QUANTUM FIELD THEORY
FOR THE EARLY UNIVERSE

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<td>$0\nu\beta\beta$</td>
<td>neutrinoless double beta decay</td>
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<td>1PI</td>
<td>1-particle irreducible</td>
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<td>2PI</td>
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<td>BAU</td>
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<td>CL</td>
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<td>EEV</td>
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<td>LFV</td>
<td>lepton flavour violation</td>
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<td>LH</td>
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<td>LHC</td>
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<td>LHS</td>
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<td>NWA</td>
<td>narrow-width approximation</td>
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<td>RG</td>
<td>renormalization group</td>
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<tr>
<td>RH</td>
<td>right-handed</td>
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<td>right-hand side</td>
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<td>RIS</td>
<td>real intermediate state</td>
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<td>RL</td>
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<td>SM</td>
<td>Standard Model</td>
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<td>UV</td>
<td>ultraviolet</td>
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<td>VEV</td>
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In this thesis we discuss the quantum field-theoretical techniques that allow a consistent, complete and unified description of the generation of the observed matter/antimatter asymmetry in the Early Universe, through the Resonant Leptogenesis mechanism. After reviewing the basic formalism of thermal field theory at equilibrium, we present the paradigm of leptogenesis, with particular emphasis on the resonant case, in which the presence of quasi-degenerate heavy Majorana neutrinos can enhance significantly the $CP$ asymmetry generated by their decays in the Early Universe.

After these introductory discussions, we develop a fully flavour-covariant formalism for transport phenomena, capable of describing the time-evolution of particle number-densities in a statistical ensemble with arbitrary flavour content. By using this formalism for a semi-classical analysis of Resonant Leptogenesis, in which the resummation of resonant heavy-neutrino absorptive transitions is performed effectively at zero temperature, we obtain the flavour-covariant rate equations, that provide a complete and unified description of this phenomenon, capturing three relevant physical effects: (i) the resonant mixing between the heavy-neutrino states, (ii) coherent oscillations between heavy-neutrino flavours, and (iii) quantum decoherence effects in the charged-lepton sector.

Subsequently, we discuss the consistent field-theoretical formulation of non-equilibrium phenomena and use it to develop a flavour-covariant analysis of Resonant Leptogenesis in a fully thermal-field-theoretical framework. This formalism allows us to confirm the results of the semi-classical analysis in a more first-principles approach, and provides the consistent thermal description of the resonant enhancement of the asymmetry due to heavy-neutrino mixing. Finally, we present an explicit model of Resonant Leptogenesis, with electroweak-scale Majorana neutrinos and observable signatures in current and near-future experiments. We study the predictions of this model by means of the flavour-covariant rate equations, and demonstrate the numerical significance of the formalism developed in this thesis for an accurate prediction of the baryon asymmetry generated in the Early Universe.
In the last fifty years, with the development of the Standard Model of particle physics, our understanding of nature has reached an unprecedented level. This beautiful achievement of humankind tells us about the fundamental laws that govern our Universe, in perfect agreement with an extraordinary number of experiments that have been built to test it, such as the Large Hadron Collider at CERN, in Geneva.

However, the very presence of matter in our Universe cannot be explained by the Standard Model alone. Matter and antimatter, as described by this theory, are completely symmetric, to a very good extent. In the Early Universe these would annihilate into radiation, leaving our Universe almost completely devoid of the ordinary matter, which we are instead made of. This, in addition to some other unexplained phenomena, often related to the Early Universe, tells us that there must exist physics beyond the Standard Model.

A huge experimental effort is being put in the search for such new physics. However, in order to understand precisely the predictions of new theories beyond the Standard Model, one often needs to progress in the formal description of these theories, developing new theoretical tools that allow to obtain accurate predictions in regimes particularly difficult to study, such as the Early Universe, with its extremely high temperatures and densities.

The so-called Resonant Leptogenesis is one of the most attractive mechanisms of new physics, capable of explaining the origin of the asymmetry between matter and antimatter in the Universe. Its attractiveness stems from the fact that this phenomenon is directly testable in current and near-future experiments, such as the Large Hadron Collider. At the same time, since its discovery twenty years ago, the theoretical description of this mechanism has revealed to be particularly challenging. The subject of this thesis is the development of the theoretical techniques that allow a consistent, complete and unified description of the Resonant Leptogenesis phenomenon in the Early Universe.
DECLARATION

I declare that no portion of this work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Signed: 

Date: 19th May 2015

[Signature: Daniele Ferani]
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The material discussed in this thesis is based on the following publications:


During my PhD, I have authored also the following works on related topics, not covered in this thesis:


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ai miei genitori, perché questa tesi è anche loro.

*to my parents, because this thesis is also theirs.*
Beauty is truth, truth beauty.

John Keats
CHAPTER

ONE

INTRODUCTION

The Standard Model (SM) of particle physics has provided an extraordinarily successful description of the fundamental phenomena of nature [1], bringing our understanding of the Universe to an unprecedented level. With the recent discovery of the Brout-Englert-Higgs boson at the Large Hadron Collider (LHC), all its constituents have been observed experimentally. However, we already have observational evidence for Beyond the Standard Model (BSM) physics, namely: dark matter and dark energy in the composition of the Universe, cosmic inflation (with its successful description of the Cosmic Microwave Background), neutrino oscillations and their non-zero masses and, most importantly to this thesis, the very presence of matter in the Universe, due to an asymmetry between matter and antimatter generated in the primordial evolution after inflation.

Most of these pieces of evidence are related to the evolution of the Universe in its primordial phase. Predictions about this evolution, and the related BSM physics, rely upon a successful quantum-field-theoretical (QFT) description of the content of the Universe in its primordial exotic state, characterized by extremely high temperatures and densities. In this thesis we discuss the QFT techniques that allow a consistent, complete and unified description of the generation of the observed Baryon Asymmetry of the Universe (BAU) in the Early Universe through the so-called Resonant Leptogenesis (RL) mechanism.

As anticipated above, the observed matter-antimatter asymmetry in the Universe [2, 3] and the observation of non-zero neutrino masses and mixing (for a review, see [1]) provide
two of the strongest experimental pieces of evidence for BSM physics. Leptogenesis [4] is an elegant framework that explains both phenomena in a single framework. According to the standard paradigm of leptogenesis (for reviews, see e.g. [5–8]), there exist heavy Majorana neutrinos in minimal extensions of the SM, whose out-of-equilibrium decays in an expanding Universe create a net excess of lepton number $L$, which is reprocessed into the observed baryon number $B$ through the equilibrated $(B + L)$-violating electroweak sphaleron interactions [9]. In addition, these heavy SM-singlet Majorana neutrinos $N_{\alpha}$ allow to explain the observed smallness of the light neutrino masses by the seesaw mechanism [10–14]. Hence, this phenomenon provides an attractive link between two seemingly disparate pieces of evidence for new physics at or above the electroweak scale.

In the original scenario of thermal leptogenesis [4], the heavy Majorana neutrino masses are typically close to the Grand Unified Theory (GUT) scale, $M_{\text{GUT}} \sim 2 \times 10^{16}$ GeV. In a vanilla leptogenesis scenario [15], where the heavy neutrino masses are hierarchical ($m_{N_1} \ll m_{N_2} < m_{N_3}$), the solar and atmospheric neutrino oscillation data impose a lower limit on $m_{N_1} \gtrsim 10^9$ GeV [16–19]. As a consequence, such leptogenesis models are difficult to test in foreseeable laboratory experiments. Moreover, these high-scale thermal leptogenesis scenarios, when embedded within supergravity models of inflation, could potentially lead to a conflict with the upper bound on the reheating temperature of the Universe, $T_R \lesssim 10^6$–$10^9$ GeV, required to avoid overproduction of gravitinos whose decays may otherwise spoil the success of Big Bang Nucleosynthesis (for a related discussion, see e.g. [20]).

The aforementioned problems may be solved within the framework of RL [21–23]. When two or more heavy Majorana neutrinos are nearly degenerate, the leptonic $CP$-asymmetry gets resonantly enhanced [21, 22, 24–26], thus allowing to lower the heavy-neutrino mass scale up to about the electroweak scale [27]. There is a vast literature on viable constructions of RL models (see [28] and references therein). However, since its discovery twenty years ago, this phenomenon has provided a constant theoretical challenge for its consistent QFT description, which has led to a significant progress in the
interrelated description of CP violation, mixing of unstable particles and non-equilibrium phenomena, in regimes particularly difficult to study theoretically.

Flavour effects play an important role in determining the final lepton asymmetry predicted by leptogenesis. These may be important both in the heavy-neutrino [29–34] and SM-lepton [35–40] sectors of the theory. However, only flavour-diagonal or partially-flavoured treatments were present in the literature, before the appearance of the material covered in this thesis in [28, 41–44]. A fully flavour-covariant formalism is required in order to consistently capture all the flavour effects present in the evolution of the system, that are relevant for the generation of the asymmetry. These include heavy-neutrino flavour mixing, oscillations and charged-lepton (de)coherence. These intrinsically quantum effects can be accounted for by extending the classical Boltzmann equations for particle number-densities, which are typically used to describe the evolution of the system in the Early Universe, to evolution equations for matrices of number-densities [45]. The consistent formulation of this flavour-covariant formalism, both in the semi-classical and thermal QFT approaches, constitutes the main subject of this thesis.

We should emphasize that our flavour-covariant formalism is rather general, and its applicability is not limited only to the RL phenomenon. This can be used, potentially, to study other physical phenomena, in which flavour effects may play an important role, such as the evolution of multiple jet flavours in a dense QCD medium in the quark-gluon plasma (see e.g. [46]), the evolution of neutrino flavours in a supernova core collapse (see e.g. [47]), or the scenario of CPT violation induced by the propagation of neutrinos in gravitational backgrounds [48].

The plan of the thesis is as follows. After this introduction, in Chapters 2 and 3 we review the QFT description of thermal phenomena at equilibrium. This provides the basis for the non-equilibrium QFT approach described later on in the thesis, and allows us to derive heuristically the Boltzmann equations typically used to study the statistical evolution in the Early Universe. In Chapter 4, we discuss the paradigm of leptogenesis, with a particular focus on the resonant case. There, we present the original description of this
phenomenon in terms of classical Boltzmann equations, and semi-classically effective $CP$-violating vertices. After these preliminary discussions, in Chapter 5 we develop the flavour-covariant formulation of transport phenomena, discussing also the role of flavour covariance for the discrete symmetries $C$, $P$ and $T$. Then, we derive the quantum transport equation that describes the evolution of particle number densities in the Markovian regime. In Chapter 6, we apply the flavour-covariant formalism to study RL in a semi-classical approach, in which the QFT resummation of absorptive transitions, leading to the resonant enhancement of the asymmetry, is performed semi-classically at zero temperature. In this chapter, we include all the effects that are important for an accurate description of the RL phenomenon, and obtain the flavour-covariant rate equations for RL, describing the generation of the asymmetry in the Early Universe. This semi-classical analysis has the advantage of being constructed with physical observables, i.e. the particle number-densities, in mind. However, a non-equilibrium QFT treatment is needed to go beyond this semi-classical approximation (with its related technical issues that will be discussed later on), at the same time confirming the results of Chapter 6 within a fully field-theoretical treatment. In Chapter 7, we present the field-theoretical ingredients needed for such consistent QFT formulation of transport phenomena. We develop the flavour-covariant generalization of this formalism in Chapter 8, where we use it to study RL. Within this field-theoretical framework, we confirm the results of the semi-classical approach of the previous chapters, and provide the consistent QFT analysis of the heavy-neutrino mixing phenomena in a thermal non-equilibrium framework. As anticipated at the beginning of this introduction, the results of Chapters 5, 6 and 8 provide a consistent, complete and unified description of the RL phenomenon. In Chapter 9, we present a particular RL model with electroweak-scale heavy neutrinos, that has observable signatures in current and near-future experiments both at the energy and intensity frontiers, and discuss the involved model-building and phenomenological aspects. There, we use the flavour-covariant rate equations to study the various effects involved in the generation of the asymmetry, and show the numerical significance of the formalism developed in this
thesis, for an accurate prediction of the final BAU. Finally, in Chapter 10 we summarize the results of this thesis and draw our conclusions.
EQUILIBRIUM THERMAL FIELD THEORY

In this chapter, we present the basic formalism used to describe quantum field-theoretical phenomena at finite temperature. Although we mainly work at thermodynamic equilibrium, the concepts introduced here will be important for more general non-equilibrium formulations, which will be described in Chapter 7. We first introduce, in an unified way, the two most popular formulations of thermal QFT at equilibrium, the so-called imaginary-time and real-time formalisms and, in strict relation to the latter, the closed-time path (CTP) formulation, which is the natural framework to develop consistent approaches to non-equilibrium thermal QFT.

2.1 Quantum Field Theory on functional contours

In the ordinary zero-temperature functional formulation of QFT, the time variable \( t \) is integrated from \(-\infty\) to \(+\infty\) along the real axis, in order to obtain amplitudes between the asymptotic \( in \) and the \( out \) vacuum states. In this way, by means of the Lehmann–Symanzik–Zimmermann (LSZ) formalism, one can extract the elements of the scattering matrix, and obtain predictions for scattering experiments.

However, in many situations one is interested in matrix elements between the same quantum state. This happens, for instance, when one wants to calculate the expectation value of a quantum observable in a given state, or also when one is interested in calculating
CHAPTER 2. EQUILIBRIUM THERMAL FIELD THEORY

thermal traces. In these situations, it is necessary to modify conveniently the contour of integration. This modified path-integral formulation provides the basis for a consistent formulation of thermal QFT, both in and out of equilibrium. Therefore, we allow the time coordinate \( t \) to be complex, integrating it along a given contour \( C \) in the complex plane.

2.1.1 Closed-time path formalism

First of all, we want to introduce the so-called closed-time path formalism [49]. Let us suppose that we want to calculate the expectation value of a scalar field \( \phi_H(x_1) \) in an eigenstate \( |\phi_A(x)\rangle \) of the field in the Heisenberg picture:

\[
\langle \phi_A | \phi_H(t_1) | \phi_A \rangle = \langle \phi_A(-T) | e^{-iH(-T)} e^{iH_1} \phi_S e^{-iH_1} e^{iH(-T)} | \phi_A(-T) \rangle .
\] (2.1)

Here, \( \phi_S \) and the states with time dependence are meant to be in the Schrödinger picture, we have suppressed the spatial dependence of states and operators and \( x_1 = (t_1, x_1) \). We take the quantum-mechanical Schrödinger, Heisenberg and interaction pictures to coincide at \( t_{in} = -T \). Finally, the Hamiltonian of the system is denoted by \( H \). Inserting a decomposition of the identity at the time \( t_{1} \), (2.1) becomes

\[
\int D\phi_1 \phi_1 \langle \phi_A(-T) | e^{-iH(-T-t_1)} | \phi_1(t_1) \rangle \langle \phi_1(t_1) | e^{-iH(t_1+T)} | \phi_A(-T) \rangle .
\] (2.2)

Both amplitudes have a path integral representation, from the state \( \phi_A \) at the time \(-T\) to the state \( \phi_1 \) at \( t_1 \) and then from \( \phi_1 \) at \( t_1 \) to \( \phi_\alpha \) at \(-T\) again. Taking into account the integration over the intermediate state \( \phi_1 \), we can express the expectation value as a single path integral:

\[
\langle \phi_A | \phi_H(x_1) | \phi_A \rangle = \int D\phi(x) \phi(x_1) e^{i \int_C d^4x L} ,
\] (2.3)

where the time is integrated on a contour \( C \) (see Figure 2.1) that runs from \(-T\) to some given point (for example \( T \)) and then back to \(-T\), and that must contain the time \( t_1 \). In general, one takes the limit \( T \to \infty \). Clearly, the above expressions can be generalized to
2.1. QUANTUM FIELD THEORY ON FUNCTIONAL CONTOURS

Figure 2.1: Closed-time path contour

give the expectation value of a time-ordered product of operators, as long as they live on
the first line $C_1$, which runs from $-\infty$ to $+\infty$. If, instead, the operators are taken on the
second line $C_2$, the path-integral on $C$ gives the expectation value of the anti-time-ordered
product.

This should be contrasted with the ordinary in-out formulation, which allows to cal-
culate the amplitude of time-ordered products between in and out states. This is achieved
by integrating the time from $-T$ to $+T$. Typically, one projects the in and out states onto
the asymptotic vacua by taking the limit $T \to \infty (1 - i \epsilon)$. As we will see in Chapter 7,
the CTP contour is particularly useful in the description of non-equilibrium phenomena.

In the rest of this chapter, instead, we will be focused on the equilibrium case.

2.1.2 Thermal equilibrium: imaginary-time contour

Let us consider the thermal average of an operator $A$ for a system at thermal equilib-
rium. Let us denote the inverse temperature by $\beta = 1/T$ (here $T$ is the temperature, which
should not be confused with the boundary of the integration contour above). The canonical
ensemble is described by the density matrix $e^{-\beta H}/Z(\beta)$, where $Z(\beta) = \text{Tr} e^{-\beta H}$ is
the partition function. The thermal expectation value of $A$ is given by

$$
\langle A \rangle_{\beta} = \frac{1}{Z(\beta)} \text{Tr}(e^{-\beta H} A) = \frac{1}{Z(\beta)} \sum_{\phi(x)} \langle \phi(t, x) | e^{-\beta H} A | \phi(t, x) \rangle
$$

$$
= \frac{1}{Z(\beta)} \sum_{\phi(x)} \langle \phi(t - i \beta, x) | A | \phi(t, x) \rangle ,
$$

(2.4)
since the Boltzmann operator can be considered as an evolution operator in imaginary
time. Here, $|\phi(t, x)\rangle$ denotes the eigenstate of the field operator in the Heisenberg picture,
with eigenvalue $\phi(t, x)$. Thus, we need to consider a functional integral in which the time
is integrated from $t$ to $t - i\beta$ along a contour $C$ and the field configurations are periodic
in the imaginary direction, with period $\beta$. The simplest choice for $C$ is a straight line in
the complex plane, parallel to the imaginary axis. This choice leads to the imaginary-time
formulation of thermal QFT [50, 51]. This formalism allows one to obtain correlation
functions of the field with imaginary time-arguments relatively easily. However, a major
drawback is that one often needs to analytically continue the results to real time in order
to obtain the physical functions of interest.

2.1.3 Thermal equilibrium: real-time contour

A real-time formulation can be obtained by including the real axis in the contour
$C$. The most common choice is the so-called Schwinger-Keldysh contour [51, 52] (Figure 2.2). By virtue of the Riemann-Lebesgue lemma, the path-integral along the lines
parallel to the imaginary axis factorizes in the limit $T \to \infty$ [52]. Therefore, if there
is at least one insertion of the field in the path-integral (on the real axis), we can ignore
the vertical part of the contour, because it simply gives a multiplicative constant in the
generating functional. Therefore, this real-time formulation will be completely analogous
to the CTP one, and most of the analysis given below will be automatically valid also for
the non-equilibrium CTP formalism, used in Chapters 7 and 8.

2.2 Imaginary-time formalism

As discussed above, in the imaginary-time formalism the time variable must be con-
sidered imaginary, and the field configurations are periodic in the imaginary time $\tau$, with
period $\beta$. Hence, the theory is defined on an Euclidean spacetime, instead of Minkow-
skian, and only discrete (time) frequencies are allowed in Fourier space, because of the
periodicity requirement. For the rest, the theory is analogous to the zero-temperature one. Thus, in Fourier space the free propagator for a real scalar theory (also called the Matsubara propagator) is given by

\[
\Delta^0(i\omega_n, k) = \frac{1}{\omega_n^2 + \omega_k^2}, \quad \omega_k \equiv \sqrt{k^2 + m^2},
\]  
(2.5)

with the discrete Matsubara frequencies given by \( \omega_n = 2\pi n/\beta \).

Let us define the thermal Wightman propagators:

\[
i\Delta_>(x) \equiv \langle \phi(x)\phi(0) \rangle_\beta, \quad i\Delta_<(x) \equiv \langle \phi(0)\phi(x) \rangle_\beta,
\]  
(2.6)

where the time coordinate is in general complex. Inserting twice a complete set of eigenstates |\( n \rangle \) of the Hamiltonian, with eigenvalues \( E_n \), we have

\[
i\Delta_>(x) = \text{Tr} \left( e^{-\beta H} \phi(x)\phi(0) \right) = \text{Tr} \left( e^{-\beta H} e^{iHt} \phi(0, x) e^{-iHt} \phi(0) \right)
= \sum_{m,n} e^{-E_n(\beta-it)} \langle n | \phi(0, x) | m \rangle e^{-iE_m t} \langle m | \phi(0) | n \rangle,
\]  
(2.7)

so that, if we assume that the convergence is controlled by the exponentials, we find that \( \Delta_>(x) \) is defined on the strip \(-\beta \leq \text{Im}(t) \leq 0 \). Analogously we can obtain that \( \Delta_<(x) \) is defined on \( 0 \leq \text{Im}(t) \leq \beta \). The imaginary-time propagator \( \tilde{\Delta}(\tau, x) \) is defined as the
expectation value of the (imaginary) time-ordered product of two fields

\[
\bar{\Delta}(\tau, x) \equiv \langle T \{ \phi(-i\tau, x) \phi(0) \} \rangle = \begin{cases} 
  i\Delta_>(-i\tau, x) , & \tau > 0 \\
  i\Delta_<(-i\tau, x) , & \tau < 0 
\end{cases} . \tag{2.8}
\]

We see that, a priori, the Matsubara propagator is defined on the interval \(-\beta \leq \tau \leq \beta\). However, since the field configurations are periodic in imaginary time, with period \(\beta\), it can be extended to all imaginary times, by means of the periodicity condition

\[
\bar{\Delta}(\tau - \beta, x) = \bar{\Delta}(\tau, x) , \quad 0 \leq \tau \leq \beta . \tag{2.9}
\]

In terms of the Wightman functions, this relation becomes \(\Delta_>(t, x) = \Delta_< (t + i\beta, x)\), which in Fourier space reads

\[
\Delta_< (k) = e^{-\beta k_0} \Delta_>(k) . \tag{2.10}
\]

This is the so-called Kubo–Martin–Schwinger (KMS relation [53–55]). Physically, this condition is a statement of thermodynamic equilibrium. We point out that the relations obtained so far in this section, and in particular the KMS condition, are valid not only for free propagators, but also for the dressed propagators of interacting theories.

### 2.2.1 Analytic continuations

The imaginary-time propagator can be conveniently written in terms of the spectral function

\[
\rho(k) \equiv i\Delta_>(k) - i\Delta_< (k) = (1 - e^{-\beta k_0}) i\Delta_>(k) , \tag{2.11}
\]

where we have used the KMS relation (2.10). We find

\[
\Delta_>(k) = -i \rho(k) (1 + f_+(k_0)) , \tag{2.12a}
\]

\[
\Delta_< (k) = -i \rho(k) f_+(k_0) , \tag{2.12b}
\]

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where we have introduced the positive-frequency Bose–Einstein distribution

\[ f_+(k_0) \equiv \frac{1}{e^{\beta k_0} - 1}. \]  

By choosing \( \tau \) in the interval \([0, \beta]\) and making use of (2.12), in Fourier space one finds

\[
\bar{\Delta}(i\omega_n, k) = \int_0^\beta d\tau e^{i\omega_n \tau} \int_{-\infty}^\infty dk_0 e^{-k_0 \tau} i \Delta_>(k) \\
= \int_0^\beta d\tau e^{i\omega_n \tau} \int_{-\infty}^\infty dk_0 e^{-k_0 \tau} \rho(k) (1 + f_+(k_0)) \\
= -\int_{-\infty}^{+\infty} dk_0 \frac{\rho(k)}{i\omega_n - k_0},
\]

having introduced the shorthand notation \( \int dk_0 \equiv \int dk_0/(2\pi) \). This equation is the spectral representation of the imaginary-time propagator. We want to study the analytic continuation \( \bar{\Delta}(z = k_0, k) \) of (2.14) towards real frequencies in order to relate it with the real-time propagators. Here, \( z \) denotes the generally complex zeroth component of the momentum. Requiring that \( \bar{\Delta}(z, k) \) is analytic outside the real axis and with vanishing modulus at infinity, the unique analytic continuation is given by

\[
\bar{\Delta}(z, k) \equiv -\int_{-\infty}^{+\infty} dk_0' \frac{\rho(k_0', k)}{z - k_0'},
\]

Because of the pole in this expression, if we are interested in real frequencies we need to shift them infinitesimally from the real axis. First of all, let us consider \( -\bar{\Delta}(k_0 + i\epsilon, k) \). We want to show that this continuation gives the Fourier transform of the retarded propagator

\[
i\Delta_R(x) \equiv \langle \theta(t)[\phi(x)\phi(0)] \rangle_\beta,
\]

where \( \theta(t) \) is the Heaviside step function. We have

\[ \Delta_R(k) = \int d^4 x e^{ikx} \theta(t)(\Delta_>(x) - \Delta_<^c(x)) = \int dk'_0 \frac{\rho(k'_0, k)}{k_0 - k'_0 + i\epsilon}. \]
\( = - \bar{\Delta}(k_0 + i\epsilon, \mathbf{k}). \) \hspace{1cm} (2.17)

Notice that we have not used the KMS relation, so that this relation can be easily generalized to non-equilibrium propagators. For the advanced propagator

\[ i\Delta_A(x) \equiv -\langle \theta(-t)[\phi(x)\phi(0)] \rangle, \] \hspace{1cm} (2.18)

we find, instead,

\[ \Delta_A(k) = -\bar{\Delta}(k_0 - i\epsilon, \mathbf{k}). \] \hspace{1cm} (2.19)

It can be shown that the Feynman propagator is not an analytic continuation of \( \bar{\Delta}(z, \mathbf{k}) \) (see e.g. [51]) and this is one of the reasons why this propagator plays a less prominent role in thermal QFT, as compared to the zero-temperature theory.

As we will show in the following (see (2.36) below), the spectral function of a free scalar field is given by

\[ \rho^0(k) = \varepsilon(k_0) \frac{2\pi\delta(k^2 - m^2)}{2\pi}, \] \hspace{1cm} (2.20)

i.e. essentially a Dirac delta function on the mass-shell of the particles. Here, we have denoted the sign function as \( \varepsilon(k_0) \). More generally, \( \rho(k) \) contains information on the spectral structure of the thermal quasiparticles, such as the location of their quasiparticle masses and widths. Clearly, the same information is encoded in its real-time analytic continuations, i.e. the retarded and advanced propagators. On the other hand, the Wightman propagators contain also information on the statistical content of the system, i.e. the number-density of quasiparticles (see (2.12)). At equilibrium, statistical and spectral properties are not independent, since they are related by the KMS relation, which can be rewritten as the fluctuation-dissipation theorem

\[ \Delta_1(k) = (1 + 2f_+(k_0)) \Delta(k), \] \hspace{1cm} (2.21)
where we have introduced, respectively, the spectral and Hadamard propagators

\[
\Delta(k) \equiv \Delta_>(k) - \Delta_< (k) = -i \rho(k), \quad (2.22a)
\]

\[
\Delta_1 (k) \equiv \Delta_>(k) + \Delta_<(k). \quad (2.22b)
\]

### 2.3 Real-time formalism

We want now to introduce the functional formalism on the contour \( \mathcal{C} \). The analysis applies to both the CTP formalism and the real-time contours describing the equilibrium theory. We start defining the Heaviside \( \theta \) and Dirac \( \delta \) functions on \( \mathcal{C} \) as

\[
\theta_\mathcal{C}(t-t') \equiv \begin{cases} 
1 & \text{if } t \succ_\mathcal{C} t' \\
0 & \text{if } t \prec_\mathcal{C} t' 
\end{cases}, \quad \delta_\mathcal{C}(t-t') \equiv \begin{cases} 
\delta(t-t') & t, t' \in \mathcal{C}_1 \\
-\delta(t-t') & t, t' \in \mathcal{C}_2 \\
0 & \text{otherwise}
\end{cases}, \quad (2.23)
\]

where \( \succ_\mathcal{C} \) denotes the order on \( \mathcal{C} \). The signs in the definition of \( \delta_\mathcal{C} \) can be justified noting that

\[
\int_{\mathcal{C}} dt' \delta_\mathcal{C}(t-t') f(t') = f(t), \quad (2.25)
\]

for the time \( t \) in either of the branches of \( \mathcal{C} \). Time ordering on the contour is denoted by \( T_\mathcal{C} \), so that contour-ordered correlation functions are defined as

\[
G_\mathcal{C}(x_1, \ldots, x_N) \equiv \langle T_\mathcal{C} \{ \phi(x_1, \ldots, x_N) \} \rangle. \quad (2.26)
\]
These can be obtained by functional differentiation on the contour of a generating functional $Z_C[J(x)]$

$$G_C(x_1, \ldots, x_N) = \frac{1}{Z_C} \frac{\delta^N Z_C[J(x)]}{i \delta J(x_1) \ldots i \delta J(x_N)} \bigg|_{J=0}, \quad (2.27)$$

with $\frac{\delta J(x)}{\delta J(x')} = \delta_C(t - t') \delta^{(3)}(x - x')$. By analogy with the ordinary in-out formalism, the generating functional can be expressed as a path-integral

$$Z_C[J(x)] \equiv \int D\phi e^{i \int_C d^4 x [L + J(x)\phi(x) \pm i\epsilon \phi^2(x)]}, \quad (2.28)$$

where a convergence term $\pm i\epsilon \phi^2$ has been added. The plus sign applies to the integration along $C_1$, whereas the correct sign along $C_2$ is minus (see (2.24)). For a free field, a Gaussian integration gives

$$Z^0_C[J(x)] = \mathcal{N} e^{-\frac{1}{\sqrt{2}} \int_C d^4 x \int_C d^4 x' J(x) i \Delta^0_C(x-x')J(x')}, \quad (2.29)$$

where $\Delta^0_C(x-x')$ is the free contour-ordered propagator

$$i \Delta^0_C(x-x') \equiv \theta_C(t - t') \langle \phi(x)\phi(x') \rangle^0 + \theta_C(t' - t) \langle \phi(x')\phi(x) \rangle^0. \quad (2.30)$$

We see that there is a doubling of the degrees of freedom, because we can consider the field living on the two lines $C_1$ and $C_2$ as two different fields $\phi_1$ and $\phi_2$ with a real-time coordinate. Thus, the free generating functional on the contour can be written as

$$Z^0_C[J(x)] = \mathcal{N} e^{-\frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} d^4 x \int_{-\infty}^{\infty} d^4 x' J_a(x) i \Delta^0_{ab}(x-x')J_b(x')}, \quad (2.31)$$

where $J_a(x)$ is the source on the line $a = 1, 2$ and $\Delta^0_{ab}(x-x')$ is the free propagator between a point on the line $a$ and one on the line $b$. Therefore, in the real-time formalism the propagator naturally acquires a matrix structure. Using the definition (2.30) of the contour-ordered propagator (valid not only in the free case), we see that the matrix
2.3. REAL-TIME FORMALISM

Propagator $\Delta(k)$ has the structure

$$\Delta(k) = \begin{pmatrix} \Delta_F(k) & \Delta_<(k) \\ \Delta_>(k) & \Delta_D(k) \end{pmatrix}$$

(2.32)

where $\Delta_F$ and $\Delta_D$ are, respectively, the Feynman (time-ordered) and Dyson (anti-time-ordered) propagators. In addition, the retarded and advanced propagators are, respectively, given by

$$\Delta_R(k) = \Delta_F(k) - \Delta_<(k),$$

(2.33a)

$$\Delta_A(k) = \Delta_F(k) - \Delta_>(k).$$

(2.33b)

The free propagator is a matrix Green function of the Klein-Gordon operator, i.e. it satisfies

$$\begin{pmatrix} -\partial^2_x - m^2 + i\epsilon & 0 \\ 0 & -(-\partial^2_x - m^2 - i\epsilon) \end{pmatrix} \begin{pmatrix} \Delta^0_{11}(x-x') & \Delta^0_{12}(x-x') \\ \Delta^0_{21}(x-x') & \Delta^0_{22}(x-x') \end{pmatrix} = \delta^{(4)}(x-x') \mathbb{I}. \quad (2.34)$$

In Fourier space this is easily inverted, in the distributional sense, as

$$\Delta^0(k) = \begin{pmatrix} \frac{1}{k^2 - m^2 + i\epsilon} - iA(k) \delta(k^2 - m^2) \\ -iC(k) \delta(k^2 - m^2) \end{pmatrix} \begin{pmatrix} -iB(k) \delta(k^2 - m^2) \\ -\frac{1}{k^2 - m^2 - i\epsilon} - iA'(k) \delta(k^2 - m^2) \end{pmatrix},$$

(2.35)

since $x\delta(x) = 0$ in the sense of distributions. Therefore, even if the inverse propagator is diagonal, $\Delta$ in general is not, and a matrix formulation is necessary. In Appendix A we will determine the functions $A$, $A'$, $B$ and $C$ by imposing a number of physical require-
ments on the free propagator, finding
\begin{equation}
\Delta^0(k) = \left( \begin{array}{cc}
\frac{1}{k^2 - m^2 + i\epsilon} - i f(k) 2\pi \delta(k^2 - m^2) & -i [f(k) + \theta(\bar{k})] 2\pi \delta(k^2 - m^2) \\
-i [f(k) + \theta(k)] 2\pi \delta(k^2 - m^2) & \frac{-1}{k^2 - m^2 - i\epsilon} - i f(k) 2\pi \delta(k^2 - m^2)
\end{array} \right) .
\end{equation}
(2.36)

The function \(f(k)\) depends on the boundary conditions on the path-integral. At thermal equilibrium, this is easily obtained by means of the KMS relation (2.10), which gives
\begin{equation}
[f(k) + \theta(-k)] \delta(k^2 - m^2) = e^{-\beta k_0} [f(k) + \theta(k)] \delta(k^2 - m^2) ,
\end{equation}
(2.37)

obtaining
\begin{equation}
f(k) = f(k_0) = \frac{1}{e^{\beta|k_0|} - 1} ,
\end{equation}
(2.38)

which is the Bose–Einstein distribution. This can also be written in terms of its positive- and negative-frequency parts as
\begin{equation}
f(k_0) = \theta(k_0) f_+(k_0) + \theta(-k_0) f_-(k_0) ,
\end{equation}
(2.39)

with
\begin{equation}
f_\pm(k_0) \equiv \frac{1}{e^{\pm\beta k_0} - 1} .
\end{equation}
(2.40)

The analysis above can be repeated for a complex scalar field. In this case the system at equilibrium admits a chemical potential \(\mu\) for the Nöther charge of the field. It can be shown (see e.g. [50, 51]) that the KMS relation (2.10) becomes
\begin{equation}
\Delta_<(k) = e^{-\beta(k_0 - \mu)} \Delta_>(k) ,
\end{equation}
(2.41)

and the positive- and negative-frequency Bose–Einstein distributions (2.40) are
\begin{equation}
f_\pm(k_0) \equiv \frac{1}{e^{\pm\beta(k_0 - \mu)} - 1} .
\end{equation}
(2.42)
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As a final remark, we note that also at zero temperature the real-time propagator (2.36) is not diagonal, so that its matrix structure emerges also in $T = 0$ QFT when one is interested in expectations values of observables in a given state, and not in asymptotic \textit{in-out} amplitudes.
In this chapter, we discuss the inclusion of interactions in the description of quantum fields at thermal equilibrium. We first present the thermal Feynman rules for cubic and quartic scalar self-interactions and then discuss formal aspects of the resummation of self-energy insertions, giving the dressed thermal propagator. By looking at specific examples we show how thermal corrections entail the appearance of a thermal mass and width for the quasiparticles. Finally, we discuss how the equilibrium results can be used to obtain, heuristically, Boltzmann equations that describe the semi-classical evolution of systems out of equilibrium. These semi-classical equations will be used to describe quantitatively the generation of the BAU by RL in the next chapter. We will go beyond this semi-classical treatment in Chapter 7.

3.1 Feynman rules

3.1.1 Imaginary time

In the last chapter we showed that the momentum-space free propagator for a scalar field has the Matsubara form (2.5). The only differences with respect to the zero-temperature theory is that spacetime is Euclidean and only the discrete Matsubara frequencies \( \omega_n = \)
$2\pi n/\beta$ must be included. Thus, momentum integrals are replaced by

$$\int d^4k \rightarrow T \sum_n \int d^3k .$$

(3.1)

Interaction terms $-\frac{\lambda}{4!} \phi^4$ and $-\frac{g}{3!} \phi^3$ in the Lagrangian generate the following standard Euclidean vertices:

$$= -\lambda ,$$

$$= -g .$$

(3.2)

Renormalization is carried out in the standard way, by introducing a set of wavefunction, mass and coupling counterterms $\delta Z$, $\delta m^2$, $\delta \lambda$, $\delta g$. These must be the same as in the $T = 0$ theory, because the ultraviolet (UV) behaviour of the theory cannot be modified by the presence of the thermal bath, which cannot affect real or virtual processes at energies much larger than the temperature $T$.

### 3.1.2 Real time

In the last chapter we showed that in the real-time formalism there is the appearance of both type-1 and type-2 fields, living on the two branches of the real-time contour, thus yielding the matrix structure (2.36) for the free propagator.

As regards the vertices, from the contour path-integral (2.28), we see that the integration along the branch $C_1$, forward in time, generates a perturbation theory with the same vertices as in the usual QFT; we will refer to these as type-1 vertices. Instead, the integration along $C_2$, being backward in time, generates a set of type-2 vertices with the signs reversed with respect to the usual ones. Hence, for cubic and quartic interactions we have

$$1 = -i \lambda ,$$

$$2 = +i \lambda .$$

(3.3a)
In a Feynman graph, type-\(a\) and type-\(b\) vertices are connected by the element \(ab\) of the free matrix propagator \(i\Delta^0_{ab}\). Again, a set of counterterms is also generated, both of type 1 and 2, by writing the bare field and the parameters in the Lagrangian in terms of renormalized ones.

### 3.2 Dyson equation

Let us define the self-energies in imaginary and real time in terms of the Dyson equation

\[
\bar{\Delta}(i\omega_n, k) = \bar{\Delta}^0(i\omega_n, k) - \bar{\Delta}^0(i\omega_n, k) \bar{\Pi}(i\omega_n, k) \bar{\Delta}(i\omega_n, k), \tag{3.4a}
\]

\[
i\Delta(k) = i\Delta^0(k) + i\Delta^0(k) \cdot i\Pi(k) \cdot i\Delta(k), \tag{3.4b}
\]

where the real-time self-energy matrix is\(^1\):

\[
\Pi \equiv \begin{pmatrix}
\Pi & -\Pi_< \\
-\Pi_> & -\Pi^+
\end{pmatrix}.	ag{3.5}
\]

It is well-known that the Dyson equation describes the resummation of self-energy insertions in the free propagator. This resummation is performed easily in the imaginary-time formalism:

\[
\bar{\Delta}(i\omega_n, k) = \frac{1}{\Delta^0(i\omega_n, k)^{-1} + \Pi(i\omega_n, k)}. \tag{3.6}
\]

In real time, it is useful to introduce the Keldysh representation \(\tilde{\Delta}\) as

\[
\tilde{\Delta} \equiv Q \cdot \Delta \cdot Q^{-1}, \quad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad Q^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{3.7}
\]

\(^1\)Later, we will prove the relation \(\Pi_{22} = -\Pi_{11}^1\).
An explicit calculation shows that \( \tilde{\Delta} \) has the following form:

\[
\tilde{\Delta} = \begin{pmatrix}
0 & \Delta_A \\
\Delta_R & \Delta_1
\end{pmatrix},
\]

(3.8)

where the 11 element is 0 because of the unitarity condition (A.10), i.e. \( \Delta_F + \Delta_D = \Delta_\geq + \Delta_\leq \), valid also for the full propagators. Therefore, the Dyson equation can be recast in the form

\[
i\tilde{\Delta} = i\tilde{\Delta}^0 + i\tilde{\Delta}^0 \cdot i\tilde{\Pi} \cdot i\tilde{\Delta},
\]

(3.9)

where \( \tilde{\Pi} \equiv Q \cdot \Pi \cdot Q^{-1} \) takes on the following form:

\[
\tilde{\Pi} = \begin{pmatrix}
\Pi_1 & \Pi_R \\
\Pi_A & 0
\end{pmatrix},
\]

(3.10)

with

\[
\Pi_R \equiv \Pi - \Pi_\leq, \quad \Pi_A \equiv \Pi - \Pi_\geq, \quad \Pi_1 = \Pi - \Pi^*. \quad (3.11)
\]

We thus obtain the set of Dyson equations

\[
\Delta_R = \Delta_R^0 - \Delta_R^0 \Pi_R \Delta_R,
\]

(3.12a)

\[
\Delta_A = \Delta_A^0 - \Delta_A^0 \Pi_A \Delta_A,
\]

(3.12b)

\[
\Delta_1 = \Delta_1^0 - \Delta_1^0 \Pi_A \Delta_A - \Delta_R^0 \Pi_R \Delta_1 - \Delta_R^0 \Pi_1 \Delta_R.
\]

(3.12c)

Finally, since the advanced and retarded propagators are analytic continuations of the imaginary-time one, the same holds true for the self-energies:

\[
\Pi_R(k) = -\bar{\Pi}(k_0 - \mu + i\epsilon, k), \quad (3.13a)
\]

\[
\Pi_A(k) = -\bar{\Pi}(k_0 - \mu - i\epsilon, k) = \Pi_R(k)^*.
\]

(3.13b)
3.2.1 Resummation in the real-time formalism

Both the retarded and advanced propagators satisfy a Dyson equation not involving the other elements. Thus, the resummed propagators are easily obtained as

\[
\Delta_R(k) = \frac{1}{\Delta_{R,0}(k)^{-1} + \Pi_R(k)} , \quad \Delta_A(k) = \frac{1}{\Delta_{A,0}(k)^{-1} + \Pi_A(k)} .
\] (3.14)

The free propagators are \( \Delta_{R/A,0}(k) = (k^2 - m^2 \pm i\epsilon k_0)^{-1} \). Hence, the resummed propagators are given by

\[
\Delta_R(k) = \frac{1}{k^2 - m^2 + \Pi_R(k) + i\epsilon k_0} , \quad \Delta_A(k) = \frac{1}{k^2 - m^2 + \Pi_A(k) - i\epsilon k_0} .
\] (3.15)

If the self-energies are slowly-varying with \( k \) near the mass-shell, this yields, in general, a complex mass shift (see the discussion below).

The equation satisfied by the Hadamard function is more complicated, since it involves also the other components of the matrix propagator. We thus find convenient to solve the Dyson equation in the original representation (3.4b), which can be recast in the form

\[
\Delta^{-1}(k) = \begin{pmatrix}
  k^2 - m^2 + i\epsilon + \Pi(k) & -\Pi_<(k) \\
  -\Pi_>(k) & -(k^2 - m^2 - i\epsilon + \Pi(k)^*)
\end{pmatrix}.
\] (3.16)

The conditions \( \Delta_D = -\Delta_F^*, \ \text{Re} \ \Delta_\leq = \text{Re} \ \Delta_\geq = 0 \) (see Appendix A), valid also for the full propagators, are directly transferred into the corresponding relations for the self-energies. These relations, in addition to the unitarity condition (A.10), allow to write all the self-energies in terms of the real quantities \( \text{Re} \ \Pi_R, \ \text{Im} \ \Pi_R \) and \( \text{Im} \ \Pi \), as

\[
\Pi_R = \text{Re} \ \Pi_R + i \ \text{Im} \ \Pi_R , \quad \Pi_A = \text{Re} \ \Pi_R - i \ \text{Im} \ \Pi_R , \\
\Pi_1 = 2i \ \text{Im} \ \Pi , \quad \Pi = \text{Re} \ \Pi_R + i \ \text{Im} \ \Pi , \\
\Pi_\leq = i (\text{Im} \ \Pi - \text{Im} \ \Pi_R) , \quad \Pi_\geq = i (\text{Im} \ \Pi + \text{Im} \ \Pi_R) .
\] (3.17)
Hence, a direct matrix inversion of (3.16) gives
\[
\Delta F(k) = \frac{k^2 - m^2 + \Pi(k)^*}{[k^2 - m^2 + \text{Re} \Pi_R(k)]^2 + [\text{Im} \Pi_R(k)]^2} = -\Delta D(k)^*, \quad (3.18a)
\]
\[
\Delta \leq(k) = \frac{-\Pi \leq(k)}{[k^2 - m^2 + \text{Re} \Pi_R(k)]^2 + [\text{Im} \Pi_R(k)]^2}, \quad (3.18b)
\]
\[
\Delta_1(k) = \frac{-\Pi_1(k)}{[k^2 - m^2 + \text{Re} \Pi_R(k)]^2 + [\text{Im} \Pi_R(k)]^2}. \quad (3.18c)
\]

If the imaginary part of the retarded self-energy is sufficiently small, we can interpret these results as describing a system of unstable long-lived quasiparticles, with a dispersion (or gap) relation given in terms of a thermally-corrected squared mass \( M^2(k) \). This is approximated by its on-shell (OS) form, obtained as the solution of
\[
\varepsilon^2(k) \equiv k^2 + M^2(k) = k^2 + m^2 - \lim_{\epsilon \to 0^+} \text{Re} \Pi_R(\varepsilon(k) + i\epsilon, k). \quad (3.19)
\]

The quasiparticles have a thermal width \( \Gamma(k) \) given by
\[
\Gamma(k) \equiv \frac{|\text{Im} \Pi_R(\varepsilon(k), k)|}{M(k)}. \quad (3.20)
\]

### 3.2.2 KMS condition

At equilibrium, we can still impose the KMS relation, which relates spectral and statistical properties of the system. These are not independent, for a system at thermodynamic equilibrium. The KMS relation for the self-energies
\[
\Pi_> = e^{\beta(k_0 - \mu)} \Pi_<, \quad (3.21)
\]
is inherited from (2.10). Since
\[
2i \text{Im} \Pi = \Pi_> + \Pi_< = (1 + e^{\beta(k_0 - \mu)}) \Pi_<, \quad (3.22)
\]
we have

\[ \text{Im} \Pi_R = \text{Im} \Pi - \text{Im} \Pi_\times = \frac{e^{\beta (k_0 - \mu)} - 1}{e^{\beta (k_0 - \mu)} + 1} \text{Im} \Pi, \]

which can be rewritten as

\[ \frac{\text{Im} \Pi}{\text{Im} \Pi_R} = \varepsilon (k_0) (2 f (k_0) + 1). \]

Finally, using this relation in (3.17) we may obtain all the equilibrium self-energies in terms of the retarded one:

\[ \Pi_R = \text{Re} \Pi_R + i \text{Im} \Pi_R, \quad \Pi_R = \text{Re} \Pi_R - i \text{Im} \Pi_R, \]
\[ \Pi_\times = 2 i f_+ (k_0) \text{Im} \Pi_R, \quad \Pi_\times = -2 i f_- (-k_0) \text{Im} \Pi_R. \]

3.2.3 Narrow-width approximation

When the thermal width \( \Gamma \) is much smaller than the typical scales on which the other quantities in the problem vary significantly, we may work in the narrow-width approximation (NW A), by taking the limit \( \varepsilon (k_0) \text{Im} \Pi_R \to 0^+ \). The presence of the sign function can be understood in terms of causality arguments, which require that \( \Delta_R \) cannot have poles in the semiplane \( \text{Im} k_0 > 0 \). The retarded and advanced propagators in this limit become

\[ \Delta_{R/A}(k) \sim \frac{1}{k^2 - M^2 (k) \pm i \varepsilon (k_0) M(k) \Gamma(k)}, \]

whereas for the Wightman propagators we find

\[ \Delta_\times \sim -i [f(k_0) + \theta(\pm k_0)] 2 \pi \delta (k^2 - M^2 (k)), \]
where we have made use of the representation

\[ \delta(x) \sim \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}, \quad \text{for } \epsilon \to 0^+. \]  

As anticipated above, these formulas describe a system of long-lived quasiparticles with thermal mass and width \( M(k) \) and \( \Gamma(k) \), respectively, and the NWA implies the quasiparticle approximation, in this single-field model at equilibrium.

