CLASSICALLY SPINNING AND ISOSPINNING NON-LINEAR $\sigma$-MODEL SOLITONS

A THESIS
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MAREIKE KATHARINA HABERICHTER

SCHOOL OF PHYSICS AND ASTRONOMY
JODRELL BANK CENTRE FOR ASTROPHYSICS
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ABSTRACT

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Candidate Name: Mareike Katharina Haberichter

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We investigate classically (iso)spinning topological soliton solutions in (2+1)- and (3+1)-dimensional models; more explicitly isospinning lump solutions in (2+1) dimensions, Skyrme solitons in (2+1) and (3+1) dimensions and Hopf soliton solutions in (3+1) dimensions. For example, such soliton types can be used to describe quasiparticle excitations in ferromagnetic quantum Hall systems, can model spin and isospin states of nuclei and may be candidates to model glueball configurations in QCD.

Unlike previous work, we do not impose any spatial symmetries on the isospinning soliton configurations and we explicitly allow the isospinning solitons to deform and break the symmetries of the static configurations. It turns out that soliton deformations clearly cannot be ignored. Depending on the topological model under investigation they can give rise to new types of instabilities, can result in new solution types which are unstable for vanishing isospin, can rearrange the spectrum of minimal energy solutions and can allow for transitions between different minimal-energy solutions in a given topological sector.

Evidently, our numerical results on classically isospinning, arbitrarily deforming solitons are relevant for the quantization of classical soliton solutions.
DECLARATION

The work in this thesis is based on research carried out at the Jodrell Bank Centre for Astrophysics, University of Manchester, England. No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.
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The author obtained her diploma in Theoretical Physics from the University of Karlsruhe in 2009. Her diploma research project was carried out at the Institute for Theoretical Physics working under the supervision of Prof. Klinkhamer on $SU(3)$ Sphaleron solutions. In November 2009, she continued her research on topological solutions in classical field theories starting a PhD at the University of Manchester under the supervision of Prof. Battye. Her PhD research focused on the study of classically spinning and isospinning Skyrme-type soliton solutions. She joined the School of Mathematics at the University of Kent as a Research Associate in September 2013. Currently she is working under the supervision of Dr. Krusch on a project entitled Skyrmion-Skyrmion scattering and nuclear physics.
The work presented in this thesis is based on the following published articles and manuscripts under preparation:


5.) M. Haberichter and Y. Shnir, *Isospinning baby Skyrmion Solutions for a Class of Potentials* (in prep.).

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1.1 Introduction


Recall that we can distinguish two different types of soliton solutions (both of them will appear in this thesis):

(a) Topological Solitons:
These solitons can be characterized by some topologically nontrivial mapping from the space manifold into some target manifold. Topological soliton configurations can be classified according to the homotopy groups of spheres (Compare the classification scheme given in Table 1.1.) Field configurations can be labelled by a conserved, integer-valued topological charge, usually given by the winding number of the mapping.

(b) Nontopological Solitons:
Their stability is due to the existence of a conserved Noether charge [84].

This introductory Chapter is structured as follows. In Section 1.2 we briefly review some of the basic properties of topological soliton solutions. In particular, we use the static, particle-like solutions in the one-dimensional sine-Gordon model to point out some of the crucial features of topological soliton solutions. Section 1.3 introduces the $O(3)$ sigma model and its lump solutions. In Section 1.4 we recall Derrick’s scaling argument and we give in Section 1.5 a brief introductory overview of the different scalar field theories in which we will construct (iso)spinning soliton solutions in the following thesis chapters. Section 1.5 describes how the soliton solutions in our models acquire spin and isospin. We end this chapter by giving a brief thesis outline in Section 1.7.
Table 1.1: Classification table of topological soliton solutions in $D = 1, 2, 3$ spatial dimensions. We list examples of classical solutions in scalar and gauge field theories together with their associated homotopy groups and possible areas of applications.

<table>
<thead>
<tr>
<th>Model</th>
<th>$D$</th>
<th>Homotopy group</th>
<th>name</th>
<th>applications</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sine-Gordon (SG)</td>
<td>1</td>
<td>$\pi_1(S^1) = \mathbb{Z}$</td>
<td>SG kink</td>
<td>toy model</td>
</tr>
<tr>
<td>$U(1)$ YMH</td>
<td>2</td>
<td>$\pi_1(S^1) = \mathbb{Z}$</td>
<td>vortex</td>
<td>superconductivity</td>
</tr>
<tr>
<td>$O(3)$</td>
<td>2</td>
<td>$\pi_2(S^2) = \mathbb{Z}$</td>
<td>$O(3)$ lump</td>
<td>ferromagnets</td>
</tr>
<tr>
<td>$O(3) +$ Skyrme + pot.</td>
<td>2</td>
<td>$\pi_2(S^2) = \mathbb{Z}$</td>
<td>baby Skyrmion</td>
<td>condensed matter</td>
</tr>
<tr>
<td>$O(4) +$ Skyrme</td>
<td>3</td>
<td>$\pi_3(S^3) = \mathbb{Z}$</td>
<td>Skyrmion</td>
<td>nuclear physics</td>
</tr>
<tr>
<td>$SU(2)$ YMH</td>
<td>3</td>
<td>$\pi_2(SU(2)/U(1)) = \mathbb{Z}$</td>
<td>monopoles</td>
<td>cosmology</td>
</tr>
<tr>
<td>$O(3) +$ Skyrme</td>
<td>3</td>
<td>$\pi_3(S^2) = \mathbb{Z}$</td>
<td>Hopf soliton</td>
<td>condensed matter, QCD</td>
</tr>
</tbody>
</table>

1.2 Sine Gordon Model as an Introductory Example

Before moving on to more complicated higher dimensional theories, we will briefly illustrate in this section the basic properties of topological soliton solutions by discussing the one-dimensional sine-Gordon model [130] and its particle-like solutions. Originally, the sine-Gordon model was studied by Skyrme [130] as a simplified version of his effective meson field theory [127, 128, 129]. It can be interpreted as a lower-dimensional version of the sigma and modified sigma models which are introduced in the following sections.

The sine-Gordon model is described by the $(1+1)$-dimensional Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \cdot \partial^{\mu} \phi - (1 - \phi^2), \quad (1.2.1)$$

where $\phi = (\phi_1, \phi_2) = (\sin \alpha, \cos \alpha)$ denotes a two-component real vector field of unit length $\phi \cdot \phi = 1$ and can be parametrized in terms of an angular field variable $\alpha(x, t)$. At any fixed time $t$, the scalar field $\phi$ can be identified with a mapping from the real $\mathbb{R}^1$ axis into the internal, unit circle $S^1$, so that $\phi(x) : \mathbb{R}^1 \mapsto S^1$.

A field configuration of finite total energy

$$E_{SG} = \int_{-\infty}^{\infty} \left\{ \frac{1}{2} (\partial_\phi)^2 + (1 - \phi^2) \right\} \, dx, \quad (1.2.2)$$

requires that $\phi$ goes to a constant vacuum value at spatial infinity, that is $\phi \to (0, 1)$ as $|x| \to \infty$. Alternatively, the boundary condition can be expressed in terms of the angular variable $\alpha(x)$ as $\alpha(x) \to q_+ 2\pi$ for $x \to +\infty$ and $\alpha(x) \to q_- 2\pi$ for $x \to -\infty$ with $q_+$ and $q_-$ being integers. This choice of boundary conditions naturally compactifies the domain $\mathbb{R}^1$ to the unit circle $S^1$, as the points at spatial infinity $x = \pm\infty$ take the same value on the target 1-sphere. As a result of this any static finite-energy configuration $\phi(x) : S^1 \mapsto S^1$ can be topologically classified by the number of times the image of the real axis winds
around the target $S^1$. This is given by $Q = q_+ - q_-$. The conserved winding number or the topological index $Q \in \pi_1(S^1) = \mathbb{Z}$ labels the different homotopy classes in which the soliton solutions fall according to the boundary conditions chosen at spatial infinity. For example, if the angular field variable $\alpha(x)$ is required to vanish at both boundaries $q_+ = q_- = 0$, then the image of the real axis on the target 1-sphere is a closed loop which can be continuously deformed to a point. The $Q = 0$ sine-Gordon solution associated with this topologically-trivial mapping is known as a breather and is a time-dependent, travelling wave solution. However, if $\alpha(x)$ is chosen so that it interpolates between two different vacuum states with $\alpha(-\infty) = 0$ and $\alpha(+\infty) = 2\pi$, then the mapping of the $x$ axis winds once around the target circle $S^1$. Therefore it cannot be contracted by a continuous deformation to a point.

In our calculations we take the angular field $\alpha(x)$ to be periodic in $x$ with period $L$ (modulo a $2\pi$ shift), that is $\alpha(x + L)$ is constructed being the same as $\alpha(x) + 2\pi n$ for any integer $n$. In other words, we solve the sine-Gordon equation derived from (1.2.1) with the spatial domain $\mathbb{R}$ replaced by a circle $S^1$ of finite radius. We display in the upper row of Fig. 1.1 the corresponding, numerically calculated angular field function $\alpha(x)$ which solves in the topological $Q = 1$ sector the Euler-Lagrange equation following from (1.2.1). Here, we choose the period $L$ to be 20 and $x \in [-L/2, +L/2]$ with boundary conditions $\alpha(-L/2) = 0$ and $\alpha(+L/2) = 2\pi$. This sine-Gordon one-soliton is called a $2\pi$-kink solution and represents a static, localised finite-energy solution in the scalar field model (1.2.1). Its energy density is given in the upper row of Fig. 1.1 (middle). As illustrated graphically in Fig. 1.1, the associated mapping is parametrized by the angle variable $\alpha$ and covers the whole target circle once.

In the charge-2 sector we calculate in Fig. 1.1 (lower row) a static $4\pi$-kink solution with boundary conditions $\alpha(-\infty) = 0$ and $\alpha(+\infty) = 4\pi$. Such a static 2-soliton solutions can be constructed if the sine-Gordon theory (1.2.1) is defined on a compact and periodic domain (see above). Here the period $L$ of our numerical solution is set to 10 and we restrict $x$ to the interval $[-L, +L]$ with boundary conditions $\alpha(-L) = 0$ and $\alpha(+L) = 4\pi$. With this choice of boundary conditions the image of the $\mathbb{R}$ axis is wound twice around the circle $S^1$ (see illustration in Fig. 1.1). Consequently, the associated mapping cannot be transformed continuously neither into a breather solution nor a $2\pi$-kink solution. Therefore, the interval contains exactly two 1-kink solutions which can be thought of being placed opposite to each other on a circle. The energy density plot in Fig. 1.1 of the sine-Gordon 2-kink solution shows that the total number of solitons – the “number of particles” [127, 128, 129] – is given by the topological charge $Q$.

We can define a conserved topological current

$$J^\mu = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\nu \partial_\nu} \partial_\mu \alpha \, dx,$$

(1.2.3)
so that the topological index $Q$ is given by the corresponding charge

$$Q = \int_{-\infty}^{\infty} J^0 \, dx = \frac{1}{2\pi} [\alpha(\infty) - \alpha(-\infty)] , \quad (1.2.4)$$

which can take only integer values in $\mathbb{Z}$.

In each topological sector $Q$ the energy $E_{SG}$ of a static sine-Gordon kink solution is topologically bound from below by the Bogomolny bound

$$E_{SG} \geq 8|Q| . \quad (1.2.5)$$

The numerically calculated energy values given in Fig. 1.1 confirm that our field configurations saturate (1.2.5). The bound (1.2.5) follows by expanding and rearranging (compare [100]) the inequality

$$\left( \frac{1}{\sqrt{2}} (\partial_x \phi)^2 \pm \sqrt{1 - \phi^2} \right)^2 \geq 0 , \quad (1.2.6)$$

and then identifying the expressions for the energy (1.2.2) and topological charge (1.2.4).
1.3 \(O(3)\) SIGMA MODEL

By integrating the first order Bogomolny equations derived from (1.2.5)

\[
\frac{d\alpha}{dx} = \pm 2 \sin \frac{\alpha}{2},
\]

we can easily obtain an explicit 1-kink solution

\[
\alpha(x) = 4 \tan^{-1} e^{x-x_0}.
\]

Here, the integration constant \(x_0\) denotes the position of the kink. The energy (1.2.2) associated with the 1-soliton solution (1.2.8) is found to be \(E_{SG} = \int_{-\infty}^{+\infty} 4 \text{sech}^2(x-\alpha_0) \, dx = 8\).

Note that in contrast to the soliton solutions discussed in this thesis, the sine-Gordon system (1.2.1) in one space dimension is an integrable system [1]. In particular, time-dependent multi-kink solutions of the sine-Gordon field equations can be generated analytically with inverse scattering methods and Bäcklund transformations. Unlike the static soliton solutions obtained in this subsection when taking the real sine-Gordon field \(\alpha(x)\) to be periodic in the spatial coordinate \(x\), the sine-Gordon model does not allow for static, stable multi-kink solutions when studied on the real line \(\mathbb{R}\) [107].

1.3 \(O(3)\) Sigma Model

\(O(3)\) sigma models are relevant for a huge class of phenomena both in solid state physics and particle physics. They often appear as classical limits or lower dimensional analogues of more complicated, nonlinear field theories. For example, Heisenberg’s classical model of antiferromagnets can be reduced in the limit of large spins and small distances between the spin sites to the \(O(3)\) sigma model. Furthermore, sigma models can be regarded as two-dimensional analogues of pure Yang-Mills theories defined in four-dimensional Euclidean spacetime. They play a major role in high-temperature superconductivity [44] and are relevant for the fractional quantum Hall effect [117]. Finally, they admit static, topologically-nontrivial finite-energy configurations which are known as lumps. Lumps are simple examples of classical solutions which can be studied with very limited numerical effort and are more analytically tractable than soliton solutions in more complex models. Finally, all the models discussed in this thesis are extensions of sigma models, so the sigma model can be used in some cases as a simplified toy model to guide investigations in the more complex models.

The Lagrangian density of the simplest, Lorentz-invariant \(O(3)\) sigma model in (2+1) dimensional Minkowski spacetime is given by

\[
\mathcal{L} = \frac{1}{2} \left( \partial_{\mu} \phi \cdot \partial^{\mu} \phi \right),
\]

where \(\phi = (\phi_1, \phi_2, \phi_3)\) is a three-component, real vector field subject to the unit vector constraint \(\phi \cdot \phi = 1\). In other words, \(\phi(x, t)\) is restricted to lie on a two-dimensional target
sphere $S^2_{iso}$ of unit radius, so that $\phi : \mathbb{R}^{2,1} \mapsto S^2_{iso}$. Note that by convention the dot product in (1.3.1) denotes the use of the Euclidean metric of $\mathbb{R}^3$. The “$O(3)$” in the name of the model refers to the invariance of (1.3.1) under global $O(3)$ rotations.

A convenient, alternative formulation [116] of the model (1.3.1) can be obtained by projecting the target 2-sphere $S^2_{iso}$ via a stereographic projection (see Fig. 1.2) onto the two-dimensional, spatial complex plane. We parametrize the domain, the $\mathbb{R}^2$ plane, by the complex coordinate $z = x + iy$ and denote a point on the target unit sphere $S^2$ using its complex Riemann sphere coordinate $R = \phi_1 + i\phi_2 / (1 + \phi_3)$. Alternatively, the complex Riemann sphere coordinate $R$ can be expressed in terms of the usual spherical polar coordinates $\theta$ and $\varphi$: $R = \tan (\theta/2) \exp (i\varphi)$ with $R = 0, \infty$ representing the north and south pole, respectively.

In this complex representation the Lagrangian density (1.3.1) becomes

$$L = \frac{\partial \mu R \partial \nu R}{(1 + |R|^2)^2},$$

which is known as the $\mathbb{C}P^1$ sigma model. The formulation (1.3.3) in terms of the single complex function $R(z)$ turns out to be very useful [116, 154] when constructing classical lump solutions in the $O(3)$ sigma model. In particular, analytical lump solutions [100, 119] of topological charge $N \in \mathbb{Z}$ can be obtained by taking $R$ to be a rational function of
where $p$ and $q$ are polynomials in $z$ and have no common factors. The topological degree $\max\{\deg(p), \deg(q)\}$ of the rational map (1.3.4) is identified with the topological charge $N$ of the lump solution. Recall that the rational map approach has proven to be a very powerful tool not only in the analysis of pure sigma model solitons but as well in the study of Skyrmions [67] and monopoles [43].

### 1.3.1 Belavin-Polyakov Lump

Belavin-Polyakov lumps [116] emerge as static, topologically-nontrivial finite-energy solutions in the $(2+1)$-dimensional, non-linear $O(3)$ sigma model (1.3.1). The energy $M_N$ of a static $O(3)$ lump solution is given by

$$M_N = \frac{1}{2} \int \partial_i \phi \cdot \partial_i \phi \, d^2 x,$$

(1.3.5)

where the unit vector $\phi = (\phi_1, \phi_2, \phi_3)$ must tend towards the same constant unit vector $\phi^{(0)}$ as $|x| \to \infty$. Consequently, the physical coordinate plane $\mathbb{R}^2$ is topologically equivalent to another 2-sphere $S^2_{\text{phys}}$, so that any finite-energy solution $\phi(x)$ is given by a mapping $\phi : S^2_{\text{phys}} \mapsto S^2_{\text{iso}}$ at a fixed time $t$. Hence, each lump configuration can be classified by its winding number $N = \pi_2(S^2) = \mathbb{Z}$ which counts the number of times the lump field $\phi$ winds around $S^2_{\text{iso}}$. The topological charge $N$ can be calculated via

$$N = \frac{1}{8\pi} \int \epsilon_{ij} \phi \cdot (\partial_i \phi \times \partial_j \phi) \, d^2 x,$$

(1.3.6)

and is often interpreted as the number of well separated charge-1 lump solutions which form the field configuration $\phi(x)$.

![Figure 1.3: The energy density $M_1$ (1.3.5) of an one-lump solution (1.3.11) as a function of the parameter $\lambda$.](image)
Rearranging the inequality [116]

\[
\left( \partial_i \phi \pm \epsilon_{ij} \phi \times \partial_j \phi \right)^2 \geq 0,
\]

and substituting the formulas for the energy \( M_N \) (1.3.5) and topological charge \( N \) (1.3.6), we can rederive a lower bound on the energy in each homotopy class

\[
M_N \geq 4\pi N.
\]

(1.3.8)

The equality in the Bogomolny equation (1.3.8) is satisfied if and only if the field configuration \( \phi(x) \) solves the first order Bogomolny equations

\[
\partial_i \phi \pm \epsilon_{ij} \phi \times \partial_j \phi = 0.
\]

(1.3.9)

When expressed in terms of the \( \mathbb{C}P^1 \) sigma model notation, (1.3.9) simplifies for the upper sign to the Cauchy-Riemann condition

\[
\partial_z R(z) = 0,
\]

(1.3.10)

where \( \partial_z \equiv \frac{1}{2} (\partial_1 + i \partial_2) \) and \( R(z) \) is an holomorphic, rational function of the general form (1.3.4). The simplest, topologically-nontrivial, finite energy solution of charge \( N = 1 \) takes the “hedgehog form”

\[
R(z) = \frac{\lambda}{z},
\]

(1.3.11)

where \( \lambda \in \mathbb{R} \) controls the size of the lump solution: Increasing \( \lambda \) results in spikier, bell-shaped soliton solutions. Here we choose the boundary condition \( \phi(\infty) \to \phi(0) = (0, 0, 1) \), i.e. the rational function (1.3.11) has to satisfy the spatial boundary condition \( R(\infty) = 0 \).

For winding number \( N = 2 \) a rational map solution [100] of (1.3.10) is given by

\[
R(z) = \frac{1}{z^2 + \epsilon},
\]

(1.3.12)

where \( \epsilon \) is a real parameter. The boundary condition at spatial infinity \( \phi(\infty) = \phi(0) = (0, 0, 1) \) forces the rational function \( R(z) \) (1.3.12) to vanish as \( |z| \to \infty \).

Changing \( \epsilon \) continuously from \(-1\) to \(+1\), describes the right-angle scattering of two initially well-separated, single lumps through the \( N = 2 \) ring-like lump solution. Recall that this type of right-angle scattering in head-on collisions has also been observed in more complicated soliton systems, e.g. for vortex solutions [123], monopoles [13] and Skyrme solutions [95].

The lump solutions (1.3.11) and (1.3.12) are topologically stable since their energy \( M_N \) is bounded from below by their topological charge \( N \) due to the existence of the Bogomolnyi bound (1.3.8). However, lump solutions in the sigma model are not dynam-
Derrick’s Scaling Argument

In this section we briefly recall Hobart-Derrick’s no-go theorem for soliton solutions in higher dimensional scalar field theories [42, 65, 121].

Derrick’s Theorem. Let

1. \( \phi \) be a set of \( N \) real scalar fields \( \phi_a \) with field index \( a = 1, \ldots, N \) and defined in \((D + 1)\)-dimensional spacetime.

2. the Lagrangian \( \mathcal{L} \) be of the relativistic, general standard form

\[
\mathcal{L} = \sum_{a=1}^{N} \left\{ \frac{1}{2} \left[ (\partial_0 \phi_a)^2 - (\nabla_D \phi_a)^2 \right] - V(\phi_a) \right\},
\]

where the potential term \( V(\phi_a) \) is assumed to be a non-negative function.

3. the set of scalar fields \( \bar{\phi}_a(\mathbf{x}) \) be a static, time-independent classical solution of finite energy with \( V(\bar{\phi}_a) \geq 0 \) and \( \min V(\bar{\phi}_a) = 0 \).

Then, the only non-singular, time-independent finite energy solutions in \( D \geq 2 \) space dimensions are the ground states \( \bar{\phi}_a = \text{const.} \) with \( V(\bar{\phi}_a) = 0 \).
Proof. Consider the total energy functional of a static, finite-energy solution \( \bar{\phi}_a \) of the classical field equations

\[
E = E_2 + E_0 \quad \text{with} \quad E_2 = \frac{1}{2} \sum_{a=1}^{N} (\nabla D \bar{\phi}_a)^2 \, d^D x \quad \text{and} \quad E_0 = \int V(\bar{\phi}_a) \, d^D x. \tag{1.4.2}
\]

If we deform the soliton solution \( \bar{\phi}_a \) by rescaling the space coordinate \( x \mapsto \lambda x \), the classical energy functional (1.4.2) scales as

\[
E(\lambda) = \lambda^{2-D} E_2 + \lambda^{-D} E_0.
\]

Since \( \bar{\phi}_a \) was assumed to be a solution, \( E(\lambda) \) must be stationary with respect to variations in \( \lambda \):

\[
\frac{\partial E(\lambda)}{\partial \lambda} \bigg|_{\lambda=1} = (2-D)E_2 - DE_0 = 0, \tag{1.4.3}
\]

where \( E_2 \) and \( E_0 \) are non-negative. Further, positive definiteness of the second variation of the energy functional gives

\[
\frac{\partial^2 E(\lambda)}{\partial \lambda^2} \bigg|_{\lambda=1} = (2-D)(1-D)E_2 + D(D+1)E_0 > 0, \tag{1.4.4}
\]

Thus, the Hobart-Derrick necessity conditions (1.4.3) and (1.4.4) impose the following constraints on the existence of time-independent, finite-energy solutions in pure scalar field theories:

\( D = 1 \): (1.4.3) is equivalent to the virial theorem and static soliton solutions can exist.

\( D = 2 \): Condition (1.4.3) implies that \( E_0 = 0 \). If we assume that \( V(\{\phi_a\}) \) has only a discrete set of minima, then \( \phi_a \) must be equivalent to the same minimum for all \( x \). Hence, \( \bar{\phi}_a \) is equal to the trivial, space-independent solution \( \bar{\phi}_a = \text{const.} \), such that \( V(\{\bar{\phi}_a\}) = 0 \) for all \( x \). Note that there exists a loophole: Finite-energy solutions in models with a potential function \( V(\{\phi_a\}) \) that has a continuous set of minima, e.g. scalar field theories with the potential term in (1.4.1) missing, are not ruled out by (1.4.3). An example are the sigma model lump solutions described in the previous section. However sigma model lumps are dynamically unstable: if perturbed, they shrink or expand indefinitely. In other words, they do not fulfill the stability condition (1.4.4). Summarized, for Lagrangians of the form (1.4.1) in two spatial dimensions the only static, stable finite-energy solutions are given by the classical vacuum solution.

\( D > 2 \): Condition (1.4.3) gives \( E_2 = E_0 = 0 \). This can only be satisfied if \( \partial_i \bar{\phi}_a = 0 \) and \( \bar{\phi}_a \) corresponds to an absolute minima of the potential \( V(\phi_a) \), i.e. the classical vacuum is the only static, stable solution.

\( \square \)

Recall that there are three main methods to evade Derrick’s theorem on the absence of non-trivial, static higher-dimensional soliton solutions in models only involving scalar
1.4. DERRICK’S SCALING ARGUMENT

1.) Look for *time-dependent*, multi-dimensional soliton solutions:

Note that Derrick’s theorem only excludes the existence of time-independent soliton solutions; it does not rule out time-dependent solutions. The $Q$-balls [40, 84] and $Q$-lumps [85, 148] which will be discussed further in the following chapter are examples of classical solutions which are not covered by Derrick’s theorem due to their time-dependent, internal phase. $Q$-balls arise in the $U(1)$ Goldstone model, while $Q$-lumps are massive $O(3)$ lumps stabilized by a steady rotation in internal space. In both cases the stability of the solution is guaranteed by the conservation of a nontopological charge $Q$ coming from a global $U(1)$ Noether symmetry.

2.) Include either *higher powers* of derivatives of $\phi_a$ or *higher order* derivatives in scalar theories with $D \geq 2$:

Derrick’s theorem can be circumvented by the addition of a potential term and a fourth-order term of the general form $\int d^Dx \sum_{a=1}^N (\partial^i \phi_a)^4$ to the nonlinear sigma model (1.3.1). This modification leads in $D = 2$ spatial dimensions to the appearance of baby Skyrmions (see Chapter 3) and in $D = 3$ to the existence of Skyrmions (see Chapter 5) and Hopf solitons (see Chapter 4). Applying Derrick’s scale argument to these Skyrme-type models modifies (1.4.3) to

\[(4 - D)E_4 + (2 - D)E_2 - DE_0 = 0, \quad (1.4.5)\]

where $E_4$ is the energy contribution from the fourth-order term, $E_2$ the one from the second order term and $E_0$ denotes the energy contribution due to the potential term. With a fourth-order term included (1.4.4) takes the form

\[(4 - D)(3 - D)E_4 + (2 - D)(1 - D)E_2 + D(D + 1)E_0 > 0. \quad (1.4.6)\]

Solving (1.4.5) for $E_4$ and substituting into (1.4.6) gives the stability condition

\[2(D - 2)E_2 + 4DE_0 > 0, \quad (1.4.7)\]

which is satisfied for $D = 2, 3$. Analogously, one can check that Skyrme models with terms of even higher order in the field derivatives [89, 102] included allow for classical solutions which are stable under scale transformations.

