi(x_0).$$

Here we are assuming that the random parameter $w_k$ can only take a finite or countable number of values, and also that given an admissible policy $\pi$, the expected values are well-defined and finite. Also, “min” is used to denote the infimum.

The Principle of Optimality (Bertsekas, 2005) can be expressed as the following. Given an optimal policy $\pi^* = \{\varphi_0^*, \varphi_1^*, \ldots, \varphi_{N-1}^*\}$ for the basic problem, consider the occurrence of state $x_i$ at time $i$ with positive probability. If one seeks to solve the sub-problem at time $i$ for minimising the ‘cost-to-go’ from time $i$ to time $N$

$$E \left\{ c_N(x_N) + \sum_{k=i}^{N-1} c_k(x_k, \varphi_k(x_k), w_k) \right\},$$

then one would find that the optimal policy for such sub-problem is the truncated sequence $\{\varphi_i^*, \varphi_{i+1}^*, \ldots, \varphi_{N-1}^*\}$.

The algorithm for dynamic programming (Bertsekas, 2005) proceeds from the time period $N - 1$ backwards to time period 0,

$$J_N(x_N) = g_N(x_N)$$

(2.28)
\[ J_k(x_k) = \min_{d_k \in D_k(x_k)} E \left\{ c_k(x_k, d_k, w_k) + J_{k+1}(f_k(x_k, d_k, w_k)) \right\}, \quad k = 0, 1, \ldots, N - 1, \]  
(2.29)

where the expectation is calculated with respect to the probability distribution of \( w_k \), and the last step \( J_0(x_0) \) obtains the optimal cost \( J^*(x_0) \) for every initial state \( x_0 \). And the policy \( \pi^* = \{ \varphi^*_0, \ldots, \varphi^*_N \} \) is optimal if for every state \( x_k \) and time \( k \) the control \( d^*_k = \varphi^*_k(x_k) \) minimises the right hand side on equation (2.29), under the assumption of the function \( J_k \) being well-defined and finite.

### 2.1.7 The Hamilton-Jacobi-Bellman equation

Heuristically, the Hamilton-Jacobi-Bellman (HJB) equation is the continuous-time analog of the Dynamic Programming algorithm (Bertsekas, 2005). In the context of a maximisation problem, the supremum replaces the infimum, or correspondingly “max” replaces “min,” and cost terms are rephrased as utility terms.

Let \( \Phi \) be a bounded function and \( d = d(t, X_t(w)) \) be a Markov control, i.e. a function that depends only on the current state of the system,

\[ \Phi(y) = \sup \{ J^d(y); d = d(Y) \text{ Markov control} \}, \]

assuming the existence of an optimal control \( d^* \), then the HJB equation (Øksendal, 2003) is defined by

\[ \sup_{v \in D} \{ F^v(y) + (L^v \Phi)(y) \} = 0 \quad \text{for all } y \in G \]  
(2.30)

and

\[ \Phi(y) = K(y) \quad \text{for all } y \in \partial G, \]  
(2.31)

where \( F \) is a continuous utility rate function, \( K \) is a continuous bequest function,

\[ (L^v f)(y) = \frac{\partial f}{\partial s}(y) + \sum_{i=1}^n b_i(y, v) \frac{\partial f}{\partial x_i} + \sum_{i,j=1}^n a_{ij}(y, v) \frac{\partial^2 f}{\partial x_i \partial x_j}, \]

with \( a_{ij} = \frac{1}{2}[\sigma \sigma^T]_{ij} \), \( y = (s, x) \) and \( x = (x_1, \ldots, x_n) \).

If \( v = d^*(y) \) and \( u^* = u^*(y) \) is optimal, then the supremum in (2.30) is attained, i.e.

\[ F(y, u^*(y)) + (L^{d^*(y)} \Phi)(y) = 0 \quad \text{for all } y \in G. \]
2.2 Mathematical background to Asian options

Asian options resemble the very essence of the stockpiling problem, thus we devote a review on the field before reviewing the storage literature.

2.2.1 Literature

Kemna and Vorst (1990) used Monte Carlo simulation for pricing options based on average asset values; they stated that it is impossible to obtain a formula to explicitly calculate the value of this kind of option, and thus the need to use numerical methods, such as their simulations; finite-difference methods were discarded at that time because of the number of variables involved. Kemna and Vorst (1990) assume a market that offers a constant riskless interest rate \( r \) for valuing average value options that are related to a stock with price \( S(t) \), it is assumed that the stock price follows the stochastic differential equation

\[
dS(t) = \mu S(t) \, dt + \sigma S(t) \, dW(t),
\]

where \( \mu \) and \( \sigma \) are constants and \( W(t) \) is a Wiener process. The function \( A(t) \) is used to represent a part of the final average up to time \( t \), i.e.,

\[
A(t) = \beta \int_{T_0}^{t} S(\tau) \, d\tau,
\]

where the coefficient \( \beta = \frac{1}{T - T_0} \), \( T_0 \) is the starting time from which the average value of the stock is calculated, and \( T \) is the maturity date. Then

\[
dA(t) = \beta S(t) \, dt.
\]

Kemna and Vorst (1990) worked using the Black-Scholes hedging arguments and derived a PDE to price the average value Asian option \( \tilde{C} \) for a stock with price \( S(t) \),

\[
\tilde{C}_t + \frac{1}{2}\sigma^2 S^2 \tilde{C}_{SS} + \beta S \tilde{C}_A + r (S \tilde{C}_S - \tilde{C}) = 0, \tag{2.32}
\]

The boundary conditions are:

At expiry

\[
\tilde{C}(S(T), A(T), T) = \max(A(T) - K, 0), \tag{2.33}
\]

On \( S = 0 \)

\[
\tilde{C}(0, A(t), t) = \max(e^{-r(T-t)}(A(T) - K), 0), \tag{2.34}
\]

As \( S \to \infty \)

\[
\tilde{C}_S(\infty, A(t), t) = \beta (T - t) e^{-r(T-t)}, \tag{2.35}
\]

and for \( A(t) \geq K \)

\[
\tilde{C}(S(t), A(t), t) = (A(t) - K) e^{-r(T-t)} + \frac{\beta}{r} (1 - e^{-r(T-t)}) S(t), \tag{2.36}
\]
The interpretation of (2.36) is that when $A(t) \geq K$, the payoff of the option is guaranteed to be positive. At time $t$, the payoff can be written as the amount already earned plus the amount to be increased after $t$:

$$ (A(t) - K) + \beta \int_t^T S(\tau) \, d\tau. $$

The strategy is also explained from a portfolio $\mathcal{P}$ containing $(A(t) - K)e^{-r(T-t)}$ riskless bonds to guarantee the return expressed by the first term $(A(t) - K)$. In order to obtain the return from the second term $(\beta \int_t^T S(\tau) \, d\tau)$, the portfolio also contains a fraction of the stock equal to $\beta e^{-r(T-t)}\Delta \tau$, which is converted into a riskless bond every time interval $(\tau, \tau + \Delta \tau)$, thus this portion of the stock becomes

$$ \int_t^T \beta e^{-r(T-t)} \, d\tau = \frac{\beta}{r}(1 - e^{-r(T-t)}). $$

The boundary was imposed at $A(t) = K$ as $\tilde{C}(S(t), K, t) = \frac{\beta}{r}(1 - e^{-r(T-t)})S(t)$. Then, to obtain numerical calculations, Kemna and Vorst (1990) used the Kolmogorov backward equation to price the average value Asian option,

$$ \tilde{C}(S(t), A(t), t) = e^{-r(T-t)}E^{S(t), A(t), t} \max((A(T) - K), 0), $$

with $E^{S(t), A(t), t}$ being the conditional expectation with respect to $S(t), A(t)$ and $t$. Since the combined process $(S(t), A(t))$ is not Gaussian then they implied that no formula could be explicitly found when $A(t) < K$, and thus the need for using numerical methods for valuation. They present a series of results for $\sigma \in \{0.2, 0.3, 0.4\}$ and $r \in \{0.03, 0.05, 0.07\}$.

Zvan et al. (1998) developed robust numerical methods to solve Asian option PDEs. They enhanced flux-limiting techniques to treat convection-dominated PDEs, a regime that occurs when the velocity term is larger than the diffusion term, in other words when the magnitude of the coefficient of a first-order term is large compared to the magnitude of the coefficient of the related second-order term, a condition that is prevalent with the absence of diffusion in one dimension (characteristic of Asian options). Their many examples produced oscillatory solutions, showing how erroneous diffusion may be introduced by using standard methods (even for more routine European options), thus suggesting the use of more sophisticated techniques. Using a finite-volume discretisation, that consists of defining cells (see Figure 2.3) so that a point in the cell
is selected to contain a spatial domain discretisation for the $i$-th cell, e.g., $S_i$, and the cell $i$ shares ‘interfaces’ with adjacent cells (the interface $i - \frac{1}{2}$ is shared with the cell on the left, and the interface $i + \frac{1}{2}$ is shared with the cell on the right), Zvan et al. (1998) represent the (forward) Black-Scholes equation, where $\tau = T - t$

$$\frac{\partial V}{\partial \tau} = \frac{1}{2} \sigma^2 S^2 \frac{\partial V}{\partial S} + r S \frac{\partial V}{\partial S} - r V$$

(2.38)

as

$$\frac{V_i^{n+1} - V_i^n}{\Delta \tau} = \vartheta \left( F_{i-\frac{1}{2}}^{n+1} - F_{i+\frac{1}{2}}^{n+1} + f_i^{n+1} \right)$$

(2.39)

$$(1 - \vartheta) \left( F_{i-\frac{1}{2}}^n - F_{i+\frac{1}{2}}^n + f_i^n \right),$$

(2.40)

where $n$ represents the $n$-th time-step, $F_{i-\frac{1}{2}}$ is the flux entering cell $i$ by the interface $i - \frac{1}{2}$, $F_{i+\frac{1}{2}}$ is the flux leaving cell $i$ by the interface $i + \frac{1}{2}$, and $f_i$ is a source (or sink) term.

With a weight $\vartheta = 1$ the fully implicit method is defined, $\vartheta = \frac{1}{2}$ defines the Crank-Nicolson method, and $\vartheta = 0$ produces a the fully explicit method. Then, the equivalence between the flux discretisation and the finite-difference discretisation is obtained by the substitutions

$$F_{i-\frac{1}{2}}^{n+1} = \frac{1}{\Delta S_i} \left[ -\frac{1}{2} \sigma^2 S_i^2 \frac{V_{i-\frac{1}{2}}^{n+1} - V_{i-\frac{1}{2}}^n}{\Delta S_{i-\frac{1}{2}}} - r S_i V_{i-\frac{1}{2}}^{n+1} \right],$$

(2.41)

$$F_{i+\frac{1}{2}}^{n+1} = \frac{1}{\Delta S_i} \left[ -\frac{1}{2} \sigma^2 S_i^2 \frac{V_{i+\frac{1}{2}}^{n+1} - V_{i+\frac{1}{2}}^n}{\Delta S_{i+\frac{1}{2}}} - r S_i V_{i+\frac{1}{2}}^{n+1} \right],$$

(2.42)

and

$$f_i^{n+1} = -r V_i^{n+1},$$

(2.43)

where

$$\Delta S_i = \frac{S_{i+1} - S_{i-1}}{2},$$
\[ \Delta_{S_{i-\frac{1}{2}}} = S_i - S_{i-1}, \]

and

\[ \Delta_{S_{i+\frac{1}{2}}} = S_{i+1} - S_i. \]

Zvan et al. (1998) verified that in order to ensure solutions free of oscillations, the Peclet condition

\[
\frac{1}{\Delta_{S_{i-\frac{1}{2}}}} > \frac{r}{\sigma^2 S_i} \quad (2.44)
\]

and the extra condition

\[
\frac{1}{(1 - \vartheta)\Delta_{\tau}} > \frac{\sigma^2 S_i^2}{2} \left( \frac{1}{\Delta_{S_{i-\frac{1}{2}}} \Delta_{S_i}} + \frac{1}{\Delta_{S_{i+\frac{1}{2}}} \Delta_{S_i}} \right) + r \quad (2.45)
\]

must be satisfied. Then, if \( r \) is relatively larger than \( \sigma \) in (2.38), i.e. the equation becomes convection-dominated, the conditions (2.44) and (2.45) would require a prohibitively fine grid spacing. If the PDE (2.38) is log transformed, it is possible to obtain an equation that contains constant coefficients for convection and diffusion, but even for this case Zvan et al. (1998) verified that such a transformation requires an excessively fine grid to carry out the calculations. Although the case where \( r \) is relatively larger than \( \sigma \) is financially unrealistic, this combination is used to show the difficulties in implementing numerical PDE methods for solving convection-dominated PDEs, and particularly since these difficulties are prevalent for valuing three-dimensional Asian options, where one space dimension possesses no diffusion.

Applying flux-limiting techniques Zvan et al. (1998) were able to rapidly obtain accurate values for Asian options. The flux \( F_{i+\frac{1}{2}}^{n+1} \) in (2.42) is then adjusted with the non-linear van Leer flux limiter

\[
V_{i+\frac{1}{2}}^{n+1} = V_{i+1}^{n+1} + \frac{\phi(q_{i+\frac{1}{2}}^{n+1})}{2} (V_{i+1}^{n+1} - V_{i+1}^{n+1}), \quad (2.46)
\]

where

\[
q_{i+\frac{1}{2}}^{n+1} = \frac{(V_{i+1}^{n+1} - V_{i+1}^{n+1})}{(S_{i+2} - S_{i+1})}, \quad (2.47)
\]

and

\[
\phi(q_{i+\frac{1}{2}}^{n+1}) = \frac{|q_{i+\frac{1}{2}}^{n+1}| + q_{i+\frac{1}{2}}^{n+1}}{1 + |q_{i+\frac{1}{2}}^{n+1}|}. \quad (2.48)
\]
Zvan et al. (1998) showed that these flux-limiting methods are total variation diminishing (TVD), ensuring that the calculated solutions are oscillation free. These procedures are related to slope limiters, that selectively define systems of equations to keep the slope of a function under realistic values. From a homogeneity point of view, Zvan et al. (1998) stated that the fixed-strike Asian option may not be solved by a similarity reduction.

Večer (2001) observed that the Asian option is a special case of an option on a traded account, where the holder is entitled to switch among various positions over the stock, and receives the call option payoff with zero strike after trading the stock. Since the value of the option depends on the trading strategy, the problem was characterised by the corresponding Hamilton-Jacobi-Bellman (HJB) equation, which is a problem of stochastic optimal control, with a resulting PDE similar to forms considered in this thesis.

An option on a traded account assume a random price for the stock \( S_t \), which follows a risk-neutral measure given by the equation 

\[
dS_t = S_t (r \, dt + \sigma \, dW_t),
\]

where the constants \( r \) and \( \sigma \) are the interest rate and the volatility of the stock (assuming no dividends payment). \( q_t \) represents the trading strategy for the number of shares on the stock owned at time \( t \), which is limited by the contract on the option to \( q_t \in [\alpha_t, \beta_t] \); \( \alpha_t \leq \beta_t \).

Then, for an initial investment \( X_0^q = X_0 \), the holder of the option observe the evolution of the traded account as 

\[
dx^q_t = q_t \, dS_t + \mu (X^q_t - q_t S_t) \, dt,
\]

with \( \mu \) being the interest rate paid on the reinvested amount \( (X^q_t - q_t S_t) \), with \( \mu \) being possible different to \( r \) (there is a self-financing strategy if \( \mu = r \)). At maturity time of the option \( T \) the holder then receives the payoff \( [X_T^q]^+ \). The value \( V^{[\alpha,\beta]} \) of this option is such that the seller of the option will be able to offset any strategy of the holder of the option, this by maximising the expected return over all strategies \( q_u \) under the risk-neutral probability \( \mathbb{P} \), namely

\[
V^{[\alpha,\beta]}(t, S_t, X_t) = \max_{q_u \in [\alpha,\beta]} e^{-r(T-t)} \mathbb{E} [[X_T^q]^+ | \mathcal{F}_t], t \in [0,T]
\]

which is characterised by the HJB equation (Večer, 2001)

\[
V_t + \max_{q_u \in [\alpha,\beta]} \left[ (\mu x + q(r - \mu)) V_x + \frac{1}{2} \sigma^2 s^2 (V_{ss} + 2q V_{sx} + q^2 V_{xx}) \right] + r s V_s - r V = 0 \quad (2.49)
\]
and the boundary condition becomes

$$V(T, s, x) = x^+$$

The change in variable

$$Z_t^q = \frac{X_t^q}{S_t},$$

develops (2.49) into

$$u_t + \max_{q \in [\alpha, \beta]} [(r - \mu)(q - z)u_z + \frac{1}{2}(q - z)^2\sigma^2u_{zz}] = 0,$$  \hspace{1cm} (2.50)

the boundary condition develops to

$$u(T, z) = z^+,$$

where

$$V(0, S_0, X_0) = S_0u \left( 0, \frac{X_0}{S_0} \right).$$

The value of an Asian option in terms of $u$ requires no control and becomes the PDE

$$u_t + r(q_t - z)u_z + \frac{1}{2}(q_t - z)^2\sigma^2u_{zz} = 0,$$  \hspace{1cm} (2.51)

with boundary condition $u(T, z) = z^+$. The fixed-strike Asian call option is defined with the fixed strategy $q_t = 1 - (t/T)$, no additional interest ($\mu = 0$) and $X_0 = S_0 - K$ (Večer, 2001). This because of the observation that $d(tS_t) = t dS_t + S_t dt$, which has an integral notation

$$TS_T = \int_0^T t \, dS_t + \int_0^T S_t \, dt.$$

After rearrangement the last expression can be used to identify the average price of the stock at $T$, namely

$$\bar{S}_T = \frac{1}{T} \int_0^T S_t \, dt = \int_0^T \left( 1 - \frac{t}{T} \right) \, dS_t + S_0,$$

and then the trading account just need to evolve accordingly to

$$dX_t = \left( 1 - \frac{t}{T} \right) \, dS_t,$$

to obtain at expiry

$$X_T = \int_0^T \left( 1 - \frac{t}{T} \right) \, dS_t + S_0 - K = \bar{S}_T - K.$$

(Večer, 2002) then assume that the stock price $S_t$, follows a risk-neutral measure given by the equation $dS_t = S_t(\delta - \delta) dt + \sigma dW_t)$, where the constants $r$, $\delta$ and $\sigma$
are the interest rate, a continuous dividend yield and the volatility of the stock. Using the previous model (2.49) (Večer, 2002) develops a strategy

\[ q_t = \frac{1}{(r-\delta)T} \left( e^{\delta (T-t)} - e^{-r(T-t)} \right) \]

and the initial wealth

\[ X_0 = \frac{1}{(r-\delta)T} \left( e^{\delta T} - e^{-rT} \right) S_0 - Ke^{-rT}. \]

The change in variable is in this case

\[ Z_t = \frac{X_t}{e^{\delta t} S_t} \]

The value of the fixed-strike Asian option \( V(t, S_t, K) \) satisfies the relationship

\[ V(0, S_0, K) = S_0 u(0, Z_0) \]

where

\[ Z_0 = \frac{X_0}{S_0} = \frac{1}{(r-\delta)T} \left( e^{\delta T} - e^{-rT} \right) - e^{-rT} \frac{K}{S_0} \]

Then \( u \) follows the partial differential equation (Večer, 2002)

\[ u_t + \frac{1}{2} (z - e^{-\delta t} q_t)^2 \sigma^2 u_{zz} = 0 \] (2.52)

with \( u(T, z) = (z)^+ \).

D’Halluin et al. (2005) followed a partial integro-differential (PIDE) equation approach to value fixed-strike Asian options of both European and American style; a hybrid semi-Lagrangian method was used in conjunction with fully implicit, Crank-Nicolson and BDF time stepping to value the price of fixed-strike Asian options with jump diffusion processes. By assuming a finite probability density function \( g(\eta) \) of a jump amplitude \( \eta \) in a time interval \([t, t + dt]\) such that for all \( \eta, g(\eta) \geq 0 \) and \( \int_0^\infty g(\eta) \, d\eta = 1 \), the stock price is modelled to follow the potential paths given by the stochastic differential equation

\[ \frac{dS}{S} = (\xi - \kappa \lambda) \, dt + \sigma \, dW + (\eta - 1) \, dq, \] (2.53)

where \( \xi \) and \( \sigma \) are the drift rate and the volatility on the stock price, \( dW \) is a standard Wiener process, \( dq \) is an independent Poisson process of value 0 with probability \( 1 - \lambda dt \), and 1 with probability \( \lambda dt \), \( \lambda \) is the mean arrival of the Poisson process,
(η − 1) is an impulse function that produces a jump on the price from $S$ to $S\eta$, and $\kappa$ is the expectation, $E[\eta − 1]$, for the impulse function $(\eta − 1)$.

In this case the arithmetic average

$$A = \frac{\int_0^t S(u) \, du}{t}, \quad \text{with} \quad dA = \frac{(S − A)}{t} \, dt,$$

(2.54)

is used in the model formulation. D’Halluin et al. (2005) were then able to represent the value of Asian options $V$ with no jumps ($\lambda = 0$) as

$$V_\tau = \frac{1}{2} \sigma^2 S^2 V_{SS} + \frac{(S − A)}{T − \tau} V_A + rSV_S − rV,$$

(2.55)

for a forward solution with $\tau = T − t$ and $r$ being the risk-free interest rate. The representation for the value of an Asian option $V$ with jumps is

$$V_\tau = \frac{1}{2} \sigma^2 S^2 V_{SS} + \frac{(S − A)}{T − \tau} V_A + (r − \lambda \kappa)SV_S − rV + \left(\lambda \int_0^\infty V(S\eta)g(\eta) \, d\eta − \lambda V\right).$$

(2.56)

And the Asian option valuation with jumps and American payoff style is

$$\min(\mathcal{H}V; V − V^*) = 0, \quad \text{for}$$

$$\mathcal{H}V \equiv V_\tau - \left[\frac{\sigma^2 S^2}{2} V_{SS} + \frac{S − A}{T − \tau} V_A + (r − \lambda \kappa)SV_S − (r + \lambda)V + \lambda \int_0^\infty V(S\eta)g(\eta) \, d\eta\right].$$

(2.57)

As a benchmark reference, we have identified that spectral expansions have been used to provide high precision results to a series of test cases (a set of cases introduced by Fu et al., 1998 which is a standard reference, see for example Shaw, 2000; Dufresne, 2000; Večer, 2002; Dewynne and Shaw, 2008) for valuing Asian options. Linetsky (2004) derived two analytical formulae for the value of continuously sampled arithmetic Asian options; the first with an infinite series of terms involving Whittaker functions $M$ and $W$, and the second is defined as a single real integral of an expression involving the Whittaker function $W$ and the addition of a finite number of terms from incomplete gamma functions and Laguerre polynomials. Linetsky has been able to compute the value of fixed-strike Asian options with ten digits of accuracy; we use this standard reference to compare our results in Chapter 3.
2.2.2 Development of the PDE for Asian options

An Asian option is a contract whose payoff depends on some kind of price average, over a time period, of the underlying asset to be traded. Additionally, the payoff may depend on the final asset price $S$ or it may depend on a predefined strike price $K$. The contract is written at time $t = 0$ giving the holder the right to exercise it in the future at a predefined time $T$, the expiry time.

Define $V$ as the value of the Asian option, which depends on a dividend-paying asset price $S$ with constant yield $\delta$, the running sum $I$ and on time $t$, $V(S, I, t)$. $S$ is assumed to follow geometric Brownian motion

$$dS = (\mu - \delta)S dt + \sigma S dW.$$ 

Accordingly to the payoff at the expiry time, Asian options may be classified as

- fixed-strike call option, $V(S, I, T) = \max \left( \frac{I}{T} - K, 0 \right)$.
- fixed-strike put option, $V(S, I, T) = \max \left( K - \frac{I}{T}, 0 \right)$.
- floating-strike call option, $V(S, I, T) = \max \left( S - \frac{I}{T}, 0 \right)$.
- floating-strike put option, $V(S, I, T) = \max \left( \frac{I}{T} - S, 0 \right)$.

Alternatively to the average, the running sum $I$ is defined and it can be divided by the elapsed time $t - t_0$; there is no loss of generality by assuming that $t_0 = 0$.

$$I = \int_{t_0}^{t} f(S(\tau), \tau) d\tau,$$

The quantity $I(t + dt)$ is found as an interpretation of the proof of the fundamental theorem of calculus (Marsden and Tromba, 2003)

$$I(t + dt) = I + dI = \int_{0}^{t + dt} f(S(\tau), t) d\tau,$$

as $dt \to 0$

$$I + dI = \int_{0}^{t} f(S(\tau), \tau) d\tau + f(S(t), t) dt,$$

by subtraction of $I$, $dI$ is found,

$$dI = f(S(t), t) dt. \quad (2.58)$$
Taking the assumptions from the Black-Scholes framework, the partial differential equation for the value of the Asian option \( V \) is developed using Itô’s Lemma on a function of three independent variables \( V(S, I, t) \).

\[
dV = \sigma S \frac{\partial V}{\partial S} \, dW + \left( (\mu - \delta) S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} + f(S,t) \frac{\partial V}{\partial I} \right) \, dt. \tag{2.59}
\]

Since \( dI \) has no random term, no additional randomness is introduced. A portfolio \( \mathcal{P} \), with value \( \Pi \), is constructed buying one option \( V \) and selling \( \Delta \) units of a borrowed underlying asset \( S \)

\[
\Pi = V - \Delta S, \tag{2.60}
\]

and the change in the portfolio with respect to slight changes in the value of the option and the asset price is

\[
d\Pi = dV - \Delta dS - \delta \Delta S \, dt \tag{2.61}
\]

The last term is subtracted because of the payment of dividends. Using the stochastic differential equation for the change in \( S \) \( (2.1) \) and the change of \( V \) \( (2.59) \) in the equation for the change in the value of the portfolio \( (2.61) \):

\[
d\Pi = \sigma S \left( \frac{\partial V}{\partial S} - \Delta \right) \, dW + \left( (\mu - \delta) S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} + f(S,t) \frac{\partial V}{\partial I} \right)
- (\mu - \delta) \Delta S - \delta \Delta S \right) \, dt. \tag{2.62}
\]

The randomness can be eliminated again by selecting

\[
\Delta = \frac{\partial V}{\partial S}. \tag{2.63}
\]

Then, the resulting increment in the value of the portfolio behaves deterministically:

\[
d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f(S,t) \frac{\partial V}{\partial I} - \delta \Delta S \right) \, dt \tag{2.64}
\]

Claiming that there cannot be any riskless profit, an amount \( \Pi \) can only have a growth, as if it were deposited on a riskless account, then \( d\Pi = r\Pi \, dt \) and

\[
r\Pi \, dt = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f(S,t) \frac{\partial V}{\partial I} - \delta \Delta S \right) \, dt \tag{2.64}
\]

Finally, bringing the definitions for \( \Delta \) \( (2.63) \) and \( \Pi \) \( (2.60) \) into \( (2.64) \)

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} + f(S,t) \frac{\partial V}{\partial I} - rV = 0. \tag{2.65}
\]
This development can be generalised for a non-constant risk-free interest rate \( r(t) \), volatility \( \sigma(t) \) and dividends \( \delta(t) \) as follows (Hugger, 2006)

\[
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(t)S^2 \frac{\partial^2 V}{\partial S^2} + (r(t)S - \delta(t)S) \frac{\partial V}{\partial S} + f(S, t) \frac{\partial V}{\partial I} - r(t)V = 0
\]

Hugger (2006) developed the following boundary conditions for the Asian average value call option. The approach is to use financial information to find a bound to the value of the option.

At expiry, the value of the option is equal to the payoff as if the contract were written in the same time, then

\[
V(S, I, T) = \max \left\{ \frac{I}{T} - K, 0 \right\} \quad \text{for } 0 \leq S \leq S_{\text{max}}, \quad 0 \leq I \leq I_{\text{max}}.
\] (2.66)

At \( S = 0 \), the asset price is, and remains, zero at any point in time, and the running sum is unchanged. Then the value of the option is in some sense independent of the \( S \) and \( I \) variables, thus \( V \) develops deterministically on the time interval \([0, T]\). Since \( V \) is known at expiry, the resulting ordinary differential equation

\[
\frac{\partial V}{\partial t} = rV
\]

is solved, as when the terminal value is \textit{back discounted} with risk-free interest rate \( r \):

\[
V(0, I, t) = e^{-\int_0^t r(\tau)d\tau} \max \left\{ \frac{I}{T} - K, 0 \right\} \quad \text{for } 0 \leq I \leq I_{\text{max}}, \quad 0 \leq t \leq T.
\] (2.67)

At \( I = I_{\text{max}} \) the value of the option can be found by constructing a portfolio \( \mathcal{P} \) (Hugger, 2006). The portfolio is made up one Asian option, \( \Delta \) units of risky assets and \( b \) units of risk-free assets, where \( b \) is defined by

\[
\frac{db}{dt}(t) = r(t)b(t) + \delta(t)S(t)\Delta(t) - \frac{d\Delta}{dt}(t)S(t),
\]

and then

\[
b(t) = \left( b(T) + \int_t^T \left[ S \frac{d\Delta}{dt} - \delta S\Delta \right](\tau)e^{\int_{\tau}^T r(\tau')d\tau'}d\tau \right) e^{-\int_0^T r(\tau)d\tau}, \quad 0 \leq t \leq T.
\]

In order to have a self financing portfolio, the value \( \Pi \) of the portfolio \( \mathcal{P} \) is

\[\Pi(S(t), I(t), t) = V(S(t), I(t), t) + \Delta(S(t), I(t), t) + b(S(t), I(t), t).\]

Setting \( \Delta = -\partial V/\partial S \), the portfolio has the property of being riskless and so

\[
\frac{d\Pi}{dt}(S(t), I(t), t) = r(t)\Pi(S(t), I(t), t).
\]
By integration

$$II(S(t), I(t), t) = II(S(T), I(T), T)e^{-\int_T^t r(u) du}, \quad 0 \leq t \leq T.$$ 

After the substitution of the value of $II$, the result is

$$V(S(t), I(t), t) = V(S(T), I(T), T)e^{-\int_T^t r(u) du} - (b(t) - b(T)e^{-\int_T^t r(u) du})$$

$$- (S(t)\Delta(t) - S(T)\Delta(T)e^{-\int_T^t r(u) du}), \quad 0 \leq t \leq T.$$ 

Substituting $b(t)$ in the last expression

$$V(S(t), I(t), t) = V(S(T), I(T), T)e^{-\int_T^t r(u) du} - \int_t^T S(\tau) \left[ \frac{d\Delta}{dt} - \delta\Delta \right] (\tau)e^{-\int_T^\tau r(u) du} d\tau$$

$$- (S(t)\Delta(t) - S(T)\Delta(T)e^{-\int_T^t r(u) du}), \quad 0 \leq t \leq T.$$ 

Letting $g = \left[ \frac{d\Delta}{dt} - \delta\Delta \right]$, and solving a differential equation as the one solved for $b$, then

$$\Delta(t) = \left( \Delta(T) + \int_t^T -g(\tau)e^{\int_T^\tau r(u) du} d\tau \right) e^{-\int_T^\tau r(u) du}, \quad 0 \leq t \leq T.$$ 

In this case $\Delta(t)$ and $\Delta(T)$ cannot be eliminated simultaneously. $\Delta(t)$ is eliminated and letting $\Delta(T) = -\partial V/\partial S(S(T), I(T), T)$ then

$$V(S(t), I(t), t) = V(S(T), I(T), T)e^{-\int_T^t r(u) du}$$

$$- \frac{\partial V}{\partial S}(S(T), I(T), T) \left( S(T)e^{-\int_T^t r(u) du} - S(t)e^{-\int_T^t r(u) du} \right)$$

$$+ \int_t^T g(\tau)(S(t)e^{-\int_T^\tau r(u) du} - S(\tau)e^{-\int_T^\tau r(u) du}) d\tau, \quad 0 \leq t \leq T.$$ 

Since $V(S(T), I(T), T)$ and $-\partial V/\partial S(S(T), I(T), T)$ are unknown, it may not be possible to continue with full generality. For the particular case of the fixed-strike Asian call option, the following observations arise: If for every $t \in [0, T]$ it is found that $I(t) > KT$ then $I(T) > KT$ since the average is non-decreasing with time, and then the payoff for the Asian option is $V(S(T), I(T), T) = (I(T)/T - K) > 0$.

Defining now the portfolio $\mathcal{P}$ comprising one Asian option (Hugger, 2006), $\Delta = -\partial V/\partial S$ units of risky assets and $b$ units of risk-free assets, then

$$\Delta(S(T), I(T), T) = \frac{\partial V}{\partial S}(S(T), I(T), T) = -\frac{\partial}{\partial S} \left( \frac{I(T)}{T} - K \right) = 0.$$ 

Since the terminal condition is not dependent of the stock price $S$ when $I > KT$, then

$$b(S(t), I(t), t) = \left( b(T) + \int_t^T g(\tau)S(\tau)e^{\int_T^\tau r(u) du} d\tau \right) e^{-\int_T^\tau r(u) du}, \quad 0 \leq t \leq T,$$
\[ \Delta(S(t), I(t), t) = - \int_t^T g(\tau) e^{\int_\tau^T \delta(u) du} d\tau, \quad 0 \leq t \leq T, \]

\[ V(S(t), I(t), t) = \left( \frac{I(T)}{T} - K \right) e^{-\int_t^T r(u) du} \]

\[ + \int_t^T g(\tau) \left( S(t) e^{-\int_t^\tau \delta(u) du} - S(\tau) e^{-\int_\tau^T r(u) du} \right) d\tau, \quad 0 \leq t \leq T. \]

Now \( b, \Delta \) and \( V \) may be expressed as deterministic functions of \( S, I \) and \( t \). For \( V \), the equivalence \( I(T) = I(t) + \int_t^T f(S(\tau), \tau) d\tau = I(t) + \int_t^T S(\tau) d\tau \), leads to

\[ V(S(t), I(t), t) = \left( \frac{I(t)}{T} - K \right) e^{-\int_t^T r(u) du} \]

\[ + \int_t^T \left\{ g(\tau) \left( S(t) e^{-\int_t^\tau \delta(u) du} - S(\tau) e^{-\int_\tau^T r(u) du} \right) \right\} d\tau, \quad 0 \leq t \leq T. \]

With \( g(\tau) = (1/T)e^{-\int_t^T r(u) du} \), \( V \) may be written in the form

\[ V(S(t), I(t), t) = \left( \frac{I(t)}{T} - K \right) e^{-\int_t^T r(u) du} \]

\[ + \frac{S(\tau)}{T} \int_t^T e^{-\int_t^\tau \delta(u) du + \int_\tau^T r(u) du} d\tau, \quad 0 \leq t \leq T. \]

Furthermore

\[ b(S(t), I(t), t) = \left( b(T) + \int_t^T \frac{S(\tau)}{T} d\tau \right) e^{-\int_t^T r(u) du} \]

\[ = \left( b(T) + \frac{1}{T}(I(T) - I(t)) \right) e^{-\int_t^T r(u) du}, \quad 0 \leq t \leq T. \]

Using \( b(T) = -I(T)/T + c \) to solve the first-order differential equation of \( b(t) \) and selecting \( c = K \) in order that \( II(T) = V(T) + \Delta(T) + b(T) = V(T) + b(T) = 0 \), then

\[ b(S(t), I(t), t) = - \left( \frac{I(t)}{T} - K \right) e^{-\int_t^T r(u) du}, \quad 0 \leq t \leq T. \]

Finally, choosing \( I_{\text{max}} \geq KT \):

\[ V(S, I, t) = \left( \frac{I}{T} - K \right) e^{-\int_t^T r(u) du} + \frac{S}{T} \int_t^T e^{-\int_t^\tau \delta(u) du + \int_\tau^T r(u) du} d\tau \]

\[ \text{for } 0 \leq S \leq S_{\text{max}}, \quad KT \leq I \leq I_{\text{max}}, \quad 0 \leq t \leq T. \] (2.68)

The derivatives with respect to \( I \) are then

\[ \frac{\partial V(S, I, t)}{\partial I} = \frac{1}{T} e^{-\int_t^T r(u) du}, \quad \frac{\partial^p V(S, I, t)}{\partial I^p} = 0 \quad \text{for } p \geq 2, \]
for \(0 \leq S \leq S_{\text{max}}\), \(KT \leq I \leq I_{\text{max}},\ 0 \leq t \leq T\).  \hfill (2.69)

As \(S_{\text{max}} \to \infty\), the boundary is obtained directly from (2.68)

\[
V(S_{\text{max}}, I, t) = \left(\frac{I}{T} - K\right) e^{-\int_t^T r(u) \, du} + \frac{S_{\text{max}}}{T} \int_t^T e^{-\int_t^\tau \delta(u) \, du + \int_t^\tau r(u) \, du} \, d\tau
\]

for \(S_{\text{max}} \to \infty,\ 0 \leq I \leq I_{\text{max}},\ 0 \leq t \leq T\).  \hfill (2.70)

At \(S = S_{\text{max}}\), the value is approached deriving approximate boundary conditions:

\[
V(S_{\text{max}}, I, t) \simeq \max\left\{ \left(\frac{I}{T} - K\right) e^{-\int_t^T r(u) \, du} + \frac{S_{\text{max}}}{T} \int_t^T e^{-\int_t^\tau \delta(u) \, du + \int_t^\tau r(u) \, du} \, d\tau, 0 \right\}
\]

for \(S_{\text{max}} < \infty,\ 0 \leq I \leq I_{\text{max}},\ 0 \leq t \leq T\).  \hfill (2.71)

\[
\frac{\partial V(S, I, t)}{\partial S} = \frac{1}{T} e^{-\int_t^\tau \delta(u) \, du + \int_t^\tau r(u) \, du} \, d\tau,
\]

\[
\frac{\partial^p V(S, I, t)}{\partial S^p} = 0 \quad \text{for} \quad p \geq 2, \quad \text{for} \quad KT \leq I \leq I_{\text{max}},\ 0 \leq t \leq T.  \hfill (2.72)
\]

With \(\delta(u) = 0\) (zero dividends) and a fixed interest rate \(r\), the boundary conditions become:

At expiry

\[
V(S, I, T) = \max\left\{ \left(\frac{I}{T} - K\right) \right\} \quad \text{for} \quad 0 \leq S \leq S_{\text{max}},\ 0 \leq I \leq I_{\text{max}}.  \hfill (2.73)
\]

On \(S = 0\)

\[
V(0, I, t) = e^{-r(T-t)} \max\left\{ \left(\frac{I}{T} - K\right) \right\} \quad \text{for} \quad 0 \leq I \leq I_{\text{max}},\ 0 \leq t \leq T.  \hfill (2.74)
\]

At \(I = I_{\text{max}}\)

\[
V(S, I, t) = \left(\frac{I}{T} - K\right) e^{-r(T-t)} + \frac{S}{rT} (1 - e^{-r(T-t)}) \quad \text{for} \quad 0 \leq S \leq S_{\text{max}},\ KT \leq I \leq I_{\text{max}},\ 0 \leq t \leq T.  \hfill (2.75)
\]

At \(S = S_{\text{max}}\)

\[
V(S_{\text{max}}, I, t) = \max\left\{ \left(\frac{I}{T} - K\right) e^{-r(T-t)} + \frac{S_{\text{max}}}{rT} (1 - e^{-r(T-t)}), 0 \right\}
\]

for \(S_{\text{max}} < \infty,\ 0 \leq I \leq I_{\text{max}},\ 0 \leq t \leq T.  \hfill (2.76)

The agreement with the boundary conditions from Kemna and Vorst (1990) is at hand. With the equivalence of \(A(t) \equiv I(t)/T\), and setting \(T_0 = 0\). The set of boundary conditions for the fixed-strike Asian option match completely; (2.33), (2.34) and (2.36) are equal to (2.73), (2.74) and (2.75), respectively. Only the boundary condition as
$S \to \infty$ needs some work; from the partial derivative of $V$ with respect to $S$, (2.76) may be written as
\[
\frac{\partial V}{\partial S} = \frac{e^{r(T-t)} - 1}{rT} e^{-r(T-t)},
\]
if now $e^{r(T-t)}$ is expressed to a first order Taylor approximation
\[
\frac{\partial V}{\partial S} = \frac{(T-t)}{T} e^{-r(T-t)},
\]
with the final substitution $\beta = T^{-1}$ the boundary conditions (2.35) and (2.76) are consistent.

### 2.2.3 Similarity Reduction

Rogers and Shi (1995) exploited a scaling property on the problem to reduce the dimensions from three to two dimensions, for both the fixed-strike and the floating-strike Asian options. They followed a martingale approach to derive their equation; thus the problem is stated in terms of expectations. In addition to the PDE formulation, lower and upper bounds for the value of the option were introduced, with a really accurate lower bound. First they state the objective of the formulation as
\[
\mathbb{E}(Y - K)^+
\]
defining
\[
Y \equiv \int_0^T S_u \eta(du)
\]
where, at time $t$, a dividends paying asset price $S_t$ with dividend yield $\delta$ follows a geometric Brownian motion in the form $S_t = S_0 \exp \left( \sigma W_t - \frac{1}{2} \sigma^2 t + (c - \delta) t \right)$; where $W_t$ is a Wiener process, $c$ is assumed to be equal to the riskless interest rate $r$ with a dividend yield $\delta = 0$, $\sigma$ is the volatility of the asset, $T$ is the maturity time of the option, $K$ is the strike price of the option, and the measure $\eta$ is defined as $\eta(du) = T^{-1} I_{[0,T]}(u) du$ for a fixed-strike Asian option, and for a floating-strike Asian option $\eta(du) = T^{-1} I_{[0,T]}(u) du - \gamma(u) du$ and $K = 0$; $I_{[0,T]}(u)$ is implied to be a sum function, and the expectation operator is implied to be taken under risk-neutrality ($c = r$ above).