### 3.3 Thermal mass and width

#### 3.3.1 Thermal mass

Let us apply the discussion above to specific examples in a scalar model. First, let us calculate the thermal mass due to 1-loop corrections in the neutral \( \phi^4 \) theory. Let us work in the real-time formalism.

The contributions to \( \Delta_{11}(k) \) are shown in Figure 3.1. For one self-energy insertion we have, in general,

\[ i\Delta_{11}(k) = i\Delta_{11}^0(k) + \sum_{\substack{a=1,2 \ b=1,2}} i\Delta_{1a}^0(k) [i\Pi_{ab}(k)] i\Delta_{b1}^0(k), \]  

but in this particular case the off-diagonal self-energies vanish and only \( \Pi_{11} = \Pi \) and
3.3. THERMAL MASS AND WIDTH

$\Pi_{22} = -\Pi^*$ give a nonzero contribution to the 1-loop dressed propagator. Let us calculate the Feynman self-energy:

$$i\Pi(k) = i\Pi_{11}(k) = \frac{-i\lambda}{2} \int p_i \delta\Pi_{11}(p) - i \delta m^2$$

$$= \frac{-i\lambda}{2} \int \left( \frac{i}{p^2 - m^2 + i\epsilon} + f(p_0) \frac{2\pi}{2}(p^2 - m^2) \right) - i \delta m^2 . \quad (3.30)$$

having defined $\int p_0 \equiv \int d^4 p$. We choose the $T = 0$ OS renormalization scheme (for a review, see e.g. [57]), so that the renormalization condition is

$$-i\lambda \int p \frac{i}{p^2 - m^2 + i\epsilon} - i \delta m^2 = 0 ,$$

thus obtaining

$$\Pi(k) = -\frac{\lambda}{2} \int \frac{f(\omega_p)}{\omega_p} . \quad (3.31)$$

We have introduced the shorthand notation $\int p_0 \equiv \int d^3 p$. The integral is easily calculated in the high-temperature case $T \gg m$, finding

$$\Pi(k) \simeq -\frac{\lambda}{24} T^2 . \quad (3.32)$$

By comparing this result with (3.19) and (3.25) we see that the effect of the 1-loop correction is the appearance of a momentum-independent thermal mass

$$M^2 \simeq \frac{\lambda}{24} T^2 , \quad (3.33)$$

for the quasiparticles in the high-temperature regime.

3.3.2 Cancellation of pinch singularities

Since the real-time propagator (2.36) contains terms proportional to $\delta(k^2 - m^2)$, the loop corrections can potentially be plagued by ill-defined products of $\delta$ functions having the same argument. The same problem can be rephrased as the appearance of products of $\Delta_R$ and $\Delta_A$ carrying the same momentum, so that the integration contour is pinched between two infinitesimally-near poles of the form $\frac{1}{x + i\epsilon} \frac{1}{x - i\epsilon}$. These potential pathologies,
known as pinch singularities, cannot be circumvented by means of a Wick rotation, and could, a priori, invalidate the field-theoretical formulation presented here. At equilibrium, a delicate interplay between the different matrix elements of the matrix propagator, in conjunction with the KMS relation, leads to the complete cancellation of the pinch singularities at any loop order [52]. As will be discussed in detail in Chapter 7, these potential pathologies make a consistent formulation of non-equilibrium thermal QFT more sophisticated and particular care will be needed to obtain a consistent formalism, free of pinch singularities, either in the Heisenberg or interaction pictures.

Here, we calculate explicitly the dressed 1-loop Feynman propagator in order to show how this cancellation occurs in the case of a real self-energy, such as the one at 1-loop order in the $\phi^4$ theory (cf. (3.32)). For one self-energy insertion, (3.29) and Figure 3.1 give

$$-\Delta_F^{(1)} = \Delta_{11}^0 \Pi_{11} \Delta_{11}^0 + \Delta_{12}^0 \Pi_{22} \Delta_{21}^0 = \Pi \left( \Delta_F^0 \Delta_F^0 - \Delta^0 \Delta^0 \right). \tag{3.34}$$

Explicitly, we have

$$-\frac{\Delta_F^{(1)}}{\Pi} = \left[ \frac{1}{k^2 - m^2 + i\epsilon} - i f(k_0) 2\pi \delta(k^2 - m^2) \right]^2$$

$$- \left[ -i \left( f(k_0) + \theta(-k_0) \right) 2\pi \delta(k^2 - m^2) \right] \left[ -i \left( f(k_0) + \theta(k_0) \right) 2\pi \delta(k^2 - m^2) \right]$$

$$= \left( \frac{1}{k^2 - m^2 + i\epsilon} \right)^2 + f(k_0) 2\pi \delta(k^2 - m^2) \left[ 2\pi \delta(k^2 - m^2) - \frac{2i}{k^2 - m^2 + i\epsilon} \right]. \tag{3.35}$$

The first pinch singularity $f(k_0)^2(2\pi \delta(k^2 - m^2))^2$ has cancelled out. By using the regularization (3.28) for the $\delta$ function, we obtain

$$-\frac{\Delta_F^{(1)}}{\Pi} = \left( \frac{1}{k^2 - m^2 + i\epsilon} \right)^2$$

$$+ f(k_0) \frac{4\epsilon}{(k^2 - m^2)^2 + \epsilon^2} \left[ \frac{\epsilon}{(k^2 - m^2)^2 + \epsilon^2} - i \frac{k^2 - m^2 - i\epsilon}{(k^2 - m^2)^2 + \epsilon^2} \right]. \tag{3.36}$$
3.3. THERMAL MASS AND WIDTH

Figure 3.2: 1-loop corrections to the imaginary-time propagator in the $\phi^3$ theory. Counterterm graphs are not shown.

The second potential pinch singularity disappears as well, because the $O(\epsilon)$ terms inside the square brackets cancel out and the remaining numerator is exactly 0 when $k^2 = m^2$. After some algebra we explicitly find

$$\Delta_F^{(1)} = -\Pi \left[ \left( \frac{1}{k^2 - m^2 + i\epsilon} - i f(k_0) \right) \frac{2\pi\delta'(k^2 - m^2)}{k^0} \right], \quad (3.37)$$

which is free of pinch singularities.

For the general case of one complex self-energy insertion, a rather lengthy calculation gives, again, the well-defined result:

$$\Delta_F^{(1)}(k) = -\Pi(k) \left( \frac{1}{(k^2 - m^2 + i\epsilon)^2} - i f(k_0) \right) \frac{2\pi\delta'(k^2 - m^2)}{k^0}. \quad (3.38)$$

We point out that in this general case the KMS relation is explicitly needed to demonstrate the cancellation of pinch singularities. Therefore, the same argument would not hold true out of equilibrium.

3.3.3 Thermal width

We now calculate the absorptive thermal self-energy $\text{Im}\Pi_R(k)$, essentially giving the off-shell thermal width $\Gamma(k) = \text{Im}\Pi_R(k)/M(k)$, in a scalar $\phi^3$ model. We work in the imaginary-time formalism. At 1-loop order we have two diagrams contributing to the self-energy, shown in Figure 3.2, plus the ones involving counterterms. It is easy to convince oneself that the first diagram does not give absorptive contributions. The second diagram,
including the mass counterterm, gives

\[-\Pi^{(b)}(i\omega_n, k) = \frac{(-g)^2}{2} T \sum_m \int_p \frac{1}{\omega_m + \omega_p^2} \frac{1}{(\omega_n - \omega_m)^2 + \omega_{k-p}^2} + \delta m^2. \tag{3.39}\]

The summation is standard and can be performed, for example, using the spectral representation of the Matsubara propagator \[51, eq. 2.82\]. One obtains:

\[\Pi^{(b)}(i\omega_n, k) = \frac{-g^2}{2} \int_p \frac{-1}{4 \omega_p \omega_{k-p}} \times \left\{ \left[1 + f(\omega_p) + f(\omega_{k-p})\right] \left(\frac{1}{i\omega_n - \omega_p - \omega_{k-p}} - \frac{1}{i\omega_n + \omega_p + \omega_{k-p}}\right) \right. \\
- \left. \left[f(\omega_p) - f(\omega_{k-p})\right] \left(\frac{1}{i\omega_n - \omega_p + \omega_{k-p}} - \frac{1}{i\omega_n + \omega_p - \omega_{k-p}}\right) \right\} - \delta m^2. \tag{3.40}\]

This can be easily continued analytically to real time, giving

\[\text{Im} \Pi_R(k) = \text{Im} \Pi^{(b)}_R(k) = -\text{Im} \Pi^{(b)}(k_0 + i\epsilon, k) = \frac{g^2}{2} \int_p \frac{\pi}{4 \omega_p \omega_{k-p}} \\
\times \left\{ \left[1 + f(\omega_p) + f(\omega_{k-p})\right] \left[\delta(k_0 - \omega_p - \omega_{k-p}) - \delta(k_0 + \omega_p + \omega_{k-p})\right] \right. \\
- \left. \left[f(\omega_p) - f(\omega_{k-p})\right] \left[\delta(k_0 - \omega_p + \omega_{k-p}) - \delta(k_0 + \omega_p - \omega_{k-p})\right] \right\}. \tag{3.41}\]

The evaluation of the momentum integral in (3.41) is rather lengthy and will be given in Appendix B. Here, we just quote the final result:

\[\text{Im} \Pi_R(k) = \begin{cases} \\
\frac{g^2}{16\pi|k|} \left[ \frac{1}{\beta} \log \frac{\sinh[\beta\omega_+/2]}{\sinh[\beta\omega_-/2]} - \frac{\omega_+ - \omega_-}{2} \theta(|k| - |k_0|) \right], & \text{if } \frac{4m^2}{k^2} < 1, \\
0, & \text{otherwise,} \end{cases} \tag{3.42}\]

having defined

\[\omega_\pm \equiv \sqrt{\frac{|k|}{2} \pm \frac{k_0}{2} \sqrt{1 - \frac{4m^2}{k^2}}} + m^2. \tag{3.43}\]
3.3. THERMAL MASS AND WIDTH

In Figure 3.3 we plot the typical behaviour of $\text{Im} \Pi_R(k)$. As we will discuss more in detail below, the absorptive self-energy vanishes in the kinematically forbidden region $0 \leq k^2 \leq 4m^2$, which includes the mass-shell. At $T = 0$, also the region $k^2 < 0$ is kinematically forbidden, but here we obtain a purely-thermal nonzero result. We have also checked that in the low-temperature and small-mass limit, plotted in Figure 3.4, the imaginary part of the self-energy approaches the simple result of the massless $T = 0$ theory $\text{Im} \Pi_R(k) \propto \epsilon(k_0) \theta(|k_0| - |k|)$.

Finite-temperature cutting rules [58] allow to interpret physically (3.41): each $\delta$ func-
Figure 3.5: Thermal decay processes obtained by cutting the 1-loop self-energy in the $\phi^3$ model. Lines attached to a grey blob denote particles in the thermal bath.

The process corresponds to a thermal process in which the incoming particle ($k_0 > 0$) or antiparticle ($k_0 < 0$) disappears (Figure 3.5). For definiteness, let us consider the case $k_0 > 0$, i.e. a particle. The first $\delta$ function in (3.41) corresponds to the decay of a particle into two on-shell particles thermalized in the bath (Figure 3.5a). This process can also take place at $T = 0$, but clearly the presence of Bose-Einstein particles in the thermal bath enhances the decay rate. The threshold for this process is at the energy corresponding to the creation of two on-shell particles of mass $m$, i.e. $k^2 > 4m^2$. This gives rise to the right nonzero branch in Figure 3.3, which is present also at $T = 0$ (see Figure 3.4). The second $\delta$ function in (3.41) corresponds to the analogous process for antiparticles.

Both the third and the fourth $\delta$ functions in (3.41) correspond to the scattering in the plasma: the off-shell particle under consideration interacts with an on-shell particle in the bath generating another on-shell particle that thermalizes in the bath (Figure 3.5b). This process can be written as

$$\phi(k_0, k) + \tilde{\phi}(\omega_{p-k}, p - k) \rightarrow \tilde{\phi}(\omega_p, p).$$

(3.44)

Here, the tilde ($\tilde{\cdot}$) denotes on-shell particles in the bath. Notice that the third $\delta$ function in (3.41) precisely enforces the conservation of energy for this process. It is possible to show that this requires $k$ to be spacelike. Clearly, this is a thermal-only process, because it can take place only if a plasma of particles is available for the scattering. This set of processes gives rise to the left branch in Figure 3.3, which disappears at $T = 0$ (see Figure 3.4).
In summary, we have shown how the presence of a thermal bath can significantly modify the absorptive properties of the field, either enhancing (in the case of Bose-Einstein statistics) or suppressing (Fermi-Dirac case) processes allowed also at $T = 0$, but also allowing absorptive transitions with particles in the plasma that are kinematically forbidden at zero temperature.

### 3.4 Boltzmann equations

We now use the results obtained so far to derive, heuristically, the Boltzmann-like equation that describes the evolution of the number-density of particles for small deviations from equilibrium. At this stage, we do not give a precise definition of the number-density, because this requires more detailed discussions and more powerful techniques, which will be presented in Chapters 5 and 7.

For the moment, we proceed heuristically by promoting, in the thermal QFT relations of interest, the equilibrium Bose-Einstein distribution $f(k_0)$ to a non-equilibrium number-density $n(k, t)$

$$f(\mathcal{E}(k)) \rightarrow n(k, t), \quad (3.45)$$

where we recall that $\mathcal{E}(k)$ is the energy of the thermal quasiparticles with 3-momentum $k$. This procedure can be justified, in many cases of interest, when one is within the range of applicability of quasiparticle approximations (see the discussion at the end of Section 3.2.1). A more detailed discussion will be given in Chapters 7 and 8.

In the last section, we showed that the thermal absorptive self-energy $\text{Im} \Pi_R$ can be interpreted in terms of decay processes of the particles in the presence of a thermal plasma (see (3.41) and the subsequent discussion). When one considers also the $k_0 < 0$ modes, it is possible to show that the description of creation processes is also contained in (3.41). We note that the statistical factors appearing in (3.41), promoted to non-equilibrium as
described above, have the form

\begin{align}
1 + n_1 + n_2 &= (1 + n_1)(1 + n_2) - n_1 n_2, \\
n_1 - n_2 &= n_1(1 + n_2) - (1 + n_1)n_2,
\end{align}

(3.46a, 3.46b)

where the subscripts 1 and 2 label, in a simplified notation, the particles in the thermal bath, interacting with the particle under consideration (see also Figure 3.5). Note that the combinations appearing in (3.46) are precisely the quantum statistical factors present in the decay processes in Figure 3.5, minus the ones in the corresponding creation processes. Notice that from (3.17) we have the relation

\[2 \text{Im} \Pi_R = \text{Im} \Pi_> - \text{Im} \Pi_>, \]

(3.47)

and an explicit calculation (see e.g. [51]) gives that \( \text{Im} \Pi_> \) and \( \text{Im} \Pi_< \) contain precisely the decay and creation statistical factors, respectively. By including also the quantum statistical factors for the particle under consideration, we may now infer the evolution equation for the total number-density \( n(t) = \int_k n(k, t) \):

\[
\frac{dn}{dt} = - \int \frac{d^3k}{2 \mathcal{E}(k)} \left[ i\Pi_< (\mathcal{E}(k), k; t) \left( 1 + n(k, t) \right) - n(k, t) \text{Im} \Pi_>(\mathcal{E}(k), k; t) \right],
\]

(3.48)

which has the form of a quantum Boltzmann equation\(^1\). Notice that the Wightman self-energies are now taken as time dependent, via the number-densities appearing in them. The Boltzmann-like equation (3.48) will be re-obtained more rigorously, in the Kadanoff–Baym formalism that will be presented in Chapter 7, after performing a particular quasi-particle approximation, known as Kadanoff-Baym ansatz.

In the following chapters we will often work in the classical-statistics Maxwell-Boltzmann (rather than Bose-Einstein or Fermi-Dirac) regime, and also neglect thermal masses. Generalizing the discussion to \( aX \leftrightarrow Y \) processes, and performing these simplifying

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\(^1\)The overall minus sign appearing in (3.48) is related to the one appearing in the off-diagonal entries of (3.5).
approximations, the evolution equation (3.48) for the number-density $n^a$ of the species $a$ takes on the Boltzmann form [59, 60]

$$\frac{dn^a}{dt} = -\int d\Pi_a \sum_{aX \rightarrow Y} d\Pi_X d\Pi_Y (2\pi)^4 \delta^4(p_a + p_X - p_Y) \times \left( n^a(p_a, t) n^X(p_X, t) |\mathcal{M}(aX \rightarrow Y)|^2 - n^Y(p_Y, t) |\mathcal{M}(Y \rightarrow aX)|^2 \right),$$

(3.49)

where the sum runs over all the allowed processes, $|\mathcal{M}|^2$'s are the relevant squared matrix elements and the phase-space measure for the multiparticle state $X$, containing $N_X$ particles, is defined as

$$d\Pi_X = \frac{1}{N_{id}!} \prod_{i=1}^{N_X} \frac{d^4 p_i}{(2\pi)^4} 2\pi \delta(p_i^2 - m_i^2) \theta(p_i^0),$$

(3.50)

where $\delta(x)$ and $\theta(x)$ are the usual Dirac delta and Heaviside step functions, respectively, and $N_{id}!$ is a symmetry factor in the case that the multiparticle state $X$ contains $N_{id}$ identical particles. We will use the Boltzmann form (3.49) in Chapter 4 to obtain the \textit{flavour-diagonal rate equations for RL} discussed there.
In this chapter, we present one of the most attractive mechanisms for the generation of the observed BAU, the so-called *Resonant Leptogenesis*, which provides a common origin for two of the strongest pieces of evidence for BSM physics, namely the asymmetry between matter and antimatter in the Universe and the observed light neutrino masses. Very interestingly, this phenomenon can easily provide observable signatures at present and near-future experiments, both at the energy and the intensity frontiers. We first introduce the basic framework of baryogenesis via *leptogenesis* in the Early Universe, and then immediately focus on the resonant case, in which the masses of the heavy Majorana neutrinos, responsible for the generation of the asymmetry, can be as low as the electroweak scale. Here, we present the classical description of this phenomenon, as developed in [22, 23, 27, 61], which captures the generation of the asymmetry due to the mixing of the heavy neutrinos. In the rest of this thesis we will go beyond this analysis, based on the classical Boltzmann equations, and provide a complete and unified description of all relevant effects involved in the generation of the BAU. The discussion in Sections 4.2 and 4.3 follows the presentation given in [28].
4.1 Thermal leptogenesis

Almost sixty years ago [62], Sakharov gave the necessary conditions for the dynamical generation of the baryon-antibaryon asymmetry in the Universe from a symmetric initial state. Assuming \( CPT \) invariance, these three basic conditions are:

- violation of the baryon number \( B \); otherwise a net baryon number could not be generated from a symmetric initial condition.

- \( C \) and \( CP \) violation; since a state with a nonzero baryon asymmetry is not invariant under these transformations, as opposed to the symmetric initial condition.

- non-equilibrium dynamics; since in thermal equilibrium no preferred time direction exists. Thus, in an equilibrium thermal background and a \( CPT \)-symmetric theory, \( CP \)-conjugate states are produced at the same rate. (see also [59] for a detailed proof).

At this point it is worth mentioning briefly that a net baryon number as initial condition for the evolution of the Universe would be exponentially diluted by cosmic inflation, thus making this potential argument unsatisfactory for practical, in addition to epistemological, reasons.

The SM alone does not satisfy the Sakharov conditions since, although \( B, C \) and \( CP \) are all violated (the former by nonperturbative processes), the electroweak phase transition is not strong enough, so that there is not sufficient deviation from equilibrium (for a recent review see, e.g. [63]). Therefore, a successful generation of the BAU requires \( BSM \) physics. Another strong piece of evidence for BSM physics is provided by the SM neutrino masses. Although the SM could be minimally modified, in principle, to incorporate neutrino Dirac masses generated via the Higgs mechanism, the extreme lightness of the neutrinos would required tiny Yukawa couplings, of the order of \( 10^{-13} \). The huge hierarchy between these Yukawa couplings and the SM ones would remain unexplained.

*Leptogenesis* [4] provides a common mechanism to explain both these pieces of evidence for BSM physics. In its simplest variant, a set of heavy right-handed (RH) Majorana
neutrinos \( N_{\alpha} \), singlet under the SM gauge group, is added to the SM. These heavy particles generate naturally light neutrino masses via the type-I seesaw mechanism \([10–14]\) (see also Chapter 9 for more details). All the Sakharov conditions are satisfied, since:

- \( C \) and \( CP \) are violated for generic complex heavy-neutrino Yukawa couplings \( \hat{h}_{l\alpha} \) (see also Section 5.2). Here, \( h_{l\alpha} \) is the Yukawa coupling of the heavy neutrino \( N_{\alpha} \) with the lepton-doublet \( L_l \), and the caret (\(^\wedge\)) denotes the fact that we work in the basis in which the heavy-neutrino mass matrix is diagonal (and positive).

- The \( CP \)-violating decays \( N_{\alpha} \rightarrow L_l \Phi \), with \( \Phi \) being the SM Higgs doublet, proceed out of equilibrium, in the early Universe, if their rate is not excessively larger than the Hubble expansion rate.

- The decays \( N_{\alpha} \rightarrow L_l \Phi \) can generate a net lepton asymmetry, since the total lepton number \( L \) is not conserved in the presence of the Majorana mass term (see (5.1) in the next chapter). This lepton asymmetry is converted into a net baryon asymmetry by SM non-perturbative processes, known as sphalerons \([9]\) (see Section 4.3.1), which violate \( B \) and \( B+L \) (but not \( B-L \)). The sphalerons are at equilibrium, in the early Universe, up to a temperature close to the one of the electroweak phase transition \( T_c \approx 150 \text{ GeV} \) (cf. (4.40) below).

The leptonic \( CP \)-asymmetries \( \varepsilon_{l\alpha} \) are defined in terms of the partial decay widths \( \Gamma_{l\alpha} \equiv \Gamma(N_{\alpha} \rightarrow L_l \Phi) \) and their \( CP \)-conjugates \( \Gamma^c_{l\alpha} \equiv \Gamma(N_{\alpha} \rightarrow L_l^c \Phi^c) \):

\[
\varepsilon_{l\alpha} = \frac{\Gamma_{l\alpha} - \Gamma^c_{l\alpha}}{\sum_k (\Gamma_{k\alpha} + \Gamma^c_{k\alpha})} \equiv \frac{\Delta \Gamma_{l\alpha}}{\Gamma_{N_{\alpha}}}, \tag{4.1}
\]

where \( \Gamma_{N_{\alpha}} \) is the total decay width of the heavy Majorana neutrino \( N_{\alpha} \) and we have introduced the shorthand superscript \( c \) to denote \( CP \) conjugation. The physical \( CP \)-violating observable defined in (4.1) receives contributions from two different sources (see Figure 4.1): (i) \( \varepsilon \)-type \( CP \) violation due to the interference between the tree-level and the absorptive part of the self-energy graphs in the heavy-neutrino decay, and (ii) \( \varepsilon^t \)-type \( CP \).
violation due to the interference between the tree-level graph and the absorptive part of the one-loop vertex. This terminology is analogous the one used for the $K^0\bar{K}^0$-system (for a review, see e.g. [1]), where $\varepsilon$ represents the indirect $CP$ violation through $K^0-\bar{K}^0$ mixing, while $\varepsilon'$ represents the direct $CP$ violation entirely due to the decay amplitude.

If one assumes that the masses of the heavy neutrinos are hierarchical, i.e. $m_{N_1} \ll m_{N_2} \ll m_{N_3}$, the total $CP$-asymmetry in the decay of $N_\alpha$, i.e. $\varepsilon_\alpha \equiv \sum_l \varepsilon_{l\alpha}$, is found to be [4, 24–26]

$$
\varepsilon_\alpha = \frac{1}{8\pi} \sum_{\beta \neq \alpha} \text{Im}\left[\frac{\left(\hat{h}^\dagger \hat{h}\right)_{\alpha\beta}^2}{\left(\hat{h}^\dagger \hat{h}\right)_{\alpha\alpha}}\right] \frac{m_{N_1}}{m_{N_\alpha}} \left[\frac{m_{N_\alpha}^2}{m_{N_\alpha}^2 - m_{N_\beta}^2} + 1 - \left(1 + \frac{m_{N_1}^2}{m_{N_\alpha}^2}\right) \log \left(1 + \frac{m_{N_\alpha}^2}{m_{N_\beta}^2}\right)\right].
$$

(4.2)

In this case, the final asymmetry is typically dominated by the decays of the lightest RH neutrino $N_1$. The relevant asymmetry $\varepsilon_1$, obtained from (4.2) in the limit $m_{N_1}/m_{N_{2,3}} \to 0$, is

$$
\varepsilon_1 \approx -\frac{3}{16\pi} \sum_{\beta=2,3} \text{Im}\left[\frac{\left(\hat{h}^\dagger \hat{h}\right)_{1\beta}^2}{\left(\hat{h}^\dagger \hat{h}\right)_{11}}\right] \frac{m_{N_1}}{m_{N_\beta}}.
$$

(4.3)

The final lepton asymmetry is proportional to this value, with the proportionality constant depending on the non-equilibrium evolution of the system, in particular on the amount of washout of the generated asymmetry (in the case of strong washout, cf. (4.34) below).

In the original scenario [4], the masses of the heavy neutrinos were taken close to the GUT scale $M_{\text{GUT}} \sim 10^{16}$ GeV. For Yukawa couplings smaller than 1, the seesaw

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[1] There is an exception, due to a particular flavour structure of the neutrino Yukawa couplings, when the contribution from $N_2$ is not washed out by $N_1$ interactions [64, 65].
4.2. RESONANT ENHANCEMENT OF THE ASYMMETRY

The mechanism requires $M_N \lesssim 10^{14} - 10^{15}$ GeV. Making use of the seesaw relation (cf. (9.10) in Chapter 9), it is possible to relate (4.3) to the light-neutrino mass scale and, requiring $\varepsilon_1 \gtrsim 10^{-6}$ as a typical value for successful leptogenesis, one obtains the Davidson-Ibarra bound [16]

$$m_{N_1} \gtrsim 4 \times 10^9 \text{GeV}.$$ (4.4)

As a consequence, such hierarchical models are difficult to test in foreseeable laboratory experiments. In addition, the thermal production of $N_1$ requires a reheating temperature $T_R$, at the end of inflation, at least of this order. This would be in conflict with supergravity theories, which require an upper bound on the reheating temperature of the Universe, $T_R \lesssim 10^6 - 10^9$ GeV, required to avoid overproduction of gravitinos, whose decays may spoil the success of Big Bang Nucleosynthesis (for a list of references, see e.g. [20]). Although the bound (4.4) can be mildly alleviated, e.g. by taking into account flavour effects [36], the tension with supergravity persists. As we will see in the next sections and in Chapter 9, these issues are avoided when the heavy neutrinos become nearly degenerate, i.e. in the Resonant Leptogenesis scenario, which is the main focus of this thesis.

4.2 Resonant enhancement of the asymmetry

When the heavy neutrinos become nearly degenerate, the denominator of the first term inside square brackets in (4.2) threatens to vanish, signaling the breakdown of perturbation theory. In this regime, $\varepsilon$-type effects get resonantly enhanced and become dominant, so that one needs to resum the insertions of absorptive self-energies in the decays of the heavy neutrinos. RL allows one to have very large $CP$ asymmetries, potentially of order 1, without the need of considering RH neutrinos heavier than $10^9$ GeV, such as to satisfy the bound (4.4). This bound is valid only in the hierarchical regime, where the maximal asymmetry is proportional to $m_{N_1}$ [17], once the light-neutrino constraints coming from the seesaw mechanism are imposed in (4.3). Therefore, RL provides a powerful mechanism for the generation of the observed BAU with BSM physics at scales
much lower than the ones required in the hierarchical case. We will give an example of such models in Chapter 9, where the observed BAU is generated by electroweak-scale Majorana neutrinos which provide, in addition, a rich phenomenology at the LHC and in low-energy neutrino experiments at the intensity frontier.

The heavy neutrinos are unstable particles, and thus they cannot be described by asymptotic in and out states of an $S$-matrix theory. Instead, following [23], their properties can be inferred from the transition matrix elements of $2 \leftrightarrow 2$ scatterings of stable particles, by identifying their resonant part corresponding to the propagation of a heavy-neutrino mass eigenstate. This allows one to perform an effective resummation of the heavy-neutrino self-energy diagrams contributing to the $\varepsilon$-type $CP$-asymmetry [22, 23], neglecting thermal effects. In Chapter 8 we will study this problem within the context of a non-equilibrium formulation of thermal QFT, generalizing the resummation outlined here to take into account thermal effects (see also Appendix C for technical details).

Neglecting thermal effects, the absorptive part of the heavy Majorana neutrino 1-loop self-energy transitions $N_\beta \rightarrow N_\alpha$ can be written as follows:

$$
\Sigma_{\alpha\beta}^{\text{abs}}(p) = A_{\alpha\beta}(p^2) P_L + A_{\alpha\beta}^*(p^2) P_R ,
$$

(4.5)

where $P_{L,R} = (1_4 \mp \gamma_5)/2$ are the left- and right-chiral projection operators respectively and $A_{\alpha\beta}$ is the absorptive transition amplitude, summed over all charged-lepton flavours running in the loop:

$$
A_{\alpha\beta}(\hat{h}) = \frac{(\hat{h}^\dagger \hat{h})_{\alpha\beta}}{16\pi} = \frac{1}{16\pi} \sum_l \hat{h}_{l\alpha} \hat{h}_{l\beta}^* \equiv \sum_l A_{l\alpha\beta}(\hat{h}) .
$$

(4.6)

The tree-level decay width of the heavy Majorana neutrino $N_\alpha$ is obtained as

$$
\Gamma_{N_\alpha}^{(0)} = 2m_{N_\alpha} A_{\alpha\alpha}(\hat{h}) = \frac{m_{N_\alpha}}{8\pi} (\hat{h}^\dagger \hat{h})_{\alpha\alpha} .
$$

(4.7)

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1For a critical appraisal of the alternative approaches in [66–68], see Appendix A of [28].
To account for unstable-particle mixing effects between the heavy Majorana neutrinos, we introduce $CP$-violating effective vertices, described by the resummed Yukawa couplings, denoted by (bold-faced Latin) $h_{l\alpha}$, and their $CP$-conjugates $h_{c l\alpha}$, related to the matrix elements $\mathcal{M}(N_\alpha \to L_l \Phi)$ and $\mathcal{M}(N_\alpha \to L^c_l \Phi^c)$, respectively. This formalism captures all dominant effects of heavy-neutrino mixing and $CP$-violation, and is equivalent \cite{23} to an earlier proposed resummation method \cite{22} based on a generalized LSZ reduction formalism \cite{69}. Working in the heavy neutrino mass eigenbasis, in the 3 heavy-neutrino case the resummed effective Yukawa couplings are given by \cite{23, 70}

\[
\hat{h}_{l\alpha} = \hat{h}_{l\alpha} - i \sum_{\beta, \gamma} |\epsilon_{\alpha\beta\gamma}| \hat{h}_{l\beta} \times \\
m_\alpha (m_\alpha A_{\alpha\beta} + m_\beta A_{\beta\alpha}) - i R_{\alpha\beta} [m_\alpha A_{\gamma\beta} (m_\alpha A_{\alpha\gamma} + m_\gamma A_{\gamma\alpha}) + m_\beta A_{\beta\gamma} (m_\alpha A_{\gamma\alpha} + m_\gamma A_{\alpha\gamma})] \\
m_\alpha^2 - m_\beta^2 + 2i m_\alpha^2 A_{\beta\beta} + 2i \text{Im}(R_{\alpha\gamma}) [m_\alpha^2 |A_{\beta\gamma}|^2 + m_\beta m_\gamma \text{Re}(A_{\beta\gamma})],
\]

(4.8)

where $\epsilon_{\alpha\beta\gamma}$ is the usual Levi-Civita anti-symmetric tensor, $m_\alpha^2 \equiv m_{N_\alpha}^2$ is a shorthand notation introduced for brevity, and

\[
R_{\alpha\beta} = \frac{m_\alpha^2}{m_\alpha^2 - m_\beta^2 + 2i m_\alpha^2 A_{\beta\beta}}.
\]

(4.9)

All the transition amplitudes $A_{\alpha\beta} \equiv A_{\alpha\beta} (\hat{h})$ in (4.8) are evaluated OS with $p^2 = m_{N_\alpha}^2$.

The respective $CP$-conjugate resummed effective Yukawa couplings $\hat{h}_{c l\alpha}$ can be obtained from (4.8) by replacing the tree-level Yukawa couplings $h_{l\alpha}$ with their complex conjugates $h^*_l$.\footnote{Note that $h^c \neq h^*$ in general, whereas for the tree-level Yukawa couplings, $h^c = h^*$ by $CPT$-invariance of the Lagrangian.}

In the two heavy-neutrino case, (4.8) simplifies considerably, taking the form

\[
\hat{h}_{l\alpha} = \hat{h}_{l\alpha} - i \hat{h}_{l\beta} \frac{m_{N_\alpha} (m_{N_\alpha} A_{\alpha\beta} + m_{N_\beta} A_{\beta\alpha})}{m_{N_\alpha}^2 - m_{N_\beta}^2 + 2i m_{N_\alpha}^2 A_{\beta\beta}} ,
\]

(4.10)

with $\beta \neq \alpha$ and having neglected $\epsilon'$ effects, which are numerically insignificant in RL.

The partial decay widths $\Gamma_{\alpha l}$ and $\Gamma^*_{\alpha l}$ appearing in (4.1) can be expressed in terms of the
effective Yukawa couplings \( \hat{h}_{\alpha \alpha} \) and \( \hat{h}_{c \alpha} \), and the flavour-dependent absorptive transition amplitudes \( A_{\alpha \beta}^l(\hat{h}) \), as follows:

\[
\Gamma_{l\alpha} = m_{N_\alpha} A_{\alpha \alpha}^l(\hat{h}), \quad \Gamma_{c\alpha} = m_{N_\alpha} A_{\alpha \alpha}^c(\hat{h}^c).
\]

(4.11)

Hence, the total decay width of the heavy neutrino is obtained by summing over all lepton flavours:

\[
\Gamma_{N_\alpha} = \sum_l (\Gamma_{l\alpha} + \Gamma_{c\alpha}) = \frac{m_{N_\alpha}}{16\pi} \left[ (\hat{h}^\dagger \hat{h})_{\alpha\alpha} + (\hat{h}^c \hat{h}^c)_{\alpha\alpha} \right].
\]

(4.12)

Substituting (4.11) in (4.1), the flavour-dependent leptonic CP-asymmetry in RL can be written as

\[
\varepsilon_{l\alpha} = \frac{|\hat{h}_{\alpha\alpha}|^2 - |\hat{h}_{c\alpha}|^2}{\sum_k (|\hat{h}_{k\alpha}|^2 + |\hat{h}_{c\alpha}|^2)} = \frac{|\hat{h}_{\alpha\alpha}|^2 - |\hat{h}_{c\alpha}|^2}{(\hat{h}^\dagger \hat{h})_{\alpha\alpha} + (\hat{h}^c \hat{h}^c)_{\alpha\alpha}}.
\]

(4.13)

Note that (4.13) encodes both \( \varepsilon \)- and \( \varepsilon' \)-type CP asymmetries, although the latter is numerically negligible in resonant scenarios. In the two heavy-neutrino case, (4.13) takes on the simple form:

\[
\varepsilon_{l\alpha} \approx \frac{\text{Im}[\hat{h}_{\alpha\alpha}]^2}{(\hat{h}^\dagger \hat{h})_{\alpha\alpha} (\hat{h}^c \hat{h}^c)_{\alpha\alpha}} f_{\text{reg}}
\]

(4.14)

where \( \alpha, \beta = 1, 2 \) (\( \alpha \neq \beta \)), and the self-energy regulator is given by [22, 23]

\[
f_{\text{reg}} = \frac{\left( m_{N_\alpha}^2 - m_{N_\beta}^2 \right) m_{N_\alpha} \Gamma_{N_\beta}^{(0)}}{\left( m_{N_\alpha}^2 - m_{N_\beta}^2 \right)^2 + \left( m_{N_\alpha} \Gamma_{N_\beta}^{(0)} \right)^2}.
\]

(4.15)

---

1The Yukawa structure in (4.14) agrees with [26], but differs from that given in [61, 71], which is instead proportional to \( \text{Im}[\hat{h}_{\alpha\beta}]^2 \). This difference will be crucial for the discussion in Section 9.1. The second term in the numerator in the first line of (4.14) vanishes if one sums over \( l \).
In the degenerate heavy-neutrino mass limit $\Delta m_N \equiv (m_{N_1} - m_{N_2}) \to 0$, the would-be singular behaviour of the $CP$-asymmetry is regularized by the absorptive term $(m_{N_\alpha} \Gamma_{N\beta}^{(0)})^2$ in the denominator on the right-hand side (RHS) of (4.15). Based on the simplified expression (4.14), the following necessary conditions for maximal resonant enhancement of the leptonic $CP$-asymmetry may be derived:

\begin{align}
&\text{(i) } \Delta m_N \sim \frac{\Gamma_{N_{1,2}}}{2} \ll m_{N_{1,2}}, \quad \text{(ii) } \left| \frac{2 \text{Im} [\hat{h}_{l\alpha}^* \hat{h}_{l\beta}] \text{Re} [(\hat{h}^\dagger \hat{h})_{\alpha\beta}]}{(\hat{h}^\dagger \hat{h})_{\alpha\alpha} (\hat{h}^\dagger \hat{h})_{\beta\beta}} \right| \sim 1.
\end{align}

(4.16)

4.3 Flavour-diagonal rate equations for Resonant Lepto-genesis

The time-evolution of the number-density $n^a$ of any particle species $a$ can be derived from the Boltzmann equation (3.49), obtained neglecting thermal masses and quantum-statistical factors in (3.48). In order to obtain rate equations, written in terms of the total number-densities, we assume that all species are in kinetic equilibrium, i.e.

\begin{equation}
n^a = g_a \int_p n^a(p) = g_a \int_p \frac{1}{\exp [(E_a(p) - \mu_a)/T] \pm 1},
\end{equation}

(4.17)

where the $-$ ($+$) sign in the denominator corresponds to particles obeying Bose-Einstein (Fermi-Dirac) quantum statistics, $E_a(p) = (|p|^2 + m_a^2)^{1/2}$ is the relativistic energy of the species $a$, $m_a$ being its rest mass, $g_a = g_h^a g_{iso}^a$ is the total degeneracy factor of the internal degrees of freedom, $g_h^a$ and $g_{iso}^a$ being the degenerate helicity and degenerate isospin degrees of freedom respectively, and $\mu_a \equiv \mu_a(T)$ is the temperature-dependent chemical potential. It will prove convenient, for our later discussion, to define an in-equilibrium number density $n_{eq}^a$ as the limit $\mu_a \to 0$ in (4.17). We note however that the true equilibrium number-density in general will have a nonzero chemical potential $\mu_{eq}^a$.

The limit of (4.17) of interest here is the Maxwell-Boltzmann classical-statistical
CHAPTER 4. LEPTOGENESIS

regime, in which we can drop the ±1 term in the denominator of (4.17), giving

\[ n^a(T) = g_a \int_\mathbf{p} e^{-|E_a(p) - \mu_a(T)|/T} = \frac{g_a m_a^2 T}{2\pi^2} K_2 \left( \frac{m_a}{T} \right) \frac{e^{\mu_a(T)/T}}{T} , \]  

(4.18)

where \( K_n(x) \) is the \( n \)th-order modified Bessel function of the second kind. In addition, in the following we will also need the relativistic limit \( (T \gg m_a, \mu_a) \), in which case

\[ n^a(T) = \frac{\sigma_X \zeta(3)}{\pi^2} g_a T^3 , \]  

(4.19)

where \( \sigma_X = 1 \ (3/4) \) for bosons (fermions), and \( \zeta(x) \) is the Riemann zeta function, with \( \zeta(3) \approx 1.20206 \).

Following [59, 60], the Boltzmann equation (3.49), generalized to an expanding Friedmann-Robertson-Walker Universe, can be written, under the assumptions above, as a rate equation

\[ \frac{dn^a}{dt} + 3Hn^a = - \sum_{aX \leftrightarrow Y} \left[ \frac{n^a n^X}{n_{eq}^X n_{eq}^Y} \gamma(aX \rightarrow Y) - \frac{n^Y}{n_{eq}^Y} \gamma(Y \rightarrow aX) \right] , \]  

(4.20)

where the drift terms on the left-hand side (LHS) include the dilution of the number-density due to the expansion of the Universe, parametrized by the Hubble expansion rate \( H \). The thermally averaged rate \( \gamma(X \rightarrow Y) \) is given by

\[ \gamma(X \rightarrow Y) = \int d\Pi_X d\Pi_Y (2\pi)^4 \delta^{(4)}(p_X - p_Y) e^{-p_X^2/T} |\mathcal{M}(X \rightarrow Y)|^2 \]  

\[ \equiv \int_{X_Y} |\mathcal{M}(X \rightarrow Y)|^2 . \]  

(4.21)

Here, the squared matrix element \( |\mathcal{M}(X \rightarrow Y)|^2 \) is summed, but not averaged, over the internal degrees of freedom of the initial and final multiparticle states \( X \) and \( Y \) [23]. We have also introduced an abbreviated notation \( \int_{X_Y} \) in (4.21) for the thermal average over \( X \) and \( Y \). The RHS of (4.20) comprises the collision terms accounting for the interactions that change the number density \( n^a \). If the species \( a \) is unstable, it can occur as a real
4.3. FLAVOUR-DIAGONAL RATE EQUATIONS FOR RL

intermediate state (RIS) in resonant processes of the form \( X \rightarrow a \rightarrow Y \), which must be properly taken into account in order to avoid double-counting the decays and inverse decays of \( a \) [59]. At this point, it is important to note that the formalism leading to (4.20) neglects the purely-quantum effect of coherences between different flavours (or species), which will be described, in Chapter 5, by off-diagonal entries in the matrix of number-densities. For this reason, we refer to (4.20) as a set of flavour-diagonal Boltzmann equations.

The Hubble expansion rate in the early Universe is given as a function of the temperature \( T \) by [72]

\[
H(T) \approx 1.66 g_*^{1/2} \frac{T^2}{M_{\text{Pl}}} ,
\]

(4.22)

where \( M_{\text{Pl}} = 1.2 \times 10^{19} \) GeV is the Planck mass and \( g_*(T) \) is the effective number of relativistic degrees of freedom. Following [23], we define the \( CP \)-conserving collision rate for a generic process \( X \rightarrow Y \) and its \( CP \)-conjugate \( X^c \rightarrow Y^c \) as

\[
\gamma_X^Y \equiv \gamma(X \rightarrow Y) + \gamma(X^c \rightarrow Y^c) .
\]

(4.23)

By virtue of \( CPT \) invariance, the \( CP \)-conserving collision rates must obey \( \gamma_X^Y = \gamma_Y^X \). Analogous to (4.23), a \( CP \)-violating collision rate can be defined as [23]

\[
\delta\gamma_X^Y = \gamma(X \rightarrow Y) - \gamma(X^c \rightarrow Y^c) ,
\]

(4.24)

which obeys \( \delta\gamma_X^Y = -\delta\gamma_Y^X \), following \( CPT \) invariance.