3.) Include *gauge fields* (abelian or non-abelian) in scalar field theories with $D \geq 2$:

Simple examples of higher-dimensional solitons in gauge theories are the vortex solutions [2, 106] in $(2 + 1)$-dimensional $U(1)$ Yang-Mills-Higgs theory and the magnetic monopoles [139, 115] in $(3 + 1)$-dimensional $SU(2)$ Yang-Mills-Higgs theories. The energy functional of a gauge theory with scalar fields $\phi$, vector field $A$, covariant derivative $D_i$ and field strength tensor $F_{ij}$ takes in $D$ space dimensions
the general form
\[
E[A, \phi] = \int \left\{ -\frac{1}{2} \text{Tr} F_{ij} F_{ij} + \frac{1}{2} (D_i \phi)^\dagger (D_i \phi) + V(\phi) \right\} \, d^D x
= E_4 + E_2 + E_0 ,
\]
where all energy contributions are assumed to be positive and \( V(\phi) \) is taken to be a non-negative function. Performing a scale transformation on (1.4.8)
\[
\phi(x) \mapsto \phi(\lambda x) , \quad A(x) \mapsto \lambda A(\lambda x) ,
\]
we obtain the stability condition (1.4.5). Consequently, non-trivial classical solutions are allowed for \( D = 2, 3 \).

In this thesis we will consider soliton solutions whose stability is guaranteed by 1.) and 2.).

1.5 Models

In this section, we give a brief, introductory overview of the different scalar field theories in which we will construct (iso)spinning soliton solutions in the following chapters. As pointed out in Section 1.1, a soliton configuration \( \phi(x) \) can be classified topologically by its homotopy group structure. In each of the models considered in this thesis the finite-energy configuration \( \phi(x) \) defines a mapping from \( S^D \) – the sphere at spatial infinity in \( \mathbb{R}^{D+1} \) – into the vacuum manifold \( S^{N-1} \). All the discussed models possess an \( SO(N) \) internal symmetry which is broken by the inclusion of a potential term to \( SO(N - 1) \). Hence, each field configuration represents a mapping between Riemannian spheres and can be labelled by the homotopy classes of the homotopy group \( \pi_D(S^{N-1}) \). Fig. 1.5 shows the topological classification of all the soliton solutions that shall be studied in more detail in this work. Concretely, we investigate (iso)spinning soliton solutions in the following \((2 + 1)\)- and \((3 + 1)\)-dimensional scalar field theories:
1.5 MODELS

1.5.1 Massive $O(3)$ Lumps

When modified by a potential function $V(\phi_3)$, the $O(3)$ sigma model (1.3.1) supports spinning and isospinning soliton solutions, so-called $Q$-lump solutions [85, 148]. In Chapter 2 we perform numerical simulations on isospinning, stable finite-energy solutions in $(2 + 1)$-dimensional massive $O(3)$ sigma models defined by the Lagrangian density

$$
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi \cdot \partial^\mu \phi \right) - V(\phi_3),
$$

where $\phi = (\phi_1, \phi_2, \phi_3)$ is a three-component unit vector and we take various $SO(2)$-symmetric functions as our potential. The model (1.5.1) has the same homotopy group structure as the pure $O(3)$ sigma model (1.3.1), so that the soliton solutions carry a conserved topological charge $N \in \pi_2(S^2) = \mathbb{Z}$, which is given by the integral (1.3.6). Non-spinning, time-independent finite-energy solutions in model (1.5.1) with non-vanishing potential term $V$ are ruled out by Derrick’s Theorem (1.4.3). However, isospinning, time-dependent soliton solutions exist. They are dynamically stabilised by the existence of a non-topological, Noether charge $Q$ due to their steady rotation in isospace. The allowed frequency range and the soliton’s shape depend crucially on the concrete choice of the potential term $V$ in (1.5.1). We display as an example in Fig. 1.6 the energy density iso-

![Figure 1.6: The energy densities of topological charge 1 (left) and 2 (right) Q-lump solutions isospinning at a constant angular frequency $\omega \approx 0.77$.](image)

surfaces for $N = 1$ and $N = 2$ $Q$-lump solutions for the potential $V(\phi_3) = \frac{1}{4} \mu^2 \left( 1 - \phi_3^4 \right)$ [148] with mass parameter $\mu = 1$ and for fixed angular frequency $\omega$ and fixed isospin $Q$.

1.5.2 Baby Skyrmions

Another way to break the conformal invariance of the $O(3)$ lump solutions which were introduced in Sec. 1.3.1 is to modify the $O(3)$ sigma model (1.3.1) by adding an extra term quartic in its first field derivatives – the Skyrme term – and a mass term $V$. This $(2 + 1)$-dimensional version of the full $(3 + 1)$-dimensional Skyrme model [129] is well-known
as the baby Skyrme model [111, 114] and its Lagrangian density is given by

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{4} (\partial_\mu \phi \times \partial_\nu \phi)^2 - V(\phi), \]  

(1.5.2)

where \( \phi = (\phi_1, \phi_2, \phi_3) \) is a unit 3-vector with \( \phi \cdot \phi = 1 \) and the potential term \( V(\phi) \) is chosen so that the global \( O(3) \) symmetry of the model is broken to \( O(2) \). Field configurations \( \phi(x) \) with finite potential energy require that \( \phi \) approaches a constant value at spatial infinity and as a consequence the fields can be regarded at a given time \( t \) as maps from compactified physical space \( S^2_{\text{phys}} \) to \( S^2_{\text{Iso}} \). Topological soliton solutions in the model (1.5.2), so called baby Skyrmions, exist due to \( \pi_2(S^2) \) being non-trivial. They can be labelled by an integer-valued, topological charge \( B \in \pi_2(S^2) \). The topological charge \( B \) is given by the integral (1.3.6) and is called baryon number in analogy to the Skyrme model in three spatial dimensions.

If one restricts to static fields, \( \phi(x) \), then the energy functional of a baby Skyrme soliton with baryon number \( B \) derived from the Lagrangian (1.5.2) is:

\[ M_B = \int \frac{1}{2} \partial_i \phi \cdot \partial_i \phi + \frac{1}{4} \left[ (\partial_i \phi \cdot \partial_i \phi)^2 - \left( \partial_i \phi \cdot \partial_j \phi \right) \left( \partial_i \phi \cdot \partial_j \phi \right) \right] + V(\phi) \, d^2 x, \]  

(1.5.3)

and can be shown [41, 6, 133] to be bound from below in each topological sector \( B \) by the Bogomoln\'y bound

\[ M_B \geq 4\pi B (1 + C), \]  

(1.5.4)

where \( C \) is a constant which depends on the potential term \( V(\phi) \). The energy bound (1.5.4) can only be saturated by \( O(3) \) lumps (see Section 1.3.1), that is when the Skyrme and potential terms are absent. The lower bound (1.5.4) can be verified by using the inequality (1.3.7) for the \( O(3) \) sigma model contribution in (1.5.3) together with the following inequality for the Skyrme and potential terms [111]

\[ \left( \frac{1}{2} \epsilon_{ij} \cdot \left( \partial_i \phi \times \partial_j \phi \right) + \sqrt{2V(\phi)} \right)^2 \geq 0, \]  

(1.5.5)

Rearranging (1.3.7) implies

\[ \partial_i \phi \cdot \partial_i \phi \geq \epsilon_{ij} \phi \cdot \left( \partial_i \phi \times \partial_j \phi \right), \]  

(1.5.6)

and (1.5.5) gives the following inequality:

\[ \frac{1}{4} \left[ (\partial_i \phi \cdot \partial_i \phi)^2 - \left( \partial_i \phi \cdot \partial_j \phi \right) \left( \partial_i \phi \cdot \partial_j \phi \right) \right] + V(\phi) \geq \sqrt{\frac{V(\phi)}{2}} \epsilon_{ij} \phi \cdot \left( \partial_i \phi \times \partial_j \phi \right). \]  

(1.5.7)

Identifying the right-hand-sides of (1.5.6) and (1.5.7) with the topological charge \( B \) (1.3.6)
and setting \( C = \sqrt{2V(\phi)} \) results in (1.5.4). Note that for the conventional potential term \( V(\phi) = \mu^2 (1 - \phi^3) \) with mass parameter \( \mu \) the constant \( C \) has been calculated [41, 6, 133] to be given by \( C = 4/3\mu \).

Recall that baby Skyrmions are not excluded by Derrick’s scaling argument since they satisfy the stability condition (1.4.7) in \( D = 2 \) spatial dimensions. Note that the stability condition (1.4.7) for \( D = 2 \) requires the inclusion of a positive, non-zero mass term \( V(\phi) \) in (1.5.2). Baby Skyrmions are of a preferred size at which the positive energy contributions from the quartic Skyrme term and from the mass term are equal (see Eq. (1.4.5) in two spatial dimensions). Different to their analogues in three spatial dimensions – known as Skyrmions – the mass term is essential in the two-dimensional model (1.5.2) and its choice is largely arbitrary and can result in very different qualitative behaviour (see Chapter 3). In Chapter 3 we will consider minimal energy configurations for a wide range of potentials \( V(\phi) \). However, our simulations on isospinning baby Skyrme configurations will only be performed with mass terms that depend just on \( \phi_3 \), so that the \( O(2) \) symmetry between \( \phi_1 \) and \( \phi_2 \) is maintained.

We display as an example in Fig. 1.7 the energy distributions of baby Skyrmion solutions with baryon number \( 1 \leq B \leq 6 \) in the “old baby Skyrme model” with potential \( V(\phi_3) = \mu^2 (1 - \phi_3) \) [110, 45, 55] and mass parameter \( \mu = \sqrt{0.1} \). In Chapter 3 we discuss isospinning baby Skyrme solitons for various choices of the potential term \( V(\phi) \).

![Figure 1.7: The energy densities of charge 1−6 soliton solutions in the old baby Skyrme model [110, 45, 55]. Note that there exist a \( B = 2+3 \) and \( B = 2+2+2 \) soliton configuration [110] with energies very close to those of the shown chain-5 and -6 configurations.](image-url)
1.5.3 Skyrmions with massive Pions

The Skyrme model [129] is an extension of the nonlinear \( O(4) \) sigma model in three spatial dimensions. Its Lagrangian density is given by

\[
\mathcal{L} = \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} \left( \partial_\mu \phi \cdot \partial^\nu \phi \right)^2 + \frac{1}{2} \left( \partial_\mu \phi \cdot \partial_\nu \phi \right) (\partial^\mu \phi \cdot \partial^\nu \phi) - V(\phi),
\]

where \( \phi = (\sigma, \pi) \) is a four-component unit vector composed of the scalar meson field \( \sigma \) and the pion isotriplet \( \pi = (\pi_1, \pi_2, \pi_3) \). The inclusion of the fourth order Skyrme term in the \( S^3 \) sigma model balances the energy contributions coming from the quadratic term by scaling in the opposite way (1.4.5). The topologically stable, finite-energy solutions in the \( (3 + 1) \)-dimensional Skyrme model (1.5.8) are called Skyrmions. Skyrme’s original idea [129, 131] was that baryons might be modelled as Skyrmion solutions in the nonlinear pion field theory (1.5.8). Further interest in the model arose when it was realized that in the large \( N_C \) limit QCD reduces to an effective theory of stable mesons [137, 153] governed by a Skyrme-type Lagrangian.

Note that the Lagrangian (1.5.8) is written in terms of “Skyrme units”, in which energy is measured in units of \( F_\pi / 4e \) and length in units of \( 2 / e F_\pi \). The Skyrme parameters \( F_\pi \) and \( e \) are called the pion decay constant and the Skyrme constant, respectively. Both parameters can be fixed by calibrating the model to spinning nucleons [9, 8, 21] and to the ground and excited states of larger nuclei [101, 23].

The model (1.5.8) is invariant under the six-dimensional symmetry group \( SO(3) \times SO(3) \): rotations in physical space and rotations in internal space. Here, we ignore the translational degrees of freedom. These symmetries are important for the collective coordinate quantization of Skyrme solitons [9, 8].

In contrast to the massive \( O(3) \) lumps (see Sec. 1.5.1) and the baby Skyrme solutions (see Sec. 1.5.2) discussed earlier the existence and dynamical stability of finite-energy configurations (1.4.5) in \( D = 3 \) spatial dimensions does not require the addition of a potential term \( V(\phi) \). However, since for this thesis we are primarily interested in the numerical investigation of isospinning soliton solutions, we have to include a potential function. Our numerical computations will be performed with the potential term \( V(\phi) = 2\mu^2 (1 - \sigma) \) which is one of the simplest, standard choices. Here, \( \mu \) is a rescaled, dimensionless mass parameter with \( \mu = 2m_\pi / (eF_\pi) \), where \( m_\pi \) is interpreted as a renormalized pion mass [21] and hence can be chosen to be larger than its experimental value \( m_\pi = 138 \text{ MeV} \). When considering small fluctuations around the vacuum expectation value \( \phi = (1, 0, 0, 0) \), the Lagrangian (1.5.8) with the commonly used, chiral-symmetry breaking potential term \( V(\phi) \) yields a theory of massive pions

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \pi \cdot \partial^\mu \pi - \frac{1}{2} \mu^2 \pi \cdot \pi + O(\pi^4).
\]

Furthermore, the standard mass term disfavors shell-like soliton configurations for sufficiently large mass parameters \( \mu \) and favors higher baryon and energy densities at the
Skyrmion’s center – a property relevant for the baryon interpretation of Skyrmion solutions. Alternative, more general mass terms which show the same asymptotic behaviour of the pion fields are discussed in [74, 113].

Each Skyrmion solution is associated with a conserved, integer-valued topological charge $B$ which can be calculated at a given time $t$ as the degree of the mapping $\phi : \mathbb{R}^3 \mapsto S^3_{\text{iso}}$. Constrained by the $SU(2)$ condition $\phi \cdot \phi = 1$ the Skyrme fields are restricted to live on the three-dimensional sphere $S^3_{\text{iso}}$ of unit radius. Since a configuration of finite energy requires that the Skyrme field $\phi(x)$ has to approach the same constant value $\phi(0) = (1, 0, 0, 0)$ at spatial infinity, the domain $\mathbb{R}^3$ can be compactified into $S^3_{\text{phys}}$. Consequently, each mapping can be characterized by its winding number $B \in \pi_3(S^3_{\text{iso}}) = \mathbb{Z}$ which is identified with the baryon number $B$:

$$B = -\frac{1}{12\pi^2} \int \epsilon^{abcd} \epsilon^{\nu\alpha\beta} \phi_a \partial_i \phi_b \partial_i \phi_c \partial_i \phi_d \, d^3x.$$  \hfill (1.5.10)

The static energy $M_B$ of a Skyrmion of given charge $B$ takes the form

$$M_B = \int (\partial_i \phi \cdot \partial_i \phi) + \frac{1}{2} \left( (\partial_i \phi \cdot \partial_i \phi)^2 - (\partial_i \phi \cdot \partial_j \phi)^2 \right) + 2\mu^2 (1 - \sigma) \, d^3x,$$ \hfill (1.5.11)

and has to satisfy the Faddeev-Bogomolny lower energy bound:

$$M_B \geq 12\pi^2 B.$$ \hfill (1.5.12)

The inequality (1.5.12) can be conveniently deduced within the $SU(2)$ formulation of the Skyrme model (1.5.8). To convert to the $SU(2)$ notation, we combine the sigma field $\sigma$ and the triplet of pion fields $\pi$ to form the $SU(2)$-valued scalar $U(t, x) = \sigma 1_2 + i\tau \cdot \pi$, where $\tau$ denotes the triplet of standard Pauli matrices. When rewritten in terms of the
matrix field $U(t, x)$ and its associated $su(2)$-valued right currents $R_\mu = (\partial_\mu U) U^\dagger$, the soliton’s energy (1.5.11) is given in the massless limit ($\mu = 0$) by the sum of a perfect square with the topological charge $B$:

$$
M_B = \int \left\{ \frac{-1}{2} \text{Tr}(R_i R_i) - \frac{1}{16} \text{Tr}\left( [R_i, R_j][R_i, R_j] \right) \right\} \, d^3 x \\
= -\int \left[ \frac{1}{\sqrt{2}} R_i - \frac{1}{4 \sqrt{2}} \epsilon_{ijk} [R_j, R_k] \right]^2 \, d^3 x - \int \frac{1}{2} \text{Tr}\left( \epsilon_{ijk} R_i R_j R_k \right) \, d^3 x \geq 0. \quad (1.5.13)
$$

Here, we used in the last line of (1.5.13) that the topologically conserved charge (1.5.10) is represented in terms of the $SU(2)$-valued scalar field $U(t, x)$ by

$$
B = -\frac{1}{24\pi^2} \int \epsilon_{ijk} \text{Tr}\left( R_i R_j R_k \right) \, d^3 x. \quad (1.5.14)
$$

Recall that the inequality (1.5.12) cannot be saturated since the first order Bogomolny equation

$$
R_i = \frac{1}{4} \epsilon_{ijk} [R_j, R_k] = \frac{1}{2} \epsilon_{ijk} \partial_j R_k, \quad (1.5.15)
$$

following from the square in (1.5.13) never holds. The reason here being that the relation (1.5.15) implies $\partial_i R_i = 0$ which contradicts the Skyrme field equations

$$
\partial_i \bar{R}_i = \partial_i \left( R_i - \frac{1}{4} [R_j, [R_j, R_i]] \right) = 0. \quad (1.5.16)
$$

We display in Fig. 1.8 the numerically computed baryon density isosurfaces and symmetry groups of the lowest energy Skyrme solitons for baryon numbers $1 \leq B \leq 8$, with the pion mass parameter set to $\mu = 1$. The charge-1 Skyrmion is of the well-known spherically-symmetric hedgehog form:

$$
U(x) = \exp\{i f(r) \mathbf{\hat{r}} \cdot \tau \} = \cos f(r) \mathbf{1}_2 + i \sin f(r) \mathbf{\hat{r}} \cdot \tau, \quad (1.5.17)
$$

where the radial profile function $f(r)$ is subject to the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$. This ensures a configuration of finite energy and a well-defined baryon number. Here, spherical symmetry refers to the equivariance property that is the hedgehog configuration (1.5.17) is strictly invariant under combined spatial $SO(3)$ rotations and isospin $SU(2)$ rotations. In other words, an isospin rotation is equivalent to a rotation in physical space. The $B = 2$ Skyrmion has axial symmetry; the $B = 3, 4, 7$ Skyrmion solutions have platonic symmetries (tetrahedral, cubic and icosahedral) and the $B = 5, 6, 8$

---

1Here we used that the $su(2)$-valued right currents $R_\mu = (\partial_\mu U) U^\dagger$ satisfy the Maurer-Cartan equations

$$
\partial_\mu R_\nu - \partial_\nu R_\mu + [R_\mu, R_\nu] = 0.
$$
Skyrme solitons possess dihedral symmetry. For $B \leq 8$ the symmetries of the massive, lowest energy Skyrme configurations [30] do not differ from those in the massless case [29]. Mass affects the Skyrme solutions in that they decrease in size and show a Yukawa-type falloff.

To create approximate Skyrme configurations of given baryon number $B$ we use primarily the single rational map ansatz [67]. The rational map approach allows us to set up initial Skyrme fields which share the same symmetries and have energies very close to the true, numerically calculated Skyrmion solutions. These initial Skyrme configurations can then be relaxed using a full three-dimensional relaxation algorithm to yield minimum energy Skyrme solutions of given topological charge $B$. Skyrme fields $U(x)$

![Figure 1.9: Schematic representation of the rational map approximation (compare [97]). The mapping $\phi : \mathbb{R}^3 \mapsto S^3$ can be realized by rational maps $R(z) : S^2 \mapsto S^2$. Here the domain $S^2$ of the rational map is given by 2-spheres of radius $r$ in $\mathbb{R}^3$ and the target $S^3$ is identified with 2-spheres of fixed latitude on $S^3$.](image)

– mappings from compactified $\mathbb{R}^3$ into $S^3$ – can be constructed from rational maps $R(z)$ (1.3.4) between two-spheres in the domain and target space. As illustrated in Fig. 1.9, a point $x$ in the domain $\mathbb{R}^3$ is given by the coordinate pair $(r, z)$, where $r = |x|$ denotes the radial distance from the origin and $z$ is the complex Riemann sphere coordinate $z = \tan(\theta/2) \exp(i\phi)$. The value $R(z)$ is related to a point in the unit target two-sphere of the $SU(2)$ Lie algebra by the unit vector

$$\vec{n}_R = \frac{1}{1 + |R|^2} \left( R + \bar{R}, i\left(\bar{R} - R\right), 1 - |R|^2 \right).$$

(1.5.18)

Generalization of the spherically-symmetric, hedgehog Skyrme configuration (1.5.17) then gives the rational map ansatz for the Skyrme fields:

$$U(r, z) = \exp \{if(r) \vec{n}_{R(z)} \cdot \tau\} = \cos f(r) \mathbb{1}_2 + i \sin f(r) \vec{n}_{R(z)} \cdot \tau.$$

(1.5.19)

When substituted in the static Skyrme energy functional (1.5.11), the ansatz (1.5.19) decouples the angular and radial dependent energy contributions. An approximate Skyrme configuration of given $B$ can be found by first minimizing the angular dependent terms with respect to the coefficients of the rational map and then solving for the minimized angular value the Euler-Lagrange equation for the radial profile function $f(r)$. Our extensive,
numerical simulations of isospinning, massive Skyrme solutions, presented in Chapter 5, use mainly the rational map ansätze given in [67, 99, 29, 94] to create appropriate initial conditions.

1.5.4 Massive Hopf Solitons

The massive Skyrme-Faddeev model [46, 47] is a modified $O(3)$ sigma model in $(3 + 1)$-dimensional, Minkowskian spacetime described by the Lagrangian density

$$\mathcal{L} = \frac{1}{32\pi^2\sqrt{2}} \left\{ \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} \left( \partial_\mu \phi \times \partial_\nu \phi \right)^2 - V(\phi) \right\},$$

where $\phi = (\phi_1, \phi_2, \phi_3)$ is a real three-component vector of unit length $\phi \cdot \phi = 1$ and $V(\phi)$ is a $SO(2)$-symmetric potential term. Hence, the field of the Skyrme-Faddeev model represents a mapping $\phi : \mathbb{R}^3 \mapsto S^2$. Any static, finite energy solution $\phi(x)$ requires that the vector $\phi$ takes for all time $t$ a constant value at spatial infinity, which here is selected to be the north pole of the target two-sphere, $\phi^{(0)} = (0, 0, 1)$. This finite energy boundary conditions results in an one-point compactification of the domain $\mathbb{R}^3$ to $S^3$, so that each field can be characterized by the equivalence classes of the homotopy group $\pi_3(S^2) = \mathbb{Z}$. The associated, integer-valued topological charge $N$ is called Hopf charge or Hopf invariant and the stringlike, finite energy field configurations in the model (1.5.20) are well-known as Hopf solitons.

Note that the model (1.5.20) can be derived from the $O(4)$ Skyrme model described in the previous subsection by a consistent truncation [150, 118]: One can construct an one-parameter family of generalized Skyrme models [150] which includes the $SU(2)$ Skyrme models as well as the Skyrme-Faddeev model as special cases.

Figure 1.10: Interpretation of the Hopf charge $N$ as the linking number of field lines (compare [27]). The preimages of any two distinct points $\phi_1$ and $\phi_2$ on the target space $S^2$ are two loops. The Hopf charge $N$ of the mapping $\phi(x)$ is given by the number of linkings of these two loops. The figure shows schematically a configuration with Hopf charge $N = 1$.

The Hopf charge $N$ can be expressed as the integral of a Chern-Simons 3-form over $\mathbb{R}^3$:

$$N = \frac{1}{4\pi^2} \int F \wedge A \, d^3 x,$$

(1.5.21)
where the 2-form $F = \phi^* \omega$ is the pullback of the area 2-form $\omega$ on the target two-sphere under $\phi$ to the domain three-sphere. Since the second cohomology group of $S^3$ is trivial, $H^2(S^3) = 0$, the 1-form $A$ can be defined so that the pullback is an exact, closed form $F = dA$. Recall that different to the topological soliton solutions discussed in the previous subsections, the Hopf charge cannot be interpreted as a simple winding number or as the analytic degree of the mapping $\phi$. In particular, there does not exist a local expression for the Hopf charge $N$ \eqref{eq:hopf_charge}. However, there exists a simple, heuristic interpretation of $N$ in terms of the linking number of field lines: The preimage of a generic point on the target two-sphere is given by a closed loop in $S^3$. If a field configuration $\phi(x)$ has Hopf number $N$ then the preimages of any two distinct points on the target $S^2$ will be linked $N$ times. We illustrate the linking curves for a charge-1 Hopf configuration in Fig. 1.10.

Similar to the previously described topological soliton solutions, there does not only exist a conserved topological charge but also a lower bound on the static energy

$$
M_N = \frac{1}{32\pi^2 \sqrt{2}} \int \left\{ \partial_i \phi \cdot \partial_i \phi + \frac{1}{2} \left[ (\partial_i \phi \cdot \partial_j \phi)^2 - (\partial_i \phi \cdot \partial_j \phi)(\partial_i \phi \cdot \partial_j \phi) \right] + V(\phi) \right\} d^3 x,
$$

(1.5.22)

of a Hopf configuration in terms of the Hopf charge $N$. However, this bound cannot be derived from a Bogomolny-type argument. Explicitly, the Vakulenko-Kapitanski bound \cite{79,141} is given by

$$
M_N > c|N|^{3/4},
$$

(1.5.23)

where $c = (3/16)^{3/8} \approx 0.534$. The non-trivial derivation of the topological lower energy bound \eqref{eq:v_k_bound} is based on the use of Sobolev-type inequalities and can be found in \cite{79,141}. Recall that Ward \cite{146} conjectured that $c = 1$ might be a more optimal value for the universal constant $c$ in \eqref{eq:v_k_bound}. This tighter energy bound follows when considering the Skyrme-Faddeev theory \eqref{eq:skyrme_faddeev} defined on a unit three-sphere $S^3$ rather than in $\mathbb{R}^3$ and with the normalization factor chosen as in \eqref{eq:skyrme_faddeev}. Note that Ward’s conjectured bound has proven to be in good agreement with the known numerical energy values of charge $N$ Hopf solitons \cite{27,26,64,136,56,19}.

The potential choice in \eqref{eq:skyrme_faddeev} is largely arbitrary. We perform our calculations on massive Hopf solutions with the “pion mass term” $V_I = 2\mu^2 (1 - \phi_3)$ and with the two-vacua potential $V_{II} = \mu^2 (1 - \phi_3^2)$. $V_I$ is the usual mass term in the $SU(2)$ Skyrme model \cite{129} and $V_{II}$ is the potential term known from the “new baby Skyrme model”\cite{151}. Both potentials are very simple but by no means general choices. In particular, it is convenient to perform numerical simulations with these potential terms. Recall that a potential term is not required to support the stability of non-spinning Hopf soliton solutions, see the stability condition \eqref{eq:stability_condition} in $D = 3$ spatial dimensions. It affects mainly structure and interactions of the solitons. However, isospinning solutions exist only for a non-zero mass term. Potential terms (like $V_I$ and $V_{II}$) which break the global $SO(3)$ symmetry
CHAPTER 1. PRELIMINARIES

Figure 1.11: The position (blue) and linking (red) curves for the lowest energy solitons with Hopf charges $1 \leq N \leq 8$ and potential $V(\phi) = 2\mu^2 (1 - \phi_3)$. We label each configuration with its Hopf charge and solution type. The numerical simulations were performed with the rescaled mass parameter $\mu = 1$.

of the non-linear $\sigma$-model and Skyrme-Faddeev term arise in the derivation [50] of the Skyrme-Faddeev model as an infrared limit of $SU(2)$ Yang-Mills theory. The potential functions $V_I$ and $V_{II}$ both satisfy the finite energy criteria for $\phi_3 \to +1$ and show the same asymptotic behaviour for $|x| \to \infty$, a Yukawa-type fall-off.