Continuing, Rogers and Shi (1995) define
\[
\phi(t, x) \equiv \mathbb{E} \left[ \left( \int_t^T S_u \eta(du) - x \right)^+ \middle| S_t = 1 \right],
\]
with $x$ being the scaled variable ($\xi$ on the probability notation), then they develop a martingale technique and arrive at the result

$$0 = \phi_t + r\phi + \frac{1}{2} \sigma^2 \xi^2 \phi_{\xi\xi} - (\rho_t + r\xi)\phi_{\xi},$$

with

$$\xi_t = \frac{K - \int_0^t S_u \eta(du)}{S_t}.$$  \hspace{1cm} (2.77)

After defining $f(t, x) \equiv e^{-r(T-t)}\phi(t, x)$, Rogers and Shi (1995) solve

$$f_t + \mathcal{G} f = 0,$$

with the operator

$$\mathcal{G} \equiv \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} - (\rho_t + rx) \frac{\partial}{\partial x}.$$

The boundary conditions imposed for a fixed-strike Asian option are, at expiry

$$f(T, x) = x^-, \hspace{1cm} \text{together with} \hspace{1cm} \phi(t, x) = \frac{e^{r(T-t)} - 1}{r} - x, \hspace{1cm} \text{for} \hspace{0.5cm} x \leq 0,$$

and for a floating-strike Asian option, at expiry

$$f(T, x) = (1 + x)^-, \hspace{1cm} \text{together with} \hspace{1cm} \phi(t, x) \approx \frac{e^{r(T-t)} - 1}{rT} - e^{r(T-t)} - x, \hspace{1cm} \text{as} \hspace{0.5cm} x \to -\infty.$$  \hspace{1cm} (2.81)

Rogers and Shi (1995) derived upper and lower bounds using an expectations approach, and it was shown that for typical values of $\sigma$ and $r$, of the order of $10^{-1}$, the error will be of the order of $10^{-2}$ at worst; the results presented include examples with

$$\sigma \in \{0.05, 0.1, 0.2, 0.3\}, \hspace{0.5cm} r \in \{0.05, 0.09, 0.15\}.$$

Wilmott (1995) presented a similarity reduction for the case when the payoff has the form $S^\alpha H(I/S, t)$. For the floating-strike Asian call option with continuous averaging, $V = S^\alpha H(R, t)$, where $R = I/S$, and

$$\frac{\partial H}{\partial t} + \frac{1}{2} \sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 + (\sigma^2(1-\alpha) - r) R) \frac{\partial H}{\partial R} - (1-\alpha)(\frac{1}{2} \sigma^2 \alpha + r) H = 0.$$  \hspace{1cm} (2.82)
From the boundary condition at expiry \( V(S,I,t) = \max\{S - I/T,0\} \), as there is only a linear term of \( S \), \( \alpha = 1 \), and the PDE becomes

\[
\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR) \frac{\partial H}{\partial R} = 0.
\]

(2.83)

The payoff at expiry is

\[ H(R,t) = \max\{1 - R/T,0\} \]

The condition as \( R \to \infty \), on a bounded domain, occurs only when \( S \to 0 \) and then

\[ H(\infty,t) = 0. \]

For the boundary on \( R = 0 \), first the stochastic differential equation of \( R(t) = I(t)/S(t) \) is required. \( I(t) \) is replaced with \( \int_0^t S(\tau) \, d\tau \), then the derivatives \( \partial R/\partial S = -I/S^2 \), \( \partial^2 R/\partial S^2 = 2I/S^3 \) and \( \partial R/\partial t = 1 \) are substituted into Itô’s Lemma (2.7), and lead to

\[
dR = -\sigma R \, dW + (1 + (\sigma^2 - \mu)R) \, dt,
\]

(2.84)

which gives the result \( dR = dt > 0 \) when \( R = 0 \), and means that the value of \( R \) instantaneously separates from \( R = 0 \) into \( R > 0 \) and the value of the option cannot be known with certainty, but must be finite. Now, as \( R \to 0 \), the term \( R\partial H/\partial R \ll \partial H/\partial R \) and thus is negligible. \( R^2\partial^2 H/\partial R^2 \) must be also negligible, because if not it would lead to an unrealistic large value, suppose by contradiction that

\[
\lim_{R \to 0} R^2 \frac{\partial^2 H}{\partial R^2} = O(1).
\]

As \( R \to 0 \), then

\[
\frac{\partial^2 H}{\partial R^2} = O\left(\frac{1}{R^2}\right),
\]

implying that \( H = O(\log R) \), which contradicts the fact that the option must have a finite value (Wilmott, 1995), then

\[
\frac{\partial H}{\partial t} + \frac{\partial H}{\partial R} = 0
\]

(2.85)

Dewynne and Shaw (2008) used the fact that the fixed-strike Asian PDE is autonomous in \( I \) and derived a simplified equation with \( I' = I - KT \).

Now the reduction of the problem for the fixed-strike Asian option from three to two dimensions will be explored. Here we use a similarity reduction to obtain
a (novel) variation on the fixed-strike Asian PDE formulation. For the fixed-strike Asian option, at expiry the payoff is \( V(S,I,T) = \max(I/T - K,0) \). There is no possibility to express the payoff in terms of \( R = I/S \), but an appropriate offset of the \( I \) axis will allow us to express the function in terms of \( R \) alone. We exploit the fact that \( V(S,I,T) = \max(I/T - K,0) \) can be written as \( V = S^\alpha H(R,t) \), where \( R = (I - TK)/S \).

The partial derivatives of \( V(S,I,t) \) when \( V = S^\alpha H((I - TK)/S,t) \) take the form

\[
\frac{\partial V}{\partial t} = S^\alpha \frac{\partial H}{\partial t},
\]

\[
\frac{\partial V}{\partial S} = \alpha S^{\alpha-1} H - S^{\alpha-2} (I - TK) \frac{\partial H}{\partial R},
\]

\[
\frac{\partial^2 V}{\partial S^2} = \alpha(\alpha - 1)S^{\alpha-2} H - 2(\alpha - 1)(I - TK) S^{\alpha-3} \frac{\partial H}{\partial R} + S^{\alpha-4} (I - TK)^2 \frac{\partial^2 H}{\partial R^2}
\]

\[
\frac{\partial V}{\partial I} = S^{\alpha-1} \frac{\partial H}{\partial R}
\]

Substituting the previous derivatives into the Asian option PDE (2.65), lead to

\[
\frac{\partial H}{\partial t} + \frac{1}{2} \sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 + (\sigma^2(1 - \alpha) - r)R) \frac{\partial H}{\partial R} - (1 - \alpha)(\frac{1}{2} \sigma^2 \alpha + r) H = 0.
\]

The PDE (2.82) for the function \( H(R,t) \) is not modified by letting \( R = (I - TK)/S \).

Taking \( \alpha = 1 \), as there is only a linear term of \( S \), the equation becomes

\[
\frac{\partial H}{\partial t} + \frac{1}{2} \sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR) \frac{\partial H}{\partial R} = 0. \tag{2.86}
\]

This can written in forward Lagrangian form, with \( \tau = T - t \), as

\[
\frac{DH}{D\tau} = \frac{1}{2} \sigma^2 R^2 \frac{\partial^2 H}{\partial R^2}, \tag{2.87}
\]

where

\[
\frac{DH}{D\tau} = \frac{\partial H}{\partial \tau} + \frac{\partial X}{\partial R} \frac{dR}{d\tau},
\]

and

\[
\frac{dR}{d\tau} = -(1 - rR).
\]

The equation requires boundary conditions and several alternatives are available. Boundary conditions of Dirichlet (first type), Neumann (second type) and Robin (third type) can be derived.

In the proposed similarity reduction, the boundary conditions derived by Hugger (2006) are interpreted as follows when \( \tau = T - t \).
Let \( R = \frac{I - TK}{S} \), \( V(S, I, t) = SH(R, t) \), then:

At expiry
\[
H(R, T) = \max\{R/T, 0\}, \quad \text{for } -\infty \leq R \leq \infty.
\]  
(2.88)

As \( R \to -\infty \),
\[
H \to 0, \quad \text{for } 0 \leq \tau \leq T
\]  
(2.89)

(the crucial section of the domain to solve occurs when \( R \) is evaluated to a negative number i.e. whenever \( I/T < K \), and \( R \to -\infty \) when simultaneously \( S \to 0 \)).

On \( R = 0 \), for half domain, a large value of \( S \) or when \( I = TK \), (2.76), (2.75) imply respectively that
\[
H = \frac{(1 - e^{-r\tau})}{rT}, \quad \text{for } 0 \leq \tau \leq T.
\]  
(2.90)

As \( R \to \infty \), for full domain,
\[
H \to \frac{Re^{-r\tau}}{T}, \quad \text{for } 0 \leq \tau \leq T,
\]
in the unbounded domain, but for the bounded domain \( S \to 0 \), but \( S \neq 0 \) and to adjust the bound the second term in the strategy (2.37) corrects the value to
\[
H = \frac{Re^{-r\tau}}{T} + \frac{1 - e^{-r\tau}}{rT}, \quad \text{for } 0 \leq \tau \leq T.
\]  
(2.91)

The Neumann boundary condition as \( R \to \infty \) is
\[
\frac{\partial H}{\partial R} = \frac{e^{-r\tau}}{T}.
\]  
(2.92)

The Robin boundary conditions are
\[
\frac{\partial H}{\partial \tau} = \frac{\partial H}{\partial R}, \quad \text{on } R = 0.1
\]  
(2.93)
\[
\frac{\partial H}{\partial \tau} = (1 - rR)\frac{\partial H}{\partial R}, \quad \text{as } R \to \infty.
\]  
(2.94)

Agreement with the formulation from Rogers and Shi (1995) can be derived. Following their equivalence \( f(t, x) \equiv e^{-r(T-t)}\phi(t, x) \), their bound (2.79), and \( x = -R/T \), the Dirichlet boundary conditions can be obtained.

The boundary conditions for the floating-strike Asian put option are:

At expiry
\[
H(R, T) = \max\{R/T - 1, 0\}, \quad \text{for } 0 \leq R \leq \infty.
\]  
(2.95)

\(^1\) \( R = (I - TK)/S \) does not alter the stochastic differential equation for \( dR \) (2.84).
Examining this expression, and given that the floating and the fixed-strike options are defined by the same PDE. The floating-strike Asian option has an unitary shift to the right of the $R$ axis with respect to the fixed-strike Asian option, then their boundary conditions as $R \to \infty$ are related, but with an unitary displacement. The interpretation is consistent with that of Rogers and Shi (1995), as (2.81) is declared to be an approximate value, if (2.81) is substituted in their equivalence $f(t,x) \equiv e^{-r(T-t)} \phi(t,x)$, and $x = -R/T$, the Dirichlet boundary condition is found. Then as $R \to \infty$

$$H \approx \frac{R e^{-r\tau}}{T} + \frac{1 - e^{-r\tau}}{rT} - 1.$$ (2.96)

The Neumann boundary condition is again

$$\frac{\partial H}{\partial R} = \frac{e^{-r\tau}}{T}. $$ (2.97)

The Robin boundary conditions are again

$$\frac{\partial H}{\partial \tau} = \frac{\partial H}{\partial R}, \text{ on } R = 0.$$ (2.98)

$$\frac{\partial H}{\partial \tau} = (1 - rR) \frac{\partial H}{\partial R}, \text{ as } R \to \infty.$$ (2.99)

### 2.3 Storage system formulation

Thompson et al. (2009) introduced a partial integro-differential equation (PIDE) model for the valuation and optimisation of natural gas storage facilities. The PIDE model accounts for operational constraints of storage capacity, rates of injection and extraction, as well as the possibility of alternating between injecting and extracting, the objective being to maximise the cash flow from buying and selling gas under the assumption of leakage

$$\max_{c(P,I,t)} E \left[ \int_0^T e^{-r\tau} (c - a(I,c))P \, d\tau \right],$$ (2.100)

subject to

$$c_{\min}(I) \leq c \leq c_{\max}(I).$$ (2.101)

Here $t$ denotes chronological backwards time from the horizon time $T$, and $\tau$ denotes forward time ($\tau = T - t$), $I$ is the amount of gas inside the storage facility, $I_{max}$ is the maximum capacity of the storage facility, $c$ is a control variable representing the
amount of gas extracted \((c > 0)\) or injected \((c < 0)\) into the storage, \(c_{\text{min}}\) and \(c_{\text{max}}\) are the maximum allowed injection and extraction rates respectively, \(a(I, c)\) is the amount of gas that is lost due to leakage, \(\rho\) is the risk-free interest rate, \(T\) is the horizon time, and \(E[\cdot]\) is the expectation under the risk-free measure taken over the random price of gas \(P\). The amount of gas in storage \(I\) is governed by the dynamic

\[
dI = -(c + a(I, c))
dt.\]

The stochasticity introduced into the model by Thompson et al. (2009) was that for the spot price of gas \(P\); a continuous time Itô process with jumps was used.

\[
dP = \mu(P, t)\ dt + \sigma(P, t)\ dW + \sum_{k=1}^{N} \gamma(P, t, J)\ dq, \tag{2.102}
\]

where \(\mu\), \(\sigma\) and \(\gamma\) are arbitrary functions of price, time or both. \(J\) is drawn from an arbitrary distribution \(Q_k(J)\). \(dW\) represents a standard increment of Brownian motion. \(dq\) is a Poisson process of value 0 with probability \(1 - \epsilon(P, t)\ dt\), and of value 1 with probability \(\epsilon(P, t)\ dt\).

Thompson et al. (2009) derived a Bellman-type equation to approximate value \(V\) of the storage facility

\[
\max_c \left[ \int_t^{t+\ dt} e^{-\rho(\tau-t)}(c - a(I, c))P \ d\tau + e^{-\rho dt}V(P + dP, I + dI, t + dt) \right], \tag{2.103}
\]

that can be stated as

\[
\max_c \left[V_t + \frac{1}{2}\sigma^2 V_{PP} + \mu V_P - (c + a(I, c))V_I - \rho V + (c - a(I, c))P \\
+ \sum_{k=1}^{N} \epsilon_k E[V_k^+ - V] \right] = 0. \tag{2.104}
\]

The optimal rate \(c\) is obtained from the terms involving it, by solving

\[
\max_c \left[- (c + a(I, c))V_I + (c - a(I, c))P \right] \tag{2.105}
\]

subject to

\[
c_{\text{min}}(I) \leq c \leq c_{\text{max}}(I). \tag{2.106}
\]

By neglecting the leakage assumption, the interpretation is that if the marginal change in \(V\) with respect to a small change in \(I\) is greater than the current gas spot price then
the optimal decision is to use the smallest value allowed for the control \( c_{\text{min}} \), otherwise the optimal decision is to use the greatest value allowed for the control \( c_{\text{max}} \), so that under the leakage operational constraint a gap (between the previous decisions) is generated where the optimal decision is not to inject or withdraw gas (\( c = 0 \)). Once the optimal value of \( c \) is determined, the optimal solution of (2.104) is implied.

Total variation diminishing (TVD) methods can be used to solve the related PDE effectively, preventing oscillations in the numerical solution, but since the control is a function of derivative terms, in this case \( V_t \), the implied (approximated) values for these derivative terms cannot be guaranteed to be oscillation-free, thus a parallel PDE must be solved to ensure a smooth control. Thompson et al. (2009) then solved

\[
\max_c \left[ W_t + \frac{1}{2} \sigma^2 W_{PP} + \mu W_P - \left( c + a(I, c) \right) W_I - a_I(I, c) W - \rho W + a_I(I, c) \right] P \\
+ \sum_{k=1}^{N} \epsilon_k E\left[ W_k^+ - W \right] = 0
\]  

(2.107)

where \( W = V_t \), the objective being to determine the control values to be substituted in PDE (2.104). The numerical solution was approached with an explicit finite-difference scheme and the derivative terms with respect to \( I \) were evaluated using slope limiting methods (where selectively systems of equations are defined to keep the slope under realistic values and therefore obtaining solutions that are free of oscillations).

Thompson et al. (2004) derived models for the valuation and optimal operation of electricity power plants. Their models feature the stochastic behavior of electricity spot prices and were defined to incorporate a second source of stochasticity, an unpredictable amount of inflow water is incorporated to the model for the hydroelectric power plant, and a random price of the fuel is incorporated to the model of thermal generators. The nature of storage is present in the hydroelectric generator model, as water is released from a reservoir to generate electricity and water is pumped back to the reservoir to accumulate potential energy. Even though the formulations were made with respect to two sources of randomness, Thompson et al. (2004) solved the hydroelectric problem assuming a closed system with no random inflow water, and assumed a fixed fuel price to solve the thermal generator problem, thus solving only for the case of random prices of electricity and no stochasticity in the resource variable. For the numerical solution, explicit finite-difference schemes with second-order slope limiters
were utilised, although those techniques are time-step restricted due to stability issues (in our case time steps can be decoupled from the space resolution, such that a fully four-dimensional model can be efficiently solved).

2.4 Conclusions

Real world systems have to deal with uncertain factors at all times, and depending on the nature of these random variations an impact on supply or demand may be effected. Storage systems prove their usefulness by smoothing fluctuations and then ensuring the availability of resources, then we find that there must be a value for operating a storage, as resources are stockpiled at some point in time and resources are released from the storage at a different time. From this chapter, we found that storage problems can be formulated as a function of variables with stochastic nature to take into account the effects of uncertainty; mathematical concepts and frameworks have been provided for the development.

Here we have identified financial models for Asian options that are akin to the stochastic storage valuation problem. Taking as a basis the mathematical models obtained, we identify that both problems can be represented by advection-diffusion partial differential equations, with the marked presence of a space dimension with the absence of diffusion, namely the first-order term related to the average $V_A$, see (2.55), or equivalently the running sum $V_I$, see (2.65), and the first-order term related to the level of the storage $V_I$, see (2.104). Then if we can find a technique to value Asian options, then we can extend these techniques to find the value of storage.

We followed a PDE approach to find a similarity reduction for the fixed-strike Asian option, a model that is comparable to that obtained with the Martingale technique. It is found for the fixed-strike Asian option that the bounds from Kemna and Vorst (1990), Rogers and Shi (1995) and Hugger (2006) are consistent, and then full formulations for the associated boundary value problems were presented. We can also identify that our similarity reduction is relatively equivalent to the formulation obtained for a PDE model for Asian options based on a traded account, and then one can see that (2.86) and (2.51) are close related, as (2.87) and (2.52) are.
We select the fixed-strike Asian option since its mathematical complexity precludes analytical solutions and challenges numerical solution methods; mainly because the mathematical model for Asian options resemble the model of stochastic storage. Linetsky (2004) has been able to compute highly accurate results for valuing fixed-strike Asian options using spectral expansions techniques, and compiles a series of examples with results obtained with the numerical Laplace transformation, Monte Carlo simulations and bounds for the value of the option derived with expectations arguments; we use this standard benchmark to compare models and for testing the effectiveness and efficiency of numerical solution methods that we develop in the next chapter.
Chapter 3

Numerical methods for financial and storage-related PDEs

The mathematical complexity of the partial differential equations for modeling financial Asian options and stochastic storage problems preclude analytical solutions and then numerical methods need to be implemented to obtain a numerical approximation to the solution. The approach followed here for numerically approximating partial differential equations solutions is that of finite-differences. Basically the method relies on substituting the derivatives at a point, with approximate difference quotients over a small interval, which means replacing \( \frac{\partial \phi}{\partial x} \) with \( \frac{\Delta \phi}{\Delta x} \) where \( \Delta x \) is small (Smith, 1965).

The partial differential equations to be treated here can be referred as a convection-diffusion equations (Zvan et al., 1998) or advection-diffusion equations (Spiegelman and Katz, 2006). Here, the terms advection and convection will be used as equivalents under the assumption that convection can be interpreted as advection when these describe effects from first-order (hyperbolic-like) terms. The term diffusion will be used to describe the effects from second-order terms. Also, we explore the inclusion of a mixed derivative term to introduce correlation effects.

In order to numerically solve the PDEs to be derived, classical methods for solving diffusion type problems with a finite-differences approach can be implemented, namely the fully implicit (Smith, 1965), the Crank-Nicolson (Crank and Nicolson, 1996) and the alternating-direction implicit (ADI) method (Peaceman and Rachford,
Although these methods are known to be prone to oscillations when the magnitude of the advection (first-order) terms is larger than the magnitude of the diffusion (second-order) terms, a condition prevalent in Asian options and stochastic storage PDE models with the absence of diffusion in one spatial dimension. Therefore we combine standard finite-difference techniques for solving the diffusion equation with the method of characteristics for solving advection-type problems, namely the semi-Lagrangian scheme (Staniforth and Coté, 1991).

We present existing hybrid methodologies that include the semi-Lagrangian fully implicit method and the semi-Lagrangian Crank-Nicolson method. Here we introduce and define a novel semi-Lagrangian alternating-direction implicit methodology to efficiently treat mixed advection and diffusion problems with the objective to approximate the valuation of stochastic storage problems with two sources of uncertainty.

Once that hybrid algorithms have been defined, we proceed to detail the implementation of these schemes to approximate the valuation of the fixed-strike Asian option, next we assess the accuracy of the results with respect to benchmark examples to show the merits of these techniques.

### 3.1 Classical methods for the diffusion equation

This section presents standard finite-difference methodologies that have been developed for approximating the numerical solution of diffusion-type problems.

#### 3.1.1 The fully implicit and Crank-Nicolson methods

The fully implicit and Crank-Nicolson methods are schemes used to find the approximate value of the diffusion equation

\[
\frac{\partial u}{\partial \tau} = \nabla^2 u
\]

which can be written as

\[
\frac{u^{n+1} - u^n}{\Delta \tau} = \vartheta Lu^{n+1} + (1 - \vartheta)Lu^n
\]

where (in our terminology) \( \tau = T - t \), \( u^n \) is the solution to the PDE on a regular mesh point at time step \( n \) and \( L \) is the discrete diffusion operator. If \( \vartheta = 0 \) then
an explicit scheme is defined, $\vartheta = \frac{1}{2}$ gives the Crank-Nicolson method (Crank and Nicolson, 1996), and $\vartheta = 1$ results in the fully implicit backward time-differencing method (Smith, 1965).

### 3.1.2 ADI method

This operator splitting methodology solves the initial value equation

$$\frac{\partial u}{\partial \tau} = Lu. \tag{3.1}$$

The operator $L$ acts in a $d$-dimensional space and can be written as a linear sum of $d$ pieces, one for each dimension:

$$Lu = L_1u + L_2u + \cdots + L_du. \tag{3.2}$$

The idea is to approximate $\partial u/\partial \tau$ in $d$ steps, each step progressing a fraction of time $\frac{1}{d}\Delta \tau$ until a full time $\Delta \tau$ is completed. At the $l$th step the piece $L_l$ is approximated implicitly in terms of the unknown values of $u$ at time $\tau + \frac{l-1}{d}\Delta \tau$, and all the other pieces $L_m, m \neq l$, are approximated explicitly in terms of the known values of $u$ at time $\tau + \frac{l-1}{d}\Delta \tau$.

$$\frac{u^{n+\frac{l}{d}} - u^n}{\frac{1}{d}\Delta \tau} = L_1u^{n+\frac{l}{d}} + L_2u^n + \cdots + L_du^n,$$

$$\frac{u^{n+\frac{1}{d}} - u^{n+\frac{l-1}{d}}}{\frac{1}{d}\Delta \tau} = L_1u^{n+\frac{1}{d}} + L_2u^{n+\frac{l-1}{d}} + \cdots + L_du^{n+\frac{l-1}{d}},$$

$$\vdots$$

$$\frac{u^{n+1} - u^{n+1-\frac{l}{d}}}{\frac{1}{d}\Delta \tau} = L_1u^{n+1-\frac{l}{d}} + L_2u^{n+1-\frac{l}{d}} + \cdots + L_du^{n+1}. \tag{3.3}$$

The alternating-direction implicit method is a standard procedure in the solution of diffusion-like equations in two dimensions (Peaceman and Rachford, 1955), and can also be applied to solve diffusion type equations in three dimensions (Douglas, 1962), and also for $d$-dimensional problems (Douglas Jr and Gunn, 1962).

On their own, these standard methods for solving diffusion type problems are prone to oscillations when the magnitude of the advection terms is large compared to the magnitude of the diffusion terms. Financially this combination is unrealistic, but is
a condition prevalent in PDE formulations where one of the dimensions possess no
diffusion. In our case, the situation is somewhat more complicated, on account of the
system comprising two hyperbolic-type dimensions, namely the time variable and the
storage variable (the latter exhibiting both forward and backward characteristics due
to the possibility of consumption or replenishment), and this issue is addressed next.

3.2 Lagrangian schemes

In an Eulerian approach, the PDE is solved at mesh points on a fixed grid, but is limited
in the time step that it can handle due to computational stability. In fully Lagrangian
methods for advection (also known as particle tracking schemes), the spatial grid
moves with the ‘flow’, and thus the PDE is solved at the ending position of particles
with known origin. Lagrangian schemes may start from a regular grid and evolve
into a generally highly irregular grid, but it can handle larger time steps (see for
example, Staniforth and Coté, 1991; Alam and Lin, 2008; Bowman et al., 2014).

A Lagrangian scheme solves (for example) the passive-advection problem
\[
\frac{\partial u}{\partial \tau} + \mathbf{v} \cdot \nabla u = 0.
\]
for virtual particles following corresponding flow characteristics. The PDE is solved
for every virtual particle, and large time steps can be handled by determining the
ending position of these virtual particles.

In a Lagrangian scheme the equation
\[
\frac{D u(\tau, \mathbf{x})}{D \tau} = \frac{\partial u}{\partial \tau} + \frac{d \mathbf{x}}{d \tau} \cdot \nabla u = \frac{\partial u}{\partial \tau} + \mathbf{v} \cdot \nabla u = 0
\]
is solved on a spatial grid that is distorted from the original position $\mathbf{x}$ of virtual
particles at time $\tau$ to the final position $\mathbf{x}_*$ of virtual particles at time $\tau + \Delta \tau$, and
$u_{*, n+1} = u^n$ where $u_{*, n+1} = u^{n+1}(\tau, \mathbf{x}_*)$ is the solution to the PDE for a virtual particle
arriving to the ending position at time step $n + 1$, $u^n = u^n(\tau, \mathbf{x})$ is the solution to
the PDE at the original position. The ending position is approximated by solving the
ordinary differential equation for the particle tracking problem
\[
\frac{d \mathbf{x}}{d \tau} = \mathbf{v}
\]
starting at the mesh point \( x \) and moving forward in time to the ending position \( x^* \), and then
\[
x^* = x + \int_\tau^{\tau+\Delta \tau} v \, dt.
\]

In a Lagrangian scheme, the principal components of the algorithm are basically a method for particle tracking (via the flow characteristics), and a method for visualisation (as the Lagrangian solution may need to be represented in an Eulerian framework), and it is common practice to redefine the spatial grid after a finite number of time steps (Bowman et al., 2014). A Lagrangian scheme will retain, naturally, flow properties, e.g. velocity, density, concentration or temperature (Alam and Lin, 2008).

### 3.3 Semi-Lagrangian schemes

These schemes are also based on the method of characteristics and solve the advection-reaction problem
\[
\frac{\partial u}{\partial \tau} + v \cdot \nabla u = f(u, x, \tau).
\]
for every mesh point on a regular grid.

A semi-Lagrangian advection solves the PDE for every mesh point on a regular grid, and is (also) able to handle larger time steps by tracking back the position of a value in the previous time step (Staniforth and Coté, 1991).

The semi-Lagrangian scheme then, treats the equation
\[
\frac{D u}{D \tau} = f(u, x, \tau)
\]
as an ordinary differential equation along the trajectory joining the departure position \( x^* \) at time \( \tau \) to a regular mesh point \( x \) at time \( \tau + \Delta \tau \), this can be written as
\[
u^{n+1} = u^n + \int_\tau^{\tau+\Delta \tau} f(u(t), x(t), \tau) \, dt,
\]
where \( u^{n+1} \) is the solution to the PDE on a regular mesh point at time step \( n + 1 \), \( u^n = u^n(\tau, x^*) \) is the solution to the PDE at the departure position, and the last term is the line integral of the source terms along the trajectory. The departure position is approximated by solving the ordinary differential equation for the particle tracking problem
\[
\frac{dx}{d\tau} = -v,
\]
starting at the mesh point $x$ and moving back in time. To complete the procedure, an approximation to the final integral is required; when a trapezoidal rule is used, the algorithm can be written

$$u^{n+1} = u^n + \frac{\Delta \tau}{2} \left[ f^n + f^{n+1} \right],$$

where $f^n = f(u^n, \tau)$ is the value of the source term at the departure position (Spiegelman and Katz, 2006). Usually the departure position is not on the uniform grid, and then the value of the solution and the value of the source term are obtained by interpolation.

The interpolation for the associated values of $u^n$ and $f^n$ is crucial for the accuracy and efficiency of the scheme. If the order of interpolation is improved, the accuracy is increased, but at additional computational cost, with the corresponding law of diminishing returns. Cubic interpolation has a good balance between accuracy and computational cost, its spatial truncation error is of fourth order with small damping; cubic splines have, additionally, the property of conserving mass for divergence-free flows (Staniforth and Coté, 1991).

Alone, a semi-Lagrangian scheme cannot solve a diffusion-type equation, but may contribute to the accuracy and efficiency when solving convection-diffusion PDEs, basically, the class of models to be considered here.

### 3.4 Hybrid semi-Lagrangian schemes

Hybrid semi-Lagrangian algorithms have been used to solve problems of an advection-diffusion nature. D’Halluin et al. (2005) used a hybrid method of this type in conjunction with fully implicit, Crank-Nicolson and backward differencing (BDF) time stepping to value the price of Asian options of both European and American style, following a partial integro-differential (PIDE) approach.

A semi-Lagrangian Crank-Nicolson method has been used on a problem arising in geology for calculating the thermal structure of a subduction zone (a region where one tectonic plate moves under another tectonic plate and sinks into the mantle as the plates converge). A hybrid scheme of this kind combines the methods for advection and diffusion, the objective being to approximate the solution for the scaled, constant
diffusivity advection-diffusion problem (Spiegelman and Katz, 2006)

\[ \frac{\partial u}{\partial t} + v \cdot \nabla u = \nabla^2 u, \]

solving the equation

\[ \frac{D u}{D t} = \nabla^2 u. \]

Here our objective is to approximate the solution of advection-diffusion-reaction type problems, in particular storage systems where the availability and spot price of the underlying are both subject to stochasticity

\[ \frac{\partial u}{\partial \tau} + v \cdot \nabla u = \mathcal{L} u + f(u, x, \tau) \]

(3.4)

by solving the equation

\[ \frac{D u}{D \tau} = \mathcal{L} u + f(u, x, \tau), \]

(3.5)

where \( \mathcal{L} \) is a diffusion operator.

In our notation, the superscript \( ^{(m)} \) will indicate the number \( m \) of first-order terms incorporated in the Lagrangian derivative with respect to the temporal variable, always starting with the non-diffusive (storage) dimension, and after that all first-order terms are incorporated (including those from diffusive dimensions).

**The semi-Lagrangian Crank-Nicolson and the semi-Lagrangian fully implicit methods**

These schemes can be written as

\[ \frac{u^{n+1} - u^n}{\Delta \tau} = \vartheta \mathcal{L} u^{n+1} + f^{n+1} + (1 - \vartheta) \mathcal{L} u^n + f^n, \]

(3.6)

where \( u^{n+1} \) is the approximated solution to the PDE on a regular mesh point at time step \( n + 1 \), \( u^n = u^n(t, x) \) is the solution to the PDE on the departure point \( x \) that is tracked back solving \( \frac{dx}{d\tau} = -v \) at time step \( n \), and \( \mathcal{L} \) is the discrete diffusion operator, \( f^n = f(u^n, \tau) \) is the value of the source term at the departure position and \( f^{n+1} = f(u^{n+1}, \tau + \Delta \tau) \) is the value of the source term at the final position (on a regular mesh point). \( \vartheta = \frac{1}{2} \) gives the Crank-Nicolson stepping method (SLCN), and \( \vartheta = 1 \) results in the fully implicit backward time-differencing for the hybrid scheme (SLFI).
The semi-Lagrangian alternating-direction implicit method (SLADI)

The scheme can be written as

$$\frac{Du}{D\tau} = \{L_1 u + L_2 u + \cdots + L_d u\} + f(u, x, \tau); \quad (3.7)$$

where $L$ is a $d$-dimensional diffusion operator.

The semi-Lagrangian scheme coupled with the ADI method can be implemented following two approaches. In the first, it can be implemented by following only one flow characteristic per time-step and the solution progresses from the time level $\tau$ to the time level $\tau + \Delta \tau$ requiring $d$ steps to be completed, and in conjunction with the semi-Lagrangian approach particles are tracked back every time increment $\Delta \tau$.

$$u^{n+1} - u^n = \{L_1 u^{n+\frac{1}{d}} + L_2 u^{n+\frac{2}{d}} + \cdots + L_d u^{n+\frac{d}{d}}\} + f^{n+1},$$

$$u^{n+\frac{2}{d}} - u^{n+\frac{1}{d}} = \{L_1 u^{n+\frac{1}{d}} + L_2 u^{n+\frac{2}{d}} + \cdots + L_d u^{n+\frac{d}{d}}\} + f^{n+1},$$

$$\vdots$$

$$u^{n+\frac{d}{d}} - u^{n+\frac{d-1}{d}} = \{L_1 u^{n+\frac{1}{d}} + L_2 u^{n+\frac{2}{d}} + \cdots + L_d u^{n+\frac{d}{d}}\} + f^{n+1}. \quad (3.8)$$

where $u^{n+1}$ is the approximation to the solution of the PDE on a regular mesh at time step $n + 1$, $u^n$ is the solution to the PDE on the departure point $x^*$ that is tracked back solving $\frac{dx}{d\tau} = -v$ at time step $n$, $u^{n+\frac{1}{d}} = u^{n+\frac{1}{d}}(\tau + \frac{1}{d}\Delta \tau, x^*)$ is an intermediate solution to the PDE at the intermediate location $x^*$ at time $\tau + \frac{1}{d}\Delta \tau$ (which we do not explicitly calculate, as we are following only one characteristic), and $L$ is the discrete diffusion operator, $f^{n+1} = f(u^{n+1}, \tau + \Delta \tau)$ is the value of the source term (for example) at the final position at regular mesh point.

In the second implementation, the method can be carried out as a repetition of $d$ self-defined steps that must be completed one after the other, each one advancing $\frac{1}{d}\Delta \tau$ in time. Correspondingly, in conjunction with the semi-Lagrangian approach,
particles are backtracked for every fraction of time-step increment \( \frac{1}{d} \Delta \tau \).

\[
\frac{u^{n+\frac{1}{2}} - u^n}{\frac{1}{d} \Delta \tau} = \{ \mathcal{L}_1 u^{n+\frac{1}{2}} + \mathcal{L}_2 u^n + \cdots + \mathcal{L}_d u^n \} + f^{n+\frac{1}{2}},
\]

\[
\frac{u^{n+\frac{3}{2}} - u^{n+\frac{1}{2}}}{\frac{1}{d} \Delta \tau} = \{ \mathcal{L}_1 u^{n+\frac{3}{2}} + \mathcal{L}_2 u^{n+\frac{1}{2}} + \cdots + \mathcal{L}_d u^{n+\frac{1}{2}} \} + f^{n+\frac{3}{2}},
\]

\[
\vdots
\]

\[
\frac{u^{n+1} - u^{n+1-\frac{1}{2}}}{\frac{1}{d} \Delta \tau} = \{ \mathcal{L}_1 u^{n+1-\frac{1}{2}} + \mathcal{L}_2 u^{n+1-\frac{1}{2}} + \cdots + \mathcal{L}_d u^{n+1} \} + f^{n+1},
\] (3.9)

where \( u^{n+\frac{1}{2}} \) is the approximate solution on a regular mesh at time-step fraction \( n + \frac{1}{d} \),

\[
u_{x}^{n+\frac{1}{2}} = u^{n+\frac{1}{2}}(\tau + \frac{1}{d} \Delta \tau, x) \]

is the solution on the departure point \( x \) that is tracked back solving \( \frac{d\mathbf{x}}{d\tau} = -\mathbf{v} \) at time step \( n + \frac{1}{d} \) (following \( d \) independent characteristics),

\( \mathcal{L} \) is the discrete diffusion operator, and \( f^{n+\frac{1}{2}} = f(u^{n+\frac{1}{2}}, \tau + \frac{1}{d} \Delta \tau) \) is the value of the source term at the intermediate position (a regular mesh point).

**The semi-Lagrangian alternating-direction implicit method (SLADI) with the presence of mixed derivatives**

We assume that \( \mathcal{L}_0 \) denotes the cross derivative terms in the discrete diffusion operator \( \mathcal{L} = \mathcal{L}_0u + \mathcal{L}_1u + \mathcal{L}_2u + \cdots + \mathcal{L}_d u \) and we make an explicit approximation of the piece \( \mathcal{L}_0 \) in every fractional time-step of our SLADI schemes. Correspondingly, \( \mathcal{L}_1u, \mathcal{L}_2u, \cdots, \mathcal{L}_d u \) denote \( d \) unidirectional diffusion terms of \( \mathcal{L} \) and are approximated implicitly accordingly with the ADI operator splitting methodology.

Then, in the first approach where only one flow characteristic is followed per time-step, the solution progresses from the time level \( \tau \) to the time level \( \tau + \Delta \tau \) requiring \( d \) steps to be completed, and in conjunction with the semi-Lagrangian approach, particles are tracked back every time increment \( \Delta \tau \).

\[
\frac{u^{n+\frac{1}{2}}_x - u^n_x}{\frac{1}{d} \Delta \tau} = \{ \mathcal{L}_0 u^n_x + \mathcal{L}_1 u^{n+\frac{1}{2}}_x + \mathcal{L}_2 u^n_x + \cdots + \mathcal{L}_d u^n_x \} + f^{n+1},
\]

\[
\frac{u^{n+\frac{3}{2}}_x - u^{n+\frac{1}{2}}_x}{\frac{1}{d} \Delta \tau} = \{ \mathcal{L}_0 u^{n+\frac{3}{2}}_x + \mathcal{L}_1 u^{n+\frac{1}{2}}_x + \mathcal{L}_2 u^{n+\frac{3}{2}}_x + \cdots + \mathcal{L}_d u^{n+\frac{1}{2}}_x \} + f^{n+1},
\]

\[
\vdots
\]

\[
\frac{u^{n+1} - u^{n+1-\frac{1}{2}}_x}{\frac{1}{d} \Delta \tau} = \{ \mathcal{L}_0 u^{n+1-\frac{1}{2}}_x + \mathcal{L}_1 u^{n+1-\frac{1}{2}}_x + \mathcal{L}_2 u^{n+1-\frac{1}{2}}_x + \cdots + \mathcal{L}_d u^{n+1} \} + f^{n+1},
\] (3.10)
where $u^{n+1}$ is the approximation to the solution of the PDE on a regular mesh at time step $n + 1$, $u^n = u^n(\tau, x_*)$ is the solution to the PDE on the departure point $x_*$ that is tracked back solving $\frac{dx}{d\tau} = -v$ at time step $n$, $u^{n+\frac{1}{d}}_* = u^{n+\frac{1}{d}}(\tau + \frac{1}{d}\Delta\tau, x_*)$ is an intermediate solution to the PDE at the intermediate location $x_*$ at time $\tau + \frac{1}{d}\Delta\tau$ (which we do not explicitly calculate, as we are following only one characteristic), and $f^{n+1} = f(u^{n+1}, \tau + \Delta\tau)$ is the value of the source term (for example) at the final position at a regular mesh point.

Then, in the second implementation, the method can be carried out as a repetition of $d$ self-defined steps that must be completed one after the other, each one advancing $\frac{1}{d}\Delta\tau$ in time. Correspondingly, in conjunction with the semi-Lagrangian approach, particles are backtracked for every fraction of time-step increment $\frac{1}{d}\Delta\tau$.

\[
\begin{align*}
\frac{u^{n+\frac{1}{d}} - u^n}{\frac{1}{\Delta\tau}} &= \left\{ L_0 u^n + L_1 u^{n+\frac{1}{d}} + L_2 u^n + \ldots + L_d u^n \right\} + f^{n+\frac{1}{d}}, \\
\frac{u^{n+\frac{2}{d}} - u^{n+\frac{1}{d}}}{\frac{1}{\Delta\tau}} &= \left\{ L_0 u^{n+\frac{1}{d}} + L_1 u^{n+\frac{2}{d}} + L_2 u^{n+\frac{2}{d}} + \ldots + L_d u^{n+\frac{1}{d}} \right\} + f^{n+\frac{2}{d}}, \\
&\vdots \\
\frac{u^{n+1} - u^{n+\frac{1}{d}}}{\frac{1}{\Delta\tau}} &= \left\{ L_0 u^{n+\frac{1}{d}} + L_1 u^{n+1} + L_2 u^{n+1} + \ldots + L_d u^{n+1} \right\} + f^{n+1}, \quad (3.11)
\end{align*}
\]

where $u^{n+\frac{1}{d}}$ is the approximate solution on a regular mesh at time-step fraction $n + \frac{1}{d}$, $u^{n+\frac{1}{d}}_* = u^{n+\frac{1}{d}}(\tau + \frac{1}{d}\Delta\tau, x_*)$ is the solution on the departure point $x_*$ that is tracked back solving $\frac{dx}{d\tau} = -v$ at time step $n + \frac{1}{d}(\tau + \frac{1}{d}\Delta\tau)$ (following $d$ independent characteristics), $L$ is the discrete diffusion operator, and $f^{n+\frac{2}{d}} = f(u^{n+\frac{2}{d}}, \tau + \frac{1}{d}\Delta\tau)$ is the value of the source term at the intermediate position (a regular mesh point).

In our notation for the SLADI methodology, a subscript $I$ indicates that only one characteristic is followed per time step and a subscript $II$ indicates that $d$ independent characteristics are followed, one for each fraction of time-step increment $\frac{1}{d}\Delta\tau$.