The relevant Boltzmann equations for describing leptogenesis are those involving the number-densities \( n_{\alpha}^{N} \) (with \( \alpha = 1, \ldots, N_N \)) of the heavy Majorana neutrinos, and \( n_l^L \) and \( \bar{n}_l^L \) (with \( l = 1, \ldots, N_L \)) of the lepton and anti-lepton doublets. It is convenient to introduce a new variable \( z = m_{N_1}/T \) to parametrize the evolution of the system. In the radiation-dominated epoch, relevant to the production of the asymmetry, \( z \) is related to
the cosmic time $t$ via the relation $t = z^2 / 2 H_N$, where

$$H_N \equiv H(z = 1) \simeq 17 \frac{m_{N_1}^2}{M_{Pl}}$$

is the Hubble parameter (4.22) at $z = 1$, assuming only SM relativistic degrees of freedom. We also introduce the normalized number-densities with respect to the one of photons $\eta^a(z) = n^a(z)/n^\gamma(z)$, with $n^\gamma$ given by (4.19) for $\sigma_\chi = 1$ and $g_\gamma = 2$, i.e.

$$n^\gamma(z) = \frac{2T^3\zeta(3)}{\pi^2} = \frac{2m_{N_1}^3\zeta(3)}{\pi^2 z^3}. \quad (4.26)$$

With these definitions, we may write down the flavour-diagonal Boltzmann equations (4.20) in terms of the normalized number-densities of heavy neutrinos $\eta_{\alpha}^N$ and the normalized lepton asymmetries $\delta \eta^l_\alpha = (n^l_\alpha - \bar{n}^l_\alpha)/n^\gamma$ as [27, 61]:

$$\frac{n^\gamma H_N}{z} \frac{d\eta_{\alpha}^N}{dz} = \left(1 - \frac{\eta_{\alpha}^N}{\eta_{\text{eq}}^N}\right) \sum_L \gamma_{L\alpha}^{N\Phi}, \quad (4.27a)$$

$$\frac{n^\gamma H_N}{z} \frac{d\delta \eta^L_\alpha}{dz} = \sum_{\alpha} \left(\frac{\eta_{\alpha}^N}{\eta_{\text{eq}}^N} - 1\right) \delta \gamma_{L\alpha}^{N\Phi} - \frac{2}{3} \delta \eta^L_\alpha \sum_k \left(\gamma_{Lk}^{\Phi\phi} + \gamma_{Lk}^{L\phi}\right) - \frac{2}{3} \sum_k \delta \eta^L_k \left(\gamma_{Lk}^{L\phi\phi} - \gamma_{Lk}^{L\phi}\right), \quad (4.27b)$$

where $\eta_{\text{eq}}^N \approx z^2 K_2(z)/2$ is the normalized equilibrium number-density of the heavy neutrinos. The explicit expression of the various collision rates appearing in (4.27a) and (4.27b) will be given below. Here, we have included only the dominant contributions arising from the $1 \to 2$ decays and $2 \to 1$ inverse decays of the heavy neutrinos, involving the rate $\gamma_{L\alpha}^{N\Phi}$, and the resonant part of the $2 \leftrightarrow 2$ $\Delta L = 0$ and $\Delta L = 2$ scatterings, proportional to $\gamma_{L\alpha}^{L\phi}$ and $\gamma_{L\alpha}^{L\phi}$ respectively. We have ignored the sub-dominant chemical-potential contributions from the RH charged-lepton, quark and the Higgs fields, as well as the $\Delta L = 1$ Yukawa and gauge scattering terms, whose effect is studied in [27]. In (4.27b), the washout (i.e. proportional to the asymmetry itself) contribution due to the decay term has been combined with the RIS-subtracted one due to scattering processes.
Thus, in (4.27b) the total scattering rates appear, rather than the RIS-subtracted ones, whereas the decay washout rate does not appear explicitly, being coincident with the RIS part of the scattering rates. This point is discussed in more detail below and in Section 6.4 (see also [61]).

The \( CP \)-odd collision rate \( \delta \gamma_{N_{\alpha}\Phi}^{L_{\Phi}} \) in (4.27b) can be obtained as

\[
\delta \gamma_{L_{\Phi}^{\Phi}}^{N_{\alpha}} = \varepsilon l_{\alpha} \sum_{k} \gamma_{L_{k}^{\Phi}}^{N_{\alpha}},
\]

(4.28)

with the \( CP \)-conserving decay rate given by [23]

\[
\gamma_{L_{\Phi}^{\Phi}}^{N_{\alpha}} \equiv \sum_{l} \gamma_{L_{l}^{\Phi}}^{N_{\alpha}} = \frac{m_{N_{\alpha}}^{3}}{\pi^{2} z} K_{1}(z) \Gamma_{N_{\alpha}}.
\]

(4.29)

The collision rates for the \( \Delta L = 0 \) and \( \Delta L = 2 \) scatterings are found to be [61]

\[
\gamma_{L_{l}^{\Phi}}^{N_{\alpha} \Phi} = \sum_{\alpha,\beta} \left( \gamma_{L_{l}^{\Phi}}^{N_{\alpha} \Phi} + \gamma_{L_{l}^{\Phi}}^{N_{\beta} \Phi} \right) \frac{2 \left( \hat{h}_{l_{\alpha}}^{\Phi} \hat{h}_{l_{\alpha}}^{c} \hat{h}_{l_{\beta}} \hat{h}_{l_{\beta}}^{c} + \hat{h}_{l_{\alpha}}^{c} \hat{h}_{l_{\alpha}}^{\Phi} \hat{h}_{l_{\beta}} \hat{h}_{l_{\beta}}^{c} \right)}{\left( 1 - 2i \frac{m_{N_{\alpha}} - m_{N_{\beta}}}{\Gamma_{N_{\alpha}} + \Gamma_{N_{\beta}}} \right) \left( \hat{h}_{l_{\alpha}}^{\Phi} \hat{h}_{l_{\alpha}}^{c} + \hat{h}_{l_{\beta}}^{\Phi} \hat{h}_{l_{\beta}}^{c} \right)^{2}},
\]

(4.30a)

\[
\gamma_{L_{l}^{\Phi} c l_{\Phi}^{c}}^{N_{\alpha} \Phi} = \sum_{\alpha,\beta} \left( \gamma_{L_{l}^{\Phi} c l_{\Phi}^{c}}^{N_{\alpha} \Phi} + \gamma_{L_{l}^{\Phi} c l_{\Phi}^{c}}^{N_{\beta} \Phi} \right) \frac{2 \left( \hat{h}_{l_{\alpha}}^{\Phi} \hat{h}_{l_{\alpha}}^{c} \hat{h}_{l_{\beta}} \hat{h}_{l_{\beta}}^{c} + \hat{h}_{l_{\alpha}}^{c} \hat{h}_{l_{\alpha}}^{\Phi} \hat{h}_{l_{\beta}} \hat{h}_{l_{\beta}}^{c} \right)}{\left( 1 - 2i \frac{m_{N_{\alpha}} - m_{N_{\beta}}}{\Gamma_{N_{\alpha}} + \Gamma_{N_{\beta}}} \right) \left( \hat{h}_{l_{\alpha}}^{\Phi} \hat{h}_{l_{\alpha}}^{c} + \hat{h}_{l_{\beta}}^{\Phi} \hat{h}_{l_{\beta}}^{c} \right)^{2}},
\]

(4.30b)

where we have used the NWA for the resummed heavy neutrino propagators:

\[
\frac{1}{(s - m_{N_{\alpha}}^{2})^{2} + m_{N_{\alpha}}^{2} \Gamma_{N_{\alpha}}^{2}} \approx \frac{\pi}{m_{N_{\alpha}} \Gamma_{N_{\alpha}}} \delta(s - m_{N_{\alpha}}^{2}) \theta(\sqrt{s}),
\]

(4.31)

since we are only interested in the resonant part of these \( 2 \leftrightarrow 2 \) scatterings in the RL case. The RIS contribution is given by the diagonal \( \alpha = \beta \) elements in the sums in (4.30), and its contribution to the rate equation (4.27b) coincides with the washout due to heavy-
neutrino inverse decays. Separating this from $\alpha \neq \beta$ terms, we obtain [27]

$$\frac{n^\gamma H_N}{z} \frac{d\delta \eta_L^L}{dz} = \sum_{\alpha} \left( \frac{n^\gamma_N}{n^\gamma_{eq}} - 1 \right) \varepsilon_{la} \gamma_{N\Phi}^{N_{\alpha}} - \frac{2}{3} \delta \eta_L^L \left[ \sum_{\alpha} B_{la} \gamma_{L\Phi}^{N_{\alpha}} + \sum_{k} \left( \gamma_{L_k^L \Phi}^{\nu_{\alpha}} + \gamma_{L_k^L \Phi}^{\mu_{\alpha}} \right) \right]$$

$$- \frac{2}{3} \sum_{k} \delta \eta_L^L \left[ \sum_{\alpha} \varepsilon_{la} \delta \gamma_{L_k^L \Phi}^{N_{\alpha}} + \left( \gamma_{L_k^L \Phi}^{\nu_{\alpha}} - \gamma_{L_k^L \Phi}^{\mu_{\alpha}} \right) \right], \quad (4.32)$$

which shows manifestly that only washout due to inverse decays and RIS-subtracted scattering is present, with no double counting of the former. In (4.32), we have introduced the heavy-neutrino decay branching ratio $B_{l\alpha} = \left( \Gamma_{l\alpha} + \Gamma_{l\alpha}^c \right)/\Gamma_{N_{\alpha}}$, and $\gamma_{l\alpha}^X \equiv \gamma_{l\alpha}^X - (\gamma_{l\alpha}^X)_{RIS}$ denote the RIS-subtracted collision rates, which can be obtained from (4.30a) and (4.30b) taking $\alpha \neq \beta$.

In the strong washout regime, i.e. when the heavy-neutrino and charged-lepton $K$-factors

$$K_{l\alpha} = \frac{\Gamma_{N_{\alpha}}}{\zeta(3) H_N}, \quad (4.33a)$$

$$K_{l\alpha}^{\text{eff}} = \frac{\pi^2 z}{\zeta(3) H_N m_N^3 (H_1)^{1/2}(z)} \left[ \sum_{k} \left( \gamma_{L_k^L \Phi}^{\nu_{\alpha}} + \gamma_{L_k^L \Phi}^{\mu_{\alpha}} \right) + \gamma_{L_k^L \Phi}^{\nu_{\alpha}} - \gamma_{L_k^L \Phi}^{\mu_{\alpha}} \right], \quad (4.33b)$$

are much greater than 1, the evolution of the system is very close to thermal equilibrium, and the rate equations (4.27a) and (4.27b) possess an attractor solution, which is reached after the system evolves for a sufficient amount of time. Introducing $\varepsilon_l = \sum_{\alpha} \varepsilon_{l\alpha}$, the attractor solution for the total lepton asymmetry, in the regime $z > z_l^1 \approx 2(K_{l\alpha}^{\text{eff}})^{-1/3}$, is found to be [61]

$$\delta \eta_L^L = \delta \eta_{\text{mix}}^L \equiv \frac{3}{2z} \sum_l \varepsilon_l / K_{l\alpha}^{\text{eff}}, \quad (4.34)$$

up to a point $z = z_l^3 \approx 1.25 \ln(25 K_{l\alpha}^{\text{eff}})$, beyond which the lepton asymmetry freezes out and approaches a constant value $\delta \eta_{\text{mix}}^L = (3/2) \sum_l \varepsilon_l / (K_{l\alpha}^{\text{eff}} z_l^3)$ [61].
4.3. FLAVOUR-DIAGONAL RATE EQUATIONS FOR RL

4.3.1 Observed lepton asymmetry

Now, we need to convert the net lepton asymmetry $\sum \delta \eta^L_l$ to the asymmetry in the total baryon-to-photon ratio $\delta \eta^B = (n^B - \bar{n}^B)/n^\gamma$ via $(B + L)$-violating sphaleron interactions. These are non-perturbative saddle-point solutions (instantons) of the electroweak path-integral that interpolate between different vacua related by a topological gauge transformation [73] (i.e. a gauge transformation that cannot be continuously deformed to the identity). They can be seen as tunneling transitions between the topologically nonequivalent vacua and their rate is exponentially suppressed at zero temperature. However, at high temperatures they become effective [9], once the thermal energy is sufficient to pass over the potential barrier between the different vacua, rather than tunnel through it. In each sphaleron transition, the baryon and lepton numbers are violated by one unit per generation because of the Adler-Bell-Jackiw anomaly [74, 75], whereas $B - L$ is conserved, since it is not anomalous in the SM.

In a sphaleron transition, an $SU(3)_c$ and $SU(2)_L$-singlet neutral object from each generation is created out of the vacuum [9, 73, 76, 77]. The operator responsible for sphaleron transitions can be written as

$$O_{B+L} = \prod_{i=1}^{3} \epsilon_{klm} \epsilon_{def} [Q^d_k Q^f_l Q^e_m L_n]_i ,$$  

(4.35)

where $i$ is the family index; $d,e,f$ are the $SU(3)_c$ colour indices; $k,l,m,n$ are the $SU(2)_L$ isospin indices; and $Q = (u_L \ d_L)^T$ is the $SU(2)_L$ quark doublet. The operator $O_{B+L}$ is invariant under both gauge transformations and $U(3)$ flavour rotations. Above the electroweak phase transition, all the SM processes are in chemical equilibrium. In particular, the sphaleron interactions in (4.35) and the reactions $\Phi \leftrightarrow \bar{Q} + u_R$ and $\Phi \leftrightarrow Q + \bar{d}_R$ give the following relations among the chemical potentials:

$$9 \mu_Q + \sum_l \mu_{L_l} = 0 , \quad \mu_\Phi = -\mu_Q + \mu_{u_R} , \quad \mu_\Phi = \mu_Q - \mu_{d_R} ,$$  

(4.36)
where \( u_R \) and \( d_R \) are the up- and down-type \( SU(2)_L \) quark singlets. We stress here that the chemical potentials appearing in (4.36) are per isospin and colour degrees of freedom. The total chemical potentials for the baryon and lepton numbers are then given by

\[
\mu_B = 3(2\mu_Q + \mu_{u_R} + \mu_{d_R}) = -\frac{4}{3} \sum_q \mu_{L_q}, \quad \mu_L = 2 \sum_q \mu_{L_q}.
\]

(4.37)

The factors of 2 are due to the sum over isospin of the LH doublets, whereas the overall factor of 3 in \( \mu_B \) comes from summing over 3 colours and 3 flavours, times a factor of 1/3 for the baryon number of each quark. Using the relations (4.37) into (4.18) at \( \mathcal{O}(\frac{\mu}{T}) \), we obtain the conversion of the lepton asymmetry in the LH doublet \( \delta \eta^L \) to the baryon asymmetry\(^1\)

\[
\delta \eta^B = -\frac{2}{3} \delta \eta^L.
\]

(4.38)

This is the key relation that allows one to relate the asymmetry generated in leptogenesis scenarios with the observed BAU. This relation is valid at temperatures \( T > T_c \), where \( T_c \) is the critical temperature for the electroweak phase transition, given at one loop by \[78\]

\[
T_c^2 = \frac{1}{4D_c} \left[ m_H^2 - \frac{3}{8\pi^2 v^2} \left( 2m_W^4 + m_Z^4 - 4m_t^4 \right) - \frac{1}{8\pi^2 v^4 D_c} \left( 2m_W^3 + m_Z^3 \right)^2 \right].
\]

(4.39)

Here, \( D_c \equiv (2m_W^2 + m_Z^2 + 2m_t^2 + m_H^2)/8v^2 \), \( v = 2^{-1/4}G_F^{-1/2} = 246.2 \) GeV (with \( G_F \) being the Fermi coupling constant), is the electroweak vacuum expectation value (VEV), \( m_H \) is the zero-temperature Higgs boson mass and \( m_W, m_Z, m_t \) are the \( W, Z \) boson masses and top-quark mass respectively, defined at the electroweak scale. Using the recent experimental values of the SM mass parameters \( m_W = 80.385(15) \) GeV, \( m_Z = 91.1876(21) \) GeV [1], \( m_t = 173.34 \pm 0.76 \) GeV [79], and \( m_H = 125.5^{+0.5}_{-0.6} \)

---

\(^1\)Note that since we are converting the asymmetry stored in the lepton doublet, the conversion coefficient derived here is different from 28/51 used elsewhere (see e.g. [61]), which corresponds to the total lepton asymmetry, including the RH leptons.
4.3. FLAVOUR-DIAGONAL RATE EQUATIONS FOR RL

GeV [80], we find

\[ T_c = 149.4^{+0.7}_{-0.8} \text{ GeV}. \] (4.40)

For \( T < T_c \), the sphalerons become ineffective, and the baryon asymmetry decouples from the lepton one. At this point, the value obtained in (4.38) gets diluted by standard photon interactions until the recombination epoch at temperature \( T_0 \). Assuming that there is no mechanism for significant entropy release while the Universe is cooling down from \( T_c \) to \( T_0 \), the ratio \( n_B/s \) is constant during this epoch. Here, \( s = (2\pi^2/45)g_sT^3 \) is the entropy density and \( g_s \) is the corresponding effective number of relativistic degrees of freedom. Thus, the baryon-to-photon ratio \( \eta^B = \pi^4 g_s/(45\zeta(3)) n_B/s \) at the recombination epoch is given by

\[ \delta\eta^B_0 = g_s(T_0)/g_s(T_c) \delta\eta^B \approx \frac{\delta\eta^B}{27.3}, \] (4.41)

where we have used \( g_s(T_c) = 106.75 \) and \( g_s(T_0) = 3.91 \) [72].

The theoretical prediction (4.41) should be compared with the observed value today, which remains approximately unchanged from the end of recombination epoch until the present. The latter can be expressed in terms of the baryon density \( \Omega_B h^2 \) and the primordial \( ^4\text{He} \) mass fraction \( Y_P \), as follows [81]:

\[ \frac{\delta\eta^B_0}{\Omega_B h^2} = \left[ 273.9 \pm 0.3 + 1.95 (Y_P - 0.25) \right] \times 10^{-10}. \] (4.42)

Using the recent Planck temperature power-spectrum data, combined with the WMAP polarization data at low multipoles, which give \( \Omega_B h^2 = 0.02205 \pm 0.00028 \) and \( Y_P = 0.24770 \pm 0.00012 \) at 68% CL [3], we infer from (4.42) the observed value of the baryon-to-photon ratio at 68% CL

\[ \delta\eta^B_{\text{obs}} = (6.04 \pm 0.08) \times 10^{-10}, \] (4.43)
which, using (4.38) and (4.41), gives

\[ \delta \eta^L_{\text{obs}} = -(2.47 \pm 0.03) \times 10^{-8}. \] (4.44)

The numerical value of the total lepton asymmetry in a given leptogenesis model should be compatible with the observed value (4.44), thus constraining the relevant model parameter space.
CHAPTER
FIVE

FLAVOUR-COVARIANT FORMALISM FOR TRANSPORT PHENOMENA

As discussed in the previous chapter, the semi-classical Boltzmann equations (4.27a) and (4.27b) do not take into account quantum flavour effects such as the coherent oscillations between different flavours of heavy neutrinos and the quantum-statistical decoherence of flavour off-diagonal matrix number-densities. In order to capture these effects consistently, in this chapter we derive a set of fully flavour-covariant transport equations for the matrix number-densities describing the statistical content of the system. In the next chapter, we will apply the general formalism presented here to the specific case of RL, subsequently demonstrating the importance of the flavour effects captured here, but missed in earlier treatments of the subject. Keeping this particular application in mind, we consider a specific system of $N_L$ Dirac lepton isospin doublets $L_l$, $N_N$ heavy Majorana neutrinos $N_\alpha$, and an $SU(2)_L$ Higgs doublet $\Phi$. However, we emphasize once more that the analysis of this chapter can be easily generalized to other physical situations involving flavour effects, such as the evolution of jet flavours in a quark-gluon plasma or of neutrino flavours in a core-collapse supernova. The material covered in this chapter has been presented extensively in [28], where more details can be found. For executive summaries of the formalism presented here, see also [41, 43].

In a general flavour basis, the relevant part of the Lagrangian, involving the heavy
Majorana neutrinos, is given by

\[- \mathcal{L}_N = h_l^\alpha \bar{L} l \tilde{\Phi} N_{R,\alpha} + \frac{1}{2} N_{R,\alpha}^C [m_N]^{\alpha\beta} N_{R,\beta} + \text{H.c.} , \tag{5.1} \]

where \( N_{R,\alpha} \equiv (P_R N)_\alpha \) are the RH heavy neutrino fields and \( \tilde{\Phi} \equiv i\sigma_2 \Phi^* \). The Einstein summation convention is implied henceforth in the summations over the lepton flavour indices (lower-case Latin) \( l, m, \ldots \) and the heavy-neutrino flavour indices (lower-case Greek) \( \alpha, \beta, \ldots \). We first discuss the flavour-covariant transformations of the field operators appearing in (5.1), together with the flavour-covariant generalizations of the discrete symmetry transformations \( C, P \) and \( T \). Subsequently, we derive general Markovian master equations for the matrix number-densities, and use them to describe the statistical evolution of our model system due to the out-of-equilibrium decays and inverse decays of the heavy neutrinos.

### 5.1 Flavour transformations

Under \( U(N_L) \otimes U(N_N) \) flavour transformations, the lepton fields transform in the fundamental representation as

\[
L_l \to L'_l = V_l^m L_m , \\
L^I \equiv (L_l)^I \to L'^I = V_m^I L^m , \tag{5.2a} \\
N_{R,\alpha} \to N'_{R,\alpha} = U_\alpha^\beta N_{R,\beta} , \\
N^I_R \equiv (N_{R,\alpha})^I \to N'^I_R = U_\alpha^\beta N_{R,\beta} , \tag{5.2b}
\]

with the unitary transformation matrices \( V_l^m \in U(N_L) \) and \( U_\alpha^\beta \in U(N_N) \). In our notation the operation of complex conjugation exchanges subscripts and superscripts, i.e. \( V^*_m \equiv (V_l^m)^* \) and \( U_\alpha^\beta \equiv (U_\alpha^\beta)^* \). We point out that the RH part of the Majorana neutrino fields \( N_{R,\alpha} \) transforms covariantly, as shown in (5.2b). The left-handed (LH) part, on the other hand, transforms contravariantly and, as such, the Majorana fields \( N_\alpha \) do not have definite flavour-transformation properties. The Lagrangian (5.1) is invariant under \( U(N_L) \otimes U(N_N) \), provided the heavy-neutrino Yukawa couplings and the Majorana
masses transform appropriately, as denoted by the relative position of the indices in (5.1), i.e.

\[ h_{\alpha}^l \rightarrow h_{\alpha}^l' = V_{l}^m U_{l}^{\alpha \beta} h_{m}^\beta, \quad [m_N]^{\alpha \beta} \rightarrow [M'_N]^{\alpha \beta} = U_{\alpha}^\gamma U_{\beta}^{\beta} [m_N]^{\gamma \delta}. \]

(5.3)

In the physical mass eigenbasis, the Dirac field can be expanded in a basis of plane waves:

\[
\hat{L}_l(x; \tilde{t}_i) = \sum_s \int_{\mathbf{p}} (2\hat{E}_{L,l}(\mathbf{p}))^{-1/2} \times \left( e^{-i\mathbf{p} \cdot \mathbf{x}} \hat{b}_l(\mathbf{p}, s, 0; \tilde{t}_i) + e^{i\mathbf{p} \cdot \mathbf{x}} \hat{v}_l(\mathbf{p}, s) \hat{d}_l^\dagger(\mathbf{p}, s, 0; \tilde{t}_i) \right),
\]

where \( \hat{b}_l(\mathbf{p}, s, \tilde{t}; \tilde{t}_i) \) and \( \hat{d}_l^\dagger(\mathbf{p}, s, \tilde{t}; \tilde{t}_i) \) are respectively the interaction-picture particle annihilation and antiparticle creation operators evaluated at the time \( \tilde{t} = 0 \). Hereafter, for notational convenience, we will suppress the dependence of the operators on the boundary time \( \tilde{t}_i \) at which the three pictures of quantum mechanics, viz. Schrödinger, Heisenberg and interaction pictures, are coincident. The index \( s = \pm \) denotes the two helicity states with the unit spin vector \( \mathbf{n} = ss = sp/|\mathbf{p}| \) aligned parallel or anti-parallel to the three momentum \( \mathbf{p} \), respectively. Here and in the following, we have suppressed the isospin index of the lepton doublet.

It is now important to note that \( b_k(\mathbf{p}, s, \tilde{t}) \) and \( d_k^\dagger(\mathbf{p}, s, \tilde{t}) \) transform under the same representation of \( U(N_L) \) and so also do \( b^k(\mathbf{p}, s, \tilde{t}) \equiv (b_k(\mathbf{p}, s, \tilde{t}))^\dagger \) and \( d^{l,k}(\mathbf{p}, s, \tilde{t}) \equiv (d_l^\dagger(\mathbf{p}, s, \tilde{t}))^\dagger \). The equal-time anti-commutation relations for these operators are obtained by flavour-transforming from the mass eigenbasis the corresponding flavour-diagonal anti-commutators, i.e.

\[
\{ b_l(\mathbf{p}, s, \tilde{t}), b'^m(\mathbf{p}', s', \tilde{t}) \} = \{ d^l,m(\mathbf{p}, s, \tilde{t}), d_l^\dagger(\mathbf{p}', s', \tilde{t}) \} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta_{ss'} \delta_{l,l'}^m.
\]

(5.5)

1Hence, \( d^{l,k}(\mathbf{p}, s, \tilde{t}) \) is an annihilation operator, i.e. \( d^{l,k}(\mathbf{p}, s, \tilde{t})|0\rangle = b_k(\mathbf{p}, s, \tilde{t})|0\rangle = 0 \).
By applying a flavour transformation (5.2a) to a general basis, we obtain the flavour-covariant expansion of the lepton doublet (5.4):

\[
L_l(x) = \sum_s \int_p \left[ \left( (2E_L(p))^{-1/2} \right)^l_i \left( [e^{-ip \cdot x}]^j_i [u(p, s)]^k_j b_k(p, s, 0) 
+ [e^{ip \cdot x}]^j_i [v(p, s)]^k_j d^l_k(p, s, 0) \right) \right],
\] (5.6a)

\[
L_l^i(x) = \sum_s \int_p \left[ \left( (2E_L(p))^{-1/2} \right)^l_i \left( [e^{ip \cdot x}]^j_i [\bar{u}(p, s)]^j_k b^l_k(p, s, 0) 
+ [e^{-ip \cdot x}]^j_i [\bar{v}(p, s)]^j_k d^{l,k}(p, s, 0) \right) \right],
\] (5.6b)

where the rank-2 tensors in (5.6a) and (5.6b) are defined by flavour-transforming (5.2a) from the mass eigenbasis, i.e.

\[
[(E_L(p))^2]_{l}^m \equiv V_l^k V_m^k \left( \hat{E}_{L,k}(p) \right)^2 = |p|^2 \delta_l^m,
\] (5.7)

where \( \delta_l^m \) is the usual Kronecker delta. Since \( E_L(p) \) is Hermitian, \( [E_L(p)]_m^l = ([E_L(p)]_l^m)^* = [E_L(p)]^l_m \). For the Dirac spinors, our notation is such that

\[
[u(p, s)]_l^m = V_l^k V_m^k \hat{u}_k(p, s), \quad [\bar{u}(p, s)]_l^m = V^l_k V^m_k \hat{\bar{u}}_k(s, p).
\] (5.8)

In order to write down the flavour-covariant expansion of the Majorana field, we recall that, in the mass eigenbasis, the expansion of a Majorana fermion can be obtained from a Dirac one by imposing the Majorana condition

\[
\hat{d}^{l,\alpha}(k, -r, \tilde{t}) = \hat{b}_\alpha(k, r, \tilde{t}) \equiv \hat{a}_\alpha(k, r, \tilde{t}).
\] (5.9)

Since \( b_\alpha(k, r, \tilde{t}) \) and \( d^{l,\alpha}(k, -r, \tilde{t}) \) transform differently under (5.2b), this condition cannot hold true in a general flavour basis. Instead, writing the mass-eigenbasis operators in terms of the ones in a general basis, we have \( U^{\beta}_\alpha b_\beta(k, r, \tilde{t}) = U^{\gamma}_\alpha d^{l,\gamma}(k, -r, \tilde{t}), \) where \( U^{\beta}_\alpha \) is the flavour transformation that connects the mass eigenbasis to the flavour basis.
5.1. FLAVOUR TRANSFORMATIONS

under consideration. Hence, we obtain the flavour-covariant Majorana condition

\[ d^{1,\alpha}(k, -r, \tilde{t}) = (U^* U^\dagger)^{\alpha\beta} b_\beta(k, r, \tilde{t}) \equiv G^{\alpha\beta} b_\beta(k, r, \tilde{t}) , \]  

(5.10)

where \( G^{\alpha\beta} \) denote the elements of a unitary and symmetric matrix. It can be shown that this matrix \( G \) is a rank-2 contravariant tensor,\(^1\) which coincides with the identity matrix \( 1 \) in the mass eigenbasis. Combining the constraint (5.10) with the expansions (5.6a) and (5.6b), the flavour-covariant expansion of the RH part of the Majorana neutrino field and its Dirac conjugate are found to be

\[
N_{R, \alpha}(x) = \sum_r \int_k \left[ (2E_N(k))^{-1/2} \right]^{\beta}_\alpha \left[ e^{-ik \cdot x} \right]^{\gamma}_\beta P_R [u(k, r)]^\delta a_\delta(k, r, 0) \\
+ \left[ e^{ik \cdot x} \right]^{\gamma}_\beta P_R [\bar{v}(k, -r)]^\delta G^- a_\delta(k, r, 0),
\]

(5.11a)

\[
\overline{N}^\alpha_R(x) = \sum_r \int_k \left[ (2E_N(k))^{-1/2} \right]^{\alpha}_\beta \left[ e^{ik \cdot x} \right]^{\gamma}_\beta [\bar{u}(k, r)]^\gamma P_L a^\gamma(k, r, 0) \\
+ \left[ e^{-ik \cdot x} \right]^{\gamma}_\beta [\bar{v}(k, -r)]^\gamma P_L G^- a^\gamma(k, r, 0).
\]

(5.11b)

Notice that the helicity of the \( v \) spinors is different from those of the corresponding creation and annihilation operators (see e.g. [82]). The rank-2 tensors in (5.11a) and (5.11b) can be defined using the flavour transformations (5.2b) from the mass eigenbasis. For instance, the energy matrix is explicitly given by

\[
[(E_N(k))^2]^{\beta}_\alpha = |k|^2 \delta^{\beta}_\alpha + [m_1^2 + m_N^2]^\beta_\alpha.
\]

(5.12)

The anti-commutation relation for the heavy-neutrino creation and annihilation operators

\[^1\text{Performing a flavour transformation } U' \text{ on } G^{\alpha\beta} \text{ defined in (5.10), we get:} \]

\[ G^{\alpha\beta} \rightarrow G'^{\alpha\beta} = [(U'U^*)^\dagger(U'U)^\dagger]^\alpha\beta = [U'^\ast G^\ast U'^\dagger]^\alpha\beta = U'^\ast a^\beta \delta^\gamma G'^{\gamma\delta}, \]

which is the transformation law of a rank-2 contravariant tensor.
are given by
\[
\{ a_\alpha(k, r, \tilde{t}), a^\beta(k', r', \tilde{t}) \} = (2\pi)^3 \delta^{(3)}(k - k') \delta_{rr'} \delta_\alpha^\beta .
\] (5.13)

From (5.11a) and (5.11b), we see that the elements $G_{\alpha\beta}$ play the role of generalized Majorana creation phases. Finally, the complex scalar field in (5.1) can be expanded as
\[
\tilde{\Phi}(x) = \int_q (2E_\Phi(q))^{-1/2} \left( e^{-iq \cdot x} \tilde{c}(q, 0) + e^{iq \cdot x} c^ \dagger(q, 0) \right),
\] (5.14)

where the interaction-picture creation and annihilation operators for the scalar field satisfy the commutation relations
\[
[c(q, \tilde{t}), c^\dagger(q', \tilde{t})] = [\tilde{c}(q, \tilde{t}), \tilde{c}^\dagger(q', \tilde{t})] = (2\pi)^3 \delta^{(3)}(q - q').
\] (5.15)

In a general flavour basis, the free Hamiltonians of the lepton doublet and heavy neutrino fields are
\[
H^0_L = \sum_s \int_p [E_L(p)]_m^l \left( b^m(p, s, \tilde{t}) b_l(p, s, \tilde{t}) + d^\dagger_l(p, s, \tilde{t}) d^{l,m}(p, s, \tilde{t}) \right),
\] (5.16a)
\[
H^0_N = \sum_r \int_k [E_N(k)]_\alpha^\beta a^\dagger_{\alpha}(k, r, \tilde{t}) a_{\alpha}(k, r, \tilde{t}),
\] (5.16b)
as can be readily verified by flavour transformations from the mass eigenbasis in which the Hamiltonians are flavour-diagonal.

The flavour populations and coherences in the evolution of our multiparticle system can be described in terms of flavour-covariant matrix number-densities, analogous to the ones for light-neutrino flavours introduced in [45]. For the lepton doublets, we may define
\[
[n^L_{s_1 s_2}(p, t)]_l^m = \frac{1}{V_3} \langle b^m(p, s_2, \tilde{t}) b_l(p, s_1, \tilde{t}) \rangle_t,
\] (5.17a)
\[
[\bar{n}^L_{s_1 s_2}(p, t)]_l^m = \frac{1}{V_3} \langle d^\dagger_l(p, s_1, \tilde{t}) d^{l,m}(p, s_2, \tilde{t}) \rangle_t,
\] (5.17b)
where \( \mathcal{V}_3 = (2\pi)^3 \delta^{(3)}(0) \) is the infinite three-volume of the system and the macroscopic time \( t = \tilde{t} - \tilde{t}_i \) is the interval of microscopic time between the specification of the initial conditions \((\tilde{t}_i)\) and the observation of the system \((\tilde{t})\). We stress, here, the relative reversed order of indices in the lepton and anti-lepton number-densities, which guarantees that the two quantities transform in the same representation of the flavour group and thus can be combined to form a flavour-covariant lepton asymmetry. Similarly, we define the heavy-neutrino number-densities

\[
[n^{\alpha}_{\gamma \delta}(k, r_1, r_2)(\tilde{t}, t)]_{\beta} \equiv \frac{1}{\mathcal{V}_3} \langle a^{\alpha}(k, r_1, \tilde{t}) a^{\beta}(k, r_2, \tilde{t}) \rangle_{t},
\]

(5.18a)

\[
[\bar{n}^{\alpha}_{\gamma \delta}(k, r_1, r_2)(\tilde{t}, t)]_{\beta} \equiv \frac{1}{\mathcal{V}_3} \langle G_{\alpha \gamma}(k, r_1, \tilde{t}) G^{\beta \delta}(k, r_2, \tilde{t}) \rangle_{t},
\]

(5.18b)

and the scalar number-densities

\[
n^{\Phi}(q, t) \equiv \frac{1}{\mathcal{V}_3} \langle \bar{c}^\dagger(q, \tilde{t}) c(q, \tilde{t}) \rangle_{t}, \quad \bar{n}^{\Phi}(q, t) \equiv \frac{1}{\mathcal{V}_3} \langle \bar{c}(q, \tilde{t}) \bar{c}(q, \tilde{t}) \rangle_{t}.
\]

(5.19)

The total number densities \( n^{X} \) (without three-momentum argument) are obtained by integrating (5.17a)–(5.19) over the corresponding three-momenta and tracing over helicity and isospin, i.e.

\[
n^{N}(t) \equiv \sum_{r=\pm} \int k n^{N}_{r}(k, t), \quad n^{L}(t) \equiv \text{Tr}_{\text{iso}} \sum_{s=\pm} \int p n^{L}_{s}(p, t),
\]

\[
n^{\Phi}(t) \equiv \text{Tr}_{\text{iso}} \int q n^{\Phi}(q, t),
\]

(5.20)

where we have denoted explicitly that the traces are taken in isospin space. Analogous definitions are adopted for the antiparticle number-densities. Note that all the matrix number-densities defined above, as well as the energy matrices (5.7) and (5.12), are Hermitian in flavour space.
5.2 Flavour covariant discrete symmetries

We now study the discrete symmetries $C$, $P$ and $T$ in the flavour-covariant formalism. We assume that the action of these operators in the mass eigenbasis is the standard one (see e.g. [83]), and find its generalization to an arbitrary flavour basis by means of the appropriate flavour transformations. In the mass eigenbasis, the action of the unitary charge-conjugation operator $U_C$ on elements of the Fock space is given by [83]

$$\tilde{b}_l(p, s, \tilde{t})^C \equiv U_C^{\dagger} \tilde{b}_l(p, s, \tilde{t}) U_C^\dagger = -i \tilde{d}^{\dagger, l}(p, s, \tilde{t}). \quad (5.21)$$

Note that the phase convention for the operators used here is in accordance with those used for the spinors (see [28] for a detailed discussion). By writing the mass-eigenbasis operators in terms of those in a general basis, i.e. $\tilde{b}_l(p, s, \tilde{t}) = V^m_l b_m(p, s, \tilde{t})$, $\tilde{d}^{\dagger, l}(p, s, \tilde{t}) = V_n^l \tilde{d}^{\dagger, n}(p, s, \tilde{t})$, we find the flavour-covariant $C$-transformation

$$b_l(p, s, \tilde{t})^C = U_C b_l(p, s, \tilde{t}) U_C^\dagger = -i (V V^T)_{lm} d^{\dagger, m}(p, s, \tilde{t}) \equiv -i G_{lm} d^{\dagger, m}(p, s, \tilde{t}), \quad (5.22)$$

where we have been led to introduce the matrix $G$ for the charged leptons, analogous to $G$ in (5.10) for the heavy neutrinos. Therefore, we see that in a flavour-covariant formulation the action of $C$ necessarily involves the appropriate flavour rotation. We find it convenient to define the generalized $C$-transformation $\tilde{C}$, i.e.

$$b_l(p, s, \tilde{t})^{\tilde{C}} \equiv G^{lm} b_m(p, s, \tilde{t})^C = -i d^{\dagger, l}(p, s, \tilde{t}), \quad (5.23)$$

which is a combination of the $C$-transformation and the appropriate flavour rotation.\(^1\)

Thus we see that, in a general flavour basis, the particle and antiparticle operators are related by a $\tilde{C}$-transformation, which reduces to the usual charge-conjugation operation only in the mass eigenbasis. The action of $C$ on the fermion fields is obtained similarly.

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\(^1\)These are equivalent to the transformations considered in [84, 85]. However, in our case, the appropriate flavour rotations are *forced* by the flavour-covariance of the formalism, once the canonical discrete transformations are defined in the mass basis.
as

\[ L_l(x)^C = U_C L_l(x) U_C^\dagger = G_{lm} C \bar{L}_m^T(x) , \] (5.24)

with \( C = i\gamma^0\gamma^2 \) (in the helicity basis), whereas the action of \( \tilde{C} \) gives the more familiar result

\[ L_l(x)^\tilde{C} = G_{lm} L_m(x)^C = C \bar{L}_m^T(x) , \] (5.25)

using the fact that \( G_{lm} \) is symmetric.

In a similar fashion, the parity transformation, generated by the unitary operator \( U_P \) in Fock space, can be generalized straightforwardly from the mass eigenbasis, i.e.

\[ b_l(p, s, \tilde{t})^P \equiv U_P b_l(p, s, \tilde{t}) U_P^\dagger = -s b_l(-p, -s, \tilde{t}) , \] (5.26a)

\[ L_l(x_0, x)^P \equiv U_P L_l(x_0, x) U_P^\dagger = P L_l(x_0, -x) , \] (5.26b)

with \( P = \gamma^0 \). Under CP, the action of the heavy-neutrino interaction Lagrangian (5.1) transforms as

\[ U_C U_P \left( \int_x h_{\alpha}^{l} N_{\alpha}^R \Phi^\dagger L_l + \text{H.c.} \right) U_C^\dagger U_P^\dagger = \int_x G_{lm} h_{m}^\beta G_{\beta\alpha} \bar{L}_m^\dagger \Phi N_{\alpha}^R + \text{H.c.} , \] (5.27)

where we have introduced the short-hand notation \( \int_x \equiv \int d^4x \) for the integration over spacetime. Equation (5.27) gives the CP-transformations of the Yukawa couplings:

\[ (h_{l}^{\alpha})^{CP} = G_{lm} h_{m}^\beta G_{\beta\alpha} , \quad (h_{l}^{\alpha})^{\tilde{C}P} \equiv G_{lm} (h_{m}^\beta)^{CP} G_{\beta\alpha} = h_{l}^{\alpha} . \] (5.28)

For a general matrix element, the relation (5.28) can be generalized to

\[ \mathcal{M}(X \rightarrow Y)^{\tilde{C}P} = \mathcal{M}(X^\tilde{C} \rightarrow Y^\tilde{C}) , \] (5.29)

where \( X^\tilde{C} \equiv X^{\tilde{C}P} \) is the generalized CP-transformation of the state \( X \), which can, for instance, be obtained from (5.25) and (5.26b).
The action of the time-reversal transformation \( T \) is described by an anti-unitary operator \( \mathcal{A}_T \) in the Fock space. Once more, starting from the mass-eigenbasis relation

\[
\hat{b}_l(p, s, \tilde{t})^T \equiv \mathcal{A}_T \hat{b}_l(p, s, \tilde{t}) \mathcal{A}_T^\dagger = \hat{b}_l(-p, s, -\tilde{t}) ,
\]

we find, because of the anti-linearity of \( \mathcal{A}_T \),

\[
b_l(p, s, \tilde{t})^T = \mathcal{A}_T b_l(p, s, \tilde{t}) \mathcal{A}_T^\dagger = G^l m b_m(-p, s, -\tilde{t}) ,
\]

\[
L_l(x_0, x)^T = \mathcal{A}_T L_l(x_0, x) \mathcal{A}_T^\dagger = G^l m T L_m(-x_0, x) ,
\]

with \( T = i\gamma^1\gamma^3 \). Therefore, we may introduce the generalized \( T \)-transformations \( \tilde{T} \) as follows:

\[
b_l(p, s, \tilde{t})^{T_{\tilde{T}}} \equiv G^l m b_m(p, s, \tilde{t})^{T_{\tilde{T}}} = b_l(-p, s, -\tilde{t}) ,
\]

\[
L_l(x_0, x)^{T_{\tilde{T}}} \equiv G^l m L_m(x_0, x)^{T_{\tilde{T}}} = T L_l(-x_0, x) .
\]

Hence, in a general basis, incoming and outgoing states are exchanged by a \( \tilde{T} \) operation. This can be seen by transforming the interaction Lagrangian (5.1) under \( \mathcal{A}_T \), from which we obtain

\[
(h^\alpha_l)^T = G^l m h^m \beta G^{\beta \alpha} , \quad (h^\alpha_l)^{T_{\tilde{T}}} \equiv G^l m (h^m \beta)^T G^{\beta \alpha} = h^l \alpha .
\]

Generalizing the above transformations to matrix elements gives

\[
\mathcal{M}(X \rightarrow Y)^{T_{\tilde{T}}} = \mathcal{M}(Y \rightarrow X) .
\]

From (5.25) and (5.32b), we obtain an important equivalence relation:

\[
L_l(x)^{CP_{\tilde{T}}} = G^l m L_m(x)^{CP_{\tilde{T}}} = G^l m G_{mn} L_n(x)^{CP_{\tilde{T}}} = \delta^l n L_n(x)^{CP_{\tilde{T}}} = L_l(x)^{CP_{\tilde{T}}}.
\]
5.2. FLAVOUR COVARIANT DISCRETE SYMMETRIES

As a consequence, we have the identity

\[ \tilde{CPT} = CPT. \] \hspace{1cm} (5.36)

Combining the results (5.29) and (5.34), and using the \( CPT \)-invariance of the Lagrangian (5.1), the identity (5.36) allows us to relate the matrix elements as

\[ \mathcal{M}(X \rightarrow Y)\tilde{CPT} = \mathcal{M}(Y^\tilde{c} \rightarrow X^\tilde{c}) = \mathcal{M}^*(X \rightarrow Y). \] \hspace{1cm} (5.37)

The number-density matrices defined in (5.17a)–(5.18b) have simple transformation properties under \( \tilde{C} \). Since \( d^l_{\mu}(p, s, \tilde{t}) = i b_l(p, s, \tilde{t})^\tilde{C} \), for the lepton number-densities (5.17a) and (5.17b), we have

\[ n^L(p, t)^\tilde{C} \equiv G \langle U_C \tilde{n}^L(p, \tilde{t}) U_C^\dagger \rangle_t G^\dagger = \tilde{n}^L(p, t)^T, \] \hspace{1cm} (5.38)

where the transposition on the far RHS of (5.38) acts on both flavour and helicity indices. Analogously, for the RH neutrinos we have \( a^C(\mathbf{k}, r) = -i a^C(\mathbf{k}, r) \), and hence, the transformation relation

\[ n^N(\mathbf{k}, t)^\tilde{C} \equiv G \langle U_C \tilde{n}^N(\mathbf{k}, \tilde{t}) U_C^\dagger \rangle_t G^\dagger = \tilde{n}^N(\mathbf{k}, t)^T. \] \hspace{1cm} (5.39)

Thus, \( \tilde{n}^N(\mathbf{k}, t)^T \) is the \( \tilde{C} \)-conjugate of \( n^N(\mathbf{k}, t) \). For Majorana neutrinos, the two \( \tilde{C} \)-conjugate quantities are not independent, being related by

\[ [\tilde{n}^N_{r_1 r_2}(\mathbf{k}, t)]^\beta_\alpha = G_{\alpha \mu} [n^N_{r_2 r_1}(\mathbf{k}, t)]^\mu_\lambda G^\lambda\beta. \] \hspace{1cm} (5.40)

Using the \( \tilde{C} \)-transformation relations (5.38) and (5.39), we can construct the following quantities with definite \( \tilde{C} \) transformation properties:

\[ [\delta n^L_{s_1 s_2}(p, t)]^m_l = [n^L_{s_1 s_2}(p, t)]^m_l - [\tilde{n}^L_{s_1 s_2}(p, t)]^m_l, \] \hspace{1cm} (5.41a)
\[ \begin{align*}
\delta n^N_{r_1 r_2}(k, t)_{\alpha}^\beta &= \left[ n^N_{r_1 r_2}(k, t) \right]_{\alpha}^\beta - \left[ \bar{n}^N_{r_1 r_2}(k, t) \right]_{\alpha}^\beta, \quad \text{(5.41b)} \\
\mathcal{L}^{N}_{r_1 r_2}(k, t)_{\alpha}^\beta &= \frac{1}{2} \left( \left[ n^N_{r_1 r_2}(k, t) \right]_{\alpha}^\beta + \left[ \bar{n}^N_{r_1 r_2}(k, t) \right]_{\alpha}^\beta \right), \quad \text{(5.41c)}
\end{align*} \]

which transform as

\[\begin{align*}
\delta n^L(p, t)^C &= -\delta n^L(p, t)^T, \\
\delta n^N(k, t)^C &= -\delta n^N(k, t)^T, \\
n^N(k, t)^C &= n^N(k, t)^T.
\end{align*} \quad \text{(5.42)}\]

Therefore, the quantities \(\delta n^X\) are \(\tilde{C}\)-“odd”, and \(n^N\) is \(\tilde{C}\)-“even”, where the quotation marks refer to the fact that this is not meant to be in the usual sense due to the transposition involved. The definite \(\tilde{C}\)-properties of the above quantities can be extended to \(\tilde{C}P\), once the total unpolarized number densities defined by (5.20) are considered. We did not define a \(\tilde{C}\)-even quantity for lepton number densities (analogous to \(n^N\)) since, in the following chapters, this will be approximated by the equilibrium number-density \(n^L_{eq}\), i.e.

\[ n^L(t) + \bar{n}^L(t) = 2 n^L_{eq} + O(\mu_L^2/T^2). \quad \text{(5.43)} \]

However, this is not always true for the heavy neutrinos, i.e. \(n^N(t) \neq n^N_{eq}\).