Similar to the Skyrme solitons described in the previous section, a rational map ansatz [136] can be used to provide suitable initial conditions of given Hopf charge $N$ for numerical relaxation simulations. Briefly, a Hopf configuration $\phi$ – a mapping from physical space $\mathbb{R}^3$ into the unit two-sphere $S^2$ – can be represented in terms of a rational map $W : S^3 \mapsto \mathbb{CP}^1$, a map from the unit three-sphere to the complex projective line. To achieve this, one identifies $\mathbb{R}^3$ with $S^3 \subset \mathbb{C}^2$ via a map

$$ (Z_1, Z_0) = \left( \frac{x_1 + ix_2}{r} \sin f, \cos f + \frac{1}{r} \sin f x_3 \right), \quad (1.5.24) $$

where $f(r)$ is a monotonically decreasing radial profile function with boundary conditions $f(0) = \pi$ and $f(\infty) = 0$ and $(Z_1, Z_0) \in \mathbb{C}^2$. The target two-sphere of the Hopf field $\phi$ is parametrized in terms of the Riemann sphere coordinate $W = (\phi_1 + i\phi_2)/(1 + \phi_3)$, which is taken to be a suitable rational function in the variables $Z_1$ and $Z_0$. We shall describe in more detail the techniques used to set up appropriate initial conditions of specific Hopf charge and solution type in Section 4.3.

Our relaxed Hopf configurations can be graphically visualized by plotting the associated linking structure. To display the linking number, we plot the preimages of the two cylinders defined by $\phi_3 = -1 + \epsilon$ and $\phi_1 = -1 + \epsilon$, where $\epsilon$ is a small parameter. Here, the preimage of the cylinder defined by $\phi_3 = -1 + \epsilon$, which is antipodal to the boundary vacuum value $\phi_\infty = (0, 0, 1)$ on the target two-sphere, is defined to be the soliton’s posi-
1.6. CLASSICALLY SPINNING AND ISOSPINNING SOLITONS

In this thesis, we consider spinning soliton solutions in (2+1)- and (3+1)-dimensional models defined by the Lagrangian density [129]

\[ \mathcal{L} = -c_2 \text{Tr} \left( R_\mu R^\mu \right) + c_4 \text{Tr} \left( [R_\mu, R_\nu]^2 \right) + c_0 V(U), \]  (1.6.1)

where \( c_i \) are free model parameters and \( R_\mu = \left( \partial_\mu U \right) U^\dagger \) is the \( su(2) \)-valued right-handed chiral current of the meson fields represented by the \( SU(2) \) matrix \( U \). The terms in (1.6.1) are commonly known as the \( O(n) \) sigma model, Skyrme and potential term in \( D+1 \) dimensions, respectively. The Lagrangian (1.6.1) admits topologically-nontrivial configurations due to the non-trivial homotopy group \( \pi_D \left( S^{n-1} \right) \). For example, if \( D = 2 \) the Lagrangian (1.6.1) allows for unstable Belavin-Polyakov lump solutions [116] (see Section 1.3.1) as well as baby Skyrme solitons [110] (see Section 1.5.2). In both cases configurations are classified by the homotopy group \( \pi_2 \left( S^2 \right) \in \mathbb{Z} \) which gives lumps and baby Skyrmions a conserved topological charge, known as the soliton number \( N \) and baryon number \( B \), respectively. With the homotopy group \( \pi_3 \left( S^3 \right) \) being non-trivial Skyrmions [129] (see Section 1.5.3) arise in (1.6.1) as topological solitons in \( D = 3 \) spatial dimensions. Finally, another example of topologically-nontrivially field configurations in \( D = 3 \) space
dimensions are the Hopf solitons [46, 47] (see Section 1.5.4) whose stability is due to the nontrivial homotopy group $\pi_3\left(S^2\right)$. In contrast to the aforementioned topological soliton solutions the associated topological charge $N \in \pi_3\left(S^2\right)$ is not given by the winding number of the mapping but can be interpreted as a linking number of field lines.

Solitons acquire spin and isospin in the moduli space approximation for low-energy soliton dynamics [96]. Here the infinite dimensional field configuration space of the theory is truncated to a finite dimensional one generated by the rotational and isorotational zero modes of the minimal energy solution $U_0(x)$. For the sake of generality, we treat the topological soliton solutions in $D = 2$ as embedded in the 3-dimensional space and isospace. Acting with the symmetry group $SO(3) \times SO(3)$ on a classical static soliton configuration $U_0(x)$ yields the ansatz for the dynamical soliton field

$$\tilde{U}(x, t) = A_1(t)U_0[D(A_2(t))(x)]A_1^\dagger(t),$$

(1.6.2)

where we promoted the collective coordinates $A_1, A_2 \in SU(2)$ to dynamical variables. $A_1(t)$ and $A_2(t)$ describe the isorotational and rotational fluctuations about the classical minimum-energy solution $U_0(x)$, respectively.

Substituting (1.6.2) in the Lagrangian (1.6.1) yields the kinematical additions [35, 94]

$$T = \frac{1}{2} a_i U_{ij} a_j - a_i W_{ij} b_j + \frac{1}{2} b_i V_{ij} b_j,$$

(1.6.3)

to the classical soliton mass

$$M = \int \left\{ c_2 \text{Tr}(R_i R_i) - c_4 \text{Tr}\left([R_i, R_j][R_i, R_j]\right) - c_0 V(U) \right\} \, d^Dx,$$

(1.6.4)

where $a_i$ and $b_j$ are the body-fixed angular velocities [35, 94, 86] for rotations in isospace and in physical space respectively,

$$a_j = -i \text{Tr}\left(\tau_j A_1^\dagger \dot{A}_1\right), \quad b_j = i \text{Tr}\left(\tau_j A_2^\dagger \dot{A}_2\right).$$

(1.6.5)

The inertia tensors $U_{ij}, V_{ij}$ and $W_{ij}$ are explicitly given by

$$U_{ij} = -2 \int \text{Tr}\left(c_2 T_i T_j + 2 c_4 [R_i, T_i][R_k, T_k]\right) \, d^Dx,$$

(1.6.6a)

$$V_{ij} = -2 \int \epsilon_{ijm} x_m \text{Tr}\left(c_2 R_i R_j + 2 c_4 [R_i, R_m][R_k, R_m]\right) \, d^Dx,$$

(1.6.6b)

$$W_{ij} = -2 \int \epsilon_{ijm} x_m \text{Tr}\left(c_2 T_i R_m - 2 c_4 [R_i, T_i][R_k, R_m]\right) \, d^Dx,$$

(1.6.6c)

where $R_k = (\partial_k U_0) U_0^\dagger$ is the right-invariant $su(2)$ current and

$$T_i = \frac{i}{2} [\tau_i, U_0] U_0^\dagger,$$

(1.6.7)
is another \( su(2) \) current. The momenta conjugate to \( a_i \) and \( b_i \) are the body-fixed isorotation and rotation angular momenta \( K_i \) and \( L_i \) defined via

\[
K_i = \frac{\partial T}{\partial a_i} = U_{ij} a_j - W_{ij} b_j , \quad (1.6.8a)
\]
\[
L_i = \frac{\partial T}{\partial b_i} = -W^T_{ij} a_j + V_{ij} b_j . \quad (1.6.8b)
\]

Following [94], we can rewrite the kinematical contribution \( T \) (1.6.3) as

\[
T = \frac{1}{2} c^T \mathcal{W} c , \quad (1.6.9)
\]

where \( c^T = (a, b) \) and we defined [94] the \( 6 \times 6 \) symmetric matrix:

\[
\mathcal{W} = \begin{pmatrix} U & -W \\ -W^T & W \end{pmatrix} . \quad (1.6.10)
\]

Consequently, the total energy \( H \) of a rotating and isorotation soliton configuration in the model (1.6.1) is of the form

\[
H = M + \frac{1}{2} G^T \mathcal{W}^{-1} G , \quad (1.6.11)
\]

where \( G^T = (K, L) \) and we used the relation \( G^T = c^T \mathcal{W} \).

Classically spinning and isospinning soliton solutions in scalar field theories of the form (1.6.1) can be constructed by solving one of the following variational problems [60]:

1. Extremize the Hamiltonian \( H \) for fixed spin and isospin angular momenta \( L_i \) and \( K_i \),

2. Extremize the pseudo-energy functional \( -L = M - T \) for fixed angular velocities \( b_i \) and \( a_i \).

For the sake of completeness, we explicitly state in the following the inertia tensors \( U_{ij}, V_{ij}, W_{ij} \) (1.6.6), the classical soliton masses \( M \) (1.6.4) and the kinematical contributions \( T \) (1.6.3) expressed in terms of the nonlinear sigma model notation for the four different two- and three-dimensional topological soliton solutions discussed in this thesis. The expressions given here (and the normalizations used) are the ones implemented in our numerical routines. Recall that we only give here a brief introductory overview of classically spinning and isospinning soliton solutions in the models introduced in Section 1.5. Later, we will dedicate a separate, full chapter to each of these topological soliton solutions.
A) Massive $O(3)$ Lumps

Classically spinning and isospinning topological $O(3)$ sigma model solitons (see Section 1.5.1) in $D = 2$ spatial dimensions can be obtained by setting

$$c_0 = c_2 = \frac{1}{4}, \quad c_4 = 0,$$

(1.6.12)

in Lagrangian (1.6.1). A suitable potential choice which allows for stable, (iso)spinning lump solutions is given for example by [148]

$$V(U) = \mu^2 \text{Tr}\left((\tau_3 U)^4 - 12\right),$$

(1.6.13)

where $\mu$ is a rescaled mass parameter and $\tau_3$ denotes the third standard Pauli matrix.

A finite energy configuration requires the boundary condition $U(x) \to \tau_3$ at spatial infinity. Hence this boundary condition on the $SU(2)$-valued hermitian matrix field $U$ defines a mapping $U : S^2 \to S^2$ and the field configurations can be classified topologically by the homotopy group $\pi_2(S^2)$ (see Section 1.5.1). For the parameter choice (1.6.12) and the potential term (1.6.13) the Lagrangian (1.6.1) is invariant under the six-dimensional symmetry group $\mathbb{R}^2 \times SO(2)^J \times SO(2)^I$: translations and rotations in $\mathbb{R}^2$ and isospin transformations.

For numerical calculation it is convenient to express the effective Lagrangian of the isorotational and rotational zero modes in terms of the three-component unit vector $\phi = (\phi_1, \phi_2, \phi_3)$ of the nonlinear $O(3)$ sigma model. Concretely, substituting $U = \phi \cdot \tau$ in (1.6.3) and (1.6.4), where $\tau$ denotes the triplet of standard Pauli matrices and choosing the $z$-axis as our rotation axis gives for the classical lump mass

$$M = \frac{1}{2} \int (\partial_i \phi \cdot \partial_i \phi) + \frac{1}{2} \mu^2 \left(1 - \phi_3^4\right) d^2 x,$$

(1.6.14)

and the rotational and isorotational energy contributions simplify to

$$T = \frac{1}{2} \omega^2 U_{33} - \omega W_{33} \Omega + \frac{1}{2} \Omega^2 V_{33},$$

(1.6.15)

where we defined $\omega \equiv a_3$, $\Omega \equiv b_3$. The associated moments of inertia (1.6.6) are given by

$$U_{33} = \int \left(\phi_1^2 + \phi_2^2\right) d^2 x,$$

(1.6.16a)

$$V_{33} = \int \epsilon_{lm} \epsilon_{np} x_l x_n \left(\partial_m \phi \cdot \partial_p \phi\right) d^2 x,$$

(1.6.16b)

$$W_{33} = \int \epsilon_{ln} \epsilon_{de} x_l x_d \phi^e \partial_m \phi^m d^2 x.$$

(1.6.16c)

Recall that for finite-energy configurations of the hedgehog form – fields manifestly invariant under combined rotations in physical space and isospace – the scalar field isotriplet
\( \phi \) can be parametrized as
\[
\phi = (\sin f(r) \cos (n \theta), \sin f(r) \sin (n \theta), \cos f(r)),
\]
where \( n \in \mathbb{Z} \) is a non-zero winding number, \((r, \theta)\) are polar coordinates in the \( \mathbb{R}^2 \)-plane and \( f(r) \) denotes a radial profile function subject to the boundary conditions \( f(0) = \pi \) and \( f(\infty) = 0 \). We verify that for static solutions of the hedgehog form (1.6.17) the moments of inertia (1.6.16) satisfy the relation \( U_{33} = V_{33} = W_{33} = \Lambda \), where \( \Lambda \) is given by [148]
\[
\Lambda = 2\pi \int r \sin^2 f \, dr.
\]
Finally, the classical \( O(3) \) lump mass (1.6.14) takes for an hedgehog configuration (1.6.17) the form
\[
M = \pi \int r \left( f'^2 + \frac{n^2}{r^2} \sin^2 f + \frac{\mu^2}{2} (1 - \cos^4 f) \right) dr,
\]
where \( f' \) denotes differentiation with respect to \( r \).

**B) Baby Skyrmions**

The baby Skyrme model [111, 114] (see Section 1.5.2) arises when setting
\[
c_0 = \frac{1}{2}, \quad c_2 = \frac{1}{4}, \quad c_4 = \frac{1}{32},
\]
in the \((2 + 1)\)-dimensional Lagrangian (1.6.1). Localised field configurations with finite energy require in \( 2 + 1 \) dimensions the inclusion of a mass term (see Section 1.4), which can be conveniently chosen to be [109, 110]
\[
V(U) = \mu^2 \text{Tr} (\tau_3 U - 1_2).
\]
The potential term (1.6.21) is known as the \((2 + 1)\)-dimensional analogue of the “pion mass term” of the full \((3 + 1)\)-dimensional Skyrme model (see Section 1.5.3).

Finite energy imposes the same asymptotic behaviour on the \( SU(2) \) matrix \( U \) as for the lump configurations described in the previous paragraph, in particular \( U(x) \to \tau_3 \) for \(|x| \to \infty \). Consequently, baby Skyrmie configurations can be characterized by a topologically conserved charge \( B \in \mathbb{Z} = \pi_2(S^2) \). Parametrizing the baby Skyrme field \( U \) in terms of the parameter triplet \( \phi = (\phi_1, \phi_2, \phi_3) \) by using \( U = \phi \cdot \tau \) in (1.6.4) results in the classical baby Skyrmion mass (1.5.3) with \( V(\phi) = \mu^2 (1 - \phi_3) \). With the \( z \)-axis chosen as our rotation axis the kinematical contribution (1.6.3) for stationary spinning and isospinning baby Skyrmion solutions takes the simple form (1.6.15), where the moments of inertia
(1.6.6) are given by

\[ U_{33} = \int \left( \phi_1^2 + \phi_2^2 \right) \left( 1 + \partial_k \phi \cdot \partial_k \phi \right) - \left( \phi \times \partial_k \phi \right)_3 \left( \phi \times \partial_k \phi \right)_3 \, d^2x, \tag{1.6.22a} \]

\[ V_{33} = \int \epsilon_{lmn} x_l x_n \left( \left( \partial_m \phi \cdot \partial_n \phi \right) \left( 1 + \partial_k \phi \cdot \partial_k \phi \right) \right. \]
\[ \left. - \left( \partial_k \phi \cdot \partial_m \phi \right) \left( \partial_k \phi \cdot \partial_n \phi \right) \right) \, d^2x, \tag{1.6.22b} \]

\[ W_{33} = \int \epsilon_{lmn} \epsilon_{de} \phi_d \partial_m \phi_e \left( \left( 1 + \partial_k \phi \cdot \partial_k \phi \right) \right. \]
\[ \left. - \left( \partial_k \phi \cdot \partial_m \phi \right) \epsilon_{de} \phi_b \partial_k \phi_e \right) \, d^2x. \tag{1.6.22c} \]

We can verify that for rotationally-symmetric configurations (1.6.17) of topological charge \( B = n \) the moments of inertia (1.6.22) simplify to \( U_{33} = V_{33} = W_{33} = \Lambda \), where \( \Lambda \) is given by [109]

\[ \Lambda = 2\pi \int r \sin^2 f \left( 1 + f'^2 \right) \, dr. \tag{1.6.23} \]

Here, \( f' \) denotes, as usual, differentiation with respect to \( r \). The associated baby Skyrmion mass (1.5.3) with the potential choice \( V(\phi) = \mu^2 \left( 1 - \phi_3 \right) \) is found to be

\[ M = \pi \int r \left( f'^2 + \frac{n^2}{r^2} \left( 1 + f'^2 \right) \sin^2 f + 2\mu^2 \left( 1 - \cos f \right) \right) \, dr. \tag{1.6.24} \]

which is in agreement with [109].

C) Skyrmions with massive Pions

The Lagrangian (1.6.1) in (3 + 1) dimensions with the parameter choice

\[ c_0 = 1, \quad c_2 = -\frac{1}{2}, \quad c_4 = \frac{1}{16}, \tag{1.6.25} \]

and the mass term

\[ V(U) = \mu^2 \text{Tr} (U - 1_2), \tag{1.6.26} \]

subject to the finite energy boundary condition \( U(x) \to 1_2 \) as \( |x| \to \infty \) supports Skyrmion solutions [129] (see Section 1.5.3) classified by a conserved, integer-valued topological charge \( B \in \mathbb{Z} = \pi_3 \left( S^3 \right) \). In the nonlinear sigma model notation \( U = \sigma 1_2 + i \pi \cdot \tau \) the
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classical Skyrmion mass (1.6.4) gives (1.5.11) and the inertia tensors (1.6.6) are given by

\[ U_{ij} = 2 \int \left( \pi d^n \delta_{ij} - \pi^n \right) (1 + \partial_k \phi \cdot \partial_k \phi) - \epsilon_{i j k} \epsilon_{f j k} \left( \pi^i \partial_k \pi^e \left( \pi^f + \partial_k \pi^e \right) \right) d^3 x, \]  

\[ V_{ij} = 2 \int \epsilon_{ilm} \epsilon_{jnp} x^l \left( \partial_m \phi \cdot \partial_p \phi - (\partial_k \phi \cdot \partial_m \phi) (\partial_k \phi \cdot \partial_p \phi) \right) \]  

\[ + (\partial_k \phi \cdot \partial_k \phi) \left( \partial_m \phi \cdot \partial_p \phi \right) d^3 x, \]  

\[ W_{ij} = 2 \int \epsilon_{jlm} \epsilon_{nm \tau i} \left( \epsilon_{i j f g} \pi^f \partial_k \pi^g \right) \]  

\[ d^3 x, \]

where \( \phi = (\sigma, \pi) \) is a four-component unit-vector. Spherically-symmetric hedgehog Skyrme fields of topological charge \( B = 1 \) take the form

\[ \sigma = \cos f(r), \quad \pi = \sin f(r) \hat{r}, \]  

with \( f(r) \) being the usual radial profile function satisfying \( f(0) = \pi \) and \( f(\infty) = 0 \).

We verify that for the hedgehog ansatz (1.6.28) the tensors of inertia \( U_{ij}, V_{ij} \) and \( W_{ij} \) (1.6.27) are all proportional to the unit matrix with \( U_{ij} = V_{ij} = W_{ij} = \Lambda \delta_{ij} \), where \( \Lambda \) is given by [9, 8]

\[ \Lambda = \frac{16 \pi}{3} \int r^2 \sin^2 f \left( 1 + f'^2 + \frac{\sin^2 f}{r^2} \right) dr. \]  

Finally, the Skyrmion mass (1.5.11) for the hedgehog field (1.6.28) can be verified to be [9, 8]

\[ M = 4\pi \int \left( r^2 f'^2 + 2 \sin^2 f \left( 1 + f'^2 \right) + \frac{\sin^4 f}{r^2} + 2 \mu^2 (1 - \cos f) r^2 \right) dr. \]  

D) Massive Hopf Solitons

Massive Hopf soliton solutions [46, 47, 56] (see Section 1.5.4) can be obtained with the parameter choice

\[ c_0 = 1, \quad c_2 = 1, \quad c_4 = \frac{-1}{2}, \]  

in the \((3 + 1)\) dimensional Lagrangian (1.6.1), where we include a mass term (1.6.21) to stabilize spinning and isospinning soliton solutions. The \( SU(2) \)-valued Hermitian scalar field \( U(t, x) = \phi \cdot \tau \) with \( \phi = (\phi_1, \phi_2, \phi_3) \) being a triplet of real scalar fields satisfying the unit vector constraints \( \phi \cdot \phi = 1 \) – must tend to the third standard Pauli matrix \( \tau_3 \) at spatial infinity to ensure a configuration of finite energy. Thus, each Hopf configuration (see Section 1.5.4) is characterized topologically by the Hopf invariant \( N \in \pi_3(S^2) = \mathbb{Z} \).

The static soliton mass (1.6.4) is found to be (1.5.22) and the spin and isospin inertia
tensors (1.6.6) in the kinematical contribution (1.6.3) take the form

\begin{align}
U_{ij} &= \frac{1}{16\pi^2} \sqrt{2} \int \left( \phi^2 \delta_{ij} - \phi_i \phi_j \right) (1 + \partial_k \phi \cdot \partial_k \phi) - (\phi \times \partial_k \phi)_i (\phi \times \partial_k \phi)_j \, d^3 x, \\
V_{ij} &= \frac{1}{16\pi^2} \sqrt{2} \int \epsilon_{ilm} \epsilon_{jnp} x_l x_m \left( \left( \partial_m \phi \cdot \partial_p \phi \right) - (\partial_k \phi \cdot \partial_m \phi) \left( \partial_k \phi \cdot \partial_p \phi \right) \\
&\quad + (\partial_k \phi \cdot \partial_k \phi) \left( \partial_m \phi \cdot \partial_p \phi \right) \right) \, d^3 x, \\
W_{ij} &= \frac{1}{16\pi^2} \sqrt{2} \int \epsilon_{ilm} \epsilon_{jnp} \left( \epsilon_{ide} \phi^e \partial_m \phi^e \right) (1 + \partial_k \phi \cdot \partial_k \phi) \\
&\quad - (\partial_k \phi \cdot \partial_m \phi) (\phi \times \partial_k \phi)_j \, d^3 x.
\end{align}

As done previously for \( O(3) \) lumps, baby Skyrmions and Skyrmion solutions, we can further check the analytic expressions (1.6.32) for the isorotational, rotational and mixed inertia tensors by substituting a standard hedgehog configuration. Following the construction in [27], we write the Skyrme field \( U(x) : \mathbb{R}^3 \mapsto SU(2) \) of baryon number \( B \) in terms of the rational map ansatz \( U = \exp(\mathbf{f} \mathbf{R}(c) \cdot \mathbf{\tau}) \) (1.5.19), where the unit threecomponent \( \mathbf{R}(c) : S^2 \mapsto S^2 \) determines the direction within the \( SU(2) \) Lie algebra. To generate a field configuration \( \phi \) of non-zero Hopf charge \( N = B \) the Skyrme field \( U : S^3 \mapsto S^3 \) (1.5.19) of winding number \( B \) is combined with a standard Hopf map \( H : S^2 \mapsto S^2 \). Parametrizing the \( SU(2) \) Skyrme field \( U \) in terms of a complex two-vector of unit length \( Z = (Z_0, Z_1)^T \)

\begin{equation}
U = \begin{pmatrix} Z_0 & -Z_1 \\ Z_1 & Z_0 \end{pmatrix},
\end{equation}

the Hopf map \( H \) takes the form \( \phi = Z^T \mathbf{\tau} Z \). Combining this Hopf map with the Skyrme field (1.5.19), we obtain [27] for the real three-component Hopf field \( \phi = (\phi_1, \phi_2, \phi_3) \):

\begin{align}
\phi_1 &= 2 (n_3 n_1 \sin f - n_2 \cos f) \sin f, \\
\phi_2 &= 2 (n_3 n_2 \sin f + n_1 \cos f) \sin f, \\
\phi_3 &= 1 - 2 \left( 1 - n_3^2 \right) \sin^2 f,
\end{align}

where \( \mathbf{R}(c) = (n_1, n_2, n_3) \) is an angular dependent vector of unit length and \( f(r) \) denotes a monotonically decreasing radial profile function with boundary conditions \( f(0) = \pi \) and \( f(\infty) = 0 \). Recall (see Section 1.5.3) that the unit three-vector \( \mathbf{R}(c) : S^2 \mapsto S^2 \) can be represented in terms of the stereographic projection (1.5.18) of the complex-valued point \( R(z) \) on the target two-sphere. Similarly the complex coordinate \( z \) on the domain-2 sphere can be related via stereographic projection \( z = \tan(\theta/2) \) exp (\( i \varphi \)) to the standard spherical polar coordinates. When choosing the rational map \( R(z) = z \) in (1.6.34), an axiallysymmetric \( N = 1 \) Hopf configuration is achieved. Note [27] that the \( B = 1 \) Skyrme field \( U \) from which the \( N = 1 \) Hopf configuration has been constructed is spherically-
symmetric, whereas the resulting Hopf field possesses only toroidal symmetry.

For such a toroidal charge-1 Hopf solution, the classical soliton mass (1.5.22) simplifies to

\[
M_1 = \frac{1}{3\pi \sqrt{2}} \int dr \left( r^2 f'^2 + 2 \left( 1 + 2 f'^2 \right) \sin^2 f + 2 \frac{\sin^4 f}{r^2} + \mu^2 r^2 \sin^2 f \right),
\]

which is in agreement with equation (4) in [134]. We verify that the associated isorotational inertia tensor \(U_{ij}\) (1.6.32a) is diagonal and satisfies the relation \(U_{11} = U_{22} = \Lambda\), where

\[
\Lambda = \frac{\sqrt{2}}{120\pi} \int dr \left\{ 16 \left( r^2 - 1 + 4 f'^2 r^2 \right) \sin^2 f - 4(16 f'^2 + 5)r^2 \right. \\
+ 40 \left. \right\} \sin^2 f + 5r^2 (4 f'^2 + 3). \]

Similiar the moment of inertia \(U_{33}\) for isorotations around the \(z\)-axis simplifies for a toroidal charge-1 Hopf soliton solution to:

\[
U_{33} = -\frac{\sqrt{2}}{15\pi} \int dr \left( 4 \left( r^2 - 1 + 4 f'^2 r^2 \right) \sin^2 f - (16 f'^2 + 5)r^2 \right) \sin^2 f. \]

1.7 Outline of this Thesis

This thesis is structured as follows. In Section 2 we present our results on isospinning \(O(3)\) sigma model lumps. Then, in Chapter 3 we investigate isospinning soliton solutions in baby Skyrme models for different choices of the potential term. Isospinning Hopf solitons are discussed in Chapter 4 and some of our results on isospinning Skyrmion solutions are given in Chapter 5. Chapter 6 contains a very brief summary of our results.
2.1 Introduction

Generally speaking, soliton solutions – stable, localized energy distributions – arising in classical field theories can be classified based on their stability mechanism as either topological or nontopological. Topological solitons are stabilized by the existence of a conserved, integer-valued topological charge, which can be computed as a winding number of a topologically nontrivial mapping from the space manifold into some target manifold. For example, for Skyrme solitons [129] the compactified Euclidean space $\mathbb{R}^3$ is mapped into the group space $SU(2)$ with the winding number counting the times the Skyrme field winds around the internal space $SU(2)$. Topological soliton configurations can be characterized according to the homotopy groups of spheres. In the case of Skyrme solitons the underlying mapping is given by $S^3 \mapsto SU(2) \approx S^3$ and the set of classical solutions divides into the different homotopy classes of $\pi_3(S^3) = \mathbb{Z}$ labelled by the topological index $B \in \mathbb{Z}$. Other well-known examples of solitons whose existence is ensured by nontrivial topology are kinks [119, 124], vortices [2, 106], Hopf solitons [46, 47] and magnetic monopoles [115, 139].