Detailed implementations of the four variants of this novel SLADI methodology, to treat mixed advection and diffusion problems, is presented in Chapter 4. The variants of the scheme we devised are dependent on specific characteristics followed by virtual particles in the coupled semi-Lagrangian approach.
3.5 Numerical methods for Asian options

Within this section we implement numerical methods to approximate the value of financial Asian options and assess the quality of the results obtained with respect to benchmark examples.

3.5.1 Hybrid scheme for the full-dimensional Asian option

Here we propose to solve the PDE (2.65) using a hybrid algorithm, namely the semi-Lagrangian scheme in conjunction with the fully implicit and the Crank-Nicolson methods. The schemes are detailed below and the accuracy of the results is compared with benchmark examples in section 3.5.3.

The discretisation is on a regular grid in $\tau$, $S$, and $I$, with $\tau_i = i\Delta \tau$, $S_j = j\Delta S$ and $I_k = k\Delta I$. The value of the option at the mesh point $(i,j,k)$ is then $V(S_j, I_k, \tau_i)$ or $V_{j,k}$. After reversing time, $\tau = T - t$, and moving the term $S\frac{\partial V}{\partial I}$ to the left hand side yield

$$\frac{\partial V}{\partial \tau} - S\frac{\partial V}{\partial I} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV,$$

it can be solved in the form

$$\frac{DV}{D\tau} = \vartheta LV^{n+1} + (1 - \vartheta)LV^n,$$

(3.12)

where $L$ is the operator

$$LV \equiv \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV,$$

$\vartheta = \frac{1}{2}$ gives Crank-Nicolson stepping, and $\vartheta = 1$ results in the fully implicit stepping; $V^{n+1}$ and $V^n$ are explained below.

The Lagrangian derivative along a trajectory $I = I(S, \tau)$, for $S$ fixed is

$$\frac{DV}{D\tau} = \frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial I} \frac{dI}{d\tau}.$$

Along the trajectory

$$\frac{dI}{d\tau} = -S,$$

thus, for the total difference $DV = V(S_j, I_k, \tau_i) - V(S_j, I_{k*}, \tau_{i-1})$, the value of the option $V(S_j, I_{k*}, \tau_{i-1})$, where $I_{k*} = I_k + S_j\Delta \tau$, is found at $k* = k + S_j\Delta \tau / \Delta I$. This completes the formulation (3.12) for $i = n + 1$ and $DV = V^{n+1} - V^n$. 

The approximations of the derivatives are

\[
\frac{DV}{D\tau} = \frac{V_{j,k}^i - V_{j,k}^{i-1}}{\Delta \tau},
\]

\[
\frac{\partial^2 V}{\partial S^2} = \frac{V_{j-1,k}^i - 2V_{j,k}^i + V_{j+1,k}^i}{\Delta^2 S},
\]

\[
\frac{\partial V}{\partial S} = \frac{V_{j+1,k}^i - V_{j-1,k}^i}{2\Delta S},
\]

for a semi-Lagrangian fully implicit scheme.

And

\[
\frac{DV}{D\tau} = \frac{V_{j,k}^i - V_{j,k}^{i-1}}{\Delta \tau},
\]

\[
\frac{\partial^2 V}{\partial S^2} = \frac{1}{2} \left( \frac{V_{j-1,k}^i - 2V_{j,k}^i + V_{j+1,k}^i}{\Delta^2 S} + \frac{V_{j-1,k}^{i-1} - 2V_{j,k}^{i-1} + V_{j+1,k}^{i-1}}{\Delta^2 S} \right),
\]

\[
\frac{\partial V}{\partial S} = \frac{1}{2} \left( \frac{V_{j+1,k}^i - V_{j-1,k}^i}{2\Delta S} + \frac{V_{j+1,k}^{i-1} - V_{j-1,k}^{i-1}}{2\Delta S} \right),
\]

for a semi-Lagrangian Crank-Nicolson scheme.

The fixed-strike Asian option is solved using the boundary conditions (2.73), (2.74), (2.75) and (2.76). If \(k_* \leq k_{\max}\), \(V_{j,k_*}^{i-1}\) is interpolated between the values on the mesh points \(V_{j,k_*}^{i-1}\) and \(V_{j,k_*}^{i-1}\), otherwise \(V_{j,k_*}^{i-1}\) is evaluated using the appropriate boundary condition.

The stencils followed to carry out the calculations are presented in Figure 3.1 for the Semi-Lagrangian fully implicit scheme and in Figure 3.2 for the Semi-Lagrangian Crank-Nicolson scheme. Bullets (\(\bullet\)) represent mesh points on the grid with known values of \(V\) at \(\tau = (i-1)\Delta \tau\), stars (\(\ast\)) represent the approximate departure position along the trajectory \(I = I(S, \tau)\), for \(S\) fixed, and circles (\(\circ\)) represent mesh points on the grid with unknown values of \(V\) to calculate at time \(\tau = i\Delta \tau\).

### 3.5.2 Hybrid scheme for the Asian option similarity reduction

Here we propose to solve our similarity reduction PDE (2.86) using a hybrid algorithm, namely the semi-Lagrangian scheme in conjunction with the Crank-Nicolson method. The scheme is detailed below and the accuracy of the results is compared with benchmark examples in section 3.5.3.

The discretisation is on a regular grid in \(\tau\) and \(R\), with \(\tau_i = i\Delta \tau\) and \(R_j = j\Delta R\). The value of the option at the mesh point \((i, j)\) is then \(H(R_j, \tau_i)\) or \(H^i_j\). With the presence
Figure 3.1: Semi-Lagrangian fully implicit stencil I

Figure 3.2: Semi-Lagrangian Crank-Nicolson stencil I
of only first-order and second-order terms, the PDE for the similarity reduction (2.83),
fits completely the formulation (3.6). After reversing time, $\tau = T - t$, and moving the
term $(1 - rR)\partial H/\partial R$ to the left hand side yield

$$\frac{\partial H}{\partial \tau} - (1 - rR)\frac{\partial H}{\partial R} = \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2},$$

it can be solved in the form

$$\frac{DH}{D\tau} = \vartheta LH^{n+1} + (1 - \vartheta)LH^n,$$

(3.13)

where $L$ is the operator

$$LH \equiv \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2},$$

$\vartheta = \frac{1}{2}$ gives Crank-Nicolson stepping, and $\vartheta = 1$ results in the fully implicit stepping;

$H^{n+1}$ and $H^n_*$ are explained below.

The Lagrangian derivative along a trajectory $H = H(R, \tau)$ is

$$\frac{DH}{D\tau} = \frac{\partial H}{\partial \tau} + \frac{\partial H}{\partial R} \frac{dR}{d\tau}.$$

Along the trajectory

$$\frac{dR}{d\tau} = -(1 - rR),$$

thus, for the total difference $DH = H(R_j, \tau_i) - H(R_{j*}, \tau_{i-1})$, the value of the function

$H(R_{j*}, \tau_{i-1})$, where $R_{j*} = R_j + (1 - rR_j)\Delta \tau$, is found at $j* = j + (1 - rR_j)\Delta \tau/\Delta R$.

This completes the formulation (3.13) for $i = n + 1$ and $DH = H^{n+1} - H^n_*$.

The approximations of the derivatives are

$$\frac{DH}{D\tau} = \frac{H_j^i - H_{j^*}^{i-1}}{\Delta \tau},$$

$$\frac{\partial^2 H}{\partial R^2} = \frac{H_{j-1}^i - 2H_j^i + H_{j+1}^i}{\Delta R^2},$$

for a semi-Lagrangian fully implicit scheme.

And

$$\frac{\partial^2 H}{\partial R^2} = \frac{H_j^{i}}{\Delta \tau} - \frac{H_{j^*}^{i-1}}{\Delta \tau},$$

$$\frac{\partial^2 H}{\partial R^2} = \frac{1}{2} \left( \frac{H_{j-1}^{i} - 2H_j^{i} + H_{j+1}^{i}}{\Delta R^2} + \frac{H_{j^*}^{i-1} - 2H_{j^*}^{i-1} + H_{j^*+1}^{i-1}}{\Delta R^2} \right),$$

for a semi-Lagrangian Crank-Nicolson scheme.
The fixed-strike Asian call option is solved using the boundary conditions (2.88), at expiry, (2.89), as $R \to -\infty$. (2.90) or (2.93), on $R = 0$, for half domain, and (2.91), (2.92) or (2.94) as $R \to \infty$, for full domain.

The stencils followed to carry out the calculations are presented in Figure 3.3 for the Semi-Lagrangian fully implicit scheme and in Figure 3.4 for the Semi-Lagrangian Crank-Nicolson scheme. Bullets ($\bullet$) represent mesh points on the grid with known values of $H$ at $\tau = (i - 1)\Delta\tau$, stars ($\ast$) represent the approximate departure position, and circles ($\circ$) represent mesh points on the grid with unknown values of $H$ to calculate at time $\tau = i\Delta\tau$.

### 3.5.3 Accuracy assessment

Here we compare the accuracy of the results obtained for valuing Asian options with respect to results obtained with spectral expansions theory (Linetsky, 2004), and with respect to upper and lower bounds obtained with expectations arguments (Rogers and Shi, 1995; Thompson, 1999). The results presented by Linetsky (2004), which are calculated up to ten digits of accuracy, can be used as a standard reference to benchmark results; for example Večer (2002) and Dewynne and Shaw (2008) have implemented numerical approaches to solve the related PDEs and have matched up to six digits of accuracy. The lower and upper bounds from Linetsky (2004) are credited
to Thompson (1999), in turn Thompson credited Rogers and Shi (1995) for the lower bound.

Here we value newly written contracts of fixed-strike Asian call options with strike price $K = 2$ and European-style payoff. The parameter values used in this accuracy assessment are presented in Table 3.1 (Linetsky, 2004), $\sigma$ is the volatility, $r$ is the interest rate, $T$ is the time to expiry, and $S_0$ is the initial asset price.

Table 3.2 presents the results obtained for the value of the fixed-strike Asian option using the full-dimensional PDE (2.65). The columns FI, SLFI, SLCN, EE, SLT, MC, LB and UB, give respectively, the values calculated using the fully implicit method, the values obtained with the semi-Lagrangian fully implicit scheme, the values obtained with the semi-Lagrangian Crank-Nicolson scheme, the value of the option obtained with spectral expansions, the value of the option obtained with numerical Laplace transform inversion (Shaw, 2002), the value of the option obtained with Monte Carlo simulation (Dufresne, 2000), the lower bound for the value of the option, and the upper bound for the value of the option (these last five columns, i.e. EE, SLT, MC, LB and UB, can be found in Linetsky, 2004). We observe up to four digits of accuracy for the fully implicit method (FI), the semi-Lagrangian fully implicit method (SLFI) attains basically five digits of accuracy, and the semi-Lagrangian Crank-Nicolson obtains from six to ten digits of accuracy. In brackets beneath is the percentage of error with respect to the result from spectral expansions (EE). Table 3.3 presents the results obtained for the value of the fixed-strike Asian option using the similarity reduction PDE (2.86). Here, the columns CN and SLCN give respectively, the values calculated using the Crank-Nicolson method, and the value obtained with the semi-Lagrangian Crank-Nicolson scheme. In brackets beneath is the percentage of error with respect to the result from spectral expansions (EE) (columns EE, SLT, MC, LB and UB as above). We observe from seven to ten digits of accuracy for the Crank-Nicolson method (CN), and the semi-Lagrangian Crank-Nicolson method (SLCN) obtains from five to seven digits of accuracy.

Results in Tables 3.2 and 3.3 were obtained after implementing Richardson extrapolation (Smith, 1965). Assuming that an approximate solution $v(h)$, at the same grid-point for variable grid-size $h$, can be expressed as an exact solution $v$ plus an
error, \( v(h) = v + O(h^\xi) \), then

\[
v = \frac{(h/2)^\xi v(h) - (h)^\xi v(h/2)}{(h/2)^\xi - (h)^\xi},
\]

where \( \mathcal{R} = 2^\xi \) is the rate of convergence

\[
\mathcal{R} = \frac{v(h/2) - v(h)}{v(h/4) - v(h/2)},
\]

for linear convergence \( \mathcal{R} = 2 \) (\( \xi = 1 \)), and for quadratic convergence \( \mathcal{R} = 4 \) (\( \xi = 2 \)).

Detailed valuations \( V \) and calculations for Richardson extrapolations for different grid sizes \( h \) are presented in Tables 3.4, 3.5, 3.6, 3.7 and 3.8. The column for ‘Extrapolation\(^*\)’ presents the results with assuming a constant rate of convergence \( \mathcal{R} = 2^\xi, \xi \in \{1, 2\} \), the column for \( \mathcal{R} \) shows the observed rate of convergence, and the column for ‘Extrapolation(\( \mathcal{R} \))’ gives the results after calculating the Richardson extrapolation with the observed rate of convergence \( \mathcal{R} = 2^\xi \). The last extrapolation entry, using the observed numerical rate of convergence \( \mathcal{R} = 2^\xi \), for each case are corresponding entries in Tables 3.2 and 3.3; this to present the best approximation obtained with the calculated data, although with assuming a constant rate of convergence the extrapolations produce results with half the error. For all valuations and extrapolations we give in brackets the percentage of error [\%err] with respect to the correspondent case value obtained with spectral expansions (EE).

For the full-dimensional problem we set \( S_{\text{max}} = 5 \) and \( I_{\text{max}} = 5 \). For the similarity reduction we set \( R_{\text{max}} = 20 \). We use linear interpolation for the semi-Lagrangian fully implicit scheme (as this finite-difference method is globally first-order accurate Smith, 1965) and we use cubic spline interpolation for the semi-Lagrangian Crank-Nicolson schemes (as this method can attain global second-order of convergence, see Staniforth and Coté, 1991). With respect to the interpolation procedure utilised, we identified that grids with small spacing give similar results. If cubic splines are implemented, a larger spacing in the grid does not deteriorate the solution, whereas with linear interpolation a larger spacing deteriorates the solution; the expense of using cubic spline interpolation is computing time. The definition of a cubic spline interpolation is that the resulting interpolating curve is smooth in the first derivative, and is continuous in the second derivative Press et al. (1992). All calculations were performed using an Intel(R) Xeon(R) CPU E5-2643 0 @ 3.30GHz processor.
When solving the full-dimensional problem, the fully implicit method (see Table 3.4) presents difficulties for obtaining numerical solutions within the lower and upper bounds, even with the smaller grid-sizes (except for case 6) the calculations are out of the bounds, the same difficulty is observed for the small volatility $\sigma$ examples (cases 1 and 2). Apart from the previous cases, the extrapolation calculations can be within the lower and upper bounds using grid-sizes with medium spacing. As expected the order of convergence is of first-order and this is verified by the percentage of error in the calculations.

The semi-Lagrangian fully implicit method (see Table 3.5) can obtain numerical solutions within the lower and upper bounds, for solving the full-dimensional problem, using the smaller grid-sizes except for the small volatility examples (cases 1 and 2). Most of the extrapolation calculations can be within the lower and upper bounds except for the case 1 with low volatility $\sigma$. The observed order of convergence is of first-order (as expected), also the percentage of error confirms this rate of convergence.

For the solution of the full-dimensional problem, the semi-Lagrangian Crank-Nicolson scheme (see Table 3.6) obtains valuations and extrapolations calculations within the lower and upper bounds, we only observe difficulties to obtain solutions within the lower and upper bounds for the case 1 with low volatility $\sigma$. Second-order of convergence can be observed and the percentage of error confirm this rate.

When solving the similarity reduction problem, the Crank-Nicolson method (see Table 3.7) presents difficulties to obtain numerical solutions and extrapolation calculations within the lower and upper bounds for the case 1, but it still can match five digits of accuracy for directly solving the PDE (and seven digits of accuracy are matched after Richardson extrapolation). All other results can be replicated after Richardson extrapolation (essentially these match the ten digits of accuracy). Second-order of convergence can be observed for the Crank-Nicolson scheme and this is verified by the percentage of error.

For solving the similarity reduction, the semi-Lagrangian Crank-Nicolson (see Table 3.8) has slightly more difficulties to obtain numerical solutions within the lower and upper bounds for case 1 (comparing with the previous Cranks-Nicolson method results), and can only match three digits of accuracy from directly solving the PDE
(still it matches five digits of accuracy after Richardson extrapolation). Only first-order of convergence is observed for the semi-Lagrangian Crank-Nicolson (this verified by the percentage of error) this is the main reason for not achieving the best results but we can identify a convergent solution.

We have verified that the errors are large when only standard finite-difference methods are used for solving the full-dimensional model of Asian option. When the method of characteristics is combined with standard finite-difference methods, the errors are significantly reduced and we found that the semi-Lagrangian Crank-Nicolson method produced the best approximation for numerically solving these particular fixed-strike Asian options (full-dimensional model). When solving the similarity reduction the errors were also small and produced accurate solutions, the Crank-Nicolson method produced the best results almost reproducing totally the results from spectral expansion and achieving second-order of accuracy. The semi-Lagrangian Crank-Nicolson only achieved first-order of accuracy and a convergent solution but would require longer calculations to obtain results with more accuracy. In all cases the methods present difficulties to obtain accurate solutions for the case of low volatility (unsurprisingly).

We identify that the magnitude of the volatility \( O(\sigma) \) is more relevant than the ratio \( r/\sigma \); the numerical schemes present more difficulties for solving the case number 1 with the smaller volatility, even it is not the case with the larger ratio \( r/\sigma \) (the case number 2 has the higher ratio \( r/\sigma \)). Still it can be identified that the larger this ratio \( r/\sigma \), the larger the error obtained with the numerical methods (compare errors for the case number 2 with respect to the errors in the other cases in Tables 3.5 - 3.8).

We have verified the accuracy of our results with respect very stringent bounds from expectations arguments and with respect to high precision valuations obtained with spectral expansion. And we have verified that the numerical methods developed here perform as well or can improve the results produced with the numerical Laplace transform inversion or with the Monte Carlo simulation method, as can be identified in Tables 2.65 and 2.86 (all these references values are presented by Linetsky, 2004).

### 3.5.4 Asian options valuation

In this section we present graphical results for the valuation of fixed-strike Asian options; parameters values as in the cases from Table 3.1. Figures 3.5 and 3.6 present
### Table 3.1: Benchmark parameters values.

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<th>$\sigma$</th>
<th>$r$</th>
<th>$T$</th>
<th>$S_0$</th>
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### Table 3.2: Fixed-strike Asian option value (European style); 3-dimensional PDE.

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<th>SLFI</th>
<th>SLCN</th>
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### Table 3.3: Fixed-strike Asian option value (European style); Similarity reduction.

<table>
<thead>
<tr>
<th>Case</th>
<th>CN</th>
<th>SLCN</th>
<th>EE</th>
<th>SLT</th>
<th>MC</th>
<th>LB</th>
<th>UB</th>
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<tbody>
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Table 3.5: Richardson extrapolation; Semi-Lagrangian fully implicit method.

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Table 3.6: Richardson extrapolation; Semi-Lagrangian Crank-Nicolson method.

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### Table 3.7: Richardson extrapolation; Crank-Nicolson method.

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Table 3.8: Richardson extrapolation; semi-Lagrangian Crank-Nicolson.

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the value $V$ and the delta value $\Delta V/\Delta S$ correspondingly for the case 1 (lowest volatility \( \sigma \) example); the valuation curve at $I = 0$ shows a sharp curvature about the strike price $K = 2$ and the delta value is free of oscillations. For case 2, the value $V$ is presented in Figure 3.7 and the corresponding delta value $\Delta V/\Delta S$ is presented in Figure 3.8. Figures 3.9 and 3.10 present corresponding results for cases 3, 4 and 5 for the value $V$ and the delta value $\Delta V/\Delta S$, as well as Figures 3.11 and 3.12 present the value $V$ and the delta value $\Delta V/\Delta S$ for the case 6, and for the final case (7) the value $V$ is presented in Figure 3.13 and the corresponding delta value $\Delta V/\Delta S$ is presented in Figure 3.14.

The fixed-strike Asian option with European exercise style receives a better valuation with the highest spot prices for relatively lower volatility, but we identify increased curvatures with relative larger volatility $\sigma$ for the region about the strike price ($K = 2$), and thus receiving a better valuation in this region for larger volatility coefficients. For all cases we have verified that the solutions are free of oscillations as indicated in corresponding delta valuations $\Delta V/\Delta S$. 

Figure 3.5: Fixed-strike Asian option value $V$ (European style); $\sigma = 0.10$, $r = 0.02$, $T = 1$. 

![Figure 3.5](image-url)
Figure 3.6: Fixed-strike Asian option delta value $\frac{\Delta V}{\Delta S}$ (European style); $\sigma = 0.10$, $r = 0.02$, $T = 1$.

Figure 3.7: Fixed-strike Asian option value $V$ (European style); $\sigma = 0.30$, $r = 0.18$, $T = 1$. 
Figure 3.8: Fixed-strike Asian option delta value $\frac{\Delta V}{\Delta S}$ (European style); $\sigma = 0.30$, $r = 0.18$, $T = 1$.

Figure 3.9: Fixed-strike Asian option value $V$ (European style); $\sigma = 0.50$, $r = 0.05$, $T = 1$. 
Figure 3.10: Fixed-strike Asian option delta value $\frac{\Delta V}{\Delta S}$ (European style); $\sigma = 0.50$, $r = 0.05$, $T = 1$.

Figure 3.11: Fixed-strike Asian option value $V$ (European style); $\sigma = 0.50$, $r = 0.05$, $T = 2$. 
Figure 3.12: Fixed-strike Asian option delta value $\Delta V / \Delta S$ (European style); $\sigma = 0.50$, $r = 0.05$, $T = 2$.

Figure 3.13: Fixed-strike Asian option value $V$ (European style); $\sigma = 0.251$, $r = 0.0125$, $T = 2$. 
3.6 Conclusions

The PDE models for Asian options and for stochastic storage problems present a mixed advection-diffusion nature. Additionally if a space dimension is known to have no diffusion, as in the case for the PDEs to be solved, the related equations are known to be difficult to solve accurately; erroneous diffusion may be introduced and oscillatory solutions may be produced. Given the previous sensitivities, we have presented hybrid methods to treat mixed advection-diffusion PDEs. We have presented methods found in the literature (semi-Lagrangian Crank-Nicolson and semi-Lagrangian fully implicit methods) and we have introduced a novel approach, the semi-Lagrangian alternating-directions implicit (SLADI) methodology for advection-diffusion equations.

In the present chapter we have given detailed implementations of hybrid finite-difference methods combined with the semi-Lagrangian method for numerically solving the valuation of fixed-strike Asian options with European style, the accuracy and convergence for the methods is tested and verified. The results from alternative methodologies such as the Laplace transformation, Monte Carlo simulations, but mainly from highly accurate (up to ten digits) analytical approximations from spectral expansions and very stringent bounds from expectations arguments were used as the yardstick for the benchmark.
By numerically approximating the valuation of a fixed-strike Asian option in full-dimensional form (2.65), we have verified that standard finite-difference methods produce large errors when they are utilised for solving advection-diffusion equations, such as resulted for our implementation of a fully implicit scheme. When the method of characteristics is combined with standard finite-difference methods the errors can be significantly reduced, as employed in our semi-Lagrangian fully implicit and in our semi-Lagrangian Crank-Nicolson implementations, with the latter resulting the technique producing the most accurate results and exhibiting second-order of convergence.

When solving the similarity reduction problem for the fixed-strike Asian option we obtained reasonably accurate results with implementing a semi-Lagrangian Crank-Nicolson method but we obtained slightly larger errors as the method only produced a first-order of convergence. Second-order of convergence can be observed for the Crank-Nicolson scheme, with higher accuracy and efficiency.

Based on the higher dimensionality of the model formulation, we conclude that the semi-Lagrangian Crank-Nicolson method produced the best approximation for numerically solving these particular fixed-strike Asian options (three-dimensional model); this demonstrates the quality and efficiency of hybrid semi-Lagrangian schemes. With accurate and efficient valuation algorithms at hand, we now proceed to extend this knowledge to value (four-dimensional) stochastic storage problems.
Chapter 4

Storage valuation, an application to wind power generation

Systems in the real world operate by consuming or processing resources, and this can be subject to variations as there can be different requirements at any point in time, this effected by internal or external factors to the system. A variable price of a commodity can limit the number of items that can be purchased with a fixed budget at some point in time. If the asset is consumed continuously at a constant rate, a storage can be used to keep reserves when low prices prevail as more assets can be purchased, and then compensate the lower number of assets that can be purchased when high prices of the commodity prevail. Rainfall has an inherently random behavior and has a direct effect on the amounts of food that can be produced on a given land. Water reservoirs can ensure the availability of water in dry seasons by releasing water collected during wet seasons, and thus ensure the production of food. We can identify many other systems where stockpiles can be used to ensure the availability of resources when they are more valuable, but requiring to stockpile the resource when it is more convenient.

In this chapter we define a model for valuing a storage system subject to uncertainty from both economic (commodity price, interest rates, etc) and physical (temperature, rainfall, snow/ice, etc) factors. Renewable energy systems are of special interest as the amount of resource availability varies randomly, making necessary a back-up when the resource is scarce or null, on the other hand an excess of resources presents a potential

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1Part of the material in this chapter has been accepted for publication in The Journal of Computational Finance.
advantage if the surplus can be exploited, in this case by creating a reserve. Furthermore, the underlying generated can be sold in a volatile market, that stockpiling it when prices are low and selling it when the prices are high is appealing. Thus, finding the value of storage devices for renewable energy systems is beneficial for appraising investments in such facilities. This serve both objectives, on one side storages mitigate the fluctuations on the amount of energy generated, and on the other side the management makes the storage economically beneficial.

The model we introduce is for a wind farm with a storage facility, the farm *naturally* generates a random output and is subject to variable electricity prices, the farm is committed to produce a fixed output rate of electricity, and is receiving a fixed price for the energy generated. We assume that the operational and maintenance costs can be neglected and then we basically estimate the income flow. With respect to the market, we assume that electricity can be bought and sold any time without restriction and no transaction costs are associated. With respect to the use of resources, we assume that energy is stockpiled if there is surplus on generation, and energy is taken from reserves if there is deficit in generation, thus we are essentially using the storage device to balance the output by reducing fluctuations produced by the resource, but we then implement strategies to make the storage economically beneficial. And we complement the model with the definition of boundary conditions; the linear behavior imposed on the domain extrema is verified in the appendix at the end of the chapter.

The resulting advection-diffusion PDE model in four dimension is then approached numerically, four variants of a semi-Lagrangian alternating-direction implicit (SLADI) methodology are detailed and the results obtained are compared for effectiveness and efficiency with respect to more conventional semi-Lagrangian schemes combined with the fully implicit and the Crank-Nicolson method. With the results we analyse the sensitivities that this algorithms need to deal with, essentially a ‘non-linearity error’ is identified because of a discontinuous control of the storage.

We then proceed to investigate the sources of value for the proposed system. We compare configurations with and without storage facilities, and also with and without a revenue guarantee. Then we analyse in detail the configuration with storage and a revenue guarantee, here we identify stockpiling advantages and motivate the requirement for an optimal operation of the storage device. We also explore the introduction
of a seasonal electricity spot price and a correlation effect between the amounts of energy generated and the spot price of electricity.

4.1 The Model

Consider as an example a wind farm with a back-up battery installed on an electrical network; the term battery is merely analogical for a suitable energy storage device (ESD). The farm will produce electrical energy by transforming eolian energy, which is fundamentally highly stochastic in nature, resulting in a non-fully predictable source of energy, but nonetheless these supply fluctuations can be (effectively) smoothed by a storage device (Howell et al., 2011).

The unpredictable nature of the source makes investment in building a new wind farm inherently risky, so governments looking at ways to incentivise investment in renewable energies could try to attract investors by insuring long term delivery contracts at a fixed price for the wind farm producer. We therefore introduce a simplified contract in which the seller receives a constant fixed price for delivering a constant rate of electricity (regardless of the production level). Then, given an unpredictable production rate, they must trade electricity in the market to deliver the constant rate of electricity in the contract.

The system that we analyse comprises three space variables, \( X(t), Y(t), Q(t) \), and time, \( t \). Let \( t \in (0, T) \) be the chronological time, where \( T \) is a typical time scale of the wind farm. Next, let \( X(t) \in (X_{\min}, X_{\max}) \) be the surplus or deficit in production of electricity at time \( t \); then under the simplified contract the wind farm commits to produce a constant rate of \( C \) units of electricity, which we take as being half the capacity of the wind farm (\( 2C \) is the maximum capacity) so that we may then notionally set \( X_{\min} = -C \), and \( X_{\max} = C \) for a bounded domain on \( X \), where we assume \( C \gg 1 \); with this setting under and over commitment is avoided. Let \( Y(t) \in (0, \infty] \) be a non-negative electricity spot price at time \( t \) with long-term mean value \( \theta \), although numerically we take \( Y(t) \in (0, Y_{\max}) \) as the truncated bounded \( Y \) domain, where we must choose \( Y_{\max} \) sufficiently large so that the errors introduced from truncation are sufficiently small. Also let \( Q(t) \in (0, Q_{\max}) \) be the effective charge stored in the battery at time \( t \), then a nominal minimum capacity of zero corresponds to the depth of
discharge and $Q_{\text{max}}$ is the maximum effective capacity of this storage device. These define the system’s domain $H$.

In this section we assume that there are some fixed rules by which the wind farm operator can manage the battery, by choosing whether or not to fill or to empty the battery. These are given as follows: if there is a surplus in the production of energy, electricity is completely sold when the electricity price is at least $\theta$ monetary units (the long-term mean), otherwise surplus electricity is stockpiled at the prescribed rate of charge, noticing that non-stored electricity is still sold at time $dt$; if there is a deficit of energy, the battery is discharged as long as the electricity price is at least $\theta$ monetary units, otherwise electricity is completely bought from the market, likewise the discharge of the battery is at the prescribed rate and the complementary deficit is bought from the market. With this strategy, energy is stockpiled when the electricity price is low, and sold when the electricity price is high with respect to $\theta$, the main use of the storage is then for reducing the variation in the rate of energy generated, but we also implement a control to make the storage economically beneficial. Later, as an extension of the methodology application, we explore the definition of the optimal regime for operating the storage device.

At every time, the rate of charge, as well of discharge, is a function of $X$, $Y$ and $Q$, denoted by $\mathcal{L}(X,Y,Q)$. These rates basically follow those from Howell et al. (2011), namely

$$
\mathcal{L}(X,Q) = \begin{cases} 
\min\{X, X_c, \lambda_c (Q_{\text{max}} - Q)\} & \text{if } X > 0, \\
-\min\{|X|, X_d, \lambda_d Q\} & \text{if } X < 0,
\end{cases}
$$

where $\mathcal{L}(X,Q)$ is the rate for charging or discharging the storage, respectively. $X > 0$ is the (charging) region with surplus of electricity generated and $X < 0$ is the (discharging) region with deficit of energy generated, $X_c$ and $X_d$ are correspondingly the maximum rates at which the storage device can be charged or discharged, $Q$ is the amount of energy present in the storage device, and $\lambda_c$ and $\lambda_d$ are constants that correspondingly set levels of $Q$ at which the rates of charge and discharge decrease linearly to gradually reduce to zero. Although these rates are given for conceptual storage (Howell et al., 2011) we justify using these rates as they are feasible for Li-ion batteries (Thounthong et al., 2009), and these kinds of batteries are compatible with wind farm applications (see Hall and Bain, 2008; Beaudin et al., 2010).
The maximum rates of charge and discharge may be dropped as new materials (see Kang and Ceder, 2009) and safe procedures for fast charging and discharging are becoming available (see Chen and Rincon-Mora, 2006). Thus, in incremental time $dt$ the rate of charge is constrained to be the smaller of the surplus in energy production and a proportion $\lambda$ of the remaining capacity on the battery, and the rate of discharge is restricted to be the smaller of the deficit of energy and a proportion $\lambda$ of the remaining charge in the battery; this smooths the linear decreasing rates of charge and discharge to zero when the battery becomes full and empty, respectively, and physically means that overcharging and over-discharging states are never attained (feasible for e.g. Li-ion batteries, see Thounthong et al., 2009).

A parameter $\omega$ is introduced to represent the alternation between switching on and off the battery i.e., switching on the battery for the energy surplus region, $X > 0$, means charging it, and switching on the battery for the energy deficit region, $X < 0$, means discharging it, otherwise the battery is switched off ($\omega$ is therefore basically linked to a Heaviside function). Hence we take

$$\mathcal{L}(X,Y,Q) = \begin{cases} 
-\omega \min\{|X|, \lambda Q\} & \text{if } X < 0, \\
\omega = 0 & \text{if } Y < \theta, \\
\omega = 1 & \text{if } Y \geq \theta; \\
\omega \min\{X, \lambda Q_{\text{max}} - Q\} & \text{if } X > 0, \\
\omega = 1 & \text{if } Y < \theta, \\
\omega = 0 & \text{if } Y \geq \theta; 
\end{cases} \quad (4.1)$$

The dynamics of the amount of charge in the battery are then given by

$$dQ = \mathcal{L}(X,Y,Q) \, dt. \quad (4.2)$$

As noted earlier, under the terms of the contract the seller receives a continuous fixed payment. If $X = 0$ the farm is producing exactly the right amount of electricity then the seller should receive an amount

$$\Theta \, dt \quad (4.3)$$

in payment, since there are no penalties, in time $dt$. If $X < 0$ there is a deficit in production of electricity, then we first discharge the battery and after that, we pay a
penalty by having to buy electricity from the market at price $Y$, with *ask price* factor $(1 + \beta)$, i.e.

$$-(X - \mathcal{L})Y(1 + \beta) \, dt,$$

this occurring in time $dt$. If $X > 0$ there is a surplus of electricity, then we first charge the battery and after that, we can sell electricity to the market at price $Y$, with *bid price* factor $(1 - \beta)$

$$(X - \mathcal{L})Y(1 - \beta) \, dt,$$

this is also over a time period $dt$. Specifically, for this version of our model, we regard $\Theta$ as a constant, and a symmetric bid-ask price, but it would be a simple matter to relax these constraints.

We now derive the partial differential equation (PDE) describing the value of the contract for a wind farm with a storage $V(X, Y, Q, t)$ following a Real Options valuation approach (see Trigeorgis, 1996). Here, a modified Feynman-Kac equation (derived with probabilistic methods, and based on hedging arguments, see Øksendal, 2003) is used to define the stochastic dynamics of the risk-adjusted price process for the underlying (see section 2.1.5).

We assume that the risk-adjusted (or risk-neutral) electricity spot price, $Y$, follows a Cox-Ingersoll-Ross stochastic process (see Janczura and Weron, 2009), to reproduce characteristics of mean reversion and spikes with a simple model\(^2\), i.e.,

$$dY = \kappa(\theta - Y) \, dt + \sigma_Y \sqrt{Y} \, dW_1,$$

where $\kappa$ and $\sigma_Y$ are the speed of mean reversion and the volatility of the electricity price and $dW_1$ is a standard Wiener processes.

\(^2\)The random behavior of electricity prices is characterised by exhibiting spike distributions where extreme price movements occur. The volatility of electricity prices increases with high prices and therefore there is a volatility that is proportional to prices. Mean reversion is another property of electricity prices as they fluctuate about a long-term mean price level; with the latter possibly presenting seasonality. Janczura and Weron (2009) implemented models based on the Cox-Ingersoll-Ross (CIR) process (see equation (2.14)) and also used the generalised Ornstein-Uhlenbeck (OU) process (see equation (2.13)) as a basis, and concluded that the price-dependent-volatility CIR process approximates better real data distributions than the constant-volatility OU process. Bodily and Del Buono (2002) proposed the use of a mean-reverting model with proportional volatility for a better replication of higher volatility with high prices of electricity (see equation (2.15) with $v = 1$), but Janczura and Weron (2010) compared results obtained for models based on generalised mean-reversion processes (2.15) and in the model calibration with respect to real data the conclusion is that the coefficient $v$ results $v = \mathcal{O}(1/2)$ in half of the examples, this makes the CIR process a standard choice to represent satisfactorily electricity spot prices.
Figure 4.1 presents three random walks for the electricity spot price. The realisations start at the long-term mean value for electricity $\theta$, we can observe the effect of price spikes and trends reverting to $\theta$.

The supply or deficit in the production of electricity, $X$, is considered to be proportional to the wind speed on the site where the wind farm is located, here $X$ follows an Ornstein-Uhlenbeck stochastic process (see Zárate-Miñano et al., 2013; Benth and Benth, 2009; Edwards and Hurst, 2001), in order to approximately replicate real wind speed\(^3\), namely

$$dX = -\alpha X \, dt + \sigma_X \, dW_2,$$

(4.7)

where $\alpha$ and $\sigma_X$ are the speed of mean reversion and the volatility on the generation of energy and $dW_2$ is a standard Wiener processes.

In Figure 4.2 three random walks for the generation of energy are presented. The simulations start at the mean level of production ($X = 0$); for increasing time, we observe that if the amount of energy produced is far from the mean level, at some point there is a tendency of reversion towards zero, also constant variation can be

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\(^3\)The Ornstein-Uhlenbeck (OU) stochastic process (see equation (2.13)) has been used as a basis to develop models representing real wind speed. With this stochastic process, Zárate-Miñano et al. (2013) replicated the marginal distribution and autocorrelation functions of real data, although seasonal effects and diurnal cycles are not incorporated to the model it is found that the OU process can be used for different time scales. Benth and Benth (2009) valued futures contracts on a wind speed index, with the wind speed dynamics incorporating the OU process. Edwards and Hurst (2001) also use this OU process (albeit two-dimensional) to reproduce real wind-speed fluctuations.
identified.

We assume that the penetration of wind energy into the grid is small and thus the correlation between the processes $dW_1$ and $dW_2$ is taken to be zero, and so (4.7) needs not to be risk-adjusted.

We now employ a Feynman-Kac framework for boundary value problems, as described by Evatt et al. (2010), thus we consider a class of expectations for future cash flows of the form

$$u(z) = E_z \left[ e^{-r\tau} f(Z^{\tau}) + \int_0^{\tau} e^{-rt} g(Z^t) \, dt \right],$$

(4.8)

where $q(z)$ in (2.22) has been replaced by $r$, the constant riskless interest rate; the function $f$ is the value of the solution at the domain boundary $\partial H$, the function $g$ is the sum of the corresponding instantaneous source terms (4.3), (4.4) and (4.5). $Z^t \in \mathbb{R}^4$ is an Itô diffusion given by

$$dZ^t = \begin{pmatrix} 1 \\ \mathcal{L} \\ \kappa(\theta - Y) \\ -\alpha X \end{pmatrix} \, dt + \begin{pmatrix} 0 \\ 0 \\ \sigma_Y \sqrt{Y} \, dW_1 \\ \sigma_X \, dW_2 \end{pmatrix},$$

(4.9)

$\tau$ is the first exit time from the solution domain $H$ of $Z^t$, and $E_z$ represents the expected value for $Z^0 = z \in \mathbb{R}^4$. Equation (4.8) is then the solution to the boundary value problem (2.23), with discount rate $r$, where the partial differential operator...
is \((2.24)\). The drift coefficients are \(b^T = (1, \mathcal{L}, \kappa(\theta - Y), -\alpha X)\), and the diffusion coefficients are \(\sigma^T = \left(0, 0, \sigma_Y \sqrt{Y}, \sigma_X\right)\), and so \([a_{ij}] = \frac{1}{2}\sigma \sigma^T\). This leads to the PDE for the value \(V\) of a wind farm with a storage option, which accounts for the back discounted profit flow (income and outcome),

\[
\frac{\partial V}{\partial t} + \mathcal{L} \frac{\partial V}{\partial Q} - \alpha X \frac{\partial V}{\partial X} + \frac{1}{2} \sigma^2_X \frac{\partial^2 V}{\partial X^2} + \kappa(\theta - Y) \frac{\partial V}{\partial Y} + \frac{1}{2} \sigma^2_Y \frac{\partial^2 V}{\partial Y^2} - rV = -\Theta - \begin{cases} (X - \mathcal{L})Y(1 + \beta) & \text{if } X < 0, \\ (X - \mathcal{L})Y(1 - \beta) & \text{if } X > 0. \end{cases} \tag{4.10}
\]

The above equation has a parabolic form of a four-dimensional advection-diffusion-reaction problem with no diffusion in the storage dimension, \(Q\), or time. The second-order derivative terms are the contribution to the value of the system from the stochastic diffusion of \(X\) and \(Y\). The first-order derivative terms in \(X\) and \(Y\) arise from the deterministic changes to capital value that the production of electricity drift (reverting to zero) makes to the system by augmenting or shortening the availability of energy, and the change that the electricity price drift (reverting to \(\theta\)) makes to the system by the increment or decrement in price, correspondingly. The first-order derivative term in \(Q\) originates from the locally deterministic changes to capital that the rate \(\mathcal{L}\) makes to the system value by charging or discharging the battery, producing a path-dependent effect, akin to Asian options. The linear term captures the effects from the system’s capital movements on all its future options, and all instantaneous income from every space state \((X,Y,Q)\) over every time increment \(dt\), to produce a rate of return \(r\) on \(V\). The independent terms are the instantaneous income (outcome). Equation (4.10) progresses uniformly in \(t\) but not in the \(Q\) dimension as it occurs at the stochastic rate \(\mathcal{L}\).

These kinds of systems can be referred as “compounded American options” since a storage option is creating the option to sell units of electricity in a future time. For this version of the model we implement a fixed control, but in the next chapter we determine optimal rates for injection and extraction of energy from the storage device and so the “American” style for deciding the right time to stockpile and to sell is present in the model. A storage confers to the owner the right to buy units at low prices, and sell units at (possible) high market prices in the future, which can be referred as a calendar straddle option (Carmona and Ludkovski, 2010). A second
calendar option is given by allowing the owner of the storage to stockpile when there is surplus of energy generated and selling electricity when there is a deficit of energy generated.