In the heavy-neutrino mass eigenbasis the transformation matrix \(\hat{G}\) reduces to the identity matrix \(\mathbf{1}\), and the transformations \(C\) and \(\tilde{C}\) are identical for the heavy neutrinos. In this basis, the heavy Majorana neutrino number densities (5.41b) and (5.41c) reduce to

\[ \begin{align*}
\hat{n}^N(k, t) &= \Re \left[ \hat{n}^N(k, t) \right], \\
\delta \hat{n}^N(k, t) &= 2i \Im \left[ \hat{n}^N(k, t) \right].
\end{align*} \quad \text{(5.44)} \]

We point out that both \(\hat{n}^N(t)\) and \(\delta \hat{n}^N(t)\) are \(even\) under the usual charge-conjugation operation in the mass eigenbasis, as expected for Majorana fermions: \(^1\)

\[ \hat{n}^N(k, t)^C = \hat{n}^N(k, t)^C = + \hat{n}^N(k, t), \quad \text{(5.45a)} \]

\(^1\)This is consistent with the \(\tilde{C}\) transformations in a general basis, as given by (5.42), due to the transposition involved.
\[ \delta \hat{n}^N(k, t)^C = \delta \hat{n}^N(k, t)^C = + \delta \hat{n}^N(k, t). \] (5.45b)

In addition, we note that the total lepton asymmetry \( \delta n^L(t) \equiv \text{Tr}[\delta n^L(t)] \) is \( CP \)-odd in any basis:

\[ \delta n^L(t)^{CP} \equiv \text{Tr}[\delta n^L(t)]^{CP} = \text{Tr}[\delta n^L(t)]^T = - \text{Tr}[\delta n^L(t)] = - \delta n^L(t). \] (5.46)

### 5.3 Markovian master equation

In this section, we derive the master equation governing the evolution of the matrix number-densities \( n^X(p, t) \) in the Markovian approximation. We work in the interaction picture, starting from the (picture-independent) definition of the number density in terms of the quantum-mechanical density operator \( \rho(\tilde{t}; \tilde{t}_i) \):

\[ n^X(t) \equiv \langle \hat{n}^X(\tilde{t}; \tilde{t}_i) \rangle_t = \text{Tr} \left\{ \rho(\tilde{t}; \tilde{t}_i) \hat{n}^X(\tilde{t}; \tilde{t}_i) \right\}, \] (5.47)

where the trace is over Fock space and, for notational simplicity, we leave the momentum dependence implicit. Here we have used the accent (\( \hat{\cdot} \)) to distinguish the quantum-mechanical number-density operator \( \hat{n}^X(p, \tilde{t}; \tilde{t}_i) \) from its expectation value \( n^X(p, t) \), where recall that \( t = \tilde{t} - \tilde{t}_i \) is the macroscopic time. In this section it is convenient to reintroduce the explicit dependence of the operators on the microscopic boundary time \( \tilde{t}_i \).

Differentiating (5.47) with respect to time, we have

\[ \frac{dn^X(t)}{dt} = \text{Tr} \left\{ \rho(\tilde{t}; \tilde{t}_i) \frac{d\hat{n}^X(\tilde{t}; \tilde{t}_i)}{dt} \right\} + \text{Tr} \left\{ \frac{d\rho(\tilde{t}; \tilde{t}_i)}{dt} \hat{n}^X(\tilde{t}; \tilde{t}_i) \right\} \equiv \mathcal{I}_1 + \mathcal{I}_2, \] (5.48)

where we have used the fact that \( d/dt = d/d\tilde{t} \) for fixed \( \tilde{t}_i \). In the interaction picture, the time evolution of the quantum-mechanical operator \( \hat{n}^X(\tilde{t}; \tilde{t}_i) \) is governed by the free Hamiltonian \( H^X_0 \) given by (5.16a) or (5.16b). Hence, we may use the Heisenberg equation...
of motion to write the first term in (5.48) as

\[ \mathcal{I}_1 = i \text{Tr} \left\{ \rho(\tilde{t}; \tilde{t}_i) [H_0^X, \tilde{\mathbf{n}}^X(\tilde{t}; \tilde{t}_i)] \right\} \equiv i \langle [H_0^X, \tilde{\mathbf{n}}^X(\tilde{t}; \tilde{t}_i)] \rangle_{\tilde{t}}. \]  

(5.49)

This term generates flavour oscillations in the case of a non-diagonal energy matrix. The second term in (5.48) involves the interaction Hamiltonian, e.g.

\[ H_{\text{int}} = \int_x h_\alpha^l \bar{\ell}^l \Phi N_{R,\alpha} + \text{H.c.} \]  

(5.50)

As we will see below, this contribution will generate the collision terms for the generalized Boltzmann equations (in addition to dispersive corrections). The starting point is the Liouville–von Neumann equation (see e.g. [86])

\[ \frac{d\rho(\tilde{t}; \tilde{t}_i)}{d\tilde{t}} = -i [H_{\text{int}}(\tilde{t}; \tilde{t}_i), \rho(\tilde{t}; \tilde{t}_i)]. \]  

(5.51)

Rewriting (5.51) in the form of a Volterra integral equation of the second kind, iterating once and subsequently differentiating with respect to time, we obtain the integral form

\[ \frac{d\rho(\tilde{t}; \tilde{t}_i)}{d\tilde{t}} = -i [H_{\text{int}}(\tilde{t}; \tilde{t}_i), \rho(\tilde{t}; \tilde{t}_i)] - \int_{\tilde{t}_i}^{\tilde{t}} d\tilde{t}' [H_{\text{int}}(\tilde{t}; \tilde{t}_i), [H_{\text{int}}(\tilde{t}'; \tilde{t}_i), \rho(\tilde{t}'; \tilde{t}_i)]] \].  

(5.52)

Inserting (5.52) into (5.48), we obtain

\[ \mathcal{I}_2 = -i \text{Tr} \left\{ [H_{\text{int}}(\tilde{t}; \tilde{t}_i), \rho(\tilde{t}; \tilde{t}_i)] \tilde{\mathbf{n}}^X(\tilde{t}; \tilde{t}_i) \right\} \]

\[ - \int_{\tilde{t}_i}^{\tilde{t}} d\tilde{t}' \text{Tr} \left\{ [H_{\text{int}}(\tilde{t}; \tilde{t}_i), [H_{\text{int}}(\tilde{t}'; \tilde{t}_i), \rho(\tilde{t}'; \tilde{t}_i)] \tilde{\mathbf{n}}^X(\tilde{t}; \tilde{t}_i) \right\}. \]  

(5.53)

The first term on the RHS of (5.53) vanishes for the \( H_{\text{int}} \) term given by (5.50), since it contains the product of an odd number of fields. For the second term on the RHS of
(5.53), we use the cyclicity of the trace to obtain

\[
\mathcal{I}_2 = -\int_{\tilde{t}_i}^{\tilde{t}} d\tilde{t}' \quad \text{Tr} \left\{ \rho(\tilde{t}', \tilde{t}_i) [H_{\text{int}}(\tilde{t}', \tilde{t}_i), [H_{\text{int}}(\tilde{t}; \tilde{t}_i), \hat{\mathbf{n}}^X(\tilde{t}; \tilde{t}_i)]] \right\} \\
\equiv -\int_{\tilde{t}_i}^{\tilde{t}} d\tilde{t}' \quad \langle [H_{\text{int}}(\tilde{t}', \tilde{t}_i), [H_{\text{int}}(\tilde{t}; \tilde{t}_i), \hat{\mathbf{n}}^X(\tilde{t}; \tilde{t}_i)]] \rangle_{\tilde{t}' - \tilde{t}_i} . \tag{5.54}
\]

When used in (5.48), this gives an exact and self-consistent time-evolution equation, which captures the entire evolution of the system, including any non-Markovian memory effects.

We may now perform a set of Wigner-Weisskopf approximations [87] to obtain the leading-order Markovian form of (5.48). Let us define the \( \tilde{t} \)-dependent function

\[
F(\tilde{t}; \tilde{t}_i) = [H_{\text{int}}(\tilde{t}; \tilde{t}_i), \hat{\mathbf{n}}^X(\tilde{t}; \tilde{t}_i)] . \tag{5.55}
\]

Inserting the Fourier transforms of \( H_{\text{int}}(\tilde{t}; \tilde{t}_i) \) and \( F(\tilde{t}; \tilde{t}_i) \) with respect to \( \tilde{t}' - \tilde{t}_i \) and \( \tilde{t} - \tilde{t}_i \) in (5.54), we obtain

\[
\mathcal{I}_2 = -\int_{\tilde{t}_i}^{\tilde{t}} d\tilde{t}' \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} e^{-i\omega'(\tilde{t}' - \tilde{t}_i)} e^{-i\omega(\tilde{t} - \tilde{t}_i)} \quad \langle [H_{\text{int}}(\omega'), F(\omega'')] \rangle_{\tilde{t}' - \tilde{t}_i} . \tag{5.56}
\]

Making the change of variables \( \omega = \omega'' - \omega' \), this can be recast in the form

\[
\mathcal{I}_2 = -\int_{\tilde{t}_i}^{\tilde{t}} d\tilde{t}' \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} e^{-i\omega'(\tilde{t}' - \tilde{t}_i)} e^{-i\omega''t} \quad \langle [H_{\text{int}}(\omega'), F(\omega'' - \omega')] \rangle_{\tilde{t}' - \tilde{t}_i} . \tag{5.57}
\]

As long as \( F(\omega'' - \omega') \) remains dynamical on inverse Fourier transformation, i.e. \( \omega'' \neq \omega' \), the dominant contribution to the integral (5.57) occurs for \( \tilde{t}' \sim \tilde{t} \). We thus replace \( \rho(\tilde{t}', \tilde{t}_i) \rightarrow \rho(\tilde{t}; \tilde{t}_i) \), or \( \langle \cdots \rangle_{\tilde{t}' - \tilde{t}_i} \rightarrow \langle \cdots \rangle_t \) in (5.57). We now make the change of variables \( t' = \tilde{t}' - \tilde{t} \) and take the limit \( \tilde{t}_i \rightarrow -\infty \). Herein, we assume that the statistical evolution is slow compared to the quantum-mechanical evolution (i.e. a separation of time scales), such that the system remains out-of-equilibrium in spite of the quantum-mechanical boundary time being in the infinitely distant past. With this approximation,
we may replace the interaction-picture creation and annihilation operators in $H_{\text{int}}(\tilde{\tau}; \tilde{t}_i)$ and $\tilde{n}^{X}(\tilde{t}; \tilde{t}_i)$ by their asymptotic ‘in’ counterparts via

$$c_{\text{in}}^{(t)}(p) \equiv Z^{-1/2} \lim_{\tilde{t}_i \to -\infty} e^{(-)^{iE(p)}t_i} c_{\text{in}}^{(t)}(p, \tilde{t}; \tilde{t}_i) = Z^{-1/2} \lim_{\tilde{t}_i \to -\infty} c^{(t)}(0, \tilde{t}_i),$$

where $Z = 1 + O(\hbar)$ is the wavefunction renormalization factor. Notice that, in the replacement (5.58), we must account for the free phase evolution of the interaction-picture operators. Hence, the contribution $\mathcal{I}_2$ takes the form

$$\mathcal{I}_2 \simeq -\int_{-\infty}^{0} dt' \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} e^{-i\omega't'} e^{-i\omega''t} \langle [H_{\text{int}}(\omega'), F(\omega'' - \omega')] \rangle_t.$$  

Performing the $t'$ integration, we find

$$\int_{-\infty}^{0} dt' e^{-i\omega''t'} = \pi \delta(\omega') + i \mathcal{P} \frac{1}{\omega'} = \frac{1}{2} \int_{-\infty}^{+\infty} dt' e^{-i\omega''t'} + i \mathcal{P} \frac{1}{\omega'},$$

where $\mathcal{P}$ denotes the Cauchy principal value. We are then able to write the result

$$\mathcal{I}_2 \simeq -\frac{1}{2} \int_{-\infty}^{+\infty} dt' \langle [H_{\text{int}}(t'), [H_{\text{int}}(t), \tilde{n}^{X}(t)]] \rangle_t$$

$$- \mathcal{P} \int \frac{d\omega}{2\pi} \frac{ie^{-i\omega t}}{\omega} \langle [H_{\text{int}}(\omega), [H_{\text{int}}(t), \tilde{n}^{X}(t)]] \rangle_t,$$

where objects constructed from asymptotic operators depend only on the time $t$ and the $O(\hbar)$ corrections from the wavefunction renormalization have been neglected. The first term in (5.61) is the collision term $C^X$ and the second term represents the dispersive self-energy corrections arising from vacuum contributions (Lamb shift) and thermal contributions (Stark shift), which we neglect here and in the next chapter.

Restoring the momentum dependence of the number-densities, we finally obtain the
leading-order master equation in the Markovian approximation:

\[
\frac{d}{dt} \hat{n}^X(k, t) \simeq i \langle [H_0^X, \hat{n}^X(k, t)] \rangle_t - \frac{1}{2} \int_{-\infty}^{+\infty} dt' \langle [H_{\text{int}}(t'), [H_{\text{int}}(t), \hat{n}^X(k, t)]] \rangle_t .
\]

(5.62)

This is the central equation governing the time-evolution of the particle number densities with arbitrary flavour content that will be used here and in the next chapter. In Chapters 7 and 8, instead, we will use a full field-theoretical treatment of transport phenomena, which includes the resummation of resonantly enhanced contributions \textit{ab initio}, to describe the statistical evolution of the system in a RL scenario. As will be discussed in detail in the next chapter, such resummation effects need to be included separately in the formalism that we have outlined here. With (5.62) no longer in integro-differential form [cf. (5.52)], we are free to specify the initial conditions at any finite macroscopic time \(t_0\).

We note that although the form (5.62) was also used in [45] to describe the time-evolution of active neutrinos in a thermal bath, the full flavour structure contained in (5.62) to describe the simultaneous time-evolution of multiple species, e.g. heavy neutrinos and SM leptons as in the context of leptogenesis, was not discussed before in the literature. In order to elucidate these flavour effects, in the following section we will explicitly derive \textit{fully} flavour-covariant transport equations for the system described by the Lagrangian (5.1).

---

1This approximation consists of two major assumptions: (i) separation of time scales, i.e. the QFT processes governed by \(H_{\text{int}}\) occur at time scales much smaller than the coarse-grained statistical evolution of \(\rho(t)\); and (ii) molecular chaos, i.e. the quantum-mechanical correlations that continuously form between different species are lost on time-scales relevant for the statistical evolution, so that the background can be factorized at all times.

2This formalism is often referred to as the ‘density matrix formalism’ in the literature. In our opinion, this terminology is misleading, since \(\hat{n}^X(k, t)\) is actually a \textit{matrix of densities} [45], which should be distinguished from the quantum-statistical \textit{density matrix} \(\rho(t)\). Such confusion could potentially lead to incorrect results, since, e.g., there is a crucial sign difference in the time evolution equations for the two quantities, as can be seen by comparing (5.51) with the first term on the RHS of (5.62).
5.4 Transport equations

Using the expressions (5.16a) and (5.50) for the free and interaction Hamiltonians respectively, we explicitly calculate the oscillation and collision terms in (5.62). In detail, we obtain the following evolution equations for the charged lepton and anti-lepton number densities:

\[
\frac{d}{dt} [n_{s_1s_2}^L(p,t)]_l^m = -i \left[ E_L(p), n_{s_1s_2}^L(p,t) \right]_l^m + \left[ C_{s_1s_2}^L(p,t) \right]_l^m , \tag{5.63a}
\]

\[
\frac{d}{dt} [\bar{n}_{s_1s_2}^L(p,t)]_l^m = +i \left[ E_L(p), \bar{n}_{s_1s_2}^L(p,t) \right]_l^m + \left[ \bar{C}_{s_1s_2}^L(p,t) \right]_l^m , \tag{5.63b}
\]

where commutators carrying flavour indices are understood to act in flavour space. The collision terms are given by

\[
[C_{s_1s_2}^L(p,t)]_l^m = -\frac{1}{2} \left[ \mathcal{F} \cdot \Gamma + \Gamma^\dagger \cdot \mathcal{F} \right]_{s_1s_2,l}^m , \tag{5.64a}
\]

\[
[\bar{C}_{s_1s_2}^L(p,t)]_l^m = -\frac{1}{2} \left[ \mathcal{F} \cdot \bar{\Gamma} + \bar{\Gamma}^\dagger \cdot \mathcal{F} \right]_{s_1s_2,l}^m , \tag{5.64b}
\]

where we have suppressed the overall momentum dependence and introduced the compact notation

\[
[\mathcal{F} \cdot \Gamma]_{s_1s_2,l}^m \equiv \sum_{s,r_1,r_2} \int_{k,q} \mathcal{F}_{s_1s_2r_1r_2}^L(p,q,k,t) \Gamma_{s_2r_2r_1}^\alpha \Gamma_{s_1r_1r_2}^\beta (p,q,k) \frac{m^\alpha}{n^\beta} , \tag{5.65a}
\]

\[
[\Gamma^\dagger \cdot \mathcal{F}]_{s_1s_2,l}^m \equiv \sum_{s,r_1,r_2} \int_{k,q} \Gamma_{s_1s_2r_1}^\alpha (p,q,k,t) \frac{m^\alpha}{n^\beta} \mathcal{F}_{s_2r_2r_1}^L(p,q,k) , \tag{5.65b}
\]

It is important to emphasize that our flavour-covariant formulation requires new rank-4 tensors in flavour space: (i) the statistical number-density tensors

\[
\mathcal{F}(p,q,k,t) = n^\Phi(q,t) n^L(p,t) \otimes [1 - n^N(k,t)] - [1 + n^\Phi(q,t)] [1 - n^L(p,t)] \otimes n^N(k,t) , \tag{5.66a}
\]

\[
\mathcal{F}(p,q,k,t) = \bar{n}^\Phi(q,t) \bar{n}^L(p,t) \otimes [1 - \bar{n}^N(k,t)]
\]
can be interpreted in terms of the unitarity cut of the partial one-loop heavy-neutrino inverse decays of the heavy Majorana neutrinos to the statistical evolution of the system. The absorptive tensors (5.67a) and (5.67b) represent the contributions from decays and (ii) the absorptive tensors

\[
[\Gamma_{s_1s_2r_1r_2}(p,q,k)]_{m}^{\alpha \beta} = h_{\nu}^k h_{\nu}^j (2\pi)^4[\delta^{(4)}(k-p-q)]_{\mu \nu}^{j} \\
\times \frac{1}{2E_{\Phi}(q)} \left[(2E_{L}(p))^{-1/2}\right]_{j}^{i} \left[(2E_{L}(p))^{-1/2}\right]_{k}^{n} \left[(2E_{N}(k))^{-1/2}\right]_{\lambda}^{\mu} \left[(2E_{N}(k))^{-1/2}\right]_{\gamma}^{\nu} \\
\times \text{Tr} \left\{ [u(k,r_2)]_{\delta}^{\beta}[\bar{u}(k,r_1)]_{\gamma}^{\alpha} P_{L} [u(p,s_2)]_{n}^{m} [\bar{u}(p,s_1)]_{l}^{p} P_{R} \right\} , \tag{5.67a}
\]

\[
[\Gamma_{s_1s_2r_1r_2}(p,q,k)]_{l}^{\alpha \beta} = h_{\nu}^k h_{\nu}^j (2\pi)^4[\delta^{(4)}(k-p-q)]_{\mu \nu}^{j} \\
\times \frac{1}{2E_{\Phi}(q)} \left[(2E_{L}(p))^{-1/2}\right]_{j}^{i} \left[(2E_{L}(p))^{-1/2}\right]_{k}^{n} \left[(2E_{N}(k))^{-1/2}\right]_{\lambda}^{\mu} \left[(2E_{N}(k))^{-1/2}\right]_{\gamma}^{\nu} \\
\times \text{Tr} \left\{ [v(k,r_2)]_{\delta}^{\beta}[\bar{v}(k,r_1)]_{\gamma}^{\alpha} P_{L} [v(p,s_2)]_{n}^{m} [\bar{v}(p,s_1)]_{l}^{p} P_{R} \right\} , \tag{5.67b}
\]

\[
[\Gamma_{s_1s_2r_1r_2}(p,q,k)]_{l}^{\alpha \beta} = \left[ [\Gamma_{s_2s_1r_2r_1}(p,q,k)]_{m}^{l} \right]^{\alpha \beta}_{\alpha \beta} , \tag{5.67c}
\]

\[
[\Gamma_{s_1s_2r_1r_2}(p,q,k)]_{l}^{\alpha \beta} = \left[ \Gamma_{s_2s_1r_2r_1}(p,q,k) \right]^{\alpha \beta}_{\alpha \beta} . \tag{5.67d}
\]

In (5.67a) and (5.67b), the rank-4 delta function of on-shell four-momenta originates from the integration of tensor exponentials, such as

\[
\int_{x} \left[ e^{-ip \cdot x} \right]_{m}^{l} e^{-iq \cdot x} \left[ e^{ik \cdot x} \right]_{\alpha}^{\beta} = (2\pi)^4[\delta^{(4)}(k-p-q)]_{\mu \nu}^{j} , \tag{5.68}
\]

and is defined in terms of the appropriate flavour transformation from the mass eigenbasis, as defined in Section 5.1, i.e.

\[
[\delta^{(4)}(k-p-q)]_{m}^{\alpha \beta} = V_{k}^{l} V_{m}^{k} U_{\alpha}^{\gamma} U_{\gamma}^{\beta} \delta(\hat{E}_{N,\gamma}(k) - \hat{E}_{L,k}(p) - \hat{E}_{\Phi}(q)) \delta^{(3)}(k-p-q) . \tag{5.69}
\]

The absorptive tensors (5.67a) and (5.67b) represent the contributions from decays and inverse decays of the heavy Majorana neutrinos to the statistical evolution of the system (see Section 6.1 and Figures 6.1(a)–6.1(b)). As discussed in [28], these rank-4 objects can be interpreted in terms of the unitarity cut of the partial one-loop heavy-neutrino
self-energies, using a generalized optical theorem (see Figure 6.2). This further clarifies
the necessity of the tensorial structure of these objects, and also the form of the \( 2 \leftrightarrow 2 \) scattering terms that will be included later in the evolution equations in Chapter 6.
This formalism could be also generalized to include higher order processes involving
multiple flavour degrees of freedom, e.g. \( LN \leftrightarrow L e_R \) and \( LN \leftrightarrow LN \), by introducing the

For the heavy neutrino number-densities, we analogously find the evolution equations

\[
\frac{d}{dt} [n_{r_1 r_2}^N (k, t)]_\alpha^\beta = - i \left[ E_N (k), n_{r_1 r_2}^N (k, t) \right]_\alpha^\beta + \left[ C_{r_1 r_2}^N (k, t) \right]_\alpha^\beta + G_{\alpha \lambda} \left[ \overline{C}_{r_2 r_1}^N (k, t) \right]_\mu^\lambda G^\mu_\beta, \tag{5.70a}
\]

\[
\frac{d}{dt} [\bar{n}_{r_1 r_2}^N (k, t)]_\alpha^\beta = i \left[ E_N (k), \bar{n}_{r_1 r_2}^N (k, t) \right]_\alpha^\beta + \left[ \overline{C}_{r_1 r_2}^N (k, t) \right]_\alpha^\beta + G_{\alpha \lambda} \left[ C_{r_2 r_1}^N (k, t) \right]_\mu^\lambda G^\mu_\beta, \tag{5.70b}
\]

where the heavy-neutrino collision terms are given by

\[
\left[ C_{r_1 r_2}^N (k, t) \right]_\alpha^\beta = + \frac{1}{2} \left[ \mathcal{F} \cdot \Gamma^\dagger + \Gamma \cdot \mathcal{F} \right]_{r_1 r_2, \alpha}^\beta, \tag{5.71a}
\]

\[
\left[ \overline{C}_{r_1 r_2}^N (k, t) \right]_\alpha^\beta = + \frac{1}{2} \left[ \mathcal{F} \cdot \Gamma^\dagger + \Gamma \cdot \mathcal{F} \right]_{r_1 r_2, \alpha}^\beta, \tag{5.71b}
\]

with the tensor contractions analogous to (5.65a) and (5.65b), with the role of charged-lepton and heavy-neutrino indices exchanged. Note the appearance of the matrix \( G \) in the
transport equations (5.70a) and (5.70b), and the transposition of both flavour and helicity indices. One should remember that (5.70a) and (5.70b) are not independent, because of
the relation (5.39). Notice also that the transport equations have an internal structure in
isospin space, which has been suppressed for brevity. In Chapter 6 we will derive the
rate equations for the total number-densities, and thus we will explicitly trace over these
degenerate isospin degrees of freedom.
CHAPTER SIX

FLAVOUR-COVARIANT RATE EQUATIONS FOR RESONANT LEPTOGENESIS

In this chapter, we apply the flavour-covariant formalism developed in Chapter 5 directly to RL, obtaining the flavour-covariant rate equations that describe completely the physical phenomena relevant for the generation of the lepton asymmetry. The discussion here is based on [28] (see also [41, 43]).

As already discussed in Section 4.2, there are two types of CP-violation possible in the out-of-equilibrium decay of the heavy Majorana neutrino (see Figure 4.1). In the limit when two or more heavy Majorana neutrinos become degenerate, the $\varepsilon$-type CP-violation can be resonantly enhanced by several orders of magnitude [21, 22], and finite-order perturbation theory breaks down. In this case, it is necessary to perform a consistent field-theoretic resummation of the self-energy corrections.

Within the formalism developed in Chapter 5, this resummation needs to be performed, in practice, in a heuristic zero-temperature way. From this point of view, the approach followed in this chapter may be considered as semi-classical. Its main advantage is that it is constructed with physical observables, i.e. particle number-densities, in mind. Therefore, the physical phenomena that are relevant for the generation of the BAU will be manifest and described in a clear and transparent manner. However, this approach has the disadvantage that quantum effects, such as finite particle widths, must be incorporated in an effective manner. In Chapter 8 we will confirm, within a full field-
In detail, we perform the resummation of heavy-neutrino self-energies along the lines of Section 4.2, by first flavour-rotating to the heavy neutrino mass eigenbasis, maintaining the Markovian approximation used in Section 5.3 and resumming only zero-temperature contributions, thereby neglecting thermal loop effects [88]. Finally, we will assume that the heavy-neutrino helicity states are fully decohered and equally populated [23].

In practice, we proceed by replacing the tree-level neutrino Yukawa couplings by their resummed counterparts in the transport equations given in Section 5.4. In detail, for the processes \( N \rightarrow L \Phi \) and \( L \tilde{\Phi} \rightarrow N \), we have \( h^l_{\alpha} \rightarrow \tilde{h}^l_{\alpha} \) and, for \( N \rightarrow L \tilde{\Phi} \) and \( L \Phi \rightarrow N \), we have \( h^l_{\alpha} \rightarrow [h^l_{\alpha}] \). Working in the heavy-neutrino mass eigenbasis, the resummed neutrino Yukawa couplings are given by (4.8), from which the covariant resummed Yukawa couplings may be obtained, as usual, by the appropriate flavour transformation, i.e. \( h^l_{\alpha} = V_i^m U_{j}^{\alpha} \tilde{\hat{h}}^l_{m} \), where \( \tilde{\hat{h}}^l_{m} \equiv \hat{h}^l_{m} \) in the mass eigenbasis.

## 6.1 Rate equations for decay and inverse decay

In order to obtain the rate equations from the general transport equations (5.63a), (5.63b), (5.70a) and (5.70b), we impose kinetic equilibrium. This can be ensured throughout the evolution of the system by assuming that the elastic scattering processes rapidly change the momentum distributions on time-scales much smaller than the statistical evolution time of the particle number densities. Moreover, as long as the mass splittings between different flavours inside thermal integrals are much smaller than the average momentum scale, i.e. \( |k| \sim T \gg |m_{N,\alpha} - m_{N,\beta}| \), the momentum distributions governed by the elastic processes are flavour singlets [89, 90]. Using these approximations, we introduce an average mass for the \( N_N \) quasi-degenerate heavy neutrinos:

\[
m_N^2 = \frac{1}{N_N} \text{Tr} \left( m_N^4 m_N \right),
\]
6.1. RATE EQUATIONS FOR DECAY AND INVERSE DECAY

to be used within the thermal integrals. Correspondingly, we may introduce an average energy \( E_N(k) = (|k|^2 + m_N^2)^{1/2} \). Furthermore, in the general transport equations given in Section 5.4, it is appropriate to take the classical statistical limit in which (5.66a) and (5.66b) can be approximated as

\[
\mathcal{F}(p, q, k, t) \simeq n^\Phi(q, t) n^L_p(t) \otimes 1 - 1 \otimes n^N_k(t),
\]

\[
\mathcal{F}(p, q, k, t) \simeq \bar{n}^\Phi(q, t) \bar{n}^L_p(t) \otimes 1 - 1 \otimes \bar{n}^N_k(t).
\]

The spinor traces appearing in (5.67a) and (5.67b) can be calculated easily, since we neglect the thermal mass of the lepton doublets (recall, also, that \( n^L \) is the number-density matrix for the LH lepton doublets \( L \)). One helicity index for the charged leptons can be dropped in the spinor traces, thus yielding

\[
\sum_{r=-,+} \text{Tr} \left\{ u(k, r) \bar{u}(k, r) P_L u(p, -) \bar{u}(p, -) P_R \right\} = \sum_{r=-,+} \text{Tr} \left\{ v(k, r) \bar{v}(k, r) P_L v(p, +) \bar{v}(p, +) P_R \right\} = 2 k \cdot p,
\]

where \(- (+)\) corresponds to the helicity of the massless LH leptons (RH anti-leptons).

With these approximations, the heavy-neutrino kinetic-equilibrium number density, given by the flavour-covariant generalization of (4.18) (with the chemical potential now being a rank-2 tensor), can be approximated as

\[
[n^N_k(t)]^\alpha_{\beta} \equiv \sum_r [n^N_{r\alpha}(k, t)]^\alpha_{\beta} = g_N \left[ e^{-\left( E_N(k) - \mu_N(t) \right)/T} \right]^\beta_{\alpha} \\
\simeq g_N \left[ e^{\mu_N(t)/T} \right]^\beta_{\alpha} e^{-E_N(k)/T} = g_N \frac{n^N_{\text{eq}}(t)}{n^N_{\text{eq}}} e^{-E_N(k)/T},
\]

where \( n^N(t) \) is the total heavy neutrino number density, as defined in (5.20), and \( n^N_{\text{eq}} \) is the equilibrium number-density given by (4.18) setting \( \mu_N = 0 \) and \( g_N = 2 \) for the two heavy-neutrino helicity states. For the charged-lepton number-density we analogously
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obtain

\[ [n^L(p, t)]_t^m \simeq \frac{[n^L(t)]_t^m}{n^L_{\text{eq}}} e^{-E_L(p)/T}, \]  

(6.5)

where \( E_L \) is the average energy of the lepton doublets, and \( n^L_{\text{eq}} \) is the massless equilibrium number-density given by (4.19) with \( g_L = 2 \) for the two isospin states. Notice that the factor \( g_L \) is not present in (6.5), but will appear explicitly only after the trace over isospin is performed [cf. (5.20)]. Finally, for the Higgs number density, we assume an equilibrium distribution \( n^\Phi(q, t) = e^{-E_\Phi(q)/T} \). Throughout the remainder of this chapter, we suppress the \( t \)-dependence of the number-densities for notational convenience.

We may now integrate both sides of (5.63a) and (5.63b) over the phase space and sum over the degenerate helicity and isospin degrees of freedom. We thus obtain the rate equations for the total number-densities of the charged lepton and anti-lepton doublets, accounting for the decay and inverse decay of the heavy neutrinos, as

\[
\frac{d[n^L]}{dt} = -i \left[ \mathcal{E}_L, n^L \right] - \frac{1}{2 n^L_{\text{eq}}} \left\{ n^L, \gamma(L\Phi \to N) \right\}_{\text{t}}^{m} + \frac{[n^N]_t^\alpha}{n^N_{\text{eq}}} \left[ \gamma(N \to L\Phi) \right]_{\text{t}}^m \beta,
\]

(6.6a)

\[
\frac{d[\bar{n}^L]}{dt} = +i \left[ \mathcal{E}_L, \bar{n}^L \right]_t^m - \frac{1}{2 n^L_{\text{eq}}} \left\{ \bar{n}^L, \gamma(L\tilde{\Phi} \to N) \right\}_t^m + \frac{[\bar{n}^N]_t^\alpha}{n^N_{\text{eq}}} \left[ \gamma(N \to L\tilde{\Phi}) \right]_{\text{t}}^m \beta,
\]

(6.6b)

where \( \mathcal{E}_X \) is the thermally-averaged effective energy matrix of the species \( X \)

\[
\mathcal{E}_X = \frac{g^X}{n^X_{\text{eq}}} \int_p E_X(p) e^{-E_X(p)/T} = K_1(z) \left[ m_n^X m_n^X \right]^{1/2} + 3T \mathbf{1}.
\]

(6.7)

Note that the second term on the RHS of (6.7) is isotropic in flavour space, commutes with the number-densities, and therefore, does not give any contribution to the rate equations (6.6a) and (6.6b). The \( 1 \to 2 \) and \( 2 \to 1 \) collision rates appearing in (6.6a) and
6.1. RATE EQUATIONS FOR DECAY AND INVERSE DECAY

(6.6b) are derived from the rank-4 absorptive tensors (5.67a) and (5.67b). Replacing the tree-level Yukawa couplings $h_i^\alpha$ appearing in (5.67a) and (5.67b) with their resummed counterparts $h_i^\alpha$, the $1 \rightarrow 2$ collision rates can be explicitly written as

\[
[\gamma(N \rightarrow L \Phi)]^m_\alpha \equiv \int_{NL\Phi} g_L g_{\Phi} (2 p_N \cdot p_L) h^{m}_\alpha h^\beta_i = \frac{m_N^4 K_1(z)}{16\pi} h^{m}_\alpha h^\beta_i , \quad (6.8a)
\]

\[
[\gamma(N \rightarrow L^\xi \Phi^{\bar{\xi}})]^m_\alpha \equiv \int_{NL\Phi} g_L g_{\Phi} (2 p_N \cdot p_L) [h^{\xi}]^m_\alpha [h^{\bar{\xi}}]^\beta_i = \frac{m_N^4 K_1(z)}{16\pi} [h^{\xi}]^m_\alpha [h^{\bar{\xi}}]^\beta_i , \quad (6.8b)
\]

which are the flavour-covariant generalizations of the collision rates defined in (4.21). The $2 \rightarrow 1$ collision rates in (6.6a) and (6.6b) are related to the $1 \rightarrow 2$ rates (6.8a) and (6.8b) by virtue of $\tilde{CPT}$ invariance, i.e.

\[
[\gamma(L \Phi \rightarrow N)]^m_\alpha \equiv [\gamma(L \Phi \rightarrow N)]^{m}_\alpha , \quad (6.9a)
\]

\[
[\gamma(L \Phi \rightarrow N)]^m_\alpha \equiv [\gamma(N \rightarrow L^\xi \Phi^{\bar{\xi}})]^{m}_\alpha . \quad (6.9b)
\]

The corresponding rank-2 collision rates in the anti-commutators in (6.6a) and (6.6b) are obtained from the corresponding rank-4 tensors by contracting the heavy-neutrino flavour indices as, e.g.,

\[
[\gamma(L \Phi \rightarrow N)]^{m}_\alpha \equiv [\gamma(L \Phi \rightarrow N)]^{m}_\alpha . \quad (6.10)
\]

In [28], an alternative derivation of these collision rates is also given, by considering a flavour-covariant generalization of the optical theorem in the presence of a statistical background. Thus, the necessity of the rank-4 flavour structure of these collision rates is further justified. This is illustrated in Figure 6.1 for the in-medium production of heavy neutrinos in a spatially-homogeneous statistical background of lepton and Higgs doublets. The production rates in the thermal plasma can be understood from the unitarity cut of the partial one-loop heavy-neutrino self-energy graph, as shown in Figure 6.2.

Analogously, we may obtain the flavour-covariant rate equations for the total number-densities of heavy neutrinos from the general transport equations (5.70a) and (5.70b),

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Figure 6.1: Feynman diagrams for $2 \rightarrow 1$ inverse heavy-neutrino decay, in the presence of a statistical background. The flavour indices are shown explicitly, while other indices are suppressed.

finding

$$\frac{d[n^N]_\alpha^\beta}{dt} = -i \left[ \mathcal{E}_N, n^N \right]^\beta_\alpha + \left[ C^N \right]^\beta_\alpha + G_{\alpha\lambda} \left[ \bar{C}^N \right]^\lambda_\mu G^{\mu\beta}, \quad (6.11a)$$

$$\frac{d[\bar{n}^N]_\alpha^\beta}{dt} = +i \left[ \mathcal{E}_N, \bar{n}^N \right]^\beta_\alpha + \left[ \bar{C}^N \right]^\beta_\alpha + G_{\alpha\lambda} \left[ C^N \right]^\lambda_\mu G^{\mu\beta}, \quad (6.11b)$$

where $\mathcal{E}_N$ is the thermally-averaged effective heavy-neutrino energy matrix, obtained from (6.7). The thermally averaged collision terms $C$ and $\bar{C}$ in (6.11a) and (6.11b) are explicitly given by

$$\left[ C^N \right]^\alpha_\beta = -\frac{1}{2} \frac{n_{eq}^N}{n_{eq}^L} \left\{ n^N, \gamma(N \rightarrow L\Phi) \right\}^\beta_\alpha + \frac{n_{eq}^L}{n_{eq}^L} \left[ \gamma(L\Phi \rightarrow N) \right]^m_\mu \beta, \quad (6.12a)$$

$$\left[ \bar{C}^N \right]^\alpha_\beta = -\frac{1}{2} \frac{\bar{n}_{eq}^N}{\bar{n}_{eq}^L} \left\{ \bar{n}^N, \gamma(N \rightarrow L\bar{h}\Phi) \right\}^\beta_\alpha + \frac{\bar{n}_{eq}^L}{\bar{n}_{eq}^L} \left[ \gamma(L\bar{h}\Phi \rightarrow N) \right]^m_\mu \beta, \quad (6.12b)$$

where the rank-4 collision rates are given by (6.8a) and (6.8b), and the corresponding
6.1. RATE EQUATIONS FOR DECAY AND INVERSE DECAY

![Feynman diagrams](image)

(a) Heavy-neutrino self-energy, $N \rightarrow L \Phi \rightarrow N$.

(b) Heavy-neutrino self-energy, $N \rightarrow L^e \Phi^e \rightarrow N$.

Figure 6.2: Feynman diagrams for the self-energies of the heavy neutrinos. The cut, across which positive energy flows from unshaded to shaded regions, is associated with production rates in the thermal plasma, as described by the generalized optical theorem. See also Figure 6.1.

rank-2 objects are obtained by contracting the charged-lepton indices, for instance, as

$$ \left[ \gamma(N \rightarrow L \Phi) \right]_{\alpha}^{\beta} \equiv \left[ \gamma(N \rightarrow L \Phi) \right]_{l \alpha}^{l \beta} \cdot (6.13) $$

Using the expressions (6.8a) and (6.8b), we can define the flavour-covariant generalizations of the $CP$-even and $CP$-odd rates in (4.23) and (4.24), which now have definite transformation properties under $\tilde{CP}$:

$$ \left[ \gamma^N_{L \Phi} \right]_{l \alpha}^{m \beta} = + \left[ \gamma^{L \Phi} \right]_{l \alpha}^{m \beta} = O(h^2), \quad (6.14a) $$

$$ \left[ \delta \gamma^N_{L \Phi} \right]_{l \alpha}^{m \beta} = - \left[ \delta \gamma^{L \Phi} \right]_{l \alpha}^{m \beta} = O(h^4). \quad (6.14b) $$

The corresponding rate equations for the $\tilde{CP}$-“even” and -“odd” number-densities [cf. (5.42)] are derived from (6.6a), (6.6b), (6.11a) and (6.11b):

$$ \frac{d[n^N]}{dt} = - \frac{i}{2} \left[ E_N, \delta n^N \right]_{\alpha}^{\beta} + \left[ \text{Re}(\gamma^N_{L \Phi}) \right]_{\alpha}^{\beta} - \frac{1}{2 n_{eq}} \left\{ n^N, \text{Re}(\gamma^N_{L \Phi}) \right\}_{\alpha}^{\beta}, \quad (6.15a) $$

$$ \frac{d[\delta n^N]}{dt} = - 2 i \left[ E_N, \delta n^N \right]_{\alpha}^{\beta} - 2 i \left[ \text{Im}(\delta \gamma^N_{L \Phi}) \right]_{\alpha}^{\beta} - \frac{i}{n_{eq}^2} \left\{ n^N, \text{Im}(\delta \gamma^N_{L \Phi}) \right\}_{\alpha}^{\beta} $$

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CHAPTER 6. FLAVOUR-COVARIANT RATE EQUATIONS

\[
\frac{d[\delta n^L]}{dt} = \left[ \delta \gamma^N_{\ell \phi} \right]_l^m + \frac{n^N_\alpha}{n^N_{eq}} \left[ \delta \gamma^N_{\ell \phi} \right]_l^m \alpha + \frac{[\delta n^N]_\beta}{2 n^N_{eq}} \left[ \gamma^N_{\ell \phi} \right]_l^m \beta
\]

\[\frac{d[\delta n^L]}{dt} = \left[ \delta \gamma^N_{\ell \phi} \right]_l^m + \frac{n^N_\alpha}{n^N_{eq}} \left[ \delta \gamma^N_{\ell \phi} \right]_l^m \alpha + \frac{[\delta n^N]_\beta}{2 n^N_{eq}} \left[ \gamma^N_{\ell \phi} \right]_l^m \beta
\]

where we have kept terms only up to \(O(\mu_a/T)\) and \(O(h^4)\), except the last term on the RHS of (6.15c), which is the leading washout term for the lepton asymmetry, and appears at \(O(h^6)\). In addition, we have defined, for a given Hermitian matrix \(A = A^\dagger\), its generalized real and imaginary parts, as follows:

\[
[\tilde{\text{Re}}(A)]_{\alpha}^\beta = \frac{1}{2} \left( A_{\alpha}^\beta + G_{\alpha \lambda} A_\mu^\lambda G^\mu_{\beta} \right),
\]

\[
[\tilde{\text{Im}}(A)]_{\alpha}^\beta = \frac{1}{2i} \left( A_{\alpha}^\beta - G_{\alpha \lambda} A_\mu^\lambda G^\mu_{\beta} \right).
\]

Observe that in the heavy-neutrino mass eigenbasis, these reduce to the usual real and imaginary parts:

\[
[\tilde{\text{Re}}(\hat{A})]_{\alpha}^\beta = \text{Re}(\hat{A}_{\alpha}^\beta), \quad [\tilde{\text{Im}}(\hat{A})]_{\alpha}^\beta = \text{Im}(\hat{A}_{\alpha}^\beta).
\]

Moreover, in obtaining (6.15a) and (6.15b), we have used the relations

\[
\tilde{\text{Re}}(\hat{n}^N) = \hat{n}^N, \quad i \tilde{\text{Im}}(\hat{\delta n}^N) = \delta n^N,
\]

which can be derived from (5.40), (5.41b) and (5.41c). Observe that the commutators in (6.6a) and (6.6b) would cancel to leading order in \(O(\mu_L/T)\) by virtue of (5.43), even if the thermal masses were included. On the other hand, the commutators of the thermally-averaged effective heavy-neutrino energy matrix with the number-densities in (6.15a) and (6.15b) do not vanish, and describe the coherent oscillations between different heavy-neutrino flavours.

Note that the \(\tilde{\text{CP}}\)-“odd” inverse decay terms in (6.15b) and (6.15c), i.e. \(-2i[\tilde{\text{Im}}(\gamma^N_{\ell \phi})]_{\alpha}^\beta\)
and $+|\delta\gamma_{L\Phi}^N|_l^m$, appear with the wrong sign and do not lead to the correct equilibrium behaviour. It is well known that the inclusion of $2 \leftrightarrow 2$ scattering terms (with the RIS contribution subtracted), changes the sign of these inverse decay terms and gives the correct equilibrium limit \[23, 59\]. In Section 6.4 we will explicitly prove this result in a flavour-covariant manner by including in the rate equations the RIS-subtracted collision rates for scattering, as illustrated in Figure 6.3. For the moment, we take this result at face value, and correct these signs ‘by hand’, to be able to qualitatively discuss some important physical phenomena, before including the scattering terms. Finally, we also take into account the Hubble expansion of the Universe and change $t$ in favour of $z = m_N / T$, to write down the rate equations in terms of the normalized number-densities, introduced in Section 4.3:

\[
\frac{H_N n^\gamma}{z} \frac{d[\eta^N]_\alpha}{dz} = -i \frac{n^\gamma}{2} [\mathcal{E}_N, \delta\eta^N]_\alpha + [\tilde{\mathrm{Re}}(\gamma_{L\Phi}^N)]_\alpha - \frac{1}{2\eta^N_{\text{eq}}} \{\eta^N, \tilde{\mathrm{Re}}(\gamma_{L\Phi}^N)\}_\alpha, \tag{6.19a}
\]

\[
\frac{H_N n^\gamma}{z} \frac{d[\delta\eta^N]_\alpha}{dz} = -2i \frac{n^\gamma}{2} [\mathcal{E}_N, \eta^N]_\alpha + 2i [\tilde{\mathrm{Im}}(\delta\gamma_{L\Phi}^N)]_\alpha
- \frac{i}{\eta^N_{\text{eq}}} \{\eta^N, \tilde{\mathrm{Im}}(\gamma_{L\Phi}^N)\}_\alpha - \frac{1}{2\eta^N_{\text{eq}}} \{\delta\eta^N, \tilde{\mathrm{Re}}(\gamma_{L\Phi}^N)\}_\alpha, \tag{6.19b}
\]

\[
\frac{H_N n^\gamma}{z} \frac{d[\delta\eta^L]_m}{dz} = -[\delta\gamma_{L\Phi}^N]_m + \frac{[\eta^N]_\beta}{\eta^N_{\text{eq}}} [\delta\gamma_{L\Phi}^N]_l^m [\gamma_{L\Phi}^N]_l^m
- \frac{1}{3} \{\delta\eta^L, \gamma_{L\Phi}^N\}_l^m. \tag{6.19c}
\]

In the last term on the RHS of (6.19c), we have used $\eta^L_{\text{eq}} = 3/4$, which follows from (4.19) and (4.26).