Nontopological soliton solutions owe their stability and existence not to a topological argument but to a dynamical one. Their stability is essentially due to the existence of a Noether charge $Q$ which follows from a continuous internal global symmetry of the model. $Q$-balls [40, 84, 140] are examples of this nontopological, periodical time-dependent solution type. $Q$-ball configurations can emerge in pure scalar field theories in $D \geq 1$ spatial dimensions with a scalar potential term included and with a global $U(1)$ symmetry present. The Noether charge associated with the $U(1)$ symmetry can be identified with the isospin $Q = \Lambda \omega$ of the internally rotating finite-energy solution with moment of inertia $\Lambda$ and of fixed angular frequency $\omega$. $Q$-balls are very different to topologically-nontrivial stationary, classical soliton solutions in that the charge they carry is not quantized but can take any value within a certain range. As time-dependent, non-dissipative solutions their existence is not ruled out by Derrick’s theorem [42]. Not only have spherically-symmetric, nontopological minimal-energy configurations been stabi-
lized successfully with the $Q$-mechanism, but so have been planar and axially-symmetric stationary solutions – so called $Q$-walls [142] and $Q$-vortices [90]. One of the most interesting applications of $Q$-ball type solutions is their possible relevance [81, 82] for baryogenesis via the Affleck-Dine mechanism [10].

In this chapter we focus on $Q$-lumps [85, 148] – 2-dimensional sigma model solitons\(^1\) that carry both types of conserved charges, an integer-valued topological charge $N$ together with a $U(1)$ Noether charge $Q$. $Q$-lumps are topological $O(3)$ sigma model lump solutions which are prevented from shrinking to zero size by a steady rotation in isospace – the “$Q$-ball mechanism”. $Q$-lump solutions in two spatial dimensions are characterized topologically by their winding number $N \in \pi_2(S^2) = \mathbb{Z}$. However, similar to the purely nontopological solitons described above, they are time-dependent solutions and only stable within a certain frequency range which depends strongly on the choice of the potential term. In this chapter we calculate stationary minimal-energy configurations of fixed $N$ and fixed isospin $Q$ in Ward’s lump model [148]. Different to Ward’s original calculations we do not impose that the isospinning soliton solutions will have exactly the same symmetries as the static solutions. In our fully two-dimensional numerical relaxation calculations we explicitly allow the isospinning solutions to deform and to break the symmetries of the static soliton configurations. Our numerical results are found to be in excellent agreement with the ones presented in Ref. [148] when solely assuming deformations within an axially-symmetric ansatz.

Note that in this chapter we mainly use $Q$-lump solutions as a simplified field theoretic model to check our numerical routines which will be used in the subsequent Chapters 3-5 to construct isospinning solutions in more complicated scalar field theories. However, $Q$-stabilized sigma model solitons in $D = 2$ are interesting in their own right. For example possible applications are straight, cosmic strings [148, 147] stabilized by their steady internal rotation.

This chapter is organized as follows. In Section 2.2 we review the construction of $Q$-ball solutions. Following Ward’s argumentation we use the $Q$-mechanism as an antishrinking mechanism in Section 2.3 to stabilize the Belavin-Polyakov lumps [116] in $D = 2$ spatial dimensions. Then, in Section 2.4 we perform gradient flow simulations in one and two spatial dimensions to verify that our numerical setup reproduces the rotationally-symmetric, stationary $Q$-soliton solutions constructed in Ref. [148]. Finally, we end with our concluding remarks in Section 2.5.

Note that some of the numerical results reported in this chapter form part of our conference proceeding [18].

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\(^1\) See Section 1.3.1 for a brief introduction on $O(3)$ sigma model lumps.
2.2 \( U(1) \) \textit{Q-Balls}

\( Q \)-ball solutions are time-dependent (with a rotating internal phase) nontopological soliton solutions originally introduced by Coleman in Ref. [40]. They can arise in nonlinear scalar field theories with a spontaneously broken \( U(1) \) symmetry. \( U(1) \) \textit{Q-ball} solutions carry a Noether charge \( Q \) which is associated with a global \( U(1) \) symmetry and their stability and existence are essentially due to the conservation of the \( U(1) \) charge \( Q \). The static (see [40, 84, 140] and references therein) and dynamical [28, 34] properties of \( Q \)-ball solutions have been investigated mostly using numerical methods. Numerical simulations of multi-\( Q \)-ball interactions [28] reveal interesting phenomena such as phase-dependent interaction forces, charge transfer and \( Q \)-ball fusion and fission processes.

One of the simplest models that allows for \( Q \)-ball type soliton solutions [40, 84] is a pure scalar field theory in \((D + 1)\)-dimensional Minkowskian spacetime described by the Lagrangian

\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi \cdot \partial^\mu \phi \right) - V(|\phi|) ,
\]

where \( \phi = (\phi_1, \phi_2) \) is a two-component vector of real scalar fields. The potential term in (2.2.1) is typically taken to be a polynomial with powers of \( \phi \) up to six [31, 105, 118, 142]

\[
V(|\phi|) = \lambda \left(|\phi|^6 - a|\phi|^4 + b|\phi|^2\right) .
\]

The positive parameters \( \lambda, a, b \) in the sextic potential (2.2.2) are chosen, so that the inequality

\[
V''(0) > \min_\phi \left( \frac{2V(|\phi|)}{|\phi|^2} \right) ,
\]

is satisfied. Here primes denote differentiation with respect to \(|\phi|\). Furthermore, the potential \( V(|\phi|) \) should have the global minimum \( V(|\phi|) = 0 \) at \(|\phi| = 0 \). Note that the potential choice (2.2.2) and the restriction (2.2.3) imposed on the potential term will be explained further below (see Section 2.2.1). As an example, we show in Fig. 2.1 (a) a simple potential \( V(|\phi|) - \) a polynomial in \(|\phi|^2\) – which admits \( Q \)-ball configurations.

The Lagrangian (2.2.1) is invariant under global \( SO(2) \) transformations

\[
(\phi_1, \phi_2) \mapsto (\cos \alpha \phi_1 + \sin \alpha \phi_2, -\sin \alpha \phi_1 + \cos \alpha \phi_2) .
\]

The corresponding, conserved Noether current is given by

\[
j_\mu = -\phi_1 \partial_\mu \phi_2 + \phi_2 \partial_\mu \phi_1 ,
\]
CHAPTER 2. TOPOLOGICAL $Q$-SOLITONS

(a) Potential $V(|\phi|)$ in Eq. (2.2.2).

(b) Effective potential $V_\omega(f)$ in Eq. (2.2.19).

Figure 2.1: Qualitative shape of potentials supporting $Q$-ball type configurations (analogous to the figures in [118, 142]). For illustrative purposes only, we sketch the model potential $V(|\phi|)$ (2.2.2) and the associated effective potential $V_\omega(f)$ (2.2.19) with the parameter choice $\lambda = 1$, $a = 2$, $b = 1.1$ [142]. For this potential choice, $Q$-ball solutions exist for all $\omega$ in the frequency range $0.2 < \omega^2 < 2.2$.

and the associated Noether charge which is interpreted as the conserved isospin is

\[ Q = \int j_0 \, d^D x = \int \left( -\phi_1 \partial_0 \phi_2 + \phi_2 \partial_0 \phi_1 \right) d^D x. \quad (2.2.6) \]

We can find time-dependent, isospinning soliton solutions in the model (2.2.1) by making the dynamical Ansatz

\[ \hat{\phi}(x, t) = R(\omega t) \phi(x), \quad (2.2.7) \]

where $R$ denotes the $SO(2)$ rotation matrix and $\omega$ is the field’s angular rotation frequency. Substituting (2.2.7) in (2.2.1) results in the Lagrange function

\[ L = \frac{1}{2} \Lambda \omega^2 - M, \quad (2.2.8) \]

where the classical $Q$-ball mass is given by

\[ M = \int \frac{1}{2} \partial_i \phi \cdot \partial_i \phi + V(|\phi|) \, d^D x, \quad (2.2.9) \]

and the moment of inertia of the isospinning field configuration can be calculated via

\[ \Lambda = \int \left( \phi_1^2 + \phi_2^2 \right) d^D x. \quad (2.2.10) \]

The conserved isospin (2.2.6) for the time-dependent solutions (2.2.7) is given by

\[ Q = \omega \int \left( \phi_1^2 + \phi_2^2 \right) d^D x = \omega \Lambda. \quad (2.2.11) \]
2.2. U(1) Q-BALLS

Analogous to isospinning Hopf solitons [19, 60] isospinning nontopological soliton solutions in (2.2.1) can be obtained by solving one of the following two variational problems for \( \phi \):

1. Extremize the pseudo-energy functional \( F_\omega = -L \) subject to \( \omega \) being fixed,

2. Extremize the total energy functional \( H = M + Q^2/(2\Lambda) \) for fixed conserved isospin \( Q = \Lambda \omega \).

Consequently, Q-ball solutions can be obtained by solving the variational equation

\[
\left( \Delta + \omega^2 \right) \phi = \frac{\partial V(|\phi|)}{\partial \phi},
\]

(2.2.12)

for a fixed angular frequency \( \omega \).

Note that when rescaling the space coordinate \( x \mapsto \lambda x \) (see Derrick’s scaling argument in Section 1.4) the Lagrangian (2.2.8) scales as

\[
L(\lambda) = -\lambda^{2-\nu}E_2 + \lambda^{-\nu} \left( \frac{1}{2} \omega^2 \Lambda - E_0 \right),
\]

where the energy contributions are given by

\[
E_2 = \frac{1}{2} \int \partial_i \phi \cdot \partial_i \phi \, d^Dx, \quad E_0 = \int V(|\phi|) \, d^Dx.
\]

(2.2.13)

Hence, Q-ball solutions have to satisfy the virial relation

\[
(D - 2)E_2 - D \left( \frac{1}{2} \omega^2 \Lambda - E_0 \right) = 0,
\]

(2.2.14)

which can be only fulfilled for nonzero \( \omega \) (if \( E_2 \neq 0, E_0 \neq 0 \)).

2.2.1 Q-Balls in one spatial Dimension

To illustrate the important features of Q-ball type soliton solutions, let us briefly recall the simple example of Q-ball solutions in (1 + 1) dimensions [40]. Setting \( \phi_1 = \phi_2 = f(r) \) in (2.2.12), the real radial profile function \( f(r) \) for a stationary Q-ball solution in \( D = 1, 2, 3 \) spatial dimensions has to satisfy the ordinary differential equation

\[
\frac{d^2 f}{dr^2} - \frac{(1 - D)}{r} \frac{df}{dr} + \omega^2 f = \frac{1}{2} \frac{dV(f)}{df},
\]

(2.2.15)

with boundary conditions \( f(\infty) = 0 \) and \( f'(0) = 0 \). Note that the important features of Q-ball solutions in any spatial dimension \( D \) are already captured by their analogues in one dimension, where analytical formulas are available. Hence, for the sake of simplicity, we will restrict the following discussion to the one-dimensional example and refer the reader for a detailed discussion of Q-ball solutions in higher dimensions to the literature [40, 84, 28, 140].
In \((1+1)\) spacetime dimensions the ordinary differential equation for the profile function (2.2.15) simplifies to
\[
\frac{d^2 f}{dr^2} + \omega^2 f = \frac{1}{2} \frac{dV(f)}{df}.
\] (2.2.16)

Evidently, for \(Q\)-ball solutions of the form \(\phi_1 = \phi_2 = f(r)\) the mass (2.2.9) simplifies in \((1+1)\) spacetime dimensions to
\[
M = \int_{-\infty}^{\infty} \left( (df/dr)^2 + V(f) \right) dr,
\] (2.2.17)
and the \(Q\)-ball’s moment of inertia (2.2.10) is given by
\[
\Lambda = 2 \int_{-\infty}^{\infty} f^2 dr.
\] (2.2.18)

Following Coleman’s argumentation [40] Eq. (2.2.16) can be interpreted (when taking \(f\) as the particle position and \(r\) as the time coordinate) in terms of a point particle of unit mass moving in a one-dimensional, effective potential given by
\[
V_\omega(f) = \frac{1}{2} \omega^2 f^2 - V(f).
\] (2.2.19)

Note that this simple interpretation [40, 118] imposes constraints on the potential term \(V(f)\) and on the frequency range in which stable \(Q\)-ball solutions can be found. We display in Fig. 2.1 (b) the qualitative shape of the effective potential \(V_\omega(f)\) (2.2.19) for the convenient potential choice (2.2.2) and for three different angular frequency values \(\omega\).

For a solution of finite energy the “particle” starts at time \(r = 0\) close to the maximum \(A\) (see Fig. 2.1 (b)), moves to the left dissipating its energy and comes to rest for \(r \to \infty\) at the local maximum \(B\) with zero energy \(V_\omega\). Static solutions with finite energy can only exist within the frequency range \(\omega_- < |\omega| < \omega_+\). For \(\omega \to \omega_-\) the potential maximum \(A\) approaches the position \(D\). Thus, if the particle is released at a position too close to \(D\), undershooting can happen: With the effective potential negative, the particle does not have enough energy to reach \(B\). As \(\omega \to \omega_+\) position \(C\) approaches \(B\) and the particle can overshoot, that is, it will arrive (within a finite time) at the position \(B\) with nonzero velocity. Recall that the \(\omega_-\)-limit (large charge \(Q\)-balls) is the so-called thin wall limit [40] since as \(\omega\) tends to \(\omega_-\) surface effects can be neglected and the bulk of the \(Q\)-ball becomes more and more important\(^1\). In contrast, in the limit \(\omega \to \omega_+\) (small charge \(Q\)-balls) the bulk energy contributions are much smaller than the surface contributions and

\(^1\)Note that bulk and surface energy contributions for a \(Q\)-ball of the form \(\phi_1 = \phi_2 = f(r)\) in \(D = 1\) spatial dimensions are given by:
\[
E_{\text{surface}} = E_\sigma = \frac{1}{2} \int_{-\infty}^{\infty} (df/dr)^2 dr,
\] (2.2.20a)
\[
E_{\text{bulk}} = \frac{1}{2} \Lambda(f) \omega^2 + \int_{-\infty}^{\infty} V(f) dr,
\] (2.2.20b)
2.3. \textit{Q-Solitons in the Non-linear O(3) Sigma Model}

the \( Q \)-ball configurations become increasingly thick-walled [80]. Detailed discussions of \( Q \)-ball configurations in the thin and thick wall approximation can be found in [40] and [80], respectively.

Note that the qualitative shape of the effective potential (see Fig. 2.1 (b)) implies that \( V''(0) < 0 \) (primes denote differentiation with respect to \( f \)) in order for \( Q \)-ball configurations to exist. Consequently, it follows that there exists an upper limit on the angular frequency \( \omega \)

\[
V''(0) < 0 \quad \text{(2.2.19)} \quad \omega^2 < V''(0) \equiv \omega^2_+.
\]

As \( V_\omega(f) \) must be positive for some nonzero \( f \) (see Fig. 2.1 (b)), there also exist a lower bound on the angular velocity \( \omega \) in internal space:

\[
\max_f V_\omega(f) \geq 0 \quad \text{(2.2.19)} \quad \min_f \left( \frac{2V(f)}{f^2} \right) \equiv \omega^2_- < \omega^2_+.
\]

Summarized, we impose condition (2.2.21) to avoid overshooting and condition (2.2.22) to avoid undershooting. Eq. (2.2.21) and (2.2.22) combined result in the constraint (2.2.3) on the type of potential \( V(f) \) and define the frequency range that allows for \( Q \)-balls.

Note that for the sextic potential term (2.2.2) the ordinary differential equation (2.2.16) can be solved explicitly [34, 83]. When choosing \( \lambda = 1, a = 1, b = 1/2 \) in (2.2.2), the analytic profile function \( f \) is given [34] by

\[
f(r) = \frac{4 - \omega^2}{2 + \sqrt{2\omega^2 - 4 \cosh(2r \sqrt{4 - \omega^2})}}.
\]

The conditions (2.2.21) and (2.2.22) give the allowed frequency range \( 1/\sqrt{2} < \omega < 1 \) for which \( Q \)-ball solutions exist. We display in Fig. 2.2 the charge \( Q \) and the total energy \( E_{\text{tot}} \) as function of \( \omega \). Both curves are monotonically decreasing functions of the angular frequency \( \omega \). They both diverge at the lower limit \( \omega_- \) and take finite values at the upper frequency limit \( \omega_+ \).

In the subsequent sections we will use the \( Q \)-mechanism described above to stabilize the Belavin-Polyakov solitons [116].

2.3 \textit{Q-Solitons in the non-linear O(3) Sigma Model}

Topological \( Q \)-Solitons [85, 148] arise as soliton solutions in the \((2 + 1)\)-dimensional \( O(3) \) sigma model with mass terms included. The Lagrangian density can be written in

with the moment of inertia (2.2.10) \( \Lambda(f) = 2 \int_\infty^\infty f^2 \, dr \)
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Figure 2.2: Isospin $Q = \Lambda \omega$ and total energy $E_{\text{tot}} = M + \frac{1}{2}\Lambda \omega^2$ for $Q$-ball solution as a function of angular frequency $\omega$. We use the potential term (2.2.2) with $\lambda = 1$, $a = 1$, $b = 1/2$. Charge and energy expressions are calculated with the analytically known profile function (2.2.23). (Compare Fig. 1 in Ref. [34]).

the form

$$\mathcal{L} = \frac{1}{2} \left( \partial_{\mu} \phi \cdot \partial^{\mu} \phi \right) - V(\phi_3), \quad (2.3.1)$$

where $\phi = (\phi_1, \phi_2, \phi_3)$ is a triplet of real scalar fields restricted by the unit vector constraint $\phi \cdot \phi = 1$. Unlike the theory (2.2.1) the Lagrangian (2.3.1) constrained by the condition $|\phi|^2 = 1$ admits topologically-nontrivial soliton solutions due to the homotopy group $\pi_2(S^2)$ being nontrivial: A finite-energy configuration requires that the real vector field $\phi(x)$ approaches the same constant value $\phi(0)$ at all points at spatial infinity, explicitly $\phi(x) \to \phi(0)$ as $|x| \to \infty$. This boundary condition for $\phi$ results in a one-point compactification of the physical coordinate plane $\mathbb{R}^2$ into a two-sphere $S^2_{\text{phys}}$. Meanwhile, the unit vector constraint implies that the triplet of scalar fields lives on a two-sphere $S^2_{\text{int}}$ of unit radius. Thus, each finite-energy field configuration $\phi$ can be interpreted as a mapping from $S^2_{\text{phys}}$ into the internal space $S^2_{\text{int}}$ and can be characterized by the integer-valued topological charge $N$ (1.3.6) which counts the number of times the domain sphere $S^2_{\text{phys}}$ is wrapped around the target sphere $S^2_{\text{int}}$. The associated static soliton solutions are known as $O(3)$ lumps or Belavin-Polyakov solitons [116] and have been described in more detail in Section 1.3.1.

As we pointed out in Section 1.3.1 and Section 1.4 the existence of some nontrivial topology does not guarantee the existence of stable soliton solutions. The $O(3)$ lump solutions are unstable [88, 112] which is a direct consequence of the scale invariance of the Lagrangian (2.3.1): if perturbed, $O(3)$ lumps either shrink or expand indefinitely (see Section 1.3.1). One way to cure this “rolling instability” [85] is to dynamically stabilize the $O(3)$ lump using the $Q$-mechanism described in the previous section. The resulting
2.3. \textsc{Q-solitons in the non-linear \(O(3)\) sigma model}

stable, isospinning soliton solutions of topological charge \(N\) and of Noether conserved isospin \(Q\) are well-known as \(Q\)-lumps [85, 148].

The global \(SO(3)\) symmetry of the model (2.3.1) is broken by the potential term \(V(\phi_3)\) to \(SO(2)\). The conserved currents and charge coming from the Noether symmetry are given by (2.2.5) and (2.2.11), where \(\phi = (\phi_1, \phi_2, \phi_3)\) now stands for the triplet of scalar fields with \(\sum_{a=1}^{3} \phi_a^2 = 1\) in \(D = 2\) space dimensions. Making the analogous dynamical Ansatz (2.2.7) in (2.3.1), the total energy of an isospinning lump configuration of fixed angular frequency \(\omega\) in \(D = 2\) spatial dimensions takes the form \(E_N = E_{\sigma} + E_{\text{iso}} + E_{\mu}\), where

\[
E_{\sigma} = \frac{1}{2} \int (\partial_i \phi \cdot \partial_i \phi) \, d^2 x, \quad (2.3.2a)
\]

\[
E_{\text{iso}} = \frac{1}{2} \omega^2 \Lambda = \frac{Q^2}{2\Lambda}, \quad (2.3.2b)
\]

\[
E_{\mu} = \int V(\phi_3) \, d^2 x, \quad (2.3.2c)
\]

and the moment of inertia \(\Lambda\) can be calculated via (2.2.10). To ensure a configuration of finite energy \(E_{\text{tot}}\) we impose the boundary condition \(\phi \rightarrow (0, 0, 1)\) as \(|x| \rightarrow \infty\).

Stationary \(Q\)-lump solutions of a given winding number \(N\) are constructed by minimizing either the Hamiltonian for fixed isospin \(Q\) or the pseudo-energy functional \(F_{\omega}\) for fixed angular frequency \(\omega\). Recently [60], it was argued that the numerical solution of the second formulation – the static pseudo-energy minimization problem – might be easier since the structure of the corresponding variational equation resembles the static PDE of the \(O(3)\) sigma model, whereas the second approach is related to the numerical solution of a more complicated differential-integral equation.

Concretely, \(Q\)-lump solutions can be found as critical points of the pseudo-energy functional

\[
F_{\omega}[\phi] = -L[\phi] = E_{\sigma} + E_{\mu} - E_{\text{iso}}, \quad (2.3.3)
\]

\[
= \int \left\{ \frac{1}{2} (\partial_i \phi \cdot \partial_i \phi) + V_{\omega}(\phi_3) \right\} d^2 x, \quad (2.3.4)
\]

where the effective potential \(V_{\omega}(\phi)\) is given by

\[
V_{\omega}(\phi_3) = V(\phi_3) - \frac{\omega^2}{2} \left(1 - \phi_3^2\right). \quad (2.3.5)
\]

Similiar to the “purely” nontopological soliton solutions described in the previous sections \(Q\)-lumps are only stable within a certain frequency range. Following Ward’s argumentation in [148] we can verify that the angular frequency \(\omega\) has to take values within the frequency range

\[
K\mu^2 \leq \omega^2 \leq \mu^2, \quad (2.3.6)
\]
where $\mu$ is a mass parameter and the constant $K$ is defined by

$$K = \min[V_0(\phi_3)/(\phi_1^2 + \phi_2^2)],$$  \hspace{1cm} (2.3.7)

with the normalized potential function $V_0(\phi_3) = 2V(\phi_3)/\mu^2$. From Derrick’s scaling argument it follows that an isospinning $O(3)$ lump has to satisfy in $D$ spatial dimensions

$$(2-D)E_\nu - DE_\mu + DE_{\text{iso}} = 0.$$  \hspace{1cm} (2.3.8)

If $D = 2$, Eq. (2.3.8) simplifies to the virial relation

$$E_{\text{iso}} = E_\mu.$$  \hspace{1cm} (2.3.9)

Eq. (2.3.9) implies that there only exists a solution for

$$K \leq 1.$$  \hspace{1cm} (2.3.10)

With the definition (2.3.7) and the Eq. (2.3.9) we verify that the following inequality [148] has to be fulfilled by the isospinning soliton solution

$$E_\mu = \frac{\mu^2}{2} \int V_0 \, d^2x \geq \frac{\mu^2}{2} K \Lambda = \frac{\mu^2 K}{\omega^2} E_{\text{iso}} = \frac{\mu^2 K}{\omega^2} E_\mu.$$  \hspace{1cm} (2.3.11)

Consequently, (2.3.10) and (2.3.11) result in (2.3.6).

Note that the lower frequency bound in (2.3.6) is related with the condition that the deformed potential (2.3.5) is required to be positive. Vanishing of the potential (2.3.5) results in an unstable configuration and isospinning soliton solutions cease to exist in this limit. In the following sections we will discuss $Q$-lump solutions for Ward’s potential choice [148].

### 2.4 $Q$-lumps

Isospinning $O(3)$ lumps have been previously studied [85, 148] with the following $SO(3)$ symmetry breaking potential terms:

$$V(\phi_3) = \begin{cases} V_I(\phi_3) = \frac{1}{4} \mu^2 (1 - \phi_3^4), \\ V_{II}(\phi_3) = \frac{1}{4} \mu^2 (1 - \phi_3^2). \end{cases}$$  \hspace{1cm} (2.4.1)

In the following, we set the rescaled mass parameter $\mu$ to 1. Note that for the potential choices (2.4.1) the constant (2.3.7) is given by $K = 1/2$, so that we expect stable, isospinning soliton solutions for angular frequencies $\omega$ within the range $1/\sqrt{2} \leq \omega \leq 1$. Normally the inclusion of a mass term would force the soliton solutions to shrink to zero width, but for $Q$-solitons this is balanced by the centrifugal effects due to the steady isoro-
Q-lump solutions with the potential choice \( V = V_I \) were first discussed in [148]. A Q-lump model with the potential term \( V = V_{II} \) present, has been investigated in [85]. This type of model is known as the continuum limit of the anisotropic Heisenberg model. For both models – \( V_I \) and \( V_{II} \) – a field configuration of finite energy requires that the unit 3-vector \( \phi \) has to approach either \((0,0,1)\) or \((0,0,-1)\) at spatial infinity.

Note that the double vacuum model \( V_{II} \) is a particularly convenient potential choice which will be used in Chapters 3-5 to investigate isospinning soliton solutions in models of the Skyrme family. For this particular potential choice the effective potential (2.3.5) vanishes at the critical value \( \omega_- = 1/\sqrt{2} \).

