Effective Field Theories of Heavy-Quark Mesons

A thesis submitted to The University of Manchester for the degree of Doctor of Philosophy (PhD) in the faculty of Engineering and Physical Sciences

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Contents

Abstract 10
Declaration 12
Copyright 13
Acknowledgements 14

1 Introduction 16
  1.1 Ordinary Mesons ........................................ 21
    1.1.1 Light Mesons ........................................ 22
    1.1.2 Heavy-light Mesons ................................... 24
    1.1.3 Heavy-Quark Mesons .................................. 28
  1.2 Exotic $c\bar{c}$ Mesons ................................ 31
    1.2.1 Experimental and theoretical studies of the $X(3872)$ . 34

2 From QCD to Effective Theories 41
  2.1 Chiral Symmetry ........................................... 43
    2.1.1 Chiral Symmetry Breaking ............................ 46
    2.1.2 Effective Field Theory ............................... 57
  2.2 Heavy Quark Spin Symmetry ................................ 65
    2.2.1 Motivation ........................................... 65
    2.2.2 Heavy Quark Effective Theory ....................... 69

3 Heavy Hadron Chiral Perturbation Theory 72
  3.1 Self-Energies of Charm Mesons ............................ 78
  3.2 Mass formula for non-strange charm mesons ............... 89
    3.2.1 Extracting the coupling constant of even and odd charm meson transitions .................. 92
4 HHChPT for Charm and Bottom Mesons
4.1 LECs from Charm Meson Spectrum .......................... 99
4.2 Masses of the charm mesons within HHChPT ................. 101
4.3 Linear combinations of the low energy constants .............. 106
4.4 Results and Discussion ........................................ 108
4.5 Prediction for the Spectrum of Odd- and Even-Parity Bottom
Mesons ............................................................. 115

5 Short-range interactions between heavy mesons in framework
of EFT .............................................................. 126
5.1 Uncoupled Channel .............................................. 127
5.2 Two-body scattering with a narrow resonance ............... 137
5.3 Coupled Channels .............................................. 145
5.4 Coupled channels with a narrow resonance .................. 155

6 Decays of mesonic molecules in EFT ............................ 163
6.1 Width of the $X(3872)$ .......................................... 164
6.2 Line shapes of the $X(3872)$ ................................... 183
   6.2.1 Line shapes of the strong $\bar{D}^0 D^{*0}$ scattering channel ... 185
   6.2.2 Line shapes of the resonance coupled to the $\bar{D}^0 D^{*0}$
   scattering channel ............................................. 188

7 Conclusions ......................................................... 194

Bibliography .......................................................... 200
List of Figures

1.1 The gluonic flux tube generated between two heavy color sources is observed by the lattice simulation [5]. 19
1.2 The light mesons form an octet in the limit of flavor symmetry in which all light quarks have the same mass. The labels $S$ and $I_z$ refer to the strangeness and isospin $z$-component quantum numbers respectively. 24
1.3 Colored boxes represent charmonium states that have been discovered before (blue boxes) and after (red boxes) the $B$-factories. The (missing) charmonium states predicted by potential models are presented by green boxes [6–8]. 30

2.1 The $\sigma$-model potential with $m^2 > 0$. In this case, the particles $\pi$ and $\sigma$ have same masses $m_{\pi}^2 = m_{\sigma}^2 = m^2$. However, there is no experimental evidence for parity-doubling. This Figure is taken from [31]. 49
2.2 The $\sigma$-model potential with $m^2 < 0$. In physics, this is called the Mexican hat potential. In this case, there are two modes: (1) massive mode which corresponds to small expansion about the minimum in the radial directions (oscillation in magnitude), this gives mass to $\sigma$-particle, $m_{\sigma}^2 = -2m^2$, and (2) massless mode (Goldstone mode) corresponds to the small tangential expansion around the minimum, $m_{\pi}^2 = 0$. The solid circle is known as the chiral circle. The dynamics of the Goldstone bosons at low energy are constrained by the chiral circle, read discussion on nonlinear $\sigma$-model. This Figure is taken from [31]. 50
2.3 This tilted Mexican hat potential reflects both the spontaneous symmetry breaking and the explicit symmetry breaking. The vacuum is shifted by adding the linear term in $\sigma$. This term is proportional to quark mass which is generated by its coupling to the Higgs field. This Figure is taken from [31]. 53
2.4 The pionic one-loop graph (right) derived from $L_2$ is of order $O(Q^4)$. This loop graph can be renormalized by adding the tree level terms (left) derived from $L_4$. The figure is taken from [29].

2.5 Blue and red boxes represent the charm meson states of $\frac{1}{2}^-$ and $\frac{3}{2}^+$ doublets respectively. The hyperfine splitting of the states is explained by the finite mass of the charm quark. All charm meson masses are taken from Table 1.4 excluding the mass of the charged nonstrange meson $D^+_0$ which is reported by the FOCUS collaboration with a large error [57].

3.1 The self-energy diagrams for the $H_1$ and $H^*_1$ fields.

3.2 The self-energy diagrams for the $S_1$ and $S^*_1$ fields.

4.1 The self-energy diagrams for the ground-state fields $H$.

4.2 The self-energy diagrams for the excited-state fields $S$.

4.3 The combination $\eta_S$ plotted against $g'$. The central value is represented by the solid line. The experimental errors are shown by the dashed lines. The theoretical uncertainty is a constant $\pm 5$MeV.

4.4 The combination $\xi_S$ plotted against $g'$. The experimental uncertainties are shown by dashed lines surrounding the central values and an estimate theoretical uncertainty is shown by dot-dashed line.

4.5 The combination $L_S$ plotted against $g'$. The notation is the same as in Fig. 4.4.

4.6 The combination $T_S$ plotted against $g'$. The notation is the same as in Fig. 4.4.

4.7 The solid line represents the central value of the splitting $m_{B^*} - m_B$. The associated uncertainties, which include the experimental errors of the charm meson masses and the coupling constants and the error from the input parameter $\frac{m_c}{m_b}$, are given by the dashed lines. The dot-dashed line represents an estimate theoretical uncertainty.

4.8 The solid line represents the central value of the splitting $m_{B_s} - m_B$. The notation is the same as in Fig. 4.7.

4.9 The solid line represents the central value of the splitting $m_{B^*_s} - m_B$. The notation is the same as in Fig. 4.7.

4.10 The solid line represents the central value of the splitting $m_{B_0} - m_B$. The notation is the same as in Fig. 4.7.
4.11 The solid line represents the central value of the splitting $m_{B_s^0} - m_B$. The notation is the same as in Fig. 4.7. 

4.12 The solid line represents the central value of the splitting $m_{B_0^*} - m_B$. The notation is the same as in Fig. 4.7. 

4.13 The solid line represents the central value of the splitting $m_{B_s^*} - m_B$. The notation is the same as in Fig. 4.7. 

5.1 The basic loop integral of $\bar{D}^0 D^0$. 

5.2 The contribution of contact and bubble diagrams to the scattering matrix. 

5.3 (left) The position of the pole $E_B$ in complex energy plane. As shown the cut runs from $E = 0$ to $\infty$. (right) The position of the poles in momentum complex plane. 

5.4 (left) The position of the resonance, $E_R - \frac{i}{2} \Gamma_R$, is shown in the energy plane. The cut runs from $E = 0$ to $\infty$. (right) In the momentum plane, as $p^-$ is very close to the physical axis, it corresponds to the physical resonance. The second pole $p^+$ is far from the physical axis. It corresponds to an unphysical solution, $E_R + \frac{i}{2} \Gamma_R$, in the $E$-plane. 

5.5 As we vary the mixing angle, poles move on the complex plane as shown by arrows. For $\phi = \frac{\pi}{2}$, a pole is located on sheet II. It corresponds to a virtual state of the upper channel. This pole lies below the upper elastic threshold on the real axis in the $E$-plane as shown by the solid circle. As we vary the mixing angle slightly to allow a small mixing with the lower channel, the pole which is located on the pure imaginary part on sheet II moves toward the physical axis as shown by the symbol $\oplus$ in the $p$-plane. This pole reflects the existence of a resonance state that can decay to the particles in the lower channel. In the $E$-plane, the resonance state is denoted by the symbol $\oplus$. 

5.6 The symbol $\oplus$ denotes a pole that is located on sheet II. This pole corresponds to a Breit-Wigner resonance. The symbol $\otimes$ denotes a pole that is located on sheet III. This corresponds to a Breit-Wigner resonance. 

6.1 This Figure represents the basic loop diagram for the self-energy of the $X(3872)$. The solid line represents $\bar{D}^0$ and the double solid line represents $D^{*0}$. 

6.2 In (a), the loop diagram with self-energy. In (b), the interference loop diagram. The notation is the same as in Fig. 6.1, with dashed lines representing a pion.
6.3 The functions $F^{(a)}$ (solid line) and $F$ (dashed line) represent the imaginary parts of the loop diagrams with and without a virtual pion respectively.

6.4 Line shapes of the $X(3872)$. The solid line represents line shapes for positive $\gamma_b$. The dashed line represents line shapes for negative $\gamma_b$. In our plots, we set the factor $B = 1\,\text{MeV}$ in the factorization expression given in Eq. (6.50).

6.5 Line shapes of the $X(3872)$ for $E_X = 0.5\,\text{MeV}$ and $g' = 0.05$ are plotted with $B = 1\,\text{MeV}$. The solid line represents the line shape for $a_{bg} = 0\,\text{MeV}^{-1}$, the dashed line represents the line shape for $a_{bg} = 0.005\,\text{MeV}^{-1}$, and the dot-dashed line represents the line shape for $a_{bg} = -0.005\,\text{MeV}^{-1}$.

6.6 Line shapes of the $X(3872)$ for $E_X = -0.5\,\text{MeV}$ and $g' = 0.05$ are plotted with $B = 1\,\text{MeV}$. The solid line represents the line shape for $a_{bg} = 0\,\text{MeV}^{-1}$, the dashed line represents the line shape for $a_{bg} = 0.005\,\text{MeV}^{-1}$, and the dot-dashed line represents the line shape for $a_{bg} = -0.005\,\text{MeV}^{-1}$.
### List of Tables

1.1 The six flavors of quarks with their quantum numbers. The $u$, $d$, and $s$ masses are given in the $\overline{\text{MS}}$ scheme at a renormalization scale of $\mu = 2$ GeV. The charm and bottom $\overline{\text{MS}}$-masses are evaluated at their own scale, i.e. $m_c(m_c)$ and $m_b(m_b)$. The top quark mass results from lepton and jets channels in ATLAS and CMS experiments at CERN [1]. .................................................. 17

1.2 The lightest mesons with their masses, quark contents, and spin-parity. ................................................................. 22

1.3 The $K$ mesons with their masses, quark contents, and spin-parity. ................................................................. 23

1.4 The listed charmed meson states have been used in this thesis. 26

1.5 The masses are taken from Particle Data Group [1]. ............... 29

1.6 The masses are taken from Particle Data Group [1]. ............... 31

1.7 The possible mesonic molecules in charm sector that are predicted by Tornqvist using a potential model with one-pion-exchange [109]. ................................................................. 36

2.1 In the limit $m_Q \to \infty$, the spin of the heavy quark decouples from the spin of the light degrees of freedom. In this case, the heavy-light mesons can be classified according to the parity and total angular momentum of the light degree of freedom, $S^p_l$ [36]. ................................................................. 67

3.1 The values of $h$ are obtained in the heavy quark limit. In extracting values of $h$, we have used $f = 92.4 \text{MeV}$, $m_{\pi} = 140 \text{MeV}$, $m_{D^{\pm}} = 1869.61 \text{MeV}$, and $m_{D^{*\pm}} = 2010.26 \text{MeV}$ [1]. 95

4.1 The masses of $H_1$ ($H_1^*$) are obtained by taking the isospin average of $D^0$ and $D^\pm$ ($D^{*0}$ and $D^{*\pm}$). ................................................................. 113
4.2 The charm and bottom $\overline{\text{MS}}$-masses are evaluated at their own scale, i.e. $m_c(m_c)$ and $m_b(m_b)$. In [1], the $\overline{\text{MS}}$ values are converted to the pole scheme. The ratio of charm and bottom masses obtained from the pole mass is close to the ratio of the pseudoscalar charm and bottom mesons $\frac{m_D}{m_B} = 0.35$. In the kinetic mass scheme, the charm and bottom masses are evaluated at $\mu = 1\text{GeV}$ [65].
Abstract

We study the masses of the low-lying charm and bottom mesons within the framework of heavy-hadron chiral perturbation theory. We work to third order in the chiral expansion, where meson loops contribute. In contrast to previous approaches, we use physical meson masses in evaluating these loops. This ensures that their imaginary parts are consistent with the observed widths of the D-mesons. The lowest odd- and even-parity, strange and nonstrange mesons provide enough constraints to determine only certain linear combinations of the low-energy constants (LECs) in the effective Lagrangian. We comment on how lattice QCD could provide further information to disentangle these constants. Then we use the results from the charm sector to predict the spectrum of odd- and even-parity of the bottom mesons. The predicted masses from our theory are in good agreement with experimentally measured masses for the case of the odd-parity sector. For the even-parity sector, the $B$-meson states have not yet been observed; thus, our results provide useful information for experimentalists investigating such states. The near degeneracy of nonstrange and strange scalar $B$ mesons is confirmed in our predictions using HHChPT. Finally, we show why previous approaches of using HHChPT in studying the mass degeneracy in the scalar states of charm and bottom meson sectors gave unsatisfactory results.

Interactions between these heavy mesons are treated using effective theories similar to those used to study nuclear forces. We first look at a strongly-interacting channel which produces a bound or virtual state and a dimer state which couples weakly to a weakly-interacting channel to produce a narrow resonance. We also look at the short-range interactions in two channels. We consider two cases: two channels where one has a strong $s$-wave interaction which produces bound or virtual states, and a dimer state which couples weakly to weakly-coupled channels which in turn can produce narrow resonances. For each of these systems, we use well-defined power-counting schemes. The results can be used to investigate resonances in the charmonium and bottomonium systems. We demonstrate how the method can be applied to the $X(3872)$.
The widths of the $X(3872)$ for decay processes to $\bar{D}^0 D^{*0}$ and $\bar{D}^0 D^0\pi$ are calculated. We use these results to obtain the line shapes of the $X(3872)$ under different assumptions about the nature of this state.
Declaration

The University of Manchester

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Faculty: Engineering and Physical Sciences

Thesis Title: Effective Field Theories of Heavy-Quark Mesons

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Finally, I would like to devote these lines for my father who I wished to be with me today. His love and support have made it possible for me to pursue this degree and made me the person I am today. Father, may Allah have mercy on you, for all you do I will be forever grateful.

Above all, the praises and deepest thanks are due to Allah, the Almighty, for all the blessings that have been given to me.
To my father (may Allah have mercy on him),
   To my mother,
   To my wife and my son Rakan,
   To my brothers and sisters.
Chapter 1

Introduction

Quantum Chromodynamics QCD is the field theory of the strong interaction. It describes the dynamics of quarks and gluons. The gauge symmetry of QCD is the non-Abelian $SU(3)_c$, where $c$ stands for the color charge of the strong interaction. Quarks (antiquarks) carry color (anticolor) charges and transform invariantly under $SU(3)$ rotation in color space. Quarks are spin-$\frac{1}{2}$ fundamental particles and come in six flavors with different masses and quantum numbers, see Table 1.1. The massless gluons, force carriers of strong interaction, also carry color charges themselves. As a consequence, the anti-screening of color charge is more dominant in QCD. To understand it, let us recall the lowest order of the QCD running coupling $\alpha_s(\mu^2)$,

$$\alpha_s(\mu^2) = \frac{1}{4\pi} \beta_0 \ln \left( \frac{\mu^2}{\Lambda_{\text{QCD}}^2} \right),$$

(1.1)
<table>
<thead>
<tr>
<th>Property of quark</th>
<th>up (u)</th>
<th>down (d)</th>
<th>strange (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (MeV)</td>
<td>$2.3^{+0.7}_{-0.5}$</td>
<td>$4.8^{+0.5}_{-0.3}$</td>
<td>$93.5 \pm 2.5$</td>
</tr>
<tr>
<td>Electric charge</td>
<td>$+\frac{2}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>$-\frac{1}{3}$</td>
</tr>
<tr>
<td>Isospin</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>Isospin z-component</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>Strangeness</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Property of quark</th>
<th>charm (c)</th>
<th>bottom (b)</th>
<th>top (t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (GeV)</td>
<td>$1.275 \pm 0.025$</td>
<td>$4.180 \pm 0.030$</td>
<td>$173.29 \pm 0.23 \pm 0.92$</td>
</tr>
<tr>
<td>Electric charge</td>
<td>$+\frac{2}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>$+\frac{2}{3}$</td>
</tr>
<tr>
<td>Charm</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Bottomness</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
<td>Topness</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1.1: The six flavors of quarks with their quantum numbers. The $u$, $d$, and $s$ masses are given in the $\overline{\text{MS}}$ scheme at a renormalization scale of $\mu = 2$ GeV. The charm and bottom $\overline{\text{MS}}$-masses are evaluated at their own scale, i.e. $\overline{m}_c(\overline{m}_c)$ and $\overline{m}_b(\overline{m}_b)$. The top quark mass results from lepton and jets channels in ATLAS and CMS experiments at CERN [1].

where $\mu$ is the energy-scale parameter known as the renormalization point, and $\Lambda_{\text{QCD}}$ is the QCD dynamical scale that defines nonperturbative side of the theory. The numerical value of $\Lambda_{\text{QCD}}$ can be estimated theoretically or determined from experiments$^1$. The coefficient $\beta_0$ is related to the number of active flavors as follows,

$$
\beta_0 = \frac{1}{16 \pi^2} \left(11 - \frac{2}{3} N_f\right),
$$

where $\beta_0 > 0$ for $N_f \leq 16$.

$^1$ Experimentally, $\Lambda_{\text{QCD}} \sim 200$ MeV.
Hence, QCD as a non-Abelian field theory is characterized by having two distinct behaviours depending on the energy scale:

- At high energies (short distances) \( \Lambda_{QCD} < \mu \); the strong coupling constant \( \alpha_s \) decreases logarithmically as quarks approach each other. At very high energies \( (\mu \to \infty) \), quarks behave as free particles. This is known as asymptotic freedom. In this limit, a perturbative treatment is reliable where the physical observables can be expanded in terms of \( \alpha_s \). The perturbative (high energy) side of QCD has been confirmed in many experiments.

- In sharp contrast, QCD develops a nonperturbative nature at low energies (large distances) \( \Lambda_{QCD} > \mu \). Here the coupling constant becomes strong and quarks, antiquarks, and gluons are confined. This is known as quark or color confinement. This, in fact, is a signature of gluon self-interactions that cause the field lines between (separated) quark and antiquark, for instance, to be squeezed into a flux tube. This gluonic flux tube generated between two heavy color sources is shown in Fig. 1.1 from a lattice simulation [5]. Therefore, at low energies quarks, antiquarks, and gluons are confined. Experimentally, quarks and gluons have not been seen as isolated objects. Instead colorless hadrons are detected. This indeed provides a strong evidence for the color-confinement hypothesis.

In the case of large coupling constant, the direct solution of QCD is complicated. There are, however, two complimentary approaches that have been established to circumvent this problem. One of them is more computational and it is known as Lattice QCD, usually abbreviated to LQCD. The other approach is called an effective field theory (EFT) and can be loosely considered as a phenomenological theory with a Lagrangian which contains a
Figure 1.1: The gluonic flux tube generated between two heavy color sources is observed by the lattice simulation [5].
large number of unknown bare constants. These constants are called Low Energy Constants and abbreviated to LECs. LECs can be determined by experiments and in the absence of experimental data, they can be estimated from Lattice simulation. Thus, both approaches, i.e. LQCD and EFT, are complimentary.

Effective field theories are constructed to probe and interpret the dynamics of hadrons by exploiting the symmetries of QCD. In this thesis, we will utilize effective field theories to investigate the properties of mesons containing a single heavy quark, in particular charm and bottom mesons. The interactions between these heavy mesons will be studied within the framework of effective field theories. This will enhance our understanding of the nature of some newly observed charmonium and bottomonium resonances, including the $X(3872)$.

Before proceeding, we will give a nontechnical review of the ordinary and exotic mesons. In accordance with color confinement, which is confirmed experimentally, observed hadrons must be color singlet states. Mesons are bosonic hadrons and made of a quark/antiquark pair. They have integer spins and zero baryon number.

In QCD, the combinations of the quarks and antiquarks like $q\bar{q}$ are allowed. With exception to the lightest meson states ($\pi$, $K$, $\eta$), mesons with quark contents of the form $q\bar{q}$ are well described by the Quark Model and are called ordinary mesons. The Quark Model has been applied since the mid-1960’s to classify the observed spectra of hadrons in terms of quarks [2–4]. In addition to $q\bar{q}$, other combinations of $q$ and $\bar{q}$ that give color singlet states, for example $qq\bar{q}\bar{q}$, are also allowed in QCD. Such states are called
Mesons are commonly classified based upon their masses. To this end, let us first classify their constituent quarks. As shown in Table 1.1, there are large gaps between the masses of quarks. Thus, by using QCD-scale $\Lambda_{\text{QCD}}$, quarks can be classified into three light quarks $u$, $d$, and $s$, and three heavy quarks $c$, $b$, and $t$. With regards to the heavy quarks, the charm is not that massive relative to the bottom and top quarks.

### 1.1 Ordinary Mesons

The simplest combination of quarks and anti-quarks produces what are so-called ordinary mesons. They can be classified into $J^P_C$ multiplets where $J$ is total angular momentum, $P$ is intrinsic parity, and $C$ is charge conjugation. The quantum numbers $P$ and $C$ can be defined for meson states using $(-1)^{1+l}$ and $(-1)^{l+s}$ respectively, where $l$ is the orbital angular momentum between quarks and $s$ is their total spin. The total spin is either 1 for aligned spins or 0 for antialigned spins.

For multiple quark combinations, one can produce several meson states with different spin-parities. For example, the lowest lying mesons are ones with no orbital angular momentum and with $s = 0$. This corresponds to ground state mesons that are labeled by $J^{PC} = 0^{-+}$. However, the excited states, with the same quark contents, have total spin $s = 1$. Thus, they have different spin-parity quantum number, $J^{PC} = 1^{+-}$.

Based upon the quark masses presented in Table 1.1, ordinary mesons
1.1.1 Light Mesons

Light mesons are constructed from the combinations of $u$, $d$, and $s$ quarks. As the strange quark is much heavier than the up and down quarks, light mesons (or any mesons) can also be classified into non-strange and strange mesons.

The ground state of light mesons is defined by $J^P = 0^-$, in which $l = 0$ for the system and spins of the quarks are antialigned $s = 0$. In the next Chapter, we will show that these states are thought to be the pseudoscalar Goldstone bosons resulting from the dynamical breaking of chiral symmetry.

The nonstrange ground state mesons are pions that are made up of $u$ and $d$ quarks. While the charged pions $\pi^\pm$ consist of combinations of different quark flavors, the neutral pion $\pi^0$ is built from identical combinations, see Table 1.2. Mesons containing identical combinations of $q\bar{q}$-pair are called quarkonia or unflavored mesons. The neutral pion is the lightest quarkonium.

In Table 1.2, the quantum mechanical mixture for $\pi^0$ implies that the neutral...

<table>
<thead>
<tr>
<th>Light meson states</th>
<th>Quark content</th>
<th>Mass (MeV)</th>
<th>$J^P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^+$</td>
<td>$u \bar{d}$</td>
<td>140</td>
<td>$0^-$</td>
</tr>
<tr>
<td>$\pi^0$</td>
<td>$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$</td>
<td>135</td>
<td>$0^{--}$</td>
</tr>
<tr>
<td>$\pi^-$</td>
<td>$d\bar{u}$</td>
<td>140</td>
<td>$0^-$</td>
</tr>
</tbody>
</table>

Table 1.2: The lightest mesons with their masses, quark contents, and spin-parity.
Table 1.3: The $K$ mesons with their masses, quark contents, and spin-parity.

<table>
<thead>
<tr>
<th>Light meson states</th>
<th>Quark content</th>
<th>Mass (MeV)</th>
<th>$J^P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K^0$ ($\bar{K}^0$)</td>
<td>$d\bar{s}(s\bar{d})$</td>
<td>498</td>
<td>$0^-$</td>
</tr>
<tr>
<td>$K^+$($K^-$)</td>
<td>$u\bar{s}$ ($s\bar{u}$)</td>
<td>494</td>
<td>$0^-$</td>
</tr>
</tbody>
</table>

pion spends half its time as $u\bar{u}$ and half its time as $d\bar{d}$. This mixing can only occur for quarks with a small mass splitting.

In the isospin limit ($m_u = m_d$), $\pi^\pm$ and $\pi^0$ become degenerate states that transform invariantly under isospin symmetry. This symmetry is broken by the electromagnetic interaction as $u$ and $d$ have different charges. The size of symmetry breaking is of the order of the mass splitting ($m_d - m_u \simeq 3\text{ MeV}$).

As the lowest meson states, pions cannot decay strongly into other hadrons, i.e. this is kinematically forbidden. Instead, they either decay electromagnetically with conservation of quark number, $\pi^0 \to \gamma\gamma$, or weakly with violation of the quark number, $\pi^+ \to \mu^+\nu_\mu$. Thus, they are long-lived particles.

The other sector of the ground state of the light mesons is the strange mesons, also called $K$-mesons. These mesons are formed by the coupling of the $s$-quark to one of lightest quarks ($u$ and $d$). As shown in Table 1.3, there are four combinations of the strange quark with $u$ and $d$ quark that yield four types of $K$ mesons.

As $m_u \sim m_d \ll \Lambda_{\text{QCD}}$ and $m_s \lesssim \Lambda_{\text{QCD}}$, nonstrange and strange sectors of the ground state become degenerate and transform into each other by the generators of the $SU(3)_f$ group, where $f$ refers to flavor, see Fig 1.2.

The next excited state (light vector meson) is defined with $l = 0$ and
aligned spins of the light quarks $s = 1$. Examples of the vector mesons are $\rho(776)$, $K^*(892)$, and $\omega(783)$, where numbers between parentheses represent their masses in MeV. In the limit $m_u = m_d = m_s$, the degeneracy of the light vector mesons can be shown in a graph similar to Fig 1.2 for the ground state mesons.

Figure 1.2: The light mesons form an octet in the limit of flavor symmetry in which all light quarks have the same mass. The labels $S$ and $I_z$ refer to the strangeness and isospin $z$-component quantum numbers respectively.

1.1.2 Heavy-light Mesons

The coupling of the charm and bottom quarks to one of the light quarks $u$, $d$, and $s$ form the charm and bottom mesons respectively. As these mesons are characterized by having non-zero heavy quark quantum number (charm or bottom), their ground states can only decay via weak interaction, in which
quark quantum numbers are not conserved. The properties of the heavy-light systems can be successfully described by effective field theories, either by Heavy Quark Effective Theory (HQET) which is formulated in terms of quark and gluon degrees of freedom or by one formulated in terms of hadronic degrees of freedom, Heavy Hadron Chiral Perturbation Theory (HHχPT) [9]. The later will be considered in this thesis to study the properties of charm and bottom mesons.

Let us first consider charm mesons, also called $D$-mesons or open charm mesons. Due to large mass difference between strange and nonstrange light quarks, the $D$-mesons can be classified into two sectors: non-strange and strange charmed mesons. Nonstrange $D$-mesons transform as a doublet under isospin symmetry, in the limit ($m_u = m_d$), and together with the strange $D$-meson they form an $SU(3)_f$ triplet, in the limit ($m_u = m_d = m_s$). In Table 1.4, we list the masses of charm mesons that we use in this thesis. Due to the $s$-wave strong decay of the excited nonstrange to the ground state mesons, the masses of excited nonstrange mesons are reported with large uncertainties.

By examining charm meson mass spectra, one finds that strange and nonstrange splittings of the well determined states, i.e. $J^p = 0^-$ and $J^p = 1^-$, are consistent with the size of the $SU(3)_f$ breaking, $O(100 \text{ MeV})$.

However, this is not the case for the scalar sector in which the splitting $m_{D_{s0}^*} - m_{D_{0}^*} = -0.3$ MeV is inconsistent with the size of the $SU(3)_f$ breaking. The measured mass of $D_{s0}^{*\pm}$ is in serious disagreement with quark models predictions in which the expected mass of strange scalar charm meson spread from 2400 to 2500 MeV [11–15]. The large predicted mass of $D_{s0}^{*\pm}$
Table 1.4: The listed charmed meson states have been used in this thesis.

<table>
<thead>
<tr>
<th>$J^p = 0^-$</th>
<th>$D^0$</th>
<th>$D^\pm$</th>
<th>$D_s^\mp$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (MeV)</td>
<td>1864.84 ± 0.07</td>
<td>1869.61 ± 0.10</td>
<td>1968.30 ± 0.11</td>
</tr>
<tr>
<td>Full width Γ</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$J^p = 1^-$</th>
<th>$D^*(2007)^0$</th>
<th>$D^*(2010)^\pm$</th>
<th>$D_s^{*\mp}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (MeV)</td>
<td>2006.96 ± 0.10</td>
<td>2010.26 ± 0.07</td>
<td>2112.1 ± 0.4</td>
</tr>
<tr>
<td>Full width Γ</td>
<td>&lt; 2.1 MeV</td>
<td>96 ± 22 keV</td>
<td>&lt; 1.9 MeV</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$J^p = 0^+$</th>
<th>$D_s^0(2400)^0$</th>
<th>$D_0^+$</th>
<th>$D_s^{0*(2317)^\pm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (MeV)</td>
<td>2318 ± 29</td>
<td>...</td>
<td>2317.7 ± 0.6</td>
</tr>
<tr>
<td>Ref.</td>
<td>PDG [1]</td>
<td>...</td>
<td>PDG [1]</td>
</tr>
<tr>
<td>Full width Γ</td>
<td>267 ± 40 MeV</td>
<td>...</td>
<td>&lt; 3.8 MeV</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$J^p = 1^+$</th>
<th>$D_s^+(2430)^0$</th>
<th>$D_s^{*\pm}$</th>
<th>$D_s^{*(2460)^\pm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (MeV)</td>
<td>2427 ± 36</td>
<td>...</td>
<td>2459.5 ± 0.6</td>
</tr>
<tr>
<td>Ref.</td>
<td>BELLE [55]</td>
<td>...</td>
<td>PDG [1]</td>
</tr>
<tr>
<td>Full width Γ</td>
<td>384^{+107}_{75} ± 74 MeV</td>
<td>...</td>
<td>&lt; 3.5 MeV</td>
</tr>
</tbody>
</table>

in the quark models can be lowered by including the effect of the hadronic loops, we refer the interested reader to Refs. [89, 90]. Thus, to produce the near mass degeneracy in the scalar charm meson sector, the mass shift in the strange scalar due to the self-energy contributions has to be large in comparison with the mass shift in the nonstrange scalar by an amount comparable to the strange quark mass, i.e. 80 − 100 MeV.

Since the masses of $D_s^{*\mp}$ and $D_s^{*\pm}$ lie below the $DK$ and $D^*K$ thresholds, some exotic explanations, such as hadronic molecule and $cq\bar{q}s\bar{q}$ tetraquark
state, have been introduced to understand their structures [19–24]. It is worth mentioning that HH$\chi$PT works better toward the understanding of the masses and widths of $D^{*\pm}$ and $D^{\pm}$ if they are considered as conventional states, i.e. made of $c\bar{s}$ [17].

The equality of mass differences (hyperfine splittings) between $0^\pm$ and $1^\pm$ multiplets

\[ m_{D^{*0}} - m_{D^0} = 142.12 \pm 0.1 \text{ MeV}, \]
\[ m_{D^{*+}} - m_{D^+} = 140.65 \pm 0.1 \text{ MeV}, \]
\[ m_{D^{*+}} - m_{D^{+}} = 143.8 \pm 0.4 \text{ MeV}, \]

is understood by the heavy quark symmetry of the heavy-light meson systems, for (technical) details see the last section of Chapter 2.

Experimentally, $D$-mesons can be produced either by the decay of $B$-mesons or by $e^+e^-$ annihilation experiments that are operated with energies lying at the mass thresholds of charm mesons. For example, the charm mesons $D^+$ and $D^-$ can be produced by operating an $e^+e^-$-experiment at the center of mass energy $\sqrt{s} = 4.02 \text{ GeV}$ [27]:

\[ e^+e^- \rightarrow D^+D^- + \text{anything}. \]

Furthermore, $D^+$ can also be produced by the following decay channel of the pseudoscalar charged $B$-meson

\[ B^- \rightarrow D^0 \pi^- \rightarrow D^+ \pi^- \]
In fact, many experiments (BELLE, FOCUS, CLEO, and BABAR) have studied the $B$-meson decay to optimize our understanding of $D$-mesons. Moreover, LHC$_b$ (the beauty experiment) is essentially designed to find new physics by investigating the properties of $B$-mesons. This in turn provides precious information related to the heavy charmed mesons.

For the $B$-mesons, we present the available data on the masses of the negative (odd) parity sector, see Table 1.5. While the isospin splitting is very small for the $B$-meson ($m_{B^0} - m_{B^+} = 0.32 \pm 0.06 \text{MeV}$), the flavor splitting is of expected size, ($m_{B_s^0} - m_B = 87.35 \pm 0.23 \text{MeV}$). As expected from the heavy quark symmetry, the observed hyperfine splitting within $B$ meson states, e.g. $m_{B^{*+}} - m_{B^+} = 45 \pm 0.4 \text{MeV}$, is smaller than the relevant splitting within charm meson states, e.g. $m_{D^{*+}} - m_{D^+} = 140.65 \pm 0.1 \text{MeV}$.

The positive parity bottom states, i.e. scalar and axial vector sectors, have not yet been observed. In this thesis, we use HH$\chi$PT to predict the full spectra of the odd and even parity bottom meson states. The results are in good agreement with masses given in Table 1.5, see Chapter 4 for details.

Experimentally, $B$-mesons can be produced in $B$-factories in which $e^+e^-$ annihilation experiments are operated at energies comparable to the masses of some bottomonium states. The produced $B$-mesons can decay into lighter $B$-mesons, $D$-meson states, or into charmonium states such as $J/\psi$.

### 1.1.3 Heavy-Quark Mesons

Here, we only review heavy quarkonium states, i.e. charmonium $c\bar{c}$ and bottomonium $b\bar{b}$. Such states have null charm and bottom quantum numbers.
$J^p = 0^-$

<table>
<thead>
<tr>
<th>Mass (MeV)</th>
<th>$B^0$</th>
<th>$B^+$</th>
<th>$B_s^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5279.58 ± 0.17</td>
<td>5279.26 ± 0.17</td>
<td>5366.77 ± 0.24</td>
<td></td>
</tr>
</tbody>
</table>

$J^p = 1^-$

<table>
<thead>
<tr>
<th>Mass (MeV)</th>
<th>$B^*$</th>
<th>$B^*$</th>
<th>$B_s^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5325.2 ± 0.4</td>
<td>5325.2 ± 0.4</td>
<td>5415.4$^{+2.4}_{-2.1}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1.5: The masses are taken from Particle Data Group [1].

These quantum numbers are conserved in the strong and electromagnetic interactions.

Since the charm and bottom quarks are massive, one can use nonrelativistic quantum mechanics to predict the spectrum of charmonium and bottomonium. This approach gives so-called potential models. The potentials formulated in these models reproduce the asymptotic behaviour of QCD. In recent years, the effective field theory approach, i.e. Non Relativistic QCD (NRQCD), has been used to study the heavy quarkonium systems [9].

The first charmonium $J/\psi$ was discovered by two simultaneous experiments (SLAC and Brookhaven) in November 1974. Its discovery led scientists to extend the old version of the quark model with the light flavors ($u, d, s$) to include the first heavy quark flavor named charm and labeled by $c$. In fact, ten years before its discovery, charm was introduced by Bjorken and Glashow to account for the lepton-quark symmetry [16].

As $J/\psi$ is created in the annihilation of $e^+e^-$, it has a negative charge conjugation and is identified as $1^3S_1$. The ground state of the charmonium $1^1S_0$ is produced by the radiative decay of $J/\psi$ in the following process: $J/\psi(3097) \rightarrow \eta_c(2980) + \gamma$. In Fig. 1.3, we present the charmonium states that have been discovered as of 2014. Charmoinum discovered before oper-
Three years after discovering $J/\psi$, the first bottomonium state $\Upsilon(1S)$ was reported by Fermilab at mass $9460.30 \pm 0.26$ MeV [1]. The narrow width
of $\Upsilon(1S)$ was strong evidence of existence of a fifth quark ($b$), $\Gamma_{\Upsilon(1S)} = 54.02 \pm 1.25$ keV. A list of some bottomonium states are shown in the Table 1.6.

<table>
<thead>
<tr>
<th>Name</th>
<th>$J^P$</th>
<th>Mass (MeV)</th>
<th>Width (KeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Upsilon(1S)$</td>
<td>0$^-$</td>
<td>9460.30 ± 0.26</td>
<td>54.02 ± 1.25</td>
</tr>
<tr>
<td>$\Upsilon(2S)$</td>
<td>1$^-$</td>
<td>10023.26 ± 0.31</td>
<td>31.98 ± 2.63</td>
</tr>
<tr>
<td>$\Upsilon(1D)$</td>
<td>2$^-$</td>
<td>10163.7 ± 1.4</td>
<td>...</td>
</tr>
<tr>
<td>$\Upsilon(3S)$</td>
<td>1$^-$</td>
<td>10355.2 ± 0.5</td>
<td>20.32 ± 1.85</td>
</tr>
</tbody>
</table>

Table 1.6: The masses are taken from Particle Data Group [1].

1.2 Exotic $c\bar{c}$ Mesons

As stated before, after operating the $B$-factories many states have been identified as $c\bar{c}$ mesons with spectra which do not fit into the quark models predictions, see Fig. 1.3.

However, the discovery of the charged charmonium-like state $Z_c(3900)\pm$ is the first proof for the existence of exotic mesons. This state has been simultaneously reported by two groups (BESIII [76] and Belle [77]) with mass $M_Z \simeq 3.9 \text{GeV}/c^2$. The $Z_c(3900)\pm$ state is different from the neutral charmonium-like states, which are denoted as $X$ and $Y$ in Fig. 1.3. As a charged state, $Z_c(3900)\pm$ cannot be interpreted as a hidden-charmonium, $c\bar{c}$. A charged composite state must have at least four quarks as minimal constituents. Based on that, the quark contents of the $Z_c(3900)^+$ and $Z_c(3900)^-$ states can be identified as $c\bar{c}ud$ and $c\bar{c}d\bar{u}$ respectively. The decay transition
of $Z_c(3900)^±$ into the charmonium $J/ψ$ and charged pion $π^±$ supports its explanation as a system of four quarks.