### 6.2 Lepton asymmetry via heavy-neutrino oscillations

The rate equations (6.19a)–(6.19c), describe two physically distinct mechanisms for the generation of lepton asymmetry. The first is the standard $T = 0 \varepsilon$- and $\varepsilon'$-type $CP$ violation given by (4.13) due to the mixing and decay of the heavy Majorana neutrinos. In
the semiclassical approach adopted here, this is taken into account by means of the one-loop resummed effective Yukawa couplings defined in (4.8), which appear in the collision rates (6.8a) and (6.8b). This is the only source of lepton number asymmetry in the flavour-diagonal Boltzmann equations (4.27a) and (4.27b). However, the flavour-covariant rate equations (6.19a)–(6.19c) also include a second source for the asymmetry, due to the $CP$-violating oscillations of the on-shell heavy neutrinos in the thermal bath. This contribution originates from the sequence of an on-shell production of heavy neutrinos in the bath due to inverse decays, which can oscillate between different flavours in the bath and then decay back into charged-leptons. Formally, this process has the same structure as the scattering diagrams in Figure 6.3. While the $T = 0$ QFT processes are taken into account by the resummation of the Yukawa couplings [22, 23], the oscillation phenomenon is related to the thermal off-diagonal part of the intermediate heavy-neutrino propagator. This will be confirmed by the thermal field-theoretical analysis in Chapter 8. Thus, our flavour-covariant semiclassical approach captures the leading order effect of a complete thermal resummation procedure that generalizes the analysis in [23] (this will be discussed in detail in Chapter 8).

In this section, we present a qualitative analysis of the heavy-neutrino oscillation phenomenon in the RL scenario, and show that this mechanism contributes to the total lepton asymmetry at order $O(h^4)$ around $z = 1$. Note that, although similar at first glance, this is qualitatively as well as quantitatively different from the phenomenon first proposed in [29], and studied in [30, 31, 33, 34, 91] for the light sterile neutrino case, where the final lepton number asymmetry is of order $O(h^6)$, as also recently stressed in [92]. In order to extract the heavy-neutrino oscillation effect from the flavour-covariant rate equations (6.19a)–(6.19c), we modify the rate equations by using the tree-level Yukawa couplings, instead of the resummed ones, thus artificially ‘switching off’ all $T = 0 \, \varepsilon$- and $\varepsilon'$-type $CP$ violation effects. In this case, we can drop the $\tilde{CP}$-“odd” $\delta\gamma$ collision terms in the
rate equations (6.19b)–(6.19c), which further simplify in the mass eigenbasis, as

\[ \frac{H N n^\gamma}{z} \frac{d[\delta \eta^N]_{\alpha\beta}}{dz} \supset -2i n^\gamma \left[ \hat{E}_N, \hat{\eta}^N \right]_{\alpha\beta} - \frac{1}{2\eta^N_{eq}} \left\{ \delta \hat{\eta}^N, \text{Re}(\hat{\gamma}^N_{L\Phi}) \right\}_{\alpha\beta}, \]  

(6.20a)

\[ \frac{H N n^\gamma}{z} \frac{d[\delta \hat{\eta}^L]_{lm}}{dz} \supset \frac{[\delta \hat{\eta}^N]_{\beta\alpha}}{2\eta^N_{eq}} \left[ \hat{\gamma}^N_{l\Phi} \right]_{lm\alpha\beta} - \frac{1}{3} \left\{ \delta \hat{\eta}^L, \hat{\gamma}^N_{L\Phi} \right\}_{lm}. \]  

(6.20b)

The rate equation for $\hat{\eta}^N$ is still given by (6.19a), specialized to the mass eigenbasis. Notice that, in the mass eigenbasis, the flavour rotation matrices reduce to the identity, i.e. $G = G = 1$, and we do not need to distinguish between upper and lower flavour indices in (6.20a) and (6.20b). It is useful to perform a time-stepping analysis to see the infinitesimal time evolution of the total lepton asymmetry. We choose to start with an incoherent diagonal heavy-neutrino number density matrix, and a zero initial lepton asymmetry at some $z = z_{in}$:

\[ [\hat{\eta}^N]_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta, \quad [\delta \hat{\eta}^N]_{\alpha\beta} = 0, \quad [\delta \hat{\eta}^L]_{lm} = 0. \]  

(6.21)

Interference in the inverse decays will generate off-diagonals in $\hat{\eta}^N$ at $O(h^2)$. However, we see from (6.19a), (6.20a) and (6.20b) that only $\hat{\eta}^N$, and not $\delta \hat{\eta}^N$ or $\delta \hat{\eta}^L$, receives these contributions. This is because the processes

\[ L \Phi \rightarrow \sum_\alpha c_\alpha \hat{N}_{R,\alpha}, \quad L^\dagger \Phi^\dagger \rightarrow \sum_\alpha \bar{c}_\alpha \hat{\bar{N}}_{R,\alpha}^\dagger, \]  

(6.22)

(the $c_\alpha$ being complex coefficients in the linear combination) have to proceed at the same rate for $\bar{C}P$-conserving inverse decays, if no initial lepton asymmetry is present. Hence, only $\text{Re}(\hat{\eta}^N) = \hat{\eta}^N$ can be generated. Therefore, after a small time interval, at $z = z_{in} + \delta z_1$ we have

\[ [\hat{\eta}^N]_{\alpha\beta} = O(h^2) \quad \text{for} \quad \alpha \neq \beta, \quad [\delta \hat{\eta}^N]_{\alpha\beta} \simeq 0, \quad [\delta \hat{\eta}^L]_{lm} \simeq 0. \]  

(6.23)

Now from (6.20a), we see that heavy-neutrino oscillations will induce imaginary parts of
\(\hat{\eta}^N\), and therefore, a non-zero \(\delta\hat{\eta}^N\) [cf. (6.20a)] due to the off-diagonals of \(\hat{\eta}^N\) in (6.23).

Thus, at a later time \(z = z_{in} + \delta z_2\), the number-densities will be

\[
[\hat{\eta}^N]_{\alpha\beta} = \mathcal{O}(h^2) \quad \text{for} \quad \alpha \neq \beta, \quad [\delta\hat{\eta}^N]_{\alpha\beta} = \mathcal{O}(h^2), \quad [\delta\hat{\eta}^L]_{lm} \simeq 0.
\]

Finally at \(z = z_{in} + \delta z_3\), interference in the \(\mathcal{O}(h^2)\) decays will create a non-zero lepton asymmetry of order \(\mathcal{O}(h^4)\) from the \(\delta\hat{\eta}^N\) term in (6.20b):

\[
[\hat{\eta}^N]_{\alpha\beta} = \mathcal{O}(h^2) \quad \text{for} \quad \alpha \neq \beta, \quad [\delta\hat{\eta}^N]_{\alpha\beta} = \mathcal{O}(h^2), \quad [\delta\hat{\eta}^L]_{lm} = \mathcal{O}(h^4).
\]

The physical reason for this is that the \(\tilde{CP}\)-conjugated processes

\[
\sum_\alpha c_\alpha \hat{N}_{R,\alpha} \rightarrow L \Phi, \quad \sum_\alpha c_\alpha^* \hat{N}_{R,\alpha} \rightarrow L^c \tilde{\Phi},
\]

are respectively proportional to the number densities \(\hat{\eta}^N\) and \(\hat{\eta}^N = (\hat{\eta}^N)^*\), which now differ by \(\mathcal{O}(h^2)\) in their off-diagonal elements.

Therefore, this \(\mathcal{O}(h^4)\) contribution to the total lepton asymmetry is due to a sequence of the coherent heavy-neutrino inverse decays, oscillations and decays. These effects get enhanced in the same regime as the resonant \(T = 0\ \varepsilon\)-type \(CP\) violation, namely, for \(z \approx 1\) and \(\Delta m_N \sim \Gamma_N\) [22]. For \(z \ll 1\), this effect is suppressed by the small mass of the Majorana neutrinos, and for \(z \gg 1\) the inverse decays are frozen out and thus not capable to create an asymmetry. Similarly, if the heavy-neutrino mass-splitting \(\Delta m_N\) is too large compared to \(\Gamma_N\), the oscillations are averaged out during the typical time scale of a decay process, whereas if it is too small the oscillations proceed too slowly and \(\delta\hat{\eta}^N\) produced thereof is constantly washed out.

In Section 9.4, we will show quantitatively that, in the weakly-resonant regime considered explicitly there, the lepton-asymmetry generation via the heavy-neutrino oscillation phenomenon discussed above is of the same order as the one due to their mixing in the vacuum, and hence, leads to an enhancement of the total lepton asymmetry (even in the
charged-lepton flavour diagonal case) compared to that predicted by the flavour-diagonal Boltzmann equations discussed in Section 4.3.

6.3 Decoherence in the charged-lepton sector

In this section, we include in the rate equations the effect of the charged-lepton Yukawa couplings, described by the interaction Lagrangian

\[ \mathcal{L}_y = y^l_k \bar{L}^k \Phi e_{R,l} + \text{H.c.}, \tag{6.27} \]

where \( e_{R,l} \equiv l_R \) (with \( l = e, \mu, \tau \)) and the Yukawa couplings are real and diagonal in the charged-lepton Yukawa eigenbasis, i.e. \( \hat{y}_k^l = y_l \delta_k^l \). The number-densities of the RH leptons and anti-leptons, \( n^R \) and \( \bar{n}^R \) respectively, are defined analogous to \( n^L \) and \( \bar{n}^L \) [cf. (5.17a) and (5.17b)]. The processes involving the charged-lepton Yukawa couplings are responsible for the decoherence of the LH leptons into their (would-be) mass eigenbasis. However, the interaction of the charged-leptons with the heavy-neutrinos [cf. (5.1)] induce non-zero off-diagonal elements in the charged-lepton number-density matrix, which tend to recreate a coherence between the charged-lepton flavours. Thus, there is a competition between the coherence effect induced by the heavy-neutrino Yukawa couplings and the decoherence effect due to the charged-lepton Yukawa couplings, and for large enough neutrino Yukawa couplings, the coherence effect could be significant, as we will show explicitly in Chapter 9. In particular, we will find that the decoherence is not complete in the temperature range relevant for the production of the asymmetry in the RL_{\tau} scenarios with \( 200 \text{ GeV} \lesssim m_N \lesssim 2 \text{ TeV} \), and it is important to include the off-diagonal lepton number-densities in the rate equation for the lepton asymmetry.

Let us first consider the contribution of thermal Higgs decays and inverse decays

\[ \Phi(q) \leftrightarrow L(p) \bar{e}_R(k), \tag{6.28} \]
and then generalize it to other relevant processes. The contribution of this process to the LH charged-lepton transport equation (5.63a) can be obtained in a similar manner as the heavy-neutrino decays and inverse decays. Explicitly, we obtain

$$\frac{dn^L}{dt} \supset \int_{p,k,q} \left( -\frac{1}{2} \{ n^L(p), \Gamma_{\text{dec}}(p, k, q) \} + \Gamma_{\text{back}}(p, k, q) \right). \quad (6.29)$$

In the above, we have defined the charged-lepton decoherence and back-reaction rates

$$[\Gamma_{\text{dec}}(p, k, q)]^m_l \equiv A(p, k, q) y_l^i y_l^m [\bar{n}^R(k)]^j_i, \quad (6.30a)$$

$$[\Gamma_{\text{back}}(p, k, q)]^m_l \equiv A(p, k, q) y_l^i y_l^m n^\Phi(q), \quad (6.30b)$$

where the flavour-singlet factors $A(p, k, q)$, whose explicit form is not needed here, contains the relevant kinematic factors.

In the would-be mass eigenbasis for the charged leptons (where the charged-lepton Yukawa coupling matrix is diagonal), the diagonal entries of (6.29) have the form

$$\frac{d[\hat{n}^L]_l}{dt} \supset \int_{p,k,q} A(p, k, q) y_l^2 \left( n^\Phi(q) - [\hat{n}^L(p)]_l [\hat{n}^R(k)]_l \right), \quad (6.31)$$

where the index $l$ is not summed over, and we have assumed $\hat{n}^R(k)$ to be diagonal, neglecting higher-order phenomena involving heavy-neutrino Yukawa couplings. Since the evolution equations of $\hat{n}^R$ and $n^\Phi$ contain the same term (6.31), and since the rate of the process (6.28) is much larger than the Hubble rate for the relevant time period, we can safely assume chemical (as well as kinetic) equilibrium for the reaction $\mu_\Phi = \mu_{L,l} + \bar{\mu}_{R,l}$, in addition to the complete decoherence of $\hat{n}^R$. Therefore, the evolution equations for $\Phi$ and $\bar{e}_R$ do not need to be considered explicitly and, instead, we can use the relevant KMS (or detailed balance) relation that gives the equilibrium solution of their rate equations.

For the process (6.28) under consideration, the KMS relation is simply given by

$$n^\Phi = [\hat{n}^L(p)]_l [\hat{n}^R(k)]_l, \quad (6.32)$$
for all $l$ (not summed over), which implies that the diagonal contribution (6.31) identically vanishes, and only the off-diagonal entries of the anti-commutator in (6.29) are responsible for the decoherence of charged leptons to the mass eigenbasis. Notice that at this stage it is inconsistent to approximate $\Phi$ and $\bar{e}_R$ to be at equilibrium, since this would violate the KMS relation (6.32).

It can be shown that the form (6.29) is valid for any flavour-dependent process involving one LH charged lepton, with $\Gamma_{\text{dec}}$ being the corresponding decoherence rate. Since the reactions that cause the decoherence in the LH charged-lepton sector are all fast compared to the Hubble rate, the back-reaction rate $\Gamma_{\text{dec}}^{\text{back}}$ in (6.29) can be determined from the conditions

$$\left[ \Gamma_{\text{dec}}, \Gamma_{\text{dec}}^{\text{back}} \right] = 0, \quad \left[ \hat{\Gamma}_{\text{dec}}^{\text{back}} \right]_{ll} = \left[ \hat{\Gamma}_{\text{dec}} \right]_{ll} \left[ \hat{n}^L \right]_{ll}. \quad (6.33)$$

The first condition comes from the fact that $\Gamma_{\text{dec}}^{\text{back}}$ and $\Gamma_{\text{dec}}$ are simultaneously diagonal in the charged-lepton mass eigenbasis. The second condition is the generalized KMS relation (6.32) involving any species in chemical equilibrium with the LH charged leptons.

The charged-lepton Yukawa contributions to the rate equation for anti-lepton number-density $\bar{n}^L$ is analogous to that given in (6.29). To obtain the corresponding contribution to the rate equation for the lepton asymmetry, we adopt the same set of approximations as in Section 6.1, and in particular, we use the kinetic-equilibrium number density (6.5). Taking into account the expansion of the Universe, we finally obtain

$$\frac{H_N}{z} n^L \frac{d}{dz} \delta \eta^L + \frac{1}{2} \frac{1}{\eta_{\text{eq}}^L} \left\{ \delta \eta^L, \gamma_{\text{dec}} \right\} + \delta \gamma_{\text{dec}}^{\text{back}}, \quad (6.34)$$

where $\gamma_{\text{dec}}$ and $\delta \gamma_{\text{dec}}^{\text{back}}$ are the $\tilde{C}P$-even and -odd thermally-averaged decoherence and back-reaction rates, respectively. Here we have ignored the sub-dominant $\left\{ \eta_{\text{eq}}^L, \delta \gamma_{\text{dec}} \right\}$ term, which depends on the asymmetry in the RH charged-lepton sector that is assumed to be small compared to the asymmetry in the LH sector. The $\tilde{C}P$-even rate can be expressed in terms of the charged-lepton thermal width as $\gamma_{\text{dec}} = \Gamma_T n_{\text{eq}}^L$. The thermal
width, which has the form \( \hat{\Gamma}_T = \text{diag}\{\Gamma_{T,i}\} \) in the mass eigenbasis, has been calculated explicitly in [78], taking into account the inverse Higgs decays and the relevant fermion and gauge scatterings:

\[
\Gamma_{T,i} \approx 3.8 \times 10^{-3} T y_t^2 \left[ (-1.1 + 3.0 x) + 1.0 + y_t^2 (0.6 - 0.1 x) \right],
\]

where \( y_t \) is the top quark Yukawa coupling, and \( x = M_H(T)/T = z M_H(T)/m_N, M_H(T) \) being the Higgs thermal mass. Note that while calculating the final rates for the processes involving the charged-lepton Yukawa couplings, it is important to take into account their thermal masses, which control the phase-space suppression for the decay and inverse decay of the Higgs boson [78]. In addition, note that all the chemical potentials can be consistently neglected in the calculation of the rate, as long as we satisfy the generalized KMS relations given by (6.33). After thermal averaging, (6.33) can be written as

\[
\begin{bmatrix} \gamma_{\text{dec}}, \delta \hat{\gamma}_{\text{dec}} \end{bmatrix} = 0, \quad [\delta \hat{\gamma}_{\text{dec}}]_ll = \left[ \hat{\gamma}_{\text{dec}} \right]_ll \left[ \hat{\eta}_L \right]_ll \eta_{\text{eq}}.
\]

These equations ensure that the KMS detailed balance conditions are satisfied, without having to resort to following the evolution of all the SM species involved in the processes mediated by the charged-lepton Yukawa couplings.

## 6.4 Scattering terms

In this section, we include the contribution of \( 2 \leftrightarrow 2 \) scattering processes and describe the flavour-covariant generalization of the subtraction of the RIS contributions present in the \( \Delta L = 2 \) and \( \Delta L = 0 \) scattering terms (see Figure 6.3). Specifically, we show how the sign of the \( \tilde{CP} \)-“odd” inverse decay terms in (6.15b) and (6.15c) is flipped, so that the correct approach to equilibrium is restored.

We start from the scattering contributions to the charged-lepton transport equations. The contributions of \( \Delta L = 2 \) scattering to the charged-lepton number densities can be
obtained by a flavour-covariant generalization of the relevant part of the flavour-diagonal rate equation (4.32):

\[
\frac{d[n^L]}{dt} \supset - \frac{1}{2n_{eq}^L} \left[ \gamma'(L\Phi \rightarrow L^\ast \Phi^\ast) \gamma^l_{n^L} \gamma^n_k + [n^L]^m_n \left[ \gamma'(L\Phi \rightarrow L^\ast \Phi^\ast) \right]_{n^L}^m_k \right] + \frac{[n^L]^n_k}{n_{eq}^L} \left[ \gamma'(L^\ast \Phi^\ast \rightarrow L\Phi) \right]_{n^L}^k_m , \tag{6.37a}
\]

\[
\frac{d[\bar{n}^L]}{dt} \supset - \frac{1}{2n_{eq}^L} \left[ \gamma'(L^\ast \Phi^\ast \rightarrow L\Phi) \gamma^l_{\bar{n}^L} \gamma^n_k + [\bar{n}^L]^m_n \left[ \gamma'(L^\ast \Phi^\ast \rightarrow L\Phi) \right]_{\bar{n}^L}^m_k \right] + \frac{[n^L]^n_k}{n_{eq}^L} \left[ \gamma'(L\Phi \rightarrow L^\ast \Phi^\ast) \right]_{\bar{n}^L}^k_m . \tag{6.37b}
\]

The contribution to the total lepton asymmetry is then given by

\[
\frac{d[\delta \eta^L]}{dt} \supset - 2 \left[ \delta \gamma^L_{\Phi\Phi}\gamma^L_{\Phi\Phi} \right]_{l}^m - \frac{1}{4n_{eq}^L} \left\{ \delta n^L, \gamma^L_{\Phi\Phi} \right\}_l^m - \frac{[\delta n^L]^n_k}{2n_{eq}^L} \left[ \gamma^L_{\Phi\Phi} \right]_{n}^k_m , \tag{6.38}
\]

where, by virtue of \( \tilde{C} \tilde{P} \tilde{T} \), we have defined the contractions

\[
\left[ \gamma^L_{\Phi\Phi} \right]_{l}^m = \left[ \gamma^L_{\Phi\Phi} \right]_{k}^m , \tag{6.39a}
\]

\[
\left[ \delta \gamma^L_{\Phi\Phi} \right]_{l}^m = \left[ \delta \gamma^L_{\Phi\Phi} \right]_{k}^m . \tag{6.39b}
\]

The rank-4 scattering rates introduced here can be also justified by means of the generalized optical theorem given in [28]. For consistency with the previous calculations, only their resonant parts must be kept. Using the NWA for the heavy neutrino propagator, and the same approximations as for the decay and inverse-decay terms in Section 6.1, we find the \( \tilde{C} \tilde{P} \)-“even” collision rates

\[
\left[ \gamma^L_{\Phi\Phi} \right]_{l}^m = \sum_{\alpha,\beta} \frac{\left[ \hat{\gamma}^N_{L\Phi} \right]_{\alpha\alpha} + \left[ \hat{\gamma}^N_{L\Phi} \right]_{\beta\beta}}{1 - 2i \frac{m_N - \bar{m}_N}{\Gamma_N + \Gamma_{N\beta}}} \times 2 \left( \hat{h}_t^{\beta} \hat{h}_n^{\beta} \hat{h}_m^{\alpha} \hat{h}_k^{\alpha} + \hat{h}_t^{\alpha} \hat{h}_n^{\alpha} \hat{h}_m^{\beta} \hat{h}_k^{\beta} \right) \left[ (\hat{h}_t^{\alpha} \hat{h}_n^{\alpha})_{\alpha\alpha} + (\hat{h}_t^{\beta} \hat{h}_n^{\beta})_{\beta\beta} + (\hat{h}_t^{\beta} \hat{h}_n^{\alpha})_{\alpha\beta} + (\hat{h}_t^{\alpha} \hat{h}_n^{\beta})_{\beta\alpha} \right]^2 , \tag{6.40}
\]
CHAPTER 6. FLAVOUR-COVARIANT RATE EQUATIONS

which are the flavour-covariant generalizations of those given by (4.30b). The RIS contribution is obtained by taking the diagonal \( \alpha = \beta \) elements in the summation, whereas the \( \gamma' \)-terms can be obtained by taking \( \alpha \neq \beta \). Using the fact that \( [\delta \gamma_{L^c \Phi}]^m_{l} = 0 \) up to \( \mathcal{O}(h^4) \) due to the unitarity of the scattering matrix [59], we obtain the RIS-subtracted \( \widetilde{CP} \)-“odd” collision rates

\[
[\delta \gamma_{L^c \Phi}]^m_{l} k = - \sum_{\alpha} (\gamma_{L_{\Phi} \alpha}^N_{\text{eq}}) \frac{(\hat{h}^\dagger \hat{h}^c)_{\alpha \alpha} [\hat{h}^c]^m_{l \alpha} [\hat{h}]^m_{l \alpha} - (\hat{h}^\dagger \hat{h})_{\alpha \alpha} \hat{h}_{\alpha}^l \hat{h}_{\alpha}^m}{[(\hat{h}^\dagger \hat{h})_{\alpha \alpha} + (\hat{h}^\dagger \hat{h}^c)_{\alpha \alpha}]^2}. \quad (6.41)
\]

The contributions of the \( \Delta L = 0 \) scattering to the charged-lepton transport equations are given by

\[
\frac{d[n_{l}^L]^m_{l} n_{k}}{dt} = - \frac{1}{2 n_{eq}^L} \left( [\gamma'(L\Phi \rightarrow L\Phi)]^n_{l} k [n_{l}^L]^m_{n} + [n_{l}^L]^n_{l} [\gamma'(L\Phi \rightarrow L\Phi)]^m_{l} k \right)
+ \frac{[n_{l}^L]^n_{l} k}{n_{eq}^L} [\gamma'(L\Phi \rightarrow L\Phi)]^m_{n} k l, \quad (6.42a)
\]

\[
\frac{d[\bar{n}_{l}^L]^m_{l} n_{k}}{dt} = - \frac{1}{2 n_{eq}^L} \left( [\gamma'(L^c\Phi \rightarrow L^c\Phi)]^n_{l} k [\bar{n}_{l}^L]^m_{n} + [\bar{n}_{l}^L]^n_{l} [\gamma'(L^c\Phi \rightarrow L^c\Phi)]^m_{l} k \right)
+ \frac{[\bar{n}_{l}^L]^n_{l} k}{n_{eq}^L} [\gamma'(L^c\Phi \rightarrow L^c\Phi)]^m_{n} k l, \quad (6.42b)
\]

and the corresponding contribution to the asymmetry is

\[
\frac{d[\delta n_{l}^L]^m_{l}}{dt} \supset - 2 [\delta \gamma_{L^c \Phi}]^m_{l} \frac{1}{4 n_{eq}^L} \left( [\delta n_{l}^L, \gamma_{L^c \Phi}]^m_{l} + \frac{[\delta n_{l}^L]^n_{l} k}{2 n_{eq}^L} [\gamma_{L^c \Phi}]^m_{n} k l. \right) \quad (6.43)
\]

In (6.43), using \( \widetilde{CP} \)-invariance, we have defined the contractions

\[
[\gamma_{L^c \Phi}]^m_{l} k = [\gamma_{L^c \Phi}]^m_{l} k l = [\gamma_{L^c \Phi}]^m_{k l}, \quad (6.44a)
\]

\[
[\delta \gamma_{L^c \Phi}]^m_{l} k = [\delta \gamma_{L^c \Phi}]^m_{l} k l = - [\delta \gamma_{L^c \Phi}]^m_{k l}. \quad (6.44b)
\]

Using the same set of approximations as in the \( \Delta L = 2 \) case, we obtain the flavour-
and the corresponding RIS-subtracted \( \tilde{C}_P \) -“odd” quantity

\[
\left[ \delta_{\gamma L^\Phi} \right]_{l}^{m} \equiv - \sum_{\alpha} \frac{(\gamma L^\Phi)_{\alpha \alpha}}{\alpha} \frac{\hat{h}^\beta_{l} \hat{h}^\alpha_{l} \hat{h}^{\gamma}_{m} - (\hat{h}^\beta_{l} \hat{h}^\alpha_{l} \hat{h}^{\gamma}_{m})}{2 \left( \hat{h}^\beta_{l} \hat{h}^\alpha_{l} \hat{h}^{\gamma}_{m} \right)_{\alpha}}.
\]

(6.46)

The flavour structure of the \( \gamma \)-terms in (6.40) and (6.45) can be understood diagrammatically from the unitarity cuts of partial self-energies (Figures 6.2 and 6.4), obtained by virtue of a generalized optical theorem (see [28]).

Combining (6.38) and (6.43), the total contribution of \( 2 \leftrightarrow 2 \) scattering to the lepton asymmetry can be written as

\[
\frac{d[\delta \eta^L]_{l}^{m}}{dt} = - 2 \left( \left[ \delta_{\gamma L^\Phi} \right]_{l}^{m} + \left[ \delta_{\gamma L^\Phi} \right]_{l}^{m} \right) - \frac{1}{4 \eta_{eq}^{L}} \left\{ \delta h^L, \gamma_{L^\Phi} + \gamma_{L^\Phi} \right\}_{l}^{m} - \frac{[\delta h^L]_{k}^{n}}{2 \eta_{eq}^{L}} \left[ \left[ \gamma_{L^\Phi} \right]_{n \ l}^{k \ m} - \left[ \gamma_{L^\Phi} \right]_{n \ l}^{k \ m} \right).
\]

(6.47)

Collecting the results of this section, we can also establish the following identities, valid up to \( O(h^4) \):

\[
\left[ \delta_{\gamma L^\Phi} \right]_{l}^{m} + \left[ \delta_{\gamma L^\Phi} \right]_{l}^{m} = \left[ \gamma_{N L^\Phi} \right]_{l}^{m},
\]

(6.48a)

\[
\left[ \gamma_{L^\Phi, RIS} \right]_{l}^{m} + \left[ \gamma_{L^\Phi, RIS} \right]_{l}^{m} = \left[ \gamma_{L^\Phi} \right]_{l}^{m},
\]

(6.48b)

\[
\left[ \gamma_{L^\Phi, RIS} \right]_{l}^{m} - \left[ \gamma_{L^\Phi, RIS} \right]_{l}^{m} = 0.
\]

(6.48c)

Using these identities in the scattering contribution given by (6.47), and including it in the
rate equation (6.15c), we obtain
\[
\frac{d[\delta n^L]}{dt} = -\left[\delta \gamma_{L,\Phi}^N\right]_l^m + \frac{[n^N]}{n_{eq}^N} \left[\delta \gamma_{L,\Phi}^N\right]_l^m + \frac{[\delta n^N]}{2 n_{eq}^N} \left[\gamma_{L,\Phi}^N\right]_l^m
\]
\[
- \frac{1}{4 n_{eq}^L} \left\{ \delta n^L, \gamma_{L,\Phi}^N + \gamma_{L,\Phi}^N \right\} - \frac{[\delta n^L]}{2 n_{eq}^L} \left( [\gamma_{L,\Phi}^N]_{n l}^k - [\gamma_{L,\Phi}^N]_{n l}^k \right).
\]
(6.49)

Note that, thanks to the first identity (6.48a), the sign of the inverse-decay term is now flipped with respect to that in (6.15c), as anticipated at the end of Section 6.1, so that the correct approach to equilibrium is guaranteed. The remaining identities (6.48b) and (6.48c) are important to guarantee the consistency of the formalism: following [59], we have described processes like \(L \Phi \rightarrow N \rightarrow L \Phi\) (with an on-shell \(N\)) statistically, i.e. as the successive statistical evolution of the number-density \(n^N\) due first to an inverse decay and then to a decay process. The RIS-subtracted scattering term needs to be considered in order to avoid double-counting. However, (6.48b) and (6.48c) allow us to write the washout term in terms of a complete (including RIS) scattering rate, with no inverse decay rate at all, thus describing resonant processes like \(L \Phi \rightarrow N \rightarrow L \Phi\) field-theoretically. Both these descriptions lead to the same result, as also shown in [61]. The field-theoretical non-equilibrium formalism presented in Chapters 7 and 8 will provide a unified description of these phenomena, where the correct approach to equilibrium is obtained, without the need of a RIS subtraction.

In addition to these contributions, it is shown in [28] that RIS terms also appear in the rate equation for the \(\overline{C}P\)-“odd” heavy-neutrino number-density \(\delta n^N\), which we recall is purely off-diagonal in the mass eigenbasis. These originate from the internal heavy-neutrino line in the scattering processes, which in the thermal case contains a contribution proportional to the heavy-neutrino number-density, beyond the classical statistical approximation considered here. When this “thermal” RIS subtraction is also performed, the sign of the \(\overline{C}P\)-“odd” inverse decay term in (6.19a) is also flipped, thus guaranteeing the correct approach to equilibrium.
6.5 Final rate equations

Here we put together the various contributions obtained in this chapter: heavy-neutrino decays and inverse decays, discussed in Section 6.1; processes involving charged-lepton Yukawa interactions (cf. Section 6.3); $\Delta L = 0$ and $\Delta L = 2$ resonant scatterings via heavy neutrino exchange (cf. Section 6.4). Finally, taking into account the expansion of the Universe, the following set of manifestly flavour-covariant rate equations is obtained for the $\bar{C}P$-“even” number density matrix $\eta^N$ and the $\bar{C}P$-“odd” number-density matrices $\delta\eta^N$ and $\delta\eta^L$:

\[
\frac{H_N n^\gamma}{z} \frac{d[\eta^N]}{dz} \beta = -\frac{i}{2} [\eta^N, \delta\eta^N]_\beta + \left[\text{Re}(\gamma^N_{L\Phi})\right]_\alpha, \tag{6.50a}
\]

\[
\frac{H_N n^\gamma}{z} \frac{d[\delta\eta^N]}{dz} \beta = -2i [\eta^N, \delta\eta^N]_\beta + 2i \left[\text{Im}(\delta\gamma^N_{L\Phi})\right]_\alpha - \frac{1}{2\eta_{eq}^N} \left\{\delta\eta^N, \text{Re}(\gamma^N_{L\Phi})\right\}_\alpha, \tag{6.50b}
\]

\[
\frac{H_N n^\gamma}{z} \frac{d[\delta\eta^L]}{dz}^m = -[\delta\gamma^N_{L\Phi}]^m_l + \frac{[\eta^N]_\beta^\alpha}{\eta_{eq}^N} [\delta\gamma^N_{L\Phi}]^m_l \alpha + \frac{[\delta\eta^N]_\beta^\alpha}{2\eta_{eq}^N} [\gamma^N_{L\Phi}]^m_l \alpha \\
- \frac{1}{3} \left\{[\delta\eta^L, \gamma^N_{L\Phi}]^m_l + \gamma^N_{L\Phi} \right\}_l^m \\
- \frac{2}{3} [\delta\eta^L]_n^k \left[\gamma^N_{L\Phi}\right]_n^k l - [\gamma^N_{L\Phi}]_n^k l \right) \\
- \frac{2}{3} \left\{[\delta\eta^L, \gamma_{\text{dec}}]\right\}_l^m + [\delta\gamma_{\text{dec}}]_l^m. \tag{6.50c}
\]

The flavour-covariant rate equations (6.50a)–(6.50c) are the main new results presented in this thesis. They provide a complete and unified description of the RL phenomenon, consistently capturing the following physically distinct effects in a single framework:

(i) Resonant mixing between heavy neutrinos, described by the resummed Yukawa couplings in $\gamma^N_{L\Phi}$ and $\delta\gamma^N_{L\Phi}$. This provides a flavour-covariant generalization of the
mixing effects discussed earlier in [23].

(ii) Coherent oscillations between heavy neutrinos, described by the commutators in (6.50a) and (6.50b), and transferred to the lepton asymmetry via the decay term \( [\gamma L_\Phi]_l^m \beta \) in the first line of (6.50c). We stress here that this phenomenon of coherent oscillations is an \( \mathcal{O}(h^4) \) effect on the total lepton asymmetry, and so differs from the \( \mathcal{O}(h^6) \) mechanism proposed in [29] (see Section 6.2).

(iii) Decoherence effects in the charged-lepton sector, described by the last line of (6.50c). Our description of these effects goes along the lines of [36], which has been generalized here to an arbitrary flavour basis.

As an application, we will use the rate equations (6.50a)–(6.50c) in Chapter 9 for the numerical evaluation of the lepton asymmetry in the RL\(_\tau\) model under consideration there. We will also derive approximate analytic solutions, in the strong washout regime, of these general rate equations in Section 9.4. Finally, notice that taking the limit in which the number densities are diagonal, i.e. \( [\eta^N]_\alpha^\beta = \delta_\alpha^\beta \eta^N_\beta \) and \( [\delta \eta^L]_l^m = \delta_l^m \delta \eta^L_m \) in (6.50a)–(6.50c), we recover the flavour-diagonal Boltzmann equations (4.27a) and (4.27b).
6.5. FINAL RATE EQUATIONS

(a) $\Delta L = 0$ scattering, $n^\Phi[n^L]_l^k [\gamma(L\Phi \rightarrow L\Phi)]_{k \ m}^l n$.

(b) $\Delta L = 0$ scattering, $\tilde{n}^\Phi[\tilde{n}^L]_l^k [\tilde{\gamma}(L\tilde{\Phi} \rightarrow L\tilde{\Phi})]_{k \ m}^l n$.

(c) $\Delta L = 2$ scattering, $n^\Phi[\tilde{n}^L]_l^k [\gamma(L\tilde{\Phi} \rightarrow L\tilde{\Phi})]_{k \ m}^l n$.

(d) $\Delta L = 2$ scattering, $\tilde{n}^\Phi[\tilde{n}^L]_l^k [\tilde{\gamma}(L\tilde{\Phi} \rightarrow L\tilde{\Phi})]_{k \ m}^l n$.

Figure 6.3: Feynman diagrams for $\Delta L = 0$ scattering [(a), (b)] and $\Delta L = 2$ scattering [(c), (d)], in the presence of a statistical background. The flavour indices are shown explicitly, while other indices are suppressed.
(a) Charged-lepton self-energies, with $\Delta L = 0$ internally.

(b) Charged-lepton self-energies, with $\Delta L = 2$ internally.

Figure 6.4: Feynman diagrams for the self-energies of the lepton doublets. The cut, across which positive energy flows from unshaded to shaded regions, is associated with production rates in the thermal plasma, as described by the generalized optical theorem. See also Figure 6.3.
In this chapter, we present the consistent quantum field-theoretical description of non-equilibrium phenomena. We first review the Cornwall–Jackiw–Tomboulis (CJT) formalism which, together with the real-time CTP formalism discussed in Chapters 2 and 3, provides the framework in which non-equilibrium QFT can be formulated consistently. Then, we introduce the CTP formulation of transport phenomena in the Heisenberg picture, obtaining the celebrated Kadanoff-Baym (KB) equations, and discuss the approximations typically performed to make their (semi-)analytic usage tractable. Finally, we outline the recently-developed KB formalism in the interaction picture, which allows to go beyond such approximations, in the situations where this is needed. In the next chapter we will use the latter to study Resonant Leptogenesis in a fully field-theoretical approach, thus going beyond the analysis of Chapters 5 and 6.

### 7.1 Cornwall–Jackiw–Tomboulis effective action

In many situations of interest, finite-order perturbation theory breaks down, and one needs to devise resummation methods to deal with this problem. An example is provided by equilibrium thermal QFT at high temperatures, where thermal loop corrections may become dominant, compared to tree-level contributions, thus invalidating the perturbative
expansion [93, 94]. Another situation, of particular relevance for this thesis, is the resonant enhancement of the CP asymmetry in RL, in the limit in which the heavy neutrinos become quasi-degenerate (see 4.2).

In Sections 4.3 and 6.4 we have seen that, in the semi-classical approaches adopted there, the non-equilibrium evolution equations of unstable particles are subject to potential double-counting of different processes, included both in the statistical and field-theoretical descriptions of the evolution, and one needs to subtract the RIS contributions by hand. For example, in a leptogenesis scenario, the resonant part of the field-theoretical process $L\Phi \rightarrow N \rightarrow L\Phi$ is already accounted for, as the sequence of an inverse decay $L\Phi \rightarrow N$ and a subsequent decay $N \rightarrow L\Phi$ in the statistical evolution of the RH-neutrino number-density. In order to construct a consistent field-theoretical formalism that is automatically free of this kind of pathology, one needs to resum automatically sequences of processes like these, in the field-theoretical description of the unstable particles. In other words, one needs to have sequences of decay, inverse-decay and scattering processes automatically taken into account in the propagator of the field of interest.

An elegant framework to perform this kind of resummations automatically is provided by the CJT formalism [95]. In its simplest version, the so-called 2-Particle-Irreducible (2PI) effective action is expressed in terms of the background field and the dressed propagator. The full 2PI effective action is formally equivalent to the standard 1-Particle-Irreducible (1PI) one. For practical purposes, however, one is compelled to consider truncations to the 2PI effective action, in terms of a loopwise diagrammatic expansion. At any given order of this loopwise expansion, the 2PI effective action automatically resums an infinite set of perturbation-theory diagrams, induced by partially resummed propagators. There is an extensive literature related to the CJT formalism, mainly for thermal applications ([96–100] constitutes by no means an exhaustive list). The original formalism can be modified in order to describe consistently theories with global symmetries [101], thus enhancing its range of applicability to phenomenologically-interesting non-thermal situations (see, e.g. [102]).
7.1. CORNWALL–JACKIW–TOMBOULIS EFFECTIVE ACTION

The CJT formalism is a generalization of the 1PI effective action, where in addition to the local source $J_x$, multi-local sources $K_{xy}$, $K_{xyz}$ etc, are introduced. For notational simplicity, in this section we use a DeWitt matrix-like notation, e.g. $K_{xy} \equiv K(x, y)$, with implicit sum over repeated indices. For our purposes, it is sufficient to consider its simplest version, the 2PI formalism, which contains one local and one bi-local source, i.e. $J_x$ and $K_{xy}$. In the 2PI formalism, the connected generating functional $W[J, K]$ is given by

$$W[J, K] = -i \ln \int \mathcal{D}\phi \exp \left[ i \left( S[\phi] + J_x \phi_x + \frac{1}{2} K_{xy} \phi_x \phi_y \right) \right], \quad (7.1)$$

where $S[\phi] = \int_x \mathcal{L}[\phi]$ is the classical action of the scalar field $\phi$. The formulae considered here are easily generalized to models with more than one field. The background field $\phi_x$ and the connected propagator $\Delta_{xy}$ are obtained by single and double functional differentiation of $W[J, K]$ with respect to the source $J_x$:

$$\frac{\delta W[J, K]}{\delta J_x} \equiv \phi_x, \quad -i \frac{\delta W[J, K]}{\delta J_x \delta J_y} = \langle \hat{\phi}_x \hat{\phi}_y \rangle - \langle \hat{\phi}_x \rangle \langle \hat{\phi}_y \rangle \equiv i \Delta_{xy}, \quad (7.2)$$

where we have not specified the ordering of the field operators $\hat{\phi}$, since the discussion here is easily generalized to real-time propagators with CTP structure (cf. Section 2.3). In addition, differentiating $W[J, K]$ with respect to the bi-local source $K_{xy}$ yields

$$\frac{\delta W[J, K]}{\delta K_{xy}} = \frac{1}{2} \left( i \Delta_{xy} + \phi_x \phi_y \right). \quad (7.3)$$

To obtain the 2PI effective action $\Gamma[\phi, \Delta]$, we perform a double Legendre transform of $W[J, K]$ with respect to $J$ and $K$:

$$\Gamma[\phi, \Delta] = W[J, K] - J_x \phi_x - \frac{1}{2} K_{xy} \left( i \Delta_{xy} + \phi_x \phi_y \right). \quad (7.4)$$
As derived in the pioneering article [95], $\Gamma[\phi, \Delta]$ takes on the explicit form

$$\Gamma[\phi, \Delta] = S[\phi] - \frac{i}{2} \text{Tr} \left( \ln \Delta \right) + \frac{i}{2} \text{Tr} \left( \Delta^{(0)} \right)^{-1} - i\Gamma^{(\geq 2)}, \quad (7.5)$$

where $\Delta^{(0)}^{-1} = \delta^2 S[\phi]/(\delta \phi_x \delta \phi_y)$ is the inverse tree-level propagator and $\Gamma^{(\geq 2)}$ stands for all two- and higher-loop 2PI vacuum diagrams\(^1\) in which all propagator lines are expressed in terms of the dressed propagator $\Delta$. Given $\Gamma[\phi, \Delta]$, the equations of motions are obtained by its functional derivatives

$$\frac{\delta \Gamma[\phi, \Delta]}{\delta \phi_x} = - J_x - K_{xy} \phi_y, \quad \frac{\delta \Gamma[\phi, \Delta]}{\delta \Delta_{xy}} = - \frac{i}{2} K_{xy}. \quad (7.6)$$

We see that in the limit of vanishing external sources $J$ and $K$, the physical solution is obtained by extremizing the 2PI effective action $\Gamma[\phi, \Delta]$. As the computation of the full effective action $\Gamma[\phi, \Delta]$ constitutes a formidable task, one has to rely on truncations of the diagrammatic series $\Gamma^{(\geq 2)}$. However, at any order of such truncation, one selectively resums infinite sets of perturbation-theory contributions, in an automatic manner. For vanishing external sources, the equation of motion for the dressed propagator $\Delta$ takes on the following Schwinger-Dyson form:

$$\Delta^{-1} = \Delta^{(0)}^{-1} + \Pi[\phi, \Delta], \quad (7.7)$$

where $\Pi[\phi, \Delta]$ is the 1PI self-energy, in which the propagator lines are given by the dressed propagator matrix $\Delta$. In standard perturbation theory, this corresponds diagrammatically to an infinite set of selectively resummed Feynman graphs, as shown in Figures 7.1 and 7.2. Keeping in mind the application to non-equilibrium phenomena, we can summarize the results of this section as follows: the 2PI formalism provides the equation of motion (7.7) for the dressed propagator, which automatically resums infinitely-many nested self-energy insertions in the propagators of the theory. These, in addition to the

\(^1\)These are the vacuum diagrams that cannot be disconnected by cutting 2 lines.
7.2. **Kadanoff–Baym Equations**

In Section 2.1 we showed that, when one is interested in the expectation value of observables in a given quantum state, the *in-in* CTP formalism should be used, rather than the more common *in-out* one, which is instead suitable for scattering-matrix calculations. The same holds true if, working in the Heisenberg picture, one considers the ensemble expectation value (EEV) of an observable $O(\tilde{t}; \tilde{t}_i)$ in a general mixed state, described by a density matrix $\rho(\tilde{t}_i; \tilde{t}_i)$. The EEV is given by

$$\langle O \rangle_t = \frac{1}{Z} \operatorname{Tr} \rho(\tilde{t}_i; \tilde{t}_i) O(\tilde{t}; \tilde{t}_i),$$  \hspace{1cm} (7.8)
where \( Z \) is the partition function and \( t = \tilde{t} - \tilde{t}_i \), with \( \tilde{t} \) being the time of observation. The quantum-mechanical pictures are taken coincident at the time \( \tilde{t}_i \), and we recall that the density matrix does not evolve in time, for an isolated system, in the Heisenberg picture.