### 2.4.1 Ward’s Q-lump Solutions

To construct Q-lump solutions in the \((2+1)\)-dimensional \( O(3) \) sigma model system (2.3.1) with the mass term \( V_I \) [148] included, we can either extremize the pseudo-energy \( F_\omega = -L = M_N - \frac{1}{2} \omega^2 \Lambda \) for fixed angular frequency \( \omega \) or the Hamiltonian \( H = M_N + Q^2/(2\Lambda) \) for fixed isospin \( Q = \Lambda \omega \). The Euler-Lagrange equations corresponding to the variational problem for the energy-like functional \( F_\omega \) are given by

\[
\tilde{\partial}_i^2 \phi + \mu^2 \phi^3 \hat{e}_3 + \omega^2 (\phi_1 \hat{e}_1 + \phi_2 \hat{e}_2) - \lambda \phi = 0,
\]

where the Lagrange multiplier \( \lambda \) has been added to maintain the constraint \( \phi \cdot \phi = 1 \). Contracting (2.4.2) with the scalar field \( \phi \) and using the second derivative of the unit vector contraint, the Lagrange multiplier is computed to be

\[
\begin{align*}
\lambda &= \lambda_\sigma + \lambda_\mu + \lambda_{\text{iso}}, \\
\lambda_\sigma &= -\left( \partial_j \phi \cdot \partial_j \phi \right) \\
\lambda_\mu &= \mu^2 \phi^3, \\
\lambda_{\text{iso}} &= \omega^2 (\phi_1^2 + \phi_2^2).
\end{align*}
\]

We solve the variational equations (2.4.2) by solving numerically the associated gradient flow evolution equations. The gradient flow equation is first order in a fictitious time \( t \) and can be obtained by setting the velocity \( \dot{\phi} \) of the three-component vector field \( \phi \) equal to minus the first variation of the pseudo-energy \( F_\omega \), explicitly:

\[
\dot{\phi} = -\frac{\delta M_N}{\delta \phi} + \frac{1}{2} \frac{\delta \Lambda}{\delta \phi} \omega^2 - \lambda \phi.
\]

For fixed topological charge \( N \) and angular frequency \( \omega \) a suitable initial configuration is evolved according to the flow equations (2.4.4) on grids typically containing \((401)^2\) lattice points and a lattice spacing of \( \Delta x = 0.15 \). We discretize the gradient flow equations using second-order accurate finite difference approximations for the spatial derivatives and first-
Table 2.1: Total energy $E_N$ per soliton, angular frequency $\omega$ and moment of inertia $\Lambda$ of (iso)spinning lump solutions of winding number $N = 1, \ldots, 5$ and isospin $Q$. We solved the equation of motion of an isospinning hedgehog configuration (2.4.6) by a 1D-gradient flow evolution. Our results are in perfect agreement with those obtained by solving (2.4.2) with a 2D-gradient flow algorithm. For comparison, we also list the energy values $E_N^\text{Ward}/4\pi N$ and frequency values $\omega_\text{Ward}/\mu$ stated in [148].

<table>
<thead>
<tr>
<th>$N$</th>
<th>$E_N/4\pi N$</th>
<th>$E_N^\text{Ward}/4\pi N$</th>
<th>$\omega/\mu$</th>
<th>$\omega_\text{Ward}/\mu$</th>
<th>$\Lambda/2\pi$</th>
<th>$Q/4\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.887</td>
<td>1.888</td>
<td>0.859</td>
<td>0.859</td>
<td>2.326</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>1.840</td>
<td>0.826</td>
<td>0.826</td>
<td>0.826</td>
<td>4.840</td>
<td>2.0</td>
</tr>
<tr>
<td>3</td>
<td>1.826</td>
<td>0.817</td>
<td>0.817</td>
<td>0.817</td>
<td>7.340</td>
<td>3.0</td>
</tr>
<tr>
<td>4</td>
<td>1.820</td>
<td>0.814</td>
<td>0.814</td>
<td>0.814</td>
<td>9.831</td>
<td>4.0</td>
</tr>
<tr>
<td>5</td>
<td>1.818</td>
<td>0.882</td>
<td>0.882</td>
<td>0.882</td>
<td>12.311</td>
<td>5.0</td>
</tr>
</tbody>
</table>

We calculated explicitly the Lagrange multiplier $\lambda$ at each timestep of the gradient flow evolution. A suitable initial charge $N = n$ lump configuration with the correct topology and asymptotic behaviour at spatial infinity can be generated via the hedgehog ansatz

$$\phi = (\sin f \cos (n\theta), \sin f \sin (n\theta), \cos f),$$

where $(r, \theta)$ are polar coordinates in the $xy$-plane and $f(r)$ denotes a monotonically decreasing radial profile function satisfying the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$. Substituting (2.4.5) in (2.4.2) yields the following second order nonlinear ordinary differential equation for the hedgehog profile function $f(r)$

$$f'' = -\frac{1}{r} f' + \left(\frac{n^2}{r^2} - \omega^2\right) \cos f \sin f + \mu^2 \cos^2 f \cos f \sin f,$$

where primes denote as usual differentiation with respect to $r$. For the rotationally-symmetric ansatz (2.4.5) the energy contributions (2.3.2) simplify to

$$E_\sigma = \pi \int r \left(f'' + \frac{n^2}{r^2} \sin^2 f\right) dr,$$  

$$(2.4.7a)$$

$$E_\mu = \frac{1}{2} \pi \mu^2 \int r \left(1 - \cos^4 f\right) dr,$$  

$$(2.4.7b)$$

and the associated isorotational moment of inertia (2.2.10) takes the simple form

$$\Lambda = 2\pi \int r \sin^2 f \, dr.$$  

$$(2.4.8)$$

The corresponding formula for the topological charge $N$ (1.3.6) of an hedgehog configuration (2.4.5) is given by

$$N = 2\pi n \int \sin f f' \, dr.$$  

$$(2.4.9)$$
We solve the one-dimensional, two-point boundary value problem (2.4.6) for fixed topological charge $N$ and for fixed isospin $Q = \Lambda \omega$ with a 1D gradient flow evolution algorithm on the radial interval $r \in [0, 30]$ with a grid spacing $\Delta r = 0.03$. The obtained profile functions for $1 \leq N \leq 5$ are shown in Fig. 2.3 and the corresponding energy and inertia values are listed in Table 2.1. Our numerical results are found to agree well with those given in [148].

As displayed in Fig. 2.3, the energy density of the radially-symmetric, minimal energy 1-lump configuration peaks at the origin. For increasing topological charge $N$ the energy density peak moves radially outward resulting in ring-like soliton configurations. The total energy $E_N$ per soliton decreases with increasing $N$ suggesting [148] that the multi-soliton solutions do not break up in lower charge lump solutions. Furthermore, the energy density peaks of the ring-like solutions seem to approach an asymptotic height with increasing $N$. Similarly, the energy densities of charge-$N$ Skyrmion solutions in the new baby Skyrme model [151] (compare Fig. 3.2) form rings of larger and larger radii and converge to an asymptotic height for large topological charges $N$. Asymptotically, the energy has been shown [151] to grow linearly with $N$.

![Figure 2.3: Profile functions $f(r)$ (left), energy density $E_N$ (middle) and moment of inertia density $\Lambda$ (right) for Ward’s hedgehog $Q$-lump solutions (2.4.5) of topological charge $1 \leq N \leq 5$ and isospin $Q = N$ (close to the thick-wall regime). Recall that the conserved isospin $Q$ is given in units of $4\pi$ and that our calculations are performed with the mass parameter $\mu$ set to 1. We solve (2.4.6) with a 1D gradient flow evolution code on the radial interval $r \in [0, 30]$ with a grid spacing $\Delta r = 0.03$ and boundary conditions $f(0) = \pi$ and $f(30) = 0$. The obtained numerical energy and inertia values can be found in Table 2.1.](image-url)

We display in Fig. 2.4 the profile functions, energy and inertia densities as a function of the radial coordinate $r$ for 1-lump hedgehog solutions with isospin values in the range $10 \leq Q \leq 70$. As already discussed in [148], we can observe in Fig. 2.4 as $Q$ decreases a transition from the thin-wall limit [40] to the thick-wall limit [80]: For $Q$ sufficiently large ($Q \to \infty$, $\omega \to \omega_- = 1/\sqrt{2}$) the bulk contributions $E_{\mu}$ and $E_{\text{iso}}$ to the total energy are larger than the surface energy contribution $E_{\sigma}$. In this so-called thin-wall regime the profile function $f(r)$ can be clearly distinguished into three different regions: Close to the origin the profile function decreases rapidly from $\pi$ to $\pi/2$, then it forms a plateau with $f = \pi/2$ ($\phi_3 = 0$) and finally the function $f$ takes its asymptotic value $f = 0$ ($\phi_3 = 1$). With increasing angular velocity $\omega$ ($Q \to 0$, $\omega \to \omega_+ = 1$) the thin-wall approximation starts to break down and the $E_{\sigma}$ energy term (surface tension effects) becomes increasingly
important. The profile function $f(r)$ in this so-called thick-wall regime does not exhibit any plateau (see Fig. 2.3) and drops rapidly from $\pi$ at $r = 0$ to its asymptotic value $f = 0$. 

![Figure 2.4: Profile functions $f(r)$ (left), energy density $E_N$ (middle) and moment of inertia density $\Lambda$ (right) for 1-lump hedgehog solutions in Ward’s [148] massive $O(3)$ sigma model for a range of isospin values (close to the thin-wall regime). The numerical calculations are performed for isospin values in the range $10 \leq Q \leq 70$ and for $\mu = 1$. Discretization and solution method are chosen as in Fig. 2.3.](image)

We verify that solving Eq. (2.4.2) without imposing any spatial symmetries on the isospinning 1-lump solution reproduces the behaviour expected from an isospinning, radially-symmetric lump configuration. The energy and inertia density distributions (see Fig. 2.5) calculated with our fully two-dimensional relaxation algorithm are in good agreement with those obtained when only assuming deformations within a rotationally-symmetric ansatz (see Fig. 2.3 and Fig. 2.4). In particular, the energy density of the isospinning $O(3)$ sigma model $N = 1$ lump is peaked at $r = 0$ in the thick-wall limit and as $Q$ increases a plateau is formed. For $Q = 1$ ($\omega = 0.859$) our 2D gradient flow evolution gives an energy value $E_1 = 1.888 \times 4\pi$ for the charge-1 lump configuration and the corresponding moment of inertia is given by $2.325 \times 2\pi$ which is in perfect agreement with the values given in Table 2.1. Note that the “units” are chosen for a simple comparison with literature values, in particular with the hedgehog solutions calculated in [148].

Solving Eq. (2.4.6) for topological charge $N = 2$ results in ring-like lump configurations with the profile functions, energy and moment of inertia densities displayed in Fig. 2.6. As we approach the lower frequency limit ($\omega \to \omega_- = 1/\sqrt{2}$) an increasingly larger region is formed where the profile functions $f$ take the constant value $\pi/2$. We confirm the same geometrical shape when performing full 2D-relaxation calculations (see density surface and contour plots in Fig. 2.7). Our 2D-gradient flow simulations give for the charge-2 lump for $Q = 2$ ($\omega = 0.826$) an energy value $E_2 = 1.839 \times 8\pi$ and moment of inertia $\Lambda = 4.841 \times 2\pi$. For comparison, minimizing the pseudo-energy functional $F_\omega$ within the rotationally-symmetric ansatz (2.4.5) results for the 2-lump with $\omega = 0.826$ in $E_2 = 1.840 \times 8\pi$ and $\Lambda = 4.840 \times 2\pi$.

Finally, we display in Fig. 2.8 the total energy $E_N$, the moment of inertia $\Lambda$ and the angular frequency $\omega$ as functions of isospin $Q$ for $N = 1$ and $N = 2$ lump solutions (We verified that our 1D and 2D relaxation calculations reproduce exactly the same be-
2.4. $Q$-LUMPS

$$q = 1 \quad \quad q = 5 \quad \quad q = 10 \quad \quad q = 30 \quad \quad q = 70$$

$$\omega = 0.85 \quad \quad \omega = 0.80 \quad \quad \omega = 0.78 \quad \quad \omega = 0.74 \quad \quad \omega = 0.73$$

Figure 2.5: We display the energy density $E_N$, topological charge density $N$ and moment of inertia density $\Lambda$ as function of the isospin $|Q|$ for the (iso)spinning $N = 1$ lump solution in the $O(3)$ sigma model with potential $V_I$ included. Our numerical simulations are performed on grids with $401 \times 401$ lattice points and a lattice spacing of $\Delta x = 0.15$. To find the stationary points of the Hamiltonian, we perform for fixed $Q$ and $N$ a 2D gradient flow evolution of a start configuration initially created with the hedgehog ansatz ($r_{max} = 30$). Note that the conserved isospin $Q$ is given in units of $4\pi$, i.e. we defined $q = Q/4\pi$.

Figure 2.6: Same as Fig. 2.4, but for (iso)spinning $N = 2$ lump hedgehog solutions and for isospin values in the interval $10 \leq Q \leq 90$ (close to the thin-wall regime).

The $\omega(Q)$ curve shown in Fig. 2.8 confirms that stable $Q$-lump solutions only exist for frequencies $1/\sqrt{2} < \omega < 1$. 

haviour.)
Figure 2.7: Same as Fig. 2.5, but for the (iso)spinning \( N = 2 \) lump solution in the \( O(3) \) sigma model with potential \( V_I \) included. Again, we define \( q = Q/4\pi \).

Figure 2.8: Total energy \( E_N \) (left), moment of inertia \( \Lambda \) (middle) and angular frequency \( \omega \) (right) as a function of isospin \( Q \) for topological charge-1 and -2 \( Q \)-lump solutions in model \( V = V_I \). We verify that our 1D simulations on hedgehog lump solutions (2.4.5) and our 2D gradient flow simulations on deforming lump solutions (without imposing rotational symmetry) show exactly the same behaviour. The corresponding energy density contour and surface plots for 1- and 2-lumps can be found in Fig. 2.5 and Fig. 2.7, respectively. Our results for the 1-soliton solution are in good agreement with Fig. 2 in [148].

### 2.5 Conclusions

In this chapter we performed full 2-dimensional numerical relaxation calculations of isospinning charge-1 and -2 lump solutions in the \( O(3) \) sigma model. Compared to previous calculations [85, 148], we did not make any assumptions on the spatial symmetries.
of the isospinning lump configurations. We verified that our numerical simulations reproduce the rotationally-symmetric lump solutions constructed in Ref. [148].

Note that in this chapter the $O(3)$ sigma model served us primarily as a testing ground for our numerical approach. In the following chapters we will use very similar numerical techniques to investigate isospinning soliton solutions in the baby Skyrme, Skyrme and Skyrme-Faddeev model.
3.1 Introduction

Baby Skyrmions [111, 114] are topological soliton solutions in (2 + 1)-dimensional versions of the Skyrme model [129] for nuclear physics. They can be used to describe quasiparticle excitations in ferromagnetic quantum Hall systems [132, 144] or can arise as stable skyrmion spin textures in various other condensed-matter systems such as helical magnets Fe$_{1-x}$CO$_x$Si [155] and MnSi [104].

In this chapter, our numerical study of isospinning soliton solutions in baby Skyrme models is not only motivated by their physical relevance but even more so by the similarities to their analogues in 3+1 dimensions, which are known as Skyrmions. Using the baby Skyrme model as a simplified model to guide investigations in the full (3+1)-dimensional Skyrme theory has proven to be useful in the study of various Skyrme soliton properties. For example, scattering processes of baby Skyrmions [108, 109] strongly resemble those of Skyrmions [125] and allow us to study the long-range forces between static, moving and spinning Skyrmions with moderate numerical effort.

Suitably quantized Skyrmions [9, 94] of topological charge $B$ are promising candidates to model spin and isospin states of nuclei with baryon number $B$. However, most attempts have so far been either based on a rigid-body type approximation [9, 94] – the soliton’s shape is taken to be rotation frequency independent – or only considered axially-symmetric deformations [21, 66, 54] of the spinning Skyrme configurations. As pointed out by several authors [36, 21, 66], neglecting any deformations that could arise from the dynamical terms in the Skyrme model is far from being an adequate approximation and working within an axially-symmetric solution ansatz is only a valid simplification for few Skyrmions since most Skyrme solitons are not axially-symmetric.

In this chapter, we use the baby Skyrme model as a testing ground to numerically investigate how the Skyrme soliton’s geometrical shape, its mass and its moment of inertia change when classically isorotating around the $z$-axis. We perform numerical simulations of isospinning soliton solutions in the conventional baby Skyrme model [111, 114] and in two alternative models, the double-vacuum potential [45] and the one-parametric family
of potentials [61, 62, 72] which interpolates between the “old” baby Skyrme model [111, 114] and the holomorphic model [87, 111, 135].

To our knowledge, there only exists research on isospinning, rotationally-symmetric charge-1 and charge-2 soliton solutions of the old baby Skyrme model [109, 4] and of two modified baby Skyrme models where the domain $\mathbb{R}^2$ is replaced by a 2-sphere and a unit disk, respectively [61, 62, 72]. Furthermore, isospinning rotationally-invariant solitons have been constructed analytically [57] in the BPS limit of the old baby Skyrme model. Compared to previous work, we do not impose any symmetries on the isospinning Skyrme configurations and we do not apply the rigid body approximation in our 2D numerical computations. Our calculations are performed for isospinning soliton solutions up to charge 6 in the old baby Skyrme model and up to 4 in the two alternative models.

This chapter is structured as follows. In Section 3.2 we briefly review seven different versions of baby Skyrme models: five $O(2)$-symmetric models and two models in which the potential term has only a discrete symmetry and baby Skyrme solitons acquire an internal structure – one is Ward’s $D_2$ symmetric model [149] and the other the recently proposed $D_3$ symmetric model [71]. We explain in Section 3.3 how we construct isospinning baby soliton solutions. Then, in Section 3.4 we create suitable initial field configurations of non-zero baryon number $B$ which in Section 3.5 are numerically minimized using a 2D gradient flow evolution algorithm. The resulting baby Skyrme configurations of vanishing isospin for three different, $O(2)$ symmetric models serve us as initial fields for our numerical simulations of isospinning baby Skyrmions in Section 3.6. We present our conclusions in Section 3.7.

Note that the work presented in this chapter has been done in collaboration with Richard Battye and Yakov Shnir. In particular, the majority of our results on isospinning baby Skyrmion solutions has been obtained independently by Alexey Halavanau and Yakov Shnir using completely different numerical techniques [59]. Our findings on isospinning solutions in the old baby Skyrme model form part of [20].

### 3.2 Baby Skyrme Models

The Lagrangian density of the $(2 + 1)$-dimensional baby Skyrme model is defined by

$$L = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{4} \left( \partial_\mu \phi \times \partial_\nu \phi \right)^2 - V(\phi), \quad (3.2.1)$$

where $\phi = (\phi_1, \phi_2, \phi_3)$ is a real vector field of unit length and hence the target space of $\phi$ is given by a 2-sphere $S^2$. The first term in (3.2.1) is known as the $O(3)$ sigma model term, the second is the $(2 + 1)$-dimensional analogue of the Skyrme term and the last term is a potential term which is needed to prevent the soliton from spreading out.

Finite energy solutions in the model (3.2.1) require that the vector field $\phi$ has to approach a constant value $\phi_\infty$ at spatial infinity, i.e. $\phi(x, t) \to \phi_\infty$ as $|x| \to \infty$ for all time $t$. Consequently, this boundary condition results in a one-point compactification of physical
space $\mathbb{R}^2$ into a 2-sphere $S^2_{\text{space}}$ and hence a field configuration $\phi$ can be labelled by the winding number $B$ of the map $S^2_{\text{space}} \mapsto S^2_{\text{iso}}$, explicitly given by

$$B = \deg[\phi] = \frac{1}{4\pi} \int \phi \cdot \partial_1 \phi \times \partial_2 \phi \, d^2x.$$ \hspace{1cm} (3.2.2)

The topological charge $B \in \pi_2(S^2) = \mathbb{Z}$ is called baryon number in analogy to the Skyrme model [129] and field configurations that minimize the static energy functional in a given topological sector $B$ are known as baby Skyrmions. In this chapter, the static energy $M_B$ of a baby Skyrmion solution of topological charge $B$ will be given in units of $4\pi B$. Our normalization choice is motivated by the Bogomolny lower energy bound

$$M_B \geq 4\pi B.$$ \hspace{1cm} (3.2.3)

The shape and the behaviour of baby Skyrmion solutions depend strongly on the choice of the potential term $V(\phi)$ in (3.2.1). Here we will consider the following potentials

$$V(\phi) = \begin{cases} 
V_I(\phi) &= \mu^2 (1 - \phi_3), \\
V_{II}(\phi) &= \mu^2 (1 - \phi_3^2), \\
V_{III}(\phi) &= \mu^2 (1 - \phi_3^4), \\
V_{IV}(\phi) &= \frac{1}{2} \mu^2 (1 - \phi_3^2)(1 - \phi_3^4), \\
V_V(\phi) &= \mu^2 |1 - (\phi_1 + i\phi_2)|^2 (1 - \phi_3), \\
V_{VI}(\phi) &= \mu^2 \phi_3^2 (1 - \phi_3), \\
V_{VII}(\phi) &= \mu^2 (1 - \phi_3)^2.
\end{cases} \hspace{1cm} (3.2.4)$$

where $\mu$ is a rescaled mass parameter. $V = V_I$ is the most common potential choice which was originally included in [110, 109] in analogy to the pion mass term of the full $(3+1)$-dimensional Skyrme model. To ensure a finite-energy configuration the field has to approach its vacuum value $\phi_3 = +1$ at spatial infinity. Choosing $V = V_I$ in (3.2.1) results in two massive modes ($\phi_1, \phi_2$) of mass $\mu$ and one massless ($\phi_3$). Here, the mass parameter $\mu$ is typically chosen to be $\mu = \sqrt{0.1}$ [110, 109], so that the size of the $B = 1$ soliton solution is approximately of order of the Compton wavelength of the mesons in our model. This choice was originally motivated by the full $(3+1)$-dimensional Skyrme model, in which the 1-soliton size is approximately equal to the pion’s wavelength. Static multi-soliton solutions in model (3.2.1) with potential $V_I$ have been studied in particular in [110, 45, 55].

Note that the topological lower energy bound (3.2.3) for the baby Skyrme models (3.2.4) can be further improved due to the fact that both the $O(3)$ sigma model term and the Skyrme and potential terms together obey Bogomolny bounds separately. The energy bound related to the $O(3)$ sigma model term is given by (3.2.3), whereas the second bound
for the Skyrme and potential terms depends on the specific potential choice. In the case of the standard baby Skyrme model \( V = V_I \) the soliton’s energy obeys the improved, tighter Bogomolny bound \[41, 6, 133\]

\[ M_B \geq 4\pi B \left( 1 + \frac{4}{3} \mu \right), \quad (3.2.5) \]

where the second term is due to the Skyrme and potential term in (3.2.1). In general, one can calculate for each of the models (3.2.4) an energy bound\(^1\) of the type [6]

\[ M_B \geq 4\pi B (1 + C\mu), \quad (3.2.6) \]

where the numerical factor \( C \) depends on the chosen potential term \( V \).

The potential \( V_I \) breaks the global \( O(3) \) symmetry of the Lagrangian (3.2.1) to an \( O(2) \) symmetry. Further examples of \( O(2) \) symmetric potential choices are the new baby Skyrme model \[78, 151\] with \( V = V_{II} \) and the holomorphic model \[87, 111, 135\] with \( V = V_{III} \). The minimal-energy multi-skyrmion solutions which emerge in the model \( V_{II} \) are significantly different from those supported by the mass term \( V_I \). The lowest-energy soliton solutions in the new baby Skyrme model \( V = V_{II} \) are rotationally-symmetric, ring-like configurations, whereas in the old baby Skyrme model \( V = V_I \) ring-like structures turn out to be unstable for topological charges \( B > 2 \) \[110\] and evolve into non-radially symmetric soliton solutions. The potential \( V_{II} \) has two vacua, one for \( \phi_3 = +1 \) and the other for \( \phi_3 = -1 \). The vacuum value at spatial infinity is chosen to be \( \phi_3 = +1 \) so that both models share the same asymptotic behaviour.

The simplest holomorphic model \[87, 111, 135\] is given by the potential choice \( V_{III} \) in (3.2.1). Note that for the mass term \( V_{III} \) the asymptotic behaviour does not depend in leading order on the potential term and the soliton is only polynomially localized \[87\]. In contrast to the other baby Skyrme models discussed here the holomorphic model \( V_{III} \) does not allow for stable multi-skyrmion solutions. However, in the charge-1 sector, there exists an explicit, analytic solution of the holomorphic form \( W(z) = \lambda z \) (1.3.11), where the constant \( \lambda = \sqrt{2} \) can be calculated as a ratio from the coefficients of the Skyrme and mass term \( V(\phi) \) in (3.2.1) (see \[87, 111, 135\]) and determines the soliton’s size. Here \( W(z) \) denotes the complex Riemann sphere coordinate (1.3.2) obtained via stereographic projection from the field triplet \( \phi \) and the complex variable \( z = x + iy \) parametrizes the \( \mathbb{R}^2 \)-domain (compare Fig. 1.2). Hence, the model \( V_{III} \) with its analytic static charge-1 soliton solution allows us to check our numerical algorithms. Because its stable, finite-energy solution in the charge-one sector can be represented in terms of holomorphic functions of the form \( W(z) = \lambda z = \lambda (x + iy) \), we call the baby Skyrme model \( V_{III} \) holomorphic.

\( V_{IV} \) \[149\] and \( V_{V} \) \[71\] are examples for potential terms for which the \( O(3) \) symmetry of (3.2.1) is not broken to a continuous but a discrete symmetry. For \( V_{IV} \) the unbroken

\(^1\)For an explicit derivation of Bogomolny bounds in baby Skyrme models the interested reader is referred to the discussions given in Refs. [41, 6, 133].
symmetry group is the dihedral group $D_2$. The potential $V_{IV}$ has 4 vacua on the target 2-sphere $S^2_{iso}$: $\phi = (0, 0, 1)$ at the north pole which is taken to be the boundary condition at spatial infinity, $\phi = (0, 0, -1)$ at the soliton’s centre and 2 vacua lying on the equatorial circle at $\phi_3 = 0$. The finite-energy solutions in this model take the shape of polygonal rings [149]. Similar to the old baby Skyrme model, potential choice $V_{IV}$ results in two massive fields ($\phi_1$ and $\phi_2$) of mass $\mu$ and one massless field ($\phi_3$).

The vacuum structure of $V_V$ resembles the one of Ward’s potential $V_{IV}$. However, this time the global $O(3)$ symmetry is broken to the dihedral group $D_3$ and the reflection symmetry $\phi \mapsto -\phi$ is absent. There are four vacua on the 2-sphere $S^2_{iso}$: $\phi = (0, 0, 1)$ at the north pole and three equatorial vacua with $\phi_3 = 0$. Note that again both fields $\phi_1$ and $\phi_2$ are of mass $\mu$ and $\phi_3$ is massless. To simplify comparison with [71] we set in our numerical computations $\mu = 1$ for the potential choices $V_{IV}$ and $V_V$. Stable multi-soliton solutions in a baby Skyrme model with potential $V_V$ included have been shown to correspond to polyiamonds [71].