Figure 4.3 presents an example of a random walk for the stochastic process followed by the wind farm state; for clarity, the plot on the left presents the path followed in the \((X, Q)\) plane, and the plot on the right presents the corresponding path in the \((Y, Q)\) plane. This simulation begins from a state without deficit or surplus in energy generation \((X = 0)\) and from a mean price of electricity \((Y = \theta)\), for a storage at half capacity \((Q = \frac{1}{2}Q_{\text{max}})\) at the initial time \((t = 0)\). For this realisation, the energy production immediately enters the deficit region with a low price for electricity, then the charge in battery remains the same. After that we generally observe high prices of electricity with respect to \(\theta\), thus the battery is predominantly discharged. Then we can identify that the production of energy reverts to normal levels and reaches the surplus region, then the battery is switched off and is not discharged. After some time the price of electricity generally decreases and becomes low with respect to \(\theta\), together with a generation of energy that in general remains in the surplus region, thus the battery is appropriately switched on, so that charge is mainly replenished until we observe a final increment on electricity prices and the battery is no longer recharged at the end of this random walk.
We next consider the boundary conditions, \( f(w) \) for \( w \in \partial H \), to be applied to this system. We assume that the system’s maximum deficit and surplus of energy is much greater than the charge capacity of the battery i.e., \( |X_{\text{min}}|, |X_{\text{max}}| \gg Q_{\text{max}} \) and that the valuation \( V \) grows linearly with \( X \) as \( |X| \to \infty \), so that we can impose \( \partial^2 V / \partial X^2 \to 0 \) in these limits (specifically this is the condition imposed at the extrema of the \( X \) domain, at \( X = X_{\text{max}} \) and \( X = X_{\text{min}} \)); these boundary conditions are explored and confirmed in the Appendix. The accuracy of these conditions depends explicitly on the choice of parameters, since we require that the rate of mean reversion is sufficiently large so that the probability of the process ever reaching \( X = X_{\text{max}} \) or \( X = X_{\text{min}} \) is sufficiently small. Thus we impose the reduced PDE at the extreme (truncated) \( X \) locations

\[
\frac{\partial V}{\partial t} + \mathcal{L} \frac{\partial V}{\partial Q} - \alpha X \frac{\partial V}{\partial X} + \frac{1}{2} \sigma^2_X \frac{\partial^2 V}{\partial X^2} + \kappa \theta \frac{\partial V}{\partial Y} - \frac{1}{2} \sigma^2_Y \frac{\partial^2 V}{\partial Y^2} - rV = -\Theta - \left\{ \begin{array}{l} (X - \mathcal{L})Y(1 + \beta) \text{ on } X_{\text{min}}, \\ (X - \mathcal{L})Y(1 - \beta) \text{ on } X_{\text{max}}. \end{array} \right. \tag{4.11}
\]

As the electricity price \( Y \) becomes large we will consider the absence of diffusion, and so we set \( \partial^2 V / \partial Y^2 = 0 \) at the extreme value of \( Y \), namely \( Y_{\text{max}} \), and so we impose the following reduced PDE

\[
\frac{\partial V}{\partial t} + \mathcal{L} \frac{\partial V}{\partial Q} - \alpha X \frac{\partial V}{\partial X} + \frac{1}{2} \sigma^2_X \frac{\partial^2 V}{\partial X^2} + \kappa \theta \frac{\partial V}{\partial Y} - \frac{1}{2} \sigma^2_Y \frac{\partial^2 V}{\partial Y^2} - rV = -\Theta - \left\{ \begin{array}{l} (X - \mathcal{L})Y(1 + \beta) \text{ if } X < 0, \\ (X - \mathcal{L})Y(1 - \beta) \text{ if } X > 0. \end{array} \right. \tag{4.12}
\]

When the price of electricity \( Y \) is equal to zero, the value is determined from the degenerate PDE simply by setting \( Y = 0 \), namely

\[
\frac{\partial V}{\partial t} + \mathcal{L} \frac{\partial V}{\partial Q} - \alpha X \frac{\partial V}{\partial X} + \frac{1}{2} \sigma^2_X \frac{\partial^2 V}{\partial X^2} + \kappa \theta \frac{\partial V}{\partial Y} - rV = -\Theta. \tag{4.13}
\]

When the battery is empty, \( Q = 0 \) for \( X < 0 \), it can no longer be discharged and then \( \mathcal{L} = 0 \), and so the PDE reduces to

\[
\frac{\partial V}{\partial t} - \alpha X \frac{\partial V}{\partial X} + \frac{1}{2} \sigma^2_X \frac{\partial^2 V}{\partial X^2} + \kappa \theta \frac{\partial V}{\partial Y} + \frac{1}{2} \sigma^2_Y \frac{\partial^2 V}{\partial Y^2} - rV = -\Theta - XY(1 + \beta). \tag{4.14}
\]

When the battery is full, \( Q = Q_{\text{max}} \) for \( X > 0 \), it can no longer be charged and then,
again, \( \mathcal{L} = 0 \), and so the PDE reduces to

\[
\frac{\partial V}{\partial t} - \alpha X \frac{\partial V}{\partial X} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial X^2} + \kappa (\theta - Y) \frac{\partial V}{\partial Y} + \frac{1}{2} \sigma^2 Y \frac{\partial^2 V}{\partial Y^2} - r V = -\Theta - XY(1 - \beta).
\] (4.15)

Finally, we set a final value at the end of the time scale of the wind farm

\[ V(X, Y, Q, T) = V^T. \] (4.16)

For the purposes of the calculations presented here, we always set \( V^T = 0 \), and so the forcing of the system arises from the source terms on the right-hand-sides of (4.10) - (4.15).

The value \( V \) accounts for the back discounted profit flow entitled to the owner of the wind farm, subject to both uncertain output and electricity prices, by receiving a fixed-revenue guarantee for a fixed-output rate, by receiving the income from selling surplus energy, and by paying the cost of energy in deficit. The use of the storage is essentially to balance the output; energy is injected into the storage if there is surplus of energy generated and low prices of electricity with respect to the long-term mean price of electricity, and energy is extracted from the storage if there is a deficit of the energy generated and high prices of electricity with respect to the long-term mean price of electricity.

### 4.2 SLADI

Here we propose to solve the PDE (4.10) using a hybrid algorithm, namely a semi-Lagrangian scheme in conjunction with the ADI method. Four versions of the scheme are fully explained below, accuracy and computational times are assessed and compared with more standard semi-Lagrangian Crank-Nicolson and semi-Lagrangian fully implicit methodologies results in subsection 4.4.

In our notation, the superscript \(^{(m)}\) will indicate the number \( m \) of first-order terms incorporated in the Lagrangian derivative with respect to the temporal variable, always starting with the non-diffusive (storage) dimension, and after that all first-order terms are incorporated (including those from diffusive dimensions). In the SLADI methodology, a subscript \(_{1}\) indicates that only one characteristic is followed per time.
step and a subscript $\Pi$ indicates that $d$ independent characteristics are followed, one for each fraction of time-step increment $\frac{1}{d} \Delta \tau$.

To continue the exposition, equation (4.10) is written in forward parabolic form (in $\tau$), namely

$$-\frac{\partial V}{\partial \tau} + \mathcal{L} \frac{\partial V}{\partial Q} - \alpha X \frac{\partial V}{\partial X} + \kappa (\theta - Y) \frac{\partial V}{\partial Y} + \frac{1}{2} \sigma_X^2 \frac{\partial^2 V}{\partial X^2} + \frac{1}{2} \sigma_Y^2 Y \frac{\partial^2 V}{\partial Y^2} - \tau V$$

$$= -\Theta - \begin{cases} (X - \mathcal{L})Y(1 + \beta) & \text{if } X < 0, \\ (X - \mathcal{L})Y(1 - \beta) & \text{if } X > 0. \end{cases}$$

PDE (4.17) will be approximated in two basic (ADI) steps, which are executed in immediate succession. In the first step, an implicit difference approximation of the derivative terms with respect to $X$, and an explicit difference approximation of the derivative terms with respect to $Y$ is implemented; in the second step, an implicit difference approximation of the derivative terms with respect to $Y$, and an explicit difference approximation of the derivative terms with respect to $X$ is implemented. The devised variations on the scheme are dependent on specific characteristics followed by virtual particles in the semi-Lagrangian approach.

The spatial domain is discretised as follows, let $V(X_i, Y_j, Q_k, \tau_l)$ be the value of the system at time $\tau_l = l \Delta \tau$, for $X_i = X_{\min} + i \Delta X, Y_j = j \Delta Y$ and $Q_k = k \Delta Q$, then $V_{i,j,k}^l$ corresponds to the solution $V$ at the grid point $(i, j, k, l)$. One common feature of all the schemes to be presented is that the truncation error is notionally $O(\Delta \tau + \Delta Q^2 + \Delta X^2 + \Delta Y^2)$. A detailed empirical study on the effect of truncation errors is presented in subsection 4.4 and 4.4.1.

### 4.2.1 SLADI$^{(1)}$

Here, form (3.5) is partially rewritten by incorporating (just) the non-diffusive dimension $Q$ into the total derivative, and then the Lagrangian derivative along a trajectory $Q = Q(X, Y, \tau)$, for $X$ and $Y$ fixed in PDE (4.17), is

$$\frac{DV}{D\tau} = \frac{\partial V}{\partial \tau} + \frac{dQ}{d\tau} \frac{\partial V}{\partial Q}.$$ (4.18)

Along the trajectory

$$\frac{dQ}{d\tau} = -\mathcal{L},$$ (4.19)
we then have
\[
\frac{DV}{Dr} = -\alpha X \frac{\partial V}{\partial X} + \kappa(\theta - Y) \frac{\partial V}{\partial Y} + \frac{1}{2} \sigma_X^2 \frac{\partial^2 V}{\partial X^2} + \frac{1}{2} \sigma_Y^2 Y \frac{\partial^2 V}{\partial Y^2} - rV + \Theta + \left\{ \begin{array}{ll} (X - \mathcal{L})Y(1 + \beta) & \text{if } X < 0, \\ (X - \mathcal{L})Y(1 - \beta) & \text{if } X > 0. \end{array} \right. \tag{4.20}
\]

**SLADI\(_1\)**

This first variant of the methodology comprises the following basic steps for the difference approximation to (4.20):

**Step 1:**
\[
\frac{V_{i,j,k_{\ast}}^{l+\frac{1}{2}} - V_{i,j,k_{\ast}}^l}{\frac{1}{2} \Delta r} = -\alpha X_i \frac{V_{i+1,j,k_{\ast}}^{l+\frac{1}{2}} - V_{i-1,j,k_{\ast}}^{l+\frac{1}{2}}}{2 \Delta X} + \kappa(\theta - Y_j) \frac{V_{i,j+1,k_{\ast}}^{l+1} - V_{i,j-1,k_{\ast}}^{l+1}}{2 \Delta Y} + \frac{1}{2} \sigma_X^2 \frac{V_{i+1,j,k_{\ast}}^{l+\frac{1}{2}} - 2V_{i,j,k_{\ast}}^{l+\frac{1}{2}} + V_{i-1,j,k_{\ast}}^{l+\frac{1}{2}}}{\Delta X^2} + \frac{1}{2} \sigma_Y^2 Y_j \frac{V_{i,j+1,k_{\ast}}^{l+1} - 2V_{i,j,k_{\ast}}^{l+1} + V_{i,j-1,k_{\ast}}^{l+1}}{\Delta Y^2} - r\frac{V_{i,j,k_{\ast}}^{l+\frac{1}{2}}}{\Delta r} + \Theta + \left\{ \begin{array}{ll} (X_i - \mathcal{L}_k^{l+1})Y_j(1 + \beta) & \text{if } X_i < 0, \\ (X_i - \mathcal{L}_k^{l+1})Y_j(1 - \beta) & \text{if } X_i > 0. \end{array} \right. \tag{4.21}\]

**Step 2:**
\[
\frac{V_{i,j,k}^{l+1} - V_{i,j,k_{\ast}}^{l+\frac{1}{2}}}{\frac{1}{2} \Delta r} = -\alpha X_i \frac{V_{i+1,j,k_{\ast}}^{l+\frac{1}{2}} - V_{i-1,j,k_{\ast}}^{l+\frac{1}{2}}}{2 \Delta X} + \kappa(\theta - Y_j) \frac{V_{i,j+1,k}^{l+1} - V_{i,j-1,k}^{l+1}}{2 \Delta Y} + \frac{1}{2} \sigma_X^2 \frac{V_{i+1,j,k_{\ast}}^{l+\frac{1}{2}} - 2V_{i,j,k_{\ast}}^{l+\frac{1}{2}} + V_{i-1,j,k_{\ast}}^{l+\frac{1}{2}}}{\Delta X^2} + \frac{1}{2} \sigma_Y^2 Y_j \frac{V_{i,j+1,k}^{l+1} - 2V_{i,j,k}^{l+1} + V_{i,j-1,k}^{l+1}}{\Delta Y^2} - rV_{i,j,k}^{l+\frac{1}{2}} + \Theta + \left\{ \begin{array}{ll} (X_i - \mathcal{L}_k^{l+1})Y_j(1 + \beta) & \text{if } X_i < 0, \\ (X_i - \mathcal{L}_k^{l+1})Y_j(1 - \beta) & \text{if } X_i > 0, \end{array} \right. \tag{4.22}\]

where the values \(V_{i,j,k_{\ast}}\), denoting \(V(X_i, Y_j, Q_{k_{\ast}}, \tau^l)\), are determined by (natural cubic spline) interpolation on the \(Q_{k_{\ast}}\) gridpoints,
\[
Q_{k_{\ast}}^{l+1} = Q_{k_{\ast}}^l + \Delta r L_{k}^{l+1}; \tag{4.23}\]

and the values \(V_{i,j,k_{\ast}}^{l+\frac{1}{2}}\), denoting \(V(X_i, Y_j, Q_{k_{\ast}}, \tau^{l+\frac{1}{2}})\), are located at the \(Q_{k_{\ast}}^{l+\frac{1}{2}}\) gridpoints,
\[
Q_{k_{\ast}}^{l+\frac{1}{2}} = Q_{k_{\ast}}^l + \frac{1}{2} \Delta r L_{k}^{l+1}; \tag{4.24}\]

which we take to be at half the way along the trajectory and then we do not need to explicitly determine the location because we are assuming that \(L_{k}^{l+1}\) is approximately constant along the corresponding characteristic.
The approximations (4.23) and (4.24) to (4.19) imply that we make one full and one half $\tau$ step backwards along the same trajectory, correspondingly, applying Euler’s method, and this leads to a global error of $O(\Delta\tau)$, i.e., of first order in timestep. The first step involves solving a tridiagonal system for the $X_i$ values on the *irregular* $Q_{l+\frac{1}{2}}$ grid, and the second step involves solving a tridiagonal system for the $Y_j$ values on the *regular* $Q_{l+1}$ grid.

One key point with regard to the above, which distinguishes it from the following variant, is that $L_{l+\frac{1}{2}}$ is evaluated on the regular grid $(Q_{l+1})$ throughout, i.e.

$$L_{l+\frac{1}{2}} = \begin{cases} -\omega \min\{-X_i, \lambda Q_{l+1}\} & \text{for } X_i < 0, \\ \omega \min\{X_i, \lambda (Q_{\max} - Q_{l+1})\} & \text{for } X_i > 0; \end{cases}$$

(4.25)

and (for this reason), interpolation of data from the regular grid on to the irregular grid is only necessary at the $\tau_l$ level.

**SLADI$_{11}^{(1)}$**

Here, equation (4.20) is used to calculate the solution of the system at every half time level on *regular* mesh points (this distinguishes this scheme from SLADI$_{11}^{(1)}$).

This second variant of the methodology comprises the following steps for the difference approximation to (4.20):

**Step 1:**

$$\frac{V_{i+\frac{1}{2},j,k}^l - V_{i,j,k}^l}{\frac{1}{2}\Delta\tau} = -\alpha X_i V_{i+1,j,k}^l + \frac{1}{2}\Delta\tau V_{i,j+1,k}^l - V_{i,j-1,k}^l + \kappa(\theta - Y_j) \frac{V_{i+1,j,k}^l - V_{i,j,k}^l}{2\Delta Y} + \frac{1}{2}\sigma^2 X_i V_{i+1,j,k}^l + \frac{1}{2}\sigma^2 V_{i,j+1,k}^l - 2V_{i,j,k}^l + V_{i-1,j,k}^l \frac{V_{i,j+1,k}^l - V_{i,j-1,k}^l}{\Delta Y}$$

$$- rV_{i,j,k}^l + \Theta + \begin{cases} (X_i - L_{l+\frac{1}{2}})Y_j(1 + \beta) & \text{if } X_i < 0, \\ (X_i - L_{l+\frac{1}{2}})Y_j(1 - \beta) & \text{if } X_i > 0. \end{cases}$$

(4.26)

Here

$$L_{l+\frac{1}{2}} = \begin{cases} -\omega \min\{-X_i, \lambda Q_{l+\frac{1}{2}}\} & \text{for } X_i < 0, \\ \omega \min\{X_i, \lambda (Q_{\max} - Q_{l+\frac{1}{2}})\} & \text{for } X_i > 0. \end{cases}$$

(4.27)
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Step 2:

\[
\frac{V_{i,j,k}^{l+\frac{1}{2}} - V_{i,j,k}^{l+\frac{1}{2}}}{\frac{1}{2}\Delta\tau} = -\alpha X_i \frac{V_{i+1,j,k}^{l+\frac{1}{2}} - V_{i-1,j,k}^{l+\frac{1}{2}}}{2\Delta X} + \kappa(\theta - Y_j) \frac{V_{i,j+1,k}^{l+1} - V_{i,j-1,k}^{l+1}}{2\Delta Y} \\
+ \frac{1}{4}\sigma_X^2 \frac{V_{i+1,j,k}^{l+\frac{1}{2}} - 2V_{i,j,k}^{l+\frac{1}{2}} + V_{i-1,j,k}^{l+\frac{1}{2}}}{\Delta_X^2} + \frac{1}{2}\sigma_Y^2 Y_j \frac{V_{i,j+1,k}^{l+1} - 2V_{i,j,k}^{l+1} + V_{i,j-1,k}^{l+1}}{\Delta_Y^2} \\
- rV_{i,j,k}^{l+1} + \Theta + \begin{cases} 
(X_i - \mathcal{L}_k^{l+1})Y_j (1 + \beta) & \text{if } X_i < 0, \\
(X_i - \mathcal{L}_k^{l+1})Y_j (1 - \beta) & \text{if } X_i > 0.
\end{cases}
\] (4.28)

Here

\[
\mathcal{L}_k^{l+1} = \begin{cases} 
-\omega \min\{-X_i, \lambda Q_k^{l+1}\} & \text{for } X_i < 0, \\
\omega \min\{X_i, \lambda(Q_{\max} - Q_k^{l+1})\} & \text{for } X_i > 0.
\end{cases}
\] (4.29)

The values \(V_{i,j,k}^{l}\) denote \(V(X_i, Y_j, Q_k^{l}, \tau_i)\), and are determined by (natural cubic spline) interpolation at the \(Q_k^{l}\) gridpoints,

\[
Q_k^{l} = Q_k^{l+\frac{1}{2}} + \frac{1}{2}\Delta\tau \mathcal{L}_k^{l+\frac{1}{2}}.
\] (4.30)

and the values \(V_{i,j,k}^{l+\frac{1}{2}}\), denoting \(V(X_i, Y_j, Q_k^{l+\frac{1}{2}}, \tau_i^{l+\frac{1}{2}})\), are determined by (natural cubic spline) interpolation at the \(Q_k^{l+\frac{1}{2}}\) gridpoints,

\[
Q_k^{l+\frac{1}{2}} = Q_k^{l+1} + \frac{1}{2}\Delta\tau \mathcal{L}_k^{l+1}.
\] (4.31)

The approximations (4.30) and (4.31) to (4.19) imply that we make a half \(\tau\) step backwards along independent trajectories applying Euler’s method.

The first step involves solving a tridiagonal system for the \(X_i\) values, leading to a (partial) solution at \(\tau + \frac{1}{2}\Delta\tau\), but on the regular grid. The second step involves solving a tridiagonal system to determine the solution on all \(Y_j\) grid points, also on a regular grid.

4.2.2 SLADI(3)

In this version of the scheme, all first-order derivative terms (including the \(X\) and \(Y\) derivatives) are incorporated into the total derivative of \(V\) with respect to \(\tau\) to obtain the form (3.5). The Lagrangian derivative along a trajectory in \(X(\tau), Y(\tau),\) and \(Q(\tau)\) is then

\[
\frac{DV}{D\tau} = \frac{\partial V}{\partial \tau} + \frac{dX}{d\tau} \frac{\partial V}{\partial X} + \frac{dY}{d\tau} \frac{\partial V}{\partial Y} + \frac{dQ}{d\tau} \frac{\partial V}{\partial Q}.
\] (4.32)
and the trajectory is defined by
\[
\frac{dX}{d\tau} = \alpha X, \quad \frac{dY}{d\tau} = \kappa(Y - \theta), \quad \frac{dQ}{d\tau} = -L, \quad (4.33)
\]
and then
\[
\frac{DV}{D\tau} = \frac{1}{2} \frac{\partial^2 V}{\partial X^2} + \frac{1}{2} \frac{\partial^2 V}{\partial Y^2} - rV + \Theta + \begin{cases} 
(X - L)Y(1 + \beta) & \text{if } X < 0, \\
(X - L)Y(1 - \beta) & \text{if } X > 0.
\end{cases} \quad (4.34)
\]

The solution progression is very much in the same way as with the SLADI\(_1\) variants, but with the added requirement that the interpolation between grids must be carried out in three dimensions \((X, Y, Q)\), rather than just one; for this quadratic Lagrange interpolation was implemented.

**SLADI\(_1^3\)**

This third variant of the methodology comprises the following basic steps for the difference approximation to \((4.20)\):

**Step 1:**
\[
\frac{V_{i,j,k}^{l+\frac{1}{2}} - V_{i,j,k}^l}{\frac{1}{2}\Delta\tau} = \frac{1}{2} \frac{\partial^2 V}{\partial X^2} \frac{V_{i+\frac{1}{2},j,k}^{l+\frac{1}{2}} - V_{i-\frac{1}{2},j,k}^{l+\frac{1}{2}}}{\Delta X^2} + \frac{1}{2} \frac{\partial^2 V}{\partial Y^2} \frac{V_{i,j+\frac{1}{2},k}^{l+\frac{1}{2}} - V_{i,j-\frac{1}{2},k}^{l+\frac{1}{2}}}{\Delta Y^2} - rV_{i,j,k}^{l+\frac{1}{2}} + \Theta + \begin{cases} 
(X_i - L_{k+1}^l)Y_j(1 + \beta) & \text{if } X_i < 0, \\
(X_i - L_{k+1}^l)Y_j(1 - \beta) & \text{if } X_i > 0.
\end{cases} \quad (4.35)
\]

**Step 2:**
\[
\frac{V_{i,j,k}^{l+1} - V_{i,j,k}^{l+\frac{1}{2}}}{\frac{1}{2}\Delta\tau} = \frac{1}{2} \frac{\partial^2 V}{\partial X^2} \frac{V_{i+1,j,k}^{l+\frac{1}{2}} - V_{i-1,j,k}^{l+\frac{1}{2}}}{\Delta X^2} + \frac{1}{2} \frac{\partial^2 V}{\partial Y^2} \frac{V_{i,j+1,k}^{l+1} - V_{i,j-1,k}^{l+1}}{\Delta Y^2} - rV_{i,j,k}^{l+1} + \Theta + \begin{cases} 
(X_i - L_k^{l+1})Y_j(1 + \beta) & \text{if } X_i < 0, \\
(X_i - L_k^{l+1})Y_j(1 - \beta) & \text{if } X_i > 0.
\end{cases} \quad (4.36)
\]

where the values \(V_{i,j,k}^l\), denoting \(V(X_i, Y_j, Q_k, \tau^l)\), are determined by (natural cubic spline) interpolation on the \(Q_k^l\) gridpoints,
\[
X_i^l = X_i^{l+1}(1 - \Delta\tau\alpha), \\
Y_j^l = Y_j^{l+1}(1 - \Delta\tau\kappa) + \Delta\tau\kappa\theta, \\
Q_k^l = Q_k^{l+1} + \Delta\tau L_k^{l+1}; \quad (4.37)
\]
and the values \( V_{i_*, j_*, k_*}^{l+\frac{1}{2}} \), denoting \( V(X_{i_*}, Y_{j_*}, Q_{k_*}, \tau^{l+\frac{1}{2}}) \), are located at the \( X_{i_*}, Y_{j_*}, Q_{k_*}^{l+\frac{1}{2}} \) gridpoints,

\[
\begin{align*}
X_{i_*}^{l+\frac{1}{2}} &= X_{i}^{l+1}(1 - \frac{1}{2}\Delta_x \alpha), \\
Y_{j_*}^{l+\frac{1}{2}} &= Y_{j}^{l+1}(1 - \frac{1}{2}\Delta_y \kappa) + \frac{1}{2}\Delta_y \kappa \theta, \\
Q_{k_*}^{l+\frac{1}{2}} &= Q_{k}^{l+1} + \frac{1}{2}\Delta_z \mathcal{L}_{k}^{l+1}.
\end{align*}
\] (4.38)

which we take to be at half the way along the trajectory and then we do not need to explicitly determine the location because we are assuming that \( \mathcal{L}_{k}^{l+1} \) is approximately constant along the corresponding characteristic, but in this case we have obtained solutions on an intermediate grid (in the \( X \) direction) at the half-step, and after that we use these values to complete the full time step. At the end of the full time step the solutions are now located on a totally regular grid. For this particular implementation one needs to choose a balanced grid spacing \((\Delta_x, \Delta_y)\) to avoid instabilities due to the intermediate grid at the half step.

The approximations (4.23) and (4.24) to (4.19) imply that we make one full and one half \( \tau \) step backwards along the same trajectory, correspondingly, applying Euler’s method, and this leads to a global error of \( \mathcal{O}(\Delta_\tau) \), i.e. of first order in timestep. The first step involves solving a tridiagonal system for the \( X_i \) values on the irregular \( Q_{k_*}^{l+\frac{1}{2}} \) grid, and the second step involves solving a tridiagonal system for the \( Y_j \) values on the regular \( Q_{k}^{l+1} \) grid.

Again, one key point with regard to the above, which distinguishes it from the following variant, is that \( \mathcal{L}_{k}^{l+1} \) is evaluated on the regular grid \((Q_{k}^{l+1})\) throughout, i.e.

\[
\mathcal{L}_{k}^{l+1} = \begin{cases} 
-\omega \min\{-X_i, \lambda Q_{k}^{l+1}\} & \text{for } X_i < 0, \\
\omega \min\{X_i, \lambda(Q_{\max} - Q_{k}^{l+1})\} & \text{for } X_i > 0; 
\end{cases}
\] (4.39)

and (for this reason), interpolation of data from the regular grid on to the irregular grid is only necessary at the \( \tau^l \) level.
In this final variant, the numerical solution is then progressed as follows:

**Step 1:**

\[
\frac{V_{i,j,k}^{l+\frac{1}{2}} - V_{i,j,k}^l}{\frac{1}{2}\Delta_x} = \frac{1}{2} \sigma_X V_{i+1,j,k}^{l+\frac{1}{2}} - 2V_{i,j,k}^{l+\frac{1}{2}} + V_{i-1,j,k}^{l+\frac{1}{2}} \frac{\Delta^2_X}{\Delta^2_X} + \frac{1}{2} \sigma_Y Y_{j} V_{i,j+1,k}^{l+\frac{1}{2}} - 2V_{i,j,k}^{l+\frac{1}{2}} + V_{i,j-1,k}^{l+\frac{1}{2}} \frac{\Delta^2_Y}{\Delta^2_Y} - rV_{i,j,k}^{l+\frac{1}{2}} + \Theta + \begin{cases} (X_i - \mathcal{L}_k^{l+\frac{1}{2}})Y_j (1 + \beta) & \text{if } X_i < 0, \\ (X_i - \mathcal{L}_k^{l+\frac{1}{2}})Y_j (1 - \beta) & \text{if } X_i > 0. \end{cases}
\]

(4.40)

Here

\[
\mathcal{L}_k^{l+\frac{1}{2}} = \begin{cases} -\omega \min \{-X_i, \lambda Q_k^{l+\frac{1}{2}}\} & \text{for } X_i < 0, \\ \omega \min \{X_i, \lambda (Q_{\max} - Q_k^{l+\frac{1}{2}})\} & \text{for } X_i > 0. \end{cases}
\]

**Step 2:**

\[
\frac{V_{i,j,k}^{l+1} - V_{i,j,k}^{l+\frac{1}{2}}}{\frac{1}{2}\Delta_x} = \frac{1}{2} \sigma_X V_{i+1,j,k}^{l+1} - 2V_{i,j,k}^{l+\frac{1}{2}} + V_{i-1,j,k}^{l+\frac{1}{2}} \frac{\Delta^2_X}{\Delta^2_X} + \frac{1}{2} \sigma_Y Y_{j} V_{i,j+1,k}^{l+1} - 2V_{i,j,k}^{l+\frac{1}{2}} + V_{i,j-1,k}^{l+\frac{1}{2}} \frac{\Delta^2_Y}{\Delta^2_Y} - rV_{i,j,k}^{l+1} + \Theta + \begin{cases} (X_i - \mathcal{L}_k^{l+1})Y_j (1 + \beta) & \text{if } X_i < 0, \\ (X_i - \mathcal{L}_k^{l+1})Y_j (1 - \beta) & \text{if } X_i > 0. \end{cases}
\]

(4.42)

Here

\[
\mathcal{L}_k^{l+1} = \begin{cases} -\omega \min \{-X_i, \lambda Q_k^{l+1}\} & \text{for } X_i < 0, \\ \omega \min \{X_i, \lambda (Q_{\max} - Q_k^{l+1})\} & \text{for } X_i > 0. \end{cases}
\]

The values \(V_{i,j,k}^l\) denote \(V(X_i, Y_j, Q_k, \tau^l)\), and are determined by (quadratic Lagrange) interpolation in three dimensions at the \(X_i^l, Y_j^l, Q_k^l\) gridpoints,

\[
X_i^l = X_i^{l+\frac{1}{2}} (1 - \frac{1}{2}\Delta_x \alpha),
\]

\[
Y_j^l = Y_j^{l+\frac{1}{2}} (1 - \frac{1}{2}\Delta_x \kappa) + \frac{1}{2}\Delta_x \kappa \theta,
\]

\[
Q_k^l = Q_k^{l+\frac{1}{2}} + \frac{1}{2}\Delta_x \mathcal{L}_k^{l+\frac{1}{2}}.
\]

(4.44)

and the values \(V_{i,j,k}^{l+\frac{1}{2}}\), denoting \(V(X_i, Y_j, Q_k, \tau^{l+\frac{1}{2}})\), are determined by (quadratic Lagrange) interpolation in three dimensions at the \(X_i^{l+\frac{1}{2}}, Y_j^{l+\frac{1}{2}}, Q_k^{l+\frac{1}{2}}\) gridpoints,

\[
X_i^{l+\frac{1}{2}} = X_i^{l+1} (1 - \frac{1}{2}\Delta_x \alpha),
\]

\[
Y_j^{l+\frac{1}{2}} = Y_j^{l+1} (1 - \frac{1}{2}\Delta_x \kappa) + \frac{1}{2}\Delta_x \kappa \theta,
\]

\[
Q_k^{l+\frac{1}{2}} = Q_k^{l+1} + \frac{1}{2}\Delta_x \mathcal{L}_k^{l+1}.
\]

(4.45)
CHAPTER 4. STORAGE VALUATION, A WIND POWER APPLICATION

The approximations (4.44) and (4.45) to (4.33) imply that we make two half $\tau$ steps backwards along independent trajectories $(X, Y, Q)$ applying Euler’s method. For practical computations it is convenient to calculate the field (across the $XY$ domain) for the explicit approximation to the required second-order derivative terms and then find the corresponding values using (quadratic Lagrange) interpolation in three dimensions (just) on to the $(i_*, j_*, k_*)$ location, otherwise multiple interpolants in three dimensions are required for every finite-difference equation.

The first step involves solving a tridiagonal system, to determine the solution at all $X_i$ points, on a regular grid. The second step involves solving a tridiagonal system, to determine the solution at all $Y_j$ points, also on a regular grid.

4.3 Crank-Nicolson and fully implicit schemes

Here we define implementations for the semi-Lagrangian Crank-Nicolson (SLCN) and for the fully implicit (SLFI) methods. In a first approach, again, form (3.5) is partially obtained by incorporating only the non-diffusive dimension $Q$ into the Lagrangian derivative, then proceeding from equation (4.20), we solve for every time step the following difference approximation

$$
\frac{V_{i,j,k}^{l+1} - V_{i,j,k}^l}{\Delta t} = \vartheta \left( -\alpha X_i \frac{V_{i+1,j,k}^{l+1} - V_{i-1,j,k}^l}{2\Delta X} + \kappa(\theta - Y_j) \frac{V_{i,j+1,k}^{l+1} - V_{i,j-1,k}^l}{2\Delta Y} \\
+ \frac{1}{2}\sigma_X^2 \frac{V_{i+1,j,k}^{l+1} - 2V_{i,j,k}^{l+1} + V_{i-1,j,k}^l}{\Delta^2 X} + \frac{1}{2}\sigma_Y^2 \frac{V_{i,j+1,k}^{l+1} - 2V_{i,j,k}^{l+1} + V_{i,j-1,k}^l}{\Delta^2 Y} \\
- rV_{i,j,k}^l \right) + \Theta + \begin{cases} 
(X_i - \mathcal{L}_{k+1}^{l+1})Y_j(1 + \beta) & \text{if } X_i < 0, \\
(X_i - \mathcal{L}_{k+1}^{l+1})Y_j(1 - \beta) & \text{if } X_i > 0.
\end{cases}
$$

(4.46)

Here

$$
\mathcal{L}_{k+1}^{l+1} = \begin{cases} 
-\omega \min \{-X_i, \lambda Q_k^{l+1}\} & \text{for } X_i < 0, \\
\omega \min \{X_i, \lambda(Q_{\max} - Q_k^{l+1})\} & \text{for } X_i > 0;
\end{cases}
$$

(4.47)
where the values \( V_{i,j,k}^l \), denoting \( V(X_i, Y_j, Q_{k^*}, \tau^l) \), are determined by (natural cubic spline) interpolation at the \( Q_{k^*} \) gridpoints,

\[
Q_{k^*}^l = Q_{k^*}^{l+1} + \Delta \tau \mathcal{L}_{k}^{l+1}. \tag{4.48}
\]

\( \vartheta = \frac{1}{2} \) gives Crank-Nicolson (SLCN(1)) time stepping, and \( \vartheta = 1 \) results in the fully implicit (SLFI(1)) time stepping.

Again, we assume that \( \mathcal{L}_{k}^{l+1} \) is approximately constant along the corresponding characteristic.

In this second approach, all first-order derivative terms (including the \( X \) and \( Y \) derivatives) are incorporated into the total derivative of \( V \) with respect to \( \tau \) to obtain the form (3.5). Proceeding from equation (4.34), we solve for every time step the following difference approximation

\[
\frac{V_{i,j,k}^{l+1} - V_{i,j,k}^l}{\Delta \tau} = \vartheta \left( \frac{1}{2} \sigma_X^2 \frac{V_{i+1,j,k}^{l+1} - 2V_{i,j,k}^{l+1} + V_{i-1,j,k}^{l+1}}{\Delta \tau^2} \right.
\]

\[
+ \frac{1}{2} \sigma_Y^2 \frac{V_{i,j+1,k}^{l+1} - 2V_{i,j,k}^{l+1} + V_{i,j-1,k}^{l+1}}{\Delta \tau^2}
\]

\[
- rV_{i,j,k}^{l+1} + (X_i^{l+1} - \mathcal{L}_{k}^{l+1})Y_j^{l+1}(1 \pm \beta) \bigg) \bigg) + \Theta, \text{ if } X_i \leq 0. \tag{4.49}
\]

Here

\[
\mathcal{L}_{k}^{l+1} = \begin{cases} 
- \omega \min\{X_i, \lambda Q_{k}^{l+1}\} & \text{for } X_i < 0, \\
\omega \min\{X_i, \lambda (Q_{\max} - Q_{k}^{l+1})\} & \text{for } X_i > 0;
\end{cases} \tag{4.50}
\]

The values \( V_{i,j,k}^{l} \) denote \( V(X_i, Y_j, Q_{k^*}, \tau^l) \), and are determined by (quadratic Lagrange) interpolation in three dimensions at the \( X_{i^*}^l, Y_{j^*}^l, Q_{k^*}^l \) gridpoints,

\[
X_{i^*}^l = X_i^l(1 - \Delta \alpha), \quad Y_{j^*}^l = Y_j^l(1 - \Delta \kappa) + \Delta \kappa \theta, \quad Q_{k^*}^l = Q_{k^*}^{l+1} + \Delta \tau \mathcal{L}_{k}^{l+1}. \tag{4.51}
\]
$\theta = \frac{1}{2}$ gives Crank-Nicolson (SLCN$^{(3)}$) time stepping, and $\theta = 1$ results in the fully implicit (SLFI$^{(3)}$) time stepping.

Correspondingly, each case involves solving a large, albeit sparse, system to determine the solution at all $X_i, Y_j$ points, on a regular grid. In order to solve this system, an iterative scheme, based on the GMRES algorithm (Saad and Schultz, 1986) from the SuperLU library (Li, 2005) was invoked with a residual tolerance set to $1 \times 10^{-8}$ (of course the SLADI schemes have the clear advantage of not requiring iteration).

### 4.4 Accuracy assessment and computational times

Here we assess the accuracy and efficiency of the different variants of the ‘SLADI’ scheme, along with the slightly more conventional semi-Lagrangian Crank-Nicolson (SLCN) and semi-Lagrangian fully implicit (SLFI) approaches.

Previous literature (Douglas and Russell, 1982) has indicated that semi-Lagrangian schemes have time-truncation errors that have been shown to be smaller, compared to those incurred by standard techniques, when the method of characteristics is combined with finite-difference procedures for the numerical solution of convection-dominated diffusion problems. ADI schemes for the numerical solution of convection-diffusion equations with mixed derivative terms have been shown to be unconditionally stable (In’t Hout and Welfert, 2007), when applied to finite-difference discretisation of general parabolic problems. Here we assess through computational experimentation the performance of the schemes detailed above.

We first conduct a convergence test on an equally spaced grid in all space dimensions, namely $\Delta X = \Delta Y = \Delta Q = h$, by letting $h \to 0$ along with the two choices $\Delta_r = \frac{1}{2}h$ and then $\Delta_r = \frac{1}{4}h$. Assuming that an approximate solution $V^l_{i,j,k}$ can be expressed as an exact solution $V^l_{i,j,k}$ plus an error

$$V^l_{i,j,k}(h) = V^l_{i,j,k} + O(h^{\xi}),$$

we verify the convergence order when the grid size is reduced by a factor of two on each refinement. The ratio of convergence $\mathcal{R}$ is defined as

$$\mathcal{R} = \frac{V^l_{i,j,k}(h/2) - V^l_{i,j,k}(h)}{V^l_{i,j,k}(h/4) - V^l_{i,j,k}(h/2)}; \quad (4.52)$$

for linear convergence $\mathcal{R} = 2$ ($\xi = 1$), and for for quadratic convergence $\mathcal{R} = 4$ ($\xi = 2$).
Tables 4.1 and 4.2 show the resulting converged values and the corresponding convergence ratios for $\Delta_T = \frac{1}{2} T$, at representative points in state space, whilst Tables 4.3 and 4.4 show the results for $\Delta_T = \frac{1}{4} T$. The selected points correspond to critical states where the battery’s charging/discharging regime changes ($|X| = \{0, Q_{\text{max}}\}$ and $Y = \theta$), to the extreme and half levels of charge state ($Q = \{0, \frac{1}{2} Q_{\text{max}}, Q_{\text{max}}\}$), and to a zero spot price ($Y = 0$).

Tables 4.5 and 4.6 indicate the corresponding grid sizes employed for these (typical) computations, together with the associated (illustrative) computing time necessary for each choice of grids, for $\Delta_T = \frac{1}{2} T$ and $\Delta_T = \frac{1}{4} T$, respectively. All calculations were performed using an Intel(R) Xeon(R) CPU E5-2643 0 @ 3.30GHz processor. The parameters used in this test are $\sigma_X = 0.5$, $\sigma_Y = 0.5$, $\alpha = 0.1$, $\beta = 0$, $\kappa = 1$, $\lambda = 1$, $\theta = 5$, $\Theta = 5$, $r = 0.01$, $X_{\text{max}} = 5$, $Y_{\text{max}} = 15$, $Q_{\text{max}} = 1$ and $T = 1$; as noted earlier, for a time origin of $\tau = 0$ we have a zero value for all $V$.

Figures 4.4, 4.5, 4.6 and 4.7 show the resulting heat maps for the rate of convergence $R$ on the schemes indicated. The plots correspond to the rates observed based on the final refinement namely with $\Delta_T = \frac{1}{4} T$. Here, plus and diamonds representations indicate regions where the rate of convergence oscillates outside the plotting range. Note that invariably these issues occur in regions where the schemes have obtained about four digits of accuracy, and so invariably the nonuniform convergence in the magnitude of the valuations per se is invariably rather small. Non-uniform, first-order, convergence rates have been observed on the optimal gas storage operation problem (Chen and Forsyth, 2007). Furthermore, the primary cause of the non-uniformity in the present context is due to the coefficients of the fundamental PDE (4.10) not all being smooth (analytic); this view is reinforced by inspection of Figures 4.4, 4.5, 4.6 and 4.7. Indeed, this is an aspect raised by D’Halluin et al. (2005). Indeed, in mathematical finance, the so-called ‘nonlinearity error’, i.e. nonuniform convergence, is commonplace, caused fundamentally by discontinuities not aligning perfectly with a numerical grid. Second-order global accuracy is an ongoing research topic in this general area, for example recently Ware (2013) has suggested a semi-Lagrangian time-stepping algorithm with a Fourier-cosine discrete approximation, with demonstrated second-order accuracy.
### Table 4.1: Convergence of semi-Lagrangian schemes\(^{(1)}\); \(\Delta x = \frac{1}{2} h\).