Following [103], the generating functional of EEVs can be written as (cf. (2.28) for its equilibrium real-time form)

\[
Z_C[J(x), \rho] \equiv \int \mathcal{D}\phi e^{i \int C \! d^4x [C + J \phi + i \epsilon \phi^2]} \langle \phi_1(\mathbf{x}, \tilde{t}_i) | \rho(\tilde{t}_i; \tilde{t}_i) | \phi_2(\mathbf{x}, \tilde{t}_i) \rangle ,
\]

where \( \phi_{1,2}(\mathbf{x}, \tilde{t}_i) \) are the field configurations on the two CTP branches at the boundary time \( \tilde{t}_i \). We point out that, working in the Heisenberg picture, the density matrix is inserted at the boundary time \( \tilde{t}_i \). The density-matrix elements \( \exp(iK[\phi_a]) \equiv \langle \phi_1(\mathbf{x}, \tilde{t}_i) | \rho(\tilde{t}_i; \tilde{t}_i) | \phi_2(\mathbf{x}, \tilde{t}_i) \rangle \) can be expanded in series of multi-local kernels [103]

\[
K[\phi_a] = K + \int_C d^4x K(x) \phi(x) + \frac{1}{2} \int_C d^4x \int_C d^4x' K(x, x') \phi(x) \phi(x') + \ldots ,
\]

where the multi-local sources \( K(x, x', \ldots) \) have support only at \( x_0, x'_0, \ldots = \tilde{t}_i \). Assuming Gaussian initial correlations, one typically truncates this series up to the bi-local term. The constant \( K \) can be absorbed into the partition function. The local and bi-local sources, instead, are physically non-zero at the initial boundary time. However, by using the 2PI formalism presented in the last section, one can obtain an equation of motion for the dressed propagator, in which these physical sources simply appear as the initial condition at the time \( \tilde{t}_i \).

The 2PI construction is formally unchanged in the CTP formalism considered here. One obtains the Schwinger-Dyson equation for the CTP propagator \( \Delta_{ab} \) (cf. (7.7))

\[
\Delta_{ab}^{-1}(x, y) = \Delta_{ab}^{(0)}(x, y; \phi) + \Pi_{ab}(x, y; \phi, \Delta) ,
\]

supplemented by an appropriate boundary condition at \( x_0, y_0 = \tilde{t}_i \). Assuming vanishing background field and using the explicit form of the inverse tree-level propagator of a scalar
field of mass $m$, the Schwinger-Dyson equation (7.11) can be rewritten as

$$-(\partial_x^2 + m^2)\Delta_{ab}(x, y) = \eta_{ab}\delta(x - y) - \eta^{cd}(\Pi_{ac} \ast \Delta_{db})(x, y),$$

(7.12)

where we have introduced the CTP metric $\eta^{ab} = \eta_{ab} = \text{diag}\{1, -1\}$ and $\ast$ indicates the convolution

$$(A \ast B)(x, y) \equiv \int_{z_0 = T}^{z_0 = T} d^4z A(x, z) B(z, y),$$

(7.13)

where the (fixed) half-extension $T$ of the time integration should not be confused with the temperature. By manipulating the CTP indices along the lines of Section 3.2, (7.12) gives the KB equations in position space [104–108], for the retarded and advanced propagators

$$-(\partial_x^2 + m^2)\Delta_{R/A}(x, y) = (\Pi_{R/A} \ast \Delta_{R/A})(x, y) = \delta(x - y),$$

(7.14)

and for the Wightman ones

$$-(\partial_x^2 + m^2)\Delta_{\leq}(x, y) = (\Pi_{\leq} \ast \Delta_{\leq})(x, y) + (\Pi_{\Pu} \ast \Delta_{\leq})(x, y)$$

$$= \frac{1}{2} \left[ (\Pi_{\leq} \ast \Delta_{\leq})(x, y) - (\Pi_{> \Pu} \ast \Delta_{\leq})(x, y) \right],$$

(7.15)

where we have introduced the principal-part propagator $\Delta_{\pu}$, defined as

$$\Delta_{\Pu} \equiv \frac{1}{2} (\Delta_R + \Delta_A).$$

(7.16)

By virtue of the identity $\Delta_R(x, y) = \Delta_{A\Pu}^\dagger(y, x)$, which can be proved easily using the definitions (2.17) and (2.18), we see that the principal-part propagator is the Hermitian part of $\Delta_{R,A}$, and it is sometimes referred to as the Hermitian propagator.

It is often convenient to transform the KB equations (7.14) and (7.15) to Wigner space, where the relative spacetime coordinate $x - y$ is Fourier-transformed to $k$, but the average coordinate $X = (x + y)/2$ is retained in position space. In order to do so conveniently, one needs to perform two approximations: (i) the limits of time-integration are extended
CHAPTER 7. THERMAL QUANTUM FIELD THEORY OUT OF EQUILIBRIUM

to $\pm \infty$; (ii) the so-called gradient expansion at zeroth order, i.e. one neglects $\partial_X$ compared to $k$; both these approximations are based on the assumed separation of time scales of the statistical and field-theoretical processes, and are a form of Wigner-Weisskopf approximation (cf. Section 5.3). Performing these approximations in Wigner space, and separating the Hermitian and anti-Hermitian parts of the equation for the Wightman propagator (7.15), the KB equations in Wigner space are found to be [104–107]

\[
\begin{align}
(k^2 - m^2) \Delta_\leq(k, X) &= - \left( \Pi_\leq(k, X) \Delta_\mathrm{p}(k, X) + \Pi_\mathrm{p}(k, X) \Delta_\leq(k, X) \right), \\
k \cdot \partial_X \Delta_\leq(k, X) &= \frac{i}{2} \left( \Pi_\geq(k, X) \Delta_\geq(k, X) - \Pi_\leq(k, X) \Delta_\leq(k, X) \right).
\end{align}
\]

(7.17)

The first equation is known as the constraint equation, whereas the second one is the kinetic equation. Once a quasiparticle approximation is performed, as described below, the latter describes the statistical evolution of the quasiparticle number-density.

The propagators appearing in (7.17) are the dressed ones, and their form is typically unknown. In order to obtain expressions suitable for (semi-)analytic computations, one needs to perform a quasiparticle approximation, writing the Wightman propagators in terms of a suitable number-density. This is typically done by means of the so-called KB ansatz:

\[
i \Delta_\leq(X, k) \approx 2\pi \delta(k^2 - M^2(k)) \left( \theta(\pm k_0) + n_{\text{dress}}(k, X) \right),
\]

(7.18)

where $n_{\text{dress}}(k, X)$ is the number-density of spectrally-dressed quasiparticles. This form mimics the equilibrium propagators in the NWA (3.27). Using the KB ansatz (7.18) in the kinetic equation (7.17b), and integrating over spatial momentum modes and positive frequencies, one obtains, for a spatially-homogeneous system,

\[
\frac{dn_{\text{dress}}}{dt} = - \int \frac{d^3k}{2 E(k)} \left[ i \Pi_\leq(\varepsilon(k), k; t) \left( 1 + n_{\text{dress}}(k, t) \right) - n_{\text{dress}}(k, t) i \Pi_\geq(\varepsilon(k), k; t) \right],
\]

(7.19)

which is precisely the quantum Boltzmann equation (3.48), obtained heuristically in Chap-
7.3 Closed-time path formalism in the interaction picture

In this section, we outline briefly the interaction-picture formulation of the CTP formalism introduced in [109] (see also [110]). This approach allows one to truncate perturbatively the evolution equations of interest, without the need to resort to the KB ansatz (7.18), or similar approximations.

In the Heisenberg-picture CTP formalism, as discussed in the previous section, the density operator $\rho(\tilde{t}_i; \tilde{t}_i)$ does not evolve in time, remaining fixed at the boundary time $\tilde{t}_i$. As a result, the free propagators encode the initial conditions of the statistical ensemble, e.g.

$$i\Delta^0_\varphi (x, y, 0) = \langle \phi(x)\phi(y) \rangle_0 \equiv \frac{1}{Z} \text{Tr} \rho(0)\phi(x)\phi(y), \quad (7.20)$$

where $\rho(0) \equiv \rho(\tilde{t}_i; \tilde{t}_i)$. In this case, it is well-known that a well-defined perturbative expansion is not possible (for a review, see [108]). This may be understood by considering the Taylor expansion of the exponential approach to equilibrium:

$$e^{-\Gamma t} = 1 - \Gamma t + \frac{1}{2!} (\Gamma t)^2 + \ldots . \quad (7.21)$$

Any truncation of this expansion at a finite order in the decay rate $\Gamma$ leads to secular behaviour for $t > 1/\Gamma$ [108], i.e. to unphysically growing terms, without the late-time approach to equilibrium. When one takes the limit $t \to \infty$, this problem manifests in the Feynman-Dyson series as the appearance of pinch singularities [111] (see Section 3.3.2), which result from ill-defined products of delta functions with identical arguments. At equilibrium, as discussed in Section 3.3.2, these singularities cancel after using the KMS relation. However, this does not happen out of equilibrium, in general, since the KMS relation does not hold true. As a consequence, it is necessary to work with dressed propagators, and therefore ansaetze are necessary to perform (semi-)analytic computations. In
Section 7.2 we have given an example of this, when obtaining the quantum Boltzmann equation.

On the other hand, it has been shown recently [109] that a perturbative framework of non-equilibrium thermal field theory is in fact viable, if one works instead in the interaction picture. To achieve this, one needs to modify the original CTP contour: the density operator \( \rho(\tilde{t}_f; \tilde{t}_i) \) is inserted at the time of observation \( \tilde{t}_f \), rather than at \( \tilde{t}_i \); thus, the length of the contour is not fixed, but changes in time, as illustrated graphically in Figure 7.3. Since the interaction-picture density operator \( \rho(\tilde{t}_f; \tilde{t}_i) \) evolves in time, the free positive-frequency Wightman propagator becomes

\[
i \Delta^0_n(x, y; \tilde{t}_f; \tilde{t}_i) = \langle \phi(x; \tilde{t}_i) \phi(y; \tilde{t}_i) \rangle_t = \frac{1}{Z} \text{Tr} \rho(t) \phi(x; \tilde{t}_i) \phi(y; \tilde{t}_i),
\]

where \( \rho(t) \equiv \rho(\tilde{t}_f, \tilde{t}_i) \), with \( t = \tilde{t}_f - \tilde{t}_i \), and all objects are in the interaction picture. Assuming spatial homogeneity, the free Wightman propagators are then given by (cf. (2.36) for their equilibrium form)

\[
i \Delta^0_n(k, \tilde{t}_f; \tilde{t}_i) = 2\pi \delta(k^2 - m^2) \left( \theta(\pm k_0) + n(k, t) \right),
\]

depending on the time-dependent number-density \( n(k, t) \) of spectrally-free particles. This
appears as an unknown function in the Feynman-Dyson series, with its functional form
being fixed only after the governing transport equation has been solved. Thus, the expo-
nential approach to equilibrium is present implicitly in the free propagators of the theory,
thereby avoiding the problem of secularity or pinch singularities (for a more detailed dis-
}
where * denotes the weighted convolution integral in the double-momentum space

$$A * B \equiv \int_{q, q'} (2\pi)^4 \delta_t^{(4)}(q - q') A(p, q, \tilde{t}_f; \tilde{t}_i) B(q', p', \tilde{t}_f; \tilde{t}_i) .$$  (7.27)

Here, the weight function is given by

$$(2\pi)^4 \delta_t^{(4)}(q - q') \equiv \int_{z \in \Omega_t} e^{-i(q - q') \cdot z} = (2\pi)^4 \delta_t(q_0 - q'_0) \delta^{(3)}(q - q') ,$$  (7.28)

with $\Omega_t = [\tilde{t}_i, \tilde{t}_f] \times \mathbb{R}^3$ and

$$\delta_t(q_0 - q'_0) \equiv \frac{1}{\pi} \frac{\sin[(q_0 - q'_0)t/2]}{q_0 - q'_0} .$$  (7.29)

In (7.26), we have suppressed the dependence on the two momenta $p, p'$ and time $\tilde{t}_f$, for notational simplicity.

Using (7.24) in (7.26) and following [109], the evolution equation for the spectrally-dressed number-density is found to be

$$\frac{dn_{\text{dress}}(t, X)}{dt} = - \int_{p, p'} (X) [\Pi_p, \Delta_{\langle \cdot \rangle}]_* - \frac{1}{2} \int_{p, p'} (X) \left\{ [\Pi_{\langle \cdot \rangle}, \Delta_{\langle \cdot \rangle}]_* - [\Pi_{\langle \cdot \rangle}, \Delta_{\langle \cdot \rangle}]_* + 2 [\Pi_{\langle \cdot \rangle}, \Delta_p]_* \right\} ,$$  (7.30)

where we have introduced the definitions

$$[A, B]_* \equiv A * B - B * A ,$$  (7.31a)

$$\{A, B\}_* \equiv A * B + B * A .$$  (7.31b)

Performing the Markovian approximation as will be described in detail in Section 8.2.1, one can show that this equation, at late times, formally resembles the Heisenberg-picture one, with zeroth-order gradient expansion and KB ansatz performed. However, the main difference is that this equation can be truncated, perturbatively if needed, in two indepen-
7.3. CLOSED-TIME PATH FORMALISM IN THE INTERACTION PICTURE

dent ways:

- *statistically*, by choosing an explicit approximate form for the self-energies on the RHS, i.e. for the truncation of the 2PI effective action. This truncation corresponds to the choice of statistical processes taken into account in the evolution of the number-density. In the next chapter we will retain the 1-loop CJT-resummed form for the self-energies, thus effectively resumming an infinite set of perturbation-theory processes (among which are the heavy-neutrino absorptive transitions).

- *spectrally*, since both the LHS (see (7.24)) and the RHS are formally linear in the Wightman propagator and a perturbative expansion is well-defined (see the discussion after (7.23)). Thus, (7.30) is valid separately to any order in the truncation of this *external leg*, at least when the self-energies do not depend on the propagator of the particle species under consideration\(^1\). This truncation corresponds to choosing what objects are counted by the number-density. For instance, by truncating the external leg at tree-level, one obtains the evolution equation for the number-density \(n\) of spectrally-free particles.

---

\(^1\)At high orders in the statistical truncation, the propagator of the species under consideration will in general appear in the CJT self-energies, and we expect these two truncations to not be independent anymore.
CHAPTER
EIGHT

KADANOFF–BAYM APPROACH TO RESONANT LEPTOGENESIS

In this chapter, we study the effect of mixing and oscillations on the generation of the asymmetry in a fully field-theoretical flavour-covariant treatment. In particular, we work within the KB formalism in the interaction picture, outlined in Section 7.3, generalizing it to the multi-flavour case of interest here, in a flavour-covariant way. This thermal field-theoretical analysis, performed in the weakly resonant regime $\Gamma_{N_\alpha} \ll |M_{N_\alpha} - M_{N_\beta}| \ll M_N$, will confirm that mixing and oscillations are two distinct phenomena, with an enhancement of the total asymmetry by a factor of order 2, with respect to analyses that take into account mixing or oscillations alone. The material covered in this chapter has been presented originally in [42] (see also [43]).

8.1 Flavour-covariant scalar model of Resonant Leptogenesis

In order to study the role of heavy-neutrino flavour effects without the technical complications arising from the fermionic nature of the heavy neutrinos and charged leptons, we consider a simple scalar toy model of RL with two real scalar fields $N_\alpha$ (with $\alpha = 1, 2$), one complex scalar field $L$ and a real scalar $\Phi$. This simple model includes all qualitatively important features of leptogenesis; the two real scalar fields mimic heavy Majorana
neutrinos of two flavours and the complex scalar field models charged leptons of a single flavour. Moreover, $\Phi$ plays the role of the SM Higgs field. The approximate global $U(1)$ symmetry associated with the complex scalar field $L$ acts as the lepton number. Similar toy models have been used extensively in the literature to study RL in the KB formalism [112–117].

In order to capture fully the flavour-dynamics in the heavy-neutrino sector, we adopt the flavour-covariant formulation presented in Chapter 5. The relevant part of the Lagrangian of this scalar model may then be written in the following manifestly-covariant form:

$$L_N = h^\alpha L^\dagger \Phi N\alpha + \frac{1}{4} N\alpha [m_N^2]^\alpha\beta N\beta + \text{H.c.} ,$$  

with transformation laws as in (5.2) and (5.3). In this toy model, the Sakharov conditions (cf. Section 4.1) for the generation of the BAU are satisfied as follows. First, the lepton number is explicitly broken by the $L^\dagger \Phi N$ term in (8.1). Second, the charge conjugation symmetry is violated, provided that $\text{arg}(h^1) \neq \text{arg}(h^2)$ and the heavy neutrinos are non-degenerate. In this scalar model, $C$-violation also implies $CP$-violation, since $CP$-transformations on the scalar fields are identical to $C$-transformations, up to a sign change of the spatial coordinates. Finally, the out-of-equilibrium requirement can be satisfied by the decays of $N\alpha$ in an expanding Universe.

In the weakly-resonant regime of RL, the heavy-neutrino mass eigenbasis can be defined to be that in which the thermal mass matrix, given by

$$[M_N^2(k)]_{\alpha\beta} \equiv [\vert m_N \vert^2]_{\alpha\beta} - \vert \text{Re} \Pi_N(k) \vert_{\alpha\beta} ,$$  

is diagonal in the vicinity of the two quasi-degenerate thermal mass shells. This is well-defined since the retarded heavy-neutrino self-energy $i\Pi_N^R(k)$ is a slowly-varying function of $k_0$ near the thermal mass shells. Therefore, we can approximate $[M_N^2(k)]_{\alpha\beta}$ by its OS
8.1. FLAVOUR-COVARIANT SCALAR MODEL OF RESONANT LEPTOGENESIS

form \([M_N^2(k)]_{\alpha}^{\beta}\), obtained as the solution of the thermal gap equation (cf. (3.19))

\[
[M_N^2(k)]_{\alpha}^{\beta} = k^2 \delta_{\alpha}^{\beta} + \lim_{\epsilon \to 0^+} \left( Re \Pi_R^N \left( \Phi_N(k) + i\epsilon, k \right) \right)_{\alpha}^{\beta}.
\]

(8.3)

The KB framework in the interaction picture, outlined in Section 7.3, can be easily generalized to the scalar multi-flavour case of the toy model considered here. For instance, the free positive-frequency Wightman propagator is

\[
[i \Delta N, 0^>(x, y, \tilde{t}_i)]_{\alpha}^{\beta} = \langle N_\alpha(x; \tilde{t}_i) N^\beta(y; \tilde{t}_i) \rangle_t \equiv \frac{1}{Z} \text{Tr} \rho(\tilde{t}_f; \tilde{t}_i) N_\alpha(x; \tilde{t}_i) N^\beta(y; \tilde{t}_i).
\]

(8.4)

Assuming a Gaussian and spatially-homogeneous ensemble, we may write the double-momentum representation (see Section 7.3 and [28, 109]) of the heavy-neutrino Wightman propagators in the mass eigenbasis as

\[
[i \Delta N, 0^>(k, k', \tilde{t}_f; \tilde{t}_i)]_{\alpha}^{\beta} = \frac{2\pi}{(2\pi)^3} \delta^{(3)}(k - k') \left( \theta(k_0) \theta(k'_0) \delta_{\alpha}^{\beta} + \left[ n_N^N(k, t) \right]_{\alpha}^{\beta} \right) (2\pi)^3 \delta^{(3)}(k - k').
\]

(8.5)

In general, the heavy-neutrino Wightman propagators depend explicitly on the zeroth components of two four-momenta, \(k_0\) and \(k'_0\), since the time-translational invariance of free propagators is broken in the presence of flavour oscillations. The phase \(e^{i(k_0 - k'_0)\tilde{t}_f}\) is due to the free evolution of the interaction-picture operators. We stress here that the number-density \(\hat{n}^N(k, t)\) appearing in the free propagator (8.5) counts spectrally-free quasiparticles (see Section 7.3), and thus coincides with the one introduced in Section 5.1.

In the weakly-resonant regime, we may approximate \(\hat{m}_{N, \alpha} \simeq m_N\) in the on-shell delta functions of (8.5). Thus, we obtain the free homogeneous heavy-neutrino Wightman propagators, which can be written, in a general basis, in the single momentum representation

\[
[i \Delta N, 0^>(k, t)]_{\alpha}^{\beta} = 2\pi \delta(k^2 - m_N^2) \left( \theta(\pm k_0) \delta_{\alpha}^{\beta} + \left[ n_N^N(k, t) \right]_{\alpha}^{\beta} \right).
\]

(8.6)
By resumming the dispersive self-energy corrections, we may replace $m_N$ in (8.6) by the average thermal mass $M_N(k) \equiv (\hat{M}_{N,1}(k) + \hat{M}_{N,2}(k))/2$, given by the solution to (8.3).

For our subsequent discussion, we need also the equilibrium form of the dressed Higgs and charged-lepton Wightman propagators for vanishing chemical potential. In the NWA, it will be sufficient to use the standard quasiparticle expressions (cf. (3.27), which can be generalized also to complex fields)

\begin{align}
  i\Delta_{\varphi,\text{eq}}(q) &= 2\pi\delta(q^2 - M_{\varphi}^2) \left[ \theta(\pm q_0) + n_{\text{eq}}^\varphi(q) \right], \\
  i\Delta_{L,\text{eq}}(p) &= 2\pi\delta(p^2 - M_L^2) \left[ \theta(\pm p_0) + \theta(p_0) n_{\text{eq}}^L(p) + \theta(-p_0) \bar{n}_{\text{eq}}^L(p) \right],
\end{align}

where $M_X^2$ denotes the thermal mass of the species $X$ and $n_{\text{eq}}^X(p) = (e^{E_X(p)/T} - 1)^{-1}$ is the equilibrium quasiparticle number-density of $X$, obeying Bose-Einstein statistics, with $E_X(p)$ being the OS quasiparticle energy, as determined by the thermal gap equation (3.19). Note that, by virtue of $CPT$ invariance, we have $\bar{n}_{\text{eq}}^X = n_{\text{eq}}^X$.

### 8.2 Quantum transport equations

In this section, we obtain the rate equations for the heavy neutrinos and charged leptons. In particular, we focus on the source term for the charged-lepton asymmetry in the scalar toy model described in the last section.

In the multi-flavour case, the number-density of dressed quasiparticles (7.24) is generalized to

\begin{equation}
  n_{\text{dress}}(t, X) \equiv \int_{p,p'}^{(X)} (p_0 + p'_0) i\Delta_{\varphi}(p, p', \tilde{t}_f; \tilde{t}_i).
\end{equation}

We remind the reader that this equation is valid order by order in the loopwise expansion. For instance, by inserting the free heavy-neutrino propagator on the RHS of (8.8), one can obtain the number-density $n_N$ of spectrally-free particles (with respect to absorptive transitions). In the multi-flavour case, the KB equation for the Wightman propagators (7.15)
becomes
\[
\left( -\partial_x^2 - |m|^2 \cdot + \Pi_P \ast \right) \Delta_\Xi = -\frac{1}{2} \left( \Pi_\ga \ast \Delta_\ga - \Pi_\la \ast \Delta_\la + 2 \Pi_\Xi \ast \Delta_\rho \right),
\]
(8.9)

where \( \ast \) now indicates the convolution (cf. (7.13))
\[
A \ast B \equiv \int_{z \in \Omega} A(x,z,\tilde{t}_f;\tilde{t}_i) \cdot B(z,y,\tilde{t}_f;\tilde{t}_i),
\]
(8.10)

with \( \cdot \) denoting matrix multiplication in flavour space. For the charged lepton and Higgs in the toy model, the matrix product trivially reduces to scalar multiplication, as in Section 7.3. Using (8.8) and proceeding as in Section 7.3, we may obtain the evolution equation for the dressed number-density (cf. (7.30))
\[
\frac{dn_{\text{dress}}(t,X)}{dt} - \int_{p, p'} \left( p^2 - p'^2 \right) \Delta_\la - \int_{p, p'} \left( [m]^2, \Delta_\la \right) - \left[ \Pi_P, \Delta_\la \right] \ast \right) \\
= -\frac{1}{2} \int_{p, p'} \left( \left\{ \Pi_\ga, \Delta_\la \right\} \ast - \left\{ \Pi_\la, \Delta_\ga \right\} \ast + 2 \left[ \Pi_\la, \Delta_\rho \right] \ast \right).
\]
(8.11)

Here, we have introduced the (anti-)commutators in flavour space:
\[
[A, B] \ast \equiv A \ast B - B \ast A,
\]
(8.12a)
\[
\{A, B\} \ast \equiv A \ast B + B \ast A,
\]
(8.12b)

with \( \ast \) denoting the weighted convolution integral in the double momentum space (cf. (7.27))
\[
A \ast B \equiv \int_{q, q'} (2\pi)^4 \delta_4(q-q') A(p,q,\tilde{t}_f;\tilde{t}_i) \cdot B(q',p',\tilde{t}_f;\tilde{t}_i).
\]
(8.13)

In the LHS of (8.11), the first two terms comprise the drift terms; the latter two describe mean-field effects, including oscillations. The terms on the RHS of (8.11) describe col-
lisions. We emphasize that (8.11) is obtained without the need to perform a gradient expansion or make use of a quasiparticle ansatz. Thus, (8.11) is valid at any order in perturbation theory, thereby capturing fully the flavour effects, non-Markovian dynamics and memory effects. In addition, for the sake of generality we have considered a spatial inhomogeneous ensemble, even though we will not need this in the following.

8.2.1 Heavy-neutrino rate equations

Starting from the general transport equation (8.11), we may now proceed to derive the rate equations for the heavy-neutrino number-densities. The principal-part self-energy $\Pi^N_P$ in the last term on the LHS of (8.11) combines with the tree-level heavy-neutrino mass $|m_N|^2$ to give the thermal mass: $M^2_N = |m_N|^2 - \Pi^N_P$, where we have used $\text{Re}(\Pi^N_R) = \Pi^N_P$ (see, e.g. [109]) in (8.2). In the absence of mixing, the commutator containing $\Delta^N_P$ involves a principal-value integral that we may safely neglect for quasi-degenerate heavy neutrinos. However, mixing between the RH neutrinos causes the appearance of off-diagonal entries in $\Delta^N_P$ proportional to Dirac delta functions in the NWA. It can be shown that, in the weakly-resonant regime, these are higher-order effects compared to the ones that we are taking into account in our analysis.

Therefore, assuming also spatial homogeneity, from (8.11) we obtain the following rate equation for the dressed heavy-neutrino number-density $n^N_{\text{dress}}$:

$$\frac{d n^N_{\text{dress}}}{dt} = \int_{k,k'}^{(X)} \left[ - i \left[ M^2_N, i\Delta^N_N \right] - \frac{1}{2} \left\{ i\Pi^N_N, i\Delta^N_N \right\}_* - \left\{ i\Pi^N_N, i\Delta^N_N \right\}_* \right].$$

(8.14)

Neglecting the $O(h^6)$ terms proportional to the lepton asymmetry, we approximate the charged-lepton and Higgs propagators in the heavy-neutrino self-energies by their quasiparticle equilibrium forms, as given in (8.7a) and (8.7b). Hence, the non-Markovian
heavy-neutrino self-energies may be written in the form

\[
[i \Pi^N_{<}(k, k', \tilde{t}_{f}; \tilde{t}_{i})]_{\alpha}^{\beta} = 2 \tilde{\Re} (h^\dagger h)_{\alpha}^{\beta} \times \int_{p, q} (2\pi)^4 \delta_\xi(k - p - q) (2\pi)^4 \delta_\xi(k' - p - q) \Delta^L_{\xi}(p) \Delta^\xi_{\text{eq}}(q) .
\] (8.15)

We now perform a Wigner-Weisskopf approximation analogous to the one in Section 5.3, in order to obtain the Markovian limit of (8.14). This is performed by replacing \( \Omega_t \) with \( \Omega_\infty \) in the space-time integrals, which corresponds to taking the limit \( t \to \infty \) in the vertex functions by virtue of the identity

\[
\lim_{t \to \infty} \delta_\xi(k_0 - p_0 - q_0) = \delta(k_0 - p_0 - q_0) .
\] (8.16)

Note that the free-phase contributions in (7.25) and those present in the dressed heavy-neutrino propagator (cf. (8.5)) cancel in this energy-conserving limit. Thus, we make the following replacement in the Markovian approximation

\[
\int_{k, k'}^{(X)} \Delta^N_<(k, k', \tilde{t}_{f}; \tilde{t}_{i}) \longrightarrow \int_{k, k'} \theta(k_0 + k'_0) \Delta^N_<(k, k', t) ,
\] (8.17)

where the Markovian propagator is distinguished by the form of its time argument.

With the approximations (8.16) and (8.17), we obtain from (8.14) the Markovian heavy-neutrino rate equation for the dressed number-density:

\[
\frac{d [n^N_{\text{dress}}]_{\alpha}}{dt} = \int_{k, k'} \theta(k_0 + k'_0) \left[ - i \left[ M^2_N, i \Delta^N_<(k, k', t) \right]_{\alpha}^{\beta} - \frac{1}{2} \left[ [i \Pi^N_<(k)]_{\alpha}^{\gamma} [i \Delta^N_<(k, k', t)]_{\gamma}^{\beta} + [i \Delta^N_<(k, k', t)]_{\alpha}^{\gamma} [i \Pi^N_<(k')]_{\gamma}^{\beta} \right] \right. \\
+ \left. \frac{1}{2} \left[ [i \Pi^N_<(k)]_{\alpha}^{\gamma} [i \Delta^N_<(k, k', t)]_{\gamma}^{\beta} + [i \Delta^N_<(k, k', t)]_{\alpha}^{\gamma} [i \Pi^N_<(k')]_{\gamma}^{\beta} \right] \right].
\] (8.18)
The explicit form of the Markovian heavy-neutrino self-energies is given by

\[ i [\Pi^N_\leq (k)]_{\alpha \beta} = 2 \widetilde{\text{Re}}(h^\dagger h)_{\alpha \beta} B^\text{eq}_\leq (k). \tag{8.19} \]

Herein, we have introduced the thermal loop functions

\[ B^\text{eq}_\leq (k) \equiv \int_{p,q} (2\pi)^4 \delta^{(4)}(p - k + q) \Delta^{\Phi,\text{eq}}(q) \Delta^{L,\text{eq}}(p), \tag{8.20} \]

which satisfy \( B^\text{eq}_\leq (-k) = B^\text{eq}_\leq (k) \in \mathbb{R} \). In the classical-statistical regime and restricting to positive energy \( (k_0 > 0) \), the thermal loop functions are given by

\[ B^\text{eq}_\geq (k_0 > 0, k) = \int d\Pi_\Phi \int d\Pi_L (2\pi)^4 \delta^{(4)}(k - p_\Phi - p_L), \tag{8.21a} \]

\[ B^\text{eq}_\leq (k_0 > 0, k) = \int d\Pi_\Phi \int d\Pi_L (2\pi)^4 \delta^{(4)}(k - p_\Phi - p_L) n^{\Phi}_{\text{eq}}(E_\Phi) n^{L}_{\text{eq}}(E_L). \tag{8.21b} \]

As described in Section 7.3, the rate equations (8.11) may be truncated in a perturbative loopwise manner as follows: (i) spectrally, by truncating the external leg of the KB equation and (ii) statistically, by truncating the self-energy, i.e. the set of statistical processes taken into account. In order to obtain the asymmetry at \( \mathcal{O}(h^4) \), and to make contact with the analysis in Chapters 5 and 6, it is sufficient to consider the evolution of the spectrally-free heavy-neutrino number-density. To this purpose we truncate (8.18) spectrally at zeroth loop order, replacing the external heavy-neutrino propagators by the free homogeneous propagator in (8.6). On the other hand, we may retain the full 1-loop CJT-resummed statistical evolution, described by the heavy-neutrino self-energy, which contains the dressed (quasiparticle) charged-lepton and Higgs propagators in (8.7a) and (8.7b). This truncation scheme is summarized diagrammatically in the first line of Figure 8.1. We thus obtain the rate equation for the spectrally-free number-density
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\[
\Delta^0_{N}\sim N
\]

\[
\Delta^0_{L}\sim L
\]

Figure 8.1: Diagrammatic representation of the truncation procedure. The heavy-neutrino equation is truncated spectrally at zeroth loop order, whereas the charged-lepton one is not. Both equations are truncated statistically at 1-loop CJT-resummed order.

\[
\left[n^N\right]_{\alpha}\beta:
\]

\[
\frac{d\left[n^N\right]_{\alpha}\beta}{dt} = \int \theta(k_0) \left\{ -i \left[ M^2_{N}, i\Delta^0_{N}(k,t) \right]_{\alpha}\beta - \frac{1}{2} \left\{ \{i\Pi_{N}^N(k), i\Delta^0_{N}(k,t)\}_{\alpha}\beta - \{i\Pi_{N}^N(k), i\Delta^0_{N}(k,t)\}_{\alpha}\beta \right\} \right\},
\]

(8.22)

where the \(k'\)-integral in (8.18) was carried out trivially.

Substituting explicitly for the free heavy-neutrino propagator (8.6) and assuming kinetic equilibrium along the lines of Chapter 6, (8.22) gives the rate equation for \(n^N(t)\):

\[
\frac{d\left[n^N\right]_{\alpha}\beta}{dt} = -i \left[ \mathcal{E}_{N}, n^N \right]_{\alpha}\beta + \left[ \tilde{\text{Re}}(\gamma^N_{L\Phi}(0)) \right]_{\alpha}\beta - \frac{1}{2 n^N_{\text{eq}}} \left\{ n^N, \tilde{\text{Re}}(\gamma^N_{L\Phi}(0)) \right\}_{\alpha}\beta, \quad (8.23)
\]

where the \(\tilde{C}P\)-“even” rate is defined in terms of the tree-level Yukawa couplings (compare it with (6.8))

\[
\left[\gamma^N_{L\Phi}(0)\right]_{\alpha}\beta \equiv \int_{N L\Phi} 2 h_{\alpha} h_{\beta}, \quad (8.24)
\]

The thermally-averaged effective energy matrix is given by (6.7), with the thermal mass matrix in place of the \(T = 0\) one. Separating the \(\tilde{C}P\)-“even” and “odd” parts of (8.23),
we obtain the final rate equations

\[
\frac{d\left[n^N\right]_\beta}{dt} = -\frac{i}{2} \left[ E_N, \delta n^N \right]_\alpha \beta + \left[ \text{Re}(\gamma_{L\Phi}^{N,(0)}) \right]_\alpha \beta - \frac{1}{2 n_{eq}} \left\{ n^N, \text{Re}(\gamma_{L\Phi}^{N,(0)}) \right\} _\alpha \beta,
\]

(8.25a)

\[
\frac{d[\delta n^N]_\alpha}{dt} = -2 i \left[ E_N, n^N \right]_\beta \alpha - \frac{1}{2 n_{eq}} \left\{ \delta n^N, \text{Re}(\gamma_{L\Phi}^{N,(0)}) \right\} _\alpha \beta.
\]

(8.25b)

These equations agree with those obtained in the semi-classical approach (6.50a) and (6.50b), when the effective Yukawa couplings used there are replaced by the tree-level ones. This is sufficient to obtain the form of the lepton asymmetry at \(O(h^4)\) in the weakly-resonant regime (see Sections 8.4 and 9.4), in complete agreement with the results of Chapter 6.

In particular, we draw attention to the second term on the RHS of (8.25a), as identified in Chapter 6, which induces flavour coherences in the heavy-neutrino number-density \([n^N]_\alpha \beta\), triggering oscillations in addition to mixing.

### 8.2.2 Lepton asymmetry source term

The source term for the lepton asymmetry \(\delta n^L\) is obtained by considering the contribution to the lepton transport equation that contains the \(CP\)-even part of the lepton and Higgs propagators, i.e. not proportional to the asymmetry itself. In the regime where the asymmetry is small, we may approximate these propagators as having the equilibrium forms given by (8.7a) and (8.7b) in the single-momentum representation.

Proceeding analogously to the heavy-neutrino case, replacing the charged-lepton and Higgs propagators by their quasiparticle equilibrium forms in (8.7a) and (8.7b), we obtain the following Markovian approximation of the source term for the lepton asymmetry

\[\delta n^L = n^L - \bar{C}, c. : \]

\[
\frac{d\delta n^L}{dt} \supset - i \int_{k,k',n,q} \theta(p_0 + k'_0 - q_0)(2\pi)^4 \delta^{(4)}(p - k + q)
\]

\[\text{For further details of the double-momentum structure of the self-energies, see [109, 110].}\]
\[ h_{ij} h^{\alpha \beta} \left( \frac{1}{2} [\Delta_<(^N_{k, k', t}) + \Delta_>(^N_{k, k', t})]_{\alpha \beta} \Delta_{\Phi, eq}^{L, eq}(q) \Delta_{\Phi, eq}^{L, eq}(k' - q) \\
- \frac{1}{2} [\Delta_<(^N_{k, k', t}) + \Delta_>(^N_{k, k', t})]_{\alpha \beta} \Delta_{\Phi, eq}^{L, eq}(q) \Delta_{\Phi, eq}^{L, eq}(k' - q) \right) - \tilde{C.c.}, \quad (8.26) \]

where \( \tilde{C.c.} \) denotes the generalized charge-conjugate terms.

In the next section, we will demonstrate explicitly that it is not appropriate to replace the dressed heavy-neutrino propagator in (8.26) by the free heavy-neutrino propagator given in (8.6). This would correspond to a statistical truncation of the source term for the lepton asymmetry \( \delta n_L \) and not a spectral truncation, as was the case with this replacement in the heavy-neutrino rate equations of Section 8.2.1 (cf. (8.22)). On the other hand, there is no need to consider spectral truncations of (8.26), because the charged-lepton propagators on the RHS are approximated by their equilibrium quasiparticle form (see also Figure 8.1).

### 8.3 Flavour mixing and Kadanoff–Baym ansaetze

In this section, we derive the contribution of the dressed heavy-neutrino Wightman propagators to the source term for the asymmetry in the presence of flavour mixing. Moreover, we show that the standard quasiparticle or KB ansaetze for the form of these propagators are insufficient to capture all the physically-relevant phenomena. Specifically, we demonstrate that both heavy-neutrino mixing and oscillations provide distinct contributions to the \( \mathcal{O}(h^4) \) lepton asymmetry (at least in the weakly-resonant regime studied here) and that the flavour-mixing contribution is tacitly discarded when the standard quasiparticle or KB ansaetze are used.

Working in the Markovian approximation and assuming that the charged-lepton and Higgs propagators have the equilibrium forms (8.7), one can make use of the Keldysh representation discussed in Section 3.2 to rewrite, after some algebraic manipulations, the
Schwinger-Dyson equation for the Wightman propagator in the form

\[
i \Delta_N^<(k, k', t) = i \Delta_N^{N,0}(k) + i \Pi_<(k) (2\pi)^4 \delta^{(4)}(k - k') \cdot i \Delta_N^A(k') + \int_{\Delta_N^R(k) \cdot i \Pi_<(k) \cdot i \Delta_N^A(k'), t} + i \Delta_N^{N,0}(k, k', t) \cdot i \Pi_A(k') \cdot i \Delta_N^A(k').
\]

(8.27)

Instead, the equation for the advanced propagator takes on the simple closed form (3.12b) (see also (7.14))

\[
i \Delta_A^N(k) = i \Delta_A^{N,0}(k) + i \Delta_A^{N,0}(k) \cdot i \Pi_A(k) \cdot i \Delta_N^A(k).
\]

(8.28)

As shown diagrammatically in Figure 8.2, (8.27) can be solved iteratively, obtaining

\[
[i \Delta_N^<(k, k', t)]_{\alpha}^{\beta} = [i \Delta_N^R(k)]_{\alpha}^{\gamma}[i \Pi_N^B(k)]_{\gamma}^{\delta}(2\pi)^4 \delta^{(4)}(k - k') [i \Delta_N^A(k')]_{\delta}^{\beta} + \sum_{m=0}^{\infty} \left([(i \Delta_R^0(k) \cdot i \Pi_R^N(k))^{m}]_{\alpha}^{\gamma} [i \Delta_N^0(k, k', t)]_{\gamma}^{\delta} \sum_{n=0}^{\infty} \left[(i \Pi_A^N(k') \cdot i \Delta_N^A(k'))^{n}]_{\delta}^{\beta}. \right.
\]

(8.29)

The first term on the RHS of (8.29) gives a null contribution to the source term (8.26), since the whole term has an equilibrium form, vanishing when one subtracts the \(\tilde{C}\)-conjugate in (8.26). In general, this term describes the washout due to \(\Delta L = 0\) and \(\Delta L = 2\) scatterings, when one includes the non-equilibrium part of \(\Pi_\omega\) in the rate equation (8.26). This is best seen if one inserts this term, with \(\Pi_\omega\) generalized to non-equilibrium, into the rate equation (8.26). Thus, the washout due to scattering is automatically included, with \textit{no double-counting} in the source term, and the RIS subtraction procedure (see Sections 4.3 and 6.4) is not needed, as expected on general grounds in the KB formalism [112, 113, 118].

The second term on the RHS of (8.29) can be written in terms of resummed Yukawa couplings \(h^\alpha\) that generalize the ones introduces in Section 4.2, since we have the follow-
Figure 8.2: Iterative solution to the Schwinger-Dyson equation for the dressed heavy-neutrino matrix Wightman propagator. Here, the double lines are fully dressed propagators, whereas the single lines are the propagators dressed with dispersive corrections only. Unshaded circles denote the relevant self-energies, whereas the shaded ones are the amputated self-energy corrections to the vertices, which can be identified at leading order with the resummed Yukawa couplings (see (8.30) and Appendix C).
ing equivalence in the heavy-neutrino mass eigenbasis, at leading order:

\[
\hat{h}^\alpha\left[\sum_{n=0}^{\infty} \left( i\hat{\Delta}_R^0(k) \cdot i\hat{\Pi}_R(k) \right)^n \right]_{\alpha}^\beta = \frac{L}{N} \phi \phi^* + \frac{L}{N} i\hat{\Pi}_R i\hat{\Delta}_R^{N,0} \phi \phi^* + \ldots \sim \hat{h}^\beta. \tag{8.30}
\]

In Appendix C, we prove that this equivalence holds in (8.26) at \(O(h^4)\) in the lepton asymmetry, at least in the weakly-resonant regime, and show that these \(h^\alpha\)’s have the same functional form as in (4.10), with thermal masses and widths appearing in place of the \(T=0\) ones. There, we also show that, in the mass eigenbasis, the part of the heavy-neutrino propagator contributing to the source term for the asymmetry can be written as

\[
[i\hat{\Delta}^N(k', k, t)]_{\alpha\beta} \supset [\hat{\Delta}^N_R(k)]_{\gamma\gamma} \left[ [i\hat{\Delta}_R^{N,0}(k)]_{\gamma\gamma}^{-1} [i\hat{\Delta}_A^{N,0}(k,k',t)]_{\gamma\delta} [\hat{\Delta}_A^{N,0}(k')]_{\delta\delta}^{-1} \right] [\hat{\Delta}^N_A(k')]_{\delta\beta}.
\tag{8.31}
\]

On the other hand, the KB ansatz (7.18) for the heavy-neutrino propagator, generalized to more than one flavour and restricting to positive frequencies, takes the following form in the heavy-neutrino mass eigenbasis:

\[
[i\hat{\Delta}^N_{KB, <}(k, k', t)]_{\alpha\beta} = 2\pi\delta(k^2 - \hat{M}^2_{N,\alpha}) 2\pi\delta(k'^2 - \hat{M}^2_{N,\beta}) [n^N_{KB}(k, t)]_{\alpha\beta} (2\pi)^3\delta^{(3)}(k-k'),
\tag{8.32}
\]

which satisfies the following properties

\[
(k^2 - \hat{M}^2_{N,\alpha}) [i\hat{\Delta}^N_{KB, <}(k, k', t)]_{\alpha\beta} = 0, \quad [i\hat{\Delta}^N_{KB, <}(k, k', t)]_{\alpha\beta} (k'^2 - \hat{M}^2_{N,\beta}) = 0.
\tag{8.33}
\]

It is immediately apparent that the full form of the dressed heavy-neutrino Wightman propagator in (8.29) and, equivalently, (8.31) does not satisfy (8.33), by virtue of the flavour mixing that gives rise to the resummed Yukawa couplings. We may therefore
conclude that the application of KB ansätze for the heavy-neutrino propagators discards the physical phenomena of flavour mixing.

In [119], it has been pointed out that, for the single-flavour case studied there, one needs to include explicitly the effect of the width of the heavy neutrinos in the collision terms, when performing a zeroth-order gradient expansion or, equivalently, the Markovian approximation. Our results, obtained in a different approach as compared to [119], show that in the multi-flavour case the inclusion of off-diagonal widths in the source terms is also necessary in order to describe properly the phenomenon of flavour mixing.

Other approaches in the literature [117, 120], although not relying explicitly on a KB ansatz, are still able to solve the KB equations for the dressed heavy-neutrino propagator only up to an unknown function that parametrizes the external perturbation of the system. In principle, both mixing and oscillations are present in such double-time approaches. However, it is not clear whether our predictions are in quantitative agreement, since a direct comparison is made difficult by the simplified non-equilibrium setting in a non-expanding Universe studied in [117, 120], with the non-equilibrium evolution originating from an initial instantaneous perturbation of the system.

From (8.30), we see that, in the source term for the asymmetry, the mixing effects due to the heavy-neutrino absorptive transitions can be factorized into the resummed Yukawa couplings. Moreover, we can replace the non-homogeneous free heavy-neutrino propagator $\Delta_{\sim, 0}(k, k', t)$ on the RHS of (8.29) with its homogeneous approximation (8.6). Thus, in perfect analogy with the semi-classical approach followed in Chapter 6, the contribution of the charged-lepton self-energy to the source term may be written in terms of the spectrally-free heavy-neutrino propagator and the resummed Yukawa couplings $h^{\alpha}$ as

$$\frac{d\delta n^{L}}{dt} \supset - \int_{k} \theta(k_{0}) \left[ h_{\beta} h^{\alpha} \left( [i\Delta_{\sim, 0}(k, t)]_{\alpha}^{\beta} B_{<}^{eq}(k) - [i\Delta_{\sim, 0}(k, t)]_{\alpha}^{\beta} B_{<}^{eq}(k) \right) - \tilde{C}.c. \right],$$

(8.34)

without having required a quasiparticle ansatz for the dressed heavy-neutrino propagator. This procedure is illustrated diagrammatically in Figure 8.3 and proven explicitly in
Figure 8.3: Diagrammatic representation of the source term for the charged-lepton asymmetry in terms of the resummed Yukawa couplings and the spectrally-free heavy-neutrino propagator.

Appendix C.