The potential choices $V_{VI}$ and $V_{VII}$ are both simple generalizations of the old baby Skyrme model $V_I$. The potential term $V_{VI}$ [45] is obtained by multiplying $V_I$ by an extra factor $\phi_3^2$, whereas the class of potentials $V_{VII}$ interpolates as a function of the free, real parameter $s$ between the “old” model ($s = 1$) and the holomorphic baby Skyrme model ($s = 4$) [61, 62, 72]. Due to the factor $\phi_3^2$, the potential term $V_{VI}$ has additional minima on the whole circle $\phi_3 = 0$. As assumed before, the baby Skyrme field $\phi$ is required to take its vacuum value $\phi_3 = +1$ at spatial infinity, that is the “north pole” of the target 2-sphere $S^2_{iso}$. Numerical simulations [45] of baby Skyrmion solutions in the model with potential $V_{IV}$ included revealed that their properties (e.g. binding energies) are closer to those of the standard baby Skyrme model $V_I$. However, the structure of the $B = n$ multi-Skyrmion solutions has been found [45] to resemble $n$ single solitons placed on the edges of regular polygons. We will calculate the lowest-energy field configurations in the model $V_{IV}$ for a range of mass values $\mu$.

The potential term $V_{VII}$, studied in detail in [61, 62, 72] allows for the existence of non-rotationally symmetric, static finite-energy solutions. Here, the free parameter $s$ can be interpreted [61, 62, 72] as a control parameter which determines not only if stable soliton solution exist at all but also whether rotational symmetry breaking in the static solutions of the model occurs. Generally speaking, it was shown in [61, 62, 72] that for small $s$ values the solutions tend to be rotationally-symmetric, ring-like and the single baby Skyrmions interact strongly. However, with an increasing $s$ parameter the mutual attraction between the constituents decreases and the configurations start to break rotational symmetry. Above a certain critical value of $s$, repulsion dominates so that the soliton solutions break apart in individual Skyrmions which are moving infinitely apart. The other parameter which strongly affects the appearance of the soliton solutions in a given topological sector $B$ is the mass parameter $\mu$. Increasing its value for fixed $s$ can also result in non-rotationally symmetric, spatial energy distributions.
3.3 Classically Isospinning Baby Skyrmions

To find isospinning baby Skyrmion solutions we perform a time-dependent $SO(2)$ isorotation on a static baby Skyrmion configuration $\phi$ via

$$(\phi_1, \phi_2, \phi_3) \mapsto (\cos (\alpha + \omega t) \phi_1 + \sin (\alpha + \omega t) \phi_2, -\sin (\alpha + \omega t) \phi_1 + \cos (\alpha + \omega t) \phi_2, \phi_3),$$

(3.3.1)

where the rotation axis is chosen to be $(0, 0, 1)$, $\omega$ is the angular frequency and the angle $\alpha \in [0, 2\pi)$. Substituting the dynamical ansatz (3.3.1) into the Lagrangian (3.2.1) gives

$$L = \frac{1}{2} \Lambda_B \omega^2 - M_B,$$

(3.3.2)

where $M_B$ is the classical soliton mass

$$M_B = \int \frac{1}{2} (\nabla \phi_1 \cdot \nabla \phi_1 + \frac{1}{4} \left[ (\partial_i \phi \cdot \partial_i \phi)^2 - (\partial_i \phi \cdot \partial_j \phi) (\partial_i \phi \cdot \partial_j \phi) \right] + V(\phi) \, d^2 x,$$

(3.3.3)

and $\Lambda_B$ is the moment of inertia

$$\Lambda_B = \int \left( \phi_1^2 + \phi_2^2 \right) (1 + \partial_k \phi \cdot \partial_k \phi) - (\phi \times \partial_k \phi)_3 (\phi \times \partial_k \phi)_3 \, d^2 x$$

$$= \int \left( \phi_1^2 + \phi_2^2 \right) + (\partial_k \phi_3)^2 \, d^2 x.$$  

(3.3.4)

The Noether charge associated with the $SO(2)$ transformation (3.3.1) is the conserved total isospin $K = \omega \Lambda$.

As shown in [60] the problem of constructing isospinning soliton solutions in Skyrme models can be formulated in terms of the following two variational problems for $\phi$:

1. Extremize the pseudo-energy functional $F_\omega = -L$ for fixed $\omega$.

2. Extremize the total energy functional $H = M_B + K^2/(2\Lambda_B)$ for fixed $K$.

We performed most of our numerical simulations with both formulations and verified that we obtained the same soliton shape and dependence of the soliton’s energy on the rotation frequency $\omega$.

3.3.1 Critical Angular Frequencies

As mentioned in the previous section, isospinning baby Skyrmions can be found for fixed angular frequency $\omega$ as critical points of the pseudo-energy functional $F_\omega(\phi) =$
\[ M_B - \frac{1}{2} \Lambda_B \omega^2 \] which takes the form

\[
F_\omega(\phi) = \int \left[ \frac{1}{2} \left( 1 - \omega^2 (1 - \phi^2) \right) \left( \partial_i \phi \cdot \partial_i \phi + \omega^2 (\phi \times \partial_i \phi) \right) + \frac{1}{4} \left( \partial_i \phi \times \partial_j \phi \right)^2 + V_\omega(\phi) \right] \, d^2 x, \tag{3.3.5}
\]

where the effective, deformed potential \( V_\omega(\phi) \) is given by

\[
V_\omega(\phi) = V(\phi) - \frac{\omega^2}{2} (1 - \phi^2). \tag{3.3.6}
\]

It was pointed out in [60] that isospinning soliton solutions in Skyrme-like models suffer from two different types of instabilities. One is due to the nullification of the terms quadratic in first spatial derivatives in (3.3.5) at some critical frequency value \( \omega_1 \) and the other is related with the violation of the virial relation

\[
\int \frac{1}{4} \left( \partial_i \phi \times \partial_j \phi \right)^2 \, d^2 x = \int V_\omega(\phi) \, d^2 x, \tag{3.3.7}
\]

at a second critical frequency value \( \omega_2 \). Recall\textsuperscript{2}[60] that the terms in the curly brackets in (3.3.5) effectively describe for \( 0 < \omega < 1 \) the geometry of a squashed sphere \( S^2 \) deformed along the direction \( \phi_\infty = (0, 0, 1) \). For \( \omega > \omega_1 = 1 \), the metric becomes singular and the pseudo-energy of \( \phi \) is no longer bounded from below. The second critical frequency \( \omega_2 \) is sensitive to the concrete potential choice \( V(\phi) \) in (3.2.1) and follows from the condition that both integrals in (3.3.7) must be positively defined and non-zero in order to allow for stable isospinning soliton solutions. Consequently, stable isospinning soliton solutions can only be constructed for all angular frequencies \( \omega \leq \min\{1, \omega_2\} \).

Note that the double vacuum model \( V_{II} \) is a particularly convenient choice to study the pattern of critical frequencies in baby Skyrme models. For this particular potential choice the second critical frequency value \( \omega_2 \) can then be read from (3.3.6) directly to be given by \( \omega_2 = \sqrt{2} \mu \). Consequently, stable isospinning solutions are only expected to exist for angular frequencies \( \omega \leq \min\{1, \sqrt{2} \mu\} \) in the rotationally invariant baby Skyrme model \( V_{II} \).

### 3.4 Initial Conditions

We create suitable initial field configurations with non-trivial baryon number \( B \) by linear superposition of static \( B = 1 \) hedgehog solutions. These initial baby Skyrmion fields will then be used as input for a 2D gradient flow code to search for static \((\omega = 0)\)

\textsuperscript{1}Applying Derrick’s scaling argument to the pseudo-energy functional \( F_\omega(\phi) \) (3.3.5), we can verify that stationary points of \( F_\omega(\phi) \) can only exist if the virial relation (3.3.7) is fulfilled.

\textsuperscript{2}Note that the expressions stated here are in perfect agreement with the formulae given in Section 2 of [60], the only difference being that we already specified our rotation axis to be \( \phi_\infty = (0, 0, 1) \).
minimal-energy soliton solutions in model (3.2.1) with the standard potential term (4.2.2).

Hedgehog fields – fields for which a spatial rotation can be compensated by an isospin rotation – are of the form

$$\phi(x) = (\sin f \cos (B\theta - \chi), \sin f \sin (B\theta - \chi), \cos f),$$  \hspace{1cm} (3.4.1)

where \((r, \theta)\) are polar coordinates in the plane, \(\chi \in [0, 2\pi]\) is a phase shift and \(f(r)\) is a monotonically decreasing radial profile function with boundary conditions \(f(0) = \pi\) and \(f(\infty) = 0\). Substituting (3.4.1) in the energy functional (3.3.3) \((\mu = \sqrt{0.1})\) and solving the associated Euler-Lagrange equation numerically [11], we find for the \(B = 1\) soliton mass \(1.562 \times 4\pi\) and the corresponding moment of inertia (3.3.4) is given by \(7.533 \times 2\pi\). Here, the “units” are chosen to simplify comparison with literature values, in particular with the hedgehog solutions calculated in Ref. [109].

To construct multi-soliton solutions in the baby Skyrme model it is convenient to parametrize the real 3-component unit vector \(\phi\) in terms of a single complex scalar field \(W\) [111] via the stereographic projection

$$\phi = \frac{1}{1 + |W|^2} \left( W + \bar{W}, i(W - \bar{W}), (1 - |W|^2) \right).$$  \hspace{1cm} (3.4.2)

A rotationally-symmetric, complex field \(W(n)\) is given by

$$W(n)(r, \theta) = \tan \left( \frac{f}{2} \right) \exp(-in\theta),$$  \hspace{1cm} (3.4.3)

where \(f\) is the solution of the reduced equation for a charge-\(n\) baby Skyrmion.

Initial field configurations of baryon number \(B = n\) are obtained within this complex field formalism by a linear superposition of \(n\) 1-soliton solutions [45]

$$W(x, y) = \sum_{c=1}^{n} W^{(1)}(x - x_c, y - y_c) \exp(i\chi_c),$$  \hspace{1cm} (3.4.4)

where \(W^{(1)}\) is the complex field of the \(c\)th baby Skyrmion, \((x_c, y_c)\) are the cartesian coordinates of the centre of the \(c\)th baby Skyrmion and \(\chi_c\) are the \(c\)th baby Skyrmion’s respective phase. For most of our numerical simulations we use a circular initial set-up [45] of \(n\) equally spaced baby 1-Skyrmions with relative phase shifts \(\delta\chi = \frac{2\pi}{n}\) for maximal attraction [110].

### 3.5 Static Baby Skyrmion Solutions

To find the stationary points of the energy functional \(M_B\), given in (3.3.3), we solve the associated gradient flow equation numerically. The gradient flow equation is of first order in a fictitious time and is obtained by setting the velocity of the field equal to minus
the variation of the energy functional

\[ \dot{\phi} = -\frac{\delta M_B}{\delta \phi} - \lambda \phi, \quad (3.5.1) \]

where the Lagrange multiplier \( \lambda \) imposes the unit vector constraint \( \phi \cdot \phi = 1 \). The initial configurations are evolved according to the flow equations (3.5.1) on rectangular grids typically containing \((601)^2\) lattice points and with a lattice spacing \( \Delta x = 0.2 \). Some of our relaxation calculations on 1-baby skyrmion solutions are performed on finer grids with \( \Delta x = 0.15 \) and \((401)^2\) grid points for \( V_I \) and \( \Delta x = 0.05 \) with \((801)^2\) grid points for the dihedral potential choices \( V = V_{IV}, V_{V} \). The gradient flow equations are discretized using second-order accurate finite difference approximations for the spatial derivatives and first-order ones for the time derivatives. The Lagrange multiplier \( \lambda \) is explicitly calculated at each timestep of the gradient flow evolution. In order to investigate the critical behaviour (see Section 3.3.1) of isospinning soliton solutions in \( O(2) \) symmetric baby Skyrme models (that is in models with potential terms \( V_I - V_{III} \) and \( V_{IV}, V_{VI} \)), we perform all our full field relaxation calculations for a range of mass values \( \mu \). Whenever necessary, e.g. for large values of the mass parameter \( \mu \), we increase further the resolution of our numerical simulations by working on square lattices with lattice spacing \( \Delta x = 0.02 \) and \((601)^2\) lattice points.

We list in Table 3.1-3.5 the energy and moment of inertia values which we obtained for static, minimum-energy configurations in the seven different baby Skyrme models. Whenever possible, we compare our numerical results with values found in the literature. The associated energy density contour plots are displayed in Fig. 3.1-3.7.
Table 3.1: Multi-skyrmion solutions of the old baby Skyrme model, i.e. with $V = V_I$. For comparison, we include the normalized energies $M_B^{\text{Foster}}/B$ given in [55]. "⋆" labels local energy minima.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$M_B/4\pi$</th>
<th>$M_B/4\pi B$</th>
<th>$M_B^{\text{Foster}}/4\pi B$</th>
<th>$\Lambda_B/2\pi$</th>
<th>Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.564</td>
<td>1.564</td>
<td>1.564</td>
<td>7.556</td>
<td>hedgehog</td>
</tr>
<tr>
<td>2</td>
<td>2.935</td>
<td>1.467</td>
<td>1.468</td>
<td>10.617</td>
<td>one pair</td>
</tr>
<tr>
<td>3</td>
<td>4.423</td>
<td>1.474</td>
<td>1.474</td>
<td>15.389</td>
<td>one triple</td>
</tr>
<tr>
<td>4</td>
<td>5.858</td>
<td>1.464</td>
<td>1.464</td>
<td>20.524</td>
<td>two pairs</td>
</tr>
<tr>
<td>5</td>
<td>7.323</td>
<td>1.464</td>
<td>1.464</td>
<td>25.303</td>
<td>5-chain</td>
</tr>
<tr>
<td>5*</td>
<td>7.363</td>
<td>1.472</td>
<td>1.470</td>
<td>25.917</td>
<td>triple+p</td>
</tr>
<tr>
<td>6</td>
<td>8.778</td>
<td>1.463</td>
<td>1.462</td>
<td>30.742</td>
<td>three pairs</td>
</tr>
<tr>
<td>6*</td>
<td>8.876</td>
<td>1.464</td>
<td>1.462</td>
<td>31.716</td>
<td>6-chain</td>
</tr>
</tbody>
</table>

Old baby Skyrme model ($V = V_I$):

In the old baby Skyrme model ($V = V_I$) with the rescaled mass parameter $\mu$ set to $\sqrt{0.1}$ our 2D gradient flow algorithm reproduces the axially-symmetric $B = 1$ and $B = 2$ soliton solutions [110]. Their minimal energies are found to be $M_1 = 1.564 \times 4\pi$ and $M_2 = 2.935 \times 4\pi$ which is in excellent agreement with the literature values [110, 45, 55]. The corresponding moments of inertia are given by $\Lambda_1 = 7.556 \times 2\pi$ and $\Lambda_2 = 10.617 \times 2\pi$, respectively. For comparison, minimizing the energy functional $M_B$ within the axially-symmetric ansatz (3.4.1) results for the 1-soliton in $M_1 = 1.562 \times 4\pi$ and $\Lambda_1 = 7.533 \times 2\pi$. The analogous calculation for $B = 2$ gives an energy of $M_2 = 2.934 \times 4\pi$ and a moment of inertia $\Lambda_2 = 10.593 \times 2\pi$. As already observed [23] for soliton solutions in the full $(3+1)$-dimensional Skyrme model, the numerical values for the moments of inertia are a lot less accurate than the ones for the solitons’ masses.

However, in the old baby Skyrme model, hedgehog solutions for topological charges $B > 2$ have been shown [110] to be unstable against axial perturbations. The 3-baby Skyrmion configuration forms a chain of three aligned $B = 1$ baby Skyrmions with a phase shift $\delta \chi = \pi$ between neighbouring 1-Skyrmions. Similarly, the energy density distribution for the $B = 4$ soliton is linear, but made up of two radially-symmetric 2-Skyrmions. Our obtained energies $M_3 = 4.423 \times 4\pi$ and $M_4 = 5.858 \times 4\pi$ are within 0.005% agreement with the energy values given in [55]. Relaxing a circular set-up of five $B = 1$ baby Skyrmions we reproduce the 5-chain solution [55] which we confirm to be of lower energy than the 2 + 3 configuration (labelled $5^*$) [110]. For $B = 6$ we find two configurations which can be seen as energy-degenerate within our numerical accuracy: The 6-chain solution (labelled $6^*$) [55] and a configuration with three 2-baby Skyrmions placed at the vertices of an equilateral triangle [110].

New baby Skyrmee model ($V = V_{II}$):

For the sake of completeness, we calculate soliton solutions in the baby Skyrme model (3.2.1) with the two-vacua potential $V_{II}$ [151]. The corresponding energy densities shown
in Fig. 3.2 take the form of rings for \( B \geq 2 \) and the minimal energy configurations can be accurately described within the rotationally-symmetric ansatz (1.6.17). Their radii increase with increasing topological charge \( B \). It was shown \([151]\) that asymptotically the height of the energy density peak (compare Fig. 3.2) approaches a constant value and the total energy of the soliton solution starts to grow linearly with \( B \).

![Figure 3.2: Energy densities of new baby Skyrmions (\( V = V_{\text{III}} \)). Left: Energy density \( \mathcal{E}_{\text{tot}} \) as function of the radial coordinate \( r \) for hedgehog baby Skyrmion solutions (1.6.17) in the new baby Skyrme model with mass parameter \( \mu = \sqrt{0.1} \) and for topological charges \( 1 \leq B \leq 6 \). We calculate the minimal-energy hedgehog solutions of given charge \( B = n \) by solving the second order variational equation for the profile function \( f(r) \) subject to \( f(0) = \pi \) and \( f(\infty) = 0 \) via the collocation method \([11]\). Right: We calculate new baby Skyrmion solitons of minimal energy \( M_B \) (3.3.3) with \( \mu = 0.5 \) and for topological charges \( 1 \leq B \leq 4 \) with a 2D Gradient flow code. Here, we do not assume a hedgehog configuration of the form (3.4.1).](image)

**Holomorphic Baby Skyrme model (\( V = V_{\text{III}} \)):**

As mentioned in Section 3.2 the baby Skyrme model (3.2.1) with the potential choice \( V = V_{\text{III}} \) \([87, 111, 135]\) provides us with an analytic, static, rotationally-symmetric 1-soliton solution of the form \( W(z) = \lambda z \), where the parameter \( \lambda \) – the width of the energy density peak – is given by \( \sqrt{2} \).

Note that we have been unable to construct numerically isospinning soliton solutions for the holomorphic potential \( V_{\text{III}} \). Charge-1 baby Skyrmion solutions seem to be unstable for all angular frequencies \( \omega \).

**\( D_2 \) symmetric Baby Skyrme model (\( V = V_{\text{IV}} \)):**

In the baby Skyrme model with the \( D_2 \) symmetric potential \( V_{\text{IV}} \) \([149]\) the 1-soliton solution is composed of two constituents and can be thought of as an \( 1 \times 1 \) array of two half-Skyrmions. In general, there exist two solution types for higher baryon numbers \( B \):
polygons made up of \( B \) half-baby Skyrmions and lattice configurations composed of arrays of half-Skyrmions. In the case of \( B = 2 \) both solution types are equivalent, in such that the 2-soliton solution can be described equivalently as a \( 2 \times 2 \) square array of half-Skyrmions or a ring of four half Skyrmions (see Fig. 3.3). For baryon number \( B = 3 \) we construct numerically a \( B = 3 \) ring solution which seems to be more energetically favourable than a lattice configuration. Similarly, we compute for \( B = 4 \) a ring solution formed by eight half-baby Skyrmions as the soliton solution of lowest energy. Additionally, we find a stable charge-4 Skyrme configuration of slightly higher energy (labelled \( 4^{*} \)) which can be seen as a bound state of two single \( B = 2 \) lattice cells.

Table 3.2: Same as Table 3.1 but for multi-skyrmion solutions of the baby Skyrme model with potential term \( V = V_{IV} \) [149] included.

<table>
<thead>
<tr>
<th>( B )</th>
<th>( M_{B}/4\pi )</th>
<th>( M_{B}/4\pi B )</th>
<th>( A_{B}/2\pi )</th>
<th>Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.962</td>
<td>1.962</td>
<td>4.628</td>
<td>( D_{2} )</td>
</tr>
<tr>
<td>2</td>
<td>3.476</td>
<td>1.738</td>
<td>7.482</td>
<td>( D_{4} )</td>
</tr>
<tr>
<td>3</td>
<td>5.061</td>
<td>1.687</td>
<td>10.546</td>
<td>( D_{6} )</td>
</tr>
<tr>
<td>4</td>
<td>6.673</td>
<td>1.668</td>
<td>13.559</td>
<td>( D_{8} )</td>
</tr>
</tbody>
</table>

**\( D_{3} \) symmetric Baby Skyrme model (\( V = V_{IV} \)):**

Further, we calculate static minimal energy solutions with charge \( 1 \leq B \leq 5 \) in the \( D_{3} \) symmetric version of the baby Skyrme model introduced in [71]. We verify that the 1-soliton is of triangular shape and can be viewed as composed of three partons arranged at the three vertices (see Fig. 3.4). The minimal energy multi-Skyrmion configurations have been shown to be represented by polyiamonds – configurations made up of equilateral triangles placed with coincident sides. The stable soliton solution for baryon number \( B = 2 \) and \( B = 3 \) are diamond-shaped and triamond-shaped, respectively. For \( B = 4 \) we reproduce the three tetriamond configurations of comparable energies described in [71]. We extend the numerical results of [71] to slightly larger soliton numbers \( B \) and verify that the predicted polyiamonds correspondence continues. For \( B = 5 \) we calculate pentiamonds (see Fig. 3.4) as minimal-energy multi-soliton solutions. The charge-5 soliton...
Table 3.3: Same as Table 3.1 but for multi-skyrmion solutions of the baby Skyrme model with the potential choice $V = V_V [71]$. For comparison, we include (if available) the normalized energies $M^{JSP}_B / 4\pi B$ and $M^{JW}_B / 4\pi B$ given in [71] and [152], respectively.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$M_B / 4\pi$</th>
<th>$M^{JSP}_B / 4\pi B$</th>
<th>$M^{JW}_B / 4\pi B$</th>
<th>$\Lambda_B / 2\pi$</th>
<th>Symmetry</th>
<th>figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.768</td>
<td>2.768</td>
<td>2.768</td>
<td>3.496</td>
<td>$D_3$</td>
<td>3.4 (a)</td>
</tr>
<tr>
<td>2</td>
<td>5.258</td>
<td>2.629</td>
<td>2.630</td>
<td>5.141</td>
<td>$D_2$</td>
<td>3.4 (b)</td>
</tr>
<tr>
<td>3</td>
<td>7.830</td>
<td>2.610</td>
<td>2.612</td>
<td>8.004</td>
<td>$D_1$</td>
<td>3.4 (c)</td>
</tr>
<tr>
<td>4</td>
<td>10.388</td>
<td>2.597</td>
<td>2.599</td>
<td>10.467</td>
<td>$C_2$</td>
<td>3.4 (f)</td>
</tr>
<tr>
<td>4</td>
<td>10.392</td>
<td>2.598</td>
<td>2.599</td>
<td>10.331</td>
<td>$D_1$</td>
<td>3.4 (e)</td>
</tr>
<tr>
<td>4</td>
<td>10.480</td>
<td>2.620</td>
<td>2.622</td>
<td>11.229</td>
<td>$D_3$</td>
<td>3.4 (d)</td>
</tr>
<tr>
<td>5</td>
<td>12.930</td>
<td>2.590</td>
<td>2.590</td>
<td>12.959</td>
<td>$D_2$</td>
<td>3.4 (i)</td>
</tr>
<tr>
<td>5</td>
<td>12.932</td>
<td>2.590</td>
<td>12.959</td>
<td>12.908</td>
<td>$C_1$</td>
<td>3.4 (g)</td>
</tr>
<tr>
<td>5</td>
<td>12.933</td>
<td>2.591</td>
<td>12.908</td>
<td>13.639</td>
<td>$C_1$</td>
<td>3.4 (h)</td>
</tr>
<tr>
<td>5</td>
<td>13.008</td>
<td>2.606</td>
<td>13.639</td>
<td>12.959</td>
<td>$D_2$</td>
<td>3.4 (k)</td>
</tr>
<tr>
<td>5</td>
<td>12.932</td>
<td>2.590</td>
<td>12.959</td>
<td>13.221</td>
<td>$C_1$</td>
<td>3.4 (i)</td>
</tr>
<tr>
<td>5</td>
<td>12.990</td>
<td>2.602</td>
<td>13.221</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

solutions appear to be almost energy-degenerate (see Table 3.3) within our numerical accuracy. Again it seems that the linear arrangement is of lowest energy supporting the conjecture in [71].

Figure 3.4: Energy density contour plots for soliton solution in the $D_3$ symmetric baby Skyrme model ($V = V_V$) with charges $B = 1 - 5$. The mass parameter $\mu$ is chosen (analogous to [71, 152]) to be 1. The corresponding energy values are listed in Table 3.3.
Table 3.4: Same as Table 3.1 but for multi-skyrmion solutions of the baby Skyrme model with the potential choice $V = V_{VI}$ [45]. The values stated here are calculated for mass parameter $\mu = 1$.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$M_B/4\pi$</th>
<th>$M_B/4\pi B$</th>
<th>$\Lambda_B/2\pi$</th>
<th>Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.965</td>
<td>1.965</td>
<td>4.597</td>
<td>hedgehog</td>
</tr>
<tr>
<td>2</td>
<td>3.662</td>
<td>1.831</td>
<td>8.368</td>
<td>pair</td>
</tr>
<tr>
<td>3</td>
<td>5.411</td>
<td>1.804</td>
<td>13.677</td>
<td>triangle</td>
</tr>
<tr>
<td>4</td>
<td>7.100</td>
<td>1.775</td>
<td>17.888</td>
<td>square</td>
</tr>
</tbody>
</table>

Table 3.5: Same as Table 3.1 but for multi-skyrmion solutions of the baby Skyrme model with the potential choice $V = V_{VII}$ [61, 62, 72]. The values stated here are calculated for mass parameter $\mu = 1$ and control parameter $s = 1.5$.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$M_B/4\pi$</th>
<th>$M_B/4\pi B$</th>
<th>$\Lambda_B/2\pi$</th>
<th>Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.485</td>
<td>2.485</td>
<td>3.926</td>
<td>hedgehog</td>
</tr>
<tr>
<td>2</td>
<td>4.836</td>
<td>2.420</td>
<td>5.216</td>
<td>one pair</td>
</tr>
</tbody>
</table>

Eslami Baby Skyrme Model ($V = V_{VI}$):

We display in Fig. 3.5 the spatial energy density distributions of charge one up to four Skyrmion solutions in the double-vacuum model $V = V_{VI}$ [45] for mass values $\mu \leq 1$. The associated energy and moment of inertia values can be found in Table 3.4 for $\mu = 1$. For baryon number $B = 1$ the minimal energy configuration is rotationally-symmetric and can be described by the hedgehog ansatz (3.4.1). Here, increasing the mass parameter only results in more compact, localized objects. The $B = 2$ baby Skyrme soliton solution shows the interesting phenomenon of rotational symmetry breaking. For low mass parameters the charge-2 solution is ring-like and rotationally-symmetric. With increasing mass $\mu$, the finite-energy configuration starts to break up into its weakly bound, constituent charge-1 Skyrmions. Similar, the $B = 3$ Skyrmion evolves from a slightly distorted hedgehog configuration into a configuration described by three individual Skyrmions placed on the vertices of a triangle. Finally the lowest-energy configuration for four Skyrmions is ring-like at low mass values and splits with increasing mass value into four single Skyrmions placed on the vertices of a square.