<table>
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<th>State</th>
<th>(X)</th>
<th>(Y)</th>
<th>(Q)</th>
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<th>SLADI(^{(1)})</th>
<th>SLFI(^{(1)})</th>
<th>SLCN(^{(1)})</th>
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### Table 4.2: Convergence of semi-Lagrangian schemes\(^{(3)}\); \(\Delta_t = \frac{1}{2}h\).

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### Table 4.3: Convergence of semi-Lagrangian schemes\(^{(1)}\): $\Delta \tau = \frac{1}{4} h.$

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Table 4.4: Convergence of semi-Lagrangian schemes\(^{(3)}\); \(\Delta \tau = \frac{1}{4} h\).

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Figure 4.4: Heat maps for the rate of convergence $\mathcal{R}$ for the SLADI schemes indicated above each column; from top to bottom plots correspond to $Q = 0$, $Q = \frac{1}{2}Q_{\text{max}}$ and $Q = Q_{\text{max}}$. 
Figure 4.5: Heat maps for the rate of convergence $R$ for the SLADI schemes indicated above each column; from top to bottom plots correspond to $Q = 0$, $Q = \frac{1}{2}Q_{\text{max}}$ and $Q = Q_{\text{max}}$. 

\[ \text{SLADI}^{(1)}_{\text{II}} \]

\[ \text{SLADI}^{(3)}_{\text{II}} \]
Figure 4.6: Heat maps for the rate of convergence $\mathcal{R}$ for the SLFI schemes indicated above each column; from top to bottom plots correspond to $Q = 0$, $Q = \frac{1}{2}Q_{\text{max}}$ and $Q = Q_{\text{max}}$. 
Figure 4.7: Heat maps for the rate of convergence $R$ for the SLCN schemes indicated above each column; from top to bottom plots correspond to $Q = 0$, $Q = \frac{1}{2}Q_{\text{max}}$ and $Q = Q_{\text{max}}$. 
We observe that all schemes produce largely consistent solutions with high resolution grids. In these tests the SLADI schemes were found to be faster than SLFI and SLCN approaches under the specified tolerance when comparing schemes with the same dimensional interpolation. From results in table 4.5 we identify some effects on the linear solver from a less diagonally dominant matrix at the final refinement for SLFI\(^{(3)}\), although in a relatively coarser grid, the SLADI methodology is not affected in this respect, all other times scale consistently.

We observe that SLADI\(^{(1)}\) performs reasonably and has the most economical implementation (low computational cost), although does appear to exhibit some convergence issues from the fifth significant figures (only) in parts of the domain, but overall exhibits general linear convergence with respect to the mesh size \(h\) (this is dominated by the error due to the finiteness of \(\Delta r\)). The three-dimensional version SLADI\(^{(3)}\) requires approximately double of the time of SLADI\(^{(1)}\) to complete the calculations but the truncation-error issues are significantly reduced, but we make the observation that a balanced grid spacing is required to prevent oscillations.

SLCN\(^{(1)}\) has a computational overhead for locating departure positions at every time increment \(\Delta r\) and approximation at the half time step works well, but requires an iterative approach to solve the system of equations (in the \(X\) and \(Y\) dimensions),

### Table 4.5: Grid resolution and running time; \(\Delta r = \frac{1}{2}h\).

<table>
<thead>
<tr>
<th>Grid size</th>
<th>SLADI(^{(1)})</th>
<th>SLADI(^{(1)})</th>
<th>SLFI(^{(1)})</th>
<th>SLCN(^{(1)})</th>
<th>SLADI(^{(3)})</th>
<th>SLADI(^{(3)})</th>
<th>SLFI(^{(3)})</th>
<th>SLCN(^{(3)})</th>
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<td>1</td>
<td>1</td>
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<td>0</td>
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<td>0</td>
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<td>14</td>
</tr>
<tr>
<td>(0.03125)</td>
<td>90</td>
<td>141</td>
<td>201</td>
<td>196</td>
<td>180</td>
<td>250</td>
<td>325</td>
<td>251</td>
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<tr>
<td>(0.015625)</td>
<td>1976</td>
<td>3383</td>
<td>4812</td>
<td>4697</td>
<td>2874</td>
<td>4054</td>
<td>36664</td>
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</tr>
</tbody>
</table>

### Table 4.6: Grid resolution and running time; \(\Delta r = \frac{1}{4}h\).

<table>
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<th>SLFI(^{(1)})</th>
<th>SLCN(^{(1)})</th>
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<th>SLADI(^{(3)})</th>
<th>SLFI(^{(3)})</th>
<th>SLCN(^{(3)})</th>
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</thead>
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<td>2</td>
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<td>22</td>
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<td>31</td>
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<td>27</td>
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<tr>
<td>(0.03125)</td>
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<td>385</td>
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<td>500</td>
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<tr>
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<td>7187</td>
<td>5974</td>
<td>8107</td>
<td>11222</td>
<td>8488</td>
</tr>
</tbody>
</table>
which leads to a higher (more than double) computational cost compared to that incurred by the SLADI(I) methodology. We observe that SLCN(I) has the additional cost for interpolation, but the time required by the iterative approach to solve the system of equations remains approximately the same as for SLCN(I), with significant reduction of regions with convergence issues. We have identified domain sections with second-order convergence, but this is localised to regions where uniform source terms are present e.g., at $Q = 0$ for $X < 0$ where the battery can no longer be discharged and at $Q = Q_{\text{max}}$ for $X > 0$ where the battery can no longer be charged.

The SLFI schemes were found to be most computational time demanding, with regard to the time required by the iterative solver. In these SLFI schemes, the global rate of convergence is of first-order (as expected), but nevertheless the regions with truncation-error convergence issues are significantly reduced for the one-dimensional version (with respect to those observed on the other one dimensional schemes), yet there still remain some truncation-error convergence issues for the three-dimensional version.

As noted already, unlike the SLADI(I) schemes, the SLADI(II) schemes use quadratic Lagrange interpolations every time increment $\Delta \tau$ for SLADI(I) and every half time increment $\frac{1}{2} \Delta \tau$ for SLADI(II) (on account of the fact that interpolation has to be performed in three-dimensions - higher-order interpolations incur higher computational cost). A few sample computations with natural cubic-splines were performed, but there was negligible improvement in accuracy of the valuations thus obtained. The SLADI(II) generally requires a slightly higher computational time to complete the calculations as the SLCN(I) scheme (with a lower dimensional interpolation), but the calculations are similar or cheaper than those in the SLCN(II) scheme (with the same dimensional interpolation). Additionally, the regions with truncation-error convergence issues are significantly reduced compared to those occurring in SLADI(I), these issues occurring at (only) the sixth significant digit, with a general linear convergence.

SLADI(I) produces oscillations towards the extreme values of the $X$ domain, especially for smaller values of $Q$, which we may interpret as the unlikely regime of a nearly empty battery when the wind farm is producing energy at the maximum level; here the alternating-direction methodology is affected, as the system becomes convection dominated. We verified (see entries for $X = 0$ on Tables 4.1 and 4.3) that these issues
are mitigated as the temporal variable ($\Delta \tau$) is refined. In this respect, SLADI$^{(3)}_{II}$ is superior, since it can naturally accommodate cases in which the advective coefficients dominate the diffusion coefficients, by incorporating all first-order derivative terms in the Lagrangian derivative and is therefore free of oscillations.

### 4.4.1 SLADI convergence and stability analysis

All SLADI versions that have been implemented produced robust results that compare well to those obtained with more conventional methodologies. From the schemes definition (see section 4.2) we have that for every $Q$ level computed, a couple of ADI steps are performed from the mapping dictated by the Lagrangian derivative; the mapping is executed once every time increment $\Delta \tau$ for SLADI$^{I}_{I}$ schemes, and every half-time increment $\frac{1}{2}\Delta \tau$ for SLADI$^{II}_{II}$ schemes. In the limit of no motion $v \to 0$, or drift coefficients tending to zero, every $Q$ level computed is simply an execution of the ADI method, as the location of departure position at time $\tau$ will coincide with the arriving position at time $\tau + \Delta \tau$. In the limit of no diffusion, i.e. $\sigma_X \to 0$, and $\sigma_Y \to 0$ every $Q$ level computed is a semi-Lagrangian advection step, as only source terms are integrated.

Even though non-uniform rates of convergence have been observed for control problems (Chen and Forsyth, 2007), perhaps not in disagreement with the rates observed here, we proceed with further experimentation to investigate the stability and convergence of the SLADI methodology. The previous convergence test (see Tables 4.1 - 4.4, and Figures 4.4 - 4.7) presents results for a zero bid-ask difference, i.e. $\beta = 0$, which alleviates convergence oscillations from more general cases with $\beta \neq 0$.

With the objective of reducing non-linearity errors, we now present the counterintuitive example where

$$
\mathcal{L}(X, Y, Q) = \left\{ \begin{array}{ll}
-\omega \min\{|X|, \lambda Q\} & \text{if } X < 0, \\
\omega = 1 & \text{if } Y < \theta, \\
\omega = 0 & \text{if } Y \geq \theta; \\
\min\{X, \lambda(Q_{\max} - Q)\} & \text{if } X > 0;
\end{array} \right.
$$

(4.53)

for controlling the battery and a bid-ask difference $\beta = 1$ (no income from selling electricity). The objective is to have only one region for source terms related to the
battery operation \((X < 0 \text{ and } Y < \theta)\). Sample convergence results are presented in Figure 4.8. We can identify, on the far right side of the plots, that when only the constant payment \(\Theta\) is integrated the methodologies do not introduce additional errors, but even in this case, we still observe non-linearity error on the remainder of the domain. In this counterintuitive example we are only discharging energy whenever there is deficit of energy and prices are low with respect to the long-term mean price of electricity \(\theta\), and there is no additional income for selling electricity since the bid-ask difference has been set to \(\beta = 1\), again the objective is obtaining a smooth curvature by reducing non-linearity errors. Figure 4.9 presents a decreasing \(V\) for negative and decreasing \(X\), but a constant \(V\) for positive and increasing \(X\) since we are only receiving the fixed payment \(\Theta\). At the mean level of energy generation \((X = 0)\), see Figure 4.10, along the \(Y\) direction we observe the effects of discharging the battery at low prices of electricity with respect to the long-term mean price of electricity \(\theta\) and the value \(V\) decreases as with increasing \(Y\). The expression \(Y = 0\) is used for indicating “in the limit as \(Y\) approaches zero,” since the selected parameters prevent the stochastic process for the electricity spot price to reach a zero level.

Next we continue with a simplified model that involves all processes for advection and diffusion with fixed boundary values on \(X = 0\), \(Y = 0\), and \(Q = 0\). We solve the PDE

\[
\frac{\partial V}{\partial t} - \alpha X \frac{\partial V}{\partial X} - \kappa Y \frac{\partial V}{\partial Y} - \lambda Q \frac{\partial V}{\partial Q} + \frac{1}{2} \sigma_X^2 X \frac{\partial^2 V}{\partial X^2} + \frac{1}{2} \sigma_Y^2 Y \frac{\partial^2 V}{\partial Y^2} - r V = 0, \tag{4.54}
\]

subject to boundary conditions

\[
V = 0 \text{ on } X = 0, Y = 0, \text{ and } Q = 0,
\]

\[
\frac{\partial V}{\partial X} = 0 \text{ on } X = X_{\text{max}}, \text{ and } \frac{\partial V}{\partial Y} = 0 \text{ on } Y = Y_{\text{max}},
\]

with the final condition

\[
V(X_i, Y_j, Q_k, T) = Q_k \text{ for } t = T.
\]

With this simplified version for the model we consider mean-reverting and diffusion processes with smoother coefficients. Tables 4.7 and 4.8 present the resulting converged values along with the corresponding rates observed; the convergence test is on an
Figure 4.8: Heat maps for the rate of convergence $\mathcal{R}$ for schemes indicated above. Plus and diamonds representations indicate regions where the rate of convergence oscillates outside the plotting range. The plots correspond to the level $Q = Q_{\text{max}}$. 
CHAPTER 4. STORAGE VALUATION, A WIND POWER APPLICATION

Figure 4.9: Solution at the indicated spot prices; \( \sigma_X = 0.5, \sigma_Y = 0.5, \alpha = 0.1, \beta = 0, \kappa = 1, \lambda = 1, \theta = 5, \Theta = 5, r = 0.01, X_{\text{max}} = 5, Y_{\text{max}} = 15, Q_{\text{max}} = 1 \) and \( T = 1 \).

Figure 4.10: Solution on \( X = 0 \); parameters as in Figure 4.9.
A convergent solution across all the domain and the results compare well with those obtained using more conventional methodologies (see Figure 4.14), although globally a first-order rate of convergence is observed, a result which is perhaps not in disagreement with subquadratic rates reported by D’Halluin et al. (2005) for Asian options with low volatility coefficients. We then confirm that the oscillations in the convergence rate, present in the main model, arise from non-linearity errors from a grid not perfectly aligning with discontinuities related to the battery control.

In any event, all SLADI schemes performed well overall, and the results that are presented next are within graphical accuracy. Since SLADI\textsuperscript{I} and SLADI\textsuperscript{II} exhibited the best combination of accuracy, lowest computational cost and robustness overall,

<table>
<thead>
<tr>
<th>(X)</th>
<th>(Y)</th>
<th>(\Delta L)</th>
<th>(\Delta v)</th>
<th>(\Delta Q)</th>
<th>(\Delta \tau)</th>
<th>(\Delta h)</th>
<th>(\Delta \tau)</th>
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</tr>
<tr>
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<td>0.60016806</td>
<td>0.60016806</td>
<td>0.60016806</td>
</tr>
</tbody>
</table>

Equally spaced grid in all space dimensions, namely \(\Delta X = \Delta Y = \Delta Q = h\) and \(\Delta \tau = \frac{1}{4} h\), for \(h = \{0.2, 0.1, 0.05, 0.025, 0.0125\}\). Figures 4.11 and 4.12 show sample results, along the \(X\) and \(Y\) axis at the specified levels.
Table 4.8: Convergence of semi-Lagrangian schemes\(^{(3)}\) in model PDE (4.54).

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y )</th>
<th>( \text{SLADI}_{I}^{(3)} )</th>
<th>( \text{SLADI}_{II}^{(3)} )</th>
<th>( \text{SLFI}^{(3)} )</th>
<th>( \text{SLCN}^{(3)} )</th>
</tr>
</thead>
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<td>5/3</td>
<td>0.40682397</td>
<td>n.a.</td>
<td>0.35922581</td>
<td>0.4041291</td>
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<td></td>
<td></td>
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</tr>
<tr>
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<td>n.a.</td>
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</table>

Figure 4.11: Solution on \( Q = Q_{\text{max}} = 1 \) (thick lines) and on \( Q = \frac{1}{2} Q_{\text{max}} \) (thin lines); \( \sigma_X = 0.5, \sigma_Y = 0.3, \alpha = 0.5, \lambda = 0.5, \kappa = 0.5, r = 0.01, X_{\text{max}} = 5, Y_{\text{max}} = 5, Q_{\text{max}} = 1 \) and \( T = 1 \).
Figure 4.12: Solution on $Q = Q_{\text{max}} = 1$ (thick lines) and on $Q = \frac{1}{2}Q_{\text{max}}$ (thin lines); parameters as in Figure 4.11.

Figure 4.13: Heat maps for the rate of convergence $\mathcal{R}$ for schemes indicated above. The plots correspond to the level $Q = Q_{\text{max}}$. 
Figure 4.14: Heat maps for the rate of convergence $R$ for schemes indicated above. The plots correspond to the level $Q = Q_{\text{max}}$. 
these were the preferred algorithms used in ‘production’ computations, results of which are described next.

4.5 Sources of value

Within this section we identify the source of value for the upgraded wind farm. This involves identifying the value of the wind farm by itself, when the wind farm has been granted a revenue guarantee, when the wind farm is operating a storage device without revenue guarantee and finally the wind farm operating the storage device with a revenue guarantee, All formulations are based on (4.10) and are written in forward parabolic form $\tau = T - t$.

The value $V_1$ of a wind farm operating by itself is described by the PDE (4.10), but the right-hand-side is equal to the total amount of energy generated, times the bid price of electricity $(X + C)Y(1 - \beta)$

$$- \frac{\partial V_1}{\partial \tau} - \alpha X \frac{\partial V_1}{\partial X} + \frac{1}{2} \sigma_X^2 \frac{\partial^2 V_1}{\partial X^2} + \kappa(\theta - Y) \frac{\partial V_1}{\partial Y} + \frac{1}{2} \sigma_Y^2 Y \frac{\partial^2 V_1}{\partial Y^2} - rV_1 = -(X + C)Y(1 - \beta). \quad (4.55)$$

The value $V_2$ of a wind farm operating with a revenue guarantee is described by the PDE (4.10), but the right-hand-side is equal to the surplus or deficit of energy generated, times the corresponding bid price or ask price of electricity

$$- \frac{\partial V_2}{\partial \tau} - \alpha X \frac{\partial V_2}{\partial X} + \frac{1}{2} \sigma_X^2 \frac{\partial^2 V_2}{\partial X^2} + \kappa(\theta - Y) \frac{\partial V_2}{\partial Y} + \frac{1}{2} \sigma_Y^2 Y \frac{\partial^2 V_2}{\partial Y^2} - rV_2$$

$$= -\Theta - \begin{cases} 
XY(1 + \beta) & \text{if } X < 0, \\
XY(1 - \beta) & \text{if } X > 0.
\end{cases} \quad (4.56)$$

The value $V_3$ of a wind farm operating a storage without revenue guarantee is described by the PDE (4.10), but the right-hand-side is equal to the total amount of energy generated with added or subtracted energy, times the bid price of electricity

$$- \frac{\partial V_3}{\partial \tau} + L \frac{\partial V_3}{\partial Q} - \alpha X \frac{\partial V_3}{\partial X} + \frac{1}{2} \sigma_X^2 \frac{\partial^2 V_3}{\partial X^2} + \kappa(\theta - Y) \frac{\partial V_3}{\partial Y} + \frac{1}{2} \sigma_Y^2 Y \frac{\partial^2 V_3}{\partial Y^2} - rV_3$$

$$= -(X + C - L)Y(1 - \beta). \quad (4.57)$$

Finally, our original formulation (4.10) gives the value $V$ of a wind farm operating a storage device and has been granted a revenue guarantee.
Figure 4.15: Solution at $Y = 0$ for the corresponding $V_1$, $V_2$, $V_3$ and $V$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$, $Q = \frac{1}{2}Q_{\text{max}}$ and $Q = 0$; parameters $\sigma_X = 0.5$, $\sigma_Y = 0.5$, $\alpha = 0.1$, $\beta = 0.1$, $\lambda = 1$, $\kappa = 1$, $\theta = 5$, $\Theta = 25$, $r = 0.01$, $X_{\text{max}} = 5$, $Y_{\text{max}} = 15$, $Q_{\text{max}} = 1$ and $T = 1$.

The parameters in this analysis are $\sigma_X = 0.5$, $\sigma_Y = 0.5$, $\alpha = 0.1$, $\beta = 0.1$, $\lambda = 1$, $\kappa = 1$, $\theta = 5$, $r = 0.01$, $X_{\text{max}} = 5$, $Y_{\text{max}} = 15$, $Q_{\text{max}} = 1$ and $T = 1$. Here we are analysing the sources of value, then for this set of results we have $C = 5$ and the total capacity of the wind farm is then $2C = 10$ units of energy per unit of time. Since $\theta = 5$, then we consider the fixed income to be $\Theta = \theta X_{\text{max}} = 25$.

For a comparison under equivalent conditions, we have used the same fixed control (4.1) for charging and discharging the battery for calculating the valuations $V_3$ and $V$. The reason is again with respect to the use of the storage for balancing the output rate of energy generated, we store energy only after the mean level of energy generation for low prices of electricity, and we use the reserves if the energy generated is less than the mean level of energy generation for high prices of electricity with respect to the long-term mean price of electricity $\theta$.

Figures 4.15, 4.16 and 4.17 show the solution along the $X$ direction for indicated $Y$ values, Figures 4.18, 4.19 and 4.20 show the solution along the $Y$ direction for indicated $X$ values. We observe that the cases $V_2$, and $V$ (where the wind farm has a revenue guarantee) have a higher valuation for lower prices of electricity with respect to the long-term mean price for electricity $\theta$. There is only a small region for low levels
Figure 4.16: Solution at $Y = 5$ for the corresponding $V_1, V_2, V_3$ and $V$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$, $Q = \frac{1}{2} Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.15.

Figure 4.17: Solution at $Y = 10$ for the corresponding $V_1, V_2, V_3$ and $V$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$, $Q = \frac{1}{2} Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.15.
CHAPTER 4. STORAGE VALUATION, A WIND POWER APPLICATION

Figure 4.18: Solution at $X = -1$ for the corresponding $V_1, V_2, V_3$ and $V$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$, $Q = \frac{1}{2}Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.15.

Figure 4.19: Solution at $X = 0$ for the corresponding $V_1, V_2, V_3$ and $V$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$, $Q = \frac{1}{2}Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.15.
of energy generated and low prices of electricity (about $X < -2$ and $Y < \theta$) where the wind farm setups without a revenue guarantee (cases $V_1$ and $V_3$) have a slightly higher valuation (see Figure 4.16), but for increasing $X$ we observe a higher valuation of the wind farm setups with a revenue guarantee; this certainly give the investor a security if electricity prices are expected to be reduced, and still gives additional benefits for prices of electricity slightly higher than $\theta$ (specially at higher levels of energy generation, see the region under the curves in Figure 4.21) because the revenue guarantee is not penalised, whereas a system without a revenue guarantee is always selling electricity at bid price $(1 - \beta)Y$.

For the wind farm setups without a revenue guarantee (cases $V_1$ and $V_3$) there are always positive slopes in the valuation solution, and then the higher the energy generated and the higher the electricity spot price, then the higher the valuation of the system. We can identify that for electricity prices higher than $\theta$, the wind farm setups without a revenue guarantee have in general a higher valuation than the wind farm setups with a revenue guarantee (specially at low levels of energy generation), this gives the investor more benefits if the electricity prices are being expected to increase (see the region over the curves in Figure 4.21); although there is a significant region where the wind farm setups with a revenue guarantee (cases $V_2$ and $V$) have a
higher valuation for electricity prices slightly higher than \( \theta \) (as noted in the paragraph above) since the constant payment \( \Theta \) has no penalisation, whereas as system without a revenue guarantee has to sell electricity at bid price \((1 - \beta)Y\).

Under this comparison we observe a dual effect of storage. While stockpiling (in the charging region \( X > 0 \)) there is a reduced the valuation of the wind farm (cases \( V_3 \) and \( V_5 \)) compared to a system without storage (cases \( V_1 \) and \( V_2 \)), this because the storage is building potential at the expense of a lower income, and the a system without storage will receive a better valuation as always is selling the output. On the other side, with using the reserves from the storage (in the discharging region \( X < 0 \)) the system receives a better valuation as additional potential is obtained (from more income or less expenses), whereas a system with no storage is not obtaining additional income. In the next chapter we define a criteria for deciding optimal rates for charging and discharging the battery for every system state.

### 4.6 Wind farm valuation

Now that the numerical integrity of the SLADI schemes has been confirmed, we proceed to evaluate an example with parameters \( \sigma_X = 0.5, \sigma_Y = 0.5, \alpha = 0.1, \beta = 0.1, \lambda = 1, \kappa = 1, \theta = 5, \Theta = 5, r = 0.01, X_{\text{max}} = 25, Y_{\text{max}} = 20, Q_{\text{max}} = 1 \) and \( T = 1 \). From this
section we take the value $\Theta = 5$ as being a constant utility where the total fixed cost has been subtracted from the contractual fixed revenue, and also we use the expression $Y = 0$ for indicating “in the limit as $Y$ approaches zero,” since the selected parameters prevent the stochastic process for the electricity spot price to reach a zero level.

Figures 4.22 and 4.23 show the solution along the $X$ axis for $Q = 0$ (the empty storage case), for a range of spot prices of electricity as indicated. It is noteworthy that even for (slightly) negative deficits of wind (i.e. $X < 0$), the system can still have some positive value (because of the potential for added value as a result of storage). The interpretation of this is that although there may be a deficit of energy, and even though the storage is completely empty, there still remains the potential for income generation because we are receiving the fixed income $\Theta$ all the time. Notice also that for $|X| \gg 1$, the system behaves linearly with $X$, details of which are presented in the Appendix.

Likewise, Figure 4.24 shows the solution along the $X$ axis for $Q = Q_{\text{max}} = 1$ (corresponding to full storage), and presents a similar picture to the $Q = 0$ results of the previous figure, but instead of observing effects of charging the battery with surplus energy at low prices of electricity we observe benefits from discharging the battery on the deficit of energy at high prices of electricity with respect to the long-term mean price $\theta$. 

Figure 4.22: Solution on $Q = 0$; $\sigma_X = 0.5$, $\sigma_Y = 0.5$, $\alpha = 0.1$, $\beta = 0.1$, $\lambda = 1$, $\kappa = 1$, $\theta = 5$, $\Theta = 5$, $r = 0.01$, $Q_{\text{max}} = 1$ and $T = 1$. 

![Graph](image-url)
Figure 4.23: Solution on $Q = 0$, parameters as in Figure 4.22.

Figure 4.24: Solution on $Q = Q_{\text{max}} = 1$, parameters as in Figure 4.22.
Figure 4.25: Evolution of the solution in time at $X = 0$, parameters as in Figure 4.22.

Figure 4.25 shows the variation of the valuation/solution along $X = 0$ (i.e. there is no surplus or deficit of energy generation) for both the empty store ($Q = 0$) and the full store ($Q = Q_{\text{max}}$) at selected times, and this confirms the potential for positive value to the system, even in this case. The corresponding results with $\theta = 2.5$ (i.e. a lower long-term mean price) are shown in Figure 4.26 (all other parameters remain the same). An analysis of the $\tau \gg 1$ behavior of the $|X| \to \infty$ solution is given in the Appendix, and this indicates that ultimately a $\tau$-independent state is achieved.

Now we present a comparison of the solution with respect to the simplified case when the storage facility is (basically) charged as long as there is surplus in the production of electricity $X > 0$, and discharged as long as there is deficit in the production of electricity $X < 0$; this regime implies $\omega = 1$ everywhere and so the rate of charge, as well of discharge, of the storage facility is independent of $Y$. Note that, as stated in the Appendix, the solution is (simply) linear in $Y$. Here we use ‘switching’ to represent the solution using the original charging/discharging strategy, described by (4.1), for managing the battery and ‘no switching’ to represent the solution for a regime with $\omega = 1$ everywhere.

At this maturity time $T = 1$ we observe in Figure 4.27 a decreasing or constant value for $V$ and then an increasing value $V$ for increasing $Y$, this because the system is about to enter a regime where the battery is charged for low prices of energy with respect to the long term mean value of electricity $\theta$ and energy is sold after that price
in the surplus region, also the system is about to enter a regime where the battery is discharged for high prices of electricity with respect to $\theta$ but not discharged for lower prices with respect to $\theta$.

The general increase in value for high prices of energy (see Figures 4.27 and 4.28) with ‘switching’ does not come without a cost, since the value of the system is reduced at lower energy prices (see Figure 4.27) compared to a regime where the battery is always switched on. However the added value at higher prices is greater than the reduction in value at lower prices, see Figure 4.28, indicating an overall advantage to implementing switching.

4.6.1 Parameters sensitivity analysis

In this subsection we present a sensitivity analysis to variations in the PDE parameters, varying one at a time; we consider a base case with parameters $\sigma_X = 0.5$, $\sigma_Y = 0.5$, $\alpha = 0.1$, $\beta = 0.1$, $\lambda = 1$, $\kappa = 1$, $\theta = 5$, $\Theta = 5$, $r = 0.01$, $X_{\text{max}} = 25$, $Y_{\text{max}} = 20$, $Q_{\text{max}} = 1$ and $T = 1$.

Remarkable results

Here we present results that show a marked effect due to the operation of the storage device. We consider this to be an advantage or improvement to the solution because
Figure 4.27: Solution at $Y = 0$, from top to bottom the curves correspond to $Q = Q_{\text{max}}, Q = 0.5$ and $Q = 0$, parameters as in Figure 4.22.

Figure 4.28: Solution at $X = 0$, from top to bottom the curves correspond to $Q = Q_{\text{max}}, Q = 0.5$ and $Q = 0$, parameters as in Figure 4.22.
Figure 4.29: Solution at $Y = 0$ for the corresponding $\sigma_X$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; base case with parameters $\sigma_X = 0.5$, $\sigma_Y = 0.5$, $\alpha = 0.1$, $\beta = 0.1$, $\lambda = 1$, $\kappa = 1$, $\theta = 5$, $\Theta = 5$, $r = 0.01$, $X_{\text{max}} = 25$, $Y_{\text{max}} = 20$, $Q_{\text{max}} = 1$ and $T = 1$.

of the use of reserves or elements subject to improvement such that they give an indication to search for a better management of the storage device.

The effects of small fluctuations on the energy generation can be observed with small diffusion coefficients in the $X$ dimension; the requirement of buying electricity in the deficit region, the delay of additional income in the charging region of the battery and the extra income from selling electricity are visible in Figure 4.29. Large diffusion coefficients cause these effects to become compounded (as expected since we are plotting along the direction of diffusion $\sigma_X$).

On the mean level of energy generation of energy $X = 0$ (see Figure 4.30), an empty storage receives a better valuation (close to the constant income) with lower $\sigma_X$ values, as there is a small chance to deviate from the mean generation level. On the other hand, a full storage receives a lower valuation as then, no advantage is taken at high prices of electricity. There is a slight benefit on low prices of electricity for a full storage as there will be no need of adding charge; the opposite effects are obtained for a larger $\sigma_X$ value. With these results we clearly identify the importance of considering the resource’s random nature, large diffusion coefficients increase the potential income (outcome), whereas small diffusion coefficients reduce potential income (outcome).
Figure 4.30: Solution at $X = 0$ for the corresponding $\sigma_X$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29.

Figure 4.31: Solution at $X = 0$ for the corresponding $\sigma_X$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29, except $T = 10$. 
Figure 4.32: Solution at $Y = 0$ for the corresponding $\sigma_Y$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29.

Figure 4.31 presents valuation results at a longer time scale, namely at $T = 10$. A large diffusion coefficient produces an overall reduction in the value of the system since surplus electricity is sold to the market at bid price $Y(1 - \beta)$, and electricity in deficit is purchased from the market at the ask price $Y(1 + \beta)$, and a large diffusion coefficient compounds these penalisation effects. With a low diffusion coefficient, the value of an empty storage is improved but the value of the system with a full storage is only increased for low prices of electricity and there is a reduction in value for high prices of electricity (the results have this pattern since we are plotting along the $Y$ direction which is not direction of diffusion $\sigma_X$). Then, under the present strategy for managing the storage, there is only a partial benefit obtained from randomness in the rate of energy generated.

The effects of the electricity spot price fluctuation are presented in Figures 4.32 and 4.33. In this case we observe almost no change in the value for varying the volatility when there is a zero spot price for the electricity, as any $\sigma_Y$ coefficient is multiplied by a zero coefficient ($Y = 0$) in the reduced PDE (4.13), see Figure 4.32, and almost no potential has been transferred. Along the price axis, small values of $\sigma_Y$ produce a sharp response for switching the battery on and off, whereas a high value for $\sigma_Y$ produces a smooth response to switching. The overall valuation for the system on low prices of electricity increases, but the valuation is reduced (until there is no diffusion)
Figure 4.33: Solution at $X = 0$ for the corresponding $\sigma_Y$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29.

Figure 4.34: Solution at $X = 0$ for the corresponding $\sigma_Y$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29, except $T = 10$. 
on high prices of electricity for larger coefficients $\sigma_Y$ (as expected since we are plotting along the direction of diffusion $\sigma_Y$).

Figure 4.34 presents valuation results at a longer time scale, namely at $T = 10$. A large diffusion coefficient produce and overall increase on the value of the system on the mean level of energy generation, for a low diffusion coefficient we still identify a sharp response to the switching strategy for managing the storage device.

Figures 4.35 and 4.36 show the effects for varying the $\lambda$ coefficient, which can be interpreted as a technical restriction on the amount of energy that can be injected into the battery or extracted from such storage device. Smaller values in $\lambda$ reduce the benefits from discharging the battery (see the curve for $Q = Q_{\text{max}}$ in Figure 4.36), but on the other hand the costs of charging the battery are then attenuated, as the charging process is distributed over a longer period of time (see the $Q = 0$ curve in Figures 4.35 and 4.36). An empty storage in the system will receive a higher valuation, as more surplus energy is sold directly (smaller amounts of energy are injected to the battery), but significantly there is a larger reduction in value for a full storage, as potentially smaller amounts of energy from the storage device will be used to offset future deficits.

From Figures 4.37 and 4.38 we identify the amplified effects of charging larger storages, the charging process lasts longer and additional income is then delayed in
Figure 4.36: Solution at $X = 0$ for the indicated $\lambda$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29.

Figure 4.37: Solution at $Y = 0$ for different storage capacities $Q_{\text{max}}$, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29.
the surplus region, thus reducing the system’s valuation for an empty storage (see the curve for $Q = 0$ in Figure 4.37). We clearly observe at the mean level of energy generation $X = 0$ (see Figure 4.38) that there is a saturation level for the capacity of the storage device, as no additional potential benefit is obtained from larger storages. We identify an urge to implement a better control as no additional potential is obtained as when there are more favorable technical rates for operating the storage device.

In Figures 4.39 and 4.40 we observe the effects from the magnitude of the penalisation after selling electricity cheaply, at the bid price $Y(1 - \beta)$, and buying expensive electricity, at the ask price $Y(1 + \beta)$ from the market. There is an overall increment in value of the system for low $\beta$ values, conversely the value decreases for high $\beta$ values. The effect is observed on the slope of the valuation along the $X$ direction, so that with larger $\beta$ values, benefits are reduced in the surplus region $X > 0$, and costs are increased in the deficit region $X < 0$, and the opposite effect is observed with small $\beta$ values (see Figure 4.39). Along the $Y$ direction we also observe a direct effect on the slope of the valuation, a larger magnitude for the slope is observed with low penalisation $\beta$ and a smaller magnitude for the slope is observed with high penalisation $\beta$. It is noteworthy the usefulness of the ‘switching’ strategy for operating the storage device as for example for the full storage (see curves $Q = Q_{\text{max}}$ in Figure 4.40), in this case for low prices of electricity relative to $\theta$, the slope for the valuation has changed.
Figure 4.39: Solution at $Y = 0$ for the range of $\beta$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29.

Figure 4.40: Solution at $X = 0$ for the range of $\beta$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29.
Figure 4.41: Solution at $Y = 0$ for different $\alpha$ values (indicated), from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameter as in Figure 4.29.

from positive (with low penalisation $\beta$) to negative (with high penalisation $\beta$), but for high prices of electricity with respect to $\theta$ the slope for the valuation remains positive even with high penalisation $\beta$.

**Expected results**

Here we present effects on the solution that can be anticipated because of a change in the value of the PDE parameters. These include rotations, translations, stretching, scaling or a combination of these effects on the solution.

We start with the effects of different $\alpha$ values, see Figures 4.41 and 4.42 for both the empty store ($Q = 0$) and the full store ($Q = Q_{\text{max}}$). This increases or decreases the speed of mean reversion for energy availability, such that lower values of $\alpha$ indicate that whenever a level of electricity production is reached, then this level slowly augmentates or reduces towards the mean level of energy production. Correspondingly, higher values of $\alpha$ cause the electricity production to return faster to the mean level of energy production. The value of the system is then increased in the surplus region for energy production, but reduced in the deficit region for a lower $\alpha$ value, the opposite occurs for a higher $\alpha$ value as the system’s value is reduced in the surplus region and increased in the deficit region (see Figure 4.41).

At $X = 0$ the system receives an increment in value for higher prices of energy for
Figure 4.42: Solution at $X = 0$ for different $\alpha$ values (indicated), from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29.

A lower $\alpha$ value on $Q_{\text{max}}$, as for a possible longer time, electricity can be sold in the surplus region or extracted from the store in the deficit region. The opposite occurs for brief excursions outside the normal level of energy generation, with a higher $\alpha$ value, as less advantage is obtained from using the storage or from selling the excess of energy generated at the $Q_{\text{max}}$ level. Slightly opposite effects are observed for low prices of electricity as the battery will not be discharged or electricity will be sold cheaply, and then a small benefit from a higher $\alpha$ value is observed.

Again along $X = 0$, but now at $Q = 0$ the system’s value is increased for all prices of electricity for a higher $\alpha$ value, even if the store is empty the amount of energy generated tends to return faster to normal level, thus reducing the potential time required to buy energy from the market. On the other hand, a lower $\alpha$ value reduces the system valuation as either the production is about to be in deficit, or the store is about to be charged, or electricity is about to be sold cheaply for a possible longer time (see Figure 4.42).

The results obtained for different values of $\kappa$ are shown in Figures 4.43 and 4.44 for both the empty store ($Q = 0$) and the full store ($Q = Q_{\text{max}}$). $\kappa$ affects the speed of mean reversion for electricity prices; lower values for $\kappa$ indicate that whatever electricity price is reached, then the price slowly increases or reduces towards the long-term mean value of electricity; correspondingly higher values of $\kappa$ cause the electricity
Figure 4.43: Solution at $Y = 0$ after varying kappa (indicated), from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29.

Figure 4.44: Solution at $X = 0$, after varying kappa (indicated), from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29.
price to return faster to the long-term mean value of electricity.

At zero spot price for \( Q = Q_{\text{max}} \) the system’s value increases in the surplus region but decreases in the deficit region for a higher \( \kappa \) value. This because the electricity price will potentially move back to the mean value faster, whilst an opposite effect is observed with a lower \( \kappa \) value. For \( Q = 0 \), as the battery is charged in the surplus region, the previous effects are delayed until a slightly positive value of energy is available; at low electricity prices, the higher the \( \kappa \) value, it is more expensive to transfer potential for discharging the battery (see Figure 4.43).

In the \( Y \) direction and along the mean energy production level (\( X = 0 \)) we basically observe that a lower value of \( \kappa \) produces a steeper slope for the curves, as excursions away from the mean value \( \theta \) last longer; the opposite effect is obtained with a higher value for \( \kappa \).

Figures 4.45 and 4.46 show the effects on the solution for different long-term mean values for electricity \( \theta \). The position of \( \theta \) with respect to the \( Y \) axis will basically indicate where the switching effect is ‘felt’. This effect is displaced to the left for lower \( \theta \) values, and to the right for higher \( \theta \) values. The variation in the valuation can be explained by means of a relative speed of mean reversion with respect to the base case. For lower \( \theta \) values, comparatively, the speed of mean reversion is lower for lower prices than \( \theta \), additionally the battery is discharged over a wider range, and
thus there is higher valuation of the system for a full battery $Q = Q_{\text{max}}$. On the other hand, there is a comparatively higher speed of mean reversion for higher prices than $\theta$, and so there is an electricity price level for which the valuation is reduced. For an empty storage $Q = 0$, and lower $\theta$ values, the comparatively lower speed of mean reversion for lower prices than $\theta$, and a comparatively higher speed of mean reversion for higher prices than $\theta$, with a narrower range to charge the battery, explains the general increase in value for the system. At zero spot price $Y = 0$ we observe similar effects in Figure 4.45 as in Figure 4.43 for low coefficients of mean reversion $\kappa$, again, is because comparatively there is a lower speed of mean reversion for $Y < \theta$.

To explain the effects of varying the riskless interest rate $r$, the wind farm time scale $T$, and the constant payment $\Theta$ we explore the solution as $|X| \to \infty$. Let us assume a solution of the linear form

$$ V = X(YV_{00}(\tau) + V_{01}(\tau)) + Q(YV_{10}(\tau) + V_{11}(\tau)) + V_{12}(\tau) $$

to be consistent with the boundary conditions (4.11) (an analysis of these boundary conditions are explored and confirmed in the Appendix). Here (−) superscripts will are used to represent the solution as $X \to -\infty$, and (†) superscripts are used to represent the solution as $X \to \infty$.

In particular, $\omega = 1$ is considered everywhere and an adequate change of variable
in $Q$ is made when $X \to \infty$. This explains three incomes, two of them from the different sources of energy, i.e. $X$ from the wind farm and $Q$ from the storage device, both proportional to $Y$ times a function of time (to be determined) plus an unknown function of time, and a third income from the constant payment $\Theta$.

After collecting terms of $O(Y)$ in (4.17) we obtain

$$V^-_0 = (1 + \beta) \left( \frac{1 - e^{-\rho \tau}}{\rho} \right),$$

$$V^+_0 = (1 - \beta) \left( \frac{1 - e^{-\rho \tau}}{\rho} \right),$$

which lead to

$$V^-_{01} = (1 + \beta) \theta \left( \frac{1 - e^{-(r+\alpha) \tau}}{r + \alpha} - \frac{1 - e^{-\rho \tau}}{\rho} \right),$$

$$V^+_{01} = (1 - \beta) \theta \left( \frac{1 - e^{-(r+\alpha) \tau}}{r + \alpha} - \frac{1 - e^{-\rho \tau}}{\rho} \right),$$

where $\rho = r + \alpha + \kappa$.