In this section, we will briefly mention the complicated combinations of the quarks, antiquarks, and gluons that are allowed by QCD and form the structure of the exotic mesons:

- **Charmonium hybrids ($c\bar{c}g$):** are composite of a charm quark, a charm antiquark, and a dynamical gluon. The states $Y(4260)$, $Y(4360)$, and $Y(4660)$ shown by red boxes in Fig. 1.3 are identified as charmonium hybrids. This interpretation is based on the results obtained from calculating the spectra of charmonium hybrids using lattice gauge theory, see [6–8] and references therein.

- **Tetraquarks ($c\bar{c}qq$):** contain two quarks and two antiquarks. Tetraquarks can be further classified according to color structure:

  (i) It could be a compact neutral color state of quarks and antiquarks.

  (ii) It could have substructure. This provides three possibilities:

    (1) **Di-quarks and di-antiquarks:** are composite of colored clusters, i.e. $cq$ and $\bar{c}\bar{q}$. This possibility is problematic and many of its predicted states have not been observed [6–8].

    (2) **Hadro-charmonia ($ψ(c\bar{c})h(q\bar{q})$):** are composite of a charmonium and a light meson. This possibility of tetraquark mesons was introduced by Dubynskiy and Voloshin [6–8, 18]. Some newly discovered $c\bar{c}$ mesons characterized by having a single hadronic transition to a charmonium and a light meson. For example, the transitions of the $Y(4008)$
and $Y(4260)$ states to the charmonium $J/\psi$ with emission of two pions $\pi\pi$ indicates that these states could be hadro-charmonia containing $J/\psi$. As the $Y(4360)$ and $Y(4660)$ states decaying into $\psi(2S)\pi\pi$, they might be hadro-charmonia containing $\psi(2S)$ [6–8].

(3) **Meson molecules**: contain two ordinary mesons bound by the strong force between their colored constituent particles\(^2\), i.e. quarks and antiquarks. Some $c\bar{c}$ mesons are proposed to be charmonium molecules. For example, the vector $Y(4260)$ and the axial vector $X(3872)$ charmonia have been interpreted as $\bar{D}^0D_1$ and $\bar{D}^0D^{*0}$ molecules respectively. For the $b\bar{b}$ mesons sector, the bottomonium states $Z_b(10610)^\pm$ and $Z_b(10650)$ are proposed to be $B^*\bar{B}$ and $B^{*}\bar{B}^*$ molecules respectively [6–8].

It is worth mentioning that the nature of some newly observed $c\bar{c}$ mesons is not well understood. For example, there are many possibilities to identify the nature of the vector charmonium $Y(4260)$: it has been classified as a charmonium hybrid, hadro-charmonium and a charmonium molecule. This is also the case for the $X(3872)$. In addition to its identification as a hadronic molecule, its interpretation as a charmonium state, $\chi_{c1}(2P)$, or a compact tetraquark are not excluded, for details see [78, 103] and references therein.

In this thesis, effective field theory methods have been developed to investigate the properties of such mesonic states. As the $X(3872)$ is taken as an illustrative example in our work, we will review some of its experimental facts and discuss its theoretical background.

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\(^2\)This is similar to Van der Waals force that binds atoms in molecules.
1.2.1 Experimental and theoretical studies of the $X(3872)$

The $X(3872)$ was discovered in 2003 by Belle Collaboration [67] and then confirmed by the CDF, D0, and Babar Collaborations [68, 70, 71]. The $X(3872)$ was first observed in the charged and neutral decay modes of the $B$-meson, $B^\pm \to K^\pm X$ and $B^0 \to K^0 X$. It can also be produced in abundance in proton-proton ($pp$) collisions at center of mass energy 7 TeV at LHCb [72].

The width of the $X(3872)$ is determined to be below experimental resolution, $\Gamma_X < 1.2 \text{ MeV}$ at the 90% confidence level [1]. The radiative decay of the $X(3872)$, $X \to J/\psi\gamma$, favours the spin-parity assignment $1^{++}$ as recently confirmed by LHCb [73]. Other observed decay channels of the $X(3872)$ are $J/\psi\pi^+\pi^-$, $J/\psi\pi^+\pi^-\pi^0$, $D^0\bar{D}^{*0}$, $D^0\bar{D}^0\pi^0$, and $D^0\bar{D}^0\gamma$.

The combined averaged mass of the $X(3872)$ extracted from the $J/\psi\pi\pi$ mode [1],

$$M_X = 3871.69 \pm 0.17 \text{ MeV},$$

(1.2)

is very close to the threshold of $\bar{D}^0 D^{*0}$,

$$m_{D^0} + m_{D^{*0}} = 3871.8 \pm 0.122 \text{ MeV},$$

(1.3)

which suggests that the $X(3872)$ is a mesonic molecule with constituents $D^0\bar{D}^{*0}$.

The binding energy of the $X(3872)$ is

$$E_B = M_X - (m_{D^0} + m_{D^{*0}}) = -0.11 \pm 0.21 \text{ MeV}.$$
This indicates that the $X(3872)$ could be a bound state of $D^0\bar{D}^{*0}$ with negative $E_B$, or a resonance near the $D^0\bar{D}^{*0}$-threshold with positive $E_B$ as implied by the error bars.

The interpretation of the structure of the $X(3872)$ is still disputed. Beside its interpretation as a $D^0\bar{D}^{*0}$ molecule, the decay modes of the $X(3872)$ into $J/\psi\gamma$ and $J/\psi\pi\pi$ (this comes from $J/\psi\rho$) suggest that the $X(3872)$ could be a conventional charmonium state or a compact tetraquark state respectively. However, the molecule interpretation is prompted by the following experimental facts:

(i) The small binding energy and the spin-parity assignment $1^{++}$ suggests the existence of a strong $s$-wave coupling between the $X(3872)$ resonance and $D^0\bar{D}^{*0} + \bar{D}^0D^{*0}$.

(ii) The comparable branching fraction for the decay modes $J/\psi\pi^+\pi^-\pi^0$ and $J/\psi\pi^+\pi^-$ ($\text{Br}[X \rightarrow J/\psi\pi^+\pi^-\pi^0]/\text{Br}[X \rightarrow J/\psi\pi^+\pi^-] = 1.0 \pm 0.4 \pm 0.3$ [69]) supports the molecular picture and rules out the conventional charmonium $c\bar{c}$ explanation. Note that the decay modes $J/\psi\pi^+\pi^-\pi^0$ and $J/\psi\pi^+\pi^-$ proceed through $J/\psi\omega$ (isoscalar) and $J/\psi\rho$ (isovector), respectively. This indicates strong isospin violation. The isospin violation is mainly enhanced by the broad decay width of $\rho$, i.e. $\Gamma_\rho \approx 150\text{ MeV} \gg \Gamma_\omega \approx 8\text{ MeV}$.

(iii) The average ratio for the decay modes $D^0\bar{D}^{*0}$ and $J/\psi\pi^+\pi^-$ ($\text{Br}[X \rightarrow D^0\bar{D}^{*0}]/\text{Br}[X \rightarrow J/\psi\pi^+\pi^-] = 16.7 \pm 5.8$ [75]) suggests that the component $D^0\bar{D}^{*0}$ is large in the $X(3872)$ wave function.

In fact, the existence of a $D^0\bar{D}^{*0}$ bound state, which is a candidate for the
Table 1.7: The possible mesonic molecules in charm sector that are predicted by Tornqvist using a potential model with one-pion-exchange [109].

<table>
<thead>
<tr>
<th>Constituents</th>
<th>$J^P$</th>
<th>Mass (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D^0\bar{D}^{*0}$</td>
<td>$0^-$</td>
<td>$\approx 3870$</td>
</tr>
<tr>
<td>$D^0\bar{D}^{*0}$</td>
<td>$1^+$</td>
<td>$\approx 3870$</td>
</tr>
<tr>
<td>$D^{*0}\bar{D}^{*0}$</td>
<td>$0^{++}$</td>
<td>$\approx 4015$</td>
</tr>
<tr>
<td>$D^{*0}\bar{D}^{*0}$</td>
<td>$0^{-+}$</td>
<td>$\approx 4015$</td>
</tr>
<tr>
<td>$D^{*0}\bar{D}^{*0}$</td>
<td>$1^{+-}$</td>
<td>$\approx 4015$</td>
</tr>
<tr>
<td>$D^{*0}\bar{D}^{*0}$</td>
<td>$2^{++}$</td>
<td>$\approx 4015$</td>
</tr>
</tbody>
</table>

$X(3872)$, was already predicted in the early nineties by Nils Tornqvist with a mass very close to the $D^0\bar{D}^{*0}$-threshold, a spin-parity $J^P = 1^+$, and isospin-0 [108, 109]. Inspired by the deuteron, Tornqvist used a one-pion-exchange potential to predict the mesonic molecules (deusons) shown in Table 1.7. After the observation of the $X(3872)$, Eric Swanson’s calculation using a potential model that incorporates both one-pion-exchange and quark-exchange potentials supported the presence of a $D^0\bar{D}^{*0}$ weakly bound state with $J^{PC} = 1^{++}$ [110].

In Ref. [122], P. Wang and X. G. Wang found a $DD^*$ bound state from one-pion exchange within the framework of unitarized heavy meson chiral perturbation theory. In their approach, they treated the short-range interaction perturbatively, and found that the creation of a bound state, which corresponds to the $X(3872)$, was only due to the pion exchange and is not affected by the short-range interaction. This result has led Baru et al. [123] to investigate the $X(3872)$ within the framework of a nonrelativistic three-
body equation with nonperturbative pions. The short-range interaction was included to make the divergent equations, which result from including only one-pion exchange, well defined. They found that the short-range interaction is strong and should be treated nonperturbatively, unlike the case in [122].

If the $X(3872)$ is generated by a strong short-range interaction then one can use the binding energy to determine the large scattering length $a$ through the relation

$$E_B = -\frac{1}{2 M_{D^0\bar{D}^0} a^2},$$

where $M_{D^0\bar{D}^0} = 966.65$ MeV is the reduced mass of $D^0\bar{D}^0$. For $E_B = -0.11$ MeV, the scattering length is $a = \pm 13.51$ fm. The sign of the scattering length tells us if the $X$ is a bound or virtual state. While positive $a$ corresponds to a bound state, negative $a$ indicates $X$ is a virtual state.

The large scattering length allows us to determine some universal properties of the bound state such as the wave function of the $s$-wave molecular state which can be written as

$$\psi(r) = \frac{1}{\sqrt{2\pi a}} e^{-\frac{r}{a}},$$

where $r$ is the separation between the constituents of the $X(3872)$. This universal wave function has been used by Voloshin to predict the momentum distributions for the decay channels $D^0\bar{D}^0\pi^0$ [111] and $D^0\bar{D}^0\gamma$ [112].

The separation between the constituents of the hadronic molecule can be determined from the scattering length, $r_{\text{rms}} = \frac{a}{\sqrt{2}}$. For the case of the $X(3872)$, the separation between the $\bar{D}^0D^0$-pair is $r_{\text{rms}} = 9.55$ fm. This is very large in comparison with the size of typical hadrons $\sim O(1\text{ fm})$.  

37
However, the average ratio for the decay modes $\psi(2S)\gamma$ and $J/\psi\gamma$ ($R_\phi = \text{Br}[X \rightarrow \psi(2S)\gamma]/\text{Br}[X \rightarrow J/\psi\gamma] = 2.46 \pm 0.46 \pm 0.29 [74]$) indicates that the $X(3872)$ is not a pure $D^0\bar{D}^{*0}$ molecule. The measured ratio $R_\phi$ is consistent with the interpretation of the $X(3872)$ as either a mixture of $c\bar{c}$ with a $D^0\bar{D}^{*0}$ molecule or even a pure charmonium $c\bar{c}$.

Based on the above arguments, the $X(3872)$ physical state could be a mixture with a compact $c\bar{c}$ or tetraquark $c\bar{c}q\bar{q}$ component as well as the extended structure $D^0\bar{D}^{*0}$. To determine the admixture fraction of $D^0\bar{D}^{*0}$ molecular component in the $X(3872)$ state, one can apply Weinberg’s idea on the deuteron [113, 115]. In this case, Eq. (1.6) becomes

$$\psi(r) = \sqrt{\frac{1-Z}{2\pi R}} e^{-\frac{r}{R}}, \quad (1.7)$$

where $R = \frac{1}{\sqrt{-2M_{DD^*E_B}}}$. The variable $Z$ is known as the Weinberg factor. This factor represents the probability of the physical $|X\rangle$ to be in a bare compact state $|X_{\text{elem.}}\rangle$. The values of the Weinberg $Z$-factor as a probability are bound between zero, which reflects the molecular structure of the $X$, and unity, which indicates that the $X$ is an elementary state. To leading order in the effective-range expansion, the dependence of the scattering length on the Weinberg $Z$-factor is given by [116]

$$a = \frac{2(1-Z)}{2-Z} R + O\left(\frac{1}{\Lambda}\right), \quad (1.8)$$

where $\frac{1}{\Lambda}$ is the range of the force. The scale $\Lambda$ can be identified as the pion mass, see Chapter 5. If we knew the scattering length and binding energy,
the Weinberg Z-factor could be extracted and consequently the nature of the \(X(3872)\) could be determined. However, the scattering length is not measurable.

The above mentioned approach assumes that all particles are stable (i.e. the Z-factor is real). However, this method can be generalized to the case of unstable particles (resonances) [116, 117]. This was achieved by introducing a continuum counterpart to the factor Z. This new quantity, known as the spectral density of the bare state \(w(E)\), has been used to analyse the line shapes\(^3\) of the \(X(3872)\) for the decay channels \(D^0\overline{D}^0\pi^0\) and \(J/\psi\pi^+\pi^-\) [96, 97, 102]. The line shapes of the \(X(3872)\) for other observed decay channels were analysed using different theoretical methods [98, 99, 103–106].

Using line shapes for the decay modes \(D^0\overline{D}^0\pi^0\) and \(D^0\overline{D}^0\gamma\) (which come from \(D^{*0} \rightarrow D^0\pi^0\) and \(D^{*0} \rightarrow D^0\gamma\), respectively) requires careful analysis to understand the nature of the \(X(3872)\). As an illustration, Hanhart et al. [96] used the Flatté parameterization of the line shapes to analyse the existing data at that time from Belle on the decay mode \(D^0\overline{D}^0\pi^0\) [100]. A near-threshold enhancement was discovered in this mode in the decay \(B \rightarrow KD^0\overline{D}^0\pi^0\). The position of the peak, \(M_X = 3875.2 \pm 0.7^{+0.3}_{-1.6} \pm 0.8\) MeV, is nearly 4 MeV higher than the averaged mass of the \(X(3872)\) extracted from the \(J/\psi\pi\pi\) mode, see Eq. (1.2). The authors of [96] concluded that this data favored the \(X(3872)\) being a virtual state. Later, the same approach was applied by Kalashnikova and Nefediev [97] to new data from Belle [101], in which a new analysis for the \(D^{*0}\overline{D}^0\)-system showed that the position of

\(^3\)The line shape of a resonance is defined as the invariant mass distribution of its decay products.

39
the peak is at $M_X = 3872.6^{+0.5}_{-0.4} \pm 0.4$ MeV. The authors of [97] concluded that the new data from Belle prefers that the $X(3872)$ is a bound state with a sizable $c\bar{c} \, 2^3P_1$ charmonium component. This conclusion is completely different from the conclusion of [96]. This result is largely due to differences between the new higher-statistics data from Belle [101] and the older data [100].

Exploiting the closeness of $M_X$ to the $D^0\bar{D}^{*0}$-threshold, the authors of [87] applied the pionless EFT to calculate the binding energy of the $X(3872)$. Later, the extension of EFT to include the pions effect on the properties of the $X(3872)$ was undertaken by Fleming et al. in their seminal paper [88]. They calculated the partial width $\Gamma[X \to D^0\bar{D}^{*0}\pi^0]$ and found that the effect of the pion exchange can be treated perturbatively. In Refs. [87, 88], the $X(3872)$ state was assumed to be a $D^0\bar{D}^{*0}$ molecule.

Finally, it is worth mentioning that the lattice simulation has found a candidate for the $X(3872)$ state with mass about $11 \pm 7$ MeV below the $\bar{D}^0D^{*0}$-threshold, spin-parity $J^{PC} = 1^{++}$, and 0-isospin [121]. The lattice calculation of Ref. [121] was performed with a large pion mass, $m_\pi \approx 266$ MeV. This lattice calculation cannot provide any indication about the nature of the $X(3872)$ state, although calculations for a range of pion masses could be very helpful [123, 124].
Chapter 2

From QCD to Effective Theories

This Chapter introduces the essential ideas behind constructing effective field theories, the theoretical framework that will be used in this thesis.

It is convenient to start with the general structure of the QCD Lagrangian density [29]

\[ \mathcal{L}_{\text{QCD}} = \sum_{i=u,d,s,c,b,t} \bar{\psi}_i (i\gamma^\mu D_\mu - m_i) \psi_i - \frac{1}{4} G_{\mu\nu,a} G^{\mu\nu}_a + \ldots, \] (2.1)

where \( \psi_i \) (\( \bar{\psi}_i \)) represents the Dirac spinors of \( i \)-th quark with mass \( m_i \). The covariant derivative \( D_\mu \) is \( D_\mu = \partial_\mu - i g_s \sum_{a=1}^{8} A_{\mu,a} \frac{\lambda_a}{2} \), where \( \lambda_a \) are the Gell-Mann matrices. The gauge sector in the Lagrangian is represented by the non-Abelian strength tensor \( G_{\mu\nu,a} \), e.g. it describes gluon dynamics \( A_{\mu,a} \). The ellipsis in the Lagrangian represents terms that are not necessary in our
discussion, e.g. ghost and gauge fixing terms.

The Lagrangian, $\mathcal{L}_{\text{QCD}}$, is invariant under the local gauge symmetry $SU(3)_c$ in which all quark flavors experience the same strong force. The only term that allows us to distinguish between quark flavors is the mass term. One can express the first term of the Lagrangian in the following way:

$$\mathcal{L}^m_{\text{QCD}} = \overline{q}(i\gamma^\mu D_\mu - m_q)q + \overline{Q}(i\gamma^\mu D_\mu - m_Q)Q,$$

where $q = u, d, s$, $Q = c, b, t$, and the dynamical nonperturbative QCD scale is $\Lambda_{\text{QCD}} \sim 200 \text{ MeV}$. The purpose of writing $\mathcal{L}^m_{\text{QCD}}$ in this way is to investigate other symmetry patterns of QCD that emerge at some limits of quark masses. Symmetries play a crucial role in providing explanations for many results obtained in the lab. In other words, symmetries are essential tools to probe strongly interacting systems, and to examine their dynamics.

At low energy, the QCD Lagrangian becomes invariant under the following approximate global symmetries [29]:

- **Isospin symmetry**: in the limit $m_u = m_d = m_q$, both flavors $u$ and $d$ rotate continuously into each other under $SU(2)_I$ group. The consequence of this symmetry is nearly degenerate states for pions in the absence of the electromagnetic interactions.

- **Flavor symmetry**: in the limit $m_u = m_d = m_s = m_q$, light quark flavors $u, d,$ and $s$ transform into each other under $SU(3)_f$. Since $m_s > m_{u,d}$, this symmetry is broken down to isospin symmetry.

- **Chiral symmetry**: in the limit $m_q \to 0$, the left-handed spinor disen-
tangles from the right-handed spinor. It is dynamically broken by the vacuum.

- Heavy quark symmetry: in the limit $m_Q \to \infty$, the spin and flavor of a heavy quark become conserved quantum numbers, causing degeneracy between scalar (pseudoscalar) and axial vector (vector) states of charm and bottom mesons.

Below we briefly discuss chiral and heavy quark symmetries.

### 2.1 Chiral Symmetry

To discuss chiral symmetry of light quark flavors, we consider the massless part of Eq. (2.2),

$$\mathcal{L} = i \bar{q} \gamma^\mu D_\mu q.$$  \hspace{1cm} (2.3)

For simplicity, we start by considering a two-component spinor combining up and down quarks [29],

$$q = \begin{pmatrix} u \\ d \end{pmatrix},$$  \hspace{1cm} (2.4)

which contains 24 components representing the two flavor quarks. Each flavor is associated with three colors and four Dirac components. The Dirac spinor $q$ can be decomposed into the (chiral) components $q_L$ (left-handed) and $q_R$
(right-handed) using the following projection operators\textsuperscript{1}

\begin{align*}
P_L &= \frac{1}{2}(1 - \gamma_5), \quad (2.5) \\
P_R &= \frac{1}{2}(1 + \gamma_5). \quad (2.6)
\end{align*}

Thus,

\begin{align*}
q &= P_Lq + P_Rq \\
&= q_L + q_R. \quad (2.7)
\end{align*}

By expressing the quark field $q$ in terms of its chiral components, Eq. (2.3) becomes

\begin{equation}
\mathcal{L} = i(\bar{q}_{L,f}\gamma^\mu D_\mu q_{L,f} + \bar{q}_{R,f}\gamma^\mu D_\mu q_{R,f}). \quad (2.9)
\end{equation}

The terms in Eq. (2.9) are independent of each other, and it is straightforward to identify the symmetry group of this Lagrangian. This approximate symmetry is called chiral symmetry, and it is dominant at very low energies compared to the QCD energy scale, $\Lambda_{\text{QCD}}$.

The representation of the chiral symmetry group can be defined in terms

\textsuperscript{1} The properties of these projection operators are:

\begin{equation*}
P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_LP_R = P_RP_L = 0, \quad P_L + P_R = 1.
\end{equation*}

The chiral matrix $\gamma_5$ is given by $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. In the chiral representation, $\gamma_5$ is represented by

\begin{equation*}
\gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.
\end{equation*}
of the transformation operators,

\[ R = e^{-i \frac{\tau_a}{2} \theta_\mu^a}, \quad R \in SU(2)_R, \]

\[ L = e^{-i \frac{\tau_a}{2} \theta_{\mu}^a}, \quad L \in SU(2)_L, \]

where the generators \( \tau^a \) of an \( SU(2) \) group are given by the Pauli matrices,

\[ \tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.10} \]

In the chiral limit \( m_q \to 0 \), left and right handed chiral fields transform independently as a doublet under \( SU(2)_L \) and \( SU(2)_R \) respectively,

\[ q_L \to q'_L = L q_L = L \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \tag{2.11} \]

\[ q_R \to q'_R = R q_R = R \begin{pmatrix} u_R \\ d_R \end{pmatrix}. \tag{2.12} \]

Hence \( \mathcal{L} \) in Eq. (2.9) is invariant under the semi-simple chiral symmetry group \( SU(2)_L \times SU(2)_R \).

By using Noether’s theorem, the six associated conserved currents of this symmetry are

\[ R_{\mu}^a = \overline{q}_R(t, \vec{x}) \gamma_\mu \frac{\tau^a}{2} q_R(t, \vec{x}), \quad \partial^\mu R_{\mu}^a = 0 \]

\[ L_{\mu}^a = \overline{q}_L(t, \vec{x}) \gamma_\mu \frac{\tau^a}{2} q_L(t, \vec{x}), \quad \partial^\mu L_{\mu}^a = 0, \]
with charge densities defined as

\[ Q^a_L(t) = \int d^3x \, q^\dagger_L(t, \vec{x}) \frac{\tau^a}{2} q_L(t, \vec{x}), \]  
\[ Q^a_R(t) = \int d^3x \, q^\dagger_R(t, \vec{x}) \frac{\tau^a}{2} q_R(t, \vec{x}). \]  

Furthermore, vector and axial vector conserved currents can be obtained from combinations of the independent left- and right-handed chiral currents,

\[ V^a_\mu = R^a_\mu + L^a_\mu = \bar{q} \gamma_\mu \frac{\tau^a}{2} q, \quad \partial^\mu V^a_\mu = 0, \] 
\[ A^a_\mu = R^a_\mu - L^a_\mu = \bar{q} \gamma_\mu \gamma_5 \frac{\tau^a}{2} q, \quad \partial^\mu A^a_\mu = 0, \]

and their charge operators can be constructed in the same manner\(^2\).

### 2.1.1 Chiral Symmetry Breaking

The invariance of the massless Lagrangian under \( SU(2)_L \times SU(2)_R \) symmetry predicts the existence of degenerate states with the same spin and opposite parity, this is known as parity doubling.

The expected parity doubling states have not been found in nature. For instance, the vector meson \( \rho \) (with \( J^P = 1^- \)) and the axial vector meson \( a_1 \) (with \( J^P = 1^+ \)) have vastly different masses (\( m_\rho = 776 \text{ MeV} \) and \( m_{a_1} = 1230 \text{ MeV} \)). Clearly, they do not form degenerate states as predicted by chiral symmetry.

The observed spectrum of light hadrons, however, indicates that the ground state is only invariant under the isospin vector transformation \( SU(2)_V \),

\(^2\)The axial charge operators do not form a closed algebra.
i.e. pions are nearly degenerate. This, in other words, means that the vacuum state is not invariant under the axial vector transformation. Hence, the chiral symmetry is dynamically (spontaneously) broken by the vacuum to the isospin diagonal subgroup. Thus, the size of breaking of the chiral symmetry by the ground state (vacuum) is of order $m_{a_1} - m_\rho \sim O(500 \text{ MeV})$.

This mechanism of breaking the chiral symmetry is known as the Nambu-Goldstone mode. It is one of the ways of breaking chiral symmetry besides the explicit chiral symmetry breaking in which the Lagrangian is not invariant under the exact chiral symmetry. The size of the explicit chiral symmetry breaking is very small, this will be shown at the end of this subsection.

In QCD, the non-zero quark condensate $\langle \bar{q}q \rangle \neq 0$ is thought to be responsible for the spontaneous breaking of chiral symmetry. In the following, we will introduce the dynamical breaking of chiral symmetry using a toy model. For this purpose, we consider the Lagrangian of scalar fields as in the linear $\sigma$-model [40, 41]. In this model, the proposed Lagrangian

$$L_{\sigma M} = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2} \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} - V(\pi, \sigma),$$  \hspace{1cm} (2.17)

describes dynamics of the isosinglet field $\sigma$, which represents an artificial particle, and isotriplet pseudoscalar field (e.g. pion) $\vec{\pi} \equiv \pi = (\pi_1, \pi_2, \pi_3)$. The potential energy is

$$V(\pi, \sigma) = \frac{m^2}{2}(\pi^2 + \sigma^2) + \frac{\lambda}{4}(\pi^2 + \sigma^2)^2.$$  \hspace{1cm} (2.18)

The above Lagrangian is invariant under $SO(4)$ rotation symmetry in
which the length of the vector $\Phi = \sqrt{\pi^2 + \sigma^2}$ is preserved. Since the group algebra of $SO(4)$ is isomorphic to Lie group of $SU(2)_L \times SU(2)_R$, $\mathcal{L}_\sigma$ is invariant under global chiral symmetry.

It must be stated that the sign of the coupling $\lambda$ is constrained to be positive to make $V(\pi, \sigma)$ bounded from below, whereas the sign of the coefficient $m^2$ is unconstrained and has a marked impact when minimizing the potential energy. If we minimize the potential energy, we get

$$\frac{\partial V(\pi, \sigma)}{\partial \pi} = 0 = \pi (m^2 + \lambda(\pi^2 + \sigma^2)),$$

$$\frac{\partial V(\pi, \sigma)}{\partial \sigma} = 0 = \sigma (m^2 + \lambda(\pi^2 + \sigma^2)).$$

For $m^2 > 0$, one obtains a non-degenerate minimum point that is located at $\pi = \sigma = 0$, see Fig. (2.1). This provides $m_\pi^2 = m_\sigma^2$ (parity doubling) which does not manifest in nature.

The choice $m^2 < 0$ generates classical minima which lie at

$$\pi^2 + \sigma^2 = -\frac{m^2}{\lambda} = f^2. \quad (2.19)$$

In this situation, the spontaneous symmetry breaking (SSB) of the rotation symmetry occurs whenever we (spontaneously) pick one of the classical minima. As a result, the symmetry group is broken down, $SO(4) \rightarrow SO(3)$. The constant $f$ is called the pion decay constant. It plays a crucial role in low energy physics, this will be shown after introducing nonlinear $\sigma$-model.

In quantum field theory, the ground state (corresponding to a minimum in classical physics) for any physical field $\psi$ is characterized by having zero
Figure 2.1: The $\sigma$-model potential with $m^2 > 0$. In this case, the particles $\pi$ and $\sigma$ have same masses $m^2_\pi = m^2_\sigma = m^2$. However, there is no experimental evidence for parity-doubling. This Figure is taken from [31].

vacuum expectation value, $\langle 0 | \psi | 0 \rangle = 0$. In our current problem, one can assign the following vacuum expectation values (VEVs) for $\sigma$, and $\pi$ fields$^3$:

$$\langle 0 | \sigma | 0 \rangle = f,$$  \hspace{1cm} (2.20)

$$\langle 0 | \pi | 0 \rangle = 0.$$  \hspace{1cm} (2.21)

This aspect of the linear $\sigma$-model corresponds to QCD, where $\langle \bar{q}q \rangle \sim \langle \sigma \rangle$. Thus, the continuous rotational symmetry $SO(4)$ is spontaneously broken down to $SO(3)$ by the vacuum expectation value of $\sigma$, i.e. only the components of $\pi$ transforms invariantly under $SO(3)$. Since the orthogonal group $SO(3)$ is isomorphic to $SU(2)_V$, the field, $\pi$, transforms invariantly under

$^3$ We cannot choose $\langle 0 | \sigma | 0 \rangle = 0$ and $\langle 0 | \pi | 0 \rangle = f$. This is because the vacuum is scalar while the pion is pseudoscalar under parity transformation.
Figure 2.2: The $\sigma$-model potential with $m^2 < 0$. In physics, this is called the Mexican hat potential. In this case, there are two modes: (1) massive mode which corresponds to small expansion about the minimum in the radial directions (oscillation in magnitude), this gives mass to $\sigma$-particle, $m_\sigma^2 = -2m^2$, and (2) massless mode (Goldstone mode) corresponds to the small tangential expansion around the minimum, $m_\pi^2 = 0$. The solid circle is known as the chiral circle. The dynamics of the Goldstone bosons at low energy are constrained by the chiral circle, read discussion on nonlinear $\sigma$-model. This Figure is taken from [31].

With nontrivial VEV, Eq. (2.20), $\sigma$ is an unphysical field and one has to quantize it by performing a small perturbation around its minimum,

$$\sigma = f + \sigma', \quad (2.22)$$

where $\langle 0|\sigma'|0 \rangle = 0$. Substituting Eq. (2.22) into the potential defined in Eq. (2.18) gives

$$V(\pi, \sigma) = \frac{1}{2} (-2m^2) \sigma'^2 + \lambda f \sigma' (\sigma'^2 + \pi^2) + \frac{\lambda}{4} (\sigma'^2 + \pi^2)^2 + \ldots, \quad (2.23)$$
where the ellipsis indicates constant terms.

Clearly, from Eq. (2.23), the scalar field $\sigma$ gains mass, $m_\sigma^2 = -2m^2$, while the pseudoscalar field $\pi$ becomes massless, $m_\pi^2 = 0$, see Fig. (2.2). The massless mode is a consequence of Goldstone’s theorem, which states: if the continuous (global) symmetry of the Lagrangian is not respected by the vacuum, massless particles, known as Goldstone bosons, will emerge and their number is equivalent to the number of broken generators.

The generated Goldstone bosons have the same symmetry properties of the axial generators. Thus, they are pseudoscalar. They transform under parity in a similar way to the axial currents. Since the vacuum is invariant under isospin transformations, the three emerging Goldstone bosons form an isospin multiplet representing degenerate states under $SU(2)_V$. Since the mentioned properties are consistent with those of the pions, they are thought to be the generated pseudoscalar Goldstone bosons, i.e. pions transform as pseudoscalar under parity and as a triplet under isospin transformations.

In nature, pions have smaller masses ($m_{\pi^\pm} = 140$ MeV and $m_{\pi^0} = 135$ MeV) in comparison to the hadronic scale. To give massless pions small masses, one has to explicitly break chiral symmetry of the linear $\sigma$-model. Before considering the explicit breaking of chiral symmetry, let us introduce Partial Conserved Axial Current (PCAC) hypothesis which states that the divergence of axial current is exact (zero) in the chiral limit $m_q \to 0$, and it is broken as light quarks gain masses [31]. The expression of PCAC is obtained
by taking the divergence of the following matrix element\(^4\)

\[
\langle 0 | A^{\mu a} | \pi^b \rangle = i f p^\mu \delta^{ab}, \tag{2.24}
\]

which describes the coupling of a single pion state to the vacuum.

In fact, Eq. (2.24) describes the dynamics of the Goldstone bosons (pions), e.g. pion decays into leptons \(\pi^+ \rightarrow l^+ \nu_l\). Thus, it measures the strength of pion decay into other states via the vacuum. The value of \(f\) in the chiral limit is \(f_0 = 85\, \text{MeV}\) \([32]\). Its physical value can be measured from the decay rate \([35]\)

\[
\Gamma_{\pi^+ \rightarrow \mu^+ \nu_\mu} = \frac{G_F^2 f^2}{4\pi} m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2. \tag{2.25}
\]

By substituting

\[
\tau = \Gamma^{-1} = 2.6 \times 10^{-8}\, \text{s}, \quad G_F = 1.16 \times 10^{-11}\, \text{MeV}^{-2},
\]

\[
m_\pi = 139\, \text{MeV}, \quad m_\mu = 105.6\, \text{MeV},
\]

into Eq. (2.25) one finds, \(f = 92.4\, \text{MeV}\). The difference between \(f\) and \(f_0\) is proportional to the quark masses \(f = f_0(1 + \mathcal{O}(m_q))\).

By using the normalization of the pion state \(\langle 0 | \pi^a | \pi^b \rangle = \delta^{ab}\), one can write Eq. (2.24) after taking the divergence as

\[
\langle 0 | \partial_\mu A^{\mu a} | \pi^b \rangle = f m_\pi^2 \langle 0 | \pi^a | \pi^b \rangle. \tag{2.26}
\]

\(^4\)Eq. (2.24) is constructed by using the Lorentz symmetry for spinless pions.
At the operator level, this gives the PCAC hypothesis, i.e.

\[ \partial_\mu A^{\mu a} = f m_\pi^2 \pi^a. \]  

(2.27)

The PCAC hypothesis can be used to relate the quark masses to the masses of the Goldstone bosons. To this end, let us now consider the explicit breaking of chiral symmetry which gives pions small masses. To break the chiral symmetry of the linear \( \sigma \)-model, we add a linear term in \( \sigma \), i.e. \( \epsilon \sigma \), to \( \mathcal{L}_{\sigma M} \), where \( \epsilon \) is a constant. The breaking term can be written as

\[ \mathcal{L}_B = \epsilon \sigma \approx \epsilon f + O\left( \frac{1}{f} \right), \]  

(2.28)

where \( \sigma \) is replaced by \( \sqrt{f^2 - \phi^2} \). By comparing the divergence of the axial

\[ V(\pi, \sigma) \]

\[ \sigma \]

\[ \pi \]

**Figure 2.3:** This tilted Mexican hat potential reflects both the spontaneous symmetry breaking and the explicit symmetry breaking. The vacuum is shifted by adding the linear term in \( \sigma \). This term is proportional to quark mass which is generated by its coupling to the Higgs field. This Figure is taken from [31].
current obtained from Eq. (2.28), which is $\partial_\mu A^{\mu a} = \epsilon \pi^a$, with the Partial Conserved Axial Current (PCAC) hypothesis, see Eq. (2.27), we find

$$m_\pi^2 = \frac{\epsilon}{f}.$$  \hspace{1cm} (2.29)

From above equation, one can estimate the relation between masses of light quarks and Goldstone bosons. We know that the presence of the light quark masses in QCD Lagrangian would explicitly break chiral symmetry, i.e.

$$m_q \bar{q}q = m_q (\bar{q}_L q_R + \bar{q}_R q_L).$$  \hspace{1cm} (2.30)

By comparing Eq. (2.30) with Eq. (2.28) and Eq. (2.29), one can make the following simple analogy

$$\sigma \sim \bar{q}q,$$ \hspace{1cm} (2.31)

$$\epsilon \sim m_q,$$ \hspace{1cm} (2.32)

to find $m_\pi^2 \propto m_q$. As the up and down quarks have small masses, explicit breaking of chiral symmetry is very small, i.e. of order few MeV. The mass relations between Goldstone bosons and light quarks will be represented in the next subsection after extending our discussion to three quark flavors.

The mass of $\sigma$ particle, which is a result of small oscillations around the vacuum in the radial direction as shown in Fig. 2.2, also receives a small contribution of order $\sim O(m_q)$ as we shift the vacuum by adding a linear term in $\sigma$ which explicitly breaks chiral symmetry, see Fig. 2.3.
However, the observed spectra does not show any scalar meson with a mass less than the hadronic scale 1 GeV. Thus, for dynamics at energies much smaller than the chiral symmetry breaking scale (this can be identified as $m_\rho$), one can integrate out the $\sigma$-field by sending its mass to infinity. This can be done by taking $\lambda \to \infty$ in the linear $\sigma$-model. In the limit $m/\lambda \to \infty$, $f$ is fixed and the Mexican-hat potential gets infinitely steep in the sigma direction [30]. In this situation, the dynamics is confined by what is known as the chiral circle, see Fig. 2.2. The chiral circle is defined by the condition

$$\sigma^2 + \pi^2 = f^2. \quad (2.33)$$

For the $\sigma$-field satisfies Eq. (2.33), one can define a $2 \times 2$ unitary matrix $U$ as

$$\sigma + i \tau \cdot \pi = f U. \quad (2.34)$$

The fields $\sigma$ and $\phi$ can be expressed in terms of $U$ as follow

$$\sigma = \frac{f}{2} \text{Tr}(U), \quad \pi_i = -i \frac{f}{2} \text{Tr}(\tau_i U), \quad (2.35)$$

where we have used the properties of Pauli matrices

$$\tau_i \tau_j = \delta_{ij} + i \epsilon_{ijk} \tau_k, \quad \text{Tr}(\tau_i \tau_j) = 2 \delta_{ij}, \quad \text{Tr} \tau_i = 0. \quad (2.36)$$

The quantity $U$ is a nonlinear function of the Goldstone bosons and it
transforms linearly under $SU(2)_L \times SU(2)_R$,

$$U \rightarrow U' = R U L^\dagger.$$  \hspace{1cm} (2.37)

To write the Lagrangian given in Eq. (2.17) in terms of $U$, one first substitutes Eq. (2.35), in which we express the $\sigma$ and $\pi$ fields in terms of $U$, into Eq. (2.17) and then replace the $\sigma$ field by its vacuum expectation value $\langle \sigma \rangle = f$ which is fixed in the limit $m/\lambda \rightarrow \infty$. By doing so, one defines the so-called nonlinear $\sigma$-model

$$L_{\text{NLSM}}^{\text{kin}} = \frac{f^2}{2} \text{Tr} (\partial_\mu U \partial^\mu U),$$ \hspace{1cm} (2.38)

which only describes the dynamics of the massless Goldstone bosons at low energies. The presence of the derivatives in the Lagrangian, which gives momentum-dependent vertices, guarantees the weakness of Goldstone interactions that vanish in the chiral limit. The nonlinear $\sigma$-model is compatible with the symmetries of QCD, in particular chiral symmetry.