Hen & Karliner Baby Skyrme model ($V = V_{VII}$):

Another soliton system in which rotational symmetry breaking has been observed and discussed in detail is the baby Skyrme model with the one-parametric family of potential terms $V_{VII}$ [61, 62, 72]. In our formulation the parameters that control the appearance of the soliton solutions are given by the rescaled mass parameter $\mu$ and the real parameter $s$. We show in Fig. 3.6 and Fig. 3.7 energy density contour plots for varying parameter $\mu$ and $s$, respectively. For $B = 1, 2$ we list numerical energy and moment of inertia values for fixed $\mu$ and $s$ in Table 3.5.

When increasing $\mu$ with $s$ fixed, or increasing $s$ with $\mu$ fixed, ring-like $B = n$ multi-
soliton solutions (3.4.1) split gradually into their $n$ constituent charge-1 Skyrmions which start moving further and further apart from each other. This violation of rotation invariance in the static solutions of the baby Skyrme model with potential $V_{VII}$ has been discussed in detail in [61, 62, 72]. It has been observed that the qualitative and quantitative properties of the topological soliton solutions in the one-vacuum potentials $V_{VII}$ strongly depend on the specific value of $s$. The parameter $s$ does not only control the rotational symmetry breaking in the model but also determines whether the interaction between baby Skyrmions is attractive or repulsive [61, 62, 72].
Figure 3.6: Energy density contour plots for $B = 1 - 4$ Skyrmion solutions in the Hen & Karliner baby Skyrme model ($V = V_{VII}$ [61, 62, 72]) for fixed $s$ and a range of mass values $\mu$. As $\mu$ increases, rotational symmetry breaking occurs.

Note that a particularly interesting choice for the parameter $s$ in the one-parameter family of potentials $V_{VII}$ is given by $s \in [1/2, 1]$ [5]. In this case the model supports the existence of baby Skyrme solitons of the compacton\(^1\) type [5], that is they reach the vac-

\(^1\)Recall that soliton configurations – stable, spatially localized, finite-energy solutions of nonlinear field theories – approach their constant vacuum value asymptotically as $|x| \to \infty$, usually exponentially (see for example sine-Gordon kinks, baby Skyrmions). Different to ordinary solitons, compactons [122] are soliton solutions with compact support, that is they do not possess exponential tails and they vanish exactly outside
uum value at a finite distance in a power-like manner. Recently, there has been increased interest [57, 5, 6, 133] in compact baby Skyrmion solutions, particularly in so-called pure, restricted or extreme baby Skyrme models – models (3.2.1) with the quadratic $O(3)$ sigma model term omitted [57]. The resulting models are invariant under area-preserving diffeomorphisms [57] and there exist analytical solutions [57] which saturate the Bogomolny bound (3.2.6), but with the first term$^1$ absent. Recall that exact Bogomolny-Prasad-Sommerfield (BPS) type solutions in generalized $(3 + 1)$-dimensional Skyrme models are also known analytically [7] and might result in a more accurate description of physical nuclei properties [7, 33, 32]. Features that can be reproduced by BPS soliton solutions are for example a linear relationship between the soliton energies (nuclei masses) and its baryon numbers, lower binding energies for higher charge solitons and vanishing force between solitons when sufficiently far separated.

![Energy density contour plots for charge-2 soliton solutions in the Hen & Karliner baby Skyrme model](image)

Figure 3.7: Energy density contour plots for charge-2 soliton solutions in the Hen & Karliner baby Skyrme model ($V = V_{VII}$ [61, 62, 72]) for different values of the control parameter $s$. The mass parameter $\mu$ is chosen to be 0.3. For $s \leq 2$ the minimal energy solutions are rotationally-symmetric and ring-like, whereas increasing the parameter $s$ results in the breaking of the rotational symmetry of the charge-2 configuration: the $B = 2$ baby Skyrmion breaks apart into two individual charge-1 solutions which move further and further away from each other. As a result, stable $B = 2$ solutions cease to exist for larger $s$ values. Similar violation of rotational symmetry and splitting of Skyrmion solutions in their constituents can be observed when varying the parameter $s$ for higher charge solitons [61, 62, 72].

In Fig.3.8 we display the normalized energies per $4\pi B$ as function of the baryon number $B$ for the baby Skyrme configurations, which we assume to be global energy minima in the seven different models briefly reviewed in this section.

### 3.6 Numerical Results for Isospinning Baby Skyrmions

In this section, we present the results of our 2D energy minimization simulations of isospinning soliton solutions in $O(2)$-symmetric versions of the baby Skyrme model given

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$^1$The first term in (3.2.6) follows from the $O(3)$ sigma model part of the full baby Skyrme model (3.2.1).
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Figure 3.8: Normalized energies $M_B/4\pi B$ versus baryon number $B$ for baby Skyrme solitons with the potential choices given in (3.2.4). We plot the energy values of those configurations which we believe to be global energy minima.

by the potential choices $V_I, V_{II}, V_{III}$. To find the stationary points of the total energy functional $H = M_B + K^2/(2\Lambda_B)$ for a fixed angular momentum $K$ and for a given topological charge $B$, we perform a 2D gradient flow evolution in analogy to Eqn. (3.5.1) starting with well-chosen initial configurations. We use the static configurations obtained in the previous section as our start configurations for vanishing angular momentum ($K = 0$). We then increase $K$ in a stepwise manner using previously calculated configurations as start configurations for the next value of $K$. To check our computations we also performed gradient flow calculations to minimize the pseudo-energy functional $F_\omega (3.3.5)$ for a fixed rotation frequency $\omega$. Note that all simulation parameters are chosen as stated in Section 3.5. In particular, we carry out most of our simulations on relatively large grids containing $(601)^2$ lattice points with a lattice spacing $\Delta x = 0.2$ in order to capture the asymptotic behaviour of our isospinning soliton solutions.

3.6.1 Isospinning Soliton Solutions in the old Baby Skyrme Model

The majority of our computations of isospinning baby Skyrmions with charges $B = 1 - 6$ have been performed with the rescaled mass parameter $\mu$ set to $\sqrt{0.1}$. In this case there is a maximum angular frequency $\omega_{\text{crit}} = \omega_2 = \mu = \sqrt{0.1}$ beyond which no stable isospinning baby Skyrmion solution exists.

- $B = 1, 2$: For the rotationally-symmetric 1- and 2-soliton solutions we verify that for the standard mass value $\mu = \sqrt{0.1}$ our 2D gradient flow evolution reproduces the behaviour expected from an isospinning hedgehog soliton solution [109]. Recall that for fields of the hedgehog type (3.4.1) spatial rotations and isorotations are equivalent and consequently $K$ can be interpreted as the total isospin or equivalently as the total spin of the baby Skyrme field $\phi$. We plot in Fig. 3.9 and in Fig. 3.10 the dependences of the spinning $B = 1, 2$ baby Skyrmion’s mass $E_{\text{tot}}$ and its moment of inertia $I$ on its angular frequency $\omega$ and its isospin $K$. We confirm that we obtain
3.6. NUMERICAL RESULTS FOR ISOSPINNING BABY SKYRMIONS

Figure 3.9: Isospinning $B = 1$ soliton solution in the old baby Skyrme model ($\mu = \sqrt{0.1}$). A start configuration (3.4.1) is numerically minimized using 2D gradient flow on a 401 × 401 grid with a lattice spacing of $\Delta x = 0.15$ and a time step size $\Delta t = 0.005$. To check our numerics we explicitly verify that minimization of the pseudo-energy $F_\omega$ (for fixed angular frequency $\omega$) and minimization of the Hamiltonian $H$ (for fixed isospin $K$) reproduce the same curves. Additionally, we compare our results with those we expect for an isospinning, axially-symmetric deforming 1-soliton solution (that is we solve Eq. (3.6.1) for the radial profile function $f(r)$ with boundary conditions $f(0) = \pi$ and $f(\infty) = 0$). Our results agree well with those given in [109] for an (iso)spinning 1-soliton deforming within an hedgehog ansatz (3.4.1).

The same energy and moment of inertia curves when we substitute the hedgehog field (3.4.1) in the pseudo-energy functional $F_\omega$ (3.3.5) or in the Hamiltonian $H$ and solve the associated variational equation [109]

\[
\left( r + \left( \frac{B^2}{r} - \omega^2 r \right) \sin^2 f \right) f'' + \left( 1 - \left( \omega^2 + \frac{B^2}{r^2} \right) \sin^2 f + \left( \frac{B^2}{r} - \omega^2 r \right) f' \sin f \cos f \right) f' = 0,
\]

for the radial profile function $f(r)$ with boundary conditions $f(0) = \pi$ and $f(\infty) = 0$. $E_{\text{tot}}(\omega)$ and $\Lambda(\omega)$ grow rapidly with $\omega$ and diverge at $\omega_{\text{crit}} = \omega_1 = \sqrt{0.1}$. We can see that the total energy $E_{\text{tot}}(K)$ and the moment of inertia $\Lambda(K)$ increase linearly with $K$ for $K$ sufficiently large and only depend quadratically on $K$ for slowly rotating
solutions. Consequently, the rigid body approximation is only a good approximation for small values of $K$, in particular $K \leq 1.05 \times 4\pi$ for $B = 1$ and $K \leq 1.51 \times 4\pi$ for $B = 2$. Close to the cutoff ($\omega \approx 0.31, K \approx 2.49 \times 4\pi$) the energy values given by the rigid body formula are roughly 10% larger than those for the non-rigidly rotating 1-soliton solution. Similarly, for $B = 2$ the rigid body approximation predicts an energy value at $\omega \approx 0.31, K \approx 2.49 \times 4\pi$, which is approximately 3% larger than the one calculated for the deforming charge-2 solution. The energy density contour plots in Fig. 3.11 show the deformation of the charge-1 and charg-2 solitons as function of isospin $K$ and rotation frequency $\omega$. As shown in Fig. 3.11, the isospinning charge-2 configuration preserves its rotational symmetry for all frequency values $\omega < 0.31$ and breaks spontaneously its rotational symmetry at $\omega \approx \omega_2$.

In the $(3+1)$-dimensional Skyrme-Faddeev model it was found [60] that stable, isospinning soliton solutions only exist for angular frequencies $\omega \leq \min \{1, \mu\}$ and that they can be destabilized by nonlinear velocity terms in the field equations far before the upper limit $\omega_2 = \mu$ is reached. Similarly, we find in the baby Skyrme model that for $\mu > 1$ isospinning soliton solutions become unstable far before reach-
To simplify comparison with [109], the isospin \( K \) is given in units of \( 4\pi \), i.e. we define \( k = K / 4\pi \).

For the mass range \( \mu \in (0, 1] \) the total energy, \( E_{\text{tot}} \), and the moment of inertia, \( \Lambda \), diverge at \( \omega_2 = \mu \), whereas for mass values \( \mu > 1 \) we find that they take finite values at the critical frequency \( \omega_1 = 1 \). It is interesting to compare this pattern of critical behaviour with the one calculated when only considering rotationally-symmetric deformations. As shown in Appendix A, the so obtained critical frequencies for charge-1 and -2 solitons inadvertently suggest the existence of stable isospinning baby Skyrme solutions with mass \( \mu > 1 \) for angular frequencies \( \omega > \omega_1 = 1 \). This result is simply an artefact of the hedgehog approximation (3.4.1).

Furthermore, for larger mass values \( \mu > \sqrt{0.1} \) our full 2-dimensional relaxation calculations show that at some third critical angular frequency value, \( \omega_3 \), the isospinning charge-2 soliton solutions become unstable to break up into their charge-1 components which start moving apart from each other. Generally speak-
Figure 3.12: Total energy $E_{\text{tot}}$ and isospin $K$ for $B = 1$ soliton solutions in the standard baby Skyrme model as function of angular frequency $\omega$. The mass parameter takes the values $\mu = 0.5, 1, 1.5, 2$.

Figure 3.13: Total energy $E_{\text{tot}}$ and isospin for $B = 4$ soliton solutions in the old baby Skyrme model as a function of angular frequency $\omega$ and for a range of mass values $\mu$.

Increasing the mass value $\mu$ results in increasingly larger rotational symmetry breaking at a given angular momentum $K$ (see Fig. 3.14). Rotationally-symmetric Skyrme configurations are found to be of significantly higher energy and turn out to be unstable for $\mu$ and $K$ sufficiently large. The corresponding energy density contour plots for a range of values of $\mu$ are shown in Fig. 3.15. We observe that for increasing mass parameter $\mu$, the break-up into individual charge-1 constituents occurs at increasingly higher values of $\omega_3 \leq \omega_1 = 1$ (compare middle plot in Fig. 3.14 and the corresponding break-up frequency values listed in the table). Recall that isospinning baby Skymion solutions do not minimize the total energy functional $E_{\text{tot}}(\omega)$ for fixed angular frequency $\omega$ which explains the negative deviations $\Delta E_{\text{tot}}$ from the rotationally-symmetric deforming charge-2 baby Skyrmon configuration shown in the middle plot of Fig. 3.14.

We show in Fig. 3.16 the deviations from the rigid body, plotted against the angular momentum $K$, for both the charge-1 and charge-2 solutions. As the mass value $\mu$ increases, the rigid body approximation provides more accurate results for the isospinning solutions of the model.

- $B = 3$: For the standard mass value ($\mu = \sqrt{0.1}$) the linear 3-soliton splits into three weakly bound, linearly arranged $B = 1$ hedgehog solitons when isospinning about the $z$–axis, as shown by the energy density contour plots in Fig. 3.11. The energy curves given in Fig. 3.17 show the linear dependence of the total energy
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\[ \Delta E_{\text{tot}} = \frac{(E_{\text{hedgehog}} - E_{\text{tot}})}{E_{\text{tot}}} \]

Figure 3.14: The deviation \( \Delta E_{\text{tot}} = \left( E_{\text{hedgehog}} - E_{\text{tot}} \right) / E_{\text{tot}} \) from the rotationally-symmetric deforming charge-2 baby Skyrmion configuration (3.6.1) as a function of isospin \( K \) (left) and angular frequency \( \omega \) (right) for a range of mass values \( \mu \). Approximate values for the break-up frequencies \( \omega_3 \) – the frequencies at which the charge-2 solutions start to split into its charge-1 constituents – are listed in the table. We verified that for this mass range the isospinning charge-1 baby Skyrmion solution does not deviate significantly from a rotationally-symmetric deforming \( B = 1 \) Skyrme configuration.

\[
\begin{array}{c|ccccccc}
\mu & 0.1 & 0.5 & 0.8 & 1.0 & 1.5 & 2.0 \\
\omega_3 & 0.31 & 0.41 & 0.48 & 0.49 & 0.50 & 0.53 \\
\end{array}
\]

\( E_{\text{tot}}(K) \) on the isospin \( K \) for \( K > 2.0 \times 4\pi \). For \( K < 2.0 \times 4\pi \) deformations due to centrifugal effects can be neglected and the isospinning solution can be essentially seen as a rigid rotor. We find that with increasing mass value \( \mu \) the isospinning \( B = 3 \) soliton becomes increasingly stable to decay into its constituents, i.e. the break-up frequency \( \omega_3 \) takes larger values.

- **\( B = 4 \):** The energy densities for the isospinning 4-baby Skyrme soliton are plotted for \( \mu = \sqrt{0.1} \) in the last row of Fig. 3.11. The pair of two weakly bound 2-solitons breaks into 4 single linearly arranged 1-solitons. The corresponding moment of inertia curves \( \Lambda(\omega) \) and energy curves \( E_{\text{tot}}(\omega), E_{\text{tot}}(K) \) can be found in Fig. 3.17. As \( \mu \) increases the splitting into individual charge-1 constituents happens at increasingly higher rotational frequency values.

- **\( B = 5 \):** With the mass parameter \( \mu \) set to its standard value, the two different \( B = 5 \) baby Skyrme configurations – 5-chain solution and weakly bound 3 + 2 solution – split into 5 almost undistorted 1-solitons, see Fig. 3.18. As already seen for the lower charge baby Skyrmion solutions, the deformations preserve the symmetries of the static, non-spinning Skyrmion solutions. Both Skyrme configurations are of very similiar energy and show as function of \( \omega \) and \( K \) almost identical energy curves, see Fig. 3.19.

- **\( B = 6 \):** The two (within the limits of our numerical accuracy) energy-degenerate 6-soliton configurations with \( \mu = \sqrt{0.1} \) – 6-chain solution and weakly bound 2 + 2 + 2 solution – break into 6 single 1-baby Skyrmions. As above, the deformations do not
break the symmetries of the non-spinning solutions, compare Fig. 3.18. The energy degeneracy is not removed by isospinning the charge 6-solitons, see the energy curves given in Fig. 3.20.

We display in Fig. 3.21 as a function of isospin $K$ the mean charge radii of $B = 1 - 6$ baby Skyrmions (with $\mu = \sqrt{0.1}$) defined as the square root of the second moment of the
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Figure 3.16: The deviation \( \Delta E_{\text{tot}} = \left( E_{\text{Rigid}} - E_{\text{tot}} \right) / E_{\text{Rigid}} \) from the rigid body approximation for charge-1 and charge-2 baby Skyrmions as a function of isospin \( K \) for various rescaled mass values \( \mu \).

Figure 3.17: Total energy \( E_{\text{tot}} \) as a function of angular frequency \( \omega \) and as a function of isospin \( K \) for solitons in the old baby Skyrme model \( (V = V_I) \) with baryon number \( B = 3, 4 \).

topological charge density \( B(x) \) (3.2.2)

\[
\langle r^2 \rangle = \frac{\int r^2 B(x) \, d^2 x}{\int B(x) \, d^2 x}.
\]  

(3.6.2)

The changes in the baby Skyrmions’ shapes are reflected by the changes in slopes of the mean charge radius curves in Fig. 3.21. We observe that for isospin values \( K > 1.04 \times 4\pi \) the radius \( \langle r^2 \rangle^{1/2} \) of the charge-1 solution grows approximately linear with \( K \). For \( B = 2 \) the linear growth starts at higher angular momenta \( (K \approx 1.51 \times 4\pi) \). These changes in slope are related with the rigid-body approximation only being a valid simplification for slowly isospinning Skyrme configurations, whereas for higher isospin values deformations due to centrifugal effects become increasingly important. Higher charge solutions can change their slopes several times. For example, the radius curve for the 5-chain solution can be divided by its different slopes in three different regimes: For isospin values \( K \leq 2.28 \times 4\pi \) the charge-5 chain is made up of two \( B = 2 \) tori weakly bound together by a single \( B = 1 \) baby Skyrmion. In the isospin range \( 2.28 \times 4\pi < K \leq 7 \times 4\pi \) the chain is formed by
Figure 3.18: Energy density contour plots for isospinning multi-solitons in the standard baby Skyrme model with charges $B = 5, 6$ and mass parameter $\mu$ set to $\sqrt{0.1}$. Again we define $k = K/4\pi$. Note that the tiny deviations from the linear alignment of the chain-like $B = 5, 6^*$ baby Skyrme configurations are purely numerical effects.

Figure 3.19: Total energy $E_{\text{tot}}$ as a function of angular frequency $\omega$ and as a function of isospin $K$ for solitons in the old baby Skyrme model ($V = V_I$) with baryon number $B = 5$.

the two tori moving further apart and the single $B = 1$ constituent. Further increase of $K$ results into 5 individual, linearly aligned $B = 1$ Skyrmions.
3.6. NUMERICAL RESULTS FOR ISOSPINNING BABY SKYRMIONS

Figure 3.20: Total energy $E_{\text{tot}}$ as a function of angular frequency $\omega$ and as a function of isospin $K$ for solitons in the old baby Skyrme model ($V = V_I$) with baryon number $B = 6$.

Figure 3.21: Mean Charge radii $<\hat{r}^2>^{1/2}$ (3.6.2) for baby Skyrme solitons of topological charges $1 \leq B \leq 6$ as function of isospin $K$ and for mass value $\mu = \sqrt{0.1}$.

3.6.2 Isospinning Soliton Solutions in the New Baby Skyrme Model

We display in Fig. 3.22 for a range of rescaled mass values $\mu$, the total energies $E_{\text{tot}}$ of $B = 1$ Skyrmion solutions in the new baby Skyrme model [78, 151] as functions of angular frequency $\omega$ and isospin $K$. In addition, the angular momentum $K$ is plotted as a function of $\omega$. Clearly, we observe the pattern of critical behaviour described in Section 3.3.1. If the mass parameter $\mu$ is chosen to be smaller than or equal to the first critical value $\omega_1 = 1$, the deformed potential (3.3.6) vanishes at $\omega_2 = \sqrt{2}\mu$ (violating the virial relation (3.3.7)) and the Skyrmion’s energy $E_{\text{tot}}$ as well as its angular momentum $K$ diverge. When setting $\mu > \omega_1 = 1$, we verify that the pseudo-energy functional $F_\omega$ (3.3.5) is again unbounded from below and hence, isospinning baby Skyrmion solutions cease to exist for angular frequencies $\omega > \omega_1$. Again, we observe that the Skyrmion’s total energy $E_{\text{tot}}$ and its isospin $K$ take finite values at the critical angular frequency $\omega_1$. Remarkably, the rotational symmetry of the Skyrmion configurations becomes broken as $\omega$ approaches the critical value $\omega_1 = 1$ (see energy density contour plots in Fig. 3.23).

We confirm the same pattern of critical behaviour for higher charge $B$ solutions. As displayed in Fig. 3.24 for the mass parameter $\mu = 1.5$ and topological charges $B = 1 - 4$ isospinning soliton solutions are unstable for rotation frequencies $\omega > \omega_1 = 1$. Again, we
observe that both the Skyrmions’ total energies $E_{\text{tot}}$ and their isospin $K$ remain finite up to the critical frequency value $\omega_1$. The associated energy density contour plots presented in Fig. 3.25 and Fig. 3.26 show the violation of the rotation invariance in the new baby Skyrme model for charge-2-4 solitons isospinning with frequencies close to the critical value $\omega_1 = 1$.

Note that for mass values $\mu \leq \omega_1 = 1$ the isospinning baby Skyrmion solutions in the new baby Skyrme model are well approximated by isospinning hedgehog solutions. Substituting the charge $B$ hedgehog ansatz (3.4.1) in $F_\omega(\phi)$ (3.3.5) and minimizing the pseudo-energy with respect to the radial profile function $f(r)$ results for the $B = 1$ soliton solutions in the energy curves and critical behaviour shown in Fig. 3.27. The total energies $E_{\text{tot}}$ diverge at $\omega_2 = \sqrt{2}\mu$ and isospinning solutions can only persist for frequencies $\omega < \omega_2$. To further check our numerics, we verify in Fig. 3.28 and Fig. 3.29 that our fully two-dimensional relaxation algorithm reproduces for mass parameters $\mu \leq 1$ the energy, isospin and moment of inertia curves expected from an isospinning, radially-symmetric soliton configuration. When comparing our results with the rigid-body approximation (see Fig. 3.29 (b)), we confirm that the energy values for rigidly isospinning baby Skyrmions in the model with potential $V_{\mu}$ can be significantly larger than the ones for the deforming soliton solutions.

Unlike our observations of isospinning soliton solutions in the old baby Skyrme model [110, 109], we do not find that the charge $B$ solutions in the new baby Skyrme model decay into $B$ charge one solitons at some critical value of $\omega$. The $E_{\text{tot}}(K)$ curves shown in Fig. 3.24 and the pseudo-energy functional $F_\omega$ (3.3.5) per unit charge, plotted in Fig. 3.30 as a function of the rotation frequency $\omega$ for $B = 1 - 4$ baby Skyrmions, do not exhibit any crossings. This is different from the isospinning $B \geq 2$ multiskyrmion solutions in the old baby Skyrme model for which we observe that at some third critical frequency value $\omega_3$ the pseudo-energy $F_\omega/B$ turns out to be comparable to the one of $B$ single charge-1 baby Skyrmion solutions, so that the decay of multisoliton solutions into their charge-1 components is energetically possible.
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$$\downarrow \mu$$

$$\mu = 0.5$$

$$\omega = 0, k = 0$$  $$\omega = 0.3, k = 0.78$$  $$\omega = 0.5, k = 1.54$$  $$\omega = 0.68, k = 4.64$$

$$\mu = 1.0$$

$$\omega = 0, k = 0$$  $$\omega = 0.4, k = 0.75$$  $$\omega = 0.8, k = 1.72$$  $$\omega = 1.0, k = 2.66$$

$$\mu = 1.5$$

$$\omega = 0, k = 0$$  $$\omega = 0.4, k = 0.69$$  $$\omega = 0.8, k = 1.54$$  $$\omega = 0.99, k = 2.23$$

$$\mu = 2.0$$

$$\omega = 0, k = 0$$  $$\omega = 0.4, k = 0.67$$  $$\omega = 0.8, k = 1.5$$  $$\omega = 0.96, k = 2.02$$

Figure 3.23: Energy density contour plots of isospinning $B = 1$ soliton solutions in the new baby Skyrme model $V_{II}$ for a range of mass values. The corresponding energy curves and moment of inertia curves are given in Fig. 3.22. Note that the rotational invariance of the Skyrmion configurations becomes slightly broken when approaching the critical frequency value $\omega_1 = 1$.

3.6.3 Isospinning Soliton Solutions in the Hen & Karliner Baby Skyrme Model

We display in Fig. 3.32 the energy density contour plots of isospinning $B = 2 - 4$ baby Skyrmions in the model with potential term $V_{III}$ [61, 62, 72] included. The majority of our calculations have been performed with the control parameter $s = 0.5$, so that the model allows for compact soliton solutions.

We find that $B = n$ multi-Skyrmion solutions split at some third critical frequency
Figure 3.24: Total energy $E_{\text{tot}}$ and isospin for $B = 1 - 4$ baby Skyrmions with the two-vacua potential $V_{II}$ as function of angular frequency $\omega$. The mass parameter is set to $\mu = 1.5$

\[
\downarrow \mu \\
\mu = 0.5 \\
\mu = 1.5
\]

\[
\omega = 0, k = 0 \\
\omega = 0.3, k = 1.28 \\
\omega = 0.5, k = 2.57 \\
\omega = 0.65, k = 5.32
\]

\[
\omega = 0, k = 0 \\
\omega = 0.3, k = 0.75 \\
\omega = 0.8, k = 2.33 \\
\omega = 1.0, k = 3.55
\]

Figure 3.25: Contour plots of the energy density of isospinning $B = 2$ soliton solutions of the baby Skyrme model with potential term $V_{II}$ included. The rescaled mass parameter $\mu$ is chosen to be 0.5 and 1.5, respectively.

value $\omega_3 < \omega_1 = 1$ into its charge-1 components that start moving infinitely apart from each other. Note that for baryon numbers $B = 3$ and $B = 4$, the isospinning solutions seem to pass through configurations that strongly resembles the ones found in the standard baby Skyrme model ($V = V_I$).