Next we can evaluate the PDE (4.17) when $Q = 0$ and $X \to -\infty$, or we can evaluate the PDE (4.17) when $Q = Q_{\text{max}}$ and $X \to \infty$, and obtain

$$V^-_{12} = V^+_{12} = \Theta \left( \frac{1 - e^{-\tau \lambda}}{r} \right)$$

Now, terms of $O(QY)$ are collected in (4.17) and we obtain

$$V^-_{10} = (1 + \beta) \left( \lambda \frac{1 - e^{-\tau \lambda}}{\theta} \right),$$

$$V^+_{10} = -(1 - \beta) \left( \lambda \frac{1 - e^{-\tau \lambda}}{\theta} \right),$$

where $\theta = r + \lambda + \kappa$, which lead to

$$V^-_{11} = (1 + \beta) \lambda \theta \left( \frac{1 - e^{-(r+\lambda) \tau}}{r + \lambda} - \frac{1 - e^{-\tau \lambda}}{\theta} \right),$$

$$V^+_{11} = -(1 - \beta) \lambda \theta \left( \frac{1 - e^{-(r+\lambda) \tau}}{r + \lambda} - \frac{1 - e^{-\tau \lambda}}{\theta} \right),$$

We may combine the above results to give the elements of the linear solution plus the independent term

$$V^- = (1 + \beta) X \left[ (Y - \theta) \left( \frac{1 - e^{-\tau \lambda}}{\rho} \right) + \theta \left( \frac{1 - e^{-(r+\alpha) \tau}}{r + \alpha} \right) \right]$$

$$+ (1 + \beta) \lambda Q \left[ (Y - \theta) \left( \frac{1 - e^{-\tau \lambda}}{\theta} \right) + \theta \left( \frac{1 - e^{-(r+\lambda) \tau}}{r + \lambda} \right) \right]$$

$$+ \Theta \left( \frac{1 - e^{-\tau \lambda}}{r} \right)$$

(4.67)
Then, given the $r$ and $\tau$ related discount factors $\left(1 - e^{-r\tau}\right)$, where $\tau = r + \varphi$ and $\varphi = \{0, \alpha, \lambda, \alpha + \kappa, \lambda + \kappa\}$, we identify that $r$ and $T$ must produce opposite effects.

The effects of different values of the constant riskless interest rate $r$ are shown in Figures 4.47 and 4.48. With a higher riskless interest rate $r$, lower valuations for the wind farm are obtained in the surplus region for energy generation and in parts of the deficit region (as potential is transferred with the battery operation), correspondingly lower costs in regions of high energy deficit are obtained; this can also be interpreted as a relatively reduced uncertainty whilst a slightly opposite effect is observed with low $r$ values.

The effects from varying the wind farm timescale $T$ are shown in Figures 4.49 and 4.50. With a longer lifespan on the project, higher valuations for the wind farm are obtained (as income is expected for a longer period of time) in the surplus region for energy generation and in parts of the deficit region (as potentially the battery can
Figure 4.48: Solution at $X = 0$ for indicated $r$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29.

Figure 4.49: Solution at $Y = 0$ for different time scales $T$ on the wind farm, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29.
be discharged). Correspondingly a higher cost is incurred for regions of high energy deficit. The opposite effect is observed for a shorter lifespan of the wind farm.

Figures 4.51 and 4.52 show the effects on the solution for different payments Θ. This constant payment can represent fixed costs or income. These figures basically indicate (numerically) that there is a shift of the resulting valuations along the axis $V$. As suggested by the term (4.62), the effects of modifying this parameter Θ are basically on shifting the net storage valuation by an amount $\left(\Theta \frac{1 - e^{-r\tau}}{r}\right)$ along the valuation dimension, since we are always receiving this constant payment.

We have verified that no additional errors are introduced because of domain truncation with increasing $|X_{\text{min}}|$, $X_{\text{max}}$ and $Y_{\text{max}}$, and correspondingly there is no change in the valuations thus obtained.

### 4.6.2 Introduction of seasonality effects

In this section we assume that the risk-adjusted electricity spot price $Y$ (4.6) incorporates seasonality and so

$$dY = \kappa(\mathcal{A}(t) - Y)\, dt + \sigma_Y \sqrt{Y} \, dW_1,$$

(4.69)

$$\mathcal{A}(t) = \mathcal{A}_0 + \psi_S A \sin(2\pi(t - t_S A)),$$

(4.70)
Figure 4.51: Solution at $Y = 0$ for selected $\Theta$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29.

Figure 4.52: Solution at $X = 0$ for selected $\Theta$ values, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 4.29.
where $\kappa$ is the speed of mean-reversion towards the long-term mean price of electricity $A(t)$ that incorporates seasonality; $A_0$ is then the long-term equilibrium price for electricity without seasonality effect. $\sigma$ is the volatility of the electricity price, $\psi_{SA}$ is the semianual seasonality parameter indicating the (absolute) maximum deviation from the long-term equilibrium price without seasonality, and $t_{SA}$ is a centering parameter for the seasonality effect indicating the equilibrium position.

In this formulation the long-term mean price for electricity $A(t)$ fluctuates periodically around the long-term equilibrium price $A_0$, in this case with a period of a one year ($t = 1$) (as for example described by Lucia and Schwartz, 2002 and Janczura et al., 2013) and exhibits one peak that correspond to high prices of electricity most likely to be in winter season. The $2\pi$ term can be adjusted to the form $2k\pi$ where $k$ is the number of seasonal peaks.

With the introduction of this seasonal spot price of electricity PDE (4.10) in forward parabolic for $\tau = T - t$ becomes

$$
- \frac{\partial V}{\partial \tau} + \mathcal{L} \frac{\partial V}{\partial Q} - \alpha X \frac{\partial V}{\partial X} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial X^2} + \kappa(A(T - \tau) - Y) \frac{\partial V}{\partial Y} + \frac{1}{2} \sigma^2 Y \frac{\partial^2 V}{\partial Y^2} - rV
$$

$$
= -\Theta - \left\{ \begin{array}{ll}
(X - \mathcal{L})Y(1 + \beta) & \text{if } X < 0, \\
(X - \mathcal{L})Y(1 - \beta) & \text{if } X > 0.
\end{array} \right.
$$

(4.71)

We implement two approaches for operating the storage device when the seasonality coefficient is introduced, a fixed strategy and a moving strategy with respect to the seasonal price. In the first approach we consider to discharge and charge the battery accordingly to the fixed strategy (4.1), which we refer as a 'battery control at $\theta$'. In the second approach we implement a decision based on the seasonal price of electricity $A(t)$ with $t = T - \tau$ for $t \in (0, T)$, we refer to this a 'battery control at $A(t)'$, namely

$$
\mathcal{L}(X, Y, Q) = \left\{ \begin{array}{ll}
-\omega \min\{|X|, \lambda Q\} & \text{if } X < 0, \\
\omega = 0 & \text{if } Y < A(t), \\
\omega = 1 & \text{if } Y \geq A(t); \\
\omega \min\{X, \lambda(Q_{\text{max}} - Q)\} & \text{if } X > 0, \\
\omega = 1 & \text{if } Y < A(t), \\
\omega = 0 & \text{if } Y \geq A(t); \\
\end{array} \right.
$$

(4.72)

Figures 4.53 and 4.54 present results obtained with the introduction of a seasonal electricity spot price and we compare with a solution without seasonality effect. Along
Figure 4.53: Solution at $Y = 0$ with and without seasonality effect, from top to bottom the curves correspond to $Q = Q_{\max}$ and $Q = 0$; parameters as in Figure 4.29, except $A_0 = 5$, $\psi_{SA} = 1$, $t_{SA} = 0$.

Figure 4.54: Solution at $X = 0$ with and without seasonality effect, from top to bottom the curves correspond to $Q = Q_{\max}$, $Q = \frac{1}{2}Q_{\max}$ and $Q = 0$; parameters as in Figure 4.29, except $A_0 = 5$, $\psi_{SA} = 1$, $t_{SA} = 0$. 
the $X$ direction and $Y = 0$ we observe a slight increment in the value of the system for $X > 0$ and a slight reduction in the value of the system for $x < 0$ for a solution that incorporate a seasonal electricity spot price. Along the $Y$ direction, at the mean level of energy generation ($X = 0$), for the first approach we observe a slight increment of the system valuation on the region about the equilibrium price of electricity $s_0 = \theta$ for a battery with fixed control at $\theta$, only a small increment is observed as positive and negative effects are averaged; for the second approach, there is a significant increase in the value on the region for low prices with respect to the equilibrium price of electricity $s_0 = \theta$ and a slight reduction of the value on the the region for high prices with respect to the equilibrium price of electricity $s_0 = \theta$, this indicates that there is an advantage for only stockpiling with respect to low prices and selling surplus electricity for high prices in the market.

### 4.6.3 Introduction of correlation effects

Here we study the effect after introducing a correlation between the processes that are the source of randomness in our wind farm model, i.e. we now consider that there is an impact on electricity spot prices because of the amount of energy transferred to the grid by the wind farm. We then consider that the process (4.7) has been risk-adjusted and that there exists a correlation factor $\rho_{XY}$ between the processes $dW_1$ and $dW_2$ from the stochastic processes (4.6) and (4.7). The PDE (4.10) including a mixed derivative is obtained and is written in forward parabolic form (in $\tau$), namely

$$-\frac{\partial V}{\partial \tau} + \mathcal{L} \frac{\partial V}{\partial Q} - \alpha X \frac{\partial V}{\partial X} + \frac{1}{2} \sigma_X^2 \frac{\partial^2 V}{\partial X^2} + \kappa(\theta - Y) \frac{\partial V}{\partial Y} + \frac{1}{2} \sigma_Y^2 Y \frac{\partial^2 V}{\partial Y^2} + \rho_{XY} \sigma_X \sigma_Y \sqrt{Y} \frac{\partial^2 V}{\partial X \partial Y} - \tau V = -\Theta - \begin{cases} (X - \mathcal{L})Y(1 + \beta) & \text{if } X < 0, \\ (X - \mathcal{L})Y(1 - \beta) & \text{if } X > 0. \end{cases}$$

Cross derivatives terms have been treated explicitly with ADI schemes (see for example Craig and Sneyd, 1988; In’t Hout and Welfert, 2009), and particularly this approach has been implemented for convection-diffusion equations with mixed derivatives (In’t Hout and Welfert, 2007). Here we explore the use of this strategy for every half-time step $\frac{1}{2} \Delta \tau$ of the SLADI$^{(3)}$ methodology (3.11) for a convection-diffusion PDE without diffusion in one dimension (namely $Q$) and subject to control (namely the discontinuous $\mathcal{L}$) which makes the problem less amenable.
Figure 4.55: Solution at $Y = \theta$ for different $\rho$ values (indicated), from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; other parameters as in Figure 4.29.

Figure 4.56: Solution at $X = 0$ for different $\rho$ values (indicated), from top to bottom the curves correspond to $Q = Q_{\text{max}}$, $Q = \frac{1}{2}Q_{\text{max}}$ and $Q = 0$; other parameters as in Figure 4.29.
Figures 4.55 and 4.56 present results for different correlation coefficients $\rho$ between the stochastic processes for electricity generation and the electricity spot price. With a positive correlation coefficient we identify that the valuation of the system generally increases as there is more energy availability, but there is a reduction of the valuation, for high charge levels, about the region close to the mean-reverting levels of the energy generation ($X = 0$) and the electricity price ($Y = \theta$). The opposite effect can be observed for a negative correlation coefficient, where the valuation of the system is generally reduced with more energy availability, but there is a higher valuation, for high charge levels, about the region close to the mean-reverting levels of the energy generation ($X = 0$) and the electricity price ($Y = \theta$). The unexpected behavior of the solution is effect of the fixed operation of the storage device and the cross-derivative terms mixing the information about the mean-reverting levels of energy generation and the electricity price.

In this case we observed restrictions for the time-step size that can be implemented, and also for the refinement in the non-diffusive dimension $Q$. For particular calculations where the initial grid sizes are $\Delta_X = \Delta_Y = \Delta_Q = h = \frac{1}{4}$ and $\Delta_\tau = \frac{1}{16}$ reliable numerical results were able to be obtained, but with reducing the time-step size by a factor of 2 for all dimensions, in a convergence test, produced oscillatory numerical results, namely in the 5th refinement where $\Delta_\tau = \frac{1}{256}$ and $\Delta_X = \Delta_Y = \Delta_Q = \frac{1}{64}$. With the objective to keep $\Delta_\tau = \mathcal{O}(h^2)$, implementations with reduction of the time step by a factor of 4, whereas all other space dimensions have a refinement of a factor of two have been carried out, but oscillatory results on refined grids were also produced. These instabilities can be interpreted as effects of the explicit treatment of the mixed derivative terms and the discontinuity in $\omega$ for the $\mathcal{L}$ rate (4.1). Adverse effects on the accuracy of ADI methods have been observed by Beam and Warming (1980) for solving parabolic partial differential equations with mixed derivatives. We have obtained non-oscillatory solutions with extra refinement in the time variable and the non-diffusive dimension $Q$ namely $\Delta_\tau, \Delta_Q = \mathcal{O}(h^2)$ with $\Delta_X = \Delta_Y = h$, this because we are solving a second-order PDE and we are balancing the effects from the grid spacing for first-order terms with that of second-order terms, but this requires very refined grids and long computations, nevertheless in the next chapter we obtain smoother solutions with the optimal operation of the storage device and the solutions
are free of oscillations without extra refinement of the grid.

4.7 Conclusions

We have defined a time dependent model for valuing a wind farm with an energy storage, a system subject to both uncertain energy output and electricity price. We were able to set up an operational scheme where the mean output level equals a contractual rate of electricity generation, receiving a fixed income for this. In order to meet this fixed commitment, the wind farm operator trades energy in the open market by selling surplus energy and by purchasing the amount of energy in deficit. The use we have given to storage is to reduce fluctuations in the amount of energy generated, then the storage is utilised to stockpile energy when there is a surplus of electricity generation and a low price with respect to the long-term mean price of electricity, and energy is extracted from the storage when there is a deficit of electricity generation and a high price with respect to the long-term mean price of electricity. This system is represented by a four-dimensional parabolic PDE and presents features of an advection-diffusion-reaction problem with no diffusion in the storage dimension or time, appropriate boundary conditions were defined and we have imposed linear solutions on the domain extrema for the energy generation and the truncated maximum price for electricity, and these boundary conditions are verified in the appendix at the end of the chapter.

Detailed implementations of our semi-Lagrangian alternating-direction implicit (SLADI) methodology were defined, as well as detailed implementations for semi-Lagrangian Crank-Nicolson (SLCN) and semi-Lagrangian fully implicit (SLFI) methods. For our main formulation of an advection-diffusion problem, we found that the stability and convergence of our SLADI methodology compares well with respect to that observed for more conventional SLCN and SLFI methods, we identified non-uniform convergence rates which can be attributed to a “non-linearity error” as the numerical grid does not perfectly align with discontinuities imposed on the control of the battery. Further numerical investigation allowed us to identify convergent results for PDE models with smoother coefficients for all the schemes described above.
We confirmed the effectiveness and efficiency of hybrid semi-Lagrangian and finite-difference methods for solving the advection-diffusion PDE model for the wind farm. We have extended our SLADI methodology to problems that incorporate mixed derivatives and we were able to obtain numerical solutions, but we found restrictions on the grid sizes that can be implemented as oscillatory solution can be produced; we return to this issue in the next chapter where smoother solutions allow for less restrictions in the grid size.

With accurate valuations we can conclude that a storage device can be used to effectively smooth the fluctuations in the energy generated by a random resource and then potential income can be transferred to regions with deficits in energy generation. The volatility in the electricity spot prices has been identified to be beneficial for transferring potential from high prices to low prices of electricity, and a seasonal random price of electricity provide an opportunity to gain potential with the storage. The storage device can prove useful to avoid penalties in the electricity market prices, with keeping electricity in storage, selling electricity at relatively lower prices is avoided (because of the bid price \(1 - \beta Y\)), and also with using the reserves from the storage device, purchasing electricity at relatively higher prices is avoided (because of the ask price \((1 + \beta)Y\)). Technical capacity of the storage shows a requirement to implement a better management since a fixed operation with respect electricity prices prevent additional income.

Optimal charging and discharging rates can be determined so that the system value is maximised, by choosing the ‘best’ value of the rate \(L\) for charging and discharging the battery at each point in state space, a situation that is addressed in the next chapter.
Appendix

4.A Windfarm valuation behavior on the domain extrema

Here we explore (using asymptotic analyses) the solution behavior for the wind farm valuation for large energy surplus (deficit) i.e., $|X| \to \infty$, which takes on the linear form $V = XV_0(Y, Q, \tau) + V_1(Y, Q, \tau) + \cdots$, to be consistent with both the boundary conditions (4.11) and the PDE (4.17). Throughout (-) superscripts are used to represent the solution $X \to -\infty$, and (+) superscripts are used to represent the solution $X \to \infty$. The $O(X)$ terms in (4.17) lead to

$$-\frac{\partial V_0^-}{\partial \tau} - \omega \lambda Q \frac{\partial V_0^-}{\partial Q} + \kappa (\theta - Y) \frac{\partial V_0^-}{\partial Y} + \frac{1}{2} \sigma_Y^2 Y \frac{\partial^2 V_0^-}{\partial Y^2} - (r + \alpha) V_0^- = -Y(1 + \beta), \quad (4.74)$$

and

$$-\frac{\partial V_0^+}{\partial \tau} + \omega \lambda (Q_{\text{max}} - Q) \frac{\partial V_0^+}{\partial Q} + \kappa (\theta - Y) \frac{\partial V_0^+}{\partial Y} + \frac{1}{2} \sigma_Y^2 Y \frac{\partial^2 V_0^+}{\partial Y^2} - (r + \alpha) V_0^+$$

$$= -Y(1 - \beta), \quad (4.75)$$

whilst the $O(X^0)$ terms yield

$$-\frac{\partial V_1^-}{\partial \tau} - \omega \lambda Q \frac{\partial V_1^-}{\partial Q} + \kappa (\theta - Y) \frac{\partial V_1^-}{\partial Y} + \frac{1}{2} \sigma_Y^2 Y \frac{\partial^2 V_1^-}{\partial Y^2} - r V_1^-$$

$$= -\Theta - \omega \lambda Q Y (1 + \beta), \quad (4.76)$$

and

$$-\frac{\partial V_1^+}{\partial \tau} + \omega \lambda (Q_{\text{max}} - Q) \frac{\partial V_1^+}{\partial Q} + \kappa (\theta - Y) \frac{\partial V_1^+}{\partial Y} + \frac{1}{2} \sigma_Y^2 Y \frac{\partial^2 V_1^+}{\partial Y^2} - r V_1^+$$

$$= -\Theta + \omega \lambda (Q_{\text{max}} - Q) Y (1 - \beta). \quad (4.77)$$

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The material in this appendix has been accepted for publication in The Journal of Computational Finance.
It is now possible to decompose $V_0^-$ and $V_0^+$ in the following form:

$$V_0^-(Y, Q, \tau) = V_{00}(Y) + Q e^{\Lambda_0^+ \tau} V_{01}^-(Y) + e^{\Lambda_0^+ \tau} V_{02}(Y), \quad (4.78)$$

$$V_0^+(Y, Q, \tau) = V_{00}(Y) + (Q_{\text{max}} - Q) e^{\Lambda_0^- \tau} V_{01}^+(Y) + e^{\Lambda_0^- \tau} V_{02}(Y). \quad (4.79)$$

$V_1^\pm$ can be decomposed in a not dissimilar form, but a discussion of this is omitted in the interests of brevity. The resulting equations for $V_0^\pm$ are then

$$\kappa(\theta - Y) \frac{dV_{00}^\pm}{dY} + \frac{1}{2} \sigma_Y^2 Y \frac{d^2V_{00}^\pm}{dY^2} - (r + \alpha)V_{00}^\pm = -Y(1 \mp \beta), \quad (4.80)$$

which have the simple solutions

$$V_{00}^\pm = \frac{1 \pm \beta}{r + \alpha + \kappa} \left( Y + \frac{\kappa}{r + \alpha} \theta \right). \quad (4.81)$$

The resulting equations for the $V_{01}^\pm$ are then

$$-\Lambda_0^+ V_{01}^\pm - \omega \lambda V_{01}^\pm + \kappa(\theta - Y) \frac{dV_{01}^\pm}{dY} + \frac{1}{2} \sigma_Y^2 Y \frac{d^2V_{01}^\pm}{dY^2} - (r + \alpha)V_{01}^\pm = 0, \quad (4.82)$$

whilst $V_{02}(Y)$ and $\Lambda_0$ (which are common to both $X \to \infty$ and $X \to -\infty$) are determined by

$$-\Lambda_0 V_{02} + \kappa(\theta - Y) \frac{dV_{02}}{dY} + \frac{1}{2} \sigma_Y^2 Y \frac{d^2V_{02}}{dY^2} - (r + \alpha)V_{02} = 0. \quad (4.83)$$

Hence, in the long-time limit, the far-field $|X| \to \infty$ approaches $V_{00}^\pm$, as described by (4.81).

The (temporal) rate at which the solution approaches these limits ($V_{00}^\pm$) is determined by the $\Lambda_0^\pm$, together with $\Lambda_0$ (in particular by the least negative of these values). Considering, first, the $\Lambda_0^\pm$, these must be determined through a homogeneous (eigenvalue) problem. A numerical investigation was mounted, in this respect, using (standard) second-order central differencing to approximate the $Y$ derivatives (again, the system was treated as a first-order system for $V_{01}^\pm$ and $dV_{01}^\pm/dY$). The result of this is a (generalised) eigenvalue problem of the form

$$A V_{01}^\pm = \Lambda_0^\pm V_{01}^\pm. \quad (4.84)$$

Some numerical results (obtained using a standard QZ algorithm) are shown in Figure 4.A.1 (taking the financial parameter values $\lambda = 1$, $\kappa = 0.25$, $r = 0.01$, $\alpha = 0.1$). Clearly, from this figure, the temporal rate of decay of this component of the solution
towards the ‘steady state’ is determined by the \( X \to -\infty \) (rather than the \( X \to \infty \)) behavior. It is also interesting to consider the limits as the long-term price becomes very low or very high. Consider, first, the limit \( X \to -\infty \). As \( \theta \to 0 \), for \( Y \to 0 \), but \( Y \gg O(\theta) \) it is easy to show that

\[
V_{01}^- \to A_0 + A_1 Y \log Y + A_2 Y + \cdots, 
\]

where the \( A_i \) are all constants, and in particular

\[
A_1 = \frac{2(r + \alpha + \lambda + \Lambda_0)}{\sigma_Y^2} A_0. 
\]

It is clear that we must next consider the \( Y = O(\theta) \) scale, on which the ‘switching’ effect is ‘felt’. If we set \( \hat{Y} = Y/\theta \), then we consider two regimes: \( \hat{Y} < 1 \) and \( \hat{Y} > 1 \), whilst at \( \hat{Y} = 1 \), we must ensure smooth pasting, although there must clearly be a discontinuity in \( d^2V_{01}^- / d\hat{Y}^2 \); this regime must also match (asymptotically) with (4.85) as \( \hat{Y} \to \infty \). We then find that

\[
V_{01}^- = 1 + (\theta \log \theta) B_1 \hat{Y} + \theta \left( \frac{2(r + \alpha + \Lambda_0)}{\sigma_Y^2} \right) \hat{Y} \log \hat{Y} + K_1 \hat{Y} + \cdots, 
\]

for \( \hat{Y} < 1 \), \hspace{1cm} (4.87)

\[
V_{01}^- = 1 + (\theta \log \theta) B_1 \hat{Y} + \theta \left( \frac{2(r + \alpha + \lambda + \Lambda_0)}{\sigma_Y^2} \right) \hat{Y} \log \hat{Y} + K_2 \hat{Y} + K_3 + \cdots, 
\]

for \( \hat{Y} > 1 \), \hspace{1cm} (4.88)

where we have normalised the latter eigenfunction with respect to its value at \( Y = 0 \), and \( K_1 \), \( K_2 \) and \( K_3 \) are constants, which may be chosen to ensure that \( V_{01} \) and \( dV_{01} / d\hat{Y} \) are continuous. The implication of (4.87) is that \( dV_{01}^- / d\hat{Y} (Y = 0) \) is generally unbounded, which is unacceptable (from a financial point of view). Furthermore, careful inspection of the eigenfunctions, obtained numerically, confirmed that this is not the case, the value only becoming unbounded when \( \theta = 0 \). This being the case, we must have that

\[
\Lambda_0^- = -(r + \alpha), 
\]

a result that can be seen to be fully consistent with our numerical results shown in Figure 4.A.1.
In the corresponding case for $X \to \infty$, the major difference is that $'\left( r + \alpha + \Lambda_0^- \right)'$ is replaced by $'\left( r + \alpha + \lambda + \Lambda_0^+ \right)'$ in (4.87) and $'\left( r + \alpha + \lambda + \Lambda_0^- \right)'$ in (4.88) is replaced by $'\left( r + \alpha + \Lambda_0^+ \right)'$. The reasoning above then leads us immediately to the conclusion that

$$\Lambda_0^+ = -(r + \alpha + \lambda); \quad (4.90)$$

again, this result appears to be in agreement with other results shown in Figure 4.A.1.

Let us now turn our attention to the high long-term price limit, i.e. $\theta \to \infty$. Again, focusing first on the $X \to -\infty$ case, then we write

$$-(r + \alpha + \omega \lambda + \Lambda_0^-) V_{01}^- - \kappa \hat{Y} \frac{dV_{01}^-}{dY} + \frac{1}{2} \sigma_0^2 \hat{Y} \frac{d^2V_{01}^-}{dY^2} = 0, \quad (4.91)$$

where $\hat{Y} = (Y - \theta)/\theta \hat{Y}$, corresponding to a relatively narrow spot price range, centered about $Y = \theta$; this too is an eigenvalue problem, but is now on a doubly infinite domain, subject to $V_{01}^- \to 0$ as $|Y| \to \infty$, with $\omega = 0$ for $\hat{Y} < 0$ and $\omega = 1$ for $\hat{Y} > 0$. This system was then solved in precisely the same manner as (4.83), and yielded the key result that the crucial, least negative eigenvalue $\Lambda_0^- \approx -0.455$; this is clearly consistent with the $\theta \gg 1$ results presented in Figure 4.A.1. When we consider the $\theta \gg 1$ limit for $X \to \infty$, the resulting eigenvalue problem can be reduced to (4.91) precisely, by replacing $\hat{Y}$ by $-\hat{Y}$, and as a result we have $\Lambda_0^+ \approx -0.455$ (also).
Moving on to consider the system (4.83); In this case, fortuitously, it is possible to write the eigensolution in analytic form, namely

\[ V_{02} = \mathcal{L}_n^\alpha \left( \frac{1}{2} \sigma_Y^2 Y \right), \]  
(4.92)

where \( \mathcal{L}_n^\alpha (z) \) denotes the generalised Laguerre polynomial (Abramowitz and Stegun, 1964), where \( \hat{\alpha} = \sigma_Y^2 \kappa \theta / 2 \) and \( n = -(r + \alpha + \lambda_0) / \kappa \). Furthermore, since we must insist that \( n \) must be an integer, and that \( V_{02} \) can only grow linearly (at most) as \( Y \to \infty \), it transpires that we need only consider \( n = 0 \) and \( n = 1 \). The key result is then that the least negative eigenvalue (corresponding to \( n = 0 \)) is given by

\[ \lambda_0 = -(r + \alpha). \]  
(4.93)

It is worth noting that this result is in agreement with a parallel numerical investigation that was mounted on the system (4.83). This is important, and indeed illustrates the fact that as \( \tau \to \infty \), the temporal decay towards the ‘steady state’ is governed by (4.93), this being the least negative of all the temporal eigenvalues.

Finally, it is also worth noting that if \( \omega \equiv 1 \) (or \( \omega \equiv 0 \)) everywhere, then the global solution does admit solutions of the form

\[ V = Y \tilde{V}_0(X, Q, \tau) + \tilde{V}_1(X, Q, \tau), \]  
(4.94)

a result verified by our numerical work.
Chapter 5

Storage optimisation, an application to wind power generation

In this chapter we explore the problem of determining the optimal regime for managing storage, when a given rate of production is present in the system. Utilising the reserves or replenishing the levels of the storage correspondingly mitigate deficit difficulties and take advantage of any excess in production. If the underlying spot price can be arbitraged, this will result in additional benefit by shifting units from low prices to high prices in the market. Here, based on the Dynamic Programming argument (Bertsekas, 2005), we determine optimal states for charging and discharging the storage device depending on the level of energy generated by the wind farm. The strategy basically states that storage reserves are utilised if the benefit obtained after discharging a small amount of energy from the storage is larger than the purchase cost, correspondingly energy is sold to the market as long as the sale benefit is greater than the stockpile benefit.

The wind farm model from the previous chapter is redefined to maximise the expected cash flows for managing electricity from the storage device alongside energy generation. In order to have a consistent comparison with the system defined in the previous chapter (see section 4.1), the wind farm is committed to deliver a constant

\(^1\)Part of the material in this chapter has been accepted for publication in *The Journal of Computational Finance*
rate of electricity (regardless of the production level), and is receiving a constant fixed price for that, then given an unpredictable production rate, the wind farm operator must trade electricity in the market (with random prices) to deliver the constant rate of electricity in the contract.

Additionally, we assume that the operational and maintenance costs can be neglected, then we are only accounting for income flow. With respect to the market, we assume that electricity can be bought and sold any time without restriction and no transaction costs are associated. With respect to the use of resources, we assume that energy is stockpiled if there is surplus in generation, and energy is taken from reserves if there is deficit in generation, thus we are essentially using the storage device to balance the output by reducing fluctuations produced by the resource. In the following section, we consider how to optimise the value of the system, by choosing the ‘best’ value of the rate \( L \) for charging and discharging the battery at each point in state space, and then obtain optimal economic benefits from storage under our set of assumptions.

We verify that our boundary value problem formulation for stochastic control satisfies assumptions from the Strong Comparison Result, this to justify that the solution to the problem is continuous and corresponds to the unique viscosity solution, an extensive numerical experimentation is used to show the convergence and stability of the numerical methods, this for the advection-diffusion problem. We also investigate the effects obtained with the introduction of mixed derivatives and a seasonal electricity price.

We then proceed to extend the use of the SLADI methodology to numerically optimise our wind farm model and the results are compared with solutions obtained for different wind farm setups: one with fixed control of the storage device (as the system described in the previous chapter), the second with a battery always charging if there is surplus energy and always discharging if there is deficit of energy, and a third configuration with no battery.

Finally we present results for the value of the wind farm in the long-run. Results are presented for the advection-diffusion problem for a large time-scale of the wind farm \( T \), we use our SLADI methodology and stable results are produced. In the appendix we explore the implementation of iterative schemes for obtaining steady-state solutions,
this for a stochastic storage problem subject to uncertain electricity prices and a fixed
demand; a projected successive over-relaxation method is implemented on a fixed grid,
and we explore the use of boundary-fitted coordinates to approach the solution.

5.1 Optimal regime for storage operation

In this section we illustrate an approximation to the optimal strategy for operating
the storage device. Similar methods have been used to solve models with just one
source of uncertainty; Chen and Forsyth (2007) valued and optimised the operation
of gas storage facilities. Chen and Forsyth (2008) then priced hydroelectric power
plants subject to operational restrictions with a stochastic control approach. Semi-
Lagrangian schemes have been used extensively and effectively in this research area
to solve PDE and PIDE type models. Here we use our semi-Lagrangian alternating-
direction implicit methodology and optimise values using a model with two sources of
uncertainty.

Similar models have been studied in the optimal operation of electric power plants
literature. Thompson et al. (2004) derived models incorporating two sources of stochasticity, one from the electricity spot price and the other from the resources used to run
the hypothetical power plants. However, even though the formulations were made
with respect to two sources of randomness, they solved the models by assuming a
constant value for the resource’s random variable, thus solving only for the case of
random prices of electricity. Their explicit finite-difference schemes were utilised for
the numerical solution of the model, but stability considerations imposed time-step
restrictions rendering these methods computationally expensive.

Here we approximate the solution of a full four-dimensional model efficiently, in
which the preferred algorithm for producing calculations was SLADI$^{(3)}_{II}$ as this pro-
duced the most consistent estimations of control regions.

We define the system’s domain $H$ in terms of the variables $t, Y(t), X(t)$ and $Q(t)$.
The variable $t \in (0, T)$ represents chronological time, and $T$ is a typical time scale of
the wind farm.
$Y(t) \in (0, \infty]$ is used to model a non-negative risk-adjusted (or risk-neutral) electricity spot price at time $t$, where $\theta$ is the long-term mean value of electricity. We assume that $Y$ follows a Cox-Ingersoll-Ross stochastic process (see Janczura and Weron, 2009), to reproduce characteristics of mean reversion and spikes with a simple model, i.e.,

$$dY = \kappa(\theta - Y) \, dt + \sigma_Y \sqrt{Y} \, dW_1; \quad (5.1)$$

$\kappa$ and $\sigma_Y$ are the speed of mean reversion and the volatility of the electricity price, and $dW_1$ is a standard Wiener processes. Numerically we take $Y(t) \in (0, Y_{\text{max}})$ as the truncated bounded $Y$ domain; where we must choose $Y_{\text{max}}$ sufficiently large so that the errors introduced from truncation are sufficiently small.

$X(t) \in (X_{\text{min}}, X_{\text{max}})$ represents the surplus or deficit in production of electricity at time $t$, and we assume this to be proportional to the wind speed on the site where the wind farm is located, here $X$ follows an Ornstein-Uhlenbeck stochastic process (see Zárate-Miñano et al., 2013; Benth and Benth, 2009; Edwards and Hurst, 2001), in order to approximately replicate real wind speed, namely

$$dX = -\alpha X \, dt + \sigma_X \, dW_2; \quad (5.2)$$

$\alpha$ and $\sigma_X$ are the speed of mean reversion and the volatility on the generation of energy, and $dW_2$ is a standard Wiener processes. Under the simplified contract the wind farm is committed to produce a constant rate of $C$ units of electricity, which we take as being half the capacity of the wind farm ($2C$ is the maximum capacity) so that we may then notionally set $X_{\text{min}} = -C$, and $X_{\text{max}} = C$ for a bounded domain on $X$, where we assume $C \gg 1$; with this setting under and over commitment is avoided.

$Q(t) \in (0, Q_{\text{max}})$ represents the effective charge stored in the battery at time $t$, then a nominal minimum capacity of zero corresponds to the depth of discharge and $Q_{\text{max}}$ is the maximum effective capacity of this storage device.

Under the terms of the contract the seller receives a continuous fixed payment when the farm is producing exactly the right amount of electricity ($X = 0$) and then the seller should receive an amount

$$\Theta \, dt \quad (5.3)$$

in payment, without penalties, in time $dt$. If $X < 0$ there is a deficit in production of electricity, then we decide to discharge the battery or to pay a penalty by having to
buy electricity from the market at price $Y$, with *ask price* factor $(1 + \beta)$, i.e.

$$-(X - \mathcal{L})Y(1 + \beta) \, dt,$$

this occurring in time $dt$. If $X > 0$ there is a surplus of electricity, then we decide to
charge the battery or to sell electricity to the market at price $Y$, with *bid price* factor $(1 - \beta)$

$$(X - \mathcal{L})Y(1 - \beta) \, dt,$$

this is also over a time period $dt$. As in the previous chapter, we regard $\Theta$ as a
constant, and a symmetric bid-ask price, but it would be a simple matter to relax
these constraints.

The objective is to maximise the expected cash flows for managing electricity from
the storage device alongside energy generation

$$u(z) = \max_{\mathcal{L}} \left\{ E_z \left[ e^{-r\tau} f(Z^\tau) + \int_0^\tau e^{-rt} g(Z^t) \, dt \right] \right\},$$

then we drop the control parameter $\omega$ in (4.1) to define maximum extraction rates
($\mathcal{L}_{\text{min}}$) in the energy deficit region and maximum injection rates ($\mathcal{L}_{\text{max}}$) in the surplus
region,

$$\begin{cases}
\mathcal{L}_{\text{min}} = -\min\{|X|, \lambda Q\} & \text{if } X < 0, \\
\mathcal{L}_{\text{max}} = \min\{X, \lambda (Q_{\text{max}} - Q)\} & \text{if } X > 0.
\end{cases}$$

And the dynamics of the amount of charge in the battery are then given by

$$dQ = \mathcal{L}(X, Y, Q) \, dt.$$

With the assumption that the penetration of wind energy into the grid is small
we take the correlation between the processes $dW_1$ and $dW_2$ to be zero, and so (5.2)
needs not to be risk-adjusted (correlation effects are explored later in the chapter).

Thus we search for the solution to the Hamilton-Jacobi-Bellman (HJB) equation,
where we have made the substitution $\tau = T - t$

$$\max_{\mathcal{L}|X} \left[ -\frac{\partial V}{\partial \tau} + \mathcal{L} \frac{\partial V}{\partial Q} - \alpha X \frac{\partial V}{\partial X} + \kappa (\theta - Y) \frac{\partial V}{\partial Y} + \frac{1}{2} \sigma_X^2 \frac{\partial^2 V}{\partial X^2} + \frac{1}{2} \sigma_Y^2 \frac{\partial^2 V}{\partial Y^2} \\
- rV + \Theta + (X - \mathcal{L})Y(1 \pm \beta) \right] = 0, \quad \text{if } X \leq 0. \quad (5.9)$$
Maximisation is obtained from the terms involving the control variable \( L \) and the rate of energy generation \( X \)

\[
\max_{\mathcal{L}} \left[ \mathcal{L} \frac{\partial V}{\partial Q} - \alpha X \frac{\partial V}{\partial X} + (X - \mathcal{L})Y(1 \pm \beta) \right]
\]

subject to

\[
\mathcal{L}_{\min} \leq \mathcal{L}(X, Q) \leq \mathcal{L}_{\max}, \quad \text{if } X \leq 0;
\]

A rearrangement of (5.10),

\[
\max_{\mathcal{L}} \left[ (1 \pm \beta)Y - \alpha \frac{\partial V}{\partial X} \right] X + \left( \frac{\partial V}{\partial Q} - (1 \pm \beta)Y \right) \mathcal{L}
\]

if \( X \leq 0 \),

makes it easier to identify that:

- For the deficit region \( (X < 0) \) the maximisation of equation (5.11) depends only on the control variable \( \mathcal{L} \), as the wind farm is buying electricity to meet the contract and the use of reserves in storage reduces the amount of energy that is bought.

- For the surplus region \( (X > 0) \), in a ‘stockpiling sense’ the maximisation of equation (5.11) will occur only if the coefficient of \( \mathcal{L} \), namely \( \left( \frac{\partial V}{\partial Q} - (1 - \beta)Y \right) \), is greater than the coefficient of the given rate of energy generated \( X \), namely \( (1 - \beta)Y - \alpha \frac{\partial V}{\partial X} \), otherwise stockpiling is not convenient and surplus energy is rather sold to the market.

Note that the term \( \partial^2 V/\partial X^2 \) is already incorporated in the value of the wind farm \( V \), and so the term is not included in the maximisation since only \( O(X) \) terms can have an effect on the decision to determine the rate \( \mathcal{L} \).

Here, the constrained system in section 4.1 is optimised by setting \( \mathcal{L}_{\max} = 0 \) on the deficit region \( (X < 0) \) where the battery is discharged, and by setting \( \mathcal{L}_{\min} = 0 \) on the surplus region \( (X > 0) \) where the battery is charged, the battery is used essentially to reduce fluctuations in the output rate and in this chapter we find the most economically beneficial strategy for managing the storage.

We assume that solving the PDE (5.9) with boundary conditions (4.11) - (4.16), where \( V^T = 0 \), satisfies the Strong Comparison Result (see Chen and Forsyth, 2007 for details). The Strong Comparison Result is satisfied by the viscosity solution of degenerate elliptic HJB equations with Dirichlet boundary conditions if a series of assumptions is satisfied. In particular, Barles and Rouy (1998) demonstrated that:
1. For a region on a boundary with an outgoing characteristic, if the associated coefficient for the diffusion term vanishes regardless of the control variable value, then the viscosity solution on such boundary region is the limit for the viscosity solution from interior points.

2. For a region on a boundary, related with first-order terms in the PDE, if the characteristic is incoming to the domain without regard of the control value, then (in the classical sense) the boundary values correspond to the viscosity solution on such region.

The first point above is satisfied in our case, as PDE (5.9) can be regarded as a four-dimensional degenerate elliptic PDE in the variable $z = (X, Y, Q, \tau) \in [X_{\min}, X_{\max}] \times [0, Y_{\max}] \times [0, Q_{\max}] \times [0, T]$, since it does not possess second-order derivatives with respect to $\tau$ and $Q$, i.e. there is no diffusion present in these two former dimensions.

The boundary conditions (4.11) - (4.13) indicate no diffusion on the extreme values of $X$ and $Y$, and a vanishing diffusion-coefficient to zero on $Y = 0$, these for regions with outgoing characteristics. The second point above is satisfied in our case as the final value $V^T$ acts as a Dirichlet boundary condition for the incoming characteristic to the domain.

We are making a strong assumption about the numerical scheme achieving a continuous viscosity solution for numerically solving the PDE (5.9) with boundary conditions (4.11) - (4.16), and $V^T = 0$. In the paragraph above we have verified that our boundary problem formulation is consistent with the assumptions from the Strong Comparison Result, and we conduct a numerical test in section 5.2 to assess the convergence of the solution using the SLADI methodology and we also present results towards the steady state for stability in section 5.4.