Unlike the linear $\sigma$-model, one does not need to know some details about higher scales, e.g. the mass of $\sigma$, to predict scattering amplitudes derived from Eq. (2.38). The only thing one has to know when calculating the scattering amplitude using Eq. (2.38) is the numerical value of $f$ which can be obtained from experiments.

The action of the nonlinear $\sigma$-model given in Eq. (2.38) is not just phenomenological. Since it describes the dynamics of the light (approximate) Goldstone bosons (e.g. pions) at low energies at which there is no need for
details of higher scales, it is an approximation of QCD. Thus, the nonlinear \( \sigma \)-model can be systematically improved\(^5\). This systematic improvement was achieved by introducing effective field theory (EFT).

### 2.1.2 Effective Field Theory

In this subsection, we only discuss the main points of the effective field theory and refer the interested reader to Ref. [29] and references therein.

It is convenient to start with the Weinberg’s theorem which states [33]:

For a given set of asymptotic states, perturbation theory with the most general Lagrangian containing all terms allowed by the assumed symmetries will yield the most general S-matrix elements consistent with analyticity, perturbative unitarity, cluster decomposition, and the assumed symmetries.

In this case, the asymptotic states are the low energy degrees of freedom, i.e. pions. Thus, based on Weinberg’s theorem, one can construct the general structure of effective Lagrangian with an infinite number of terms that are consistent with the symmetries of the underlying theory. In QCD, the symmetries are: global continuous symmetries (chiral, isospin), and local symmetries (Lorentz, Poincaré), and discrete symmetries (charge conjugation, parity, time reversal).

The parameters of these terms, known as the low energy constants (LECs), contain information on the dynamics of QCD, i.e. they can be expressed in terms of the QCD parameters (e.g. quark masses, \( \Lambda_{\text{QCD}} \)). Thus, their numerical values should be determined from QCD. Since, we are unable to

---

\(^5\)The Lagrangian \( \mathcal{L}_{\text{NL}} \) gives rise to both tree and loop contributions, i.e. loops could be derived from the expanded terms with the higher orders in the field \( \phi \). Thus, we have to renormalize these loop diagrams in a systematic way.
solve QCD in the nonperturbative regime, the parameters (LECs) can be
determined by fitting to experimental data [29].

However, a Lagrangian with an infinite number of terms is not predictive
and, as such, one has to implement a procedure known as power counting.
Power counting is an essential ingredient of any effective theory that are
necessary to organize terms according to their importance. Having soft scales
$Q$ well separated from hard scales $\Lambda$, the expansion parameter for the system
can be defined as the ratio of scales, i.e. $\frac{Q}{\Lambda}$. Thus, one can expand in small
scales and keep track the expansion by counting powers of low energy scales
$Q$ [29]. However, this is completely different from the case of perturbation
theories, such as QCD at high energies, in which the perturbative expansions
are performed in the small coupling constant $g_s$.

Power counting is a powerful tool not only for organizing terms in a sys-
tematic way, but also for renormalizing higher order quantum corrections
(loops) by the coefficients of the contact terms. Below, we will employ these
ideas and techniques to construct the lowest order of the dynamical La-
grangian in Chiral Perturbation Theory.

It is convenient to begin with the Lagrangian of nonlinear $\sigma$-model

$$L_{\text{NL$\sigma$M}} = \frac{f^2}{2} \text{Tr} \left( \partial_\mu U \partial^\mu U^\dagger \right), \quad (2.39)$$

where the effective field $U$ is an $SU(2)$ unitary matrix containing Goldstone
bosons. It can be expressed in terms of a traceless and Hermitian matrix $\phi$
as follows\(^6\)

\[ U = e^{i\sqrt{2} \phi}, \tag{2.40} \]

where the matrix \( \phi \) represents the pion fields in a compact way

\[ \phi = \frac{3}{\sqrt{2}} \sum_{\alpha=1}^{3} \tau_{\alpha} \phi_{\alpha} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi^0 & \sqrt{2} \pi^+ \\ \sqrt{2} \pi^- & -\pi^0 \end{pmatrix}. \tag{2.41} \]

In our conventions, we use the physical value of the pion decay constant \( f = 92.4 \text{ MeV} \). It is different from the ones used by Wise in [34] in which \( f = 135 \text{ MeV} \) was used. Thus, one has to replace \( f \) in [34] by \( \sqrt{2} f \) to account for different conventions.

From power counting, the momentum and mass of Goldstone bosons can be taken as the low energy scales,

\[ U \sim O(Q^0), \tag{2.42} \]
\[ \partial_{\mu} U \sim p_{\mu} \sim O(Q), \tag{2.43} \]
\[ m_q \propto m_\pi^2 \sim O(Q^2). \tag{2.44} \]

\(^6\) In fact, there are many possibilities to define the nonlinear representation of the Goldstone-bosons fields \( \phi \) other than the simple exponential form that we presented in Eq. (2.40). For example, one can use the square-root form [62]

\[ U = \sqrt{1 - \frac{\phi^2}{f^2} + i \frac{\tau \cdot \phi}{f}}, \]

in the chiral Lagrangian given in Eq. (2.39) to get the same prediction for the on-shell (physical) observables. Of course, off-shell predictions depend on the form of \( U \), for details see Ref. [29]. It is worth mentioning that as the quantity \( U \) transforms linearly under \( SU(2)_L \times SU(2)_R \): \( U \mapsto U' = RUL^\dagger \), the chiral Lagrangian involving \( U \) which is also invariant under \( SU(2)_L \times SU(2)_R \) would yield the same on-shell (physical) result for different forms of \( U \) [54].
The hard scales, on the other hand, can be identified as $\Lambda_\chi = 4\pi f \sim 1$ GeV (the chiral symmetry breaking scale), $m_N$ (nucleon mass), or rho mass $m_\rho$.

The chiral Lagrangian given in Eq. (2.39) is invariant under global chiral symmetry. To make it invariant under local chiral symmetry, one has to introduce the covariant derivative defined as follows

$$D_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu, \quad (2.45)$$

where $r_\mu$ and $l_\mu$ are external fields. Eq. (2.45) transforms under chiral symmetry as

$$D_\mu U \rightarrow (D_\mu U)' = R(D_\mu U)L^\dagger. \quad (2.46)$$

So, in terms of covariant derivative, Eq. (2.39) can be written as

$$L_{\text{eff}} = \frac{f^2}{4} \text{Tr} \left(D_\mu U(D^\mu U)^\dagger\right). \quad (2.47)$$

Here the purpose of writing the multiplicative factor as $\frac{f^2}{4}$ is to get the same structure as the pion kinetic energy when expanding in $\phi$, e.g. $L_{\text{eff}} \equiv \frac{1}{2} \partial_\mu \phi_\alpha \partial^\mu \phi_\alpha + L_{\text{interaction}}$.

Thus far, the quark mass has not been included in the above chiral Lagrangian $L_{\text{eff}}$ and if we do so the symmetry will be explicitly violated. Therefore let us write the quark masses in terms of a $2 \times 2$ matrix,

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \quad (2.48)$$

To keep $L_{\text{eff}}$ invariant under $SU(2)_L \times SU(2)_R$ with the presence of the

60
quark masses, $M$ should transform as follows

$$M \mapsto RML^\dagger.$$ (2.49)

Now the lowest order chiral Lagrangian that transforms invariantly under $SU(2)_L \times SU(2)_R$ transformation is the $L_{\text{eff}}$ using explicit symmetry breaking [29],

$$L_2 = \frac{f^2}{4} \text{Tr} \left( D_\mu U (D^\mu U)^\dagger \right) + \frac{f^2 B_0}{2} \text{Tr} \left( MU^\dagger + U M^\dagger \right),$$ (2.50)

where the index 2 refers to the order of chiral Lagrangian in power of low energy scales, i.e. the momentum of pion and quark mass. The coefficient $B_0$ is related to the pion decay constant and the quark condensate of $u$ and $d$ quark flavors. To this end, let us first expand the second term of Eq. (2.50) up to the leading order in $\phi$,

$$L_{s.b.}^2 = f^2 B_0 (m_u + m_d) + O(\phi^2).$$ (2.51)

By assuming isospin symmetry, the vacuum expectation values of the scalar quark densities can be obtained as follows

$$\langle 0 | \bar{u} u | 0 \rangle = \langle 0 | \bar{d} d | 0 \rangle = \frac{1}{2} \langle 0 | \bar{q} q | 0 \rangle = \frac{\partial H_{\text{QCD}}}{\partial m} = -\frac{\partial L_{s.b.}^2}{\partial m}. \quad (2.52)$$

From Eq. (2.51) and Eq. (2.52), one can get

$$B_0 = -\frac{\langle 0 | \bar{q} q | 0 \rangle}{2 f^2}. \quad (2.53)$$
It is worth mentioning that identifying the LECs $f$ and $B_0$ allow us to use $\mathcal{L}_2$ to predict other physical process of the Goldstone bosons. For instance, the measured LEC $f$ from the pion decay mode $\pi^+ \to \mu^+ \nu_\mu$ can be used to predict the amplitude of $\pi\pi$ scattering $\sigma(\pi^0\pi^0 \to \pi^+\pi^-)$, for details one can refer to [29].

In $\mathcal{L}_2$, the perturbative expansion was performed in the (small) quark masses which result from the explicit breaking of chiral symmetry. The above Lagrangian represents the lowest order of the Chiral Perturbation Theory. This Lagrangian, however, is nonrenormalizable. More precisely, the divergences that could arise from loop graphs (with higher order) derived from its vertices cannot be renormalized by the coefficients $f$ and $B_0$.

For instance, the one-loop graph shown on the right in Fig. 2.4 is of order $O(Q^4)$. Here we used $\nu = 2 + 2N_L + \sum_{n=1}^{\infty} 2(n-1)N_{2n}$ to define the order $\nu$ of Feynman graphs in four dimensions, where $N_L$ denotes the number of loops and $N_{2n}$ is the number of vertices.

In order to renormalize this one-loop diagram, one has to construct the most general Lagrangian of order $O(Q^4)$, i.e. $\mathcal{L}_4$. In principle, such Lagrangians can be constructed by considering all the terms which are proportional to $O(Q^4)$. This has been done by Gasser and Leutwyler, we refer the interested readers to Ref. [29] and references therein. With the presence of $\mathcal{L}_4$ parameters, one can renormalize the one-loop graph obtained using vertices from $\mathcal{L}_2$. To do so, one has to consider the relevant terms from both Lagrangians, i.e. $\mathcal{L}_{\text{int.}} = \mathcal{L}_2^{4\phi} + \mathcal{L}_4^{2\phi}$, where the superscript indicates the number of interacting pions. In this case, the divergence of the pionic one-loop diagram at $O(Q^4)$ derived from $\mathcal{L}_2^{4\phi}$ (diagram on the right in Fig. 2.4) can
be cancelled by the tree level terms derived from $\mathcal{L}_4^{\phi}$ (diagram on the left in Fig. 2.4) [29].

Up to now, we only considered two quark flavors $u$ and $d$ that happened to have close masses that become degenerate in the absence of the electromagnetic interaction which breaks isospin symmetry. The extension, however, to three quark flavors is straightforward. In chiral perturbation theory, we only need to increase the dimensionality of $\phi$

$$\phi = \sum_{a=1}^{8} \frac{\lambda_a \phi_a}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}} \eta & \sqrt{2} \pi^+ & \sqrt{2} K^+ \\ \sqrt{2} \pi^- & -\pi^0 + \frac{1}{\sqrt{3}} \eta & \sqrt{2} K^0 \\ \sqrt{2} K^- & \sqrt{2} K^0 & -\frac{2}{\sqrt{3}} \eta \end{pmatrix}, \quad (2.54)$$

and $M$

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \quad (2.55)$$

to account for the presence of the strange quark. Thus, $\mathcal{L}_2$ given in Eq. (2.50) with above expressions for $\phi$ and $M$ becomes invariant under local chiral symmetry represented by $SU(3)_L \times SU(3)_R$.

To derive the relations between masses of light quark flavors ($u,d,s$) and those of Goldstone bosons, one can easily expand the mass term in Eq. (2.50)
up to the second order in the Goldstone bosons fields \((\pi^0, \pi^\pm, K^\pm, K^0, \bar{K}^0)\). In the isospin limit, one can get [29]

\[
m^2_\pi = 2B_0 m, \quad m^2_K = B_0 (m + m_s), \quad m^2_{\eta} = \frac{2}{3} B_0 (m + 2m_s),
\]

with \(B_0 = -\frac{\langle 0|\bar{q}q(0)|0\rangle}{3f^2}\). The mass relations provided above is known as the Gell-Mann, Oakes, and Renner relations. For the case of two flavors, using \(m = \frac{m_u + m_d}{2} = 3.55\) MeV, \(f = 92.4\) MeV, and \(m_{\pi^\pm} = 139\) MeV, one can estimate the size of the condensate of up and down quarks,

\[
\frac{\langle \bar{u}u + \bar{d}d \rangle}{2} = -\frac{f^2 m_{\pi^\pm}^2}{2m} \sim -(200\text{ MeV})^3 \sim -\Lambda^3_{\text{QCD}}.
\]

The interaction of Goldstone bosons \((\pi, K, \eta)\) with matter fields (baryons, vector mesons and heavy mesons) can be described by the vector chiral connection \(\Gamma^\mu\) and the axial vector vielbein \(u^\mu\) that are given by

\[
\Gamma^\mu = \frac{1}{2} \left[ \xi \partial^\mu \xi^\dagger + \xi^\dagger \partial^\mu \xi \right] \approx \frac{i}{4f^2} \epsilon^{\alpha\beta\gamma} \phi_\alpha \partial^\mu \phi_\beta \tau_\gamma + O(\phi^4),
\]

\[
u^\mu = \frac{i}{2} \left[ \xi^\dagger \partial^\mu \xi - \xi \partial^\mu \xi^\dagger \right] \approx -\frac{1}{2f} \partial^\mu \phi_\alpha \tau_\alpha + O(\phi^3),
\]

where the coset field \(\xi\) is related to \(U\) by

\[
\xi^2 = U = e^{i\frac{\sqrt{2}a}{f}}.
\]

Both \(\Gamma^\mu\) and \(u^\mu\) transform as an isospin vector under \(SU(2)_V\). In terms of
the coset field, the quark mass matrix which breaks chiral symmetry can be expressed as $m^\xi = \frac{1}{2} \left( \xi M \xi + \xi^\dagger M \xi^\dagger \right)$. For the case of two quark flavors, substituting Eq. (2.48) and $\xi = e^{i\phi} \sqrt{2} f$ into $m^\xi$ gives

$$m^\xi = \frac{1}{2} \left( \xi M \xi + \xi^\dagger M \xi^\dagger \right) \approx m I_{2 \times 2} + O(\phi^2),$$

where in the isospin limit $m_u = m_d = m$.

2.2 Heavy Quark Spin Symmetry

2.2.1 Motivation

The idea of the Heavy Quark Spin Symmetry (HQSS), heavy quark symmetry for short, originally stems from an analogy with the hydrogen atom in which an electron is orbiting around the proton. The analogy can be made in the following context: any charged particle with spin has a magnetic moment that is inversely proportional to particle mass ($\mu \propto \frac{1}{m}$). As a consequence, the proton in the hydrogen atom has a tiny magnetic moment due to its large mass. Additionally, the magnetic interaction of the proton and the electron splits the energy spectrum; this tiny splitting is known as hyperfine splitting. In this situation, the shift in the hydrogen spectrum depends on the proton spin and can be reduced by considering the mass of the proton to be very large, $m_p \to \infty$, causing the energy spectrum to be degenerate and independent of the spin of the proton.

This simple idea stimulated physicists to utilize similar concepts to study the strongly interacting systems at very low energy for the heavy-light meson
systems ($Q\bar{q}$) such as the heavy charm and bottom mesons.

For the case of charm (bottom) mesons system, the heavy charm (bottom) quark ($m_Q \gg \Lambda_{QCD}$) is assumed to play the same role as the proton while the light antiquark ($m_q \ll \Lambda_{QCD}$) forms a cloud around the heavy quark. For such systems, the heavy quark has a color magnetic moment proportional to its mass in the following way $\mu^c \propto \frac{1}{m_Q}$. By taking the limit $m_Q \to \infty$, a new symmetry arises (heavy quark symmetry). Its consequence is to form degenerate multiplets of heavy charm or bottom mesons that are independent of the spin orientation of the heavy quark\textsuperscript{7}. More precisely, heavy mesons with definite parity become degenerate states of the spin symmetry group, $SU(2)_s$. Thus generators of $SU(2)_s$ allow us to transform between these degenerate states.

In the quark model, the classification of the heavy-light mesons $Q\bar{q}$ states is built on the total angular momentum $J$ and the parity, $JP$, where parity is obtained from the expression $P = (-1)^{l+1}$, for details see Chapter 1. However, in the limit $m_Q \to \infty$ the spin of heavy quark $s_Q$ has no effect, so one can use the parity and total angular momentum of the light degrees of freedom, i.e. the light anti-quark and gluons, to classify the heavy-light meson states. The total angular momentum of the light degrees of freedom is given by $S_l = l \pm \frac{1}{2}$, where $l$ is the orbital angular momentum of the system.

For the ground state, $l = 0$ (S-wave in the quark model), there are two degenerate states with $J^P = 0^-, 1^-$, which form members of the $S^P_l = \frac{1}{2}^-$ doublet. For the lowest lying excited states, $l = 1$ (P-wave in the quark

\textsuperscript{7}In the limit $m_Q \to \infty$, the dynamics of such systems ($D$ and $B$ mesons) are completely independent of spin and flavor of the heavy quark.
model), the spin of the light degrees of freedom

\[ S_l = 1 \pm \frac{1}{2} = \begin{cases} \frac{3}{2}, \\ \frac{1}{2} \end{cases} \]  

(2.63)

has two possible values. For the \( S_l^p = \frac{1}{2}^+ \)-doublet, the degenerate states are \( 0^+, 1^+ \). The degenerate states \( 1^+, 2^+ \) are members of the \( S_l^p = \frac{3}{2}^+ \) doublet, see Table 2.1.

<table>
<thead>
<tr>
<th>( S_l^p )</th>
<th>Heavy Quark Doublet (( J^p ))</th>
<th>Quark Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2}^- )</td>
<td>( 0^-, 1^- )</td>
<td>( \bar{q} ) in S-wave</td>
</tr>
<tr>
<td>( \frac{1}{2}^+ )</td>
<td>( 0^+, 1^+ )</td>
<td>( \bar{q} ) in P-wave</td>
</tr>
<tr>
<td>( \frac{3}{2}^+ )</td>
<td>( 1^+, 2^+ )</td>
<td>( \bar{q} ) in P-wave</td>
</tr>
</tbody>
</table>

**Table 2.1:** In the limit \( m_Q \to \infty \), the spin of the heavy quark decouples from the spin of the light degrees of freedom. In this case, the heavy-light mesons can be classified according to the parity and total angular momentum of the light degree of freedom, \( S_l^p \) [36].

Although \( 1^+ \) states of \( \frac{1}{2}^+ \) and \( \frac{3}{2}^+ \) doublets (see Eq. (2.63)) can mix, they can be distinguished and labeled by analyzing their decay angular distributions. The state \( 1^+ \) of the \( \frac{3}{2}^+ \)-doublet can only decay by d-wave\(^8\) (\( \Gamma \) is narrow\(^9\)). It can be discriminated from \( 1^+ \) of the \( \frac{1}{2}^+ \)-doublet which decays by s-wave (\( \Gamma \) is broader).

The masses of the nonstrange charm mesons are shown in Fig 2.5. The

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\(^8\)The allowed strong decay transitions can be determined from the spin and parity of the light degrees of freedom, for details see section 3.2.1.

\(^9\)The decay width \( \Gamma \) is related to the momentum \( p \) of the emitted light meson as \( \Gamma \propto |p|^{2l+1} \). For case of low energies (small \( p \)), the width \( \Gamma \) of the d-wave transition \( (l = 2) \) is narrow in comparison with the s-wave transition \( (l = 0) \) [60].
$D_0^0$ 2427 MeV $D_1^{++}$ Not Seen
$D_0^0$ 2318 MeV $D_0^+$ 2403 MeV
$D^{*0}$ 2006 MeV $D^{*+}$ 2010 MeV
$D^0_{c\bar{u}}$ 1864 MeV $D^+_{c\bar{d}}$ 1869 MeV

Figure 2.5: Blue and red boxes represent the charm meson states of $\frac{1}{2}^-$ and $\frac{1}{2}^+$ doublets respectively. The hyperfine splitting of the states is explained by the finite mass of the charm quark. All charm meson masses are taken from Table 1.4 excluding the mass of the charged nonstrange meson $D_0^+$ which is reported by the FOCUS collaboration with a large error [57].

equality of the hyperfine splittings for the well determined masses, i.e.

\[
m_{D^{*0}} - m_{D^0} = 142.12 \pm 0.1 \text{ MeV},
\]
\[
m_{D^{*+}} - m_{D^+} = 140.65 \pm 0.1 \text{ MeV},
\]
can be understood by the heavy quark symmetry where $\frac{\Lambda_{QCD}^2}{m_c} = m_{1^-} - m_{0^-}$.

As expected from the heavy quark symmetry, the hyperfine splitting in $B$ meson sector, e.g. $m_{B^{*+}} - m_{B^+} = 45 \pm 0.4 \text{ MeV}$, is smaller than the splitting in $D$ meson sector, e.g. $m_{D^{*+}} - m_{D^+} = 140.65 \pm 0.1 \text{ MeV}$.

In the next subsection, we exploit the scale separation $m_Q \gg \Lambda_{QCD}$ to construct an effective theory for massive fields where heavy quark symmetry is exact.
2.2.2 Heavy Quark Effective Theory

In this subsection, we briefly discuss the heavy quark effective theory; for reviews, see [42]-[53].

For mesons containing a single heavy quark, the strong interaction between a heavy quark, with four momentum $P^\mu = m_Q v^\mu$, and the light degrees of freedom (light quarks and gluons), with momentum $k^\mu$, will be blind to heavy quark spin (and the velocity) in the limit $m_Q \to \infty$. This can be understood in the following context: the final momentum of the heavy quark becomes

$$P^\mu = m_Q v^\mu + k^\mu,$$

(2.64)

where at low energy, i.e. energy well below $m_Q$, $m_Q \gg k^\mu \sim \Lambda_{\text{QCD}}$, the color static heavy quark is almost on shell

$$P_\mu P^\mu = m_Q^2,$$

(2.65)

where in this limit, the heavy quark velocity is normalized as $v^2 = 1$.

For the static heavy quark, $v^\mu = (1, \vec{0})$. The four-velocity $v^\mu$ can be altered by a small amount of order $\frac{\Lambda_{\text{QCD}}}{m_Q}$ as the momentum of the heavy quark or the light degrees of freedom change by an order $\Lambda_{\text{QCD}}$. In this case, $v^\mu$ becomes a conserved quantum number of the heavy quark up to leading order in $\frac{1}{m_Q}$. Thus, one can construct an effective field theory for the heavy quark fields with a conserved velocity quantum number.
To this end, let us first recall the second term of Eq. (2.2),

\[ \mathcal{L} = \overline{Q}(i\gamma^\mu D_\mu - m_Q)Q, \]  

where \( Q \) is Dirac spinor of a heavy quark. It is convenient to express the heavy quark field \( Q \) in terms of large \( h^Q_v \) and small \( H^Q_v \) velocity dependent fields,

\[ Q = e^{-im_Qv \cdot x} \left( h^Q_v + H^Q_v \right), \]  

with

\[ \#h^Q_v = h^Q_v, \]  
\[ \#H^Q_v = -H^Q_v. \]  

The field \( H^Q_v \) is kinematically irrelevant due to our assumption of the heavy quark being almost on shell. To derive the relevant effective (approximated) Lagrangian, one can substitute Eq. (2.67) into Eq. (2.66) and after some algebraic steps, one can express the effective Lagrangian in terms of \( h^Q_v \),

\[ \mathcal{L}_v = i\overline{h}^Q_v v \cdot D h^Q_v + O \left( m_Q^{-1} \right). \]  

The leading term preserves heavy quark symmetry. In the effective Lagrangian, Eq. (2.70), the Dirac gamma matrices of the QCD Lagrangian are replaced by the velocity \( v \) of the heavy quark which in turn indicates that the interactions between the heavy quark and gluons do not affect the spin of the heavy quark. This term is known as the static limit and is invari-
ant under $SU(2)_s$ spin symmetry. The representation of the spin symmetry group is given by

$$U_s = e^{i\vec{\theta} \cdot \vec{S}},$$

with $\vec{\theta}$ being the infinitesimal vector of the spin transformation. In the rest frame, the spin generator $S^i$ can be defined in terms of Pauli matrices $\tau^i$ (Eq. (2.10)) as

$$S^i = \frac{1}{2} \begin{pmatrix} \tau^i & 0 \\ 0 & \tau^i \end{pmatrix}.$$  

Under an infinitesimal transformation in spin space, the field $h^Q_v$ transforms as

$$h^Q_v \rightarrow (1 + i \vec{\theta} \cdot \vec{S}) h^Q_v.$$  

Therefore, under this transformation the effective Lagrangian is invariant, i.e.

$$\mathcal{L}^{m_Q \rightarrow \infty}_v = \bar{h}_v^Q \left[ i \bar{v} \cdot D, i \vec{\theta} \cdot \vec{S} \right] h^Q_v = 0.$$  

Other terms of $\mathcal{L}_v$ are obtained by expansion in inverse powers of $m_Q$. These terms break the heavy quark spin symmetry and provide interpretation to the hyperfine splitting in the charm and bottom meson sectors.

The ideas of chiral and heavy quark symmetries can be incorporated to study the spectra and interactions of hadrons containing single heavy quarks. This is the subject of the next Chapter.
Chapter 3

Heavy Hadron Chiral
Perturbation Theory

In the previous Chapter, the approximate chiral and heavy quark symmetries of QCD have been discussed. Weinberg’s theorem was then used to construct chiral perturbation theory and the heavy quark effective theory.

In this Chapter, the chiral and heavy quark spin symmetries will be incorporated in a single effective framework to study the interaction of the Goldstone bosons with mesons containing a single heavy quark. This framework is known as the Heavy Hadron Chiral Perturbation Theory (HHChPT). In this effective theory, the chiral and heavy quark spin symmetries are synthesized by establishing the heavy hadron chiral Lagrangian which transforms linearly under $SU(2)_V \times SU(2)_s$.

We will review HHChPT and for sake of simplicity we restrict our discussion to the nonstrange charm meson in the isospin limit.
Let us first define the effective fields that represent the degenerate states in the heavy quark limit. It is common in the literature to use $\mathcal{H}_a$ and $\mathcal{S}_a$ to label the effective fields for $\frac{1}{2}^-$ and $\frac{1}{2}^+$ doublets respectively. The subscript $a$ refers to the isospin index $a = 1, 2$ for up and down quarks respectively.

The explicit expression for $\mathcal{H}_a$ is

$$\mathcal{H}_a = \left(\frac{1 + \frac{g}{2}}{2}\right)(H_{a\mu}\gamma^\mu - H_a\gamma_5),$$

(3.1)

where the pseudoscalar $J^P = 0^- (D^0, D^+)$ and vector $J^P = 1^- (D^{*0}, D^{*+})$ states are denoted by $H_a$ and $H_{a\mu}$ respectively.

The effective field $\mathcal{H}_a$ is velocity dependent. $\mathcal{H}_a$ annihilates heavy meson pseudoscalar states $H_a$ and heavy meson vector states $H^\mu_a$. The effective field $\mathcal{H}_a$ is invariant under heavy quark spin symmetry $SU(2)_s$. More precisely, the generators of $SU(2)_s$ allow us to transform between the degenerate pseudoscalar $H_a$ and vector $H_{a\mu}$ charm meson states. For an infinitesimal transformation in spin space, the heavy meson multiplet $\mathcal{H}_a$ transforms as a doublet under the heavy quark symmetry group $SU(2)_s$,

$$\mathcal{H}_a \rightarrow \mathcal{H'}_a = U_s\mathcal{H}_a.$$

(3.2)

Since the heavy mesons states contain light anti-quarks, $\mathcal{H}_a$ transforms as a doublet under $SU(2)_V$, where for $SU(2)$ the doublet and the antidoublet are equivalent representations. The invariance of $\mathcal{H}_a$ under the unbroken isospin vector symmetry controls its interaction with the pions embodied in the vector connection $\Gamma_\mu$ and axial vector $u_\mu$ vielbein.
The conjugate of the effective field $\mathcal{H}_a$ is given by

$$\overline{\mathcal{H}}_a = \gamma^0 \mathcal{H}_a^\dagger \gamma^0,$$  \hspace{1cm} (3.3)

where $\overline{\mathcal{H}}_a$ is responsible for creating heavy meson pseudoscalar and vector states. Eq. (3.3) can be written explicitly as

$$\overline{\mathcal{H}}_a = (\gamma^\mu H^\dagger_{\mu a} + \gamma^5 H^\dagger_{a}) \left( \frac{1 + \gamma^5}{2} \right).$$  \hspace{1cm} (3.4)

The explicit form of the effective field of $\frac{1}{2}^+\!$-doublet is

$$S_a = (\frac{1 + \gamma^5}{2})(S^\mu_a \gamma^\mu \gamma^5 - S_a),$$  \hspace{1cm} (3.5)

where the scalar $J^P = 0^+$ and axial vector $J^P = 1^+$ states are denoted by $S_a$ and $S_{a\mu}$ respectively. The conjugate of the effective field $S_a$ is given by $\overline{S}_a = \gamma^0 S_a^\dagger \gamma^0$. The effective field $S_a$ ($\overline{S}_a$) annihilates (creates) heavy meson axial vector states $S_{\mu a}$ and heavy meson scalar states $S_a$.

Before writing down the effective chiral Lagrangian, we have to identify the soft and hard scales for the system. Since $m_Q \gg \Lambda_{\text{QCD}}$ and $\Lambda_{\text{QCD}} \gg m_q$, it is natural to assume two double expansions, namely $Q/\Lambda_\chi$ and $\Lambda_{\text{QCD}}/m_Q$. For the first expansion parameter $Q/\Lambda_\chi$, the soft scale is identified as $Q \sim m_\pi \sim p_\pi$, and the hard scale can be identified as $\Lambda_\chi = 4\pi f \approx 1$ GeV. The hard scale is associated with chiral symmetry breaking. At the hadronic level, the second expansion parameter with the inverse of the heavy quark mass can be related to the hyperfine splitting of heavy mesons where $\frac{\Lambda_{\text{QCD}}^2}{m_Q} \sim m_1 - m_0$.

Based on the Weinberg’s theorem, one can write down the leading order
of the heavy hadron chiral Lagrangian [42, 59]

\[
\mathcal{L}_1 = -\text{Tr}[\mathcal{H}_a (i v \cdot D_{ba} - \delta_H \delta_{ab}) \mathcal{H}_b] + \text{Tr}[\mathcal{S}_a (i v \cdot D_{ba} - \delta_S \delta_{ab}) \mathcal{S}_b],
\]

that respects both chiral and heavy quark spin symmetries. The effective fields \( \mathcal{H}_a \) and \( \mathcal{S}_a \) interact with even numbers of the pion fields via their coupling to the vector connection \( \Gamma^\mu \) that is embodied in the chiral covariant derivative \( D_\mu \), i.e.

\[
D_\mu = \partial_\mu + \Gamma_\mu,
\]

where the expression of \( \Gamma^\mu \) is given in Eq. (2.60). The factors \( \delta_H \) and \( \delta_S \) are residual masses of \( \mathcal{H}_a \) and \( \mathcal{S}_a \). The residual masses \( \delta_H \) and \( \delta_S \) are counted as \( O(m_\pi) \); this is because the difference between them is of order of the pion mass, i.e. \( \delta_S - \delta_H \sim m_\pi \).

As is obvious from the chiral Lagrangian represented by Eq. (3.6), the effective fields \( (\mathcal{H}_a, \mathcal{S}_a) \) are shifted by the same amount (e.g. \( \delta_H \) and \( \delta_S \)). For instance, \( H_\mu^a \) and \( H_a \), have an equal mass when heavy quark spin symmetry is conserved.

Expressing the chiral Lagrangian in terms of the heavy meson field is done by performing the trace of the Dirac gamma matrices, the first part of \( \mathcal{L}_1 \) is

\[
\mathcal{L}_1^H = 2 \left( i H^\dagger_\mu a v \cdot \partial H_a - i H^\dagger_\mu a v \cdot \partial H^\mu_a - \delta_H [H^\dagger_\mu a H_a - H^\dagger_\mu a H^\mu_a] 
- \frac{\gamma_{ba}}{4f^2} v \cdot \phi \times \partial \phi [H^\dagger_\mu a H_b - H^\dagger_\mu a H^\mu_b] \right).
\]

Here, we follow Ref. [42] in normalizing the effective fields to unity, i.e. \( \mathcal{H}_a \)
(S_\alpha), which in turn means that H_\alpha and H_\mu^\alpha (S_\alpha and S_\alpha^\mu) must be normalized to 2.

The corresponding Feynman rules can be derived from the above Lagrangian:

- The heavy meson pseudoscalar field propagator

\[ \frac{i}{2(v.k - \delta_H)}. \] (3.9)

- The heavy meson pseudoscalar field, normalized such that

\[ \langle 0 | H_\alpha | H_\beta \rangle = \delta_{\alpha\beta}. \] (3.10)

- The heavy meson vector field propagator

\[ \frac{-i(g^{\mu\nu} - v^\mu v^\nu)}{2(v.k - \delta_H)}. \] (3.11)

- The heavy meson vector field, which can be normalized via

\[ \langle 0 | H_\mu^a | H_\nu^b \rangle = \varepsilon_\mu \delta_{ab}, \] (3.12)

where \( \varepsilon_\mu \) is the polarization of the vector meson.

- To maintain the heavy quark symmetry at quantum level, we choose to define the nonrelativistic meson field in 4 dimensions.

- The contraction of the heavy meson vector field with the velocity van-
ishes for both on-shell and off-shell heavy vector mesons

\[ v_\mu H^\mu_a = 0, \]

(3.13)

where for the choice \( v_\mu = (1, 0) \), the spatial components of the vector field are nonzero.

- From the chiral Lagrangian, heavy meson pseudoscalar and vector fields are shifted by the same amount \( \delta_H \),

\[ \mathcal{L}_{1m}^H \sim -\delta_H [H^1_a H_a + H^1_{ia} H_{ia}], \]

(3.14)

which is only valid if the heavy quark spin symmetry holds.

In addition to \( \mathcal{L}_1 \), the full expression of the heavy hadron chiral Lagrangian contains other terms describing the axial vector coupling of the heavy meson fields to the pions. Those terms provide the pion loops contribution to the self energy of \( D \)-mesons by including the interaction between axial vector vielbein \( u_\mu \) and the spin of the light degrees of freedom.

In fact, the interaction of the heavy quark spin with the axial vector vielbein is forbidden by the heavy quark spin symmetry [42]. The axial part of the heavy hadron chiral Lagrangian is [42, 59]

\[ \mathcal{L}_2 = g \text{Tr}[\bar{H}_a \mathcal{H}_b u_{ib} \gamma_5] + g' \text{Tr}[\bar{S}_a S_b u_{ib} \gamma_5] + h \text{Tr}[\bar{H}_a S_b u_{ib} \gamma_5 + h.c.], \]

(3.15)

where the coupling constant \( g \) (\( g' \)) measures the strength of transitions within odd (even) parity charm meson states. The transitions between odd and
even parity states is measured by the coupling constant $h$. The axial vector vielbein $u^\mu_{ba}$ is given by

$$u^\mu_{ba} \simeq -\frac{1}{2f} \partial^\mu \phi \cdot \tau_{ba}. \quad (3.16)$$

### 3.1 Self-Energies of Charm Mesons

Fig. 3.1 and Fig. 3.2 show the Feynman diagrams of the one-loop correction to the masses of $D$ mesons. In evaluating loop integrals for these diagrams, one has to be careful with the tensor structure to get the correct expressions. For this purpose, we will calculate loop integrals for diagrams (a) − (e) in Fig. 3.1. The results hold for diagrams with similar tensor structure of even parity sector as shown in Fig. 3.2.

Let us start with the loop diagram (a) in Fig. 3.1, which contributes to the self-energy of the $H_1$ field, i.e. the $D^+$

$$i \Sigma_{H_1}^{(a)} = 3(-2)^2 \left( \frac{g}{2f} \right)^2 \mu^{4-d} \int \frac{d^dq}{(2\pi)^d} \frac{q^\mu q^{\nu} (g_{\mu\nu} - v_\mu v_\nu)}{2(q \cdot v - \omega_a + i\epsilon)(q^2 - m_\pi^2 + i\epsilon)}$$

$$= 2 \left[ 3 \left( \frac{g}{2f} \right)^2 (g_{\mu\nu} - v_\mu v_\nu) \mu^{4-d} \int \frac{d^dq}{(2\pi)^d} \frac{q^\mu q^{\nu}}{(q \cdot v - \omega_a + i\epsilon)(q^2 - m_\pi^2 + i\epsilon)} \right], \quad (3.17)$$

where $\omega$ is the mass difference between internal and external heavy meson states, i.e. $\omega = m_{\text{off-shell}} - m_{\text{on-shell}}$. The factor 3 results from $(\tau_1^2)_{\alpha\beta} = 3 \delta_{\alpha\beta}$, where for one-loop diagrams in which a single pion is exchanged $\alpha = \beta$, so $\delta_{\alpha\alpha} = 1$.

The chiral loop integral is divergent. However, there are many ways to regulate the above loop-integral and each one introduces a new momentum
Figure 3.1: The self-energy diagrams for the $H_1$ and $H_1^*$ fields.