Finally, assuming kinetic equilibrium and separating the \( \tilde{C}P \)-“even” and “-odd” parts of the heavy-neutrino number-density \( \mathbf{n}_N \) and \( \delta n^N \), as described in Chapters 5 and 6, the equation for the asymmetry becomes

\[
\frac{d\delta n^L}{dt} = \left( \frac{[n^N_{\alpha \beta}]_{\alpha \beta}}{n_{eq}^N} - \delta_{\alpha \beta} \right) \left[ \delta \gamma^N_{L\Phi} \right]_{\alpha \beta} + \frac{[\delta n^N]_{\alpha \beta}}{2n_{eq}^N} \left[ \gamma^N_{L\Phi} \right]_{\alpha \beta} + W[\delta n^L], \tag{8.35}
\]

where \( W[\delta n^L] \) denotes the washout terms not studied explicitly in this chapter. The thermally-averaged rates are defined as in (6.8)

\[
\left[ \gamma^N_{L\Phi} \right]_{\alpha \beta} \equiv \int_{NL\Phi} \left( h^\alpha h^\beta + [h^\xi \alpha \alpha [h^\xi \alpha \beta] \right), \tag{8.36a}
\]

\[
\left[ \delta \gamma^N_{L\Phi} \right]_{\alpha \beta} \equiv \int_{NL\Phi} \left( h^\alpha h^\beta - [h^\xi \alpha \alpha [h^\xi \alpha \beta] \right). \tag{8.36b}
\]

Equation (8.35) describes the generation of the asymmetry via both heavy-neutrino mixing (proportional to \( [\delta \gamma^N_{L\Phi}]_{\alpha \beta} \)) and oscillations (proportional to \( [\delta n^N]_{\alpha \beta} \)). In particular, the source terms agree with the ones obtained in the semi-classical approach of Chapter 6, where both the phenomena are separately identified and taken into account in the calculation of the final asymmetry.
8.4 Asymmetry via heavy-neutrino mixing and oscillations

In Section 9.4 we will obtain the approximate analytic solutions of the rate equations (8.25a), (8.25b) and (8.35) in the strong washout regime, in which the evolution of the system admits an attractor solution independent on the initial conditions. Here, we only quote the final result:

$$\delta \eta^L \simeq \delta \eta^L_{\text{mix}} + \delta \eta^L_{\text{osc}},$$

(8.37)

where the neglected terms are formally at $O(h^6)$ and higher. The mixing and oscillation contributions are given explicitly by (cf. (9.25a) and (9.25b))

$$\delta \eta^L_{\text{mix}} = \frac{g_N}{2} \frac{3}{2KZ} \sum_{\alpha \neq \beta} \text{Im}(\langle \tilde{h}^\dagger \tilde{h} \rangle_{11}^2 \langle \tilde{h}^\dagger \tilde{h} \rangle_{22}) \frac{(M_{N,\alpha}^2 - M_{N,\beta}^2)M_N \tilde{\Gamma}_{\beta\beta}^{(0)}}{(M_{N,\alpha}^2 - M_{N,\beta}^2)^2 + (M_N \tilde{\Gamma}_{\beta\beta}^{(0)})^2},$$

(8.38a)

$$\delta \eta^L_{\text{osc}} = \frac{g_N}{2} \frac{3}{2KZ} \text{Im}(\langle \tilde{h}^\dagger \tilde{h} \rangle_{12}^2 \langle \tilde{h}^\dagger \tilde{h} \rangle_{22}) \frac{(M_{N,1}^2 - M_{N,2}^2)M_N (\tilde{\Gamma}_{11}^{(0)} + \tilde{\Gamma}_{22}^{(0)})}{(M_{N,1}^2 - M_{N,2}^2)^2 + M_N^2 (\tilde{\Gamma}_{11}^{(0)} + \tilde{\Gamma}_{22}^{(0)})^2 \text{Im}[\langle \tilde{h}^\dagger \tilde{h} \rangle_{12}]^2},$$

(8.38b)

where $M_N = \frac{1}{2} \text{Tr}(M_N^2)$ is the average thermal mass for the system of two quasi-degenerate heavy neutrinos, $\tilde{\Gamma}_{a\beta}^{(0)}$ is the tree-level thermal width matrix and the K-factor is given by $K = \sum_{\alpha} \tilde{\Gamma}_{\alpha\alpha}^{(0)} / (\zeta(3)H_N)$. In (8.38), we have retained explicitly the dependence on the number of internal degrees of freedom of the heavy neutrino scalars $g_N = 1$, in order to facilitate the comparison with the realistic case of Majorana fermions, with $g_N = 2$.

These $O(h^4)$ results, valid in the weakly-resonant strong-washout regime, coincide with the ones obtained from the semi-classical rate equations (6.50), for the single lepton-flavour case studied here, if one neglects thermal masses and widths. At $O(h^4)$, the contribution of mixing is governed by the diagonal entries of the CP-“even” number-density $\hat{n}^N$, whereas that of oscillations is triggered by the presence of off-diagonal CP- “odd” $\delta \hat{n}^N$. See also the discussion in Section 6.2.
We stress here that the oscillation term in (9.25b) by itself agrees with the form for the total asymmetry given in the quantum Boltzmann approach of [121] and with earlier results of [120, 122–124] in their validity limit \( \text{Re}[(\hat{h}^\dagger \hat{h})_{12}^2] \ll (\hat{h}^\dagger \hat{h})_{\alpha\alpha} \). This oscillation phenomenon does not involve any off-shell effects, since (9.25b) can be obtained from an on-shell analysis with only tree-level Yukawa couplings (for example, by replacing \( h^\alpha \to h^\alpha \) in the analytic discussion in Section 9.4). We emphasize that, unlike previous treatments, the KB approach detailed in this chapter captures the distinct phenomena of flavour mixing [21–23, 25, 26, 67] in addition to oscillation phenomena. As will be shown numerically below and in Section 9.5, the contributions of these two distinct flavour effects, (9.25a) and (9.25b), are comparable in the weakly-resonant regime. Hence, the total lepton asymmetry in (9.24) is enhanced by a factor of order two, compared to either (9.25a) or (9.25b) alone.

In order to illustrate the distinction between the two physical phenomena contributing to the generation of the asymmetry, in Figure 8.4 we plot the numerical solution of the rate equations (9.17) and (9.23), starting from the initial conditions \( \eta^N = 2 \eta^N_{eq} I_2 \) and \( \delta\eta^L = 0 \). The black continuous line denotes the solution of the full rate equations, whereas the red dotted and blue dashed lines give the separate contribution of mixing and oscillations, respectively. The former is obtained by neglecting off-diagonal number-densities, the latter by replacing \( h^\alpha \to h^\alpha \). With this choice of initial conditions, coherences between the two heavy-neutrino flavours are initially absent. Thus, we see that at early times only flavour mixing contributes to the asymmetry. On the other hand, as discussed in Section 6.2, in order to have a significant contribution from oscillations, the off-diagonal entry \( \eta^N_{12} \) needs first to be created by coherent decays and inverse decays. For this reason, as shown in Figure 8.4, this phenomenon becomes effective later than flavour mixing. At late times, both phenomena are active and give a similar contribution in the weakly-resonant strong-washout regime, providing an enhancement by a factor of order two with respect to mixing or oscillations alone. The different time behaviour outlined above, in addition

\[1\text{The limit } \text{Re}[(\hat{h}^\dagger \hat{h})_{12}^2] \ll (\hat{h}^\dagger \hat{h})_{\alpha\alpha} \text{ implies that } \text{Im}[(\hat{h}^\dagger \hat{h})_{12}]^2 \approx (\hat{h}^\dagger \hat{h})_{11}(\hat{h}^\dagger \hat{h})_{22} \text{ in the two-flavour case.} \]
8.4. ASYMMETRY VIA HEAVY-NEUTRINO MIXING AND OSCILLATIONS

Figure 8.4: The evolution of the total asymmetry (black continuous line), starting from the initial conditions $\eta^N = 2\eta^N_{\text{eq}} 1_2$ and $\delta \eta^L = 0$. The red dotted line is the contribution of flavour mixing and the blue dashed line is that of oscillations. For illustrative purposes, the parameters are chosen as follows: $m_N = 1\text{ TeV}$, $(\hat{m}_{N,2} - \hat{m}_{N,1})/m_N = 10^{-12}$, $\hat{h}^1 = 0.5 \times 10^{-6} (1 + 5 e^{i\delta}) m_N$ and $\hat{h}^2 = 0.5 i \times 10^{-6} (1 - 5 e^{i\delta}) m_N$, with $\delta = 2 \times 10^{-5}$. For simplicity, the effect of thermal masses and widths is neglected.

For their differing physical origins, confirms that mixing and oscillations are two distinct physical phenomena, so that both their contributions to the asymmetry need to be taken into account.

Before concluding this chapter, we point out that the oscillatory behaviour in Figure 8.4 does not result from non-Markovian memory effects as, for example, in [125, 126], but from the oscillations of the heavy-neutrino coherences. The non-Markovian finite-time effects are safely neglected in the strong-washout regime of interest here. Moreover, we would like to stress again that the phenomenon of coherent heavy-neutrino oscillations, discussed above and in Section 6.2, is an $\mathcal{O}(h^4)$ effect on the total lepton asymmetry [28] and so differs from the $\mathcal{O}(h^6)$ mechanism proposed in [29]. The latter effect is instead relevant at temperatures much higher than the sterile neutrino masses, such as in the models studied in [29–31, 33, 34, 91, 92, 127, 128], where the total lepton number is
not violated at leading order. On the other hand, the $\mathcal{O}(h^4)$ effect considered here is effective in the same regime as the resonant $\varepsilon$-type $CP$ violation effects, namely for $z \approx 1$, i.e. $T \approx M_N$. Finally, we note that in this thesis we have assumed that the momentum distribution in kinetic equilibrium is a flavour singlet. As discussed in Section 6.1, this approximation is valid in the resonant regime, but not in the hierarchical one. A detailed study of this phenomenon in the hierarchical regime goes beyond the scope of this thesis and may be given elsewhere.
RADIATIVE MODEL OF RESONANT $\ell$-GENESIS

In this chapter, we discuss a radiative $RL_\ell$ model in which the lepton asymmetry is dominantly generated and stored in an individual lepton flavour $\ell$ [129], and use the flavour-covariant rate equations (6.50) to describe the generation of the BAU in this model. The discussion in this chapter is based on [28] and [44].

We start with the heavy-neutrino sector Lagrangian given by (5.1) with $N_N = 3$ heavy neutrinos. Above the scale of the electroweak phase transition, only the singlet neutrinos are massive, whose origin is assumed to lie in some UV-complete extension of the SM. Within this setup, the masses of these heavy neutrinos $N_\alpha (\alpha = 1, 2, 3)$ are naturally nearly degenerate, if one assumes an approximate $O(3)$ symmetry in the heavy neutrino sector at some high energy scale $\mu_X$:

$$m_N(\mu_X) = m_N 1 + \Delta m_N,$$

(9.1)

with $||\Delta m_N|_{\alpha\beta}|/m_N \ll 1$. The corresponding boundary values for the Yukawa coupling matrix elements $[h(\mu_X)]_{\ell}^\alpha$ depend on the particular RL$_\ell$ model under consideration [27, 129]. The Majorana neutrino mass matrix at the phenomenologically relevant low energy scale can thus be written as

$$m_N = m_N 1 + \Delta m_N + \Delta m_N^{RG},$$

(9.2)
where $\Delta m_{N}^{\text{RG}}$ is an $O(3)$-breaking perturbation matrix induced by the renormalization group (RG) evolution of the heavy-neutrino mass matrix $m_{N}$ from the high scale $\mu_{X}$ down to the scale of $m_{N}$:

$$
\Delta m_{N}^{\text{RG}} = \frac{-m_{N}}{8\pi^{2}} \ln \left( \frac{\mu_{X}}{m_{N}} \right) \text{Re} \left[ h^{\dagger}(\mu_{X})h(\mu_{X}) \right].
$$

(9.3)

9.1 No-go theorem for minimal radiative RL

The inclusion of an explicit $\Delta m_{N}$ is due to the following no-go theorem for minimal radiative RL at $O(h^{4})$ [44]. In the minimal framework, the only source of flavour in the heavy-neutrino sector is the Yukawa mass matrix $h$ and no additional breaking $\Delta m_{N}$ is present. Thus, the only $O(3)$ breaking term in the heavy-neutrino mass matrix is given by (9.3) where, at leading order $O(h^{2})$, the Yukawa couplings can be taken at the scale $m_{N}$. Thus, the RH neutrino mass matrix is real and symmetric and, as long as $|\Delta m_{N}^{\text{RG}}|_{\alpha\beta}/m_{N} \ll 1$, it can be diagonalized with positive eigenvalues by a real orthogonal matrix $O \in O(3) \subset U(3)$:

$$
m_{N} = O \hat{m}_{N} O^{T},
$$

(9.4)

Since $O$ is orthogonal, $O^{T} \Delta m_{N}^{\text{RG}} O$ is separately diagonal too, as well as

$$
\text{Re}(\hat{h}^{\dagger}\hat{h}) = \text{Re} \left[ (O^{T}h^{\dagger})(hO) \right] = O^{T} \text{Re}(h^{\dagger}h) O \propto O^{T} \Delta m_{N}^{\text{RG}} O.
$$

(9.5)

On the other hand, the asymmetry in a single lepton flavour $l$ is proportional to the quantity (cf. (4.14))

$$
\text{Im} \left[ \hat{h}_{\alpha}^{\dagger} \hat{h}_{\beta} (\hat{h}^{\dagger}\hat{h})_{\alpha\beta} \right] + \frac{m_{N,\alpha}}{m_{N,\beta}} \text{Im} \left[ \hat{h}_{\alpha}^{\dagger} \hat{h}_{\beta} (\hat{h}^{\dagger}\hat{h})_{\beta\alpha} \right] = 2 \text{Im} \left[ \hat{h}_{\alpha}^{\dagger} \hat{h}_{\beta} \right] \text{Re} \left[ (\hat{h}^{\dagger}\hat{h})_{\alpha\beta} \right] + O(h^{6}),
$$

(9.6)
with $\alpha \neq \beta$. Therefore, the asymmetry vanishes at $O(h^4)$ in minimal radiative models, where no other source of $O(3)$ flavour breaking is present.

### 9.2 Next-to-minimal model of resonant $\tau$-genesis

As an explicit example of the RL$\ell$ scenario, we consider an RL$\tau$ model with an approximate $O(3)$ symmetry at the GUT scale, $\mu_X \sim 2 \times 10^{16}$ GeV, broken to the $U(1)_l \sim O(2)$ subgroup of lepton-flavour symmetries by a neutrino Yukawa coupling matrix of the form [27, 129]

$$
\mathbf{h} = \begin{pmatrix}
0 & a e^{-i\pi/4} & a e^{i\pi/4} \\
0 & b e^{-i\pi/4} & b e^{i\pi/4} \\
0 & c e^{-i\pi/4} & c e^{i\pi/4}
\end{pmatrix} + \delta \mathbf{h},
$$

(9.7)

where $\delta \mathbf{h}$ vanishes in the flavour-symmetric limit. In order to protect the $\tau$ asymmetry from an excessive washout, and at the same time guarantee observable effects in low-energy neutrino experiments (see Appendix D), we take $|c| \ll |a|, |b| \approx 10^{-3} - 10^{-2}$.

The lightness of the SM neutrinos is protected by this leptonic symmetry, since they remain massless to all orders in perturbation theory in the flavour-symmetric limit $\delta \mathbf{h} = 0$ [130]. In order to give masses to the light neutrinos, we consider the following form of $\delta \mathbf{h}$ as a minimal departure from the flavour-symmetric limit [27, 129]:

$$
\delta \mathbf{h} = \begin{pmatrix}
\epsilon_e & 0 & 0 \\
\epsilon_\mu & 0 & 0 \\
\epsilon_\tau & 0 & 0
\end{pmatrix}.
$$

(9.8)

In addition, as discussed in the previous section, an additional breaking of the $O(3)$ symmetry in the heavy-neutrino mass eigenbasis is needed. We take this breaking term mini-
mally as \[44\]
\[\Delta m_N = \begin{pmatrix} \Delta M_1 & 0 & 0 \\
0 & \Delta M_2/2 & 0 \\
0 & 0 & -\Delta M_2/2 \end{pmatrix}, \quad (9.9)\]
where \(\Delta M_2\) breaks the \(O(2) \sim U(1)_l\) symmetry\(^1\) in the \(N_{2,3}\) sector and, as we are going to see below, allows to successfully fit the low-energy neutrino data. Instead, \(\Delta M_1\) governs the mass difference between \(N_1\) and \(N_{2,3}\) and its inclusion is sufficient to obtain successful leptogenesis.

To leading order in the symmetry-breaking parameters \(\Delta m_N\) and \(\delta h\), the tree-level light neutrino mass matrix is given by the seesaw formula

\[m_\nu \simeq \frac{v^2}{2} h m_N^{-1} h^\dagger \simeq \frac{v^2}{2 m_N} \begin{pmatrix}
\frac{\Delta m_N}{m_N} a^2 - \epsilon_e^2 & \frac{\Delta m_N}{m_N} a b - \epsilon_e \epsilon_\mu & -\epsilon_e \epsilon_\tau \\
\frac{\Delta m_N}{m_N} a b - \epsilon_e \epsilon_\mu & \frac{\Delta m_N}{m_N} b^2 - \epsilon_\mu^2 & -\epsilon_\mu \epsilon_\tau \\
-\epsilon_e \epsilon_\tau & -\epsilon_\mu \epsilon_\tau & -\epsilon_\tau^2
\end{pmatrix}, \quad (9.10)\]

where

\[\Delta m_N \equiv 2 [\Delta m_N]_{23} + i \left( [\Delta m_N]_{33} - [\Delta m_N]_{22} \right) = -i \Delta M_2, \quad (9.11)\]

and we have neglected subdominant terms \(\hat{\Delta}_{m_N} c \times (a, b, c)\). Notice that, without the \(\Delta M_2\) breaking, (9.10) would give a rank-1 matrix, thus not satisfying the low-energy neutrino data. Instead, a nonzero \(\Delta M_2\) makes (9.10) rank-2, thus allowing to fit the neutrino data, and predicting, in addition, the masslessness of the lightest neutrino.

The light-neutrino mass matrix in (9.10) is diagonalized by the usual Pontecorvo–Maki–Nakagawa–Sakata (PMNS) mixing matrix

\[m_\nu = U_{\text{PMNS}} \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) U_{\text{PMNS}}^\dagger, \quad (9.12)\]

where \(m_{\nu_i}\) are the light-neutrino mass eigenvalues. In Appendix D we will show that, for the benchmark points considered in the following, the non-unitarity of the \(3 \times 3\) PMNS

\(^1\)In the flavour-symmetric limit \(\delta h = 0\) we have \(h^\dagger h = (|a|^2 + |b|^2 + |c|^2) \text{diag}\{0, 1, 1\}.\)
mixing matrix due to the light-heavy neutrino mixing \cite{131} is very small and can be neglected. Hence, we can assume that \( U_{\text{PMNS}} \) in (9.12) is unitary, and write it in terms of the three experimentally known light neutrino mixing angles \( \theta_{ij} \) and the yet unconstrained Dirac phase \( \delta \) and Majorana phases \( \varphi_{1,2} \):

\[
U_{\text{PMNS}} = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix} \times \text{diag}(e^{i\varphi_{1}/2}, e^{i\varphi_{2}/2}, 1),
\]

(9.13)

with \( c_{ij} \equiv \cos \theta_{ij}, s_{ij} \equiv \sin \theta_{ij} \). Assuming a particular mass hierarchy between the light-neutrino masses \( m_{\nu_i} \) and for given values of the CP-phases \( \delta, \varphi_{1,2} \), we can fully reconstruct the light-neutrino mass matrix using (9.12) and, by means of (9.10), determine the following model parameters appearing in the Yukawa coupling matrix (9.7):

\[
a^2 = \frac{2m_N}{v^2} \left( m_{\nu,11} - \frac{m_{\nu,13}^2}{m_{\nu,33}} \right) \frac{m_N}{\Delta m_N}, \quad b^2 = \frac{2m_N}{v^2} \left( m_{\nu,22} - \frac{m_{\nu,23}^2}{m_{\nu,33}} \right) \frac{m_N}{\Delta m_N},
\]

\[
\epsilon_c^2 = -\frac{2m_N}{v^2} \frac{m_{\nu,13}^2}{m_{\nu,33}}, \quad \epsilon_\mu^2 = -\frac{2m_N}{v^2} \frac{m_{\nu,23}^2}{m_{\nu,33}}, \quad \epsilon_\tau^2 = -\frac{2m_N}{v^2} m_{\nu,33}.
\]

(9.14)

Therefore, the Yukawa coupling matrix (9.7) in the RL_\( \tau \) model can be completely fixed in terms of the heavy neutrino mass scale \( m_N \) and the input parameters \( c \) and \( \Delta M_2 \).

### 9.3 Benchmark points

In this section, we choose a set of benchmark points to study the generation of the asymmetry in the RL_\( \tau \) model presented in Section 9.2. For illustration, we take a set of neutrino Yukawa couplings satisfying the neutrino oscillation data for a normal hierarchy of light-neutrino masses with the lightest neutrino mass \( m_{\nu_1} = 0 \). We use the best-fit values of the light-neutrino oscillation parameters from a recent three-neutrino global
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<table>
<thead>
<tr>
<th>Parameters</th>
<th>BP1</th>
<th>BP2</th>
<th>BP3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_N$</td>
<td>120 GeV</td>
<td>400 GeV</td>
<td>5 TeV</td>
</tr>
<tr>
<td>$c$</td>
<td>$2 \times 10^{-6}$</td>
<td>$2 \times 10^{-7}$</td>
<td>$2 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\Delta M_1/m_N$</td>
<td>$-5 \times 10^{-6}$</td>
<td>$-3 \times 10^{-5}$</td>
<td>$-4 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\Delta M_2/m_N$</td>
<td>$(-1.59 - 0.47 i) \times 10^{-8}$</td>
<td>$(-1.21 + 0.10 i) \times 10^{-9}$</td>
<td>$(-1.46 + 0.11 i) \times 10^{-8}$</td>
</tr>
<tr>
<td>$a$</td>
<td>$(5.54 - 7.41 i) \times 10^{-4}$</td>
<td>$(4.93 - 2.32 i) \times 10^{-5}$</td>
<td>$(4.67 - 4.33 i) \times 10^{-5}$</td>
</tr>
<tr>
<td>$b$</td>
<td>$(0.89 - 1.19 i) \times 10^{-3}$</td>
<td>$(8.04 - 3.79 i) \times 10^{-3}$</td>
<td>$(7.53 - 6.97 i) \times 10^{-3}$</td>
</tr>
<tr>
<td>$\epsilon_e$</td>
<td>$3.31 i \times 10^{-8}$</td>
<td>$5.73 i \times 10^{-8}$</td>
<td>$2.14 i \times 10^{-7}$</td>
</tr>
<tr>
<td>$\epsilon_\mu$</td>
<td>$2.33 i \times 10^{-7}$</td>
<td>$4.30 i \times 10^{-7}$</td>
<td>$1.50 i \times 10^{-6}$</td>
</tr>
<tr>
<td>$\epsilon_\tau$</td>
<td>$3.50 i \times 10^{-7}$</td>
<td>$6.39 i \times 10^{-7}$</td>
<td>$2.26 i \times 10^{-6}$</td>
</tr>
</tbody>
</table>

Table 9.1: The numerical values of the free ($m_N$, $c$, $\Delta M_{1,2}$) and derived parameters ($a$, $b$, $\epsilon_{e,\mu,\tau}$), with $\text{Re}(a) > 0$, in the RL$\tau$ model for three chosen benchmark points.

For definiteness, we choose the leptonic CP phases $\delta = 0$, $\varphi_1 = -\pi$ and $\varphi_2 = 0$, and reconstruct the light neutrino mass matrix using the definition (9.12). Thus, we can determine the parameters $a, b, \epsilon_{e,\mu,\tau}$ of the RL$\tau$ model using the relations (9.14) and also find the heavy-neutrino mass matrix, as given by (9.3). Note that for a given light neutrino mass matrix $m_\nu$, the solutions for $a$ and $b$ obtained using (9.14) are unique up to a sign factor, and we choose the solution with $\text{Re}(a) > 0$. There is a similar sign freedom for $\epsilon_{e,\mu,\tau}$, but this is irrelevant since it applies to the whole third column of the Yukawa coupling matrix $\delta h$ and can be rotated away. Thus, the only free parameters in this model, apart from the neutrino-sector CP phases, are $m_N, c$ and $\Delta M_{1,2}$. Below we present some benchmark values for these free parameters.

We choose three benchmark scenarios with the heavy neutrino mass scales $m_N = 120$ GeV, 400 GeV and 5 TeV, in order to illustrate the physics involved in the generation of the asymmetry in different temperature regimes. The other free model parameters are
chosen such that they satisfy all the relevant experimental constraints\(^1\), as discussed in Appendix D. The values of the free model parameters \(m_N, c\) and \(\Delta M_{1,2}\) for our benchmark scenarios are given in Table 9.1, together with the corresponding values of the derived model parameters, viz. \(a, b, \epsilon_{e,\mu,\tau}\), with \(\text{Re}(a) > 0\). The low-energy lepton flavour violating (LFV) and lepton number violating (LNV) observables are briefly discussed in Appendix D for completeness, and their predicted values for the benchmark points are shown in Table D.1, along with the current experimental limits.

9.4 Analytic strong-washout solutions

In this section we find some analytic solutions of the evolution equations in the strong-washout regime. In the first part, we study the role of the heavy-neutrino coherences, and the generation of lepton number asymmetry via heavy-neutrino oscillations (see also Section 6.2). There, we neglect the \(\gamma'\) effects and charged-lepton off-diagonal number-densities, for a simplified case with two heavy neutrinos. Even though this may not be a realistic approximation, this will allow us to estimate the relative magnitude of the different effects present in the statistical evolution of the system. In the second part, we obtain a quantitatively accurate semi-analytic solution for the case of diagonal heavy-neutrino number-densities (hence no oscillations), but retaining the full off-diagonal number-density matrix for the charged leptons, thereby capturing the charged-lepton (de)coherence effects.

9.4.1 Analytic solution for the asymmetry via heavy-neutrino mixing and oscillations

Let us find the approximate analytic solution to the rate equations (6.50) in the strong washout regime, in the simplified setting obtained by neglecting the charged-lepton off-
diagonal number-densities. For simplicity, we take the $N_N = 2$ case and neglect the contribution of scattering in the charged-lepton rate equation (6.50c). We introduce the relative deviation from equilibrium $\hat{\eta}^N$ for the RH neutrinos:

$$\hat{\eta}^N \equiv \frac{\hat{\eta}^N}{\eta^N_{\text{eq}}} - 1 = \frac{1}{\eta^N_{\text{eq}}} \left(\hat{\eta}^N + \frac{\delta \hat{\eta}^N}{2}\right) - 1,$$

(9.16)

At $\mathcal{O}(h^4)$, the $\tilde{C}P$-"odd" number-density $\delta \hat{\eta}^N$ is involved in the generation of the asymmetry via heavy-neutrino oscillations (see Sections 6.2 and 8.4). Working at $\mathcal{O}(h^4)$, it is sufficient to approximate, in the rate equations (6.50a) and (6.50b), the resummed Yukawa couplings as at tree level, thus recovering the simplified forms (6.19a) and (6.20a) (see also Chapter 8 and [42]). Combining these, we find

$$\frac{d\hat{\eta}^N}{dz} = \frac{K_1(z)}{K_2(z)} \left(1 + \hat{\eta}^N - iz \left[\frac{\hat{m}_N}{\zeta(3) H_N}, \hat{\eta}^N\right] - \frac{z}{2} \left\{\text{Re}(\hat{K}^{(0)}_N), \hat{\eta}^N\right\}\right),$$

(9.17)

where the heavy-neutrino K-factor matrix

$$\hat{K}_N \equiv \frac{\hat{\Gamma}}{\zeta(3) H_N} = \frac{1}{8\pi} \frac{m_N}{\hat{h}^\dagger \hat{h} + \hat{h}^{\dagger} \hat{h}}$$

(9.18)

is given, at tree level, by

$$\left[\hat{K}^{(0)}_N\right]_{\alpha\beta} = \frac{1}{\zeta(3) H_N} \frac{m_N}{8\pi} \hat{h}^\dagger \hat{h}_{\alpha\beta},$$

(9.19a)

$$\left[\hat{K}^{(0)}_N\right]_{l\alpha\beta} \equiv \frac{1}{\zeta(3) H_N} \frac{m_N}{8\pi} \hat{h}^{\dagger}_{\alpha\beta} \hat{h}_{l\beta},$$

(9.19b)

having also defined the rank-4 K-factor $\hat{K}_{l\alpha\beta}$.

In the strong washout regime, i.e. for $|\left[\hat{K}^{(0)}_N\right]_{\alpha\beta}| \gg 1$, the system evolves towards the attractor solution, obtained by setting the RHS of (9.17) to zero:

$$i \left[\frac{\hat{m}_N}{\zeta(3) H_N}, \hat{\eta}^N\right] + \frac{1}{2} \left\{\text{Re}(\hat{K}_N), \hat{\eta}^N\right\} \approx \frac{1}{z},$$

(9.20)
where we have neglected $\hat{\eta}^N$ compared to 1. From (9.20), it is clear that all the elements of $\hat{\eta}^N$ will have the usual $1/z$ behaviour (as obtained, for the diagonal rate equations, in [61]). The exact numerical solutions of the fully flavour-covariant rate equations (6.50a)–(6.50c) also exhibit this behaviour, as shown explicitly in Section 9.5 (see Figure 9.1). The elements needed in what follows are found to be

$$[\hat{\eta}^N]_{\alpha\alpha} = \frac{1}{[\hat{K}_N^{(0)}]_{\alpha\alpha} z} \left( \frac{(m_{N,1} - m_{N,2})^2 + (\hat{\Gamma}_1^{(0)} + \hat{\Gamma}_2^{(0)})^2/4}{(m_{N,1} - m_{N,2})^2 + \frac{(\hat{\Gamma}_1^{(0)} + \hat{\Gamma}_2^{(0)})^2 \Im[(\hat{h}^\dagger h)_{12}]}{4(\hat{h}^\dagger h)_{11}(\hat{h}^\dagger h)_{22}}} \right) \simeq \frac{1}{[\hat{K}_N^{(0)}]_{\alpha\alpha} z} ,$$

(9.21)

$$\Im[\hat{\eta}^N]_{12} = \frac{\zeta(3) H_N}{z} \frac{[\hat{\Gamma}^{(0)}]_{12}}{([\hat{\Gamma}^{(0)}]_{11}[\hat{\Gamma}^{(0)}]_{22})} \left( \frac{(m_{N,1} - m_{N,2})(\hat{\Gamma}_1^{(0)} + \hat{\Gamma}_2^{(0)})/2}{(m_{N,1} - m_{N,2})^2 + \frac{(\hat{\Gamma}_1^{(0)} + \hat{\Gamma}_2^{(0)})^2 \Im[(\hat{h}^\dagger h)_{12}]}{4(\hat{h}^\dagger h)_{11}(\hat{h}^\dagger h)_{22}}} \right) ,$$

(9.22)

where the tree-level width matrix is given by $\hat{\Gamma}^{(0)} = \frac{m_{N}}{8\pi} (\hat{h}^\dagger h)$. Neglecting the charged-lepton off-diagonal coherences, as well as the RIS-subtracted contribution of scattering processes, the rate equation for the lepton asymmetry (6.50c) takes the form

$$\frac{d[\delta\eta^L]_\mu}{dz} = z^3 K_1(z) \times \left[ -\frac{1}{3} [K_L]_{\mu\mu} [\delta\eta^L]_\mu + \frac{\pi^2 z}{m_N^2 K_1(z) 2\zeta(3) H_N} \left( \Im[\hat{\eta}^N]_{12} \Im[\hat{\gamma}^N_{L\Phi}]_{112} + [\hat{\eta}^N]_{\alpha\beta} [\delta\gamma^N_{L\Phi}]_{\beta\alpha} \right) \right] ,$$

(9.23)

where $[K_L]_{\mu\mu} = \sum_\alpha K_{l\alpha\alpha}$ are the charged-lepton washout parameters. The attractor solution is found by setting the RHS of (9.23) to zero. We also neglect the $O(h^6)$ off-diagonal entries in the last term, and approximate $[\hat{\gamma}^N_{L\Phi}]_{112}$ as its $O(h^2)$ tree-level form, finally obtaining

$$\delta\eta^L \simeq \delta\eta^L_{\text{mix}} + \delta\eta^L_{\text{osc}} ,$$

(9.24)

where the neglected terms in (9.24) are formally at $O(h^6)$ and higher. Here, the mixing
and oscillation contributions are explicitly given by

\[
\delta \eta_{\text{mix}}^L = \sum_l \frac{3}{2z |K_{Ll}|^2} \sum_{\alpha \neq \beta} 2 \text{Im}\left( \hat{h}_{\alpha} \hat{h}_{\beta} \right) \text{Re}\left[ \left( \hat{h}^\dagger \hat{h} \right)_{11} \left( \hat{h}^\dagger \hat{h} \right)_{22} \right] \frac{(m_{N,\alpha}^2 - m_{N,\beta}^2) m_N \hat{\Gamma}_{\beta\beta}^{(0)}}{(m_{N,\alpha}^2 - m_{N,\beta}^2)^2 + (m_N \hat{\Gamma}_{\beta\beta}^{(0)})^2},
\]

(9.25a)

\[
\delta \eta_{\text{osc}}^L = \sum_l \frac{3}{2z |K_{Ll}|^2} 2 \text{Im}\left( \hat{h}_{\alpha} \hat{h}_{\beta} \right) \text{Re}\left[ \left( \hat{h}^\dagger \hat{h} \right)_{12} \right] \times
\]

\[
\frac{(m_{N,1}^2 - m_{N,2}^2) m_N (\hat{\Gamma}_{11}^{(0)} + \hat{\Gamma}_{22}^{(0)})}{(m_{N,1}^2 - m_{N,2}^2)^2 + m_N^2 (\hat{\Gamma}_{11}^{(0)} + \hat{\Gamma}_{22}^{(0)})^2} \frac{\text{Im}\left( \left( \hat{h}^\dagger \hat{h} \right)_{12} \right)^2}{\left( \hat{h}^\dagger \hat{h} \right)_{11} \left( \hat{h}^\dagger \hat{h} \right)_{22}}.
\]

(9.25b)

Comparing (9.24) with the lepton asymmetry due to mixing effects (9.25a) [cf. also (4.34)], we see that the total lepton asymmetry due to heavy-neutrino oscillations around \( z \sim 1 \) is of the same sign and of the same order in magnitude, as compared to that obtained from the standard \( \varepsilon \)-type \( CP \) asymmetry due to heavy-neutrino mixing. Even though some of the approximations leading to this result may not be realistic in certain cases, we expect (9.24) to be qualitatively correct, at least in the weakly resonant regime, and in Section 9.5 we will numerically verify this (see Figures 9.2–9.4) for the RL_\tau model under consideration. Finally, we point out that (9.25a) and (9.25b) are also valid when one includes thermal corrections, as is done in Chapter 8. In this case, as is shown in [42], the matrices \( \hat{m}_N \) and \( \hat{\Gamma}^{(0)} \) appearing in (9.25a) and (9.25b) are replaced, respectively, by the thermal mass and width matrices for the RH neutrinos.

### 9.4.2 Semi-analytic results for the charged lepton decoherence effect

We will now obtain the attractor solution for the charged-lepton asymmetry neglecting the heavy-neutrino off-diagonal number-densities. Neglecting the heavy-neutrino coherences, but retaining the full flavour structure for the charged leptons, the rate equation for the asymmetry (6.50c) can be written in the would-be mass eigenbasis for charged-leptons
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as follows:

\[
\frac{d}{dz} [\hat{\eta}^L]_{lm} = \frac{z^3 K_1(z)}{2} \left( \sum_\alpha [\hat{N}]_{\alpha \alpha} [\hat{\delta K}^N_{L\Phi}]_{l\alpha \alpha} - \frac{1}{3} \left\{ \hat{\eta}^L, \hat{K}_+^{\text{eff}} \right\}_{lm} \right)
- \frac{2}{3} [\hat{\eta}^L]_{kn} [\hat{K}^{\text{eff}}]_{nklm} - \frac{2}{3} \left\{ \hat{\eta}^L, \hat{K}_{\text{dec}} \right\}_{lm} + \frac{2}{3} \left\{ \hat{\eta}^{\text{back}}_{L\Phi} \right\}_{lm},
\]

(9.26)

where the various effective K-factors are defined as

\[
\hat{K}^{\text{eff}}_+ = \kappa \left( \hat{\gamma}^{L\Phi}_{L\Phi} + \hat{\gamma}^{L\Phi}_{\phi\phi} \right), \quad [\hat{K}^{\text{eff}}]_{nklm} = \kappa \left[ \hat{\gamma}^{L\Phi}_{L\Phi} - \hat{\gamma}^{L\Phi}_{\phi\phi} \right]_{nklm},
\]

\[
\hat{\delta K}^N_{L\Phi} = \kappa \hat{\gamma}^N_{L\Phi}, \quad \hat{K}_{\text{dec}} = \kappa \hat{\gamma}_{\text{dec}}, \quad \hat{\delta K}^{\text{back}}_{\text{dec}} = \kappa \hat{\gamma}^{\text{back}}_{\text{dec}},
\]

(9.27)

with \( \kappa = \pi^2 z/(\zeta(3) H N m_N^3 K_1(z)) \).

The CP asymmetries in the charged-lepton flavour space can be defined as

\[
\hat{\varepsilon}_{lm} \equiv \sum_\alpha \frac{[\hat{\delta K}^N_{L\Phi}]_{l\alpha \alpha}}{[\hat{K}^N]_{\alpha \alpha}}.
\]

(9.28)

Notice that, in general, this is a tensor in the charged-lepton flavour space, even though here we are working in the would-be mass eigenbasis for charged leptons. In the 2 heavy-neutrino mixing case, it can be approximated as

\[
\hat{\varepsilon}_{lm} \approx \sum_{\alpha \neq \beta} \frac{-i \left( \hat{h}^*_{\alpha \alpha} \hat{h}^{\alpha \beta} - \hat{h}_{\alpha \beta} \hat{h}^*_{\beta \beta} \right) \text{Re} \left[ (\hat{h}^\dagger \hat{h})_{\alpha \beta} \right]}{(\hat{h}^\dagger \hat{h})_{\alpha \alpha} (\hat{h}^\dagger \hat{h})_{\beta \beta}} \frac{(m_{N\alpha}^2 - m_{N\beta}^2) m_{N\alpha} \Gamma_{N\beta}^{(0)}}{(m_{N\alpha}^2 - m_{N\beta}^2)^2 + (m_{N\alpha} \Gamma_{N\beta}^{(0)})^2}.
\]

(9.29)

The analytic solution for the diagonal heavy-neutrino evolution equation is found from (9.20) (written in the 3 heavy-neutrino case), by neglecting off-diagonal elements, as \([\hat{N}]_{\alpha \alpha} \approx 1/(\hat{K}^N_{\alpha \alpha} z)\). By using this in (9.26), we find the attractor solution in the strong washout regime by setting the RHS of (9.26) to zero, obtaining

\[
\frac{1}{3} \left\{ \hat{\eta}^L, \hat{K}_+^{\text{eff}} + 2 \hat{K}_{\text{dec}} \right\}_{lm} - [\hat{\delta K}^{\text{back}}_{\text{dec}}]_{lm} + \frac{2}{3} [\hat{\eta}^L]_{kn} [\hat{K}^{\text{eff}}]_{nklm} \approx \frac{\hat{\varepsilon}_{lm}}{z},
\]

(9.30)

It is not easy to perform further approximations in this equation and so it is convenient
to solve the linear system for the variables $[\delta \hat{\eta}_L^L]_{lm}$ numerically. We then obtain the semi-analytic contribution of mixing and charged-lepton (de)coherence to the asymmetry in the strong-washout regime

$$\delta \hat{\eta}_L^L \supset \delta \hat{\eta}_\text{mix}^L + \delta \hat{\eta}_\text{dec}^L \simeq \sum_l [\delta \hat{\eta}_L^L]_l,$$

where the diagonal asymmetries $[\delta \hat{\eta}_L^L]_l$ are obtained by solving the linear system (9.30).

In Section 9.5, we will show that the approximate analytic solution (9.31) reproduces the exact numerical solution of the rate equations remarkably well for the case with diagonal heavy-neutrino number-densities, thus providing a fairly good estimate for the total lepton number asymmetry predicted by the fully flavour-covariant rate equations.

9.5 Numerical results for the lepton asymmetry

Using the parameter values given in Table 9.1, we numerically solve the flavour-covariant rate equations (6.50a)–(6.50c) for the evolution of the charged-lepton and heavy-neutrino number-densities. In this section we work in the heavy-neutrino and charged-lepton (would-be) mass eigenbasis. First, we discuss the results for the heavy-neutrino number densities, as shown in Figure 9.1 for the three benchmark points given in Table 9.1. Here we have chosen the initial conditions with zero lepton asymmetry, i.e. $\delta \eta_L^L = 0$, and the heavy neutrinos all in thermal equilibrium, i.e. $\eta_{\text{in}}^N = \eta_{\text{eq}}^N \mathbf{1}$. As we will see below, other choices of initial conditions lead to essentially the same results, since we work in the strong washout regime. The vertical dotted line indicates the critical value $z_c = m_N/T_c$, where $T_c$ is the critical temperature [cf. (4.40)]. The number-densities are shown in terms of the deviation from their equilibrium values: $\eta_{\text{in}}^N = \eta_{\text{in}}^N/\eta_{\text{eq}}^N - \delta_{\alpha\beta}$ [cf. (9.16)]. The evolution of the diagonal elements is shown as solid lines and that of the off-diagonal elements as dashed lines. As discussed in Section 9.4 [cf. (9.20)], both diagonal and off-diagonal heavy-neutrino number densities rapidly go to their attractor solution. Numerically, the value of $\eta_{11}^N$ is several orders of magnitude larger than the
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Figure 9.1: The deviation of the heavy-neutrino number-densities $\hat{\eta}_{\alpha\beta}^N = \frac{\hat{\eta}_{\alpha\beta}^N}{\eta_{\alpha\beta}^{eq}} - \delta_{\alpha\beta}$ from their equilibrium values for the three benchmark points given in Table 9.1. The different lines show the evolution of the diagonal (solid lines) and off-diagonal (dashed lines) number densities in the fully flavour-covariant formalism. The numerical values of $\hat{\eta}_{22}^N$ and $\hat{\eta}_{33}^N$ coincide with each other in all three cases.
other elements, whereas the values of $\eta_{22}^N$ and $\eta_{33}^N$ overlap for all the benchmark points. In all three cases, unlike $\eta_{23}^N$, the off-diagonal elements $\eta_{12}^N$ and $\eta_{13}^N$ are larger than the diagonal elements $\eta_{22}^N$, $\eta_{33}^N$. Thus, the effect of the off-diagonal contributions of $\eta_{\alpha\beta}^N$ to the lepton asymmetry cannot be neglected, as we will illustrate below.

Our results for the asymmetries in the SM lepton-doublet sector for the three benchmark points given in Table 9.1 are shown in Figures 9.2–9.4. The horizontal dotted line shows the value of $\delta\eta_L$ [cf. (4.44)] required to explain the observed BAU. The conversion of the asymmetry stops for $z > z_c$ (vertical dotted line), since the sphaleron processes are no longer in action. As such, the observed value for $\delta\eta_L$ should be compared with the model prediction at $z = z_c$. The top panels in Figures 9.2–9.4 show the evolution of the total lepton asymmetry, $\delta\eta_L \equiv \text{Tr}(\delta\eta^L)$, obtained using the fully flavour-covariant rate equations (6.50a)–(6.50c), for three different initial conditions (thick solid lines):

(i) zero lepton asymmetry $\delta\eta_{\text{in}}^L = 0$, heavy neutrinos in thermal equilibrium $\eta_{\text{in}}^N = \eta_{\text{eq}}^N 1$ (thick black line);

(ii) zero lepton asymmetry $\delta\eta_{\text{in}}^L = 0$, heavy neutrinos strongly out-of-equilibrium $\eta_{\text{in}}^N = 0$ (thick grey line);

(iii) extremely large lepton asymmetry with opposite sign compared to the observed one $\delta\eta_{\text{in}}^L = 1$; heavy neutrinos strongly out-of-equilibrium $\eta_{\text{in}}^N = 0$ (thick yellow line).

It is evident that the final lepton asymmetry $\delta\eta^L(z \gg 1)$ is independent of the initial conditions, which is a general consequence of the RL mechanism in the strong washout regime. Even starting with extremely large initial lepton asymmetry with the wrong sign, this primordial asymmetry is rapidly washed out and the final asymmetry is set by the RL mechanism itself for $z \sim 1$.

We now compare the results of our fully flavour-covariant rate equations with their diagonal [23, 27] and partially flavour-diagonal counterparts. To this end, we show in the top panels of Figures 9.2–9.4 the solution obtained when either the heavy-neutrino number-density (red dashed line) or the charged-lepton number density (green dash-
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Figure 9.2: Lepton flavour asymmetries as predicted by the BP1 parameters given in Table 9.1. The top panel shows the comparison between the total asymmetry obtained using the fully flavour-covariant formalism (thick solid lines, with different initial conditions) with those obtained using the flavour-diagonal formalism (dashed lines). Also shown (thin solid line) is the semi-analytic result (9.31). The bottom panel shows the diagonal (solid lines) and off-diagonal (dashed lines) elements of the total lepton number asymmetry matrix in the fully flavour-covariant formalism. For details, see the text.
Figure 9.3: Lepton flavour asymmetries as predicted by the BP2 minimal RL_τ model parameters given in Table 9.1. The labels are the same as in Figure 9.2.
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Figure 9.4: Lepton flavour asymmetries as predicted by the BP3 minimal RL\(_{\tau}\) model parameters given in Table 9.1. The labels are the same as in Figure 9.2.
dotted line\(^1\) or both (blue dotted line) are treated as diagonal in flavour space. For these cases, we have chosen the initial conditions \(\delta \eta_{\text{in}}^L = 0\) and \(\eta_{\text{in}}^N = \eta_{\text{eq}}^N 1\) for concreteness. As discussed above, the final lepton asymmetry is insensitive to the initial conditions in each case. We also show the semi-analytic solution (9.31), discussed in Section 9.4, for the case with diagonal heavy-neutrino number densities, which agrees well with the corresponding exact numerical solution, after a sufficient amount of time to reach the attractor solution has passed. At late times, as discussed in Section 4.3, the asymmetry freezes out and the attractor solution is no longer followed.