The solution to the problem (5.10) is

$$
\begin{align*}
\text{if } (1 + \beta)Y > \frac{\partial V}{\partial Q}, & \quad \mathcal{L} = \mathcal{L}_{\text{min}}, \\
\text{otherwise } \mathcal{L} = 0, & \quad \text{for } X < 0;
\end{align*}
$$

$$
\begin{align*}
\text{if } \left( (1 - \beta)Y - \alpha \frac{\partial V}{\partial X} \right) < \left( \frac{\partial V}{\partial Q} - (1 - \beta)Y \right), & \quad \mathcal{L} = \mathcal{L}_{\text{max}}, \\
\text{otherwise } \mathcal{L} = 0, & \quad \text{for } X > 0.
\end{align*}
$$

The approach that we follow to approximate the solution of the PDE (5.9), consists of implementing the numerical scheme SLADI$_{II}^{(3)}$ (see section 4.2.2) with control
evaluation every half time step. The following control criteria determines extraction and injection rates $L$ (which we assume to be approximately constant along the corresponding characteristic) as follows. Consider a virtual particle arriving at the regular mesh point $(i, j, k)$ at time $\tau + \frac{1}{2} \Delta \tau$, with potential rate $L^{l+\frac{1}{2}}$, and using a discrete approximation to the solution (5.12) at the foot of the characteristics (for a virtual particle departing potentially from the irregular mesh point $(i^*, j^*, k^*)$ at time $\tau$):

\begin{align}
\text{Control criteria} \quad \left\{ \begin{array}{ll}
\text{For } X < 0 & \text{if } (1 + \beta)Y_{j^*} > \frac{V^l_{i^*, j^*, k^*+1} - V^l_{i^*, j^*, k^*-1}}{2\Delta Q} \\
& \text{then } L^{l+\frac{1}{2}} = L_{\min}, \\
& \text{otherwise } L^{l+\frac{1}{2}} = L_{\max} = 0.
\end{array} \right.
\end{align}

\begin{align}
\text{For } X > 0 & \quad \text{if } 2(1 - \beta)Y_{j^*} < \frac{V^l_{i^*, j^*, k^*+1} - V^l_{i^*, j^*, k^*-1}}{2\Delta Q} + \frac{V^l_{i^*+1, j^*, k^*} - V^l_{i^*-1, j^*, k^*}}{2\Delta X}, \\
& \text{then } L^{l+\frac{1}{2}} = L_{\max}, \\
& \text{otherwise } L^{l+\frac{1}{2}} = L_{\min} = 0.
\end{align}

(5.13)

With these criteria, we discharge energy in the deficit region $(X < 0)$ if it is more expensive to buy a small amount of energy from the market (with a penalty $1 + \beta$) than the marginal value obtained from discharging the same small amount of energy from the storage device, regardless of the energy generation level. We sell all surplus energy $(L = 0)$ on the region $(X > 0)$ if the benefit from selling a small amount of surplus energy is greater than the benefit from stockpiling the same small amount of energy; otherwise we stockpile energy at the maximum charging rate allowed by the storage device.

### 5.2 Accuracy Assessment

Here we conduct a ratio of convergence test $R$, as defined previously in (4.52), for the valuation $V$ of the system with optimal charging and discharging rates of the storage device. The test is carried on a finite-difference grid with spacings: $\Delta_x = h$, $\Delta_Y = \frac{2}{3} h$, and $\Delta_Q = \frac{1}{2} h$ by letting $h \to 0$ along with the two choices $\Delta_r = \frac{1}{4} h$ and $\Delta_r = \frac{1}{8} h$, for $h = \{0.5, 0.25, 0.125, 0.0625, 0.00125\}$.

Table 5.1 shows the resulting converged values and the corresponding convergence ratios for $\Delta_r = \frac{1}{4} h$, at representative points in state space, whilst Table 5.2 shows
the results for $\Delta \tau = \frac{1}{8} h$. The selected points correspond to states where the battery’s charging/discharging regime changes ($|X| = \{0, Q_{\text{max}}\}$), to the extreme and half levels of charge state ($Q = \{0, \frac{1}{2}Q_{\text{max}}, Q_{\text{max}}\}$), to a zero spot price ($Y = 0$) and the long-term mean price of electricity $\theta$.

The parameters used in this test are $\beta = \{0, 0.1, 0.5, 0.9\}$, $\sigma_X = 0.5$, $\sigma_Y = 0.5$, $\alpha = 0.1$, $\kappa = 1$, $\lambda = 1$, $\theta = 5$, $\Theta = 5$, $r = 0.01$, $X_{\text{max}} = 25$, $Y_{\text{max}} = 20$, $Q_{\text{max}} = 1$ and $T = 10$; as noted earlier, for a time origin of $\tau = 0$ we have a zero value for all $V$.

Figures 5.1 and 5.2 present heat maps for the rate of convergence of scheme SLADI$_{\text{II}}^{(3)}$ with an implemented control criteria (5.13) for determining optimal $\mathcal{L}$ rates for charging and discharging the battery, and using different bid-ask difference factors. The plots correspond to the rates observed based on the final refinement namely with $\Delta \tau = \frac{1}{8} h$. Plus and diamonds representations are used to indicate regions where the rate of convergence oscillates outside the plotting range. We observe that invariably these oscillation issues occur in regions where the scheme has (already) obtained about four or five digits of accuracy, and so the nonuniform convergence in the magnitude of the valuations is rather small (as observed in the previous chapter). This result is perhaps not in disagreement with non-uniform convergence rates (of first-order) reported by Chen and Forsyth (2007) on the optimal gas storage operation. Again, the primary cause of the non-uniformity observed is due to the coefficients of the fundamental PDE (5.9) not all being smooth (analytic); this view is reinforced by inspection of Figures 5.1 and 5.2. Since we still have discontinuities not aligning perfectly with the numerical grid, we still observe 'nonlinearity error', i.e. nonuniform convergence. As a comparison, the gradients along the $Y$ direction are smoother in the optimal controlled operation than the gradients on a regime with fixed switching (as considered in the previous chapter), then we observe more stable results along the $Y$ direction, although the rates of convergence obtained here are basically of first-order. As mentioned in the previous chapter, second-order global accuracy is an ongoing research topic in this general area, and for example recently Ware (2013) has suggested a semi-Lagrangian time-stepping algorithm with a Fourier-cosine discrete approximation, with demonstrated second-order accuracy.

Here we also present convergence results for using a risk-adjusted (or risk-neutral) electricity price $Y_2$ with mean-reversion and proportional variance, which in this case
Table 5.1: Convergence of SLADI\(^{(3)}\) for optimally controlling storage; \(\Delta_x = \frac{1}{4}h\).

<table>
<thead>
<tr>
<th>State</th>
<th>Value (\beta = 0)</th>
<th>Value (\beta = 0.1)</th>
<th>Value (\beta = 0.5)</th>
<th>Value (\beta = 0.9)</th>
</tr>
</thead>
<tbody>
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<td>(X) (Y) (Q)</td>
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<td>(\beta = 0.1)</td>
<td>(\beta = 0.5)</td>
<td>(\beta = 0.9)</td>
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| -1 5 1 | 25.835123 [n.a.] | 22.427246 [n.a.] | 8.798375 [n.a.] | -4.676695 [n.a.] |
| 25.979467 [n.a.] | 22.609926 [n.a.] | 9.152410 [n.a.] | -4.043684 [n.a.] |

| 0 0 \(\frac{1}{\sqrt{2}}\) | 61.993188 [n.a.] | 59.005225 [n.a.] | 47.206080 [n.a.] | 35.730334 [n.a.] |
| 62.101461 [n.a.] | 59.130474 [n.a.] | 47.482046 [1.71] | 36.416903 [1.63] |
| 62.115844 [-0.1] | 59.186922 [2.37] | 47.690900 [1.98] | 36.629296 [1.98] |

| 0 5 0 | 50.519736 [n.a.] | 47.918968 [n.a.] | 37.517788 [n.a.] | 27.228632 [n.a.] |
| 50.705209 [n.a.] | 48.083707 [n.a.] | 37.616053 [n.a.] | 27.373312 [n.a.] |
| 50.749699 [4.17] | 48.140481 [2.90] | 37.734885 [0.83] | 27.587359 [0.68] |

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| 64.175763 [n.a.] | 60.873915 [n.a.] | 47.986041 [n.a.] | 36.099177 [n.a.] |
| 64.200015 [5.00] | 60.916930 [3.04] | 48.135149 [1.78] | 36.368239 [1.65] |
| 64.203723 [-31.] | 60.940466 [2.37] | 48.248823 [2.01] | 36.578369 [1.99] |

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### Table 5.2: Convergence of SLADI<sup>(3)</sup> for optimally controlling storage; $\Delta \tau = \frac{1}{8} h$.

<table>
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Figure 5.1: Heat maps for the rate of convergence $\mathcal{R}$ for different penalisation $\beta$ values (indicated above each column); from top to bottom plots correspond to $Q = 0$, $Q = \frac{1}{2}Q_{\text{max}}$ and $Q = Q_{\text{max}}$. 
Figure 5.2: Heat maps for the rate of convergence $\mathcal{R}$ for different penalisation $\beta$ values (indicated above each column); from top to bottom plots correspond to $Q = 0$, $Q = \frac{1}{2}Q_{\text{max}}$ and $Q = Q_{\text{max}}$. 
follows
\[ dY_2 = \kappa(\theta - Y_2) \, dt + \sigma Y_2 \, dW \] (5.14)
where \( dW \) is a standard Wiener process, \( \kappa \) is the speed of mean reversion, \( \theta \) is the long-term mean price of electricity and \( \sigma \) is the volatility of the price of electricity (Bodily and Del Buono, 2002). The resulting HJB to solve in forward time \( \tau = T - t \) results
\[
\max_{\mathcal{X}} \left[ -\frac{\partial V}{\partial \tau} + \mathcal{L} \frac{\partial V}{\partial Q} - \alpha X \frac{\partial V}{\partial X} + \kappa(\theta - Y_2) \frac{\partial V}{\partial Y_2} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial X^2} + \frac{1}{2} \sigma^2 \frac{Y_2 \partial^2 V}{\partial Y_2^2} \right. \\
\left. - rV + \Theta + (X - \mathcal{L})Y_2(1 \pm \beta) \right] = 0, \quad \text{if } X \equiv 0. \quad (5.15)
\]
The boundary conditions (4.11) - (4.16) and the control criteria (5.13) are consistent with the PDE (5.15). As an extension of the convergence test, the parameters values are as indicated above. The heatmaps in Figure 5.3 present patterns that are consistent with those observed in Figures 5.1 and 5.2 (for corresponding \( \beta \) values). We conclude again that non-uniform convergence rates are produced by a “nonlinearity error” that is effected by discontinuities, since the control does not perfectly align with the numerical grid. In the stochastic price (5.14) there is a larger diffusion coefficient (compared to (5.1)) for \( Y > 1 \) and even in this case we identify irregularities in the convergence rate, but this can be an indicator of slightly less irregularities in the charging region for the stochastic price (5.1) as it has a larger diffusion coefficient than (5.15) for small time \( \tau \to 0 \) and \( Y < 1 \) (when the charging regime develops). From Table 5.3 we identify a first-order of convergence for the solution and also a better valuation of the system (compared to results on Table 5.2) which is an effect of a larger diffusion coefficient.

### 5.3 Windfarm optimisation

Sample resulting valuations for the proposed wind farm with optimal rates of energy injection and extraction are presented in Figures 5.1 and 5.2. For selected spot prices (and for legibility) Figure 5.1 shows sample valuations along the \( X \) direction; by comparing the separation between the \( Q = 0 \) (thin lines) and the \( Q = Q_{\max} \) (thick lines) curves at the same spot price, we observe the added value of storage at higher prices of electricity for an optimal operation of the storage device (from this section we use the expression \( Y = 0 \) for indicating “in the limit as \( Y \) approaches zero,” since
Table 5.3: Convergence assessment for a stochastic electricity price (5.14); $\Delta x = \frac{1}{8}h$.

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Figure 5.3: Heat maps for the rate of convergence $\mathcal{R}$ for different penalisation $\beta$ values (indicated above each column), for a stochastic electricity price (5.14); from top to bottom plots correspond to $Q = 0$, $Q = \frac{1}{2}Q_{\text{max}}$ and $Q = Q_{\text{max}}$. 
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Figure 5.1: Solution for an optimally controlled storage operation at selected spot prices, for $Q = 0$ (thin lines) and $Q = Q_{\text{max}}$ (thick lines); parameters as in Figure 4.22, except $T = 10$.

the selected parameters prevent the stochastic process for the electricity spot price to reach a zero level). Figure 5.2 shows sample valuations at the mean level of energy generation $X = 0$, a non-decreasing valuation is obtained for a full storage ($Q = Q_{\text{max}}$) as potentially in this regime energy can be released for deficits on generation, and no more energy is injected to the storage device (any potential surplus generation is sold to the market) causing a ‘call option effect’ (from the point of view of the holder of the option) with respect to the prices of electricity with higher valuation for increasing $Y$.

For an empty storage $Q = 0$ we observe (Figure 5.2) a ‘put option effect’ (from the point of view of the holder of the option) on the price of electricity as potentially in this regime energy is purchased for deficits on energy generation and also there is potential for storing energy (steeper slope for low electricity spot prices) then we observe higher valuation for decreasing $Y$. A storage device at half capacity $Q = \frac{1}{2}Q_{\text{max}}$ shows a mixed ‘call-put option effect’ (financial straddle) with relatively higher value on lower prices of electricity and also higher valuation for high prices of electricity.

Figure 5.3 illustrates the advantage for the optimally controlled operation of the battery, compared to the fixed regime (as considered in the previous chapter) when the battery is switched at $\theta$, at selected times; all parameters remain the same as in Figure 4.22, except $T = 10$. For small $\tau$ values, the system about to reach the end of the timescale for the wind farm and in the optimal operation is not necessary to keep
stockpiling energy as it will become worthless at expiry, as assumed by the boundary condition (4.16), and surplus energy is rather sold to the market. Additionally, the discharging region is extended to even lower prices of electricity as, again, any stored units of energy will become worthless. The fixed switching operation at $\theta$ receives a reduced valuation because of this forced operation with lower income on both the charging and the discharging regions. A combined strategy for optimally charging and discharging the storage device is obtained after determining the profitability by following the criteria (5.13).

Figures 5.4, 5.5 and 5.6 show the difference in the solution along the $X$ axis on $Q = 0$, $Q = \frac{1}{2}Q_{\text{max}}$, and $Q = Q_{\text{max}} = 1$ respectively, for the specified spot prices, at $\tau = 10$ (Figure 5.7 shows a sample of the numerical/truncation error convergence). For an empty storage $Q = 0$ (Figure 5.4) there is no difference in value between the two systems for large deficits of energy, but the optimal management of the battery substantially increases the value on the surplus region and also even at small deficits. At half capacity (see Figure 5.5) an optimally controlled battery still obtains important improvements in value, especially in the surplus region ($X > 0$) and some additional value in the deficit region ($X < 0$) at low prices for electricity. The optimal control of the battery, now at full capacity level (see Figure 5.6), produces notable benefits...
about the mean level of energy generation ($X = 0$) and some improvements in the deficit region (particularly at low prices of electricity), but there is no additional benefit when high levels of surplus energy are generated. The optimal switching becomes more significant as the battery level approaches capacity.

The evolution in time for the optimally controlled operation of the battery, at half capacity, for a range of energy generation levels and a range of electricity spot prices can be appreciated in Figure 5.8. The pattern is canonical to other battery states; the white region indicates the maximum charging rate, the light grey region is the regime in which the battery is turned off, the dark grey region is for the maximum discharging regime, and the faded regions are for the linear increasing/decreasing rate of charge/discharge with respect to $X$. For decreasing $\tau$ the charging region is reduced.
Figure 5.4: Difference between the solution with an optimally controlled operation and that with a fixed switching at $\theta$ for $Q = 0$; parameters as in Figure 4.22, except $T = 10$.

Figure 5.5: Difference between the solution with an optimally controlled operation and that with a fixed switching at $\theta$ for $Q = \frac{1}{2}Q_{\text{max}}$; parameters as in Figure 4.22, except $T = 10$. 
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Figure 5.6: Difference between the solution with an optimally controlled operation and that with a fixed switching at $\theta$ for $Q = Q_{\text{max}} = 1$; parameters as in Figure 4.22, except $T = 10$.

Figure 5.7: Convergence for the difference between the solution with an optimally controlled operation and the solution with a fixed switching at $\theta$ for $Q = \frac{1}{2}Q_{\text{max}}$, and $Y = 0$; parameters as in Figure 4.22, except $T = 10$. 
and the discharging region extended towards lower prices of electricity. The regions close to expiry ($\tau \to 0$) indicate that there is no need to stockpile energy, and the remaining charge must be emptied even at low prices of electricity. At $\tau = 0$ occurs a singularity where all remaining charge in the battery is valued to zero and no more charge is extracted.

Figures 5.9, 5.10, 5.11 and 5.12 present a comparison of the optimally controlled storage device with respect to a system where the battery is always being charged if there is surplus on energy generation, and the battery is always being discharged if there is a deficit on energy generation, a regime which we refer to as system with a battery that is always turned ON. For an empty battery (see Figure 5.10) we identify a substantial increase in value in the surplus generation region $X > 0$ as the storage device will only be charged when stockpiling is more convenient; therefore the cash
flows are maximised by selling surplus energy to the market, and so we also identify some benefits for small deficits in energy generation. For a full storage (see Figure 5.12) and for low prices of electricity, we observe an important increase in value in the region with a deficit in energy generation $X < 0$, there are also some important benefits about the mean level of energy generation ($X = 0$) and this can also be observed along the $Y$ direction in Figure 5.9. For a half-charged battery (see Figure 5.11) we (also) identify a substantial increase in value from mixed effects from the optimal recharge of an empty storage (for the surplus region $X > 0$) and from the optimal discharge of a full storage (for the deficit region $X < 0$).

A comparison with respect to a system with no battery is presented in Figures 5.13, 5.14, 5.15 and 5.16 for a range of spot prices. Although here we are neglecting the fixed and variable costs associated with operating the storage device, in practice one has to find if the benefits obtained from reserves consumption are greater than the operational costs of the storage device. We basically observe the benefits from the optimal operation of the storage device in the deficit region $X < 0$ and there are also benefits when there are low surplus levels of energy generation, but there are parts of the domain, especially for low prices of electricity and low storage levels, see Figures 5.15 and 5.16, where the value of the optimally controlled storage device is lower than...
Figure 5.10: Difference between the solution with an optimally controlled operation and the solution of a system with a battery always turned ON for $Q = 0$; parameters as in Figure 4.22, except $T = 10$.

Figure 5.11: Difference between the solution with an optimally controlled operation and the solution of a system with a battery always turned ON for $Q = \frac{1}{2}Q_{\text{max}}$; parameters as in Figure 4.22, except $T = 10$. 
Figure 5.12: Difference between the solution with an optimally controlled operation and the solution of a system with a battery always turned ON for $Q = Q_{\text{max}}$; parameters as in Figure 4.22, except $T = 10$.

The value of the system with no battery. Complementary to our assumptions that the battery is only charged in the surplus region and discharged in the deficit region, this effect can be explained by means of the following: the storage device needs to build-up potential from units of energy that are not sold after generation, the optimal operation will then accumulate potential at the expense of a lower income. From equation (5.11) we obtain a better potential as long as the coefficient of $\mathcal{L}$ is greater than the coefficient of $X$; this means that we can have a relative lower cost even in the case when these coefficients are negative. A system with no storage has the only possibility to sell surplus electricity to the market and this is the main reason for obtaining a better valuation at low prices of electricity as a system with no storage has no possibility to build-up a potential (the storage device accumulates potential that is transferred to the rest of the domain).

5.3.1 Parameters sensitivity analysis

In this section we present the sensitivity to variations in the input parameters, varying one at a time; we consider a base case with parameters $\sigma_X = 0.5$, $\sigma_Y = 0.5$, $\alpha = 0.1$, $\beta = 0.1$, $\lambda = 1$, $\kappa = 1$, $\theta = 5$, $\Theta = 5$, $r = 0.01$, $X_{\text{max}} = 25$, $Y_{\text{max}} = 20$, $Q_{\text{max}} = 1$ and $T = 10$. Each figure within this section comprises three plots. The first plot, at the
Figure 5.13: Comparison of the solution for an optimally controlled storage (solid lines) and the solution of a system with no battery (dashed line) at the mean level of energy generation, from top to bottom curves correspond to $Q = Q_{\text{max}}$, $Q = \frac{1}{2}Q_{\text{max}}$, and $Q = 0$; parameters as in Figure 4.22, except $T = 10$.

Figure 5.14: Difference between the solution with an optimally controlled operation and the solution of a system with no battery for $Q = 0$; parameters as in Figure 4.22, except $T = 10$. 
Figure 5.15: Difference between the solution with an optimally controlled operation and the solution of a system with no battery for $Q = \frac{1}{2}Q_{\text{max}}$; parameters as in Figure 4.22, except $T = 10$.

Figure 5.16: Difference between the solution with an optimally controlled operation and the solution of a system with no battery for $Q = Q_{\text{max}}$; parameters as in Figure 4.22, except $T = 10$. 
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top, presents the valuation of the system along the X direction at Y = 0 (showing the transferred potential to a zero spot price on the limit Y → 0). The second plot, in the middle, presents the valuation of the system along the Y direction at the mean level of energy generation $X = 0$. The third plot, at the bottom, presents the corresponding optimal charging and discharging rates for a half capacity storage $Q = \frac{1}{2}Q_{\text{max}}$ for a range of energy availability and spot prices for electricity.

For the first and second plot, showing the valuation of the system, we present with different line types the results obtained after varying one particular parameter, and for each parameter value there are two curves, the one above for a full storage ($Q = Q_{\text{max}}$) and the one below for an empty storage ($Q = 0$). On the plot showing optimal charging and discharging rates of the battery (at the bottom), the white region indicates the implementation of a maximum charging rate, the light grey region indicates a battery turned off ($L = 0$), the dark grey region indicates the implementation of a maximum discharging rate, and the faded regions are for the linearly increasing/decreasing rate of charge/discharge with respect to X.

**Remarkable results**

These are results showing a marked effect due to the optimal operation of the storage device. We consider this to be an advantage or improvement to the valuation because of the use of the optimal management of the rates for charging and discharging the storage device.

The effects of fluctuations on the energy generation, for selected $\sigma_X$ values, can be observed in Figure 5.17. There is a general reduction in the value of the system for higher volatility values $\sigma_X$, because of the bid-ask penalisation for relatively selling electricity at a lower price and buying electricity at a higher price. On the other hand for a full storage, lower volatility values $\sigma_X$ can reduce very slightly the value of the system for the surplus region $X > 0$, but improve the value in the deficit region $X < 0$, this because the narrow discharging region close to $X = 0$ projects towards low prices of electricity for a full storage, and then reducing the amount of energy in deficit. Along the mean level of energy generation $X = 0$ (see centre plot in Figure 5.17), an empty storage receives a better valuation (close to the constant income) with lower $\sigma_X$ values, as there is a small chance to deviate from the mean generation level but a full storage
Figure 5.17: Solution at $Y = 0$ (top), solution at $X = 0$ (centre) and optimal charging and discharging rates at $Q = \frac{1}{2}Q_{\text{max}}$ (bottom), for indicated $\sigma_X$ values. Base case parameters are: $\sigma_X = 0.5$, $\sigma_Y = 0.5$, $\alpha = 0.1$, $\beta = 0.1$, $\lambda = 1$, $\kappa = 1$, $\theta = 5$, $\Theta = 5$, $r = 0.01$, $X_{\text{max}} = 25$, $Y_{\text{max}} = 20$, $Q_{\text{max}} = 1$ and $T = 10$. 
receives a lower valuation, as then no advantage is taken at high prices of electricity. With these results we clearly identify the importance of considering the resource’s random nature, large diffusion coefficients increase the potential income (outcome), whereas small diffusion coefficients reduce potential income (outcome). The charging region of the battery exhibits a constant response with respect to the electricity price for a low $\sigma_X$ value, and the discharging region exhibits a sharp curvature, and almost constant response with respect to the electricity price, and thus a sharp transition between these regions. On the other hand a high $\sigma_X$ value produce a smooth transition between the charging and discharging regions.

In comparison with the results from the fixed operation of the storage device (see Figure 4.31, the optimal operation of the battery can improve the value of the system with a full battery, and always the optimal operation generates a better valuation that the strategy with a fixed control at $\theta$. Still surplus electricity is sold and purchased at penalised prices and then higher diffusion coefficients produce reduced valuation of the system.

The effects of the electricity spot price fluctuation are presented in Figure 5.18. Even if there is a zero coefficient ($Y = 0$) in the reduced PDE (4.13), some potential income is transferred with higher $\sigma_Y$ values (see top and centre plots on Figure 5.18). The charging region remains basically the same with varying $\sigma_Y$, but with smaller volatility values $\sigma_Y$, the discharging region is increased towards (or even further than) the long-term value of electricity; an opposite effect in the discharging region is observed with a higher $\sigma_Y$ value.

In comparison with the results from the fixed operation of the storage device (see Figure 4.34, the optimal operation of the battery improves the value of the system with larger diffusion coefficients, thus taking advantage from conveniently discharging stored energy.

Figure 5.19 shows the effects for selected $\lambda$ values. In practice this can represent a technical restriction on the battery for amounts of energy that can be injected or extracted. Smaller values in $\lambda$ reduce the benefits from discharging the battery (see the curve for $Q = Q_{\text{max}}$ in Figure 5.19), and there is also a reduction in value for the empty storage ($Q = 0$) for charging the battery. Significantly, there is a large reduction in value for a full storage, as potentially smaller amounts of energy from
Figure 5.18: Solution at $Y = 0$ (top), solution at $X = 0$ (centre) and optimal charging and discharging rates at $Q = \frac{1}{2}Q_{\text{max}}$ (bottom), for different $\sigma_Y$ values; parameters as in Figure 5.17.
Figure 5.19: Solution at $Y = 0$ (top), solution at $X = 0$ (centre) and optimal charging and discharging rates at $Q = \frac{1}{2}Q_{\text{max}}$ (bottom), for varying $\lambda$ as indicated; parameters as in Figure 5.17.
the storage device will be used to offset future deficits. The charging region is slightly reduced with lower $\lambda$ values, but the discharging region is increased.

From Figure 5.20, we identify the effects of larger storages. With an optimal operation of the storage device we observe a higher valuation of the system in the deficit region ($X < 0$) and then an increment in value for larger storages. In this case with larger storages, the charging region is reduced, but the discharging region is increased. The technical rates for operating the storage and the capacity of the storage device are included within this section, to verify that with the optimal management of storage both factors describe equivalent effects; more favorable rates operation describe equivalent effects as the optimal management of storage.

In Figure 5.21 we observe the effects from the magnitude of the penalisation with selling electricity, at bid price $Y(1 - \beta)$, and buying electricity, at ask price $Y(1 + \beta)$ from the market. There is an overall increment in value of the system for low $\beta$ values, conversely the value decreases for high $\beta$ values. This affects the slope of the valuation along the $X$ direction, so that with larger $\beta$ values, benefits are reduced in the surplus region $X > 0$, and costs are increased in the deficit region $X < 0$; the opposite effect is observed with small $\beta$ values (see Figure 5.21). With a high $\beta$ value we observe a ‘backup’ effect by increasing the charging region and not modifying the discharging region as it will be more convenient to recharge the storage for potentially entering into deficit under this highly penalised condition.

**Expected results**

These are effects on the solution that can be anticipated because of changing the parameter values of the PDE. These include rotations, translations, scaling or a combination of these effects on the solution.

Here we have the effects for varying the speed of mean reversion for energy availability $\alpha$, see Figure 5.22. A low value for $\alpha$ indicates a slow return towards the mean level of energy production and a high value for $\alpha$ indicates a fast return to the mean level of energy production. With a low $\alpha$ value the valuation of the system is generally increased in the surplus region but generally reduced in the deficit region, the opposite occurs for a high $\alpha$ value as the system’s value is generally reduced in the surplus region and generally increased in the deficit region (see top plot on Figure 5.22), but
Figure 5.20: Solution at $Y = 0$ (top), solution at $X = 0$ (centre) and optimal charging and discharging rates at $Q = \frac{1}{2}Q_{\text{max}}$ (bottom), for different storage capacities $Q_{\text{max}}$; parameters as in Figure 5.17.
Figure 5.21: Solution at $Y = 0$ (top), solution at $X = 0$ (centre) and optimal charging and discharging rates at $Q = \frac{1}{2}Q_{\text{max}}$ (bottom), for varying the penalisation $\beta$; parameters as in Figure 5.17.
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Figure 5.22: Solution at $Y = 0$ (top), solution at $X = 0$ (centre) and optimal charging and discharging rates at $Q = \frac{1}{2}Q_{\text{max}}$ (bottom), for varying $\alpha$ (shown).
even with these effects, at $X = 0$ (centre plot on Figure 5.22) the system receives an increment in valuation for higher $\alpha$ values for both the full and the empty storage; this because of the potential transferred by the optimal battery operation. With lower values of $\alpha$ we observe a reduced region for recharging the battery, as it will be likely to take advantage of electricity prices only at very low levels and more value is obtained from directly selling surplus energy, and the discharging region is slightly reduced. The regions for operating the battery are increased with higher $\alpha$ values, again with more sensitivity on the surplus region because of the $\alpha$ term in (5.12).

The results obtained for different values of $\kappa$ are shown in Figure 5.23. Low values for $\kappa$ indicate a slow return towards the long-term mean value of electricity, correspondingly higher values of $\kappa$ cause a fast return to the long-term mean value of electricity. At zero spot price the system’s value increases in the surplus region but decreases in the deficit region for a higher $\kappa$ value. This is because the electricity price is potentially pulled back to the mean value faster, whilst an opposite effect is observed with a lower $\kappa$ value. Along the $Y$ direction, at the mean energy production level ($X = 0$) we basically observe that a lower value of $\kappa$ produces a steeper slope for the curves, as excursions away from the mean value $\theta$ last longer; the opposite effect is obtained with higher values for $\kappa$. The charging region is moved towards the long-term mean value of electricity $\theta$ with higher $\kappa$ values. With higher $\kappa$ values we also observe a movement of the discharging region towards $\theta$, but there is a small reduction of the discharging region for small deficits of energy.

Figure 5.24 shows the effects on the solution for different long-term mean values for electricity $\theta$. At zero spot price $Y = 0$, we observe that with a smaller $\theta$ value the valuation is reduced in the surplus region, and increased in the deficit region; the opposite effect is produced with a higher $\theta$ value. On the mean level of energy generation we observe a better valuation of a full storage with higher $\theta$ values, but the value of the empty storage is notably reduced. The charging and discharging regions are basically displaced with respect to the new $\theta$ value.

Effects from different values of the constant riskless interest rate $r$ are shown in Figure 5.25. With a higher riskless interest rate $r$, lower valuations for the wind farm are obtained in the surplus region for energy generation and in parts of the deficit region (as potential is transferred with the battery operation), correspondingly lower costs in
Figure 5.23: Solution at $Y = 0$ (top), solution at $X = 0$ (centre) and optimal charging and discharging rates at $Q = \frac{1}{2}Q_{\max}$ (bottom), for different values of $\kappa$; parameters as in Figure 5.17.
Figure 5.24: Solution at $Y = 0$ (top), solution at $X = 0$ (centre) and optimal charging and discharging rates at $Q = \frac{1}{2} Q_{\text{max}}$ (bottom), for different $\theta$ values; parameters as in Figure 5.17.
Figure 5.25: Solution at $Y = 0$ (top), solution at $X = 0$ (centre) and optimal charging and discharging rates at $Q = \frac{1}{2}Q_{\max}$ (bottom), using indicated $r$ values; parameters as in Figure 5.17.
high deficit of energy are obtained; this can also be interpreted as a relative reduced uncertainty and a slightly opposite effect is observed with low riskless interest rate \( r \) values. With a larger \( r \) the charging and discharging regions are notably displaced to lower prices of electricity.

The effects of varying the wind farm timescale \( T \) are shown in Figure 5.26. With a longer lifespan of the project, higher valuations for the wind farm are obtained (as income is expected for a longer period of time) in the surplus region for energy generation and in parts of the deficit region (as potentially the battery can be discharged), correspondingly a higher cost is incurred for a high deficit of energy. The opposite effect is observed for a shorter lifespan of the wind farm. In this state, with a longer timescale \( T \) the charging and discharging regions are displaced to higher prices of electricity.

Figure 5.27 shows the effects on the solution for different payments \( \Theta \). As indicated in the previous chapter, this constant payment can represent fixed costs or income. Again, the plots in the figure indicate (numerically) that the resulting curves are shifted along the valuation axis \( V \), and the translation is proportional to the magnitude of \( \Theta \). As suggested by the term (4.62), the effects of modifying this last parameter are basically of a shifting of the net storage valuation by an amount \( \left( \Theta \frac{1 - e^{-rT}}{r} \right) \) along the valuation dimension, this is because of the constant nature of the payment \( \Theta \). The regions for charging and discharging remain basically the same, we can observe some differences but they are rather small.

### 5.3.2 Introduction of seasonality effects

In this section we assume that the risk-adjusted electricity spot price \( Y \) (5.1) incorporates seasonality in the form

\[
dY = \kappa(\mathscr{A}(t) - Y) \, dt + \sigma_Y \sqrt{Y} \, dW_1, \tag{5.16}
\]

\[
\mathscr{A}(t) = \mathscr{A}_0 + \psi_{SA}(\sin 2\pi(t - t_{SA})), \tag{5.17}
\]

where \( \kappa \) is the speed of mean-reversion towards the long-term mean price of electricity \( \mathscr{A}(t) \) that incorporates seasonality; \( \mathscr{A}_0 \) is then the long-term equilibrium price for electricity without seasonality effect. \( \sigma \) is the volatility of the electricity price, \( \psi_{SA} \) is the semiannual seasonality parameter indicating the (absolute) maximum deviation from
Figure 5.26: Solution at \( Y = 0 \) (top), solution at \( X = 0 \) (centre) and optimal charging and discharging rates at \( Q = \frac{1}{2}Q_{\text{max}} \) (bottom), for varying the lifespan \( T \); parameters as in Figure 5.17.
Figure 5.27: Solution at $Y = 0$ (top), solution at $X = 0$ (centre) and optimal charging and discharging rates at $Q = \frac{1}{2}Q_{\text{max}}$ (bottom), for varying $\Theta$ (shown); parameters as in Figure 5.17.
the long-term equilibrium price without seasonality, and $t_{SA}$ is a centering parameter for the seasonality effect indicating the equilibrium position.

In this formulation the long-term mean price for electricity $\mathcal{A}(t)$ fluctuates periodically around the long-term equilibrium price $\mathcal{A}_0$, in this case with a period of a one year ($t = 1$) (as for example described by Lucia and Schwartz, 2002 and Janczura et al., 2013) and exhibits one peak that correspond to high prices of electricity most likely to be in winter season. The $2\pi$ term can be adjusted to the form $2k\pi$ where $k$ is the number of seasonal peaks.

Here we search for the solution to the Hamilton-Jacobi-Bellman (HJB) equation, where we have made the substitution $\tau = T - t$

$$\max_{\mathcal{X}|\mathcal{Y}} \left[ -\frac{\partial V}{\partial \tau} + \mathcal{L} \frac{\partial V}{\partial Q} - \alpha X \frac{\partial V}{\partial X} + \kappa (\mathcal{A}(t) - Y) \frac{\partial V}{\partial Y} + \frac{1}{2} \sigma_X^2 \frac{\partial^2 V}{\partial X^2} + \frac{1}{2} \sigma_Y^2 Y \frac{\partial^2 V}{\partial Y^2} - rV + \Theta + (X - \mathcal{L}) Y (1 \pm \beta) \right] = 0, \quad \text{if } X \leq 0. \quad (5.18)$$

Figure 5.28 presents the results with the optimal operation of the storage device for a seasonal electricity spot price. Along the $Y$ direction, at the mean level of energy generation $X = 0$ there is some increase in the value of the system for a full storage, and a slightly reduction of the value of the system for an empty storage. We observe that there is potential transferred to lower levels of electricity prices for the region with surplus on energy generation, but we observe a reduction of the value of the system for deficits of energy generation. With a seasonal electricity price the discharging region is slightly reduced and the charging region is very slightly extended.

5.3.3 Introduction of correlation effects

Here we study the effect on the optimal operation of stochastic storage after introducing a correlation between the processes that are the source of randomness. We assume that the process (5.2) has been risk-adjusted and that there exists a correlation factor $\rho_{XY}$ between the processes $dW_1$ and $dW_2$ from the stochastic processes (5.1) and (5.2). The HJB (5.9) including a mixed derivative is obtained, and is written
Figure 5.28: Solution at $Y = 0$ (top), solution at $X = 0$ (centre) and optimal charging and discharging rates at $Q = \frac{1}{2}Q_{\text{max}}$ (bottom), introducing seasonality effect; parameters as in Figure 5.17 except $\mathcal{A}_0 = 5$, $\psi_{SA} = 1$, $t_{SA} = 0$. 
in forward parabolic form (in $\tau$), namely
\[
\max_{X,W} \left[ -\frac{\partial V}{\partial \tau} + \mathcal{L}\frac{\partial V}{\partial Q} - \alpha X \frac{\partial V}{\partial X} + \kappa(\theta - Y) \frac{\partial V}{\partial Y} + \frac{\sigma_X^2}{2} \frac{\partial^2 V}{\partial X^2} + \frac{\sigma_Y^2}{2} \frac{\partial^2 V}{\partial Y^2} \\
+ \rho_{XY} \sigma_X \sigma_Y \sqrt{Y} \frac{\partial^2 V}{\partial X \partial Y} - rV + \Theta + (X - \mathcal{L})Y(1 \pm \beta) \right] = 0, \quad \text{if } X \leq 0. \quad (5.19)
\]
Again, we make an explicit treatment of the term $(\partial^2 V)/(\partial X \partial Y)$ for every half-time step $\frac{1}{2} \Delta \tau$ of the SLADI$^{(3)}$ methodology (3.11); cross derivatives terms have been treated explicitly with ADI schemes (see for example Craig and Sneyd, 1988; In’t Hout and Welfert, 2009; In’t Hout and Welfert, 2007). The explicitly treatment of cross derivative terms makes the problem less amenable as we are solving a convection-diffusion PDE without diffusion in one dimension (namely $Q$) and subject to control (namely $\mathcal{L}$).

From Figure 5.29 we observe anticipated results, with a positive correlation coefficient there is an increase in the value of the system since more energy is available to move to the market when is more economically convenient; this is also observed for an extended discharging region and a reduced charging region. On the other hand with a negative correlation coefficient, the increased amount of energy causes a reduction of the system value and then the effect is observed with an extended charging region and a reduced discharging region. With respect to the convergence of the solution, we have observed difficulties for obtaining consistent control regions, but with smoother gradients of the solution we found less restrictions on the grid size that can be implemented.

5.4 A solution towards the steady state

In this section we present a numerical approximation of results towards the steady state for the optimal operation of a storage device. We implement our SLADI methodology as described in section 5.1 for $T \gg 1$. In particular we present results for an example with parameters $\sigma_X = 0.3$, $\sigma_Y = 0.2$, $\alpha = 0.05$, $\beta = 0.1$, $\lambda = 1$, $\kappa = 0.5$, $\theta = 5$, $\Theta = 5$, $r = 0.1$, $X_{\max} = 50$, $Y_{\max} = 25$, $Q_{\max} = 1$ and $T = 250$.

Figures 5.1, 5.2, and 5.3 present results towards the steady state for the rate of change of the valuation of the system ($V$) with respect to time ($\tau$) at a sample of points in the domain. As observed in these figures we have a ratio $\Delta V/\Delta \tau$ approaching the
Figure 5.29: Solution at $Y = 0$ (top), solution at $X = 0$ (centre) and optimal charging and discharging rates at $Q = \frac{1}{2}Q_{\text{max}}$ (bottom), introducing a correlation factor $\rho$ (shown); parameters as in Figure 5.17.
zero limit (in these calculations with a tolerance $O(1 \times 10^{-9})$). We observe overshooting in the ratio $\Delta V/\Delta \tau$ on the deficit region (see Figure 5.1), but these variations are rather small and perhaps are not in disagreement with the presence of non-linearity error due to the numerical grid not perfectly aligning with discontinuities (error observed in the convergence test carried in section 5.2).

Correspondingly, Figures 5.4, 5.5 and 5.6 present the value of the slope $\Delta V/\Delta X$ settling down towards the steady state. Because of the bid-ask difference $(1 \pm \beta)$, for a given level of electricity spot price, the steeper slopes are found at the domain extreme $X = X_{\text{min}}$, the lower values for this slope are found at the domain extreme $X = X_{\text{max}}$, and at the mean level of energy generation we observe a value in transition; there are increasing values for the slope with higher prices of electricity.

The resulting solution at $\tau = 250$ can be seen in Figures 5.7 and 5.8. Along the $X$ direction (see Figure 5.7) we observe the slope transition (diffusion) between the surplus region ($X > 0$) and the deficit region ($X < 0$) due to the ask-bid difference factors $(1 \pm \beta)$ for buying and selling electricity. We also observe an increased separation between the full storage ($Q = Q_{\text{max}}$) and the empty storage ($Q = 0$) curves with respect to an increase in electricity prices, these effects arising from the benefits of storage. At the mean level of energy generation $X = 0$ (see Figure 5.8) we have a higher valuation of higher prices of electricity for a full storage as a ‘call option’ on the
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Figure 5.2: Change in value $V$ ($\Delta V$) with respect to a small change in time $\tau$ ($\Delta \tau$); parameters as in Figure 5.1.

Figure 5.3: Change in value $V$ ($\Delta V$) with respect to a small change in time $\tau$ ($\Delta \tau$); parameters as in Figure 5.1.
Figure 5.4: $\frac{\Delta V}{\Delta X}$ as a function of time $\tau$; parameters as in Figure 5.1.

Figure 5.5: $\frac{\Delta V}{\Delta X}$ as a function of time $\tau$; parameters as in Figure 5.1.
price of electricity. On the other hand there is a lower valuation with higher values of electricity for an empty storage as a ‘put option’ on the price of electricity. On intermediate storage levels we observe a mixed effects as in a ‘financial straddle’.

The charging and discharging regions towards the steady state for a storage at half capacity ($Q = \frac{1}{2} Q_{\text{max}}$) are presented in Figure 5.9. Since the parameters selected have a relatively low variability, the steady state is achieved (quicker). The convenience of discharging energy is found at relatively low prices of electricity with respect to the long term mean price of electricity ($\theta = 5$ in this example) for the discharging region, and the convenience of storage is found at even lower prices of electricity with respect to $\theta$.

These calculations can be costly (computationally) when the objective is indeed finding a steady-state solution for a partial differential equation; efficient algorithms can be then defined for such purpose. In the appendix at the end of the chapter we present procedures for the numerical approximation of steady-state solutions for the optimal operation of a particular stochastic storage system subject to fixed demand and stochastic electricity price.
Figure 5.7: A solution towards the steady state along the $X$ direction for selected electricity spot prices $Y$, from top to bottom the curves correspond to $Q = Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 5.1.