Figure 3.2: The self-energy diagrams for the $S_1$ and $S_1^*$ fields.

scale of which physical observables must be independent. In field theory, the
so-called Dimensional Regularization scheme (DR) is widely used since it pre-
serves gauge and chiral symmetries as well as Lorentz (Galilean) invariance
for relativistic (nonrelativistic) systems [81].

For loop integrals containing two or more powers of $q$ (momentum of the
internal pion) in the numerator, the standard procedure of evaluating them is
by breaking them up into simple integrals that can then be easily calculated
Thus, one can write

\[ i \mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu}{(q \cdot v - \omega + i\epsilon)(q^2 - m_\pi^2 + i\epsilon)} = g^{\mu\nu} J_2 + v^\mu v^\nu J_3, \]  

(3.18)

where

\[ J_2 = \frac{1}{d-1} [(m_\pi^2 - \omega^2) J_0 - \omega J_\pi], \]

(3.19)

and

\[ J_3 = \frac{1}{d-1} [(d \omega^2 - m_\pi^2) J_0 + \omega d J_\pi]. \]

(3.20)

The explicit expression for \( J_0 \) is

\[
J_0 = i \mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q \cdot v - \omega + i\epsilon)(q^2 - m_\pi^2 + i\epsilon)}
= \frac{\omega}{8\pi^2} [1 + R - \ln(\frac{m_\pi^2}{\mu^2}) - \frac{2}{\omega} F(\omega, m_\pi)],
\]

and the expression for \( J_\pi \) is

\[
J_\pi = i \mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - m_\pi^2 + i\epsilon)} = \frac{m_\pi^2}{16\pi^2} [\ln(\frac{m_\pi^2}{\mu^2}) - R],
\]

where \( R = \frac{2}{4-d} - \gamma_E + \ln(4\pi) + 1 \) contains a pole at \( d = 4 \). In these expressions, \( \mu \) is the renormalization scale. The function \( F(\omega, m_\pi) \) is

\[
F(\omega, m_\pi) = \begin{cases} 
-\sqrt{m_\pi^2 - \omega^2} \cos^{-1}(\frac{\omega}{m_\pi}), & m_\pi^2 > \omega^2, \\
\sqrt{\omega^2 - m_\pi^2} [i\pi - \cosh^{-1}(\frac{\omega}{m_\pi})], & \omega < -m_\pi, \\
\sqrt{\omega^2 - m_\pi^2} \cosh^{-1}(\frac{\omega}{m_\pi}), & \omega > m_\pi.
\end{cases}
\]

(3.21)

To use Dimensional Regularization consistently, one has to set \( d = 4 \) after
expanding $J_2$ and $J_3$ to first order in $4 - d$. If one sets $d = 4$ before expanding in powers of $4 - d$ as in Ref. [52], the expressions for $J_2$ and $J_3$ will be missing some finite pieces where $\frac{1}{d-1}R = \frac{1}{3}R + \frac{2}{9} \neq \frac{1}{3}R$. If there is only one integral, then the different constants can be absorbed by different renormalization schemes, i.e. this corresponds to some modified subtraction schemes. For the case of two integrals with different finite terms, there is no single consistent renormalization scheme, i.e. the differences cannot be hidden in renormalization schemes.

By expanding Eq. (3.19) and Eq. (3.20) to first order in $4 - d$ and then taking $d = 4$, we get

$$J_2 = \frac{1}{16\pi^2} \left[ \left( \frac{2}{3} \omega^3 - m^2_\pi \omega \right) \ln \left( \frac{m^2_\pi}{\mu^2} \right) + \frac{4}{3} (\omega^2 - m^2_\pi) F(\omega, m_\pi) \right. \quad \text{(3.22)}$$

$$- \frac{2}{3} \omega^3 (R + \frac{5}{3}) + \frac{1}{3} \omega m^2_\pi (3R + 4)]$$

and

$$J_3 = \frac{1}{16\pi^2} \left[ (2m^2_\pi \omega - \frac{8}{3} \omega^3) \ln \left( \frac{m^2_\pi}{\mu^2} \right) - \frac{4}{3} (4\omega^2 - m^2_\pi) F(\omega, m_\pi) \right. \quad \text{(3.23)}$$

$$+ \frac{8}{3} \omega^3 (R + \frac{7}{6}) - \frac{2}{3} \omega m^2_\pi (3R + 2)]$$

Now, by substituting Eq. (3.18) into Eq. (3.17), one gets

$$i \Sigma^{(n)}_{H_1} = 2 \left[ 3 \left( \frac{g}{2f} \right)^2 (g_{\mu\nu} - v_{\mu}v_{\nu})(-i (g^{\mu\nu} J_2 + v^\mu v^\nu J_3)) \right] \quad \text{(3.24)}$$

$$= 2 i \left[ 3 \left( \frac{g}{2f} \right)^2 (1 - g_{\mu\nu}g^{\mu\nu})J_2 \right].$$

As we have chosen to define the heavy meson fields in 4 dimensions, the
contraction of the metric tensors is $g_{\mu\nu}g^{\mu\nu} = 4$. This is quite different from regularizing gauge theories in which the components of the gauge boson fields are continued in $d$ dimensions to maintain the gauge invariance. In contrast, here it is important that regularization keeps the integrals of Figs. 3.1-(a), (c), and (d) equal. Our purpose is to preserve the heavy quark symmetry. As will be shown below, our choice of defining the meson field as 4 dimensional maintains this.

Thus, Eq. (3.24) becomes

$$i \Sigma^{(a)}_{H_1} = 2 i \left[ 3 \left( \frac{g^2}{2f} \right)^2 (-3 J_2) \right] = 2 i \left[ 3 \left( \frac{g^2}{2f} \right)^2 K_1(\omega_\pi, m_\pi) \right], \quad (3.25)$$

where in the last step we introduced the chiral function $K_1(\omega, m_\pi)$. This can be related to $J_2$ as follows

$$K_1(\omega, m_\pi) = -3 J_2 = -\frac{3}{d-1}[(m_\pi^2 - \omega^2)J_0 - \omega J_\pi]$$

$$= \frac{1}{16\pi^2} [(-2\omega^3 + 3m_\pi^2 \omega)\ln(m_\pi^2/\mu^2) - 4(\omega^2 - m_\pi^2)F(\omega, m_\pi)$$

$$+ 2\omega^3(R + \frac{5}{3}) - \omega m_\pi^2(3R + 4)], \quad (3.26)$$

where this represents the contribution to self-energy of charm mesons from one-loop diagrams with interacting particles belonging to the same doublets.

Now, we want to calculate the integral of the loop diagram in Fig. 3.1-(c),
which contributes to the self-energy of the vector charm meson

\[ i \Sigma_{H_1}^{(c)} = 3(-2)^2 \left( \frac{g}{2f} \right)^2 \left( -\mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{\epsilon \cdot q \epsilon \cdot q}{2(q \cdot v - \omega_c + i\epsilon)(q^2 - m^2 + i\epsilon)} \right) \]

\[ = 2 \left[ 3 \left( \frac{g}{2f} \right)^2 \left( -\epsilon^*_\mu \epsilon^\nu \mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu}{2(q \cdot v - \omega_c + i\epsilon)(q^2 - m^2 + i\epsilon)} \right) \right] \]

\[ = 2i \left[ 3 \left( \frac{g}{2f} \right)^2 \epsilon^*_\mu \epsilon^\nu \left( g_{\mu\nu} J_2 + \nu^\mu \nu^\nu J_3 \right) \right], \]

(3.27)

where the last line is obtained by using Eq. (3.18). Since \( v_\mu \epsilon^\mu = 0 \) and \( \epsilon^*_\mu \epsilon^\mu = -1 \), \( \Sigma_{H_1}^{(c)} \) is

\[ i \Sigma_{H_1}^{(c)} = 2i \left[ -3 \left( \frac{g}{2f} \right)^2 J_2 \right] = 2i \left[ \left( \frac{g}{2f} \right)^2 (-3 J_2) \right] \]

\[ = 2i \left[ \left( \frac{g}{2f} \right)^2 K_1(\omega_c, m_\pi) \right]. \]

(3.28)

The integral of one-loop diagram in Fig. 3.1-(d), which contributes to the self-energy of the vector charm meson, is

\[ i \Sigma_{H_1}^{(d)} = (2)^2 3 \left( \frac{g}{2f} \right)^2 \left( -\mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{\epsilon^{\mu \nu \rho \sigma} \epsilon^*_\mu \epsilon^\nu \epsilon^\rho \epsilon^\sigma (g_{\mu \sigma} - v_\mu v_\sigma) \epsilon^{\mu \nu \rho \sigma} \epsilon^*_\mu \epsilon^\nu \epsilon^\rho \epsilon^\sigma}{2(q \cdot v - \omega_d + i\epsilon)(q^2 - m_\pi^2 + i\epsilon)} \right) \]

\[ = 2i \left[ -3 \left( \frac{g}{2f} \right)^2 \epsilon^{\mu \nu \rho \sigma} \epsilon^*_\mu \epsilon^\nu \epsilon^\rho \epsilon^\sigma \epsilon^*_\mu \epsilon^\nu \epsilon^\rho \epsilon^\sigma J_2 \right]. \]

(3.29)

As \( v \cdot v = 1 \), \( \epsilon \cdot v = 0 \), and \( \epsilon \cdot \epsilon = -1 \), the contraction between indices of the
totally antisymmetric tensors yields\(^1\) \(-2\)! Thus, \(\Sigma^{(d)}_{H_1}\) becomes

\[
i \Sigma^{(d)}_{H_1} = 2i \left[ 3 \left( \frac{g}{2f} \right)^2 (-2J_2) \right] = 2i \left[ \left( \frac{g}{\sqrt{2}f} \right)^2 (-3J_2) \right]
\]

\[
= 2i \left( \frac{g}{\sqrt{2}f} \right)^2 K_1(\omega_d, m_\pi) .
\]

(3.30)

Clearly, our choice of defining meson fields in 4 dimensions, which gives \(g_\mu g^{\mu\nu} = 4\) for the loop integral of Fig. 3.2-(a), yields equal results to the loop integrals of Figs. 3.2-(c), and (d). The results of the diagrams in Figs. 3.2-(a), (c), and (d) are similar to the ones of Figs. 3.1-(a), (c), and (d) respectively.

Now, want to evaluate the loop integrals for graphs describe the interacting heavy mesons with opposite parity. To this end, let us begin with the second one-loop contribution to self-energy of \(H_1\) which is shown in Fig. 3.1-(b)

\[
i \Sigma^{(b)}_{H_1} = 3(-2)^2 \left( \frac{h}{2f} \right)^2 \mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{v \cdot q v \cdot q}{2(q \cdot v - \omega_b + i\epsilon)(q^2 - m_\pi^2 + i\epsilon)}
\]

\[
= 2 \left[ 3 \left( \frac{h}{2f} \right)^2 \nu_\mu \nu_\nu \mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{q^{\mu}q^{\nu}}{(q \cdot v - \omega_b + i\epsilon)(q^2 - m_\pi^2 + i\epsilon)} \right].
\]

(3.31)

\(^1\)To understand it, one can set \(\nu' = 0\) (for static heavy meson, i.e. \(v_{\nu'} = (1,0)\)), and \(\mu' = k = 3\) (polarization in the \(z\)-component) to find \(\varepsilon^{30ij}\varepsilon_{30k} = \varepsilon^{bij}\varepsilon_{0ij} = -2\)!. The Lorentz indices \(\sigma'\) and \(\rho'\) are replaced by ordinary indices \(i\) and \(j\) (\(i = 1, 2; j = 1, 2\)) respectively.
Similarly, substituting Eq. (3.18) into Eq. (3.31) gives

\[
i \Sigma_{H_1}^{(b)} = 2i \left[ 3 \left( \frac{h}{2f} \right)^2 v_\mu v_\nu (g^{\mu\nu} J_2 + v^\mu v^\nu J_3) \right]
= 2i \left[ 3 \left( \frac{h}{2f} \right)^2 (J_2 + J_3) \right] = 2i \left[ 3 \left( \frac{h}{2f} \right)^2 K_2(\omega_b, m_\pi) \right],
\]

(3.32)

where

\[
K_2(\omega, m_\pi) = J_2 + J_3 = [\omega^2 J_0 + \omega J_\pi] = \frac{1}{16\pi^2} \left[ (-2\omega^3 + m_\pi^2) \ln \left( \frac{m_\pi^2}{\mu^2} \right) - 4\omega^2 F(\omega, m_\pi) \right]
+ 2\omega^3 (1 + R) - \omega m_\pi^2 R.
\]

(3.33)

For the one-loop diagram with (heavy) interacting particles belonging to different doublets, the contribution to the self-energy is given by the chiral function \(K_2(\omega, m_\pi)\).

The integral of the one-loop diagram shown in Fig. 3.1-(e), which contributes to the self-energy of the vector meson, is

\[
i \Sigma_{H_1}^{(c)} = 3(-2)^2 \left( \frac{h}{2f} \right)^2 \mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{\epsilon_\mu^* v \cdot q (g^{\mu\nu} - v^\mu v^\nu) v \cdot q \epsilon_\nu}{2(q \cdot v - \omega_c + i\epsilon)(q^2 - m_\pi^2 + i\epsilon)}
= 2 \left[ 3 \left( \frac{h}{2f} \right)^2 \epsilon_\mu^* v \epsilon_\nu (g^{\mu\nu} - v^\mu v^\nu) v_\alpha v_\beta \mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{q^\alpha q^\beta}{(q \cdot v - \omega_c + i\epsilon)(q^2 - m_\pi^2 + i\epsilon)} \right]
= 2 \left[ -3 \left( \frac{h}{2f} \right)^2 v_\alpha v_\beta \mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{q^\alpha q^\beta}{(q \cdot v - \omega_c + i\epsilon)(q^2 - m_\pi^2 + i\epsilon)} \right].
\]

(3.34)
Similarly, substituting Eq. (3.18) into Eq. (3.34) gives

\[ i \Sigma_{H_1}^{(e)} = 2i \left[ 3 \left( \frac{h}{2f} \right)^2 v_\alpha v_\beta (g^{\alpha\beta} J_2 + v^\alpha v^\beta J_3) \right] = 2 \left[ \frac{3}{2} \left( \frac{h}{\sqrt{2} f} \right)^2 (J_2 + J_3) \right] \]

\[ = 2i \left[ 3 \left( \frac{h}{2 f} \right)^2 K_2(\omega, m_\pi) \right]. \]

(3.35)

The loop integrals of the diagrams in Fig. 3.2-(b) and (e) are similar to the result of Figs. 3.1-(a) and (e) respectively. The chiral functions can be defined in the $\overline{\text{MS}}$-scheme, i.e.

\[ K_1(\omega, m) = \frac{1}{16\pi^2} \left[ (-2\omega^3 + 3m^2\omega)\ln \left( \frac{m^2}{\mu^2} \right) - 4(\omega^2 - m^2)F(\omega, m) + \frac{16}{3}\omega^3 - 7\omega m^2 \right], \]

\[ K_2(\omega, m) = \frac{1}{16\pi^2} \left[ (-2\omega^3 + m^2\omega)\ln \left( \frac{m^2}{\mu^2} \right) - 4\omega^2 F(\omega, m) + 4\omega^3 - \omega m^2 \right], \]

(3.36)

where the function $F(\omega, m)$ is given in Eq. (3.21).

Before proceeding, let us add a few remarks about the expressions in other papers:

- The authors of Ref. [52] did not use Dimensional Regularization consistently. They set $d = 4$ before expanding $J_2$ and $J_3$ in powers of $4 - d$. This in turn leads to the wrong expressions for the chiral loop functions that are constructed from $J_2$ and $J_3$.

- The expression for $K_1(\omega, m_\pi)$ in Ref. [59] does not agree with our expression (Eq. (3.36)). Some finite pieces are missed due to inconsistent
use of Dimensional Regularization, as described above.

- Our expression for $K_2(\omega, m_\pi)$ agrees with the expression presented in Ref. [59] which is obtained using the $\overline{\text{MS}}$ scheme. This agreement is due to the accidental cancelation of the factor $\frac{1}{d-1}$ when adding the functions $J_2$ and $J_3$ to define the chiral loop integral $K_2(\omega, m_\pi)$, where $K_2(\omega, m_\pi) = J_2 + J_3 = \omega^2 J_0 + \omega J_\pi$.

- Moreover, our expression for $K_2(\omega, m_\pi)$ agrees with results presented in Ref. [90] when using the $\tilde{\text{MS}}$ scheme in which the factor $R$ is canceled by bare coefficients.

- In Ref. [63], the expression of $C_{20}$ in equation (C.36), which corresponds to $J_3$ in our notation, is wrong. The author used $C_{20} = m_\pi^2 J_{\pi N}(0, \omega) - C_{21}$ instead of $C_{20} = m_\pi^2 J_{\pi N}(0, \omega) - d C_{21}$. This leads to the wrong expressions for the chiral loop functions that are constructed from $J_2$ and $J_3$. In our notation, the momentum integrals $J_{\pi N}(0, \omega)$ and $C_{21}$ in Ref. [63] correspond to $J_0(-\omega)$ and $J_2(-\omega)$ respectively.

- The expression for $K_2(\omega, m_\pi)$ which is given in Ref. [89] is also wrong. The authors have used the results of Ref. [63] to define $K_2(\omega, m_\pi)$.

To cancel the infinite parts resulting from regularization (separate infinite from finite pieces) of the loop diagrams, we introduce the Lagrangian $\mathcal{L}_3$ that contains counter terms which either violate the heavy quark symmetry or the
chiral symmetry. The Lagrangian is of the form \[42, 59\]

\[
\mathcal{L}_3 = -\frac{\Delta_H}{8} \text{Tr}[\mathcal{H}_a \sigma^{\mu \nu} \mathcal{H}_a \sigma_{\mu \nu}] + \frac{\Delta_S}{8} \text{Tr}[\mathcal{S}_a \sigma^{\mu \nu} \mathcal{S}_a \sigma_{\mu \nu}]
\]

\[
+ a_H \text{Tr}[\mathcal{H}_a \mathcal{H}_b] m_{ba}^\xi - a_S \text{Tr}[\mathcal{S}_a \mathcal{S}_b] m_{ba}^\xi + \sigma_H \text{Tr}[\mathcal{H}_a \mathcal{H}_a] m_{bb}^\xi - \sigma_S \text{Tr}[\mathcal{S}_a \mathcal{S}_a] m_{bb}^\xi
\]

\[
- \frac{\Delta_H^{(a)}}{8} \text{Tr}[\mathcal{H}_a \sigma^{\mu \nu} H_b \sigma_{\mu \nu}] m_{ba}^\xi + \frac{\Delta_S^{(a)}}{8} \text{Tr}[\mathcal{S}_a \sigma^{\mu \nu} S_b \sigma_{\mu \nu}] m_{ba}^\xi
\]

\[
- \frac{\Delta_H^{(\sigma)}}{8} \text{Tr}[\mathcal{H}_a \sigma^{\mu \nu} H_a \sigma_{\mu \nu}] m_{bb}^\xi + \frac{\Delta_S^{(\sigma)}}{8} \text{Tr}[\mathcal{S}_a \sigma^{\mu \nu} S_a \sigma_{\mu \nu}] m_{bb}^\xi,
\]

(3.37)

where the hyperfine splittings of the D-meson states are measured by \(\Delta, \Delta^{(a)}, \Delta^{(\sigma)}\). These coefficients manifestly vanish in the heavy quark limit, and are counted as \(O(Q)\). The quark mass matrix \(m^{\xi}\) breaks chiral symmetry. In the isospin limit \(m_u = m_d = m, m_{\alpha \beta}^{\xi} = \frac{1}{2} (\xi m_q \xi + \xi^\dagger m_q \xi^\dagger)_{\alpha \beta} \approx (m_q)_{\alpha \beta} + O(\phi^2)\). It should be noted that \(m_{ba}^\xi\) and \(m_{bb}^\xi\) give different contributions to the D-mesons self energy. In fact, \(m_{ba}^\xi\) is responsible for mass splitting between strange and nonstrange heavy D-mesons in three quark flavors, \(SU(3)_V\). The coefficients \((a, \sigma)\) present in the chirally breaking terms are dimensionless and their terms are counted as two powers of the pion mass since \((m_q) \propto m_\pi^2\). The chiral order of the first and second lines are \(O(Q)\) and \(O(Q^2)\) respectively. The last two lines scale as \(O(Q^3)\). In terms of \(H_a, H_a^\mu, S_a,\) and \(S_a^\mu\) meson
fields, $\mathcal{L}_3$ is

$$\mathcal{L}_3 = 2 \left( \frac{\Delta H}{4} [H_{\mu a}^\dagger H_{\mu}^a + 3H_{\mu a}^\dagger H_{\mu}] + \frac{\Delta S}{4} [S_{\mu a}^\dagger S_{\mu}^a + 3S_{\mu a}^\dagger S_{\mu}] \right)$$

$$+ a_H[H_{\mu a}^\dagger H_b^a - H_{\mu a}^\dagger H_a^b]m_{ba}^\xi + a_S[S_{\mu a}^\dagger S_b^a - S_{\mu a}^\dagger S_a^b]m_{ba}^\xi$$

$$+ \sigma_H[H_{\mu a}^\dagger H_a^\mu - H_{\mu a}^\dagger H_a^\mu]m_{ba}^\xi + \sigma_S[S_{\mu a}^\dagger S_a^\mu - S_{\mu a}^\dagger S_a^\mu]m_{ba}^\xi$$

$$+ \frac{\Delta H}{4} [H_{\mu a}^\dagger H_b^a + 3H_{\mu a}^\dagger H_a^b]m_{ba}^\xi + \frac{\Delta S}{4} [S_{\mu a}^\dagger S_b^a + 3S_{\mu a}^\dagger S_a^b]m_{ba}^\xi$$

This due to breaking of the heavy quark symmetry, which leads to the hyperfine splitting of heavy meson states.

The masses of $H_a$ and $H_{\mu a}$ mesons, for instance, are shifted by an unequal amount, i.e.

$$\frac{\Delta H}{4} [H_{\mu a}^\dagger H_{\mu}^a + 3H_{\mu a}^\dagger H_{\mu}] \sim \frac{\Delta H}{4} [-H_{\mu a}^\dagger H_{\mu a} + 3H_{\mu a}^\dagger H_{\mu a}].$$

This due to breaking of the heavy quark symmetry, which leads to the hyperfine splitting of heavy meson states.

### 3.2 Mass formula for non-strange charm mesons

The physical mass of the particle is determined from the pole of the full propagator. For example, let us consider the full propagator $T_{H_a}$ for pseudoscalar
charm meson\(^2\) \(H_a\). The physical mass of \(H_a\) is determined from the pole of

\[
T_{H_a} = \frac{1}{2(v \cdot k_{H_a} - \delta_{H_a}^{\text{tree}}) - \Sigma_{H_a}},
\]

(3.40)

where the factor 2 results from the choice of normalizing the heavy meson field \(H_a\). The on-shell condition is

\[
v \cdot k_{H_a}' = m_{H_a}^{\text{phys.}} = \delta_{H_a}^{\text{tree}} + \frac{1}{2} \Sigma_{H_a},
\]

(3.41)

where \(\delta_{H_a}^{\text{tree}}\) represents the tree level contribution (without the factor of 2 which is already used in the propagator) and \(\Sigma_{H_a}\) is the sum of all one-loop contributions. We will label tree level contributions by \(m_{H_a}^{\text{tree}}\) instead of \(\delta_{H_a}^{\text{tree}}\) to avoid confusion with bare coefficients of the tree level. According to \(\mathcal{L}_1\) and \(\mathcal{L}_3\), the tree level contribution for pseudoscalar charm meson is

\[
m_{H_a}^{\text{tree}} = \delta_H - \frac{3}{4} \Delta H + a_H m_a + \sigma_H \overline{m} - \frac{3}{4} \Delta^{(a)} H m_a - \frac{3}{4} \Delta^{(\sigma)} H \overline{m},
\]

(3.42)

where \(\overline{m} = 2 m_a\). The loop contributions are

\[
\Sigma_{H_a} = 2 \left[ \frac{3g^2}{4f^2} K_1(m_{H_2} - m_{H_a}, m_a) + \frac{3h^2}{4f^2} K_2(m_{S_0} - m_{H_a}, m_a) \right].
\]

(3.43)

\(^2\)Although we are working in the isospin limit in which \(\mathcal{H}_1 = \mathcal{H}_2\), we prefer to keep the isospin indices \(a\).
By substituting Eq. (3.42) and Eq. (3.43) into Eq. (3.41), one can define the expression for the physical mass of pseudoscalar charm meson,

\[ m_{H_a} = \delta_H - \frac{3}{4} \Delta_H + a_H m_a - \frac{3}{4} \Delta^{(a)}_H m_a - \frac{3}{4} \Delta^{(\sigma)}_H \sigma_{H} - \frac{3}{4} \Delta^{(\pi)}_H \pi - 3 \frac{g^2}{4 f^2} K_1(m_{H_a} - m_{H_a}, m_\pi) + \frac{3h^2}{4 f^2} K_2(m_{S_a} - m_{H_a}, m_\pi). \]  

(3.44)

Similarly, one can define the expressions for the physical masses of other charm mesons:

- **Heavy vector meson:**

\[ m_{H^*} = \delta_H + \frac{1}{4} \Delta_H + a_H m_a + \sigma_H \sigma_{H} + \frac{1}{4} \Delta^{(\theta)}_H m_a + \frac{1}{4} \Delta^{(\phi)}_H \phi - \frac{3}{4} \Delta^{(\pi)}_H \pi - \frac{3}{4} \Delta^{(\sigma)}_H \sigma_{H} - 3 \frac{g^2}{4 f^2} K_1(m_{H^*} - m_{H^*}, m_\pi) + 2 \frac{h^2}{4 f^2} K_1(0, m_\pi) + 3 \frac{h^2}{4 f^2} K_2(m_{S^*} - m_{H^*}, m_\pi). \]  

(3.45)

- **Heavy scalar meson:**

\[ m_{S_a} = \delta_s - \frac{3}{4} \Delta_S + a_S m_a + \sigma_S \sigma_{S} - \frac{3}{4} \Delta^{(a)}_S m_a - \frac{3}{4} \Delta^{(\sigma)}_S \sigma_{S} + \frac{3g^2}{4 f^2} K_1(m_{S_a} - m_{S_a}, m_\pi) + \frac{3h^2}{4 f^2} K_2(m_{H_a} - m_{S_a}, m_\pi). \]  

(3.46)

- **Heavy axial vector meson:**

\[ m_{S^*} = \delta_s + \frac{1}{4} \Delta_S + a_S m_a + \sigma_S \sigma_{S} + \frac{1}{4} \Delta^{(\theta)}_S m_a + \frac{1}{4} \Delta^{(\phi)}_S \phi - \frac{3}{4} \Delta^{(\pi)}_S \pi + 3 \frac{g^2}{4 f^2} K_1(m_{S^*} - m_{S^*}, m_\pi) + \frac{3h^2}{4 f^2} K_2(m_{H^*} - m_{S^*}, m_\pi). \]  

(3.47)
In fact, the chiral corrections give real and imaginary contributions to the masses of heavy mesons. As an illustration, if \( \omega^2 < m_\pi^2 \), the heavy meson is stable and the chiral corrections to the particles mass come from the real part of the chiral functions, \((K_{1r}, K_{2r})\), where the subscript \( r \) stands for real. However, if \( \omega < -m_\pi \), the particle is unstable and its mass is complex, i.e. \( m = m^r - \frac{i}{2} \Gamma \), where \( m^r \) is the (physical) mass which contains all contributions from the chiral corrections and the tree level. The quantity \( \Gamma \) is the decay rate of the unstable particle that can be obtained from the imaginary parts of the chiral loop, \((K_{1i}, K_{2i})\), where the subscript \( i \) stands for imaginary.

3.2.1 Extracting the coupling constant of even and odd charm meson transitions

The rest of this Chapter is devoted to extract the numerical value of the coupling constant \( h \) using the available experimental data on the masses and the decay widths of states of the even parity \( \frac{1}{2}^+ \)-doublet. The constant \( h \) should have a unique value according to the heavy quark symmetry. Thus, the extracted value of \( h \) from different transitions provides a test of the heavy quark symmetry.

In the current problem, the strong decay of the excited charm mesons can be estimated by using \( S_l \), the spin of light degree of freedom. In general, these transitions have very large phase space since \( m_S - m_H > m_\pi \). The possible transitions will now be explained by the virtue of heavy quark symmetry.

\footnote{In our notation \( \omega \) is defined as \( \omega = m_{\text{off-shell}} - m_{\text{on-shell}} \).}
By considering $D^{1+} \rightarrow D^{1-} \pi$ decay, the angular momentum of the outgoing pion is $l_\pi = 0, 1$, which is determined from

$$l_\pi = |S_{f} - S_{i}| = |S_{f} + S_{i}|.$$  

For the transition $\frac{1}{2}^+ \rightarrow \frac{1}{2}^- + l_\pi$, both $S_{f}$ and $S_{i}$ are $\frac{1}{2}$. The parity of the pion is obtained by the relation $P_\pi = (-1)^{l+1}$. According to the conservation of the total angular momentum and parity, the pion is emitted by an $s$-wave, viz. $l_\pi = 0$. Of course, electric charge and isospin quantum numbers are conserved in the strong interactions. Thus, the allowed possible decays for the even-parity $\frac{1}{2}^+$-doublet are

$$D^0 \rightarrow D^{*+} \pi^- \text{ and } D^0 \rightarrow D^{+} \pi^-.$$  

The decay rate for above decay channels can be calculated using Fermi’s Golden rule

$$\Gamma_{D_i \rightarrow D_f \pi} = \frac{3\hbar^2}{8\pi f^2} \frac{m_{D_f}}{m_{D_i}} E_\pi^2 |\vec{q}|,$$  

where the subscripts $i, f$ stand for the initial and the final $D$-meson states respectively. $|\vec{q}|$ is the three-momentum vector of the outgoing pion and $E_\pi$ is its energy.

Now, we want show how the formula of the Fermi’s Golden rule is approximately included in self energies of mesons. In our calculation, the expression of the decay rate of any unstable particle is obtained by comparing terms of the mass $m$ derived in the previous section, i.e. by adding the tree and the chiral corrections, with the mass formula of the unstable particle.
\[ m = m^r - \frac{i}{2} \Gamma. \]

For the transition \( D_{1/2}^+ \rightarrow D_{1/2}^- \pi \), the only loop contribution arises from the \( K_2 \) function, where

\[ K_2 = K_{2r} + K_{2i} = K_{2r} - \frac{i}{4\pi} \omega^2 \sqrt{\omega^2 - m^2_\pi}. \tag{3.49} \]

Thus, the complex mass of the unstable particle reads

\[ m = m^{\text{tree}} + \Sigma = m^r - \frac{3i}{16\pi \hbar^2} \omega^2 \sqrt{\omega^2 - m^2_\pi}. \tag{3.50} \]

By comparing Eq. (3.50) with \( m = m^r - \frac{i}{2} \Gamma \), one concludes

\[ \Gamma = \frac{3\hbar^2}{8\pi f^2} \omega^2 \sqrt{\omega^2 - m^2_\pi} \]
\[ = \frac{3\hbar^2}{8\pi f^2} E^2_\pi |\vec{q}|, \tag{3.51} \]

where the second line is obtained by writing

\[ \omega^2 \sqrt{\omega^2 - m^2_\pi} = (\omega^2 - m^2_\pi + m^2_\pi)|\vec{q}| = (|\vec{q}|^2 + m^2_\pi)|\vec{q}| = E^2_\pi |\vec{q}|. \tag{3.52} \]

Obviously, Eq. (3.48) is considered as a correction to Eq. (3.51). The heavy quark limit is attained by taking \( \frac{m_{D^+}}{m_{D^+_s}} = 1 \) in Eq. (3.48). Roughly speaking, the factor \( \frac{m_{D^+}}{m_{D^+_s}} \), which reflects the physical world, measures the power of applying the heavy quark symmetry to any heavy-light meson systems. For instance, the factor of the heavy charmed meson is nearly \( \frac{m_{D^+_s}}{m_{D^+_s}} \approx 0.8 \); it gives 20% corrections to our theoretical calculation of the decay width which fits only in the heavy quark limit. Of course,
mass ratio $\frac{m_B}{m_{\pi}}$ is expected to approach unity in the $B$-meson sector where the bottom quark is massive in comparison to the charm quark and hence the heavy quark symmetry is more accurate.

From Eq. (3.51), the coupling constant $h$ satisfies

$$h^2 = \frac{8\pi f^2}{3} \frac{\Gamma}{\sqrt{\omega^2 - m^2_{\pi}}}.$$  

(3.53)

The numerical value of $h$ can be extracted by using the empirical values of the decay rates and the masses reported by BELLE, BABAR, FOCUS and CLEO experiments, see Table below.

<table>
<thead>
<tr>
<th>States</th>
<th>Experiment</th>
<th>Channel</th>
<th>Mass (MeV)</th>
<th>Width (MeV)</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_0'(2400)^0$</td>
<td>PDG [1]</td>
<td>$D^+\pi^-$</td>
<td>2318 ± 29</td>
<td>267 ± 40</td>
<td>0.47 ± 0.06</td>
</tr>
<tr>
<td>$D_1'(2430)^0$</td>
<td>BELLE[55]</td>
<td>$D^{**}\pi^-$</td>
<td>2427 ± 36</td>
<td>384$^{+130}_{-105}$</td>
<td>0.64$^{+0.14}_{-0.12}$</td>
</tr>
<tr>
<td>$D_1'(2430)^0$</td>
<td>CLEO[56]</td>
<td>$D^{**}\pi^-$</td>
<td>2461$^{+53}_{-48}$</td>
<td>290$^{+110}_{-91}$</td>
<td>0.49$^{+0.13}_{-0.11}$</td>
</tr>
</tbody>
</table>

Table 3.1: The values of $h$ are obtained in the heavy quark limit. In extracting values of $h$, we have used $f = 92.4$ MeV, $m_\pi = 140$ MeV, $m_{D^+} = 1869.61$ MeV, and $m_{D^{**}} = 2010.26$ MeV [1].

The masses and decay widths for PDG are obtained by averaging the values reported by BELLE, BABAR and FOCUS experiments. It is clear that the extracted values of the coupling constant are consistent within their uncertainties. Consequently, these results are consistent with heavy quark symmetry at this level. The measured decay widths and masses of $D_1^0$ have large experimental errors. Therefore improved experiments are needed to reduce the error bars and to test the accuracy of the symmetry.

From Table 3.1, the difference between central values of the coupling...
constant for $D_0^*(2400)^0$ and $D_1^*(2430)^0$ reported by PDG and BELLE is $\sim 17\%$, which indicates the symmetry is good in the charmed sector. It is expected to be very good in the case of the bottom sector due to the large mass of the bottom quark.

The values of $h$ shown in Table 3.1 are obtained in the heavy quark limit to examine the heavy quark symmetry in the charm sector. In the next Chapter, we will use the physical values of charm meson masses and coupling constants to determine LECs of the HHChPT Lagrangian. For this purpose it is convenient to use the coupling constants that are extracted using Fermi’s Golden rule in which the mass ratio $\frac{m_{D^*}}{m_{D_i}}$ is not unity (which corresponds to values of $h$ in Table 3.1), but is determined from the physical masses. This work has been already done by the authors of Ref. [60]. In [60], the values of the $h$ are extracted from the widths of $D_0^*(2400)^0$, $D_0^*(2400)^\pm$, and $D_1^*(2430)^0$. These are $h = 0.61 \pm 0.07$, $h = 0.50 \pm 0.06$, and $h = 0.8 \pm 0.2$ respectively. The weighted average for these values is $h = 0.56 \pm 0.04$. For the case of strong transition within states in $1^-_2$-doublet, the coupling constant $g = 0.64 \pm 0.075$ was extracted from the measured width of the charged vector meson $D^{*\pm}$ [60]. Having determined $g$, we can calculate the width of the neutral vector meson $D^{*0}$ using Fermi’s Golden rule

$$
\Gamma_{D^*} = \frac{\sqrt{2}}{3\pi} \left( \frac{g}{2\sqrt{m_\pi f}} \right)^2 \left( \frac{m_\pi M_{D^*}}{m_\pi + M_{D^*}} \right)^{\frac{1}{2}} (\Delta H - m_\pi)^{\frac{3}{2}} = 0.0453 \text{MeV, (3.54)}
$$

where we have used the following numerical values [1]:

$m_\pi = 134.97 \text{MeV}$, $M_{D^*} = 1864.84 \text{MeV}$, $M_{D^*} = 2006.96 \text{MeV}$, $f = 92.4 \text{MeV}$. 

96
As there is no experimental information about strong transition within states in $\frac{1}{2}^+\text{-doublet}$, the coupling constant $g'$ which controls such transitions cannot be measured. In the calculation undertaken in the next Chapter, we take its value to range from 0 to 1.
Chapter 4

HHChPT for Charm and Bottom Mesons

The previous Chapter gives a pedagogical introduction to the HHChPT. In this Chapter, we study the masses of the low-lying charm and bottom mesons within the framework of HHChPT. We work to third order in the chiral expansion, where meson loops contribute. The lowest odd- and even-parity, strange and nonstrange charm mesons provide enough constraints to determine only certain linear combinations of the low-energy constants (LECs) in the effective Lagrangian. We comment on how lattice QCD could provide further information to disentangle these constants. Then we use the results from the charm sector to predict the spectrum of odd- and even-parity of the bottom mesons. The predicted masses from our theory are in good agreement with experimentally measured masses.
4.1 LECs from Charm Meson Spectrum

The masses and widths of the charm mesons in the odd and even parity sectors have been experimentally determined, for summaries, see Refs. [1, 60]. These patterns and interactions of the charm mesons are governed by the spin symmetry $SU(2)_s$ of the heavy quark and the chiral symmetry $SU(3)_L \times SU(3)_R$ of the light quarks. Incorporating both approximate symmetries in a single framework was achieved by defining the heavy hadron chiral perturbation theory (HHChPT) [59]. This effective theory can be used to study dynamics of mesons containing a single heavy quark.

Using this theory, the contributions to the physical masses of odd and even parity $D$-mesons up to the one-loop chiral corrections were calculated by Mehen and Springer [59] and Ananthanarayan et al. [61]. The authors of Ref. [59, 61] fitted expressions that depend nonlinearly on these constants and found multiple solutions, often with quite different numerical values for them. As a result, no clear pattern emerged from these fits. This is because the number of unknown low energy constants (LECs) in the effective chiral Lagrangian exceeds the number of experimentally known charm meson masses. Thus getting unique numerical values of the coefficients is impossible.