From Figures 9.2–9.4, we see that the asymmetry including all flavour effects can be significantly altered in the fully flavour-covariant formalism, as compared to the predictions from the partially flavour-dependent limits. The phenomena that modify significantly the asymmetry obtained from the diagonal rate equations (4.27) are:

(i) coherent oscillations between different heavy-neutrino flavours, which create an \(\mathcal{O}(h^4)\) asymmetry in the charged-lepton sector in the RL scenario, as discussed in Sections 6.2, 8.4 and 9.4. This leads to an enhancement by a factor of order two over the heavy-neutrino flavour-diagonal limit in Figures 9.2–9.4.

(ii) the dynamics of charged-lepton flavour coherences, which are generated through the heavy-neutrino Yukawa couplings and destroyed through the charged-lepton Yukawa couplings, as discussed in Sections 6.3 and 9.4.

In the benchmark points considered here the latter phenomenon causes a suppression of the asymmetry, which impacts significantly on its final value in BP2. However, this suppression seems to be model-dependent and an enhancement is possible, in principle, in models different from the one considered here or, potentially, for values of the parameters not explored in our numerical analysis (which is not meant to be exhaustive).

The bottom panels of Figures 9.2–9.4 show the individual charged-lepton flavour contributions to the total asymmetry. Since we are considering an RL\(_\tau\) model, the asymmetry

\(^1\)In Figure 9.2, this is not visible, being coincident with the black solid line.
produced around $z = 1$ is mainly in the $\tau$-flavour ($\delta \eta_{\tau\tau}^L$), which has smaller couplings to the heavy neutrinos [cf. (9.7)], and hence, a smaller washout factor. The asymmetries generated in electron ($\delta \eta_{ee}^L$) and muon ($\delta \eta_{\mu\mu}^L$) flavours, with much larger couplings to the heavy neutrinos, would be highly suppressed by their much larger washout rates. However, for the model considered here, we have found numerically that the RIS-subtracted scattering terms discussed in Section 6.4 transfer part of the $\tau$ asymmetry to the $e$ and $\mu$ channels, especially when the charged-lepton off-diagonals are included. Thus, their suppression is smaller than what one could expect from an analysis of the respective $K$-factors.

The coherence effect in the charged-lepton sector generates off-diagonal charged-lepton number-densities. This effect may be important for $z \sim 1$, since the heavy-neutrino decays are effective in this region. Outside this region, there is a complete decoherence of the system to the charged-lepton Yukawa eigenbasis (for temperatures of the order considered here). This additional source contributes to the total lepton asymmetry $\delta \eta^L$, depending on the model parameters. In the case where the sphalerons freeze out for $z$ values not too larger than 1, this phenomenon has impact on the final asymmetry, as is the case in BP2. For BP3, where the heavy-neutrino mass scale is much larger, the coherence effect is already subdued, and the system has completely decohered to the charged-lepton Yukawa eigenbasis, well above the critical temperature, thus giving no additional contribution, as shown in Figure 9.4 (top panel). On the other hand, for BP1, the neutrino Yukawa couplings are smaller than those in BP2 and BP3, and the coherence effects are not pronounced in the total asymmetry, as can be seen from Figure 9.2 (top panel).

Before concluding this chapter, we note that there is no decoherence in the heavy-neutrino sector: both diagonal and off-diagonal heavy-neutrino number-densities go to zero with a $1/z$ behaviour, as shown in Figure 9.1. This is due to the fact that the evolution of the heavy-neutrino number-densities is entirely governed by the heavy-neutrino Yukawa couplings [cf. (6.50a) and (6.50b)], ignoring sub-dominant collision terms, such as $\Delta L = 1$ scattering processes.
In this thesis we have studied the field-theoretical aspects of the generation of the matter-antimatter asymmetry in the Early Universe, via the RL mechanism. In particular, we have presented a fully flavour-covariant formalism for transport phenomena, and applied it to the description of RL. In both the semi-classical and fully field-theoretical approaches, we have derived Markovian master equations describing the time-evolution of particle number-densities, in a quantum-statistical ensemble with arbitrary flavour degrees of freedom. Thus, we have obtained the flavour-covariant generalization of the semi-classical flavour-diagonal Boltzmann equations, and provided the consistent field-theoretical description of the resonant enhancement of the $CP$ asymmetry in RL in a thermal non-equilibrium approach. The discussion in this thesis provides a consistent, complete and unified description of the RL phenomenon in the Early Universe.

It is well-known that the RL scenario offers a unique opportunity for testing the connection between the origin of neutrino masses and the matter-antimatter asymmetry by the ongoing LHC experiments as well as by various low-energy experiments probing LFV and LNV at the intensity frontier. For this reason, it is essential to capture all the flavour effects due to the heavy neutrinos as well as the SM leptons in a consistent manner, in order to obtain an accurate prediction for the baryon asymmetry in this scenario. As we have shown in Chapter 9, for models with $m_N \simeq 10^2 - 10^3\,\text{GeV}$ that predict observable LFV and LNV signatures (thanks to appropriate flavour symmetries that allow large Yukawa
couplings), both heavy-neutrino and charged-lepton off-diagonal effects have a significant impact on the predicted asymmetry. Thus, in these situations, only the flavour-covariant formalism presented here allows to obtain accurate predictions.

The main new results of our flavour-covariant formalism for RL scenarios, presented in Chapters 5 and 6, are the following:

(i) The appearance of new rank-4 tensors in flavour space in transport equations. These are necessary to guarantee the flavour covariance of the rate equations. One can extend this, by introducing even higher rank rate tensors, to describe sub-dominant processes, such as $LN \leftrightarrow L e_R$, involving more flavour degrees of freedom.

(ii) A systematic treatment of two intrinsically quantum effects, i.e. oscillations between different heavy neutrino flavours and quantum decoherence between the charged-lepton flavours. The numerical studies given in Chapter 9 for a particular RL$_\tau$ model, reveal that these flavour off-diagonal effects could impact on the predicted asymmetry by even an order of magnitude, between different partially-flavoured treatments.

(iii) The approximate analytic solutions (see Section 9.4) to the fully flavour-covariant transport equations, which capture the relevant effects discussed above. In the strong washout regime, the total lepton asymmetry may be estimated by the sum of the contributions from flavour mixing, oscillation and decoherence effects:

$$
\delta \eta^L_{\text{tot}} \simeq \delta \eta^L_{\text{mix}} + \delta \eta^L_{\text{osc}} + \delta \eta^L_{\text{dec}},
$$

as given by (4.34), (9.25b) and (9.31). The quantitative predictions obtained from the analytic solutions for all our benchmark points agree well with the exact numerical results. The analytic expressions are presented with the aim to facilitate phenomenological studies for a given model, without the need of solving the full flavour-covariant rate equations.
The semi-classical approach presented in Chapters 5 and 6 has the advantage that it is constructed with physical observables, i.e. particle number-densities, in mind. However, it has the disadvantage that the resummation of heavy-neutrino absorptive transitions must be incorporated in an effective manner, via $CP$-violating effective vertices given by the resummed Yukawa couplings. In addition, one must subtract the RIS contributions from the collision terms, which would otherwise lead to the double-counting of decay and RIS scattering processes. Therefore, in Chapter 8 we have developed a more first-principles description of transport phenomena, in which quantum effects are incorporated consistently from the out-set.

There, we have presented a novel field-theoretical approach to the study of RL, by embedding the fully flavour-covariant formalism developed in Chapters 5 and 6 into the perturbative non-equilibrium thermal field theory formulated in [109], and outlined in Chapter 7. Within this perturbative non-equilibrium field-theoretical framework, we have confirmed the results obtained in Chapter 6 via a semi-classical formalism, reproducing them quantitatively at $O(h^4)$ in the weakly-resonant regime. The main physical result is that the mixing of different heavy-neutrino flavours and the oscillations between them are two distinct physical phenomena. The first is driven by the $CP$-“even” number density $\hat{n}^N$ and the $CP$-“odd” rate $[\delta\gamma^N_{\ell\phi}]_{\alpha\beta}$, whereas the second is mediated by the $CP$-“odd” off-diagonal coherences $[\delta\hat{n}^N]_{12}$. As identified in [28], both the phenomena contribute at $O(h^4)$ with comparable magnitude in the weakly-resonant regime. The strong-washout form of the asymmetry due to oscillations (9.25b) is in agreement with the results obtained in other KB studies [120–123].

However, the KB approach presented here includes also the effect of mixing, as given by (9.25a). This contribution agrees with the one identified in [22], and re-obtained in [23, 28], once the thermal masses and widths are used in the expressions given there. The appearance of this additional $O(h^4)$ contribution, not present as separate in previous KB studies, is due to the fact that we do not, as is typically the case, use a KB ansatz, or other equivalent approximations, for the dressed heavy-neutrino propagators. We have shown
that these approximations implicitly discard mixing effects. In the approach detailed here, such approximations are not required, since we are able to express the source term for the asymmetry in terms of the spectrally-free heavy-neutrino propagators.

In conclusion, our flavour-covariant formalism provides a complete and unified description of transport phenomena in RL models, capturing three relevant physical phenomena: (i) the resonant mixing between the heavy neutrino states, (ii) the distinct phenomenon of coherent oscillations between different heavy neutrino flavours, and (iii) quantum decoherence effects in the charged-lepton sector. The formalism developed here is rather general and may also find applications in various other transport phenomena involving flavour effects.
STRUCTURE OF THE REAL-TIME PROPAGATOR

In this appendix we impose a number of physical conditions on the free propagator (2.35) in order to determine $A$, $A'$, $B$ and $C$ as functions of the 4-momentum $k$.

**CPT transformation.** Let us perform a CPT transformation on the matrix elements of $i\Delta(x)$. On the one hand, we have

$$ (i\Delta_{11}(x))^{\text{CPT}} = (i\Delta_{11}(x))^* , $$

(A.1)

since the time-reversal operator $T$ is antiunitary. On the other hand, for a scalar field the charge-conjugation operator $C$ is trivially the identity, while $PT$ turns $x$ into $-x$. Thus, 11 and 22 matrix elements are swapped, since time-ordering is turned into anti-time-ordering. Hence, we find the relation

$$ (i\Delta_{11}(x))^* = i\Delta_{22}(-x) , $$

(A.2)

which, in Fourier space, takes on the form $\Delta_{22}(k) = -\Delta_{11}(k)^*$, so that $A'(k) = A(k)^*$. Instead, for the off-diagonal elements we have

$$ (\Delta_{12}(x))^{\text{CPT}} = (i\Delta_{12}(x))^* = i\Delta_{12}(-x) . $$

(A.3)
Here, the CTP indexes are not swapped because times on $C_1$ are always before times on $C_2$, even changing $x$ into $-x$. We thus have $-\Delta_{12}(k)^* = \Delta_{12}(k)$, finding $B(k) \in \mathbb{R}$ and, analogously, $C(k) \in \mathbb{R}$.

**Hermiticity.** Let us now impose the Hermiticity condition on the real scalar field $\phi(x)$.

For the diagonal elements we have

$$(i\Delta_{11}(x))^* = \langle T\{\phi(x)\phi(0)\}\rangle^* = \langle \theta(t)\phi(0)^+\phi(x)^+ + \theta(-t)\phi(x)^+\phi(0)^+ \rangle$$

$$= \langle \theta(t)\phi(0)\phi(x) + \theta(-t)\phi(x)^+\phi(0)^+ \rangle = i\Delta_{22}(x) . \quad (A.4)$$

In Fourier space this implies $-\Delta_{11}(-k)^* = \Delta_{22}(k)$ and, in conjunction with the results above, $A(k) = A(-k)$. Analogously, for the off-diagonal elements we find $\Delta_{12}(k) = -\Delta_{21}(-k)^*$, which leads to $B(k) = C(-k)$.

**Causality.** We impose the causality condition on the propagator by requiring that the free retarded propagator $\Delta_R^0(x) = \Delta_{11}^0(x) - \Delta_{12}^0(x)$ vanishes for $t < 0$. We have

$$\Delta_R^0(x) = \int_k e^{-i k \cdot x} \left( \Delta_{11}^0(k) - \Delta_{12}^0(k) \right)$$

$$= \int_k e^{-i k \cdot x} \left[ \frac{1}{k^2 - m^2 + i\epsilon} - i \left( A(k) - B(k) \right) \delta(k_0^2 - \omega_k^2) \right] . \quad (A.5)$$

For $t < 0$ we can close the $k_0$-integration contour above with a large semicircle, picking up the pole at $k_0 = -\omega_k$, obtaining

$$\Delta_R^0(x) = \int_k \frac{-i}{2\omega_k} \left[ e^{i(\omega_k t - k \cdot x)} + \left( A(\omega_k, k) - B(\omega_k, k) \right) e^{-i(\omega_k t - k \cdot x)} \right] \frac{2\pi}{2\pi}$$

$$+ \left( A(-\omega_k, k) - B(-\omega_k, k) \right) e^{-i(-\omega_k t - k \cdot x)} \]$$

$$= \int_k \frac{-i}{2\omega_k} \left[ e^{i\omega_k t} \left( 2\pi + A(-\omega_k, k) - B(-\omega_k, k) \right) \right. \]$$

$$\left. + e^{-i\omega_k t} \left( A(\omega_k, k) - B(\omega_k, k) \right) \right] = 0 , \quad (A.6)$$
and therefore we find the two conditions

\[ 2\pi + A(-\omega_k, k) - B(-\omega_k, k) = 0, \quad A(\omega_k, k) - B(\omega_k, k) = 0. \] (A.7)

Requiring that the advanced propagator vanishes for \( t > 0 \) does not lead to independent conditions.

**Summary.** Let us summarize the conditions obtained above:

\[ A'(k) = A(k)^*, \quad B(k), C(k) \in \mathbb{R}, \] (A.8a)

\[ A(k) = A(-k), \quad B(k) = C(-k), \] (A.8b)

\[ B(-\omega_k, k) = 2\pi + A(-\omega_k, k), \quad B(\omega_k, k) = A(\omega_k, k). \] (A.8c)

From (A.8a) and (A.8c), \( A(k) \) is real. Thus, we may set it equal to a real function \( 2\pi f(k) \); this is an even function, by virtue of (A.8b). Using (A.8c) we obtain \( B(\omega_k, k) = 2\pi f(\omega_k, k) \) and \( B(-\omega_k, k) = 2\pi + 2\pi f(-\omega_k, k) \), and finally \( C(\omega_k, k) = 2\pi + 2\pi f(\omega_k, k) \) and \( C(-\omega_k, k) = 2\pi f(-\omega_k, k) \), thanks to (A.8b). Since all these functions have support only on the mass-shell, because always multiplied by the factor \( \delta(k^2 - m^2) \), we have completely determined them in terms of an even real function \( f(k) \), obtaining the structure (2.36) for the free real-time propagator.

**Unitarity.** We may finally show that the conditions obtained above are consistent with the unitarity of the \( S \)-matrix, expressed in the form of the cutting equation (Figure A.1), valid also in the thermal case [58, 133]. For the propagator, this equation takes on the form [51]

\[ i\Delta_{11}(k) + (-1)^2 i\Delta_{22}(k) + (-1) i\Delta_{12}(k) + (-1) i\Delta_{21}(k) = 0, \] (A.9)

or, also,

\[ \Delta_F(k) + \Delta_D(k) = \Delta_< (k) + \Delta_>(k). \] (A.10)
Figure A.1: Cutting equation for the propagator. Type-2 vertices are denoted by circles, whereas standard type-1 ones are un-circled.

We remind that a factor of -1 is associated with type-2 vertices. The imaginary part of (A.9) is satisfied because the functions $A$, $A'$, $B$ and $C$ are real and $\text{Re} \, \Delta_{11}^0(k) = -\text{Re} \, \Delta_{22}^0(k)$. Its real part, coming from the imaginary part of the propagators, gives

$$- \pi - A(k) - \pi - A'(k) + B(k) + C(k) = 0 , \quad (A.11)$$

which is satisfied by virtue of (A.8).
ABSORPTIVE SELF-ENERGY IN THE SCALAR $\phi^3$ THEORY

In this appendix we perform the 3-momentum integral in (3.41), which gives the absorptive 1-loop retarded self-energy in the scalar $\phi^3$ model. The imaginary part (3.41) of the retarded self-energy is

$$\text{Im } \Pi_R(k) = \frac{g^2}{2} \int \frac{\pi}{4 \omega_p \omega_{k-p}} \times \left\{ \left[ 1 + f(\omega_p) + f(\omega_{k-p}) \right] \left[ \delta(k_0 - \omega_p - \omega_{k-p}) - \delta(k_0 + \omega_p + \omega_{k-p}) \right] - \left[ f(\omega_p) - f(\omega_{k-p}) \right] \left[ \delta(k_0 - \omega_p + \omega_{k-p}) - \delta(k_0 + \omega_p - \omega_{k-p}) \right] \right\} . \quad (B.1)$$

Let us split (B.1) in 4 integrals $I_1, I_2, I_3, I_4$, for each of the Dirac $\delta$ functions. Since the whole expression is odd in $k_0$ we can assume $k_0 > 0$. Denoting by $\theta$ the angle between $k$ and $p$ we have:

$$I_1 \equiv \frac{\pi g^2}{8} \int \frac{1}{\omega_p \omega_{k-p}} \left( 1 + f(\omega_p) + f(\omega_{k-p}) \right) \delta(k_0 - \omega_p - \omega_{k-p}) \quad (B.2)$$

with $\omega^2_{|p|} \equiv |p|^2 + m^2$. The angular integration can be recast as

$$\int_{-1}^{1} d(\cos \theta) \frac{\delta(k_0 - \omega_p - \omega_{k-p})}{\omega_{k-p}} = \int_{||k|-|p||}^{||k||+|p|} d|k-p| \frac{|k-p|}{||k|||p|} \frac{\delta(k_0 - \omega_p - \omega_{k-p})}{\omega_{k-p}}$$

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\[ I_1 = \frac{1}{|k||p|} \int_{\omega_{[k]|p|}}^{\omega_{[k]+|p|}} d\omega \delta(k_0 - \omega_p - \omega). \quad (B.3) \]

The condition $\omega_{[k]|p|} \leq k_0 - \omega_p \leq \omega_{[k]+|p|}$ can be rewritten as

\[ (k^2 - 2k_0 \omega_p)^2 \leq 4|k|^2|p|^2, \quad k_0 - \omega_p \geq 0. \quad (B.4) \]

For spacelike $k$ the first inequality is found to be

\[ |p| \geq \frac{|k|}{2} + \frac{k_0}{2} \sqrt{1 - \frac{4m^2}{k^2}} \equiv p_+, \quad (B.5) \]

but this implies

\[ 2\omega_p \geq 2|p| > |k| + k_0 > 2k_0, \quad (B.6) \]

and the second inequality in (B.4) cannot be satisfied. Therefore $I_1 = 0$ for spacelike $k$.

For timelike $k$, the first inequality in (B.4) can be satisfied for $k^2 > 4m^2$, giving

\[ p_- \equiv \left\{ \frac{|k|}{2} - \frac{k_0}{2} \sqrt{1 - \frac{4m^2}{k^2}} \right\} \leq |p| \leq p_+, \quad \text{for } k^2 > 4m^2, \quad (B.7) \]

and since

\[
4\omega_p^2 = 4|p|^2 + 4m^2 \leq |k|^2 + k_0^2 \left(1 - \frac{4m^2}{k^2}\right) + 2k_0|k| \sqrt{1 - \frac{4m^2}{k^2}} + 4m^2
\]

\[
\leq k_0^2 + k_0^2 - 4m^2 \frac{k_0^2}{k^2} + 2k_0^2 + 4m^2 = 4k_0^2 - 4m^2 \frac{|k|^2}{k^2} \leq 4k_0^2,
\]

the second inequality in (B.4) is automatically satisfied. Therefore, defining $\omega_\pm = \sqrt{p_\pm^2 + m^2}$ we finally find, for $k^2 > 4m^2$,

\[
I_1 = \frac{g^2}{32\pi|k|} \int_{p_-}^{p_+} d|p| \frac{|p|}{\omega_p} \left[1 + 2f(\omega_p)\right] = \frac{g^2}{32\pi|k|} \frac{2}{\beta} \log \frac{\sinh[\beta \omega_+/2]}{\sinh[\beta \omega_-/2]}, \quad (B.9)
\]

whereas $I_1 = 0$ otherwise. It is easy to convince oneself that $I_3 = I_4$ as well as, for
\( k_0 > 0, I_2 = 0. \) It is convenient to split \( I_3 \) in two integrals \( I_{3A} \) and \( I_{3B} \). The first one is

\[
I_{3A} = -\frac{\pi g^2}{8} \int_P \frac{1}{\omega_p \omega_{k-p}} f(\omega_p) \delta(k_0 - \omega_p + \omega_{k-p})
\]

\[
= -\frac{g^2}{32\pi |k|} \int_0^\infty d|p| \frac{|p| f(\omega_p)}{\omega_p} \int_{\omega_p}^{\omega_{k-|p|}} d\omega \delta(k_0 - \omega_p + \omega). \quad (B.10)
\]

The requirement \( \omega_{|k-|p|} \leq \omega_p - k_0 \leq \omega_{|k+|p|} \) leads again to the first inequality in (B.4), but with the condition \( \omega_p - k_0 > 0 \) this time. This implies that the integral vanishes for timelike \( k \), whereas for spacelike \( k \) we obtain

\[
I_{3A} = -\frac{g^2}{32\pi |k|} \int_{\omega_+}^\infty d\omega f(\omega). \quad (B.11)
\]

Analogously, we may find \( I_{3B} \) as

\[
I_{3B} = \frac{\pi g^2}{8} \int_P \frac{1}{\omega_{k-p}} f(\omega_{k-p}) \delta(k_0 - \omega_p + \omega_{k-p})
\]

\[
= \frac{g^2}{32\pi |k|} \int_0^\infty d|q| \frac{|q| f(\omega_q)}{\omega_q} \int_{\omega_q}^{\omega_{|k-|q|}} d\omega \delta(k_0 + \omega_q - \omega). \quad (B.12)
\]

The condition \( \omega_{|k-|q|} \leq \omega_q + k_0 \leq \omega_{|k+|q|} \) can be recast as \( (k^2 + 2 k_0 \omega_q)^2 \leq 4|k|^2 |q|^2 \), and we find that this cannot be satisfied in the timelike case, whereas for spacelike \( k \) it is equivalent \( |q| \geq p_- \). Thus, for spacelike \( k \), we obtain

\[
I_{3B} = \frac{g^2}{32\pi |k|} \int_{\omega_-}^\infty d\omega f(\omega). \quad (B.13)
\]

Collecting the results above, we have

\[
I_3 + I_4 = 2I_{3A} + 2I_{3B} = \frac{2g^2}{32\pi |k|} \int_{\omega_-}^{\omega_+} d\omega f(\omega)
\]

\[
= \frac{g^2}{32\pi |k|} \left[ \frac{2}{\beta} \log \frac{\sinh[\beta \omega_+ / 2]}{\sinh[\beta \omega_- / 2]} - (\omega_+ - \omega_-) \right], \quad (B.14)
\]
finally obtaining

\[ \text{Im} \Pi_R(k) = \begin{cases} \frac{g^2}{16\pi|k|} \left[ \frac{1}{\beta} \log \frac{\sinh[\beta\omega_+/2]}{\sinh[\beta\omega_-/2]} - \frac{\omega_+ - \omega_-}{2} \theta(|k| - |k_0|) \right], & \text{if } \frac{4m^2}{k^2} < 1, \\ 0, & \text{otherwise}. \end{cases} \]  

(B.15)
RESUMMED YUKAWA COUPLINGS IN THE KADANOFF–BAYM APPROACH

In this appendix, we show that in the KB approach described in Chapter 8, the contribution of the charged-lepton self-energy to the source term can be written in terms of resummed Yukawa couplings as in (8.34) (see also Figure 8.3), formally at $O(h^4)$ in the asymmetry. The formalism used here takes into account thermal effects, and thus generalizes the $T = 0$ resummation procedure developed in [23] and outlined in Section 4.2. The resummed Yukawa couplings that we will obtain (cf. (C.8) below), at least in the two heavy-neutrino case studied here, have the same functional form as in (4.10), with thermal masses and widths appearing in place of the $T = 0$ ones. The analysis discussed here has been presented originally in [42].

From (8.26), we see that the combination of interest, which appears in the evolution equation for the asymmetry, is

\[ T \equiv h^\alpha \left[ \Delta^< \right]_{\alpha}^\beta h_\beta \cdot \]  \hfill (C.1)

Throughout this appendix, we suppress the $N$ superscript, for notational simplicity. The contribution of the second line of (8.29) to $T$, which appears in the source term (8.26), will be denoted by $T_{src}$. By making use of (8.29) and noting that the summations there
are equal to $\Delta_{R(A)} \cdot \Delta_{R(A)}^0$, we write $T_{\text{src}}$ as

$$
T_{\text{src}} = h^\alpha \left[ \Delta_{R} \right]_\alpha^\lambda \left[ \Delta_{R}^0 \right]_\lambda^\gamma \left[ \Delta_{A}^0 \right]_\gamma^\delta \left[ \Delta_{A}^0 \right]_\delta^\mu \left[ \Delta_{A} \right]_\mu^\beta h^\beta
= \sum_{\gamma, \delta} \hat{h}^\alpha \left[ \Delta_{R} \right]_{\alpha \gamma} \left[ \Delta_{R}^0 \right]_{\gamma \delta} \left[ \Delta_{A} \right]_{\delta \beta} \hat{h}^\beta
= \hat{h}^\alpha \left[ \Delta_{R} \right]_{\alpha \gamma} \tilde{N}_{\gamma \delta} \left[ \Delta_{A} \right]_{\delta \beta} \hat{h}^\beta .
$$

(C.2)

Let us introduce the notation

$$
\phi^k \equiv \begin{cases} 
2 & \text{if } \alpha = 1 \\
1 & \text{if } \alpha = 2 
\end{cases} .
$$

(C.3)

The off-diagonal number-densities $[\hat{n}^N]_{\alpha \beta}$ are formally at $O(h^2)$, if one assumes that they are generated dynamically from an incoherent initial condition (see Section 6.2). Therefore, we have

$$
\tilde{N}_{\alpha \beta} = (s - M_{N, \alpha}^2) \left[ \Delta_{\zeta}^0 \right]_{\alpha \beta} (s - M_{N, \beta}^2) = (s - s_\alpha) \left[ \Delta_{\zeta}^0 \right]_{\alpha \beta} (s - s_\beta^*) + O(h^4) ,
$$

(C.4)

where we have used $\Gamma_{N, \alpha} [\Delta_{\zeta}^0]_{\alpha \beta} = O(h^4)$, with $\Gamma_{N, \alpha}$ being the thermal width of the heavy neutrinos. Here, $s_\alpha$ denotes the location of the complex pole of the retarded propagator (cf. (3.19) and (3.20))

$$
s_\alpha = M_{N, \alpha}^2 - i M_N \Gamma_{N, \alpha} .
$$

(C.5)

Proceeding as in [23], the resonant terms in $T_{\text{src}}$ can be expanded as

$$
T_{\text{src}} \simeq \sum_{\alpha} \hat{h}^\alpha \frac{Z_{\alpha}}{s - s_\alpha} \left( G_\alpha - \frac{D R_{12}}{D R_{\phi \phi}} G_\phi \right) ,
$$

(C.6)

with

$$
G_\alpha \equiv \hat{N}_{\alpha \delta} \left[ \Delta_{A} \right]_{\delta \beta} \hat{h}^\beta ,
$$

(C.7)
\[ D^R \equiv \Delta_R^{-1}. \] For the sake of generality, in (C.6), we have included the wavefunction renormalization \( Z_\alpha \equiv (\frac{d}{ds}\left[\Delta_R(s)\right]^{-1})^{-1}. \) even though this is a higher-order effect in the analysis given here. The resummed Yukawa couplings that appear in (C.6) are the generalization of (4.10) to the thermal case, obtained using thermal masses and widths in place of their \( T = 0 \) counterparts. Thus, they can be written explicitly as

\[
\hat{h}^\alpha = \hat{h}^\alpha - \frac{\hat{h}^\delta i \, \text{Im}[\Pi_R]_{\alpha\delta}}{M_{N,\alpha} - M_{N,\beta}^2 + i \, \text{Im}[\Pi_R]_{\beta\delta}}, \tag{C.8}
\]

where the indices are not summed over. The \( \tilde{C}P \) conjugate couplings \( [h^\gamma]_\alpha \) are obtained by using the complex-conjugate tree-level couplings in the RHS of (C.8). Equation (C.7) can be expanded, in turn, as

\[
G_\alpha \simeq \sum_{\beta} \left( \tilde{N}_{\alpha\beta} + \frac{D^R_{12}}{D^R_{\beta\beta}} \tilde{N}_{\alpha\beta} \right) \frac{Z^\beta_\alpha}{s - s^\beta_\alpha} \hat{h}_\beta. \tag{C.9}
\]

Using (C.9) in (C.6), we find

\[
T_{\text{src}} = \sum_{\alpha,\beta} \hat{h}^\alpha \frac{Z_\alpha}{s - s_\alpha} \tilde{N}_{\alpha\beta} \frac{Z^\beta_\beta}{s - s^\beta_\beta} \hat{h}_\beta \\
+ \sum_{\alpha,\beta} \frac{D^R_{12}}{D^R_{\beta\beta}} \hat{h}^\alpha \frac{Z_\alpha}{s - s_\alpha} \tilde{N}_{\alpha\beta} \frac{Z^\beta_\beta}{s - s^\beta_\beta} \hat{h}_\beta - \sum_{\alpha,\beta} \frac{D^R_{12}}{D^R_{\phi\phi}} \hat{h}^\alpha \frac{Z_\alpha}{s - s_\alpha} \tilde{N}_{\phi\beta} \frac{Z^\beta_\beta}{s - s^\beta_\beta} \hat{h}_\beta \\
+ \mathcal{O}(h^6). \tag{C.10}
\]

The contributions in the second line of (C.10) can be neglected. To show this, consider for instance the first summation; the only terms that can give contributions at \( \mathcal{O}(h^4) \) are the ones with \( \alpha = \beta \). Using (C.4), these become

\[
\frac{D^R_{12}}{D^R_{\beta\beta}} \hat{h}^\alpha \left[ \hat{\Delta}_c^0 \right]_{\alpha\alpha} \frac{s - s_\alpha}{s - s^\beta_\beta} \hat{h}_\beta + \mathcal{O}(h^6) = \frac{D^R_{12}}{D^R_{\phi\phi}} \hat{h}^\alpha \left[ \hat{\Delta}_c^0 \right]_{\alpha\alpha} i M_N \Gamma_{\alpha} \hat{h}_\beta + \mathcal{O}(h^6) \\
= \mathcal{O}(h^6), \tag{C.11}
\]
having also used (8.6) and (C.5). Therefore, we obtain the final expression

\[ T_{\text{src}} = \sum_{\alpha,\beta} h^\alpha [\Delta^0_\alpha]^{\beta} h_\beta + \mathcal{O}(h^6), \]  

(C.12)

which is the form that we use in (8.34) and Figure 8.3. We conclude that, also in the thermal case, the effect of heavy-neutrino mixing for the generation of the asymmetry can be factorized, at \( \mathcal{O}(h^4) \), into effective \( CP \)-violating vertices, given by the resummed Yukawa couplings (C.8).
LEPTON FLAVOUR AND LEPTON NUMBER VIOLATING OBSERVABLES

In this Appendix, we discuss the phenomenology of the RL_τ model presented in Section 9.2, and in particular, we provide the predictions of the three benchmark points discussed in Section 9.3 for LFV and LNV observables studied in experiments at the intensity frontier and briefly discuss heavy-neutrino searches at the LHC. The content of this appendix is based on [28].

The mixing between the light and \( N_N \) heavy Majorana neutrinos induces LFV processes such as \( \ell \to \ell'\gamma \) [10, 134–139], \( \ell \to \ell\ell_1\bar{\ell}_2 \) [138, 140, 141] and \( \mu \to e \) conversion in nuclei [142–146], through loops involving the heavy neutrinos. In general, the light-heavy neutrino mixing is parametrized in terms of a \( 3 \times N_N \) matrix \( \xi \) [147], which depends on the Yukawa coupling matrix \( h \) and the RH neutrino mass matrix \( m_N \). In the mass eigenbasis, and to leading order in \( \|\xi\| \), the mixing is given by\(^1\)

\[
B_{l_\alpha} \simeq \xi_{l_\alpha} \equiv \frac{\nu}{\sqrt{2}} \hat{h}_{l_\alpha} \hat{m}_{N_{\alpha}}^{-1},
\]

which governs the rare LFV decay rates, as discussed below.

\(^1\)The all-order expression for the mixing in terms of \( \xi \) may be found in [70, 148, 149]. Some approximate seesaw expressions are studied in [150, 151].
APPENDIX D. LFV AND LNV OBSERVABLES

The branching ratio for the $\mu \rightarrow e\gamma$ process is given by [137]

$$BR(\mu \rightarrow e\gamma) = \frac{\alpha^3 w_s^2 m^4_{\mu}}{256 \pi^2 m^4_W} \frac{m_{\mu}}{\Gamma_{\mu}} |G_{\gamma}^{\mu e}|^2,$$

(D.2)

where $m_{\mu}$ and $\Gamma_{\mu}$ are respectively the mass and width of the muon, $s_w \equiv \sin \theta_w$ is the weak mixing parameter, and $\alpha_w \equiv \frac{g_w^2}{4\pi}$ is the weak coupling strength, all evaluated at the weak scale $m_Z$. The form factor $G_{\gamma}^{\mu e}$ is defined in Appendix D of [28]. All the form factors in this appendix have the generic form $X^{\mu e}$, given by

$$X^{\mu e} = \sum \alpha B_{\alpha} B^{\star}_{\alpha} X \left( \left( \frac{m_{X}}{m_{W}} \right)^2 \right) \propto \langle \hat{h} \hat{h} \rangle_{\mu e},$$

(D.3)

with the loop functions $X$'s defined in Appendix D of [28]. The other kinematically allowed rare decay modes of this type, namely, $\tau \rightarrow \mu\gamma$ and $\tau \rightarrow e\gamma$, can be defined similar to (D.2), in terms of the mass and width of the $\tau$-lepton. The branching-ratio predictions of these rare decay modes for the three benchmark points given in Section 9.3 are given in Table D.1. For comparison, we also give the current experimental upper limits [1, 152]. The MEG limit on $\mu \rightarrow e\gamma$ branching ratio [152] is the most stringent one, and the model prediction for BP2 is within reach of the upgraded MEG sensitivity [153].

The branching ratio for the $\mu^- \rightarrow e^-e^+e^-$ process is given by [141]

$$BR(\mu \rightarrow eee) = \frac{\alpha^4 w_s}{24576 \pi^3} \frac{m^4_{\mu}}{m^4_W} \frac{m_{\mu}}{\Gamma_{\mu}} \left[ 2 \left( \frac{1}{2} F^{\mu e e}_{Box} + F^{\mu e}_{Z} - 2s_w^2 \left( F^\mu_{Z} - F^\mu_{\gamma} \right) \right)^2 
+ 4s_w^4 \left| F^\mu_{Z} - F^\mu_{\gamma} \right|^2 + 16s_w^2 \Re \left( \left( F^\mu_{Z} + \frac{1}{2} F^{\mu e e}_{Box} \right) (G_{\gamma}^{\mu e})^* \right) 
- 48s_w^4 \Re \left( (F^\mu_{Z} - F^\mu_{\gamma}) (G_{\gamma}^{\mu e})^* \right) + 32s_w^4 \left| G_{\gamma}^{\mu e} \right|^2 \left\{ \ln \left( \frac{m_{\mu}}{m_e} \right) - \frac{11}{4} \right\} \right],$$

(D.4)

where the various form factors are defined in Appendix D of [28]. The predictions for $BR(\mu \rightarrow eee)$ are shown in Table D.1 and, for comparison, we have also shown the current experimental upper limit [1]. The model predictions are well within the current
limit. One can similarly define the LFV decay rates involving the $\tau$-lepton [141]; however, the numerical values for these rates turn out to be several orders of magnitude smaller than the current experimental limits and hence we do not show them in Table D.1.

The $\mu \to e$ conversion rate in an atomic nucleus $^{A_Z}X$ is given by [146]

$$R_{\mu\to e} = \frac{2G_F^2\alpha^2 m_{\mu}^5}{16\pi^2\Gamma_{\text{capt}}} \left| 4V^{(p)} \left( 2\tilde{F}_u^{\mu e} + \tilde{F}_d^{\mu e} \right) + 4V^{(n)} \left( \tilde{F}_u^{\mu e} + 2\tilde{F}_d^{\mu e} \right) + \frac{s_w^2}{2e} C_\mu^{\mu e} D \right|^2,$$

(D.5)

where $e = g_w s_w$ is the electron charge, $\Gamma_{\text{capt}}$ is the nuclear capture rate and $V^{(p),(n)}$, $D$ are various nuclear form factors, whose numerical values for some typical nuclei of interest are given in Appendix D of [28]. The electroweak form factors $\tilde{F}_q^{\mu e}$ ($q = u, d$) in (D.5) are defined as

$$\tilde{F}_q^{\mu e} = Q_q s_w^2 F_{\gamma}^{\mu e} + \left( \frac{I_{3L}^q}{2} - Q_q s_w^2 \right) F_Z^{\mu e} + \frac{1}{4} F_{\Box}^{\mu e q q},$$

(D.6)

where $Q_q$ is the electric charge of the quark $q$ in units of $e$ ($Q_u = 2/3$, $Q_d = -1/3$), $I_{3L}^q$ is the third component of the weak isospin ($I_{3L}^u = 1/2$, $I_{3L}^d = -1/2$), and the form factors $F_{\gamma}^{\mu e}$, $F_Z^{\mu e}$, $F_{\Box}^{\mu e q q}$ are defined in Appendix D of [28]. The predictions for the $\mu \to e$ conversion rate are given in Table D.1 for certain isotopes of titanium, gold and lead nuclei, along with their experimental upper limits from SINDRUM-II [154–156]. It is worth mentioning here that the next generation experiments, such as COMET [157] and Mu2e [158] have planned sensitivities around $10^{-16}$, which could easily test the first two benchmark points. The distant future proposal PRISM/PRIME [159] could probe $\mu \to e$ conversion rates down to $10^{-18}$, thus testing also the third benchmark point.

Apart from these LFV observables, a non-zero light-heavy neutrino mixing also leads to a non-unitary PMNS mixing matrix, which can be parametrized as [147–149]

$$\tilde{U}_{\text{PMNS}} = \left( 1 + \xi^T\xi \right)^{-1/2} U_{\text{PMNS}} \equiv \left( 1 - \frac{1}{2} \Omega \right) U_{\text{PMNS}},$$

(D.7)

where $U_{\text{PMNS}}$ is the unitary matrix given by (9.13), and the non-unitarity effects are de-
APPENDIX D. LFV AND LNV OBSERVABLES

<table>
<thead>
<tr>
<th>Low-energy observables</th>
<th>BP1</th>
<th>BP2</th>
<th>BP3</th>
<th>Experimental Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{BR}(\mu \to e\gamma) )</td>
<td>( 4.5 \times 10^{-15} )</td>
<td>( 1.9 \times 10^{-15} )</td>
<td>( 2.3 \times 10^{-17} )</td>
<td>( &lt; 5.7 \times 10^{-14} ) [152]</td>
</tr>
<tr>
<td>( \text{BR}(\tau \to \mu\gamma) )</td>
<td>( 1.2 \times 10^{-17} )</td>
<td>( 1.6 \times 10^{-18} )</td>
<td>( 8.1 \times 10^{-22} )</td>
<td>( &lt; 4.4 \times 10^{-8} ) [1]</td>
</tr>
<tr>
<td>( \text{BR}(\tau \to e\gamma) )</td>
<td>( 4.6 \times 10^{-18} )</td>
<td>( 5.9 \times 10^{-19} )</td>
<td>( 3.1 \times 10^{-22} )</td>
<td>( &lt; 3.3 \times 10^{-8} ) [1]</td>
</tr>
<tr>
<td>( \text{BR}(\mu \to 3e) )</td>
<td>( 1.5 \times 10^{-16} )</td>
<td>( 9.3 \times 10^{-15} )</td>
<td>( 4.9 \times 10^{-18} )</td>
<td>( &lt; 1.0 \times 10^{-12} ) [1]</td>
</tr>
<tr>
<td>( R^\text{H}_{\mu-e} )</td>
<td>( 2.4 \times 10^{-14} )</td>
<td>( 2.9 \times 10^{-13} )</td>
<td>( 2.3 \times 10^{-20} )</td>
<td>( &lt; 6.1 \times 10^{-13} ) [154]</td>
</tr>
<tr>
<td>( R^\text{Au}_{\mu-e} )</td>
<td>( 3.1 \times 10^{-14} )</td>
<td>( 3.2 \times 10^{-13} )</td>
<td>( 5.0 \times 10^{-18} )</td>
<td>( &lt; 7.0 \times 10^{-13} ) [155]</td>
</tr>
<tr>
<td>( R^\text{Pb}_{\mu-e} )</td>
<td>( 2.3 \times 10^{-14} )</td>
<td>( 2.2 \times 10^{-13} )</td>
<td>( 4.3 \times 10^{-18} )</td>
<td>( &lt; 4.6 \times 10^{-11} ) [156]</td>
</tr>
<tr>
<td>(</td>
<td>\Omega</td>
<td>_{ee} )</td>
<td>( 5.8 \times 10^{-6} )</td>
<td>( 1.8 \times 10^{-5} )</td>
</tr>
<tr>
<td>( \langle m \rangle ) [eV]</td>
<td>( 3.8 \times 10^{-5} )</td>
<td>( 3.8 \times 10^{-5} )</td>
<td>( 3.8 \times 10^{-5} )</td>
<td>( &lt; (0.11–0.25) ) [160]</td>
</tr>
</tbody>
</table>

Table D.1: The model predictions for the low-energy observables for the three chosen benchmark points and their comparison with the current experimental limits.

scribed by the Hermitian matrix \( \Omega \), which is a function of the light-heavy neutrino mixing parameter \( \xi \) [cf. (D.1)]. To leading order in \( \|\xi\|^2 \), the non-unitarity parameters are given by \( \Omega_{\ell\ell'} \simeq \sum_{\alpha} B_{\ell\alpha}^{*} B_{\ell'\alpha} \). For the benchmark points given in Section 9.3, the predictions for \( |\Omega|_{ee} \) are shown in Table D.1, along with the current experimental limit at 90% confidence level (CL) [131]. The predictions for other elements of \( \Omega \) are much below the current experimental limits, and hence, are not shown here. Since the non-unitarity parameter \( \Omega \) is very small for all the benchmark points chosen here, we have used \( U_{\text{PMNS}} \) [cf. (9.13)] instead of \( \tilde{U}_{\text{PMNS}} \) [cf. (D.7)] as the diagonalizing matrix in (9.12).

The Majorana nature of the light and heavy neutrinos in the type-I seesaw models violate lepton number by two units, which can manifest in the neutrinoless double beta decay (0νββ) process at low energy (for a review, see e.g. [161]). In the type-I seesaw model, the 0νββ process gets contributions from diagrams mediated by both light and heavy Majorana neutrinos, and the corresponding half-life is given by

\[
\frac{1}{T^{0\nu}_{1/2}} = G_{01}^{0\nu} \left| \mathcal{M}_{\nu}^{0\nu} A_{\nu} + \mathcal{M}_{N}^{0\nu} A_{N} \right|^2, \tag{D.8}
\]

where \( G_{01}^{0\nu} \) is the decay phase-space factor, \( \mathcal{M}_{\nu}^{0\nu} \),’s are the nuclear matrix elements for 0νββ mediated by light and heavy neutrinos respectively, and the parameters \( A_{\nu,N} \) are
defined as

\[ A_\nu = \frac{1}{m_e} \sum_i (U_{PMNS})_{ei}^2 m_{\nu i}, \quad A_N = m_p \sum_\alpha \frac{B_{\alpha \alpha}^2}{m_{N\alpha}}, \]  

(D.9)

where \( m_e \) and \( m_p \) are the electron and proton masses, respectively. For all the benchmark points, the heavy neutrino contribution \( A_N \) to the half-life (D.8) turns out to be negligible compared to the light neutrino contribution \( A_\nu \), and hence, we ignore the second term on the RHS of (D.8) rewriting this in the canonical form

\[ \frac{1}{T_{1/2}^{0\nu}} = G_{01}^{0\nu} |\mathcal{M}_{0\nu}|^2 \frac{\langle m \rangle^2}{m_e^2}, \]  

(D.10)

where \( \langle m \rangle \equiv |\sum_i (U_{PMNS})_{ei}^2 m_{\nu i}| \) is known as the effective neutrino mass. For a normal hierarchy of neutrino masses with \( m_{\nu_1} = 0 \), and using the three-neutrino oscillation parameter values given in (9.15), we obtain \( \langle m \rangle = 3.8 \) meV. The current 90% CL experimental limits are \( \langle m \rangle < (0.3–0.9) \) eV from the NEMO-3 limit on \( T_{1/2}^{0\nu}(^{100}\text{Mo}) \) [162], \( \langle m \rangle < (0.2–0.4) \) eV from the GERDA+Heidelberg-Moscow+IGEX combined limit on \( T_{1/2}^{0\nu}(^{76}\text{Ge}) \) [163], and \( \langle m \rangle < (0.12–0.25) \) eV from the KamLAND-Zen+EXO-200 combined limit on \( T_{1/2}^{0\nu}(^{136}\text{Xe}) \) [160], where the range of limits is due to the nuclear matrix element uncertainties involved.

Apart from the low-energy observables discussed above, the Majorana nature of the heavy neutrinos as well as their mixing with the light neutrinos could manifest simultaneously via their ‘smoking gun’ signature of same-sign dilepton plus two jets with no missing energy [164] at the LHC [130, 165]. Within the minimal seesaw framework, both CMS [166] and ATLAS [167] experiments have derived limits on the mixing parameters \( |B_{\mu \alpha}|^2 \) between 0.001–0.1 for \( m_N = 100–400 \) GeV. Including infrared enhancement effects due to \( t \)-channel processes involving photons, these limits can be improved (by at least a factor of 5) and extended to higher heavy neutrino masses at \( \sqrt{s} = 14 \) TeV LHC [168].


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