Figure 5.8: A solution towards the steady state at the mean level of energy generation $X = 0$, from top to bottom the curves correspond to $Q = Q_{\text{max}}$, $Q = 0.5Q_{\text{max}}$ and $Q = 0$; parameters as in Figure 5.1.
Figure 5.9: Optimal charging and discharging rates towards the steady state for a storage at half capacity ($Q = \frac{1}{2}Q_{\text{max}}$); parameters as in Figure 5.1
5.5 Conclusions

In the present chapter we have defined an optimal control criteria for managing storage. We have verified that our boundary value problem formulation is consistent with the assumptions of the Strong Comparison Result, and then we can assume that a continuous and unique solution to the control problem was obtained, although we proceeded numerically to present results for convergence and stability.

We obtained enhanced valuations with respect to a regime with a fixed operation of storage and also with respect to a regime where a battery is always connected to the system. We generally obtained a better valuation from the optimal operation of the storage with respect to a system with no storage, but we identified a small region (at low electricity prices) where a system with no storage produces slightly better results; this can be explained in terms of the following, energy has to be stored in order to obtain a benefit after using those reserves, then a system with no storage will be slightly receiving a better valuation as this system will be selling electricity all the time, but the system with storage is accumulating potential to improve the solution for most of the domain.

We were able to obtain convergent solutions for the optimal control of stochastic storage systems subject to two sources of uncertainty, although the observed rate of convergence is of first-order. The solutions obtained for the optimal control of storage have smoother gradients than the solutions obtained with a fixed control operation of the storage. With smoother solutions, we observed less restrictions on the grid size and also on the time-step with the introduction of correlation effects and a seasonal spot price, another sensitivity observed is the difficulty to obtain consistent control regions.

Steady-state solutions were presented to provide numerical evidence about the stability of the SLADI methodology with a convergent solution in the long-run, these results also verify properties of the model about the stochastic dynamics admitting stationary distributions. The appendix verifies the production of steady-state solutions on fixed grids since a discontinuous control can be imposed to disjoint sets of grid points. We continue the research of suitable boundary conditions for a body-fitted coordinates technique; imposing optimality conditions on the free-boundary prevents
the method to converge because of the discontinuous control.

Notably, with the optimal operation of the storage device we have identified that some benefits can be obtained from the randomness of the energy generated, some potential can be transferred from the region with a surplus of energy to the region with a deficit of energy, but this is affected by the bid-ask difference for selling and purchasing electricity since this penalisation generally reduces the value of the system. We have observed that the variability of the electricity price significantly transfer potential from high prices to low prices of electricity. With an optimal operation the full capacity in the storage can be usefully converted into additional income, and the storage can be used to prevent adverse effects with high penalisation (bid-ask difference) of electricity prices.
Appendix

5.A Steady state solutions

In this appendix we present numerical results for the steady-state solution of a storage problem. We interpret this results as when the time scale for a project is large or the system will be running indefinitely.

Consider the following HJB equation

$$
\kappa(\theta - Y) \frac{\partial V}{\partial Y} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial Y^2} - rV + \max_{L} \left[ L \frac{\partial V}{\partial Q} - Y(D + L) \right] = 0.
$$

(5.20)

Here $D$ represents the demand of a fixed rate of energy, and the electricity spot price follows a risk-adjusted (or risk-neutral) Ornstein-Uhlenbeck stochastic process (see section 2.1.5)

$$
dY = \kappa(\theta - Y) \, dt + \sigma_Y \, dW,
$$

(5.21)

where $\kappa$ and $\sigma_Y$ are the speed of mean reversion and the volatility of the electricity spot price $Y(t)$, possibly taking negative values (we therefore select parameters that minimise the probability of zero and negative prices). We decide to use this different model for electricity spot prices to (slightly) relax boundary conditions, namely by imposing a linear solution on the extrema $X_{\text{min}}$ and $X_{\text{max}}$. The Ornstein-Uhlenbeck stochastic process has been used as a basis to develop electricity spot prices models (see for example Lucia and Schwartz, 2002; Benth et al., 2007; Barlow, 2002).

We assume here that the maximum rate $L$ of charge and discharge of the storage device is

$$
\begin{align*}
L_{\text{min}} &= -\min\{u_d, \lambda_d Q\}, \\
L_{\text{max}} &= \min\{u_c, \lambda_c (Q_{\text{max}} - Q)\},
\end{align*}
$$

(5.22)
where $u_d$ and $u_c$ are the maximum rates admitted by the storage device to be discharged and charged, correspondingly. $\lambda_d$ and $\lambda_c$ are smoothing constants that limit the discharging and charging rates correspondingly; with these rates being proportional to the amount of charge and the remaining capacity on the battery.

The boundary conditions implemented on the extrema of the $Y$ domain, i.e., $Y = 0$ and $Y = Y_{\text{max}}$ are reduced PDEs with the absence of diffusion and we impose

$$\kappa(\theta - Y) \frac{\partial V}{\partial Y} - rV + \max \left[ L \frac{\partial V}{\partial Q} - Y(D + L) \right] = 0,$$  \hspace{1em} (5.23)

we need to ensure that parameters are selected so that the errors introduced by this truncation domain are small.

When the battery is empty, $Q = 0$, it can no longer be discharged and then $L = 0$. When the battery is full, $Q = Q_{\text{max}}$, it can no longer be charged and again, $L = 0$, then the PDE reduces to

$$\kappa(\theta - Y) \frac{\partial V}{\partial Y} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial Y^2} - rV - YD = 0.$$  \hspace{1em} (5.24)

Maximisation is obtained from the terms involving the control variable $L$ in (5.20)

$$\max_L \left[ L \frac{\partial V}{\partial Q} - Y(D + L) \right].$$  \hspace{1em} (5.25)

Therefore, if $(\partial V/\partial Q) > Y$ energy is injected into the battery at the maximum rate of charge $L_{\text{max}}$, and if $(\partial V/\partial Q) < Y$ we empty the battery at the maximum discharging rate $L_{\text{min}}$. There exists then a free boundary where $(\partial V/\partial Q) = Y$ where we neither charge or discharge the battery.

### 5.A.1 A projected successive over-relaxation approach

We implement a PSOR strategy derived by Johnson (2008) to obtain a solution for the PDE (5.20) on a fixed finite-difference grid, this technique is selected to reduce the solution space by choosing the free-boundary that maximises the value of the solution. The spatial domain is discretised as follows: let $V(Y_j, Q_k)$ be the value of the system at $Y_j = j \Delta Y$ for $Q_k = k \Delta Q$, then in the iterative scheme $V_{j,k}^q$ corresponds to the solution
for the $q$-th iteration at the grid point $(j, k)$. Equation (5.20) is discretised as

$$
\kappa (\theta - Y_j) \frac{V_{j+1,k}^q - V_{j-1,k}^q}{2\Delta Y} + \frac{1}{2} \sigma_Y \frac{V_{j+1,k}^q - 2V_{j,k}^q + V_{j-1,k}^q}{\Delta Y^2} - rV_{j,k}^q \\
+ \max_{\mathcal{L}} \left[ \frac{V_{j,k}^q - V_{j,k}^{q+1}}{\Gamma} - Y_j(D + \mathcal{L}) \right] = 0, \quad (5.26)
$$

where the values $V_{j,k}$ denote $V(Y_j, Q_k)$ at the $q$-th iteration, and are determined by (natural cubic spline) interpolation on the location of $Q_k^q$ at the $q$-th iteration for $Y_j$ fixed,

$$Q_k^q = Q_k^{q+1} + \Gamma \mathcal{L}, \quad (5.27)
$$

here $\Gamma \leq 1/ \max[\lambda_c, \lambda_d]$ is time-step like relaxation parameter for the semi-Lagrangian approximation

$$\frac{V_{j,k}^{q+1} - V_{j,k}^q}{\Gamma} = \frac{V_{j,k}^{q+1} - V_{j,k}^q}{\Gamma} + \frac{Q_k^{q+1} - Q_k^q}{\Gamma} \frac{\partial V}{\partial Q},
$$

where

$$\frac{Q_k^{q+1} - Q_k^q}{\Gamma} = -\mathcal{L}.
$$

The iterative scheme is terminated when either the difference $|V_{j,k}^{q+1} - V_{j,k}^q|$ or $|V_{j,k}^{q+1} - V_{j,k}^q|$ is smaller than a prespecified tolerance.

We assume the existence of a free boundary $Y_f(Q)$ such that if $Y < Y_f$ energy is injected into the storage device at the allowed maximum rate of charge $\mathcal{L}_{\text{max}}$, whereas if $Y > Y_f$ energy is taken out from the battery at the allowed maximum discharging rate $\mathcal{L}_{\text{min}}$.

From the calculated values on the final iteration, we use the approximation of $\Delta V/\Delta Q = Y$ (using linear interpolation) to find the implied approximate location of $Y_f$.

We compare with a geometrical point of view, for which we define two regions, $H_c$ contains the resulting grid points where the battery is charged using the maximum charging rate $\mathcal{L}_{\text{max}}$ and $H_d$ contains the resulting grid points where the battery is discharged using the maximum discharging rate $\mathcal{L}_{\text{min}}$, we assume that $Y_f$ can be approximated by the average value between the two boundaries that are located next to each other on the regions.

Table 5.A.1 presents a sample of results obtained for the location of the free boundary after four refinements. The solution does not present a uniform rate of convergence.
Table 5.A.1: Free boundary location.

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<th>Geometrical</th>
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<td></td>
</tr>
<tr>
<td></td>
<td>1.18897883 [-0.2]</td>
<td>1.18887500 [ n.a. ]</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>1.16850738 [ n.a. ]</td>
<td>1.17000000 [ n.a. ]</td>
<td></td>
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<tr>
<td></td>
<td>1.16888224 [ n.a. ]</td>
<td>1.16900000 [ n.a. ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.16885641 [-0.1]</td>
<td>1.16850000 [ 0.50 ]</td>
<td></td>
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<tr>
<td></td>
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<td>1.16875000 [-0.5]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.16889724 [-1.6]</td>
<td>1.16887500 [ 0.50 ]</td>
<td></td>
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<tr>
<td>1</td>
<td>1.14779079 [ n.a. ]</td>
<td>1.15000000 [ n.a. ]</td>
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<td></td>
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<td>1.14500000 [ n.a. ]</td>
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<tr>
<td></td>
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<td>1.14875000 [ 0.50 ]</td>
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</tr>
<tr>
<td></td>
<td>1.14872015 [-0.1]</td>
<td>1.14812500 [-0.5]</td>
<td></td>
</tr>
</tbody>
</table>

$R$, but an accuracy of four to five digits was obtained. Figure 5.A.1 shows the obtained location of the free-boundary $Y_f$ at the last refinement. Sample valuation results are presented in Figure 5.A.2, and sample delta-valuation results are presented in Figure 5.A.3.

5.A.2 A boundary-fitted approach

This section presents a motivation that illustrates the use of a boundary-fitted approach for solving the storage problem (5.20). The approach is under research, and some interesting results have been obtained already.

With the definition of $Y_f(Q)$ as the free-boundary between the regions for charging
Figure 5.A.1: Free boundary ($Y_f$) location on the final refinement; $\sigma_Y = 0.2$, $\lambda_c = \lambda_d = 2$, $\kappa = 0.1$, $\theta = 2.5$, $r = 0.01$, $Q_{\text{max}} = 1$, $Y_{\text{min}} = 0$, $Y_{\text{max}} = 4$, $u_c = u_d = 1$ and $\Gamma = 0.008$, with a tolerance of $1 \times 10^{-7}$ for the difference $|V_{j,k}^{q+1} - V_{j,k}^q|$.

Figure 5.A.2: Storage valuation $V$; parameters as in Figure 5.A.1.
and discharging the battery. We then make the change in variables

\[
\hat{Y}_1 = \frac{Y - Y_{\text{min}}}{Y_f - Y_{\text{min}}}, \quad \text{for } Y < Y_f,
\]

\[
\hat{Y}_2 = \frac{Y_{\text{max}} - Y}{Y_{\text{max}} - Y_f}, \quad \text{for } Y > Y_f,
\]

that imply

\[
\frac{\partial \hat{Y}_1}{\partial Y} = \frac{1}{Y_f - Y_{\text{min}}}, \quad \frac{\partial \hat{Y}_2}{\partial Y} = -\frac{1}{Y_{\text{max}} - Y_f},
\]

and

\[
\frac{\partial \hat{Y}_1}{\partial Q} = -\frac{\hat{Y}_1}{Y_f - Y_{\text{min}}} \frac{dY_f}{dQ}, \quad \frac{\partial \hat{Y}_2}{\partial Q} = \frac{\hat{Y}_2}{Y_{\text{max}} - Y_f} \frac{dY_f}{dQ}
\]

leading to the corresponding PDEs

\[
\frac{1}{Y_f - Y_{\text{min}}} \kappa \left( \theta - (Y_{\text{min}} + \hat{Y}_1(Y_f - Y_{\text{min}})) \right) \frac{\partial V}{\partial \hat{Y}_1} + \frac{\sigma^2}{2(Y_f - Y_{\text{min}})^2} \frac{\partial^2 V}{\partial \hat{Y}_1^2} - rV
\]

\[
+ \mathcal{L} \left( \frac{\partial V}{\partial Q} - \frac{\hat{Y}_1}{Y_f - Y_{\text{min}}} \frac{dY_f}{dQ} \frac{\partial V}{\partial \hat{Y}_1} \right) - (Y_{\text{min}} + \hat{Y}_1(Y_f - Y_{\text{min}}))(D + \mathcal{L}) = 0,
\]

for \( Y < Y_f \), \( \mathcal{L} = \min\{u_c, \lambda_c(Q_{\text{max}} - Q)\} \)  \( (5.30) \)
is imposed, which in terms of the transformed variables leads to

\[- \frac{1}{Y_{\text{max}} - Y_f} \kappa \left( \theta - (Y_{\text{max}} - \hat{Y}_2(Y_{\text{max}} - Y_f)) \right) \frac{\partial V}{\partial Y_2} + \frac{\sigma^2}{2(Y_{\text{max}} - Y_f)^2} \frac{\partial^2 V}{\partial Y_2^2} - rV \]

\[+ \mathcal{L} \left( \frac{\partial V}{\partial Q} + \frac{\hat{Y}_2}{Y_{\text{max}} - Y_f} \frac{dY_f}{dQ} \frac{\partial V}{\partial Y_2} \right) - (Y_{\text{max}} - \hat{Y}_2(Y_{\text{max}} - Y_f)) (D + \mathcal{L}) = 0, \]

for \( Y > Y_f \), \( \mathcal{L} = -\min\{u_d, \lambda_d Q\} \) \hspace{1cm} (5.31)

Substituting \( Y_f' = dY_f/dQ \) and rearranging terms

\[ \frac{1}{Y_f - Y_{\text{min}}} \left( \kappa \left( \theta - (Y_{\text{min}} + \hat{Y}_1(Y_f - Y_{\text{min}})) \right) - \mathcal{L} \hat{Y}_1 Y_f' \right) \frac{\partial V}{\partial Y_1} + \frac{\sigma^2}{2(Y_f - Y_{\text{min}})^2} \frac{\partial^2 V}{\partial Y_1^2} - rV \]

\[+ \mathcal{L} \frac{\partial V}{\partial Q} - (Y_{\text{min}} + \hat{Y}_1(Y_f - Y_{\text{min}})) (D + \mathcal{L}) = 0, \]

for \( Y < Y_f \), \( \mathcal{L} = \min\{u_c, \lambda_c (Q_{\text{max}} - Q)\} \) \hspace{1cm} (5.32)

and then at the free-boundary, \( Y_f(Q) \), the optimality condition \((\partial V/\partial Q) - Y = 0\) is imposed, which in terms of the transformed variables leads to

\[ \frac{\partial V}{\partial Q} - \frac{\hat{Y}_1 Y_f'}{(Y_f - Y_{\text{min}})} \frac{\partial V}{\partial Y_1} - (Y_{\text{min}} + \hat{Y}_1(Y_f - Y_{\text{min}})) = 0, \] \hspace{1cm} (5.34)

\[ \frac{\partial V}{\partial Q} + \frac{\hat{Y}_2 Y_f'}{(Y_{\text{max}} - Y_f)} \frac{\partial V}{\partial Y_2} - (Y_{\text{max}} - \hat{Y}_2(Y_{\text{max}} - Y_f)) = 0. \] \hspace{1cm} (5.35)

We set \( V_j = \hat{V}_j + \delta V_j \) and \( Y_f = \hat{Y}_f + \delta Y_f \), and discarding the \( \mathcal{O}(\delta^2) \) terms leads to

\[ \frac{1}{Y_f - Y_{\text{min}}} = \frac{1}{\hat{Y}_f - Y_{\text{min}}} \left( 1 - \frac{\delta Y_f}{\hat{Y}_f - Y_{\text{min}}} \right) + \ldots, \]

\[ \frac{1}{Y_{\text{max}} - Y_f} = \frac{1}{Y_{\text{max}} - \hat{Y}_f} \left( 1 + \frac{\delta Y_f}{Y_{\text{max}} - \hat{Y}_f} \right) + \ldots, \]

\[ \frac{1}{(Y_f - Y_{\text{min}})^2} = \frac{1}{(\hat{Y}_f - Y_{\text{min}})^2} \left( 1 - \frac{2\delta Y_f}{\hat{Y}_f - Y_{\text{min}}} \right) + \ldots, \]
\[
\frac{1}{(Y_{\text{max}} - Y_f)^2} = \frac{1}{(Y_{\text{max}} - \tilde{Y}_f)^2} \left( 1 + \frac{2\delta Y_f}{Y_{\text{max}} - \tilde{Y}_f} \right) + \cdots,
\]

\[ Y'_f = \frac{dY_f}{dQ} = \frac{\tilde{Y}_f^k + \delta Y_f^k - \tilde{Y}_f^{k-1}}{\Delta Q}, \quad \text{for a backward difference approximation,} \]

\[ Y'_f = \frac{dY_f}{dQ} = \frac{\tilde{Y}_f^{k+1} - (\tilde{Y}_f^k + \delta Y_f^k)}{\Delta Q}, \quad \text{for a forward difference approximation,} \]

\[
\frac{\partial V}{\partial Y_1} = \frac{1}{2} \left( \tilde{V}_{j+1}^k + \delta V_{j+1}^k - (\tilde{V}_{j-1}^k + \delta V_{j-1}^k) + \frac{\tilde{V}_{j+1}^{k+1} - \tilde{V}_{j-1}^{k-1}}{2\Delta Y_1} \right),
\]

\[
\frac{\partial V}{\partial Y_2} = \frac{1}{2} \left( \tilde{V}_{j+1}^k + \delta V_{j+1}^k - (\tilde{V}_{j-1}^k + \delta V_{j-1}^k) + \frac{\tilde{V}_{j+1}^{k+1} - \tilde{V}_{j-1}^{k-1}}{2\Delta Y_2} \right),
\]

\[
\frac{\partial^2 V}{\partial Y_1^2} = \frac{1}{2} \left( \tilde{V}_j^k + \delta V_j^k - \tilde{V}_j^{k-1} \right) \frac{\Delta^2 Y_1}{}, \quad \text{for a backward difference approximation,} \]

\[
\frac{\partial^2 V}{\partial Y_2^2} = \frac{1}{2} \left( \tilde{V}_j^k + \delta V_j^k - \tilde{V}_j^{k-1} \right) \frac{\Delta^2 Y_2}{}, \quad \text{for a forward difference approximation,} \]

\[
-\rho V = -r \frac{1}{2} \left( \tilde{V}_j^k + \delta V_j^k + \tilde{V}_j^{k-1} \right), \quad \text{for a backward approximation,} \]

\[
-\rho V = -r \frac{1}{2} \left( \tilde{V}_j^k + \delta V_j^k + \tilde{V}_j^{k+1} \right), \quad \text{for a forward approximation.} \]

We then solve a system of equations of the form

\[
\alpha^k \delta V_{j-1}^k + \beta^k \delta V_j^k + \gamma_j^k \delta V_{j+1}^k + \rho_j^k \delta Y_f^k + \epsilon_j^k = 0.
\]

For the region where \( \tilde{Y}_1 \leq Y_f \)

\[
\alpha_j^k = -\frac{A_{j}^{1,k}}{4\Delta Y_1 (Y_f^j - Y_{\text{min}})} + \frac{\sigma^2}{4\Delta^2 Y_1 (Y_f^j - Y_{\text{min}})^2}, \quad (5.36)
\]

\[
\beta_j^k = -\frac{\sigma^2}{2\Delta^2 Y_1 (Y_f^j - Y_{\text{min}})^2} - \frac{1}{2r} - \frac{\mathcal{L}}{\Delta Q}, \quad (5.37)
\]

\[
\gamma_j^k = -\frac{A_{j}^{1,k}}{4\Delta Y_1 (Y_f^j - Y_{\text{min}})} + \frac{\sigma^2}{4\Delta^2 Y_1 (Y_f^j - Y_{\text{min}})^2}, \quad (5.38)
\]

\[
\rho_j^k = \frac{A_{j}^{2,k}}{2(Y_f^j - Y_{\text{min}})} \left( \tilde{Y}_1 \left( \frac{\mathcal{L}}{\Delta Q} - \kappa \right) - \frac{A_{j}^{1,k}}{Y_f^j - Y_{\text{min}}} \right) - \frac{A_{j}^{3,k} \sigma^2}{2(Y_f^j - Y_{\text{min}})^3} - \tilde{Y}_1 (D + \mathcal{L}), \quad (5.39)
\]
For the region where $\hat{Y}_2 > Y_f$

\[
\alpha_j^k = -\frac{A_6^{j,k} A_7^{j,k}}{2(Y_{\text{max}} - \hat{Y}_f^k)} + \frac{\sigma^2}{4(Y_{\text{max}} - \hat{Y}_f^k)^2},
\]

\[
\beta_j^k = -\frac{\sigma^2}{2\Delta^2 Y_2 (Y_{\text{max}} - \hat{Y}_f^k)^2} - \frac{1}{2} r + \frac{\mathcal{L}}{\Delta Q},
\]

\[
\gamma_j^k = \frac{A_5^{j,k}}{4\Delta^2 Y_2 (Y_{\text{max}} - \hat{Y}_f^k)} + \frac{\sigma^2}{4\Delta^2 Y_2 (Y_{\text{max}} - \hat{Y}_f^k)^2},
\]

\[
\rho_j^k = -\frac{A_5^{j,k} A_6^{j,k}}{2(Y_{\text{max}} - \hat{Y}_f^k)} \left( \hat{Y}_2 \left( \frac{\mathcal{L}}{\Delta Q} + \kappa \right) + \frac{A_5^{j,k}}{Y_{\text{max}} - \hat{Y}_f^k} \right) + \frac{A_7^{j,k} \sigma^2}{2(Y_{\text{max}} - \hat{Y}_f^k)^3} - \hat{Y}_2 (D + \mathcal{L}),
\]

\[
\epsilon_j^k = \frac{A_5^{j,k} A_6^{j,k}}{2(Y_{\text{max}} - \hat{Y}_f^k)} + \frac{A_7^{j,k} \sigma^2}{4(Y_{\text{max}} - \hat{Y}_f^k)^2} + A_8^{j,k},
\]

where the coefficients $A_5, A_6, A_7,$ and $A_8$ are calculated as

\[
A_5^{j,k} = \mathcal{L} \hat{Y}_2 \left( \frac{\hat{Y}_f^k - \hat{Y}_f^{k-1}}{\Delta Q} \right) - \kappa \left( \theta - (Y_{\text{max}} - \hat{Y}_2 (Y_{\text{max}} - \hat{Y}_f^k)) \right),
\]

\[
A_6^k = \frac{\hat{Y}_f^k - \hat{Y}_f^{k-1}}{2\Delta Y_2} + \frac{\hat{Y}_f^{k-1} - \hat{Y}_f^{k-1}}{2\Delta Y_2},
\]

\[
A_7^k = \frac{\hat{Y}_f^k - 2\hat{Y}_f^{k-1} + \hat{Y}_f^{k-1}}{\Delta^2 Y_2} + \frac{\hat{Y}_f^{k-1} - 2\hat{Y}_f^{k-1} + \hat{Y}_f^{k-1}}{\Delta^2 Y_2},
\]

\[
A_8^{j,k} = -\frac{1}{2} \left( \hat{Y}_f^k - \hat{Y}_f^{k-1} \right) + \mathcal{L} \left( \frac{\hat{Y}_f^k - \hat{Y}_f^{k-1}}{\Delta Q} \right) - (D + \mathcal{L}) (Y_{\text{max}} - \hat{Y}_2 (Y_{\text{max}} - \hat{Y}_f^k)).
\]
The two regions match at \( \hat{Y}_1 = \hat{Y}_2 = 1 \) and a picture for the system is

\[
\begin{pmatrix}
\beta_1 & \gamma_1 & 0 & \rho_1 \\
\alpha_2 & \beta_2 & \gamma_2 & \rho_2 \\
& \ddots & \ddots & \ddots \\
& & \alpha_{n-1} & \beta_{n-1} & \gamma_{n-1} & \rho_{n-1} \\
& & \alpha_n & \beta_n & \rho_n \\
& & \beta_{n+1} & \alpha_{n+1} & \rho_{n+1} \\
& & \gamma_{n+2} & \beta_{n+2} & \alpha_{n+2} & \rho_{n+2} \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & \gamma_{2n-1} & \beta_{2n-1} & \alpha_{2n-1} \rho_{2n-1} \\
0 & & & & \gamma_{2n} & \beta_{2n} & \rho_{2n}
\end{pmatrix}
\begin{pmatrix}
\delta V_1 \\
\delta V_2 \\
\vdots \\
\delta V_{n-1} \\
\delta V_n \\
\delta V_{n+1} \\
\vdots \\
\delta V_{2n-1} \\
\delta Y_f
\end{pmatrix}
= \begin{pmatrix}
\epsilon_1 \\
\epsilon_2 \\
\vdots \\
\epsilon_{n-1} \\
\epsilon_n \\
\epsilon_{n+1} \\
\vdots \\
\epsilon_{2n-1} \\
\epsilon_{2n}
\end{pmatrix}
\tag{5.46}
\]

which will be solved using Gaussian reductions

\[
\gamma'_i = \begin{cases} 
\gamma_i / \beta_i & \text{for } i = 1 \\
\gamma_i / (\beta_i - \gamma'_{i-1} \alpha_i) & \text{for } i = 2, \ldots, n - 1
\end{cases}
\]

\[
\rho'_i = \begin{cases} 
\rho_i / \beta_i & \text{for } i = 1 \\
(\rho_i - \rho'_i \alpha_i) / (\beta_i - \gamma'_{i-1} \alpha_i) & \text{for } i = 2, \ldots, n \\
\rho_i - \rho'_{i-1} \beta_i & \text{for } i = n + 1 \\
\rho_i - \rho'_{i-1} \beta_i / \alpha_{i-1} - \rho'_{i-2} \gamma_i & \text{for } i = n + 2, \ldots, 2n
\end{cases}
\]

\[
\epsilon'_i = \begin{cases} 
\epsilon_i / \beta_i & \text{for } i = 1 \\
(\epsilon_i - \epsilon'_i \alpha_i) / (\beta_i - \gamma'_{i-1} \alpha_i) & \text{for } i = 2, \ldots, n \\
\epsilon_i - \epsilon'_{i-1} \beta_i & \text{for } i = n + 1 \\
\epsilon_i - \epsilon'_{i-1} \beta_i / \alpha_{i-1} - \epsilon'_{i-2} \gamma_i & \text{for } i = n + 2, \ldots, 2n
\end{cases}
\]

To obtain (by backward substitution) the solutions

\[
\delta Y_f = \epsilon'_i / \rho'_i, \quad \text{for } i = 2n
\]

\[
\delta V_i = \epsilon'_i - \rho'_i \delta Y_f, \quad \text{for } i = 2n - 1, \ldots, n
\]

\[
\delta V_i = \epsilon'_i - \rho'_i \delta Y_f - \gamma'_i \delta V_{i+1}, \quad \text{for } i = n - 1, \ldots, 1
\]

The general case for PDE (5.20) is still under research as the system of equations produce instabilities and the procedure does not converge.
Figure 5.A.4: Solution $V$ with a linear behavior in $\partial V/\partial Q$; $\sigma Y = 0.2$, $\lambda_c = \lambda_d = 1$, $\kappa = 0.3$, $\theta = 2.5$, $r = 0.01$, $Q_{\max} = 1$, $Y_{\min} = 0$, $Y_{\max} = 4$, $u_c = u_d = 10$.

Briefly we present a simplified case with $u_c \gg \lambda_c(Q_{\max} - Q)$, $u_d \gg \lambda_d Q$, and $\lambda_c = \lambda_d = 1$. Which allow a linear solution along the $Q$ direction for $V \approx A(Y) + B(Y)Q$, where $B(Y) = Y$ at $Y_f$. For this example the scheme converged, and sample results are shown in Figures 5.A.4 and 5.A.5

The PSOR methodology can produce numerical results because the value for $L$ is imposed on every mesh point of a fixed grid, and the discontinuous definition of $L$ is implemented by dividing the full mesh into two disjoint sets, and convergence (with respect a particular tolerance) is attained. In the case of boundary fitted coordinates, the lack of convergence can be attributed to the impossibility to implement a discontinuous $L$ on the adjusted grid, as the boundary at $Y_f$ would require to have simultaneously imposed a regime where $L = L_{\min} = L_{\max}$, this being the cause of instabilities.
Figure 5.A.5: Linear results for the approximation of $\partial V/\partial Q$; parameters as in Figure 5.A.4.
Chapter 6

Conclusions and Further Work

Effective and efficient hybrid numerical methods have been developed and implemented for approximating the solution of advection-diffusion PDEs without diffusion in one dimension; detailed implementations are presented and the quality of the results is demonstrated. The literature indicates that the related PDEs are difficult to be solved accurately because of the absence of diffusion in one dimension, indeed standard finite-differences methods are prone to oscillations and also produce larger errors. Semi-Lagrangian methods combined with finite-difference methods were found to significantly reduce the error and produce solutions free of oscillations. The literature also indicates difficulties in solving the related PDEs when the magnitude of the coefficients for first-order terms is larger that the magnitude of coefficients for second-order terms, results indicate that indeed this condition generates slightly larger errors, but more significant sensitivities were found with respect to the magnitude of the coefficient for the second-order term and the smaller this coefficient the more difficult is to obtain an accurate solution.

For the valuation of the fixed-strike Asian option, we have compared our results with respect high precision values (ten digits accurate) obtained with spectral expansions techniques (Linetsky, 2004). We find that semi-Lagrangian methods can match five significant digits of accuracy for low volatility examples, and for larger volatility coefficients about eight significant digits of accuracy were obtained from directly solving the PDE with high resolution grids (low resolution grids reduce about two significant digits of accuracy), furthermore about ten digits of accuracy were obtained (accuracy is added with added computing time) after Richardson extrapolation. The
effectiveness and efficiency of hybrid semi-Lagrangian and finite-difference methods are demonstrated for solving advection-diffusion problems in two and three dimensions.

With respect to the boundary value problem formulation for Asian options, still all the boundary conditions for the problem in three dimensions have not been defined exactly for the bounded domain. Namely the boundary at $S_{\text{max}}$ has only been approximate (Hugger, 2006) for the fixed-strike Asian option in the full-dimensional model. As far as the research has been conducted, none was found for the floating-strike Asian option valuation problem in three dimensions for the boundary at $S_{\text{max}}$. A derivation or an approximation will be important.

We formulated an alternative (and novel) similarity reduction PDE model for valuing the Asian option with fixed strike and we find that it is equivalent to that obtained with the Martingale technique. A comprehensive set of boundary conditions have been compiled. The merits of the PDE approach for developing and solving the models are the ease for accessibility and implementation, with demonstrated quality of the results. Thus we have proceeded to extend this techniques to treat problems with an additional source of randomness.

We have presented an efficient (numerical) methodology for valuing a four-dimensional electricity storage option depending on time, random energy generation, stochastic electricity price and capacity of the storage device; hyperbolic terms were treated with a semi-Lagrangian approach and diffusion terms were treated with an alternating-direction methodology. Our ‘SLADI’ methodologies require us to solve (only) tridiagonal systems of equations to efficiently value a storage option, and are shown to be cheaper than a more conventional approach based on Crank-Nicolson and fully implicit/GMRES methodologies. Extensive numerical experimentation confirms the stability and accuracy of these methods for solving the advection-diffusion problem, but sensitivities arise with the introduction of discontinuous control; non-uniform rates of convergence were identified as an effect of a “nonlinearity error” (from a numerical grid not perfectly aligning with discontinuities). Numerical evidence has been provided for convergence tests and also steady-state solutions have been provided for stability. When mixed-derivative terms are included in the problem (additionally to solving an advection-diffusion problem with no diffusion in one space dimension, with also control implementation) we found grid-size and time-step restrictions for implementing
the SLADI methodology, still approximate solutions were presented with significant accuracy.

Notably, with the proposed switching mechanism (for deciding when to stockpile, to keep in storage or to use the reserves), there is an overall increase in the valuation of the system, generally. Then, optimisation strategies were defined to increase storage values still further. These give versatility to implement optimal control policies or, if restricted, fixed control policies. With efficient valuation now at hand, varied dynamics and more accurate models can be studied.

We have accomplished our objective of defining optimal storage policies for an upgraded wind farm (with a storage facility) under uncertain economic and physical factors. The more transcendental factors in the optimal operation of stochastic storage in our wind farm problem are: the volatility of energy generation; some potential can be transferred from the region with a surplus of energy to the region with a deficit of energy, but this is affected by the bid-ask difference for selling and purchasing electricity since this penalisation generally reduces the value of the system. The volatility of energy prices; a significant amount of potential can be transferred from high prices to low prices of electricity. The technical rates for operating the storage device and the capacity of the storage; the storage can be usefully converted into additional income. The penalisation on trading; the storage can be used to prevent adverse effects under highly penalised (bid-ask difference) electricity prices.

We have obtained some interesting steady-state results for the stochastic storage problem. We were able to obtain approximate results using a PSOR approach since the nature of a fixed grid can replicate the properties of a discontinuous control, but the boundary-fitted approach for the free-boundary problem is not able to admit a discontinuous control and the problem continues under research.

Future work is summarised as follows

- Determining the expense of a fixed-revenue contract (or alternative contract specification) and the proper value of each component in the system (electricity wind-generator, energy storage device, optimised charger).

- Evidence from the results indicates that failing to take into account the stochastic nature of any variable leads to an inaccurate valuation of the system that is being
represented, then the methodology can readily be applied to value systems where a true second source of stochasticity has been deemed as constant, see for example the electric power plants models in Thompson et al. (2004).

- Detailed dynamics for a wind farm project, we have used essential elements to develop a relevant (four-dimensional) model but for an industrial application it will be necessary to introduce specific processes and relationships, technical specifications and operational constraints.

- We found the need to continue the research with the inclusion of correlation between the stochastic dynamics, as the amount of renewable energy becomes more significant there is an impact on the prices for electricity which requires the algorithms to evolve and take into account this correlation effect. In’t Hout and Welfert (2007) have studied the stability for ADI schemes for solving advection-diffusion equations with mixed derivatives. We have obtained an approximation for this problem, but the SLADI methodology requires adjustments and corrections for treating mixed derivative terms (since we are already approximating an advection-diffusion problem with no diffusion in one dimension and with control implementation).

- Full stability and numerical analyses will strengthen the SLADI methodology applicability; in models including only first-order and second-order terms, we found that our SLADI methodology performs well and presents a stability that compares well with that observed for more conventional semi-Lagrangian techniques, but with the presence of mixed derivative terms we found restrictions on the grid size that can be implemented (extra refinement in the non-diffusive dimension and time have been implemented to approximate the solution).

- Research about convergence remedies to recover second-order convergence in the SLADI methodology.

- A projected successive over-relaxation technique coupled with semi-Lagrangian and alternating-direction implicit methodologies is a natural step forward for determining a steady-state solution for the optimal operation of a storage under
both uncertain economical and physical factors, as in our model for a wind farm with a backup battery.

- Continue the research on suitable conditions for implementing a boundary-fitted approach to obtain steady-state solutions for stochastic storage problems.

- Determining the optimal commitment is and will remain as an open problem, as there are hundreds of variables that potentially can be include in the mathematical models. This becomes intractable for a PDE approach but with the relevant factors selected accurate solutions are attainable.
Glossary

advection the movement of some material dissolved or suspended in a fluid.

advection-diffusion a process where simultaneously a material moves and spreads through or into a fluid by mixing with it.

algorithm specific set of instructions for carrying out a procedure or solving a problem.

alternating-direction implicit splitting operator approach in an $n$-dimensional diffusion space that solves a time step by dividing it into $n$ parts. In each fraction of time, only terms from one of the diffusion dimensions are implicitly calculated, while the terms from the other diffusion dimensions are treated explicitly, requiring the solution of (only) simple systems of equations.

appraising to estimate the monetary value of; determine the worth of.

boundary condition a specification of the values that a solution, or the derivative of a solution, is to take on the boundary of the domain.

boundary-fitted coordinates is the transformation of a curvilinear coordinate system into a linear coordinate system.

buffer an extra supply of materials in order to prevent a situation where none are available.

characteristic (characteristic curves or just characteristics) curves along which a PDE becomes an ordinary differential equation (ODE).

commodity an article of trade or commerce, especially a product as distinguished from a service.
correlation is the degree of linearly association between two or more quantities.

deficit a lack or shortage; deficiency.

deterministic that always produce the same output from a given starting condition or initial state.

differential equation is a mathematical equation that relates some function with its derivatives.

diffusion to spread through or into a surrounding substance by mixing with it.

diffusive characterised by spreading through or into a surrounding substance by mixing with it.

drift is the expected rate of growth of an asset price.

financial derivative (a derivative contract) is a financial contract whose fair value, at a point in time, depends on the value of one, or more, underlying assets.

finite-difference is a discrete approximation of the infinitesimal change of a function with respect to one of its variables.

fixed-strike (fixed-exercise price of an option) is a fixed amount for which the underlying asset can be traded at expiration.

floating-strike (floating-exercise price of an option) is the prevailing price of the underlying asset at expiration of the option at which the underlying asset can be traded.

hedging is an investment position intended to offset potential losses or gains that may be incurred by a companion investment.

hybrid (hybrid algorithm or hybrid method) is one combining two or more algorithms that solve the same or different problems, to combine the desired features of each, so that the overall algorithm is better than the individual components.

hyperbolic problem a processes that behave essentially like the movement of some material dissolved or suspended in a fluid.
market price of risk is the return in excess of the risk-free rate that the market 
wants as compensation for taking risk.

optimisation a mathematical technique for finding a maximum or minimum value 
of a function of several variables subject to a set of constraints.

parabolic problem a processes that behave essentially like heat diffusing through a 
solid.

prevailing existing in a particular place or at a particular time; generally current.

Real Option an alternative or choice that becomes available with a business invest-
ment opportunity.

risk-adjusted (risk-adjusted rate of return) is the total expected rate of return that 
an investor requires in order to own a project.

risk-free (risk-free rate of return) the rate given by an instrument when its future 
value is known with certainty.

risk-neutral is a term used to describe either a trading or pricing methodology which 
aims to be free of any risk, therefore the basis of comparison is the riskless rate 
of return.

riskless (riskless rate of return) the rate earned on an investment without risk as 
government bonds.

seasonal a quantity that presents periodic variations in time.

semi-Lagrangian the perspective of an observer that identifies how the flow of a flux 
evolved in order to reach the current position, thus determining the trajectory 
that had to be followed to arrive at the current position.

similarity reduction is a procedure by which it is possible to transform a partial 
differential equation having \( n \) variables into another with only \( n - 1 \) variables.

smooth (smooth function) is a function that has derivatives of all orders everywhere 
in its domain.
spectral expansion a technique involving eigenfunctions expansions, we refer the reader to Linetsky (2004).

spot price the price of a product that is being sold in a spot market.

steady-state state that is reached when numerous properties of a system are un-changing in time.

stochastic of or relating to a process involving a randomly determined sequence of observations each of which is considered as a sample of one element from a probability distribution.

stochastic differential equation is a differential equation in which one or more of the terms is a stochastic process.

stochasticity the property of a process involving a randomly determined sequence of observations each of which is considered as a sample of one element from a probability distribution.

stockpiling to accumulate (material, goods, or the like) for future use.

storage a place, as a room, to accumulate or put away a supply or stock of something, especially one for future use.

subduction zone a place at convergent boundaries by which one tectonic plate moves under another tectonic plate and sinks into the mantle as the plates converge.

successive over-relaxation iterative method to solve a linear system of equations by assigning an extra weight to the correction term.

surplus (an amount that is) more than is needed.

transient state state of a system where numerous properties are changing in time, and the system has not reached the steady-state.

tridiagonal a square matrix which non-zero values are located on the main diagonal, on the diagonal just below the main diagonal, and on the diagonal just above the main diagonal.
uncertain not clearly or precisely determined; indefinite; unknown.

underlying used to describe something on which something else is based.

valuation the act of estimating or setting the value of something; appraisal.

volatile likely to change suddenly and unexpectedly.

volatility is the degree of variation of a trading price series over time and corresponds to the standard deviation.

wind farm a group of wind turbines that are used for producing electricity.
Acronyms

**ADI** alternating-direction implicit.

**BDF** backward differencing.

**CIR** Cox-Ingersoll-Ross.

**CN** Crank-Nicolson.

**FI** fully implicit.

**HJB** Hamilton-Jacobi-Bellman.

**LSMC** Least Squares Monte Carlo.

**MC** Monte Carlo.

**OU** Ornstein-Uhlenbeck.

**PDE** partial differential equation.

**PIDE** partial integro-differential equation.

**SLADI** semi-Lagrangian alternating-direction implicit.

**SLCN** semi-Lagrangian Crank-Nicolson.

**SLFI** semi-Lagrangian fully implicit.

**TVD** total variation diminishing.
Bibliography