In this section, we attempt to remove this ambiguity by following a different approach to fit these parameters. We use the physical values of the masses in evaluating the chiral loops. As a consequence, the energy of any unstable particle is placed correctly relative to the decay threshold and the imaginary part of the loop integral can be related to the experimental decay
width. The second effect is to reduce the number of unknown parameters in comparison with the current experimental data on charm meson masses. Masses at tree level depend only on certain linear combinations of LECs. By using physical masses in chiral loops, the masses still depend linearly on these combinations. Therefore, one can express these combination of parameters directly in terms of the physical masses and loop integrals.

The numerical values generated in this manner include contributions from orders beyond $O(Q^3)$. These include divergences that we cannot cancel using counterterms in our Lagrangian. We therefore choose to use their $\beta$-functions to estimate the contributions from higher-order terms. In our fitting, we have used corrected expressions for the chiral loop functions, in contrast to the expressions presented in [52, 59] which use an inconsistent renormalization scheme.
4.2 Masses of the charm mesons within HHChPT

We begin by writing the most relevant expression of the heavy-hadron chiral Lagrangian up to the order $O(Q^3)$ [42, 59]:

$$
\mathcal{L} = -\text{Tr} [\mathcal{H}_a (iv \cdot D_{ba} - \delta_H \delta_{ab}) \mathcal{H}_b] + \text{Tr} [\mathcal{S}_a (iv \cdot D_{ba} - \delta_S \delta_{ab}) \mathcal{S}_b] \\
+ g \text{Tr} [\mathcal{H}_a \mathcal{H}_b \gamma_5] + g' \text{Tr} [\mathcal{S}_a \mathcal{S}_b \gamma_5] + h \text{Tr} [\mathcal{H}_a \mathcal{S}_b \gamma_5 + \text{h.c.}] \\
- \frac{\Delta_H}{8} \text{Tr} [\mathcal{H}_a \sigma_{\mu\nu} \mathcal{H}_b \sigma_{\rho\sigma}] + \frac{\Delta_S}{8} \text{Tr} [\mathcal{S}_a \sigma_{\mu\nu} \mathcal{S}_b \sigma_{\rho\sigma}] \\
+ a_H \text{Tr} [\mathcal{H}_a \mathcal{H}_b] m_{ba}^\xi - a_S \text{Tr} [\mathcal{S}_a \mathcal{S}_b] m_{ba}^\xi + \sigma_H \text{Tr} [\mathcal{H}_a \mathcal{H}_a] m_{bb}^\xi - \sigma_S \text{Tr} [\mathcal{S}_a \mathcal{S}_a] m_{bb}^\xi \\
- \frac{\Delta_H^{(a)}}{8} \text{Tr} [\mathcal{H}_a \sigma_{\mu\nu} \mathcal{H}_b \sigma_{\rho\sigma}] m_{ba}^\xi + \frac{\Delta_S^{(a)}}{8} \text{Tr} [\mathcal{S}_a \sigma_{\mu\nu} \mathcal{S}_b \sigma_{\rho\sigma}] m_{ba}^\xi \\
- \frac{\Delta_H^{(a)}}{8} \text{Tr} [\mathcal{H}_a \sigma_{\mu\nu} \mathcal{H}_b \sigma_{\rho\sigma}] m_{bb}^\xi + \frac{\Delta_S^{(a)}}{8} \text{Tr} [\mathcal{S}_a \sigma_{\mu\nu} \mathcal{S}_b \sigma_{\rho\sigma}] m_{bb}^\xi + ..., \quad (4.1)
$$

where $\mathcal{H}_a$ and $\mathcal{S}_a$ are the effective fields of the ground-state doublet and lowest lying excited-state doublet respectively. The index $a$ denotes the flavor of the light quark.

The members of the ground-state doublet are pseudoscalar mesons $J^P = 0^-(D^0, D^+, D_{s}^+)$ and vector mesons $J^P = 1^- (D^{*0}, D^{*+}, D_{s}^{*+})$, and the members of the excited-state doublet are scalar meson $J^P = 0^+ (D_0^{*0}, D_0^{*+}, D_{0s}^{*+})$ and axial vector mesons $J^P = 1^+ (D_1^0, D_1^+, D_{1s}^0)$.

The hyperfine splitting ($\Delta$) breaks heavy-quark symmetry and is counted as $O(Q)$, where $Q \sim m_\pi$. The residual masses $\delta_H$ and $\delta_S$ respect both symmetries. In practice, the difference $\delta_S - \delta_H$ is of the same order as the
Goldstone boson masses, i.e. $\delta_S - \delta_H \sim m_K$. Thus, we choose $\delta_H$ and $\delta_S$ to scale as $O(Q)$. The terms in the second line of Eq. (4.1) represent the axial part of the heavy hadron chiral Lagrangian. They describe the axial vector coupling of the heavy meson fields to light mesons $\pi, K, \eta$ that are contained in the axial vector vielbein $u^a$ which was defined in Eq. (2.60).

The matrix $m^{\xi}_{ba}$ in Eq. (4.1) is responsible for mass splitting between strange and nonstrange heavy $D$-mesons in three quark flavors, $SU(3)_V$. For three light quark flavors, $m^{\xi}_{\alpha\beta} = \frac{1}{2} \left( \xi M \xi + \xi^\dagger M \xi^\dagger \right)_{\alpha\beta} \approx M_{\alpha\beta} + O(\phi^2)$, where in the isospin limit $m_u = m_d = m_n$, $M$ is given by

$$M = \begin{pmatrix} m_n & 0 & 0 \\ 0 & m_n & 0 \\ 0 & 0 & m_s \end{pmatrix}. \quad (4.2)$$

From the above chiral Lagrangian, one can derive the mass formula for positive and negative parity charm mesons including the one loop self-energy.
\[ m_{H_1}^r = \delta_H + a_H m_n + \sigma_H \overline{m} - \frac{3}{4} (\Delta_H + \Delta^{(a)}_H m_n + \Delta^{(\sigma)}_H \overline{m}) + \Sigma_{H_1}, \]

\[ m_{H_3}^r = \delta_H + a_H m_s + \sigma_H \overline{m} - \frac{3}{4} (\Delta_H + \Delta^{(a)}_H m_s + \Delta^{(\sigma)}_H \overline{m}) + \Sigma_{H_3}, \]

\[ m_{H_1}^{r*} = \delta_H + a_H m_n + \sigma_H \overline{m} + \frac{1}{4} (\Delta_H + \Delta^{(a)}_H m_n + \Delta^{(\sigma)}_H \overline{m}) + \Sigma_{H_1}, \]

\[ m_{H_3}^{r*} = \delta_H + a_H m_s + \sigma_H \overline{m} + \frac{1}{4} (\Delta_H + \Delta^{(a)}_H m_s + \Delta^{(\sigma)}_H \overline{m}) + \Sigma_{H_3}, \]

\[ m_{S_1}^r = \delta_S + a_S m_n + \sigma_S \overline{m} - \frac{3}{4} (\Delta_S + \Delta^{(a)}_S m_n + \Delta^{(\sigma)}_S \overline{m}) + \Sigma_{S_1}, \]

\[ m_{S_3}^r = \delta_S + a_S m_s + \sigma_S \overline{m} - \frac{3}{4} (\Delta_S + \Delta^{(a)}_S m_s + \Delta^{(\sigma)}_S \overline{m}) + \Sigma_{S_3}, \]

\[ m_{S_1}^{r*} = \delta_S + a_S m_n + \sigma_S \overline{m} + \frac{1}{4} (\Delta_S + \Delta^{(a)}_S m_n + \Delta^{(\sigma)}_S \overline{m}) + \Sigma_{S_1}, \]

\[ m_{S_3}^{r*} = \delta_S + a_S m_s + \sigma_S \overline{m} + \frac{1}{4} (\Delta_S + \Delta^{(a)}_S m_s + \Delta^{(\sigma)}_S \overline{m}) + \Sigma_{S_3}, \]

where \( \overline{m} = 2 m_n + m_s \). The residual masses \( m^r \) are measured from \( \frac{m_H + 3m_{H^*}}{4} \), which is chosen to be a reference mass in our work.

The one-loop contributions to the self-energy \( \Sigma \) are of order \( O(Q^3) \). The Feynman diagrams of the one-loop correction to the masses of \( D \) mesons are shown in Figs. 4.1 and 4.2.

![Figure 4.1: The self-energy diagrams for the ground-state fields \( H \).](image-url)
The resulting explicit expressions for the self energies are

\[
\Sigma_{H_i} = \frac{g^2}{4f^2} \left[ 3K_1(m_{H_i} - m_{H_1}, m_\pi) + \frac{1}{3} K_1(m_{H_i} - m_{H_1}, m_\eta) + 2K_1(m_{H_3} - m_{H_1}, m_K) \right] \\
+ \frac{\hbar^2}{4f^2} \left[ 3K_2(m_{S_1} - m_{H_1}, m_\pi) + \frac{1}{3} K_2(m_{S_1} - m_{H_1}, m_\eta) + 2K_2(m_{S_3} - m_{H_1}, m_K) \right], (4.4)
\]

\[
\Sigma_{H_i^*} = \frac{g^2}{4f^2} \left[ K_1(m_{H_i} - m_{H_i^*}, m_\pi) + \frac{1}{9} K_1(m_{H_i} - m_{H_i^*}, m_\eta) + \frac{2}{3} K_1(m_{H_3} - m_{H_i^*}, m_K) \right] \\
+ \frac{g^2}{4f^2} \left[ 2K_1(0, m_\pi) + \frac{2}{9} K_1(0, m_\eta) + \frac{4}{3} K_1(m_{H_3} - m_{H_i^*}, m_K) \right] \\
+ \frac{\hbar^2}{4f^2} \left[ 3K_2(m_{S_1} - m_{H_i^*}, m_\pi) + \frac{1}{3} K_2(m_{S_1} - m_{H_i^*}, m_\eta) + 2K_2(m_{S_3} - m_{H_i^*}, m_K) \right], (4.5)
\]

\[
\Sigma_{H_3} = \frac{g^2}{4f^2} \left[ \frac{4}{3} K_1(m_{H_3} - m_{H_3}, m_\eta) + 4K_1(m_{H_3}^* - m_{H_3}, m_K) \right] \\
+ \frac{\hbar^2}{4f^2} \left[ \frac{4}{3} K_2(m_{S_3} - m_{H_3}, m_\eta) + 4K_2(m_{S_3} - m_{H_3}, m_K) \right], (4.6)
\]

\[
\Sigma_{H_3^*} = \frac{g^2}{4f^2} \left[ \frac{4}{9} K_1(m_{H_3} - m_{H_3}, m_\eta) + \frac{4}{3} K_1(m_{H_3} - m_{H_3^*}, m_K) \right] \\
+ \frac{g^2}{4f^2} \left[ \frac{8}{9} K_1(0, m_\eta) + \frac{8}{3} K_1(m_{H_3}^* - m_{H_3}, m_K) \right] \\
+ \frac{\hbar^2}{4f^2} \left[ \frac{4}{3} K_2(m_{S_3} - m_{H_3^*}, m_\eta) + 4K_2(m_{S_3} - m_{H_3^*}, m_K) \right], (4.7)
\]
The chiral loop integrals are

\[
\Sigma_{S_1} = \frac{g^2}{4f^2} \left[ 3K_1(m_{S_1} - m_{S_1}, m_\pi) + \frac{1}{3}K_1(m_{S_1} - m_{S_1}, m_\eta) + 2K_1(m_{S_1} - m_{S_1}, m_K) \right] + \frac{\hbar^2}{4f^2} \left[ 3K_2(m_{H_1} - m_{S_1}, m_\pi) + \frac{1}{3}K_2(m_{H_1} - m_{S_1}, m_\eta) + 2K_2(m_{H_1} - m_{S_1}, m_K) \right],
\]

(4.8)

\[
\Sigma_{S_1'} = \frac{g^2}{4f^2} \left[ K_1(m_{S_1} - m_{S_1'}, m_\pi) + \frac{1}{9}K_1(m_{S_1} - m_{S_1'}, m_\eta) + \frac{2}{3}K_1(m_{S_1} - m_{S_1'}, m_K) \right] + \frac{g^2}{4f^2} \left[ 2K_1(0, m_\pi) + \frac{2}{9}K_1(0, m_\eta) + \frac{4}{3}K_1(m_{S_1} - m_{S_1'}, m_K) \right] + \frac{\hbar^2}{4f^2} \left[ 3K_2(m_{H_1} - m_{S_1'}, m_\pi) + \frac{1}{3}K_2(m_{H_1} - m_{S_1'}, m_\eta) + 2K_2(m_{H_1} - m_{S_1'}, m_K) \right],
\]

(4.9)

\[
\Sigma_{S_3} = \frac{g^2}{4f^2} \left[ \frac{4}{3}K_1(m_{S_3} - m_{S_3}, m_\eta) + 4K_1(m_{S_1} - m_{S_3}, m_K) \right] + \frac{\hbar^2}{4f^2} \left[ \frac{4}{3}K_2(m_{H_3} - m_{S_3}, m_\eta) + 4K_2(m_{H_1} - m_{S_3}, m_K) \right],
\]

(4.10)

\[
\Sigma_{S_3'} = \frac{g^2}{4f^2} \left[ \frac{4}{9}K_1(m_{S_3} - m_{S_3'}, m_\eta) + \frac{4}{3}K_1(m_{S_1} - m_{S_3'}, m_K) \right] + \frac{g^2}{4f^2} \left[ \frac{8}{9}K_1(0, m_\eta) + \frac{8}{3}K_1(m_{S_1} - m_{S_3'}, m_K) \right] + \frac{\hbar^2}{4f^2} \left[ \frac{4}{3}K_2(m_{H_3} - m_{S_3'}, m_\eta) + 4K_2(m_{H_1} - m_{S_3'}, m_K) \right],
\]

(4.11)

The chiral loop integrals are

\[
K_1(\omega, m) = \frac{1}{16\pi^2} \left[ (-2\omega^3 + 3m^2\omega)\ln \left( \frac{m^2}{\mu^2} \right) - 4(\omega^2 - m^2)F(\omega, m) + \frac{16}{3}\omega^3 - 7\omega m^2 \right],
\]

(4.12)

\[
K_2(\omega, m) = \frac{1}{16\pi^2} \left[ (-2\omega^3 + m^2\omega)\ln \left( \frac{m^2}{\mu^2} \right) - 4\omega^2 F(\omega, m) + 4\omega^3 - \omega m^2 \right],
\]
renormalized in the $\overline{\text{MS}}$-scheme. The function $F(\omega, m)$ is given in Eq. (3.21).

## 4.3 Linear combinations of the low energy constants

The chiral Lagrangian given in Eq. (4.1) has twelve unknown LECs ($\delta_{H,S}$, $a_{H,S}$, $\sigma_{H,S}$, $\Delta_{H,S}$, $\Delta^{(a)}_{H,S}$, $\Delta^{(\sigma)}_{H,S}$) to describe eight masses of charm mesons. Thus finding unique values for LECs from experiments is impossible. The best we can do is to reduce their number by grouping them into linear combinations that can be determined uniquely. This procedure is based on symmetry patterns. The linear combinations that respect flavor symmetry are

$$
\begin{align*}
\eta_H &= \delta_H + \left( \frac{a_H}{3} + \sigma_H \right) \overline{m}, \\
\xi_H &= \Delta_H + \left( \frac{\Delta^{(a)}_H}{3} + \Delta^{(\sigma)}_H \right) \overline{m}, \\
\eta_S &= \delta_S + \left( \frac{a_S}{3} + \sigma_S \right) \overline{m}, \\
\xi_S &= \Delta_S + \left( \frac{\Delta^{(a)}_S}{3} + \Delta^{(\sigma)}_S \right) \overline{m},
\end{align*}
$$

where $\delta_{H,S}$ and $\Delta_{H,S}$ respect chiral symmetry, but the other terms contain the average of the quark masses $\overline{m}$ which breaks it. The parameters left after
constructing $\eta_H$, $\eta_S$, $\xi_H$, and $\xi_S$ are

\begin{align*}
L_H &= (m_s - m_n) a_H, \\
T_H &= (m_s - m_n) \Delta_H^{(a)}, \\
L_S &= (m_s - m_n) \alpha_S, \\
T_S &= (m_s - m_n) \Delta_S^{(a)}.
\end{align*}

(4.14)

The combinations $L_{H,S}$ and $T_{H,S}$ break flavor symmetry, and the latter also breaks spin symmetry. In terms of these linear combinations, the masses can be written as

\begin{align*}
m^r_{H_1} &= \eta_H - \frac{3}{4} \xi_H - \frac{1}{3} L_H + \frac{1}{4} T_H + \Sigma_{H_1}, \\
m^r_{H_3} &= \eta_H - \frac{3}{4} \xi_H + \frac{2}{3} L_H - \frac{1}{2} T_H + \Sigma_{H_3}, \\
m^r_{H'_1} &= \eta_H + \frac{1}{4} \xi_H - \frac{1}{3} L_H - \frac{1}{12} T_H + \Sigma_{H'_1}, \\
m^r_{H'_3} &= \eta_H + \frac{1}{4} \xi_H + \frac{2}{3} L_H + \frac{1}{6} T_H + \Sigma_{H'_3}, \\
m^r_{S_1} &= \eta_S - \frac{3}{4} \xi_S - \frac{1}{3} L_S + \frac{1}{4} T_S + \Sigma_{S_1}, \\
m^r_{S_3} &= \eta_S - \frac{3}{4} \xi_S + \frac{2}{3} L_S - \frac{1}{2} T_S + \Sigma_{S_3}, \\
m^r_{S'_1} &= \eta_S + \frac{1}{4} \xi_S - \frac{1}{3} L_S - \frac{1}{12} T_S + \Sigma_{S'_1}, \\
m^r_{S'_3} &= \eta_S + \frac{1}{4} \xi_S + \frac{2}{3} L_S + \frac{1}{6} T_S + \Sigma_{S'_3}.
\end{align*}

(4.15)

Now the number of parameters, $\xi_{H,S}$, $\eta_{H,S}$, $L_{H,S}$, and $T_{H,S}$ is eight, which is equal to the number of observed low-lying $D$-meson states.
4.4 Results and Discussion

In our fitting, the one-loop self energy $\Sigma$ is evaluated with physical values of the charm meson masses. This gives the energy of the decaying particle relative to the threshold in the right place. This ensures that the imaginary parts of the loop functions are correctly related to the experimental decay widths. The resulting masses depend linearly on certain combinations of parameters. However, the resulting values for these parameters contain contributions beyond $O(Q^3)$. This is a result of using empirical masses which generates higher order $\mu$-dependent terms that are not under control, i.e. they cannot be renormalized properly using $\mu$-dependence counterterms of our Lagrangian. As a result, we have to estimate a theoretical error coming from these higher-order terms. Since the $\beta$-functions of the coupling constants (or parameters) measure how these couplings (or parameters) change as $\mu$ changes, we have chosen to use their $\beta$-functions to estimate the contributions from these higher-order terms.

As an illustration, the $\beta$-function of $\eta_H$ is given by $\beta(\eta_H) = \mu \frac{\partial \eta_H}{\partial \mu}$. To derive $\beta(\eta_H)$, let us first define $\eta_H$ in terms of the physical masses and the
one-loop functions,
\[
\eta_H = \frac{1}{12} \left[ (m_{H_3} + 3m_{H_3^*}) + 2(m_{H_1} + 3m_{H_1^*}) \right] 
- \frac{h^2}{72 f^2} \left[ 18K_2 \left( m_{S_3^*} - m_{H_1^*}, m_K \right) + 18K_2 \left( m_{S_1^*} - m_{H_3^*}, m_K \right) \right] 
- \frac{h^2}{72 f^2} \left[ 3K_2 \left( m_{S_1^*} - m_{H_1^*}, m_{\eta} \right) + 6K_2 \left( m_{S_3^*} - m_{H_3^*}, m_{\eta} \right) + 27K_2 \left( m_{S_1^*} - m_{H_3^*}, m_\pi \right) \right] 
- \frac{h^2}{72 f^2} \left[ 6K_2 \left( m_{S_1} - m_{H_3}, m_K \right) + 6K_2 \left( m_{S_3} - m_{H_1}, m_K \right) + K_2 \left( m_{S_1} - m_{H_1}, m_\eta \right) \right] 
- \frac{h^2}{72 f^2} \left[ 2K_2 \left( m_{S_3} - m_{H_3}, m_\eta \right) + 9K_2 \left( m_{S_1} - m_{H_1}, m_\pi \right) \right].
\]

(4.16)

This expression respects both flavor and heavy quark symmetries, as expected from Eq. (4.13). Note that the one-loop functions of the odd parity transitions cancel each other due to the heavy quark symmetry. Without our choice of defining the meson fields as 4 dimensional (which keeps the loop integrals for transitions within states in the same doublet equal), the exact cancelation would not occur. To define the \( \beta \)-function of \( \eta_H \), one has
to differentiate Eq. (4.16) with respect to \( \mu \),

\[
\mu \frac{\partial \eta_H}{\partial \mu} = \frac{h^2}{144 \pi^2} \left[ 9 m_{S_1}^2 (5 m_{H_1} + 3 m_{H_3}) - 9 m_{S_1} (5 m_{H_1}^2 + 3 m_{H_3}^2) \\
+ 9 m_{S_3}^2 (3 m_{H_1} + m_{H_3}) - 9 m_{S_3} (3 m_{H_1}^2 + m_{H_3}^2) \\
- 24 m_{S_1}^3 - 12 m_{S_3}^3 + 15 m_{H_1} m_{S_1} + 9 m_{H_3} m_{S_1} + 9 m_{H_1} m_{S_3} \\
+ 3 m_{H_3} S_3^2 - 15 m_{H_1}^2 m_{S_1} - 9 m_{H_3}^2 m_{S_1} - 9 m_{H_1}^2 m_{S_3} - 3 m_{H_3}^2 m_{S_3} \\
+ 8 m_{H_1}^3 + 4 m_{H_3}^3 - 8 m_{S_1}^3 - 4 m_{S_3}^3 + 24 m_{H_1}^3 + 12 m_{H_3}^3 \\
+ \frac{1}{6} (111 (m_{S_1} - m_{H_1}) + 33 (m_{S_3} - m_{H_3}) + 37 (m_{S_1} - m_{H_1}) \\
+ 11 (m_{S_3} - m_{H_3})) B_0 m_n \\
+ \frac{1}{6} (33 (m_{S_1} - m_{H_1}) + 39 (m_{S_3} - m_{H_3}) + 11 (m_{S_1} - m_{H_1}) \\n+ 13 (m_{S_3} - m_{H_3})) B_0 m_s \right],
\]

(4.17)

where we have used Gell-Mann-Oakes-Renner relations to express the masses of the light mesons in terms of quark masses and \( B_0 \). This allows us to collect similar terms and simplify them. To organize terms in Eq. (4.17) according to their chiral orders, it is convenient to employ symmetry patterns to define
the following combinations of the charm meson masses,

\[
P_1 = \left( \frac{2}{3} \frac{(m_{S_1} + 3 m_{S_1})}{4} + \frac{1}{3} \frac{(m_{S_3} + 3 m_{S_3})}{4} \right) - \left( \frac{2}{3} \frac{(m_{H_1} + 3 m_{H_1})}{4} + \frac{1}{3} \frac{(m_{H_3} + 3 m_{H_3})}{4} \right),
\]

\[
P_2 = \left( \frac{2}{3} \frac{(m_{S_1} - m_{S_1})}{4} + \frac{1}{3} \frac{(m_{S_3} - m_{S_3})}{4} \right) - \left( \frac{2}{3} \frac{(m_{H_1} - m_{H_1})}{4} + \frac{1}{3} \frac{(m_{H_3} - m_{H_3})}{4} \right),
\]

\[
P_3 = \left( \frac{m_{S_3} + 3 m_{S_3}}{4} - \frac{(m_{S_1} + 3 m_{S_1})}{4} \right) - \left( \frac{(m_{H_3} + 3 m_{H_3})}{4} - \frac{(m_{H_1} + 3 m_{H_1})}{4} \right),
\]

\[
P_4 = \left( m_{S_3} - m_{S_3} - (m_{S_1} - m_{S_1}) \right) - \left( (m_{H_3} - m_{H_3}) - (m_{H_1} - m_{H_1}) \right),
\]

\[
M_1 = \frac{1}{2} \left[ \left( \frac{2}{3} \frac{(m_{S_1} + 3 m_{S_1})}{4} + \frac{1}{3} \frac{(m_{S_3} + 3 m_{S_3})}{4} \right) + \left( \frac{2}{3} \frac{(m_{H_1} + 3 m_{H_1})}{4} + \frac{1}{3} \frac{(m_{H_3} + 3 m_{H_3})}{4} \right) \right],
\]

\[
M_2 = \frac{1}{2} \left[ \left( \frac{2}{3} \frac{(m_{S_1} - m_{S_1})}{4} + \frac{1}{3} \frac{(m_{S_3} - m_{S_3})}{4} \right) + \left( \frac{2}{3} \frac{(m_{H_1} - m_{H_1})}{4} + \frac{1}{3} \frac{(m_{H_3} - m_{H_3})}{4} \right) \right],
\]

\[
M_3 = \frac{1}{2} \left[ \left( \frac{(m_{S_3} + 3 m_{S_3})}{4} - \frac{(m_{S_1} + 3 m_{S_1})}{4} \right) + \left( \frac{(m_{H_3} + 3 m_{H_3})}{4} - \frac{(m_{H_1} + 3 m_{H_1})}{4} \right) \right],
\]

\[
M_4 = \frac{1}{2} \left[ \left( (m_{S_3} - m_{S_3}) - (m_{S_1} - m_{S_1}) \right) + \left( (m_{H_3} - m_{H_3}) - (m_{H_1} - m_{H_1}) \right) \right].
\]

The combinations \( P_1 \) and \( M_1 \) respect both the flavor and heavy quark symmetries, whereas \( P_2 \) and \( M_2 \) (\( P_3 \) and \( M_3 \)) only respect flavor (heavy quark) symmetry. The combinations \( P_4 \) and \( M_4 \) break both symmetries. According to the power-counting scheme of HHChPT, they scale as

\[
M_1 \sim M_2 \sim P_1 \sim P_2 \sim O(Q), \quad M_3 \sim P_3 \sim O(Q^2), \quad M_4 \sim P_4 \sim O(Q^3).
\]

One can simply define the expressions for the charm masses in terms of these combinations. By doing so, one can write Eq. (4.17) in terms of the above
combinations

$$\frac{\partial \eta_H}{\partial \mu} = \frac{5h^2 P_3^3}{13824\pi^2 f^2} + \frac{h^2 M_4^2 P_4}{128\pi^2 f^2}$$

$$- \frac{3h^2 P_3 M_4^2}{192\pi^2 f^2} - \frac{5h^2 P_3 P_4^2}{2304\pi^2 f^2} - \frac{4h^2 M_4 P_4}{128\pi^2 f^2}$$

$$\frac{h^2 (36M_4^2 + 7P_4^2) (2P_1 - P_2)}{768\pi^2 f^2}$$

$$- \frac{36h^2 P_2 M_3 M_4}{192\pi^2 f^2} - \frac{16h^2 P_3 M_4^2}{192\pi^2 f^2} - \frac{5h^2 P_3^3}{1296\pi^2 f^2} - \frac{7h^2 P_3 P_2 P_4}{192\pi^2 f^2}$$

$$- \frac{96h^2 M_3^2 P_1}{192\pi^2 f^2} - \frac{7h^2 P_4 P_3^2}{72\pi^2 f^2}$$

$$+ \frac{5h^2 P_3}{216f^2 \pi^2} \left( m_K^2 - m_\pi^2 \right)$$

$$- \frac{h^2 (64P_3^3 - 36P_2 P_1 + 6P_3^3)}{192\pi^2 f^2} + \frac{24h^2 P_1}{216f^2 \pi^2} \left( m_K^2 + m_\pi^2 \right),$$

where we have used Gell-Mann-Oakes-Renner relations again to replace the quark masses by pion and kaon masses. From the first to the last lines, terms in Eq. (4.18) scale as $O(Q^9)$, $O(Q^8)$, $O(Q^7)$, $O(Q^6)$, $O(Q^5)$, $O(Q^4)$, and $O(Q^3)$ respectively. The theoretical uncertainty can be now estimated by looking at the $\mu$-dependent higher-order terms in the $\beta$-function.

The calculations are performed at the physical values of pion decay constant $f = 92.4$ MeV, and of the coupling constants $g$ and $h$ that are extracted from the strong decay widths: $g = 0.64 \pm 0.075$ and $h = 0.56 \pm 0.04$, for details see [60]. In Table 4.1, most of the $D$-mesons masses are taken from the particle data group [1]. The renormalization scale $\mu$ is chosen to be the average of the pion and kaon masses $\mu = 317$ MeV. The resulting numerical
\[
J^p = 0^- \quad H_1 \quad H_3
\]

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<tr>
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<tr>
<td>1867.225 ± 0.06</td>
<td>PDG [1]</td>
<td>1968.30 ± 0.11</td>
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\[
J^p = 1^- \quad H_1^* \quad H_3^*
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<td>2008.61 ± 0.06</td>
<td>PDG [1]</td>
<td>2112.1 ± 0.4</td>
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\[
J^p = 0^+ \quad S_1 \quad S_3
\]

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<td>2318 ± 29</td>
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\[
J^p = 1^+ \quad S_1^* \quad S_3^*
\]

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<td>2427 ± 36</td>
<td>BELLE [55]</td>
<td>2459.5 ± 0.6</td>
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**Table 4.1:** The masses of \( H_1 \) (\( H_1^* \)) are obtained by taking the isospin average of \( D^0 \) and \( D^\pm \) (\( D^{*0} \) and \( D^{*\pm} \)).

values of the parameters which inhabit the odd parity sector are

\[
\eta_H = 171.575 \pm 44 \pm 5 \text{ MeV},
\]

\[
\xi_H = 150.945 \pm 5 \pm 5 \text{ MeV},
\]

\[
L_H = 242.72 \pm 40 \pm 18 \text{ MeV},
\]

\[
T_H = -52.2043 \pm 18 \pm 15 \text{ MeV},
\]

where the first uncertainty is the experimental error associated with physical masses of charm mesons, and the second uncertainty is the theoretical error that we have estimated from the \( \beta \)-functions. The situation for the
Figure 4.3: The combination $\eta_S$ plotted against $g'$. The central value is represented by the solid line. The experimental errors are shown by the dashed lines. The theoretical uncertainty is a constant $\pm 5$ MeV.

Figure 4.4: The combination $\xi_S$ plotted against $g'$. The experimental uncertainties are shown by dashed lines surrounding the central values and an estimate theoretical uncertainty is shown by dot-dashed line.
even-parity parameters is different because the coupling constant $g'$ is not determined experimentally. Since the value of the odd parity coupling constant is 0.64, it is plausible to consider values for $g'$ in the range 0 to 1. The correlations between $g'$ and $\eta_S, \xi_S, L_S, T_S$ are shown in Figs. 4.3-4.6. The plots also show the associated experimental and theoretical errors.

Experimental information is not sufficient to separate the combinations of parameters into pieces that respect and break chiral symmetry, which limits their usefulness for applications to other observables. Lattice QCD calculations would be required to perform further separations of terms. For example, to disentangle chirally symmetric coefficients $\delta_H, \Delta_H$ from chiral breaking terms, lattice calculations with different quark masses would be needed for charm mesons ground and excited states.

In the following section, the LECs that have been determined from using the physical masses of the charm mesons will be used to predict the masses of the $B$-mesons.

### 4.5 Prediction for the Spectrum of Odd- and Even-Parity Bottom Mesons

Using the results from charm mesons, one can predict the spectra of the $B$ mesons. To this end, the hyperfine operators in the theory, i.e. LECs that break heavy quark symmetry, will be rescaled to define the mass formula for the odd and even parity bottom mesons. The rescaling can be achieved by multiplying these operators by the ratio of the finite charm and bottom
Figure 4.5: The combination $L_S$ plotted against $g'$. The notation is the same as in Fig. 4.4.

Figure 4.6: The combination $T_S$ plotted against $g'$. The notation is the same as in Fig. 4.4.
The charm and bottom quark masses, $m_c/m_b$, i.e.

\[
\begin{align*}
\xi_H^b &= \frac{m_c}{m_b} \xi_H, \\
T_H^b &= \frac{m_c}{m_b} T_H, \\
\xi_S^b &= \frac{m_c}{m_b} \xi_S, \\
T_S^b &= \frac{m_c}{m_b} T_S.
\end{align*}
\]

The masses of the charm and bottom quarks are not directly measured. Many theoretical and computational methods have been developed to extract their values, for a review see Refs. [1, 64]. In Table 4.2, we list the charm and bottom quark masses evaluated from different mass schemes. Clearly, the extracted masses of the charm and bottom quark are not uniquely defined. The values depend on the definition of the mass scheme used. It is not clear which is best definition for our purposes. However, as the $\overline{\text{MS}}$ definition has a small associated uncertainty, it is convenient to choose the ratio obtained

<table>
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<tr>
<th>Mass Scheme</th>
<th>Charm quark mass (GeV)</th>
<th>Bottom quark mass (GeV)</th>
<th>$\frac{m_c}{m_b}$</th>
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<td>$\overline{\text{MS}}$ [1]</td>
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<td>$4.18 \pm 0.03$</td>
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<td>Pole [1]</td>
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<tr>
<td>1 S [1]</td>
<td>....</td>
<td>$4.66 \pm 0.03$</td>
<td>..</td>
</tr>
<tr>
<td>Kinetic [65]</td>
<td>$1.077 \pm 0.074$</td>
<td>$4.549 \pm 0.049$</td>
<td>$0.237$</td>
</tr>
</tbody>
</table>

Table 4.2: The charm and bottom $\overline{\text{MS}}$-masses are evaluated at their own scale, i.e. $\overline{m}_c(\overline{m}_c)$ and $\overline{m}_b(\overline{m}_b)$. In [1], the $\overline{\text{MS}}$ values are converted to the pole scheme. The ratio of charm and bottom masses obtained from the pole mass is close to the ratio of the pseudoscalar charm and bottom mesons $\frac{m_{D}}{m_{B}} = 0.35$. In the kinetic mass scheme, the charm and bottom masses are evaluated at $\mu = 1\text{GeV}$ [65].
from it and add an extra uncertainty, of the order \( O(\Lambda_{\text{QCD}}) \), to cover the spread of \( \frac{m_c}{m_b} \) resulting from different mass schemes. Thus, the hyperfine operators in our theory can be rescaled by the factor \( \frac{m_c}{m_b} = 0.305 \pm 0.05 \).

In terms of the rescaled LECs, the mass formulae for the bottom mesons up to one-loop corrections are

\[
\begin{align*}
 m_B &= \eta_H - \frac{3}{4} \xi_H - \frac{1}{3} L_H + \frac{1}{4} T^b_H + \Sigma_{m_B}, \\
 m_{B_s} &= \eta_H - \frac{3}{4} \xi_H + \frac{1}{2} L_H - \frac{1}{2} T^b_H + \Sigma_{m_{B_s}}, \\
 m_{B^*} &= \eta_H + \frac{1}{4} \xi_H - \frac{1}{3} L_H - \frac{1}{12} T^b_H + \Sigma_{m_{B^*}}, \\
 m_{B^*_s} &= \eta_H + \frac{1}{4} \xi_H + \frac{2}{3} L_H + \frac{1}{6} T^b_H + \Sigma_{m_{B^*_s}}, \\
 m_{B_0} &= \eta_S - \frac{3}{4} \xi_S - \frac{1}{3} L_S + \frac{1}{4} T^b_S + \Sigma_{m_{B_0}}, \\
 m_{B_{s0}} &= \eta_S - \frac{3}{4} \xi_S + \frac{2}{3} L_S - \frac{1}{2} T^b_S + \Sigma_{m_{B_{s0}}}, \\
 m_{B^{*0}} &= \eta_S + \frac{1}{4} \xi_S + \frac{2}{3} L_S - \frac{1}{12} T^b_S + \Sigma_{m_{B^{*0}}}, \\
 m_{B^{*0}_s} &= \eta_S + \frac{1}{4} \xi_S + \frac{2}{3} L_S + \frac{1}{6} T^b_S + \Sigma_{m_{B^{*0}_s}},
\end{align*}
\]

where the symbols \( B, B_s, B^*, B^*_s, B_0, B_{s0}, B^{*0}, B^{*0}_s \) represent the nonstrange pseudoscalar, strange pseudoscalar, nonstrange vector, strange vector, nonstrange scalar, strange scalar, nonstrange axial vector and strange axial vector respectively. The self-energy \( \Sigma_{m_B} \) is a function of the mass difference of the \( B \) mesons and the masses of the light pseudoscalar mesons \( \pi, \eta, \) and \( K \).

To predict the masses of the bottom mesons, it is suitable to choose \( m_B \) (the ground state of the nonstrange \( B \)-meson) as the reference mass to get
the following independent splittings

\[ \omega_1 = m_{B^*} - m_B, \]  
\[ \omega_2 = m_{B_s} - m_B, \]  
\[ \omega_3 = m_{B_s^*} - m_B, \]  
\[ \omega_4 = m_{B_0} - m_B, \]  
\[ \omega_5 = m_{B_{s0}} - m_B, \]  
\[ \omega_6 = m_{B_{s0}^*} - m_B, \]  
\[ \omega_7 = m_{B_{s0}^*} - m_B. \]  

The loop functions depend on the mass differences, and so the above set form nonlinear equations. We have used an iterative method to solve them starting from the tree-level masses. The numerical values of mass splitting \( \omega \)'s are shown in Figs. 4.7-4.13.

Our theoretical prediction for masses (splittings) of the odd parity \( B \) mesons are in good agreement with the available experimental data. In the PDG [1], the splittings within odd parity \( B \) mesons are

\[ m_{B^*} - m_B = 45.78 \pm 0.35 \text{ MeV}, \]  
\[ m_{B^{*+}} - m_{B^+} = 45.0 \pm 0.4 \text{ MeV}, \]  
\[ m_{B_s} - m_B = 87.35 \pm 0.23 \text{ MeV}, \]  
\[ m_{B_{s0}^*} - m_{B_s} = 48.7 \pm 3.12 \text{ MeV}. \]  

119
The mass difference $m_{B^*_s} - m_B$ can be obtained from the above splittings as follows

$$m_{B^*_s} - m_B = (m_{B^*_s} - m_{B_s}) + (m_{B_s} - m_B) = 136.05 \pm 3.13 \text{ MeV}. \quad (4.39)$$

By comparing the results in Eq. (4.35) and Eq. (4.36) with the predicted splitting shown in Fig. 4.7, we find that the experimental measurement of hyperfine splitting of the nonstrange $B$ mesons agrees with our theoretical prediction within 1$\sigma$ standard deviation.

Similarly, the measured mass difference $m_{B_s} - m_B$ (see Eq. (4.37)) agrees with our theoretical prediction (see Fig. 4.8) within about 1$\sigma$ standard deviation. Furthermore, the measured mass difference $m_{B^*_s} - m_B$ (see Eq. (4.39)) agrees with our theoretical prediction (see Fig. 4.9) within 1$\sigma$ standard deviation.

For the even-parity sector, the $B$-meson states have not yet been observed; thus, our results, which are shown in Figs. 4.10-4.13, provide useful information for experimentalists investigating such states.

For the predicted masses (splittings) of the even-parity sector, the strong dependence on the coupling $g'$ is due to the large negative contribution from terms with

$$\frac{g'^2}{4f^2} n_f K_1(\omega, m) \simeq \frac{g'^2}{4f^2} n_f \left[ -\frac{4}{16\pi^2}(\omega^2 - m^2) F(\omega, m) + \ldots \right]$$

$$\propto -\frac{g'^2}{f^2} n_f m^2 \sqrt{m^2 - \omega^2} \cos^{-1} \left( \frac{\omega}{m} \right) + ..., $$

for $m^2 > \omega^2$ where $m = m_\eta, m_K$. The light-quark factor $n_f$ is simply ob-
**Figure 4.7:** The solid line represents the central value of the splitting $m_{B^*} - m_B$. The associated uncertainties, which include the experimental errors of the charm meson masses and the coupling constants and the error from the input parameter $\frac{m_c}{m_b}$, are given by the dashed lines. The dot-dashed line represents an estimate theoretical uncertainty.

**Figure 4.8:** The solid line represents the central value of the splitting $m_{B_s} - m_B$. The notation is the same as in Fig. 4.7.
Figure 4.9: The solid line represents the central value of the splitting $m_{B^*_s} - m_B$. The notation is the same as in Fig. 4.7.

tained from the Gell-Mann matrices and its value reflects the number of independent self-energy loop diagrams which contribute to the process.

As shown in Figs. 4.10 and 4.11, the nonstrange and strange scalar bottom mesons are nearly degenerate, the difference between their central values is $\sim 10\text{ MeV}$. This kind of mass degeneracy was first observed in the charm sector, see section 1.1.2 for details.

It is worth mentioning that the work undertaken in [89, 90] are intended to investigate the closeness of nonstrange and strange scalars in the charm and bottom sectors using, in addition to HHChPT, different potential models. They exploited the fact that the mass of $D_{s0}^*$ lying below the $DK$-threshold to include the effect of the hadronic loops to lower the bare masses predicted by the quark models. In their work, the hadronic loop contributions include only the coupling of $D_{s0}^*$ to the lowest possible intermediate states, these states form members of $\frac{1}{2}^-$-doublet in the notation of HHChPT. The self-energy
**Figure 4.10:** The solid line represents the central value of the splitting $m_{B_0} - m_B$. The notation is the same as in Fig. 4.7.

**Figure 4.11:** The solid line represents the central value of the splitting $m_{B_{s0}} - m_B$. The notation is the same as in Fig. 4.7.
Figure 4.12: The solid line represents the central value of the splitting \( m_{B_0^*} - m_B \). The notation is the same as in Fig. 4.7.

Figure 4.13: The solid line represents the central value of the splitting \( m_{B_0^*_{s0}} - m_B \). The notation is the same as in Fig. 4.7.
contributions from the coupling of $D_{s0}^*$ to the members of the $\frac{1}{2}^+$-doublet have been neglected in [89, 90] which in turn indicates their analysis within HHChPT is incomplete.

As concluded in [89, 90], the results of studying the mass degeneracy using HHChPT are not satisfactory, which is in fact not true as shown in Figs. 4.10-4.11.

Furthermore, the approach employed in [89, 90] of using bare masses in evaluating loop functions\footnote{In fact, the authors of Ref. [89] have used the incorrect expression for the chiral loop function. They used the expression for $K_2(\omega, m)$ as given in [63] which is incorrect as mentioned in Chapter 3.} is inappropriate for the case of HHChPT. For example, the predicted masses of $B_{s0}^*$ and $B_0^*$, as given in the TABLE II in [90], provide different splittings when using different bare masses in evaluating loop functions. More precisely, the mass difference $m_{B_{s0}^*} - m_{B_0^*}$ is $\sim +100$ MeV when evaluating loop functions with bare masses given in [91] and is $\sim -60$ MeV when evaluating loop functions with bare masses taken from [92]. This shows that the loop integrals are sensitive to the input mass differences of the heavy mesons, so using bare masses is not appropriate. To avoid these problems, we use the self-consistently determined masses. As a result, there is an unavoidable theoretical uncertainty, which we estimate from higher-order contributions from $\beta$-function.
Chapter 5

Short-range interactions
between heavy mesons in
framework of EFT

As illustrated in the introduction, many newly discovered quarkonium states have masses very close to the thresholds of pairs of heavy mesons. The spectra of these quarkonia fail to fit into the conventional Quark Model [1]. To interpret their origin, new structures have been proposed. Multiquark states, hybrid quarkonia, and glueballs have been considered, in addition to hadronic molecules. An example of such states is the $X(3872)$. Since its mass is very close to the threshold of $\bar{D}^0 D^{*0}$, it has been interpreted as a molecular bound state of the $\bar{D}^0 D^{*0}$ mesons. However, other identifications of the $X(3872)$ as a charmonium state, $\chi_{c1}(2P)$, or a compact tetraquark are not excluded, see [103] and references therein for more detail.
This Chapter is devoted to studying the interaction between heavy mesons at very low energy \( p \ll m_\pi \) using effective theories similar to those used to study nuclear forces. In this case, the pion is integrated out and the only relevant degrees of freedom are heavy mesons. Our approach can be used to investigate resonances in the charmonium and bottomonium systems, including the X(3872).

### 5.1 Uncoupled Channel

In this section, we apply the effective field theory method to describe short range interaction of two heavy particles with definite spin and isospin at energies near their threshold, i.e. low enough that the particles can be treated as nonrelativistic. Before proceeding further, let us recall the on-shell scattering amplitude of an uncoupled channel in nonrelativistic quantum mechanics,

\[
T = \frac{2\pi}{M} \frac{1}{p \cot \delta(p) - ip},
\]  

where \( p \) is the relative momentum of two particles and \( \delta(p) \) is the phase shift. \( M \) is the reduced mass of two particles with unequal masses\(^1\). At zero momentum, one can define the scattering length

\[
T = \frac{-2\pi}{M} a,
\]  

hence the cross section for such systems is \( \sigma \propto |a|^2 \).

\(^1\) The scattering amplitude for two particles with the equal masses is \( T = \frac{4\pi}{M} \frac{1}{p \cot \delta(p) - ip} \).
by an $s$-wave scattering. The important piece of $T$ is $p \cot \delta(p)$ which in the sufficiently low energies, $p \ll \Lambda$, can be expanded as follows [81],

$$p \cot \delta(p) = -\frac{1}{a} + \frac{1}{2} \Lambda^2 \sum_{n=0}^{\infty} r_n \left( \frac{p^2}{\Lambda^2} \right)^{n+1}$$

$$= -\frac{1}{a} + \frac{r_0}{2} p^2 + \frac{r_1}{2} \frac{p^4}{\Lambda^2} + \ldots,$$

(5.3)

where the hard scale $\Lambda$ can be taken to be the mass of the exchanged particle for the system. As an example, one can take $\Lambda \sim m_\pi$ for systems with charm or bottom meson pairs in which pions mediate the residual nuclear force. The expansion in Eq. (5.3) is known as an effective range expansion (ERE). The factor $a$ is the scattering length, $r_0$ is the effective range, and $r_1$ is called the shape parameter. The size of $r_0$ and $r_1$ are related to the interaction range between particles, $r \sim \frac{1}{\Lambda}$.

The convergence of the scattering matrix obtained from the expansion in Eq. (5.3) depends on the size of the scattering length. There are, however, two distinct limits for the size of the scattering length in comparison with the interaction range $\frac{1}{\Lambda}$ at low energy $p < \Lambda$. For the strong short-range interaction, the scattering length is large, i.e. $|\frac{1}{a}| \ll \Lambda$. In this case, the first term in Eq. (5.3) is dominant and Eq. (5.1) becomes

$$T = -\frac{2\pi}{M} \frac{1}{\frac{1}{a} + ip},$$

(5.4)

with a pole that indicates the existence of either a virtual or a real bound state.

On the other hand, the limit $\frac{1}{a} \sim \Lambda \gg p$ corresponds to small scattering
length (weak short-range interaction). Eq. (5.1) can be expanded in powers of \( a p \)

\[
T = -\frac{2\pi a}{M} (1 - i a p + ..),
\]

for systems with weak short-range interaction, the cross section is small.

The scattering amplitudes given in Eq. (5.4) and Eq. (5.5) will be reproduced in the framework of effective field theory by implementing similar ideas. Below, we consider two-body scattering (e.g. \( \bar{D}^0 D^*0 \)) at very low energy \( p < \Lambda \) where \( \Lambda \sim m_\pi \). As the relative momentum \( p \) of the \( \bar{D}^0 D^*0 \)-system is much smaller than the pion mass, it is convenient to define the expansion factor \( \frac{p}{m_\pi} \) for the system. In the current example, the low energy scale is the relative momentum of the \( \bar{D}^0 D^*0 \)-system. This is, however, not the only low energy scale as we will see in the case of the large scattering length, where its inverse provides another soft scale.

In this limit, the wavelength of the two particles is much greater than the range of interaction, so the interaction can be represented by a local operator that measures the strength of the \( s \)-wave transition of the particles. In this case, the short-range physics is integrated out and hidden in these operators. In this Chapter, we only consider the leading order term for \( s \)-wave scattering \( (^1S_0) \), that corresponds to a \( \delta \)-function potential in position-space.

It is convenient to start by writing down the lowest order of the non-relativistic version of the heavy hadron chiral Lagrangian [88]

\[
L_s = \bar{\psi}_1 (i\partial_0 + \frac{\nabla^2}{2M_1} - M_1) \psi_1 + \bar{\psi}_2 (i\partial_0 + \frac{\nabla^2}{2M_2} - M_2) \psi_2
- C_0 \left( (\psi_1^T \mathcal{J} \psi_2)^\dagger \cdot (\psi_1^T \mathcal{J} \psi_2) \right),
\]

(5.6)
where the general form of $\psi_1$ and $\psi_2$ are

$$
\psi_1 = \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} \psi_{21} \\ \psi_{22} \end{pmatrix},
$$

and the superscript $T$ denotes the transpose of $\psi$. For the case of the $X(3872)$, the components of eigenvectors represent charm mesons, $\psi_{11} = D^0$, $\psi_{12} = \bar{D}^0$, $\psi_{21} = D^{*0}$, and $\psi_{22} = \bar{D}^{*0}$. Since the $X(3872)$ is a bound state of $\bar{D}^0 D^{*0}$ and $D^0 \bar{D}^{*0}$ combination, the matrix operator $J$ can be chosen as

$$
J = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
$$

(5.8)
to provide the correct form of the $X(3872)$ wave function.

Before proceeding to derive the transition matrix, $T$, let us first consider the basic loop-integral of the diagram shown in Fig. 5.1

$$
I = -i \left( \frac{\mu}{2} \right)^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{i}{k_0 + \frac{E}{2} - \frac{k^2}{2M_1} + \Delta + i\epsilon} \frac{i}{-k_0 + \frac{E}{2} - \frac{k^2}{2M_2} + i\epsilon},
$$

(5.9)

where the external energy is given by the center of mass energy $E$ and the factor $\Delta$ defines the splitting between masses, $\Delta = M_2 - M_1$.

After performing the energy contour integral, one gets

$$
I = \left( \frac{\mu}{2} \right)^{4-D} \int \frac{d^{D-1} k}{(2\pi)^{D-1}} \frac{1}{E - \frac{k^2}{2M} + \Delta + i\epsilon},
$$

(5.10)

where $M$ is the reduced mass, $M = \frac{M_1 M_2}{M_1 + M_2}$. We are interested in energy very
For $D = 4$, four spacetime dimensions, the loop integral $I$ is linearly divergent. Here, we want to regulate $I$ using Dimensional Regularization (DR). It should be stated that DR can be successfully used to regulate logarithmic divergent integrals. By using DR, the integral of the loop

$$I = -2 \left( \frac{\mu}{2} \right)^{4-D} \int_0^\infty \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{p^2 - k^2 + i\epsilon}. \quad (5.11)$$

is well-defined in the limit of $D = 4$. However, Eq. (5.12) is ill-defined if $D = 3$. Our purpose now is to specify the appropriate subtraction schemes for the scattering length. There are two cases:

- For $a$ with a natural size ($a_\lambda \sim \Lambda$), the appropriate subtraction schemes to renormalize logarithmic divergence resulting from DR is MS, which corresponds to setting $D = 4$. This gives

$$I^{\text{MS}} = -i \frac{Mp}{2\pi}, \quad (5.13)$$
from which it can be seen there are no such divergences.

- For $a$ with an unnatural size ($\frac{1}{a} \ll \Lambda$), we need to worry about the linear (power-law) divergences which can be identified from the pole at $D = 3$. This linear divergence for $D = 4$ in DR becomes a logarithmic divergence for $D = 3$. Thus, DR can be modified in a way that allows one to subtract the pole at $D = 3$ by adding a counterterm. This procedure is known as power divergence subtraction (PDS) [79]. PDS was introduced to regulate (linear) divergent integrals at $D = 4$. In PDS, the general form of the counter term $\delta I$ can be deduced from the residue of the pole at $D = 3$ which is $\delta I = -\frac{M\mu}{2\pi(D-3)}$. The subtracted integral in PDS is [79]

$$ I^{\text{PDS}} = I + \delta I = -\frac{M}{2\pi}(\mu + ip), \quad (5.14) $$

where the renormalization scale is $\mu$. This scale should be chosen to be comparable to the magnitude of the relative momentum, $\mu \sim p < \Lambda$.

As stated before, physical observables should be independent of renormalization scale and so $\mu$-dependence terms must be canceled by adding up the contributions from tree level and loop diagrams and making the coefficients of counterterms $\mu$-dependent.

We are now in a position to calculate the T-matrix for the $s$-wave channel by solving the Lippmann-Schwinger equation:

$$ T^{-1} = -V^{-1} + I. \quad (5.15) $$
For the interaction potential $V = C_0$ and loop function $I$, the LS equation

$$T^{-1} = -C_0^{-1} + I,$$  \hspace{1cm} (5.16)

can be written as

$$T = -\frac{1}{C_0 - I}.$$  \hspace{1cm} (5.17)

For weak short-range interactions of the $\bar{D}D^*$ pair at low energy, the size of

$$\begin{array}{c}
\includegraphics[width=0.8\textwidth]{figure5.2.png}
\end{array}$$

**Figure 5.2:** The contribution of contact and bubble diagrams to the scattering matrix.

the bare coupling is small, i.e. $\frac{1}{C_0} \sim \Lambda > p$ and counted as $O(Q^0)$ according to naive Weinberg power counting. Clearly, the $T$-matrix

$$T = -\frac{1}{C_0} + \frac{1}{2\pi \mu} + \frac{M}{2\pi p},$$  \hspace{1cm} (5.18)

has a perturbative expansion where each subsequent term with increasing order provides a small contribution to the proceeding one. Since $\frac{1}{C_0} \sim \Lambda > \mu$, the terms with the expansion factor $\frac{\mu}{\Lambda}$ in Eq. (5.18) can be safely neglected and the physical observable will not be affected. Thus, one can choose to renormalize the bare coupling $C_0$ by using the MS scheme. The scattering
matrix can be written

\[ T = -C_0 \left( 1 + i \frac{Mp}{2\pi} C_0 + ... \right). \] (5.19)

By comparing Eq. (5.19) with Eq. (5.5), one gets \( C_0 = \frac{2\pi a}{M} \), which is \( \mu \)-independent. Thus, the scattering length \( a \) is natural \( a \sim \frac{1}{\Lambda} \propto O(Q^0) \). In the present example, we have applied Weinberg’s power-counting scheme and evaluated the amplitude perturbatively [79].

On the other hand, if the two body scattering generates a low energy bound or virtual state, the bare coupling \( C_0 \) is enhanced and should be counted as \( O(Q^{-1}) \). This new power-counting scheme was introduced by Kaplan, Savage, and Wise [79] and van Kolck [80] for strongly coupled systems such as \( s \)-wave scattering of \( NN \). The terms in Eq. (5.18) are now of the same order, \( O(Q^{-1}) \), as the first term in the expansion. As a consequence, one must sum up all iterated diagrams to get the transition matrix

\[ T = \frac{2\pi}{M} \frac{1}{\frac{2\pi a}{M} + \mu + ip}, \] (5.20)

where we have now employed the PDS scheme instead of the MS scheme. In the limit \( a \mu \to \infty \), \( C_0 \) scales as \( \frac{2\pi}{M\mu^2} \), see Eq. (5.21). So, if we set \( \mu \sim p \), then \( C_0 \) is counted as \( O(Q^{-1}) \). By contrast, if we apply the MS scheme for very large \( a \), then the bare coupling \( C_0 \) is large and scales as \( \frac{2\pi a}{M} \). In this case, the momentum expansion is only valid for a very small region of \( p \), i.e. \( \lvert ap \rvert < 1 \).

To express Eq. (5.20) in terms of the scattering length \( a \), one can compare
it with Eq. (5.4) to get

$$\frac{1}{C_0(\mu)} = \frac{M}{2\pi} \left( \frac{1}{a - \mu} \right).$$  \hspace{1cm} (5.21)

The most salient feature of the power counting is that the bare coupling

$C_0(\mu)$ has been fine-tuned to produce the large scattering length which in
turn leads to the large cross section.

The transition matrix has a pole indicating the existence of either a virtual
state or a real bound state. To discriminate between them, one can study
the positions of the poles in the complex momentum variable by using what
are known as Riemann surfaces. In our case, there are two Riemann sheets
crossing at the cut in the energy complex plane. They can be defined in the
complex momentum variable as, see Fig. 5.3

• $\text{Im } p > 0 \text{ sheet I (physical),}$
• $\text{Im } p < 0 \text{ sheet II.}$

The position of the pole is $p = i\gamma$ where $\gamma = \frac{1}{a}$ is the binding momentum
associated with the large scattering length. It corresponds to a shallow bound
state if the scattering length is positive where the pole is located in the upper
half plane in the complex momentum variable (sheet I), see Fig. 5.3. This
can be understood physically by studying the asymptotic behavior of the
wave function $\psi \sim e^{ipr} = e^{-\gamma r} \rightarrow 0 \text{ as } r \rightarrow \infty$. But for negative scattering
length, the pole is located on the negative part of the imaginary axis (sheet
II). This corresponds to the unphysical Riemann surface that indicates the
presence of a virtual state.

Furthermore, the large scattering length determines some universal prop-
erties of the bound state such as their binding energy

\[ E_B = \frac{p^2}{2M} = -\frac{\gamma^2}{2M}, \quad (5.22) \]

and the rms particle separation

\[ r_{rms} = \frac{a}{\sqrt{2}}. \quad (5.23) \]

![Diagram of energy and momentum planes](image)

**Figure 5.3:** (left) The position of the pole \( E_B \) in complex energy plane. As shown the cut runs from \( E = 0 \) to \( \infty \). (right) The position of the poles in momentum complex plane.

In this section, we discussed short-range interactions of two nonrelativistic heavy mesons. The scattering channel with strong coupling has a large scattering length which in turn indicates the existence of bound or virtual states. The large scattering length assists us to determine the universal properties for the system such as binding energy and particle separation. To study short-range interaction with narrow resonance, the coupling of the scattering channel to a resonance state must be taken into account and this will be our subject below.
5.2 Two-body scattering with a narrow resonance

In the previous section, the scattering amplitude of the strong short-range interaction is obtained by summing the bubble graphs with nonderivative contact interaction to all orders. This gives rise to the creation of virtual or bound states that appear as poles in the amplitude. In this section, we extend our EFT method to investigate resonances. To this end, we consider the coupling of an external field $\phi$ (dimeron) to the $s$-wave scattering channel at low energies. Here there are two limits that have a good separation of scales and thus one can build a powerful EFT. These limits are:

(i) Strong coupling between a dimer field and the scattering channel which in turn produces a virtual or bound state. This is equivalent to the physics of summing bubble graphs to all orders. In this limit the expansion parameters are the same.

(ii) Weak coupling between a dimer field and the scattering channel causes the production of a narrow resonance in the energy spectrum. In this case, $E_R (\Gamma_R \ll E_R)$ and this corresponds to a new low energy scale. This in turn provides a different expansion parameter from the strong-coupling limit.

The latter is of interest and will be considered in this section. For the case of the $X(3872)$, the dimer field represents the $X(3872)$ as being either a conventional charmonium or a tetraquark state. Throughout this thesis the
term dimeron will be used to refer to a resonance.

Now let us begin with writing down the most general Lagrangian for the system

\[ \mathcal{L} = \mathcal{L}_s + \mathcal{L}_r + \mathcal{L}_{sr}, \]  

(5.24)

where \( \mathcal{L}_s \) is already defined in Eq. (5.6) and \( \mathcal{L}_r \) is

\[ \mathcal{L}_r = \phi^\dagger \left( i\partial_0 + \frac{\nabla^2}{2M_R} - E_R \right) \phi, \]  

(5.25)

where \( \phi^\dagger (\phi) \) creates (destroys) a dimeron state and \( M_R \) is the total mass of particles in the single channel. The resonance parameter \( E_R \) defines the residual energy of the dimeron. The interaction of the dimeron with the particles is described by

\[ \mathcal{L}_{sr} = -g \left( (\psi_1^T \mathcal{J} \psi_2)^\dagger \phi + \phi^\dagger (\psi_1^T \mathcal{J} \psi_2) \right), \]  

(5.26)

where \( g \) measures the coupling strength between the dimeron and the particles in the scattering channel.

In a \( 2 \times 2 \)-matrix form, the interaction potential \( V \) of the system is

\[ V = \begin{pmatrix} C_0 & g \\ g & E_R \end{pmatrix}. \]  

(5.27)

In this section, producing a narrow resonance with a sharp peak is of interest. Thus, we must treat \( g \) as a small coupling, for strongly coupled dimer only bound or virtual states emerge and there is no resonance. The peak of the resonance becomes sharper as \( g \) gets smaller. If the coupling \( C_0 \) between
particles in the scattering channel is weak, then there will be no virtual or bound states. In this case, the effect of the s-wave transition appears as a smooth background underneath the resonance peak. To this end, we define the power-counting scheme for the system

\[ C_0 \sim O(Q^0), \]
\[ g \sim O(Q). \]  

The weak coupling \( C_0 \) produces a natural background scattering length \( C_0 = \frac{2\pi}{M} a_{bg} \) and can be renormalized in the minimal subtraction scheme, for details see the comment below Eq. (5.19). For this system, the loop integral \( I \) is defined in the minimal subtraction scheme, \( I = -i \frac{M}{2\pi} p \). At low energies, the kinetic energy of particles in the scattering channel, \( E = \frac{p^2}{2M} \), and the resonance energy \( E_R \) is taken to be of order \( O(Q^2) \). For energies lying at the vicinity of the resonance peak, we take \( \delta E = E - E_R \propto O(Q^3) \). In the current problem, we have a double expansion in \( \frac{p}{m_\pi} \) and \( \frac{\delta E}{E_R} \).

It should be stated that the power-counting scheme introduced above is similar to the one proposed in [85]. However, the author of [85] has made a wrong assumption in separating the \( T \)-matrix of the system into a sum of background and Breit-Wigner terms. As will be shown below, this assumption gives an incomplete expression for the \( T \)-matrix. Other terms, which describe the mixing between the Breit-Wigner and background, are of the same order in the power counting but were not included.
To derive the scattering amplitude for this system, it is convenient to introduce the following projection operators

\[ \mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \]

(5.30)

and

\[ \mathcal{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \]

(5.31)

where they act on two orthogonal subspaces that span the Hilbert space. The \( \mathcal{P} \)-subspace gives the physical scattering amplitude for the system. To derive the scattering matrix in the \( \mathcal{P} \)-subspace, we begin with \( T = -V + VT \) and use \( \mathcal{P} + \mathcal{Q} = 1 \) and after some algebraic steps, one gets

\[ T^{-1}_{pp} = -V^{-1}_{\text{eff}} + I, \]

(5.32)

where \( T^{-1}_{pp} = \mathcal{P}T \mathcal{P} \).

The effective potential can be written in terms of the background scattering length as

\[ V_{\text{eff}} = \frac{2\pi}{M} \left( a_{bg} + \frac{g'^2}{E - E_R} \right), \]

(5.33)

where \( g' \) is the dimensionless coupling, \( g'^2 = \frac{M}{2\pi} g^2 \). Substituting \( I \) and Eq. (5.33) into Eq. (5.32), one gets

\[ T^{-1}_{pp} = -\frac{M}{2\pi} \left[ \left( a_{bg} + \frac{g'^2}{E - E_R} \right)^{-1} + ip \right], \]

(5.34)
where its inverse can be expressed in the following way,

\[ T_{pp} = -\frac{2\pi}{M} \frac{a_{bg}(E - E_R) + g^2}{E - E_R + ipg^2 + i(E - E_R)p a_{bg}}. \] (5.35)

The physical scattering length is defined at the zero energy of the scattering matrix and its structure from the above expression is

\[ a = a_{bg} - \frac{g^2}{E_R}. \] (5.36)

where according to our convention, see Eq. (5.3), the positive (negative) sign of the scattering length indicates the existence of repulsive (attractive) interactions. From Eq. (5.35), one can identify the energy-dependent decay width of the resonance,

\[ \Gamma(E) = 2pg^2 = 2g^2 \sqrt{2ME}. \] (5.37)

At the threshold, \( \Gamma(E_R) \) is very small, while at energies well above the threshold, the number of states will increase and \( \Gamma(E) \) becomes large. At \( E = E_R \), this gives the physical width of the resonance \( \Gamma(E_R) \). In our power-counting scheme, the size of \( \Gamma(E) \) is \( O(Q^3) \) and this indicates the resonance is very narrow.

In the EFT, physical observables such as the scattering amplitude can be expanded in powers of low energy scales. For example, by expanding
Eq. (5.35) in terms of low energy scales up to $O(Q^0)$, one gets

$$T_{pp} = -\frac{2\pi}{M} \left( \frac{g^2}{E - E_R + i g^2 p} + a_{bg} - \frac{2 i a_{bg} g^2 p}{E - E_R + i g^2 p} - \frac{a_{bg} g^4 p^2}{(E - E_R + i g^2 p)^2} \right),$$

(5.38)

where the leading (first) term represents the Breit-Wigner resonance and is counted as $O(Q^{-1})$. The second term represents the background scattering and scales as $O(Q^0)$. Other terms represent the mixing between Breit-Wigner and background scattering and scale as $O(Q^0)$. The power-counting scheme implemented here leads to an enhancement of the last two terms that have been neglected in Ref. [85] due to the wrong assumption made there\(^2\). The last two terms in our $T_{pp}$-matrix (Eq. (5.38)) are not presented in the leading order $O(Q^0)$ of $T$-matrix given in Eq.(19) of Ref. [85].

The scattering matrix should have simple poles. The appearance of the double pole in Eq. (5.38) is an artifact of our expansion. This expansion is valid only for real energies. For complex energies close to the pole, we have to resum the expansion to obtain the complete expression which has a simple pole with a shifted position and a different residue. The complete expression of Eq. (5.38) is

$$T_{pp} = -\frac{2\pi}{M} \left( \frac{g^2(1 - 2 i a_{bg} p)}{E - E_R + i g^2 p} + a_{bg} \right),$$

(5.39)

where the real part of the resonance energy is shifted as $E_R \to E_R + 2 M E_R a_{bg} g^2$ to maintain the position of the pole. As we are interested in the region of physical real energies, we will consider the scattering matrix

\(^2\)The author treated the Breit-Wigner and background terms independently in calculating the $T$-matrix for the system.
given in Eq. (5.38) in our calculation of the line shapes of the \(X(3872)\) in the next Chapter. It is worth mentioning that, for real energies, both expressions of the \(T_{pp}\)-matrix given in Eq. (5.38) and Eq. (5.39) are equivalent at the order to which we are working\(^3\).

Now we want to investigate positions of the scattering matrix poles in energy and momentum planes. The denominator of the \(T_{pp}\)-matrix can be written as

\[
D(p) = \frac{p^2}{2M} - E_R + ig'^2 p. \tag{5.40}
\]

where we replaced \(E\) by \(\frac{p^2}{2M}\). To find the pole position, \(p_p\), one has to solve \(D(p_p) = 0\). The solutions are

\[
p_p^\pm = -ig'^2 M \mp \sqrt{-g'^4 M^2 + 2M E_R}, \tag{5.41}
\]

which can also be written in terms of the energy

\[
E_p^\pm = -g'^4 M + E_R \mp \sqrt{g'^8 M^2 - 2g'^4 M E_R}. \tag{5.42}
\]

According to our power-counting scheme, terms in \(E_p^\pm\) scale as

\[
g'^4 M \sim O(Q^4), \quad E_R \sim O(Q^2), \quad g'^8 M^2 \sim O(Q^8), \quad 2g'^4 M E_R \sim O(Q^6),
\]

\(^3\)Numerical comparison of Eq. (5.38) and Eq. (5.39) show excellent agreement to the accuracy we are working at.
thus terms in the square-root in Eq. (5.42) can be approximated by $i\sqrt{2g'^4ME_R}$.

At the leading orders of real and imaginary parts, $E_p^\pm$ is

$$E_p^\pm = E_R \mp ig'^2\sqrt{2ME_R}.$$  \hspace{1cm} (5.43)

As the resonance energy has a small negative imaginary part, the relevant (physical) solution is

$$E_p^+ = E_R - ig'^2\sqrt{2ME_R},$$  \hspace{1cm} (5.44)

which is located on the second sheet, close to the physical axis. From Eq. (5.37), $E_p^+$ can be written

$$E_p^+ = E_R - \frac{i}{2}\Gamma_R,$$  \hspace{1cm} (5.45)

in terms of resonance width, $\Gamma_R = \Gamma(E_R)$. The second solution

$$E_p^- = E_R + \frac{i}{2}\Gamma_R,$$  \hspace{1cm} (5.46)

is also located on sheet II, however it is far from the physical axis. In Fig. 5.4, positions of $E_p^+$ and $E_p^-$ and the corresponding momenta are shown in the complex plane.

Note that if one takes large $g$ in Eq. (5.42), i.e. of order $O(Q^0)$, then $E_p^\pm$ becomes negative and this corresponds to virtual or bound states. Thus, adding a dimeron with large $g$ does not give something new, i.e. producing resonances. That is why we demand $g$ to be small.

144
5.3 Coupled Channels

In this section, the EFT treatment will be developed to describe $s$-wave scattering for systems with more than one open channel. Such systems have been studied using different techniques such as effective theory treatment [83], and renormalization group analysis [84].

To set up the problem, let us consider two channels with different thresholds where each of them contains two heavy particles with definite spin and isospin. We label the particles in the lower channel by $A_1$ and $A_2$ and in the upper channel by $B_1$ and $B_2$. The thresholds are separated by $\Delta_{AB}$, where $\Delta_{AB} = m_{B_1} + m_{B_2} - (m_{A_1} + m_{A_2})$. For example, in the case of the $X(3872)$, the thresholds of the charged and neutral $\bar{D}D^*$ mesons are separated by $\Delta_X = m_{D^-} + m_{D^{*-}} - (m_{\bar{D}^0} + m_{D^{*+}})$.

To ensure that our EFT with only short-range interactions is valid, the size of the splitting $\Delta_{AB}$ must be smaller than the scale associated with pion
exchange, $\frac{m_2^2}{2M_{1,2}}$, where $M_{1,2}$ are the reduced masses of particles in the lower and upper channels. In this case, $\Delta_{AB}$ can be treated as a soft scale at very low energy scattering. The low energy scales in the current problem include the relative momentum in the lower channel $p = \sqrt{2M_1E}$, and in the upper channel $p' = \sqrt{2M_2(E - \Delta_{AB})} = \sqrt{\frac{M_2}{M_1}(p^2 - \delta^2)}$ where $\delta = \sqrt{2M_1\Delta_{AB}}$ is the momentum scale associated with splitting [84]. The relative momenta and splitting are, however, not the only low energy scales if we consider strong $s$-wave interaction in the scattering channels, where the inverse of the large scattering length provides another soft scale.

Now let us define the general structure of the effective Lagrangian for this system

$$\mathcal{L}_{cc} = \psi_A^\dagger \left( i\partial_0 + \frac{\nabla^2}{2m_A} \right) \psi_A + \psi_B^\dagger \left( i\partial_0 - (m_B - m_A) + \frac{\nabla^2}{2m_B} \right) \psi_B + \psi_A^\dagger \left( i\partial_0 - (m_B - m_A) + \frac{\nabla^2}{2m_B} \right) \psi_B + \mathcal{L}_{int},$$

(5.47)

where $\psi_i, \psi_i^\dagger$ are the annihilation and creation operators for a particle of type $i$. The lowest order interaction Lagrangian is

$$\mathcal{L}_{int} = -c_1 \left( (\psi_A^* \psi_A) \cdot (\psi_A^* \psi_A) \right) - c_2 \left( (\psi_B^1 \psi_B^2) \cdot (\psi_B^1 \psi_B^2) \right) - c_{12} \left( (\psi_A^* \psi_A) \cdot (\psi_B^1 \psi_B^2) + \text{h.c.} \right),$$

(5.48)

where the bare background couplings $c_1, c_2$ measure the $s$-wave transitions in the lower and upper channels respectively. The transitions between these scattering channels are given by $c_{12}$. 

146
For the coupled-channel problem, the interaction potential and loop functions can be written in a $2 \times 2$-matrix form [83]

$$V = \begin{pmatrix} c_1 & c_{12} \\ c_{12} & c_2 \end{pmatrix}, \quad I = -\frac{M^\frac{1}{2}}{2\pi} \begin{pmatrix} \mu + ip & 0 \\ 0 & \mu + ip' \end{pmatrix} M^\frac{1}{2},$$  \hspace{1cm} (5.49)

where $M$ is the diagonal matrix of the reduced masses

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$  \hspace{1cm} (5.50)

For simplicity, we will consider equal masses$^4$, $M_1 = M_2 = M$.

We are now in a position to derive the scattering matrix for the system. Here there are three possibilities for short-range interactions at very low energy:

(i) Both channels are weakly interacting.

(ii) Both channels are strongly interacting. For this system, there are two bound or virtual states.

(iii) One channel has a strong $s$-wave interaction. Here, there is a single bound or virtual state.

The last one is of interest and will be discussed in the following context. To make our discussion more general, we work in the strong and weak basis that are related to the asymptotic basis by orthogonal transformations, as we will see below. In this case, the strong and weak channels are defined as

$^4$ For the case of the $X(3872)$, the two channels ($D^{0*}\bar{D}^0$ and $D^{+*}\bar{D}^-$) are separated by 8 MeV. Their reduced masses $M_{D^{0*}\bar{D}^0} = 966.6$ MeV and $M_{D^{+*}\bar{D}^-} = 968.67$ MeV are nearly equal.
linear combinations of the asymptotic channels. To this end, let us introduce the following eigenvectors

\[ u = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}, \quad v = \begin{pmatrix} -\sin \phi \\ \cos \phi \end{pmatrix}, \]

(5.51)

that correspond to the eigenvalues \( c_u \) and \( c_v \) of the interaction potential given in Eq. (5.49). The angle \( \phi \) defines the mixing between channels. As an example, for the \( \bar{D}D^* \)-system with perfect isospin symmetry, the eigenvectors of \( V \) correspond to channels with \( \phi = \frac{\pi}{4} \); this case has been considered in Ref. [103].

From Eq. (5.51), one can construct the following operators [84]

\[ P_u = u u^\dagger, \]

(5.52)

\[ P_v = v v^\dagger, \]

(5.53)

\[ P_m = u v^\dagger, \]

(5.54)

\[ P_m^\dagger = v u^\dagger, \]

(5.55)

where \( m \) refers to the mixing between \( u \) and \( v \) channels. We consider that the strong and weak channels are represented by \( u \) and \( v \) respectively.

To find the form of the interaction potential and loop functions in the strong and weak basis, one has to perform the following orthogonal transformations

\[ V_{sw} = O^T V O, \quad I_{sw} = O^T I O, \]

(5.56)

148
where $O$ is

$$O = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}. \quad (5.57)$$

In terms of $P_u$, $P_v$, and $P_m$ operators defined above, $V_{sw}$ is

$$V_{sw} = c_u P_u + c_v P_v, \quad (5.58)$$

where

$$c_u = c_1 \cos^2 \phi + c_2 \sin^2 \phi + 2c_{12} \sin \phi \cos \phi, \quad (5.59)$$

$$c_v = c_2 \cos^2 \phi + c_1 \sin^2 \phi - 2c_{12} \sin \phi \cos \phi.$$ 

Similarly, the loop function can be written as

$$I_{sw} = -\frac{M}{2\pi} \left( (\mu + ip_u)P_u + (\mu + ip_v)P_v - ip_m(P_m + P_m^\dagger) \right), \quad (5.60)$$

where the momentum variables are defined by

$$p_u = p \cos^2 \phi + p' \sin^2 \phi, \quad (5.61)$$

$$p_v = p \sin^2 \phi + p' \cos^2 \phi,$$

$$p_m = (p - p') \sin \phi \cos \phi.$$ 

In the strong and weak basis, the $T_{sw}$-matrix can be obtained by solving the matrix form of the Lippmann-Schwinger equation to get

$$T_{sw}^{-1} = -V_{sw}^{-1} + I_{sw}. \quad (5.62)$$
The inverse of the interaction potential given in Eq. (5.58) is

\[ V_{sw}^{-1} = \frac{1}{c_u} P_u + \frac{1}{c_v} P_v. \]  

(5.63)

Now, substituting Eq. (5.63) and Eq. (5.60) into Eq. (5.62), one gets

\[ T_{sw}^{-1} = -\left[ \left( \frac{1}{c_u} + \frac{M}{2\pi} \mu + i \frac{M}{2\pi} p_u \right) P_u + \left( \frac{1}{c_v} + \frac{M}{2\pi} \mu + i \frac{M}{2\pi} p_v \right) P_v \right. \]

\[ - i \frac{M}{2\pi} p_m (P_m + P_m^\dagger) \]. \] 

(5.64)

At zero energy, the elements of \( T_{sw}^{-1} \) can be expressed in terms of the physical scattering lengths \( a_u \) and \( a_v \)

\[ \frac{M}{2\pi} a_u = \frac{1}{c_u} + \frac{M}{2\pi} \mu, \] 

(5.65)\[ \frac{M}{2\pi} a_v = \frac{1}{c_v} + \frac{M}{2\pi} \mu. \] 

(5.66)

In terms of the physical scattering lengths \( a_u \) and \( a_v \), \( T_{sw} \) is

\[ T_{sw}^{-1} = -\frac{M}{2\pi} \left[ \left( \frac{1}{a_u} + ip_u \right) P_u + \left( \frac{1}{a_v} + ip_v \right) P_v \right. \]

\[ - i p_m (P_m + P_m^\dagger) \]. \] 

(5.67)

The inverse of Eq. (5.67) is

\[ T_{sw} = - \frac{2\pi M}{D} \left[ \left( \frac{1}{a_v} + ip_v \right) P_u + \left( \frac{1}{a_u} + ip_u \right) P_v \right. \]

\[ + i p_m (P_m + P_m^\dagger) \]. \] 

(5.68)
where the determinant $D$ is

$$D = \left( \frac{1}{a_u} + ip_u \right) \left( \frac{1}{a_v} + ip_v \right) + p_m^2. \quad (5.69)$$

Without further approximation, the scattering matrix given in Eq. (5.68) is the most general structure for $u$ and $v$ scattering channels in the strong and weak basis for a momentum-independent potential.

The strongly interacting $u$ channel generates a large scattering length $a_u$ that scales as $O(Q^{-1})$. This unnatural large scattering length indicates the existence of a bound or virtual state. On the other hand, the scattering length in the $v$-channel (weakly interacting channel) is natural and scales as $O(Q^0)$ which corresponds to the Weinberg power-counting scheme.

As stated, in the EFT, one can expand the scattering matrix in terms of the low energy scales $Q$. So, at the order $O(Q^0)$, the coupled channels $T_{suw}$-matrix is

$$T_{suw} = -\frac{2\pi}{M} \left[ \left( \frac{1}{a_u} + ip_u \right) - \frac{a_v p_m^2}{\left( \frac{1}{a_u} + ip_u \right)^2} \right] P_u + a_v P_v$$

$$- \frac{ia_v p_m}{\frac{1}{a_u} + ip_u} \left( P_m + P_m^\dagger \right). \quad (5.70)$$

All that remains is to determine the poles of the elements of the expanded $T$-matrix. The denominators of the elements have the following form

$$D(p) = \frac{1}{a_u} + ip_u. \quad (5.71)$$

To find the pole position in the momentum plane, one has to solve $D(p) = 0$. 151
By expressing \( p_u \) in terms of \( p \) and \( p' \), one can express \( D(p) = 0 \) as

\[
p = \sec^2 \phi \left( \frac{i}{a_u} - p' \sin^2 \phi \right),
\]

or

\[
p' = \csc^2 \phi \left( \frac{i}{a_u} - p \cos^2 \phi \right),
\]

where the momentum variables \( p \) and \( p' \) are related by

\[
p' = \sqrt{p^2 - \delta^2}.
\]

By substituting Eq. (5.74) into Eq. (5.72), one gets the solutions

\[
p = \frac{1}{a_u (\cos^4 \phi - \sin^4 \phi)} \left[ i \cos^2 \phi \pm \sin^2 \phi \sqrt{\delta^2 a_u^2 (\sin^4 \phi - \cos^4 \phi) - 1} \right].
\]

(5.75)

Clearly, the angle \( \phi \) has a significant impact on the position of the poles.

As we have seen in the single channel case, the square root of the energy, which is the momentum, is associated with two Riemann surfaces for the energy. Here, there are two momentum variables \( (p, p') \). In the case of two open channels \( (E \geq \Delta_{AB}) \), we have four Riemann surfaces that can be classified in the following way [86]:

- \( \text{Im} \, p > 0, \, \text{Im} \, p' > 0 \), sheet I (physical),
- \( \text{Im} \, p < 0, \, \text{Im} \, p' > 0 \), sheet II,
- \( \text{Im} \, p < 0, \, \text{Im} \, p' < 0 \), sheet III,
- \( \text{Im} \, p > 0, \, \text{Im} \, p' < 0 \), sheet IV.

Poles that are located on sheet I (II) correspond to bound (virtual) states.
However, poles that are located on sheets III and IV are called Breit-Wigner resonances and shadow resonances respectively. A shadow resonance is analogous to a virtual state in a single channel and its sheet in the complex energy plane is far away from the physical energy axis. A Breit-Wigner resonance is close to the physical region in the complex energy plane and so has a large effect on the scattering amplitude [84, 86]. It is worth mentioning that our EFT treatment for the coupled channels with different thresholds is valid for energies which lie at or between these thresholds. Below, we only illustrate the types of state that can emerge from the $T$-matrix, given in Eq. (5.70), by considering the following two simple limits:

- In the limit $\cos \phi \gg \sin \phi$ Eq. (5.75), which can be approximated by

$$p \approx \frac{i}{a_u \cos^2 \phi},$$

(5.76)

gives a negative energy. Clearly, $\text{Im}(p)$ can be either positive which corresponds to a bound state (sheet I) or negative which reflects the existence of a virtual state (sheet II) of the lower channel. It is worth mentioning that this case corresponds to the kinetic energy of particles that is comparable to the lower threshold, i.e. $\Delta_{AB} \gg E$.

- In the limit $\sin \phi \gg \cos \phi$, the solution of Eq. (5.75), which is close to the physical sheet, can be approximated by

$$p \approx -\frac{i}{a_u} \left(\frac{\cos \phi}{\sin^2 \phi}\right)^2 + \frac{1}{\sin^2 \phi} \sqrt{\delta^2 - \frac{1}{a_u^2}},$$

(5.77)

where $\text{Im}(p)$ is negative (positive) for the case of positive (negative)
scattering length. By substituting Eq. (5.77) into Eq. (5.73), one can get the approximate expression for \( p' \). This is given by

\[
p' \approx \frac{1}{\sin^2 \phi} \left( \frac{i}{a_n} - \frac{\cos^2 \phi}{\sin^2 \phi} \sqrt{\delta^2 - \frac{1}{a_n^2}} \right). \tag{5.78}
\]

In this expression \( \text{Im}(p') \gg \text{Re}(p') \). Poles move on the complex momentum and energy plane as we vary the scattering mixing angle \( \phi \). Thus, if we take \( \phi = \frac{\pi}{2} \), then the particles in the upper channel do not mix with the particles in the lower channel. In this situation, for \( a_\mu > 0 \), there exists a pole on sheet II corresponds to a virtual state of the upper channel, see Fig. 5.5. For the case of negative scattering length, \( \text{Im}(p') \) becomes negative and this indicates the existence of a shadow resonance on sheet IV.

However, if there is a small mixing with the lower channel (i.e. \( \phi \) is very close to \( \frac{\pi}{2} \)), then there is a pole located on sheet II for the case of positive scattering length. This pole corresponds to a resonance in the lower channel, see Fig. 5.5. This state will decay into the particles in the lower channel.

It should be noted that this case corresponds to kinetic energy of the particles which is close to the upper threshold, i.e. \( \Delta_{AB} \gg |E - \Delta_{AB}| \).

In this section, the short-range interactions in the two scattering channels are discussed. In particular, we considered the case in which one of the coupled channels is strongly interacting at very low energy. This in turn causes the production of a bound state or a virtual state. To study the short-
Figure 5.5: As we vary the mixing angle, poles move on the complex plane as shown by arrows. For $\phi = \frac{\pi}{2}$, a pole is located on sheet II. It corresponds to a virtual state of the upper channel. This pole lies below the upper elastic threshold on the real axis in the $E$-plane as shown by the solid circle. As we vary the mixing angle slightly to allow a small mixing with the lower channel, the pole which is located on the pure imaginary part on sheet II moves toward the physical axis as shown by the symbol $\oplus$ in the $p$-plane. This pole reflects the existence of a resonance state that can decay to the particles in the lower channel. In the $E$-plane, the resonance state is denoted by the symbol $\oplus$.

range interaction with narrow resonances, we next consider the coupling of a dimer state to weakly-interacting channels.

5.4 Coupled channels with a narrow resonance

The coupling of a dimer state to two open channels will be considered in this section. Before proceeding, let us start by writing down the general structure of the effective Lagrangian

$$\mathcal{L} = \mathcal{L}_{cc} + \mathcal{L}_r + \mathcal{L}_{cr}$$

(5.79)
where $L_{cc}$ and $L_r$ are already given in Eq. (5.47) and Eq. (5.25) respectively and $L_{cr}$ is

$$L_{cr} = -g_1 \left( (\psi_{A_1} \psi_{A_2})^\dagger \cdot \phi + h.c. \right) - g_2 \left( (\psi_{B_1} \psi_{B_2})^\dagger \cdot \phi + h.c. \right), \quad (5.80)$$

where $g_1$ describes the coupling of the dimer state (resonance) to the particles in the lower channel whereas $g_2$ measures the coupling of the resonance to the particles in the upper channel.

The interaction potential for the system can be written in a $3 \times 3$-matrix form

$$V = \begin{pmatrix} c_1 & c_{12} & g_1 \\ c_{12} & c_2 & g_2 \\ g_1 & g_2 & E_R \end{pmatrix}. \quad (5.81)$$

Here, we assume that the $A$ and $B$ channels, separated by $\Delta_{AB}$, are weakly interacting and coupled weakly to the dimer state. In this case, a narrow resonance with a sharp peak will emerge in the energy spectrum. To this end, we use the same power-counting scheme that was introduced for the single channel case, i.e.

$$c_1 \sim c_2 \sim c_{12} \sim O(Q^0), \quad g_1 \sim g_2 \sim O(Q),$$

$$E \sim E_R \sim O(Q^2), \quad \delta E = E - E_R \sim O(Q^3).$$

As in the single channel problem, we have a double expansion in $\frac{p}{m_o}$ and $\frac{\delta E}{E_R}$.

By following similar steps to those shown in the previous sections, one can define the scattering matrix for this system in terms of the $u$ and $v$ basis.
The $u$ and $v$ are eigenvectors of the background potential

$$V^{b.g.} = \begin{pmatrix} c_1 & c_{12} \\ c_{12} & c_2 \end{pmatrix}. \quad (5.82)$$

The general structure of the scattering amplitude is

$$T_{pp}^{-1} = -V_{\text{eff}}^{-1} + I, \quad (5.83)$$

where the effective potential is given by

$$V_{\text{eff}} = c_u P_u + c_v P_v + G \frac{1}{E - E_R} G^\dagger. \quad (5.84)$$

Here $c_u$ and $c_v$ are the eigenvalues defined in Eq. (5.59) and $G$ is

$$G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \quad (5.85)$$

Assuming there exists some symmetry, then resonance couples to only one of the eigenvectors, e.g. $u$. For example, the isospin symmetry for the case of the $X(3872)$. This isosinglet resonance couples to the 0-isospin combinations of the neutral and charged meson states $(\bar{D}D^*)_{I=0}$. In this case, the coupling can be written as

$$g_1 = g \cos \phi, \quad g_2 = g \sin \phi, \quad (5.86)$$

where $\phi$ is mixing angle. The effective potential given in Eq. (5.84) becomes

$$V_{\text{eff}} = (c_u + \frac{g^2}{E - E_R})P_u + c_v P_v. \quad (5.87)$$
The loop function for the system is

\[ I = -\frac{M}{2\pi} \left( i p_u P_u + i P_v - i p_m (P_m + P_m^\dagger) \right), \tag{5.88} \]

where we have used the MS scheme. As in the previous section, we consider \( M_1 = M_2 = M \). In the case of weak short-range interactions, the bare couplings \( c_u \) and \( c_v \) are related to the background scattering lengths that scale as \( O(Q^0) \)

\[ a_u = \frac{M}{2\pi} c_u, \quad a_v = \frac{M}{2\pi} c_v. \]

By substituting Eq. (5.87) and Eq. (5.88) into Eq. (5.83), the general formula of the scattering matrix is

\[ T_{pp} = \frac{2\pi}{M} \left( T_u P_u + T_v P_v + T_m (P_m + P_m^\dagger) \right). \tag{5.89} \]

The explicit expressions for the elements of the \( T_{pp} \)-matrix at the order \( O(Q^0) \), in low energy scales, are

\[ T_u = \frac{g^2}{E - E_R + i g^2 p_u} + a_u - \frac{2 i a_u g^2 p_u}{(E - E_R + i g^2 p_u)} - \frac{g^4 \left( a_u p_u^2 + a_v p_v^2 \right)}{(E - E_R + i g^2 p_u)^2}, \]

where \( g' \) is the dimensionless coupling, \( g^2 = \frac{M}{2\pi} g^2 \), and \( p_u \) was defined in Eq. (5.61). As above, the double pole is an artifact of our expansion which is valid only for real energies. The element \( T_v \) is just a contact interaction,
$T_v = a_v$. The off-diagonal element is

$$T_m = \frac{ig^2a_vp_m}{E - E_R + ig^2p_u}.$$  

Our approach for the two weakly interacting channels coupled to a dimer state is valid at energies lying at the vicinity of the resonance energy, i.e. $E \sim E_R$.

Now, we want to investigate the positions of the poles of the expanded $T$-matrix. To find the pole position, $E_p$, one has to solve

$$E_p - E_R + ig^2p_u = 0. \quad (5.90)$$

For the sake of simplicity, we consider the following limits:

- In the limit $\Delta_{AB} \gg E$, the kinetic energy is comparable to the lower threshold. Here, the dimeron can decay to the continuum open channel, and also can couple to the particles in the closed channel as virtual continuum states. The momentum of the upper channel is $p' = \pm i\delta$ and Eq. (5.90) can be expressed in terms of momentum,

$$\frac{p^2}{2M} - E_R + ig^2(p\cos^2\phi \pm i\delta \sin^2\phi) = 0. \quad (5.91)$$

The solutions of the quadratic equation are

$$p^\pm = -i g_1^2 M \mp \sqrt{2M E'_R - g_1^4 M^2}, \quad (5.92)$$
where \( E'_R \) and \( g'_1 \) are resonance parameters and given by

\[
E'_R = E_R \pm g'^2 \delta \sin^2 \phi, \tag{5.93}
\]
\[
g'_1 = g' \cos \phi. \tag{5.94}
\]

Since \( E_R \sim O(Q^2) \) and \( g'^2 \delta \sim O(Q^3) \), the resonance parameter \( E'_R \) scales as \( O(Q^2) \). As the resonance has positive real part and small negative imaginary part in the \( E \)-plane, we will consider \( E_R > 0 \) (and hence \( E'_R > 0 \)) throughout this section. Since \( \text{Im} \ p^\pm < 0 \), poles are located on sheet II. From Eq. (5.92), one can express the solution for the energy up to leading order of real and imaginary parts,

\[
E^\pm_p = E'_R \mp \sqrt{-2g'^4_1 M E'_R}. \tag{5.95}
\]

The solution

\[
E'^+_p = E'_R - i g'^2_1 \sqrt{2 M E'_R}, \tag{5.96}
\]

represents a Breit-Wigner resonance. \( E'^+_p \) has a small imaginary part, of order \( O(Q^3) \), which represents the decay width of the resonance. The real part of \( E'^+_p \), which scales as \( O(Q^2) \), represents the resonance energy. This is illustrated in Fig. 5.6.

- In the limit \( \Delta_{AB} \gg |E - \Delta_{AB}| \), the kinetic energy is comparable to the upper threshold. Here, the resonance energy is close to the upper threshold \( E_R \sim \Delta_{AB} \) and the momentum of the particles in the lower channel is \( p = \sqrt{2M\Delta_{AB}} = \delta - i0 \), where we have added a small
negative imaginary part to mimic the sign of the imaginary part of the physical choice for \( p \) in the limit \( \sin \phi \gg \cos \phi \) which corresponds to our current case.

To find the positions of the poles in complex plane, one has to solve the quadratic equation

\[
\frac{p^2}{2M} + \Delta_{AB} - E_R + ig'( \delta \cos^2 \phi + P' \sin^2 \phi) = 0. \tag{5.97}
\]

The solutions are

\[
p'^\pm = -i g'_2 M \pm \sqrt{2M (E'_R - \Delta_{AB}) - g'^4_2 M^2}. \tag{5.98}
\]

The resonance parameters

\[
E'_R = E_R - ig'^2 \delta \cos^2 \phi, \tag{5.99}
\]

\[
g'_2 = g' \sin \phi, \tag{5.100}
\]

are different from the previous limit, see Eq. (5.93) and Eq. (5.94). In the current case, the coupling \( g'_2 \) measures the resonance coupling to particles in the upper channel. Here the pole

\[
E^+_p = E'_R - \Delta_{AB} - i g'^2_2 \sqrt{2M (E'_R - \Delta_{AB})}, \tag{5.101}
\]

represents a Breit-Wigner resonance that is located on sheet III, see Fig. 5.6.
The techniques developed in this Chapter are general, and can be used to investigate the properties of resonances in the charmonium and bottomonium systems. In the next Chapter, we will use the derived scattering amplitude for the single channel problem to plot the line shapes of the $X(3872)$.

**Figure 5.6:** The symbol $\oplus$ denotes a pole that is located on sheet II. This pole corresponds to a Breit-Wigner resonance. The symbol $\otimes$ denotes a pole that is located on sheet III. This corresponds to a Breit-Wigner resonance.
Chapter 6

Decays of mesonic molecules in EFT

In the previous Chapter, we studied the interactions between heavy mesons at very low energy using EFT. Our underlying goal is to investigate the nature of newly observed charmonium and bottomonium states. This Chapter takes a further step toward that by studying the decays of such states in the framework of EFT. As the decays of the $X(3872)$ are well measured, we take it as an illustrative example throughout our discussion. The properties of the decays of these mesons, such as their line shapes, are discussed for a single channel problem in the second part of this Chapter.
6.1 Width of the $X(3872)$

As described in the previous Chapter, the strong $s$-wave contact interaction of the nonrelativistic $\bar{D}^0D^{*0}$ pair is responsible for producing the $X(3872)$. The loop diagrams shown in Figs. 6.1-6.2 represent the self-energy diagrams of the $X(3872)$. In this section, we will only calculate the imaginary part of these diagrams which tells about the width and lifetime of the $X(3872)$.

The loop diagram in Fig. 6.1 represents the basic self-energy of the $X(3872)$ in an EFT without a virtual pion; we refer to it as the loop diagram without a virtual pion. In Fig. 6.2(a), the off-shell excited charm meson $D^{*0}$ emits and (re)absorbs a pion. This sub-diagram describes the self-energy of the excited charm meson. We refer to Fig. 6.2(a) as the loop diagram with self-energy. As the wave function of the $X(3872)$ is defined as a linear combination of $D^0\bar{D}^{*0}$ and $\bar{D}^0D^{*0}$, the diagram shown in Fig. 6.2(a) has a charge conjugation partner where $\bar{D}^0 \rightarrow D^0$ and $D^{*0} \rightarrow \bar{D}^{*0}$.

In Fig. 6.2(b), the intermediate charm mesons exchange a pion; this diagram describes the interference between the $D^0\bar{D}^{*0}$ and $\bar{D}^0D^{*0}$ components of the wave function. We refer to Fig. 6.2(b) as the interference loop diagram. This diagram also has a charge conjugation partner.

Here we will perform our calculation within the framework of EFT. Our approach is similar to X-EFT [88] in exploiting the experimental coincidence between the pion mass, $m_\pi$, and the hyperfine splitting within $\bar{D}^{*0}-D^0$ states, $\Delta_H$, to develop the power-counting schemes.

Within X-EFT, $m_\pi$ and $\Delta_H$ are treated as large scales. The authors of [88] exploited the experimental coincidence between them and identified their
difference as \( \delta = \Delta_H - m_\pi \sim O(Q_X) \). This leads to expanding in powers of the scale ratio \( \frac{\delta}{m_\pi} \sim 0.04 \). In addition to the effective momentum scale in the pion propagator \( \mu \left( \mu = \sqrt{\Delta_H^2 - m_\pi^2} \right) \), the momenta of nonrelativistic pion and charm mesons are identified as soft scales and are counted as \( Q_X \). Therefore energies scale as \( Q_X^2 \). The excited charm mesons \( D^{*0} (\bar{D}^{*0}) \) are treated as stable particles. Later, X-EFT has been extended to calculate the hadronic decay of the \( X(3872) \) to the \( \chi_{cJ} \) and pions \([119, 120]\). In Ref. \([124]\), Jansen et al. extended X-EFT by including the effect of the nonzero width of the excited charm meson. They used this effective theory to investigate the light quark mass dependence of the \( X(3872) \).

It is worth mentioning that the expansion parameter\(^1\) \( \frac{m_\pi}{M_D} \), which has been used in the calculations undertaken by the authors of \([88]\) and \([124]\), is not justified by the X-EFT power-counting scheme. This is due to treating \( m_\pi \) and \( \Delta_H \) as large scales; this will be shown below when introducing our power-counting scheme which is not only based on their coincidence but also on treating them as low energy scales.

The standard power-counting scheme of X-EFT reproduces the static-potential approximation\(^2\) when calculating the loop diagrams with a three-body intermediate state, \( D\bar{D}\pi \), see Fig. 6.2. The authors of Ref. \([88]\) have shown that their approach– which is based on suppressing \( \frac{m_\pi}{M_D} \)-terms, notably the kinetic energy of \( D \)-meson \( \frac{m_\pi}{M_D}p_D^2 \)– is not a good approximation for evaluating the imaginary part of the loop diagram with three-body in-

\(^1\)Note, it is natural to expand in powers of the mass ratio \( \frac{m_\pi}{M_D} \) since \( \frac{m_\pi}{M_D} \approx 0.07 \).

\(^2\)Technically speaking, this is equivalent to expanding the intermediate state propagator of the three-body state \( (D^{*0}\bar{D}^{*0}) \) in the powers of \( \frac{m_\pi}{M_D} \), which gives a pion propagator at leading order.
termediate state, see Fig. 6.2 (a). This is because neglecting the $D$-mesons’ kinetic energy in the final state gives rise to a cut with two-body structure instead of three-body, which results in incorrect behaviour near the threshold. To avoid this, the authors of Ref. [88] kept the $D$-mesons’ kinetic energy—although this is inconsistent with the second expansion in $\frac{m_\pi}{M_D}$—when calculating the imaginary parts of the loop integrals to get results in agreement with Voloshin’s calculation [111, 112]$^3$. However, the $D$-mesons’ kinetic energy are still omitted in their calculation of the real parts of the loop integrals, which contribute to the wavefunction renormalization.

In our approach, the pion mass and hyperfine splitting are treated as soft scales relative to a typical hadronic scale of $m_\rho$ (or $\Lambda_\chi$) and are counted as $O(Q)$. This is in agreement with HHChPT. We propose a new power-counting scheme exploiting the experimental coincidence between them and count their difference as $O(Q^2)$. Consequently, the pion momentum in the threshold region (and the effective momentum scale $\mu$ in the pion propagator) scales as $O(Q^\frac{3}{2})$ in our power-counting scheme. Unlike the standard power-counting scheme of X-EFT, ours reproduces a cut with the correct three-body structure when calculating the imaginary parts of diagrams with virtual pions, see Fig. 6.2. This in turn allows us to determine the differential rate for the process $X \to D^0\bar{D}^0\pi^0$ directly from evaluating the imaginary parts of the relevant loop diagrams. By setting $\Gamma_{D^*} = 0$, our theory reproduces Voloshin’s expression which was obtained using effective-range theory (ERT) [111]. The expansion parameters ($\frac{m_\pi}{M_D}$, $\frac{\delta}{m_\pi}$) that have been employed in the

$^3$ Recall that Voloshin exploited the universal wave function to predict the momentum distributions for the decay channels $D^0\bar{D}^0\pi^0$ [111] and $D^0\bar{D}^0\gamma$ [112].
work performed in [88, 124] are justified in our power-counting scheme, where they are both counted as $O(Q)$.

As will be shown, the loop integrals of the diagrams (a) and (b) in Fig. 6.2 give the same result. In our approach, the loop diagrams with and without virtual pions are of the same order for energies within $\Gamma_{D^*}$ of the $\bar{D}^0 D^{*0}$ threshold, i.e. they scale as $O(Q)$, and consequently they contribute to the leading order amplitude $A_{-1}$.

In Ref. [102], Hanhart, Kalashnikova and Nefediev have shown graphically that the loop integral of the diagram in Fig. 6.2(a) can be approximated by the loop integral of the diagram in Fig. 6.1 within a small range of energies that lie very close to the $D^{*0}\bar{D}^0$ threshold. In this Chapter, we use our power-counting scheme to justify this; we will show that the result of the loop integrals with virtual pions can be approximated by the result of the loop integral without a virtual pion in the appropriate limits.

Now let us start by evaluating the loop integral for the diagram with

---

**Figure 6.1:** This Figure represents the basic loop diagram for the self-energy of the $X(3872)$. The solid line represents $\bar{D}^0$ and the double solid line represents $D^{*0}$.
Figure 6.2: In (a), the loop diagram with self-energy. In (b), the interference loop diagram. The notation is the same as in Fig. 6.1, with dashed lines representing a pion.

The structure of this loop integral is

\[ I_{ij}^{(a)} = \left( \frac{g}{2f} \right)^2 \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \frac{q_i q_j}{q^0 - q^2 - m^2 + i\epsilon} \]
\[ \times \left( k_0 + \frac{E}{2} - \frac{k^2}{2M_{D^*}} + \frac{i}{2} \Gamma_{D^*} (k_0 + \frac{E}{2} - \frac{k^2}{2M_{D^*}}) \right) \left( -k_0 + \frac{E}{2} + \Delta_H - \frac{k^2}{2M_D} + i\epsilon \right) \]
\[ \times \left( k_0 + \frac{E}{2} - \frac{k^2}{2M_{D^*}} + \frac{i}{2} \Gamma_{D^*} (k_0 + \frac{E}{2} - \frac{k^2}{2M_{D^*}}) \right) \left( k_0 + q_0 + \frac{E}{2} + \Delta_H - \frac{(k+q)^2}{2M_D} + i\epsilon \right), \]

where we have chosen to measure the external energy \( E \) from the \( \bar{D}^*0D^{*0} \) threshold. Since the \( \bar{D}^*0D^{*0} \)-threshold is of interest, the relevant energy is the one that is measured relative to it. This relevant energy is related to \( E \) by \( E_d = E + \Delta_H \). The splitting is \( \Delta_H = M_{D^*} - M_D \), and \( \Gamma_{D^*} (k_0 + \frac{E}{2} - \frac{k^2}{2M_{D^*}}) \) is the energy-dependent width of the (virtual) excited charm meson \( D^{*0} \). The indices \( i, j \) refer to polarizations of initial and final states of the \( X(3872) \) which has \( J^P = 1^+ \). By rotational invariance, \( I_{ij}^{(a)} \) can be written as

\[ I_{ij}^{(a)} = \delta_{ij} I^{(a)}, \]
where

\[
I^{(a)} = \frac{1}{3} \left( \frac{g}{2 f} \right)^2 \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \frac{q^2}{q^2 - q^2 - m^2 + i\epsilon} \\
\times \frac{1}{k_0 + \frac{E}{2} - \frac{k^2}{2M_{D^*}} + \frac{i}{2} \Gamma_{D^*}(k_0 + \frac{E}{2} - \frac{k^2}{2M_{D^*}})} \frac{1}{-k_0 + \frac{E}{2} + \Delta_H - \frac{k^2}{2M_{D}} + i\epsilon} \\
\times \frac{1}{k_0 + \frac{E}{2} - \frac{k^2}{2M_{D^*}} + \frac{i}{2} \Gamma_{D^*}(k_0 + \frac{E}{2} - \frac{k^2}{2M_{D^*}})} \frac{1}{(k_0 + q_0) + \frac{E}{2} + \Delta_H - \frac{(k+q)^2}{2M_{D}} + i\epsilon}.
\]

(6.3)

The factor $\frac{1}{3}$ results from averaging over the three components.

It is worth mentioning that the work carried out by Hanhart, Kalashnikova and Nefediev in Ref. [102] investigated how line shapes for composite particles with unstable constituents depend on the widths of their constituents. For this, they considered two cases for unstable constituents: narrow and broad constituents. For a constituent with a narrow width (as in the case of the $X(3872)$), they found that the full expression for the self-energy diagram with virtual pion (corresponding to the loop integral given in Eq. (6.3)) agrees with the approximate results which can be obtained from the loop diagram without a virtual pion. As this agreement holds for either the physical width of the unstable constituent or its energy-dependent width, they concluded that there is no advantage to using the energy-dependent width and one can safely use the physical width instead. Thus, the loop
integral $I^{(a)}$ can be written in terms of the physical width $\Gamma_{D^*}$,

$$I^{(a)} = \frac{1}{3} \left( \frac{g}{2f} \right)^2 \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} q_i q_j \frac{1}{q^2 - q^2 - m_{\pi}^2 + i\epsilon} \frac{1}{k_0 + E - \frac{k^2}{2M_{D^*}} + i\frac{\Gamma_{D^*}}{2}}$$

$$\times \frac{1}{1 - k_0 + \frac{E + \Delta_H - k^2}{2M_D} + \frac{1}{2\Gamma_{D^*}}} \frac{1}{E_d - \frac{k^2}{2M_{D^*}} + i\Gamma_{D^*}}$$

$$\times \frac{1}{(k_0 + q_0) + \frac{E - \Delta_H - (k+q)^2}{2M_D} + i\epsilon}.$$  

(6.4)

By performing the integrals over energies $k_0$ and $q_0$, one can get

$$I^{(a)} = \frac{1}{6} \left( \frac{g}{2f} \right)^2 \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{q^2}{\sqrt{q^2 + m^2}} \frac{1}{E_d - \frac{k^2}{2M_{D^*}} + i\frac{\Gamma_{D^*}}{2}}$$

$$\times \frac{1}{E_d - \frac{k^2}{2M_{D^*}} + i\frac{\Gamma_{D^*}}{2}} \frac{1}{E_d + \Delta_H - \frac{k^2}{2M_D} - \frac{(k+q)^2}{2M_D} - \sqrt{q^2 + m^2} + i\epsilon}.$$  

(6.5)

where $E_d$ is defined above. For energies lying close to the $D^0\bar{D}^{*0}$-threshold, i.e. $E \simeq -\Delta_H$, the energy variable $E_d$ scales as in the usual way for a kinetic energy $O(Q^2)$. The loop integral $I^{(a)}$ is complicated. To simplify it, one has to determine the regions in which the momentum integrals of $k$ and $q$ have large contributions.

As already mentioned, a key ingredient in our approach is a power-counting scheme that is based on the accidental cancellation between the pion mass and the hyperfine splitting, i.e. $|\Delta_H - m_\pi| \sim 7\text{ MeV} \ll m_\pi$. Based on that we introduce the new power counting

$$\Delta_H - m_\pi \sim O(Q^2).$$  

(6.6)
As in HHChPT, the pion mass and the hyperfine splitting scale as

\[ \Delta_H \sim m_\pi \sim O(Q). \]  

(6.7)

For the \( q \)-integral, let us first rearrange the terms in the denominator of the pion propagator as

\[
E_d = \frac{k^2}{2M_D} - \frac{(k + q)^2}{2M_D} + (\Delta_H - \sqrt{q^2 + m_\pi^2}),
\]

(6.8)

where the first three terms all scale as \( O(Q^2) \). The final term is generally of order \( O(Q) \) except when the two terms in the parentheses nearly cancel. This happens for

\[
q^2 \sim \Delta_H^2 - m_\pi^2 = (\Delta_H - m_\pi)(\Delta_H + m_\pi),
\]

(6.9)

where from Eqs. (6.7)-(6.6), these terms scale as

\[
\Delta_H + m_\pi \sim O(Q),
\]

\[
\Delta_H - m_\pi \sim O(Q^2).
\]

(6.10)

As a result, \( q \) scales as

\[
q \sim O(Q^3),
\]

(6.11)

and the pion propagator is enhanced.

To understand how the width \( \Gamma_{D^*} \) depends on the low energy scales, let
us recall Eq. (3.54),

\[ \Gamma_{D^*} = \frac{\sqrt{2}}{3\pi} \left( \frac{g}{2\sqrt{m_\pi f}} \right)^2 \left( \frac{m_\pi M_D}{m_\pi + M_D} \right)^\frac{5}{2} (\Delta_H - m_\pi)^\frac{3}{2}. \] (6.12)

In this case, the width \( \Gamma_{D^*} \) scales as \( O(Q^2) \).

For the \( k \)-integral, as \( k \sim O(Q), E_d \sim O(Q^2) \), and \( \Gamma_{D^*} \sim O(Q^2) \), one naturally expects that the order of \( (k^2 - 2M_{DD^*} E_d) \) is \( O(Q^2) \). However, there is an additional enhancement occurs when the magnitude of \( k^2 - 2M_{DD^*} E_d \) becomes comparable to \( \Gamma_{D^*} \). In this case, we take

\[ k \sim O(Q), \Delta(k^2) \sim \Gamma_{D^*} \sim O(Q^2), \]
\[ k^2 - 2M_{DD^*} E_d \sim \Gamma_{D^*} \sim O(Q^2), \] (6.13)

where \( \Delta(k^2) \) is the range of \( k^2 \) around the resonance. As will be shown shortly, this additional enhancement allows us to approximate the result of the loop integral for the diagrams with virtual pion, as shown in Fig. 6.2, to the result of the loop integral for the diagram without virtual pion, as shown in Fig. 6.1.

The loop integrals for both diagrams in Fig. 6.2 give the same contribution in this approximation as will be shown at the end of this section.

Now we want to use our power-counting scheme to evaluate the loop integral \( I^{(a)} \). From Eqs. (6.13)- (6.11), one can make the approximations

\[ \frac{(q + k)^2}{2M_D} \simeq \frac{k^2}{2M_D}, \] (6.14)
\[ \Delta_H - \sqrt{q^2 + m_\pi^2} \simeq \Delta_H - m_\pi - \frac{q^2}{2m_\pi}, \] (6.15)
and the loop integral $I^{(a)}$ becomes

$$
I^{(a)} = \frac{1}{6} \left( \frac{g}{2f} \right)^2 \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{q^2}{m_\pi + \frac{q^2}{2m_\pi} E_d - \frac{k^2}{2M_{DD^*}} + i \frac{1}{2} \Gamma_{D^*}} \times \\
\frac{1}{E_d - \frac{k^2}{2M_{DD^*}} + i \frac{1}{2} \Gamma_{D^*}} \frac{1}{E_d + \Delta_H - \frac{k^2}{M_D} - m_\pi - \frac{q^2}{2m_\pi} + i \epsilon},
$$

which scales as $O(Q)$.

Since we are interested in the transition rate for the $X(3872)$ to the three-body state $\bar{D}D\pi$, we can use the following identity to pick out the imaginary part of the loop integral

$$
\frac{1}{x \pm i \epsilon} = P \frac{1}{x} \mp i \pi \delta(x),
$$

where $P$ refers to the principal value. Note that the real part can be absorbed in the physical mass of the $X(3872)$. The imaginary part of the pion propagator corresponds to putting the intermediate state $\bar{D}D\pi$ on shell. By denoting the imaginary part of $I^{(a)}$ from Eq. (6.16) by $-\mathcal{F}^{(a)}$, we get

$$
\mathcal{F}^{(a)} = \frac{1}{6} \left( \frac{g}{2f} \right)^2 \int \frac{d^3k}{(2\pi)^3} \left| \frac{1}{E_d - \frac{k^2}{2M_{DD^*}} + i \frac{1}{2} \Gamma_{D^*}} \right|^2 \int \frac{d^3q}{(2\pi)^3} \frac{q^2}{m_\pi + \frac{q^2}{2m_\pi} E_d - \Delta_H + \frac{k^2}{M_D}} \times \\
\delta \left( m_\pi + \frac{q^2}{2m_\pi} - E_d - \Delta_H + \frac{k^2}{M_D} \right).
$$

(6.18)
It is convenient to express the delta function as
\[\delta\left(m_\pi + \frac{q^2}{2m_\pi} - E_d - \Delta_H + \frac{k^2}{M_D}\right) = \frac{m_\pi}{q} \delta\left(q - \sqrt{2m_\pi \left(E_d + \Delta_H - m_\pi - \frac{k^2}{M_D}\right)}\right).\]  
\[(6.19)\]

The \(\delta\)-function with the kinetic energies of both pion and charm mesons gives the correct structure for the three-body system. All terms in the \(\delta\)-function are of order \(Q^2\) and must be considered when performing the \(q\)-integral. The \(\delta\)-function then imposes a constraint on the maximum value of the \(\bar{D}D\)-relative momentum \(k\), see Eq. (6.21). If we set \(\Gamma_{\bar{D}D^*} = 0\), Eq. (6.18) reproduces Voloshin’s calculation using effective range theory [111].

Our treatment is in contrast to the static approximation discussed in [88] in which \(\frac{m_\pi}{M_D}\) terms are suppressed, i.e. this corresponds to neglecting the kinetic energy of charm mesons. In this case, the \(\delta\)-function does not depend on both relative momenta \((q, k)\), but only on the pion momentum \(q\). This in turn generates a cut with two-body structure; in this case, the pion is moving relative to a static \(\bar{D}D\).

The authors of [88] have shown that using the static approximation in evaluating the imaginary part of \(I^{(a)}\) (i.e. Eq. (6.18)) is inappropriate. However, they used the static approximation only in evaluating the real part of the loop integral \(I^{(a)}\) which contributed to the NLO wave function renormalization. As demonstrated in [107], this is also inappropriate since using the static approximation in evaluating loop integrals that have three-body singularities leads to incorrect results for the non-analytic behaviour at threshold. In this case, both real and imaginary parts miss important contributions to \(I^{(a)}\). Therefore, the NLO contribution to the partial width \(\Gamma[X \to D^0\bar{D}^0\pi^0]\)
presented in [88] missed important contributions.

After performing the integration over the pion momentum $q$, one gets

$$
F^{(a)} = \frac{\sqrt{2} \, m_\pi M_{DD^*}^2}{3\pi^3 (E_d + \Delta_H)} \left( \frac{m_\pi}{M_D} \right)^{\frac{3}{2}} \left( \frac{g}{2 f} \right)^2 I_k. \tag{6.20}
$$

Here $I_k$ is given by

$$
I_k = \int_{0}^{\sqrt{M_D E_H}} dk \frac{k^2 (M_D E_H - k^2)^{\frac{3}{2}}}{(k^2 - 2 M_{DD^*} E_d)^2 + (M_{DD^*} \Gamma_{D^*})^2}, \tag{6.21}
$$

where $E_H = E_d + \Delta_H - m_\pi$. The relative momentum of charm mesons in the final state is bounded above by $E_H$. We have used Mathematica to solve $I_k$. The result is very long and it is convenient to write it in the form

$$
I_k = \begin{cases} 
  a \left( 2 b \Re(I_+ + M_{DD^*} \Gamma_{D^*} (2 c \Im(I_+) + d)) \right); & E_d \geq 0 \\
  a \left( -2 b \Im(I_- + M_{DD^*} \Gamma_{D^*} (2 c \Re(I_-) + d)) \right); & E_d < 0
\end{cases}, \tag{6.22}
$$

where

$$
a = \frac{\pi}{4 M_{DD^*} \Gamma_{D^*}}, \tag{6.23}
$$

$$
b = (2 M_{DD^*} E_d M_D (E_d + \Delta_H - m_\pi) + M_{DD^*} \Gamma_{D^*}^2 - 4 M_{DD^*}^2 E_d^2), \tag{6.24}
$$

$$
c = M_D (E_d + \Delta_H - m_\pi) - 4 M_{DD^*} E_d, \tag{6.25}
$$

$$
d = 8 M_{DD^*} E_d - 3 M_D (E_d + \Delta_H - m_\pi), \tag{6.26}
$$

$$
I_+ = \sqrt{\frac{-M_s \Delta_H E_d}{M_D + M_{DD^*}^2} + \frac{(\Delta_H - m_\pi) M_D + i M_{DD^*} \Gamma_{D^*}}{2 M_{DD^*} E_d - i M_{DD^*} \Gamma_{D^*}}}, \tag{6.27}
$$

$$
I_- = \sqrt{\frac{-M_s \Delta_H E_d}{M_D + M_{DD^*}^2} + \frac{(m_\pi - \Delta_H) M_D - i M_{DD^*} \Gamma_{D^*}}{2 M_{DD^*} E_d - i M_{DD^*} \Gamma_{D^*}}}. \tag{6.28}
$$
Now, we want to calculate the loop integral for the diagram without a virtual pion; this diagram is shown in Fig. 6.1. This self-energy loop integral has already been evaluated in Chapter 5 in the case $\Gamma_{D^*} = 0$. Below, we redo the calculation, this time including the physical decay width in the loop integral. The basic loop integral, $I$, is

$$I = -i \int \frac{d^4k}{(2\pi)^4} \frac{i}{k_0 + \frac{E}{2} - \frac{k^2}{2M_D} + \Delta_H + i\epsilon - k_0 + \frac{E}{2} - \frac{k^2}{2M_{D^*}} + \frac{i}{2}\Gamma_{D^*}}. \quad (6.29)$$

After performing the energy contour integral, one gets

$$I = \int \frac{d^3k}{(2\pi)^3} \frac{1}{E - \frac{k^2}{2M_{D^*}} + \Delta_H + \frac{i}{2}\Gamma_{D^*}}, \quad (6.30)$$

where the total energy $E$ of charm mesons ($\bar{D}^0$ and $D^{*0}$) is measured from the $\bar{D}^0 D^{*0}$-threshold. In terms of $E_d$, the basic loop integral, $I^b$, is given by

$$I = -2M_{D^*} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 - 2M_{D^*} E_d - iM_{D^*} \Gamma_{D^*}}. \quad (6.31)$$

Since the charm meson propagator in Eq. (6.31) has the same structure as the one in Eq. (6.5), $I$ scales in a similar way as $I^{(a)}$, i.e. $I \sim O(Q)$.

The final result for the loop integral $I$ is

$$I = \frac{M_{D^*}}{2\pi} \sqrt{-2M_{D^*} \left( E_d + \frac{i}{2}\Gamma_{D^*} \right)}. \quad (6.32)$$

This differs from the case of the loop integral with stable constituents, i.e. $\Gamma_{D^*} = 0$, see Eq. (5.13). In this case, the branch point in the $E$-plane is
located at $-\frac{i}{2} \Gamma_{D^*}$ for real energy $E_d$. It is convenient to write Eq. (6.32) as

$$I = \frac{M_{DD^*}}{2\pi} \left( \sqrt{M_{DD^*}} \left( -E_d + \sqrt{E_d^2 + \frac{\Gamma_{D^*}^2}{4}} \right) - i \sqrt{M_{DD^*}} \left( E_d + \sqrt{E_d^2 + \frac{\Gamma_{D^*}^2}{4}} \right) \right).$$

(6.33)

The imaginary part of $I$ can be defined as $-\mathcal{F}$ where

$$\mathcal{F} = \frac{M_{DD^*}}{2\pi} \sqrt{M_{DD^*}} \left( E_d + \sqrt{E_d^2 + \frac{\Gamma_{D^*}^2}{4}} \right).$$

(6.34)

Since $M_X = M_X^* - \frac{i}{2} \Gamma_X$, the partial width of the $X(3872)$ can be related to the imaginary part by

$$\Gamma_X = \frac{2}{M_{DD^*}} \mathcal{F}.$$  

(6.35)

In this case, $\Gamma_X$ depends on the energy $E_d$. At threshold, $\Gamma_X$ is very small, but at energies which lie well above the threshold, the number of states will increase and $\Gamma_X$ becomes large. According to our power-counting scheme $\Gamma_X \approx \frac{1}{\pi} \sqrt{2M_{DD^*} E_d} \gg \Gamma_{D^*}$ and hence $X(3872)$ decays into $D^0 D^{*0}$ with a much shorter lifetime than $D^{*0}$. As a result, the meson $D^{*0}$ can be treated as if it were a stable particle as it does not influence the decay of the $X(3872)$.

Now, we are in a position to illustrate how $\mathcal{F}^{(a)}$ can be approximated by $\mathcal{F}$. To this end, let us write $\mathcal{F}^{(a)}$ as

$$\mathcal{F}^{(a)} = \mathcal{A} I_k.$$  

(6.36)
where

\[ I_k = \int_0^{\sqrt{M_D E_H}} \frac{dk}{(k^2 - 2 M_{DD*} E_d)^2 + (M_{DD*} \Gamma_{DD*})^2} \]  \hspace{1cm} (6.37)

\[ \mathcal{A} = \left( \frac{\sqrt{2}}{3 \pi^2} \left( \frac{g}{2 f} \right)^2 \frac{m_\pi M_{DD*}^2}{(E_d + \Delta_H)} \left( \frac{m_\pi}{M_D} \right)^{\frac{3}{2}} \right). \]  \hspace{1cm} (6.38)

To express \( \mathcal{A} \) in terms of \( \Gamma_{DD*} \), one can write Eq. (6.12) up to the leading order in the pion mass as

\[ \Gamma_{DD*} = \frac{\sqrt{2}}{3 \pi} \left( \frac{g}{2 f} \right)^2 m_\pi \left( M_{DD*} - M_D - m_\pi \right)^{\frac{3}{2}}, \]  \hspace{1cm} (6.39)

where \( \frac{m_\pi M_{DD*}}{m_\pi + M_D} \simeq m_\pi + O \left( \frac{m_\pi^2}{M_D} \right) \). Hence, \( \mathcal{A} \) is given by

\[ \mathcal{A} = \frac{m_\pi}{(E_d + \Delta_H)} \frac{M_{DD*}^2}{\pi^2} \frac{\Gamma_{DD*}}{(M_D (\Delta_H - m_\pi))^{\frac{3}{2}}}. \]  \hspace{1cm} (6.40)

The numerator of the integral \( I_k \) can be rearranged,

\[ M_D E_H - k^2 = M_D (\Delta_H - m_\pi) + M_D E_d - k^2 \]

\[ = M_D (\Delta_H - m_\pi) + (M_D - 2 M_{DD*}) E_d + (2 M_{DD*} E_d - k^2), \]  \hspace{1cm} (6.41)

where in the second line we added and subtracted \( 2 M_{DD*} E_d \) to the last two terms of the first line. In the second line, the leading contribution comes from the first term which is counted as \( O(Q^2) \). The size of the second term \( (M_D - 2 M_{DD*}) E_d \) is \( O(Q^3) \), where \( M_D - 2 M_{DD*} = -\frac{M_D \Delta_H}{M_D + M_{DD*}} \sim O(Q) \). The last term provides a smaller contribution where \( (2 M_{DD*} E_d - k^2) \sim \Gamma \sim \)
\(O(Q^2)\). Thus, up to order \(O(Q^2)\) (in the numerator), \(I_k\) is given by

\[
I_k \simeq (M_D(\Delta_H - m_\pi))^\frac{3}{2} I_k^{(2)},
\]

where

\[
I_k^{(2)} = \int_0^{\frac{\sqrt{MD^*E_H}}{k}} \frac{k^2 dk}{(k^2 - 2 M_{DD^*} E_d)^2 + (M_{DD^*} \Gamma_{D^*})^2}.
\]

The result of the integral \(I_k^{(2)}\) is

\[
I_k^{(2)} = \frac{i}{2M_{DD^*}\Gamma_{D^*}} \left[ \sqrt{-2 M_{DD^*}} \left( E_d + \frac{i}{2} \Gamma_{D^*} \right) \right.
\]

\[
\times \arctan \left[ \sqrt{-2 M_{DD^*}} \left( E_d + \frac{i}{2} \Gamma_{D^*} \right) \right]
\]

\[
- \left. \sqrt{-2 M_{DD^*}} \left( E_d - \frac{i}{2} \Gamma_{D^*} \right) \right]
\]

\[
\times \arctan \left[ \sqrt{-2 M_{DD^*}} \left( E_d - \frac{i}{2} \Gamma_{D^*} \right) \right].
\]

As \(M_D \simeq 2 M_{DD^*}\) and \(E_H \gg |E_d \pm \frac{i}{2} \Gamma_{D^*}|\) at the \(D^0\) \(D^{*0}\)-threshold, the \(\arctan\) functions can be approximated as \(-\frac{\pi}{2}\). After some algebraic steps, one can get

\[
I_k^{(2)} = \frac{\pi}{2M_{DD^*}\Gamma_{D^*} \sqrt{M_{DD^*}}} \left( E_d + \sqrt{E_d^2 + \frac{\Gamma_{D^*}^2}{4}} \right)
\]

\[
= \frac{\pi^2}{M_{DD^*}\Gamma_{D^*} F},
\]

\[(6.45)\]
which shows that $\mathcal{F}^{(a)}$ can be approximated by

$$
\mathcal{F}^{(a)} = A I_k \simeq \frac{m_\pi}{(E_d + \Delta_H)} F.
$$

(6.46)

Note that to leading order $E_d + \Delta_H = m_\pi + O(Q^2)$, $\mathcal{F}^{(a)} = F$. In Fig 6.3, we illustrate the behaviours of $\mathcal{F}^{(a)}$ (Eq. (6.20)) and $\mathcal{F}$ (Eq. (6.34)) within the range of energy $-4\text{ MeV} \leq E_d \leq 4\text{ MeV}$. We have used the following numerical values for constants and masses:

$$
M_{D_{0}^{*+}} = 966.66\text{ MeV}, \quad M_{D_{0}^{*}} = 2006.99\text{ MeV}, \quad M_D = 1864.86\text{ MeV},
$$

$$
\Gamma_{D_{0}^{*+}} = 0.0454\text{ MeV}, \quad m_{\pi^0} = 134.97\text{ MeV}, \quad \Delta_H = 142.13\text{ MeV},
$$

$$
g = 0.64, \quad \text{and} \quad f = 92.4\text{ MeV}.
$$

(6.47)

As shown in Fig 6.3, the functions $\mathcal{F}$ and $\mathcal{F}^{(a)}$ agree very well at energies lying very close to the $\bar{D}^0D^{*0}$-threshold. For energies lying up to 4 MeV, they agree within $\sim 6\%$. Thus, the approximation given in Eq. (6.46) is acceptable at this range of energies.

Now, we want to consider the loop integral of the interference diagram shown in Fig. 6.2 (b). The structure of the loop integral is

$$
I^{(b)} = \frac{1}{3} \left( \frac{g}{2f} \right)^2 \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \frac{q^2}{q_0^2 - q^2 - m_\pi^2 + i\epsilon} \frac{1}{-k_0 + \frac{E}{2} - \frac{k^2}{2M_{D^{*+}}} + \frac{i}{2} \Gamma_{D^{*+}} k_0 + \frac{E}{2} + \Delta_H - \frac{k^2}{2M_D} + i\epsilon} \times \frac{1}{(k_0 + q_0) + \frac{E}{2} - \frac{(k+q)^2}{2M_{D^{*+}}} + \frac{i}{2} \Gamma_{D^{*+}} -(k_0 + q_0) + \frac{E}{2} + \Delta_H - \frac{(k+q)^2}{2M_D} + i\epsilon}.
$$

180
After performing the energy contour integrals, one gets

\[
I^{(b)} = \frac{1}{6} \left( \frac{g}{2f} \right)^2 \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{q^2}{\sqrt{q^2 + m^2}} \\
\times \frac{1}{E_d - \frac{k^2}{2M_{DD^*}} + \frac{i}{2} \Gamma_{D^*}} \frac{1}{E_d - \frac{(k+q)^2}{2M_{DD^*}} + \frac{i}{2} \Gamma_{D^*}} \\
\times \left[ \frac{1}{E_d + \Delta_H - \frac{k^2}{2M_D} - \frac{(k+q)^2}{2M_D} - \sqrt{q^2 + m^2 + i\epsilon}} + \frac{1}{E_d - \frac{k^2}{2M_{DD^*}} - \frac{(k+q)^2}{2M_{DD^*}} - \sqrt{q^2 + m^2 + \gamma_{D'}}} \right],
\]

where the first term inside the square brackets corresponds to the decay of the excited charm meson, \( D^{*0} \rightarrow D^0 \pi^0 \), that can occur at the low energies in which we are interested. The second term, however, does not contribute.
at low energies. Hence, we are left with

\[ I^{(b)} = \frac{1}{6} \left( \frac{g}{2f} \right)^2 \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{q^2}{\sqrt{q^2 + m^2}} E_d - \frac{1}{2M_{DD^*}} + \frac{i}{2} \Gamma_D^* \]

\[ \times \frac{1}{E_d - \frac{(k+q)^2}{2M_{DD^*}} + \frac{i}{2} \Gamma_D^*} E_d + \Delta_H - \frac{k^2}{2M_D} - \frac{(k+q)^2}{2M_D} - \sqrt{q^2 + m^2 + i\epsilon}. \]

Note that the structure of the loop integral for the interference diagram is similar to the one for the self-energy diagram. As a consequence, the loop integrals \( I^{(a)} \) and \( I^{(b)} \) scale in the same way. Now, by using the power-counting scheme defined in Eqs. (6.13)-(6.11), the loop integral \( I^{(b)} \) can be approximated as

\[ I^{(b)} = \frac{1}{6} \left( \frac{g}{2f} \right)^2 \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{q^2}{\sqrt{q^2 + m^2}} E_d - \frac{1}{2M_{DD^*}} + \frac{i}{2} \Gamma_D^* \]

\[ \times \frac{1}{E_d - \frac{k^2}{2M_{DD^*}} + \frac{i}{2} \Gamma_D^*} E_d + \Delta_H - \frac{k^2}{2M_D} - \sqrt{q^2 + m^2 + i\epsilon}. \]

This expression is now identical to Eq. (6.16). Hence, the imaginary part of \( I^{(b)} \) is

\[ \text{Im} I^{(b)} = \text{Im} I^{(a)} = \mathcal{F}^{(a)}, \quad (6.48) \]

where the function \( \mathcal{F}^{(a)} \) is given in Eq. (6.20).

The wave function of the \( X(3872) \) (as it is a bound state of neutral charm meson pairs) in flavor space is given by [78]

\[ |X\rangle = \frac{1}{\sqrt{2}} \left( |\bar{D}^0 D^{*0}\rangle - |D^0 \bar{D}^{*0}\rangle \right), \]

where the negative sign shows that the charge conjugation of the \( |X\rangle \)-wave
function is positive and this is consistent with experimental measurement of its charge conjugation. In this case, the full contribution to the width of \(X(3872)\) from the loop diagrams of Fig. 6.2 and their charge conjugate partners is

\[
\text{Im} I_X(E_d) = 2 \mathcal{F}^{(a)}.
\]  

(This result will be used in the next section to obtain the differential rate formula for the transition \(X \to \bar{D}^0 D^0 \pi^0\).

### 6.2 Line shapes of the \(X(3872)\)

The line shape of a quarkonium (or in general any particle) is the invariant mass distribution of its decay products. The comparison between theoretical predictions for the line shapes of a quarkonium state and experimental results can provide information about the nature of this particle.

The \(X(3872)\) can be produced as an intermediate state from the weak decay modes of the charged and neutral \(B\)-mesons into \(K^{\pm,0}\bar{D}^0 D^0 \pi^0\) [67, 68, 70, 71]. This three-step decay can be illustrated as

\[
\begin{align*}
B^{\pm,0} & \to K^{\pm,0} X \\
& \quad \quad \quad \quad \downarrow \bar{D}^0 D^0 \\
& \quad \quad \quad \quad \quad \quad \quad \downarrow D^0 \pi^0
\end{align*}
\]

The differential rate for this process as a function of energy leading to a \(D^0 D^0 \pi^0\) final state can be written in the form

\[
\frac{d\text{Br}(X \to \bar{D}^0 D^0 \pi^0)}{dE_d} = \mathcal{B}|T(E_d)|^2 \text{Im} I_X(E_d),
\]  

(6.50)
under the assumption of factorization, for details see [103] and references therein. The details of the $X(3872)$ production are not necessary and are buried in the constant $B$. The energy $E_d$ is measured from the $\bar{D}^0 D^{*0}$-threshold. As $E_d \sim O(Q^2)$, Eq. (6.50) is valid only at energies very close to the $\bar{D}^0 D^{*0}$-threshold. The value $\text{Im} I_X (E_d)$ is given in Eq. (6.49) and $T(E_d)$ is the scattering amplitude for the short-range interaction of the nonrelativistic $\bar{D}^0 D^{*0}$ pair, see Chapter 5 for details. In this section, we only consider the single-channel scattering problem.

For single-channel scattering, the following cases are of interest:

(i) a strong short-range interaction which can be obtained by summing the bubble graphs with nonperturbative contact interaction to all orders. This leads to a bound or a virtual state. This case corresponds to the $X(3872)$ as being a molecular state,

(ii) an external dimeron field couples weakly to an $s$-wave weakly-coupled scattering channel. This generates a narrow resonance. This case could correspond to the $X(3872)$ as being either a conventional charmonium or a tetraquark state.

In this Chapter, the line shapes of $X(3872)$ for both cases will be illustrated.
6.2.1 Line shapes of the strong $\bar{D}^0D^{*0}$ scattering channel

Let us start by writing down the $T$-matrix for the zero-range interaction of $\bar{D}^0D^{*0}$ with an unstable $D^{*0}$ at energies lying very close to $\bar{D}^0D^{*0}$-threshold,

$$T_s = -\frac{2\pi}{M_{DD^*}} \frac{1}{\gamma_b - p_*}, \quad (6.51)$$

where $p_*$ results from solving the loop integral for the single interacting channel with $\Gamma_{D^*} \neq 0$, see Eq. (6.33). The expression for $p_*$ is

$$p_* = \sqrt{\frac{M_{DD^*}}{E_d^2 + E_d^2 + \frac{\Gamma_{D^*}^2}{4}}} - i \sqrt{\frac{M_{DD^*}}{E_d^2 + E_d^2 + \frac{\Gamma_{D^*}^2}{4}}}. \quad (6.52)$$

For $\Gamma_{D^*} = 0$, the real part of $p_*$ vanishes and Eq. (6.51) reduces to the usual form given by Eq. (5.4) in Chapter 5.

The quantity $\gamma_b$ in Eq. (6.51) is the inverse of the scattering length $a$ (binding momentum). In our work, $\gamma_b$ is taken to be real. With this, we neglect the effect of other decay modes of $X(3872)$ such as $J/\psi \pi^+\pi^-$, $J/\psi \pi^+\pi^-\pi^0$ and $J/\psi \gamma$ [104].

For $\gamma_b > 0$, the scattering matrix $T_s$ has a pole near the $\bar{D}^0D^{*0}$-threshold, $\gamma_b = p_*(E_{\text{pole}})$. The position of the pole is given by [104]

$$E_{\text{pole}} = -\frac{\gamma_b^2}{2M_{DD^*}} - \frac{i}{2}\Gamma_{D^*}. \quad (6.53)$$
For the $X(3872)$ with binding energy $E_X$, $E_{\text{pole}}$ can be written as

$$E_{\text{pole}} = -E_X - \frac{i}{2} \Gamma_X,$$  \hspace{1cm} (6.54)

where in this case

$$E_X \approx \frac{\gamma_b^2}{2 M_{DD^*}}, \quad \Gamma_X \approx \Gamma_{D^*}.$$  \hspace{1cm} (6.55)

This pole is located on the first (physical) sheet. In the complex energy plane it corresponds to an unstable $\bar{D}^0 D^{*0}$ bound state with a small width related to its decay to $\bar{D}^0 D^{0} \pi^0$. In the line shapes, for positive $E_X$, a narrow Breit-Wigner resonance with a peak located at $E_X$ is produced if $\Gamma_X \ll E_X$ [104]. For $\gamma_b < 0$, the pole is located on the second sheet in the complex energy plane, and this in turn corresponds to a virtual state.

The energy $E_X$ can be extracted from experiments where it is related to the binding energy of the $X(3872)$, $E_X = -E_B$. Using the measured mass of $X(3872)$ ($M_X = 3871.69 \pm 0.17$ MeV) and $\bar{D}^0 D^{*0}$ energy threshold ($m_{D^*} + m_{D^{*0}} = 3871.8 \pm 0.122$ MeV), one gets

$$E_B = M_X - (m_{D^*} + m_{D^{*0}}) = -0.11 \pm 0.21 \text{ MeV}.$$  \hspace{1cm} (6.56)

This indicates that $X(3872)$ could be a bound state with negative $E_B$ or a resonance with positive $E_B$ as implicitly implied by the error. The binding momentum corresponding to the central value of $E_B$ is

$$\gamma_b = \sqrt{-2 M_{DD^*} E_B} = 14.58 \text{ MeV}.$$  \hspace{1cm} (6.57)
Now we are in a position to plot the line shapes of the $X(3872)$. The differential rate for the $s$-wave scattering is given by Eq. (6.50) with $T_s$ given by Eq. (6.51). Here we plot the line shape of the $X(3872)$ with $\gamma_b > 0$, this corresponds to a bound state explanation, and compare it with the case of virtual state explanation, i.e. $\gamma_b < 0$.

In Fig. 6.4, we plot the line shapes for $\gamma_b = \pm 14.58$ MeV. The position of the peak for $\gamma_b = +14.58$ MeV (solid line) is located at the central value of Eq. (6.56). This corresponds to a real bound state. In contrast, for $\gamma_b = -14.58$ MeV, which corresponds to a virtual state, there is an enhancement above the threshold (dashed line) but no peak.

![Figure 6.4](image)

**Figure 6.4:** Line shapes of the $X(3872)$. The solid line represents line shapes for positive $\gamma_b$. The dashed line represents line shapes for negative $\gamma_b$. In our plots, we set the factor $\mathcal{B} = 1$ MeV in the factorization expression given in Eq. (6.50).
6.2.2 Line shapes of the resonance coupled to the \( \bar{D}^0 D^{*0} \) scattering channel

As stated in Chapter 1, the other interpretations of the \( X(3872) \) are a charmonium state, \( \chi_{c1}(2P) \), and a compact tetraquark meson. In the following context, we will illustrate the line shape of such system. In Chapter 5, effective field theory (EFT) methods have been developed to calculate the relevant scattering matrix for the zero-range interaction with a narrow resonance. This is given by

\[
T_r = -\frac{2\pi}{M_{DD^*}} \left( \frac{g^2}{E_d - E_X - g^2 p_s} + a_{bg} \frac{2 a_{bg} g^2 p_s}{E_d - E_X - g^2 p_s} + \frac{a_{bg} g^4 p_s^2}{(E_d - E_X - g^2 p_s)^2} \right). \tag{6.58}
\]

The real part of the inverse \( T_r \)-matrix is non-zero at the resonance energy \( E_d = E_X \). To get the right expression for \( T_r \), we redefine \( E_X \) to absorb the real part of \( p_s \) at \( E_d = E_X \) as discussed in Ref. [102]. This gives

\[
p_s = \sqrt{M_{DD^*} \left( -E_d + \sqrt{E_d^2 + \frac{\Gamma_{D^*}^2}{4}} \right)} - \sqrt{M_{DD^*} \left( -E_X + \sqrt{E_X^2 + \frac{\Gamma_{D^*}^2}{4}} \right)} - i \sqrt{M_{DD^*} \left( E_d + \sqrt{E_d^2 + \frac{\Gamma_{D^*}^2}{4}} \right)}.
\]

In our EFT treatment, the dimensionless coupling \( g' \) is taken to be small \( g'^2 \sim E_X \sim O(Q^2) \). Our purpose is to produce a resonance with a sharp peak. Based on the solutions to the resonance poles that are founded in the second section of Chapter 5, the values of \( g' \) are restricted to \( g' \ll \left( \frac{2|E_X|}{M_{DD^*}} \right)^{\frac{1}{4}} \).
For resonance energy $E_X = 0.5\text{ MeV}$, we have the restriction $g' \ll 0.18$.

Here, we start by treating the $X(3872)$ as a resonance state $(E_X > 0)$. For comparative purposes, the results for $(E_X < 0)$, which corresponds to the interpretation that the $X(3872)$ is a weakly bound state, will be shown at the end of this section.

The physical solution that corresponds to the resonance (refer to Chapter 5 for more details) is

$$E^\text{pole}_d = E_X - ig'^2 \sqrt{2M_{DD} \cdot E_X}. \quad (6.59)$$

From $E^\text{pole}_d$, the physical width of $X(3872)$ resonance is

$$\Gamma_X = 2g'^2 \sqrt{2M_{DD} \cdot E_X}. \quad (6.60)$$

If we take $g' = 0.05$, which is much smaller than 0.18, the position of the $X(3872)$ resonance in the complex energy plane is

$$E^\text{pole}_d = (0.5 - 0.08\, i) \text{ MeV}, \quad (6.61)$$

and $\Gamma_X = 0.16\text{ MeV}$.

Regarding the background scattering length $a_{bg}$ in Eq. (6.58), our effective theory approach demands that $a_{bg}$ is small. This ensures that no virtual or bound states will be produced besides the resonance. To estimate the size of $a_{bg}$ in our EFT, let us write down the physical scattering length for the system,

$$a = -\frac{g'^2}{E_X} + a_{bg}, \quad (6.62)$$
where according to our power-counting scheme

\[ \frac{g^{'2}}{E_X} \sim a_{bg} \sim O(Q^0). \] (6.63)

Therefore, the relative size of \( a_{bg} \) for the \( X(3872) \) resonance is

\[ a_{bg} = \frac{g^{'2}}{E_X} = 0.005 \text{ MeV}^{-1}, \] (6.64)

where we take \( g' = 0.05 \). Here we assume that the background scattering length is of the same order as the nonperturbative QCD scale; i.e. \( a_{bg} \sim \frac{1}{\Lambda_{\text{QCD}}} \sim 1 \text{ fm}. \)

In this case, the differential rate at energies very close to the \( \bar{D}^0 D^{*0} \)-threshold is given by Eq. (6.50) with \( T_r \) given by Eq. (6.58).

For resonance energy \( E_X = 0.5 \text{ MeV}, \) and \( g' = 0.05, \) the line shapes are plotted in Fig. 6.5 for the values of the background scattering length: \( a_{bg} = 0 \text{ MeV}^{-1} \) (solid), \( a_{bg} = 0.005 \text{ MeV}^{-1} \) (dashed) and \( a_{bg} = -0.005 \text{ MeV}^{-1} \) (dot-dashed).

All peaks are located at the resonance energy \( E_X = 0.5 \text{ MeV}. \) Clearly, details of the line shapes are sensitive to the scattering length \( a_{bg}. \) The interference effects between the Breit-Wigner and background terms and their mixing are noticeable for energies lying above and below the resonance energy.

The scattering matrix given in Eq. (6.58) is a linear function of the scattering length. As shown in Fig. 6.5, the rate from the Breit-Wigner amplitude (shown by the solid line) can be slightly enhanced (decreased) for energies
Figure 6.5: Line shapes of the $X(3872)$ for $E_X = 0.5$ MeV and $q' = 0.05$ are plotted with $B = 1$ MeV. The solid line represents the line shape for $a_{bg} = 0$ MeV$^{-1}$, the dashed line represents the line shape for $a_{bg} = 0.005$ MeV$^{-1}$, and the dot-dashed line represents the line shape for $a_{bg} = -0.005$ MeV$^{-1}$.

lying above (below) the resonance energy when including the effect of background and mixing amplitudes with positive scattering length. For negative scattering length, the Breit-Wigner amplitude is enhanced (decreased) for energies lying below (above) the resonance energy. It should be noted that for energies away from the resonance, the contribution from the last term in Eq. (6.58) is suppressed by the power of $p_\pi$, where $E_d - E_X \sim O(Q^2)$.

To illustrate the difference between the resonance state and the weakly bound state assumptions, we plot the line shape of the $X(3872)$ for $E_X < 0$ in Fig. 6.6. In this case, the positions of the peaks for the values of the background scattering length: $a_{bg} = 0$ MeV$^{-1}$ (solid), $a_{bg} = 0.005$ MeV$^{-1}$ (dashed) and $a_{bg} = -0.005$ MeV$^{-1}$ (dot-dashed) are located at $E_d = E_X = -0.5$ MeV.
As shown in Fig. 6.6, the interference effects between the Breit-Wigner and background terms and their mixing are significant for energies lying above the threshold where the rate from the Breit-Wigner amplitude (shown by the solid line) is enhanced when including the effect of background and mixing amplitudes with positive and negative scattering length.

![Graph](image)

**Figure 6.6:** Line shapes of the $X(3872)$ for $E_X = -0.5$ MeV and $g' = 0.05$ are plotted with $B = 1$ MeV. The solid line represents the line shape for $a_{bg} = 0$ MeV$^{-1}$, the dashed line represents the line shape for $a_{bg} = 0.005$ MeV$^{-1}$, and the dot-dashed line represents the line shape for $a_{bg} = -0.005$ MeV$^{-1}$.

Note that the peak of the weakly-bound state (Fig. 6.6) is much narrower and higher than the peak of the strongly-bound state which is shown in (Fig. 6.4). The weakly-bound state (strongly-bound state) corresponds to a conventional charmonium (hadronic molecule) interpretation of the $X(3872)$. Because of the finite energy resolution, it will be difficult for experimentalists to distinguish between these states.

Although this method has been illustrated by application to the $X(3872)$,
our results are general and can be used to investigate line shapes of other quarkonium states with masses very close to the thresholds of pairs of heavy mesons.
Chapter 7

Conclusions

The aspects of mesons containing a single heavy quark are governed by the spin symmetry $SU(2)_s$ of the heavy quark and the chiral symmetry $SU(3)_L \times SU(3)_R$ of the light quarks. Incorporating both approximate symmetries in a single framework was achieved by defining the heavy hadron chiral perturbation theory (HHChPT). This effective theory has been used in this thesis to study the spectra and interactions of these heavy mesons.

We first studied the masses of the low-lying charm and bottom mesons using HHChPT. We expressed the masses of these heavy mesons up to third order, $O(Q^3)$, in the chiral expansion, where meson loops contribute. In evaluating loop integrals, we have applied Dimensional Regularization consistently by setting $d = 4$ after expanding in powers of $4 - d$ to first order unlike many authors. Furthermore, we have chosen to define meson fields as 4-dimensional to maintain heavy quark symmetry at loop level.

The heavy-hadron chiral Lagrangian has twelve unknown low energy con-
stants (LECs) \( \delta_{H,S}, a_{H,S}, \sigma_{H,S}, \Delta_{H,S}, \Delta_{H,S}^{(a)}, \Delta_{H,S}^{(\sigma)} \) to describe eight measured masses of charm mesons. Hence obtaining unique numerical values of the LECs is impossible. In this thesis, we have used flavor and heavy quark symmetries to construct eight linear combinations \( (\eta_{H,S}, \xi_{H,S}, L_{H,S}, T_{H,S}) \) out of the LECs. By using this method, we reduced the number of unknown LECs to be comparable with the current experimental data on meson masses. Thus, one can express these parameters directly in terms of the physical masses and loop integrals.

In contrast to previous approaches we have used physical meson masses in evaluating the heavy meson loops. As a result, the energy of any unstable particle is placed correctly relative to the decay threshold and the imaginary part of the loop integral can be related to the experimental decay width. However, the resulting values for these combinations of parameters contain contributions beyond the order \( O(Q^3) \) of heavy-hadron chiral Lagrangian. This is due to using empirical masses which generate higher order \( \mu \)-dependent terms that cannot be renormalized using \( \mu \)-dependent counterterms of our Lagrangian. To this end, we have chosen to define the \( \beta \)-functions for these combinations of parameters to estimate the contributions from higher-order terms.

To perform further separations of these linear combinations of parameters, Lattice QCD calculations with different quark masses are needed for the ground and excited state of charm mesons. For example, lattice results on the charm meson spectroscopy undertaken in Refs. [125, 126] can be used to disentangle chirally symmetric parameters \( \delta_{H,S} \) and \( \Delta_{H,S} \) from chiral breaking terms.
Having fitted the linear combinations of the LECs to the $D$-meson spectrum, we rescale the hyperfine combinations to predict the masses of odd- and even-parity bottom mesons. In our calculations, we have used a self-consistent approach to extract the $B$-meson masses, i.e. the values we started with to evaluate the mass splittings within $B$-meson states are the same as the resultant mass splittings. The predicted masses from our theory are in good agreement with experimentally measured masses for the case of the odd-parity sector. For the even-parity sector, the $B$-meson states have not yet been observed; thus, our results provide useful information for experimentalists investigating such states.

The approach developed in this thesis can be extended to predict the spectra of the other doublet of the $P$-wave states, i.e. $S^p_{l=1} = \frac{3}{2}^+$ where $S^p_{l=1}$ is the total angular momentum of the light degrees of freedom, and $l$ is the orbital angular momentum of the system. The spin-parity quantum numbers of these states are $J^p_{S^p_{l=1}} = (1^+, 2^+)$. This requires introducing a new (tensor) field to describe the dynamics of these states in the chiral Lagrangian. The general structure of the relevant chiral Lagrangian with tensor fields is represented in [52, 60] for instance.

The interaction between heavy mesons at very low energy ($p \ll m_\pi$) have been considered in this thesis by using effective theories similar to those used to study nuclear forces. Pions, in this case, are integrated out and the only relevant degrees of freedom are heavy mesons. In this limit the wavelength of the two particles is much greater than the range of interaction. Thus, the interaction can be represented by a local operator that measures the strength of the $s$-wave transition of the particles. In this case, the short-range
physics is integrated out and hidden in these operators. We only considered
the leading-order term for s-wave scattering \( (1S_0) \) which corresponds to a
\( \delta \)-function potential in position space.

We have first looked at a strongly-interacting channel which produces a
bound or virtual state and a dimer state which couples weakly to a weakly-
interacting channel to produce a narrow resonance. Our analysis has been
extended to discuss short-range interactions in two channels. We have looked
at two cases: two channels in which one has a strong s-wave interaction which
produces bound or virtual states, and a dimer state which couples weakly to
weakly-coupled channels which in turn can produce narrow resonances. For
each of these systems, we have used well-defined power-counting schemes.

We have calculated scattering amplitudes and expanded them to leading
order in low energy scales that are relevant to each case. The poles of the ex-
panded scattering amplitudes that either correspond to bound states, virtual
states, or resonances have been identified on Riemann sheets of the complex
energy plane.

The techniques developed for these systems are general, and can be used
to investigate resonances in the charmonium and bottomonium systems, in-
cluding the \( X(3872) \).

The short-range interactions of the \( \bar{D}^0 D^{*0} \) pair is responsible for produc-
ing the \( X(3872) \). The loop diagrams for processes \( \bar{D}^0 D^{*0} \) and \( \bar{D}^0 D^{0}\pi \) have
been considered in this thesis to calculate the width of the \( X(3872) \). Since the
\( X(3872) \) with \( J^{PC} = 1^{++} \) is a bound state of \( \bar{D}^0 D^{*0} \) and \( D^{0}\bar{D}^{*0} \), each loop di-
agrams for these processes has a charge conjugation partner where \( \bar{D}^0 \rightarrow D^0 \)
and \( D^{*0} \rightarrow \bar{D}^{*0} \). We have introduced a new power-counting scheme by ex-
ploiting the experimental coincidence between pion mass and the hyperfine splitting within $D^{*0}$ and $\bar{D}^0$ states. This newly proposed power-counting scheme allows us, in some appropriate limit, to approximate the results of the imaginary part of loop integrals for diagrams with virtual pions to the imaginary part of loop integrals for diagrams without virtual pions. In this limit, $\tau_D \gg \tau_X$ and hence the excited state $D^{*0}$ does not influence the decays of the $X(3872)$.

As a consequence, the power counting simplifies the calculations of loop integrals for diagrams with virtual pions where it shows that the different diagrams give the same contribution.

The full contribution from all diagrams with virtual pions are added to their charge conjugate partners to define the width of the $X(3872)$ which was used to calculate the differential rate formula for the transition $X \rightarrow \bar{D}^0 D^0 \pi$.

We then used these results to study the line shapes of the $X(3872)$ for a single channel problem with either virtual/bound state or narrow resonance/weakly bound state. For both cases, the line shapes of the $X(3872)$ are plotted.

For the interpretation that $X(3872)$ is a bound state (positive scattering length), the line shape has a peak that is located at the binding energy of the $X(3872)$, whereas the relevant line shape for the virtual state explanation (negative scattering length) just shows an enhancement above the $\bar{D}^0 D^{*0}$-threshold.

For the case of resonance explanation (positive resonance energy), we have investigated the effects of repulsive (attractive) background scattering induced by positive (negative) scattering length on the Breit-Wigner am-
plitude. We have found that the rate from the Breit-Wigner amplitude is enhanced (decreased) for energies lying above (below) the resonance energy when including the effect of background and mixing amplitudes with positive scattering length. However, for negative scattering length, the Breit-Wigner amplitude is enhanced (decreased) for energies lying below (above) the resonance energy. For comparative purposes, we plot the line shapes of the $X(3872)$ for negative resonance energy, which corresponds to a weakly-bound state interpretation. In this case, we plot the line shapes for positive and negative background scattering length. Here, sharp peaks are centered at the resonance energy below the threshold. For energies lying above the threshold, line shapes show enhancements but no peaks.

Although this method has been illustrated by application to the $X(3872)$, our results are general and can be used to investigate line shapes of other quarkonium states with masses very close to the thresholds of pairs of heavy mesons.
Bibliography