Again, we find that there exist instabilities of two different types. For mass values $\mu > 1$ isospinning solutions cease to exist for angular frequencies $\omega > \omega_1 = 1$ due to the destabilization by nonlinear velocity terms [60]. If $\mu < 1$ a second instability is found which is related with the condition of positiveness of the effective potential (3.3.6). We show in Fig. 3.31 the resulting pattern of critical behaviour for isospinning, compact ($s = 0.5$) charge-1 soliton solutions for various mass values $\mu$. 

3.7 Conclusions

We have performed full two-dimensional numerical relaxations of isospinning soliton solutions in various $O(2)$ symmetric versions of baby Skyrme models. We find that completely analogous to the recent work on internally rotating soliton solutions [60, 19] in the 3D Skyrme-Faddeev model [46, 47] there exist two types of critical frequencies. If the mass parameter $\mu$ is smaller than 1 the isospinning configurations become unstable at the critical value $\omega_2$ for which the virial relation (3.3.7) becomes violated. If the mass parameter $\mu$ is taken to be larger than 1, the energy of the isospinning solution be-
Figure 3.28: Isospinning $B = 1$ soliton solutions in the double vacuum baby Skyrme model $V_{II}$ for mass parameters $\mu = 0.2, 0.4, 0.6, 0.8$. We verify that the total energy of the isospinning Skyrme configurations can be well approximated by isospinning hedgehog solutions. Here, “1D” denotes the hedgehog results and “2D” labels the ones of our two-dimensional relaxation algorithm.

Figure 3.29: (Iso)spinning $B = 1$ soliton solution in the new baby Skyrme model ($\mu = 0.5$). We minimize a start configuration (3.4.1) numerically using 2D gradient flow on a $601 \times 601$ grid with a lattice spacing of $\Delta x = 0.2$ and a time step size $\Delta t = 0.0005$. To check our numerics, we compare our results with those we expect for an (iso)spinning, rotationally-symmetric deforming 1-soliton solution.

comes unbounded from below as $\omega$ increases above 1. Hence, stable isospinning soliton solutions can only exist for frequencies $\omega \leq \min(\omega_2, 1)$.

Summarized, we found that the critical behaviour and the qualitative shape of isospinning baby Skyrme configurations strongly depend on the specific potential choice:
3.7. CONCLUSIONS

Figure 3.30: Pseudo-energy $F_\omega (3.3.5)$ per unit charge as a function of angular frequency $\omega$ for isospinning Skyrmion solutions of topological charge $B = 1 – 4$ in the new baby Skyrme model. The rescaled mass parameter $\mu$ is chosen to be 1.5.

Figure 3.31: Isospinning, compact ($s = 0.5$) charge-1 solitons in the Hen & Karliner Baby Skyrme model ($V = V_{\text{VII}}$) for various mass values $\mu$. We display the total energy $E_{\text{tot}}$ and the isospin $K$ as a function of angular frequency $\omega$. Furthermore, the energy is plotted as a function of $\omega$. We can confirm the same pattern of critical behaviour for isospinning, higher charge soliton solutions.

- **Old Baby Skyrme Model:**
  Isospinning multi-Skyrmion solutions can break up into their constituent charge-1 Skyrmions before reaching the upper frequency limit $\omega \leq \min(\mu, 1)$. For $\mu$ sufficiently large there exists a third critical angular frequency value $\omega_3$ at which the total energy per unit charge is larger than the one of a single baby Skyrmion and the break-up into charge-1 baby Skyrmions is energetically favorable.

  This pattern of critical behaviour has been unobserved in previous work [109] on isospinning charge-1 and -2 baby Skyrme solitons. This is because the authors did not take into account deformations that break rotational symmetry, and only investigated relatively low mass values. Our numerical calculations clearly show that stable, rotationally-symmetric Skyme solitons with mass parameter $\mu > 1$ for angular frequencies $\omega > \omega_1 = 1$ are simply an artefact of the hedgehog approximation. Even for lower mass values ($\mu < 1$) we find that the hedgehog ansatz can be a very poor approximation. Here the charge-2 baby Skyrmion solution can break spontaneously its rotational symmetry when isospinning.
Further, we observe that for the conventional mass parameter choice \( \mu = \sqrt{0.1} \) the symmetries of the static, non-spinning soliton solutions are not significantly modified when isospin is added. This is in contrast to recent results on internally rotating soliton solutions [60, 19] in the 3D Skyrme-Faddeev model, where it was found that the model allows for transmutations, formation of new solution types and a rearrangement of the spectrum of minimal-energy solitons in a given topological sector when isospin is added.

However, although the soliton’s geometrical shape is largely independent of the rotation frequency \( \omega \), the soliton’s size increases monotonically with increasing \( \omega \). In general, the rigid body formula predicts for the solutions considered here total energies which are for large angular momenta roughly 1 – 10% larger than those obtained for the deformed, isospinning solutions. Naturally, the accuracy of the rigid rotator approximation improves with increasing soliton mass and topological charge \( B \).

• **New Baby Skyrme Model:**
  Isospinning charge \( B \) soliton solutions can be constructed for angular frequencies \( \omega \leq \min(\sqrt{2}\mu, 1) \). For mass values \( \mu \leq 1 \) the qualitative shape of the soliton configurations is unaffected by the addition of isospin: The minimal-energy configurations are found to form rings of increasingly larger radius as \( \omega \) increases up to its critical value \( \omega_2 = \sqrt{2}\mu \). However, for mass parameters \( \mu > 1 \) we find numerically that the rotation invariance of the isospinning baby Skyrme configurations becomes broken when \( \omega \) approaches its maximal value \( \omega_1 = 1 \). Different to the solutions in the old baby Skyrme model, the ring-like baby Skyrmions in the double vacuum model do not break up into its individual charge-1 constituents.

• **Hen & Karliner Baby Skyrme Model:**
  The geometrical shape and the critical behaviour of isospinning charge \( B \) solutions depend (with our parametrization choice) strongly on the mass value \( \mu \) and the control parameter \( s \). As the angular frequency \( \omega \) increases rotationally invariant multi-Skyrmion configurations turn out to be increasingly unstable with respect to nonaxial perturbations. The rotation invariance becomes broken and the solutions start to resemble the baby Skyrmion solutions found in the conventional model \( (V = V_I) \). Further increase of the angular frequency results for charge \( B \) solutions in a break-up into \( B \) charge-1 solitons which move apart from each other. Again, the observed second critical frequency is related with the violation of the virial relation (3.3.7).

• **Holomorphic Model:**
  Our numerical simulations suggest that there do not exist stable, isospinning soliton solutions in the holomorphic model. This agrees with the independent findings presented in the recent preprint [59].
Further lines of investigation could compute the semi-classical quantum energy spectrum of the isospinning baby Skyrmion solutions presented in this chapter. Calculations of the rotational energy spectrum [57] that take into account centrifugal deformations have been performed in the BPS limit (\(\mu \to \infty\)) of the conventional baby Skyrme model (\(V = V_f\)). When assuming rotationally-symmetric deformations the soliton’s profile function can be calculated analytically and the quantum energy spectrum can be computed using the Bohr-Sommerfeld quantization condition. Here, the quantum states allowed by our numerically calculated (higher mass) configurations are expected to differ from the ones [57] obtained when neglecting any nonaxial deformations.

Furthermore, a more systematic investigation should take into account both rotational and isosrotational degrees of freedom and should not be restricted only to radial deformations. Naturally, such an analysis would be particularly interesting for the Skyrme model [129, 131, 127] of nucleons. However, even in the (2 + 1)-dimensional versions of the Skyrme model such calculations are numerical expensive and should be the subject of future research.

Finally, it would be interesting to study Skyrmion-Skyrmion scattering processes with the Hen & Karliner one-vacuum potential term \(V_{VII}\) included. Scattering processes with different relative isospin orientations have been explored within the “dipole picture” in the full (3 + 1)-dimensional Skyrme model [95, 125] and in the standard baby Skyrme model [109, 110, 114]. For example the 90° Skyrmion-Skyrmion scattering in the most attractive channel configuration and the toroidal shape of the \(B = 2\) Skyrmion can be understood [95] when approximating a Skyrmion by a triad of dipoles. For compacton-type soliton solutions – which e.g. can be constructed in the model \(V_{VII}\) for all parameter \(s \in [1/2, 1]\) [5] – there does not exist a valid description in terms of pairs of orthogonal dipoles [109, 110, 114] to analyze the symmetries of scattering states. The compact soliton solutions reach their vacuum value at a finite distance and show a power-like falloff instead of the exponential one observed for conventional baby Skyrmion solutions. Here, it might be worth numerically studying the collision behaviour of compact baby Skyrmions, which might be quite different to the one observed for non-compact Skyrmion solutions.

**Note added**

As mentioned above, similar numerical results to those presented here on the behaviour of isospinning soliton solutions in the old and new baby Skyrme model were obtained independently by Alexey Halavanau and Yasha Shnir in a very recent preprint [59], which appeared shortly after ours [20]. The authors use a rescaled version of the conventional baby Skyrme Lagrangian (3.2.1) [110, 109]. The kinetic term differs from our notation by a factor of 2. In particular, the mass parameter \(\mu\) used in our calculations is related to the one (\(\mu_{HS}\)) used in [59] by \(\mu^2 = \mu_{HS}^2/4\). We observe for isospinning soliton solutions in the conventional baby Skyrme model the same pattern of critical be-
haviour (see Fig. 1,2 in [59]) and our results can be seen as complementary. Differences are the investigated mass range $\mu$ and the choice of initial conditions. Whereas we relax the absolute minima (especially non-rotationally symmetric configurations for $B > 2$) at $\omega = 0$ to find solutions for nonzero angular frequencies $\omega$, the authors in [59] choose rotationally-invariant ansätze as their start configurations.
Figure 3.32: Energy density contours for isospinning solitons in the Hen & Karliner model. The $B = n$ multi-Skyrmions split into $n$ single $B = 1$ Skyrmions moving infinitely apart from each other.
4.1 Introduction

Hopf soliton solutions arise as topological solitons in the Skyrme-Faddeev model [46, 47] – a non-linear $O(3)$ sigma model in $(3 + 1)$-dimensional space-time whose Lagrangian is modified by an additional term quartic in its field derivatives. Extensive numerical simulations [48, 58, 27, 26, 63, 64, 136] of the highly non-linear classical field equations have revealed a very rich spectrum of solutions which are classified by their integer-valued Hopf charge. For Hopf charges up to 16 a variety of static, stable minimum-energy solutions with the structure of closed strings, twisted tori, linked loops and knots have been identified. These string-like solitons might be candidates to model glueball configurations [49] in QCD or may arise in two-component Bose condensates [14, 70].

In this chapter we investigate the effect of isospin on classical Hopf soliton solutions. In analogy to the conventional $SU(2)$ Skyrme model we use the collective coordinate method [9] to construct Hopf solitons of well-defined, non-zero isospin: We parametrize the isorotational zero-modes of a Hopf configuration by collective coordinates which are then taken to be time-dependent. This gives rise to additional dynamical terms in the Hamiltonian which can then be quantized following semiclassical quantization rules. A simplification which is often made in the literature [9, 35, 94, 76] is to apply a simple adiabatic approximation to the (iso)rotational zero modes of the soliton by assuming that the soliton’s shape is rotational frequency independent. The limitations of this rigid body approach were pointed out by several authors [36, 21, 66, 3]. In this chapter we perform numerical computations of isospinning Hopf solitons with Hopf charges up to 8 in the full 3-dimensional classical field theory without applying the rigid body approximation and without imposing symmetry constraints on the isospinning Hopf configurations. It turns out that the Skyrme-Faddeev model with its rich topology of minimum-energy solutions, often of comparable energy, allows for “transmutations” when isospin is added and even for the formation of new, metastable Hopf solutions.

This chapter is organized as follows. In Section 4.2 we briefly review the Skyrme-Faddeev model and describe how Hopf solitons acquire isospin within the collective co-
ordinate approach. Then, in Section 4.3 we set up appropriate initial conditions which are used in Section 4.4 to compute Hopf configurations of zero isospin. The effect of isospin on these Hopf soliton solutions is studied in Section 4.5. Section 4.6 discusses the frequency range for which stable isospinning Hopf soliton solutions can be constructed. We conclude with Section 4.7.

Note that the work presented in this chapter has been done in collaboration with Richard Battye and has been published in [19].

### 4.2 Classically Isospinning Hopf Solitons

The Lagrangian density of the Skyrme-Faddeev model [46] in (3+1)-dimensions takes in terms of the real three-component unit vector \( \phi = (\phi_1, \phi_2, \phi_3) \) the form

\[
L = \frac{1}{32\pi^2} \sqrt{2} \left( \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} \left( \partial_\mu \phi \times \partial^\mu \phi \right)^2 - V(\phi) \right). \tag{4.2.1}
\]

To stabilize isospinning Hopf configurations we modified in (4.2.1) the usual Skyrme-Faddeev model by adding a mass term \( V(\phi) \) to the \( O(3) \) sigma model and Skyrme term. Here we will consider the following \( SO(3) \) symmetry breaking potentials

\[
V(\phi) = \begin{cases} 
V_I(\phi) &= 2\mu^2 (1 - \phi_3), \\
V_{II}(\phi) &= \mu^2 (1 - \phi_3^2),
\end{cases} \tag{4.2.2}
\]

where \( \mu \) is a rescaled mass parameter. The potential \( V_I \) has one vacuum for \( \phi_3 = +1 \), whereas \( V_{II} \) has two vacua; for \( \phi_3 = +1 \) and \( \phi_3 = -1 \). The planar version of (4.2.1) with \( V = V_I \) corresponds to the old Baby Skyrme model [110] and the one with \( V = V_{II} \) reproduces the new Baby Skyrme model [78, 151]. The normalization in (4.2.2) is chosen so that for \( \phi_3 \to +1 \) both potentials show the same asymptotic behaviour, explicitly given by \( \mu^2 (\phi_1^2 + \phi_2^2) \).

The Lagrangian (4.2.1) admits topologically-nontrivial, string-like, finite-energy configurations due to the third homotopy group of the 2-sphere being non-trivial, \( \pi_3(S^2) = \mathbb{Z} \). This can be seen as follows. A static finite-energy configuration requires the boundary condition \( \phi(t, x) \to (0, 0, 1) \) as \( |x| \to \infty \) for all time \( t \). Hence this boundary condition on the field \( \phi \) defines a mapping \( \phi : S^3 \to S^2 \) and the field configurations can be classified topologically by the homotopy group \( \pi_3(S^2) = \mathbb{Z} \). The topological invariant associated with each static field configuration is known as the Hopf charge \( N \). It can be interpreted geometrically as the linking number of two loops obtained as the preimages of any two generic distinct points on the target 2-sphere. The position curve of the soliton is defined as the set of points where the field is as far as possible from the boundary vacuum value \( \phi_\infty = (0, 0, 1) \). Thus it is given by the preimage of the point \( -\phi_\infty \), which is antipodal to the vacuum value. When we visualize Hopf solitons position curves, we usually display...
for clarity tube-like isosurfaces with \((0, 0, -1 + \delta)\), where \(\delta\) is chosen to be small. Similar the linking curve can be illustrated graphically by plotting an isosurface of the preimage of the vector \((-1 + \delta, 0, 0)\).

The overall factor \(1/32\pi^2/\sqrt{2}\) in (4.2.1) is motivated by Ward’s conjecture [146] that with the normalization (4.2.1) the Vakulenko-Kapitanski lower bound [79, 141] on the energy \(M_N\) of a Hopf configuration with charge \(N\) is given by

\[
M_N \geq cN^{3/4}, \quad \text{where } c = 1.
\]

The topological bound (4.2.3) has been shown to be compatible with fully 3-dimensional numerical simulations carried out in the massless Skyrme-Faddeev model [27, 26, 64] and in the massive one [56] with potential \(V_I\) included.

The Skyrme-Faddeev model (4.2.1) can be expressed in analogy to the conventional \(SU(2)\) Skyrme model [129] in terms of the \(SU(2)\)-valued Hermitian scalar field \(U(t, x) = \phi \cdot \tau\). The ansatz for the dynamical soliton field adopted in the collective coordinate quantization [9, 35, 94] is given by

\[
\hat{U}(x, t) = A_1(t)U_0(x)A_1^\dagger(t),
\]

where we have promoted the collective coordinate \(A_1 \in SU(2)\) to a time-dependent dynamical variable and ignored the translational and rotational degrees of freedom. \(A_1(t)\) describes the isorotational fluctuations about the classical minimum-energy solution \(U_0(x)\).

Substituting (4.2.4) in (4.2.1) and defining the body-fixed angular velocities via \(a_j = -i\text{Tr}(\tau_jA_1^\dagger A_1)\) the Skyrme-Faddeev Lagrangian takes the form

\[
L = \frac{1}{2}a_iU_{ij}a_j - M_N,
\]

where the Hopf soliton mass \(M_N\) is given by

\[
M_N = \frac{1}{32\pi^2 \sqrt{2}} \int \left\{ \partial_i \phi \cdot \partial_i \phi + \frac{1}{2} \left[ (\partial_i \phi \cdot \partial_j \phi) - (\partial_i \phi \cdot \partial_j \phi) \right] + V(\phi) \right\} \, d^3x,
\]

and the moment of inertia tensors is

\[
U_{ij} = \frac{1}{16\pi^2 \sqrt{2}} \int \left( \phi^2 \delta_{ij} - \phi_i \phi_j \right) (1 + \partial_k \phi \cdot \partial_k \phi) - (\phi \times \partial_k \phi) \cdot (\phi \times \partial_k \phi) \, d^3x.
\]

The momentum conjugate to \(a_i\) is the body-fixed isorotation angular momentum \(K_i\) defined via

\[
K_i = \frac{\partial L}{\partial a_i} = U_{ij}a_j.
\]

In this chapter, we choose the \(\phi_3\)-axis as our rotation axis. Using gradient-based meh-
ods we search for Hopf configurations \( \phi \) of a given topological charge \( N \) which minimize the pseudo-energy functional
\[
F_\omega(\phi) = -L = M_N - \frac{1}{2} U_{33} \omega^2 ,
\]
where the rotation frequency \( \omega = a_3 \) is calculated at each time step for a fixed \( |K| \) as follows
\[
\omega = \frac{|K|}{U_{33}} .
\]
Explicitly, the pseudo-energy \( F_\omega \) (4.2.9) takes, analogous to the energy-like functional (3.3.5) for isopinning baby Skyrmion solutions, the form
\[
F_\omega(\phi) = \frac{1}{32\pi^2 \sqrt{2}} \int \left[ \left\{ 1 - \omega^2 \left( 1 - \phi_3^2 \right) \right\} (\partial_3 \phi \cdot \partial_3 \phi) + \omega^2 (\phi \times \partial_3 \phi)_3 (\phi \times \partial_3 \phi)_3 \right]
+ \frac{1}{2} \left( \partial_i \phi \times \partial_j \phi \right)^2 + V_\omega(\phi) \right] d^3x ,
\]
where the effective potential \( V_\omega(\phi) \) is given by
\[
V_\omega(\phi) = V(\phi) - \omega^2 \left( 1 - \phi_3^2 \right) .
\]
Consequently, we expect [60] to encounter two different types of critical angular frequencies: The first critical frequency value \( \omega_1 \) is related with the first two terms in (4.2.11) which effectively define [60] the geometry of a deformed sphere \( S^2 \) squashed along the direction \( \phi_\infty \) (see Section 3.3.1). The metric on this space becomes singular at \( \omega = \omega_1 = 1 \) and as a result of this the pseudo-energy \( F_\omega \) is no longer bounded from below for \( \omega > \omega_1 \). The second critical frequency reflects the fact that Hopf solitons can only isospin up to a maximal rotation frequency \( \omega_2 \) determined by the meson masses in the specified models (4.2.2). When isospinning at angular frequencies \( \omega > \omega_2 = \mu \) the soliton solutions in the Skyrme-Faddeev models with potential \( V_I \) and \( V_{II} \) become unstable to the emission of mesons of mass \( \mu \).

Hence, isospinning Hopf soliton solutions can only be constructed for rotation frequencies \( \omega \leq \min \{1, \mu \} \) in the models with potential choice \( V_I \) and \( V_{II} \), respectively. Outside this frequency range the pseudo-energy (4.2.11) is unbounded from below and isospinning solitons of the model (4.2.1) cease to exist.

### 4.3 Initial Conditions

We create suitable initial field configurations with non-trivial Hopf charge \( N \) by using the approach presented in [136]. The basic idea is to approximate the Hopf configuration by rational maps \( W : S^3 \rightarrow \mathbb{C}P^1 \), that is, a mapping from the three-sphere to the complex
projective line. This approach enables us to set up initial conditions for knotted, linked and axial Hopf configurations with energies reasonably close to the suspected minimum energy solutions. These initial conditions can then be relaxed using a modified version of the energy minimization algorithm [29] originally designed to study Skyrmion solutions.

First we compactify $\mathbb{R}^3$ to a unit 3-sphere $S^3 \in \mathbb{C}^2$ via a degree one spherically equivariant map given by

$$ (Z_1, Z_0) = \left( \frac{x_1 + ix_2}{r} \sin f, \cos f + \frac{\sin f}{r} x_3 \right), \quad (4.3.1) $$

where $(x_1, x_2, x_3) \in \mathbb{R}^3$, $r^2 = x_1^2 + x_2^2 + x_3^2$ and $(Z_1, Z_0)$ are complex coordinates on the unit 3-sphere (with $|Z_1|^2 + |Z_0|^2 = 1$). Here $f(r)$ is a monotonically decreasing profile function with boundary conditions $f(0) = \pi$ and $f(\infty) = 0$. In our simulations we use a simple linear profile function $f(r) = \pi(r_{\text{max}} - r)/L$ for $r < r_{\text{max}}$ and $f(r) = 0$ for $r \geq r_{\text{max}}$ with $r_{\text{max}} = 6$. Approximate Hopf solutions can be obtained by writing the stereographic projection of the field $\phi$

$$ W = \frac{\phi_1 + i\phi_2}{1 + \phi_3}, \quad (4.3.2) $$

as a rational function of the complex variables $Z_1$ and $Z_0$

$$ W = \frac{p(Z_1, Z_0)}{q(Z_1, Z_0)}, \quad (4.3.3) $$

where $p$ and $q$ are polynomials in $Z_1$ and $Z_0$.

There are three different solution types which will be used as initial field configurations for our energy relaxation simulations:

- Toroidal fields of solution type $\mathcal{A}_{n,m}$ can be obtained by setting

$$ W = \frac{Z_1^n}{Z_0^m}, \quad (4.3.4) $$

where $n, m \in \mathbb{Z}$. The integer pair $(n, m)$ counts the angular windings around the two cycles of the torus. An axially-symmetric Hopf configuration of the type $\mathcal{A}_{n,m}$ can be described [27, 136] by a baby Skyrmion solution with winding number $m$ which is embedded in the $(3 + 1)$-dimensional Skyrme-Faddeev model (4.2.1) along a closed curve and with its internal phase rotated through an angle $2\pi n$ as it travels around the circle once. The Hopf charge $N$ associated with such an unlinked Hopf configuration (4.3.4) is given by $N = nm$.

- $(a, b)$-torus knots $\mathcal{K}_{ab}$ are described by the mapping

$$ W = \frac{Z_1^a Z_0^b}{Z_1^a + Z_0^b}, \quad (4.3.5) $$
where $\alpha$ is a positive integer, $\beta$ is a non-negative integer and $a, b$ are coprime positive integers with $a > b$. The rational map (4.3.5) generates a knot lying on the surface of an unknotted torus and winding $a$ and $b$ times about the torus circumferences. Fields of type $\mathcal{K}_{a, b}$ have topological charge $N = ab + \beta a$ [136].

- Linked Hopf initial configurations of the type $\mathcal{L}_{p, q}^{\alpha, \beta}$ can be constructed, when the denominator of (4.3.5) is reducible. Following the notation of [136], $p$ and $q$ label the charges of the two disconnected components that form the link and the additional linking number of each component due to its linking with the other is denoted by the superscripts $\alpha$ and $\beta$. The total Hopf charge of a field $\mathcal{L}_{p, q}^{\alpha, \beta}$ is $N = p + q + \alpha + \beta$. In particular, in this chapter we will use the rational map

$$ W = \frac{Z_{n+1}^n}{Z_1^n} = \frac{Z_1^n}{2(Z_1 - Z_0)} + \frac{Z_1^n}{2(Z_1 + Z_0)}, $$

(4.3.6)

to produce smooth initial linked configurations of solution type $\mathcal{L}_{n,n}^{1,1}$ and Hopf charge $N = 2n + 2$.

In the following section, we compute minimum-energy Hopf solutions for potential $V_I$ and $V_{II}$ using a relaxation algorithm with initial conditions constructed from the rational maps (4.3.4), (4.3.5) and from linked configurations like e.g. (4.3.6). To avoid saddle point solutions of the Skyrme-Faddeev energy functional $M_N$ we explicitly add, in a similar way to [27], symmetry-breaking, non-axial perturbations to our initial conditions.

### 4.4 Relaxed Hopf Soliton Solutions

To find the stationary points of the energy functional $M_N$ we solve the associated Euler-Lagrange equations numerically. The field equations can be implemented analogous to [29]

$$ M \ddot{\phi} - \alpha \left( \dot{\phi}, \partial_i \dot{\phi}, \partial_i \partial_j \dot{\phi} \right) - \lambda \phi + \epsilon \dot{\phi} = 0, $$

(4.4.1)

where $M$ is a symmetric matrix. The dissipation $\epsilon$ in (4.4.1) is added to speed up the relaxation process and the Lagrange multiplier $\lambda$ imposes the unit vector constraint $\phi \cdot \phi = 1$. We do not present the full field equations here since they are cumbersome and not particularly enlightening. The initial configuration is then evolved according to the flow equations (4.4.1). Kinetic energy is removed periodically by setting $\dot{\phi} = 0$ at all grid points. All the simulations presented in the following use fourth order spatial differences on grids with $(201)^3$ points, a spatial grid spacing $\Delta x = 0.1$ and time step size $\Delta t = 0.01$. The dissipation is set to $\epsilon = 0.5$ and we choose the rescaled mass parameter $\mu = 1$ throughout this chapter.

A summary of our relaxed configurations is given in Table 4.1. Each initial configuration is listed together with the final Hopf configuration it evolves to. In Fig. 4.1 we display
the linking structure of the minimum-energy configurations of charge 1 ≤ N ≤ 8 obtained for potential V_0. Here, we visualize the field configurations by plotting isosurfaces of the points (√2e−e^2, 0, −1 + e) and (−1 + e, 0, √2e−e^2) with e = 0.2. Our calculations with potential V_II produce the same Hopf solution types as for potential V_0, the main difference being that the solitons are more compact. The minimal energy solutions of both massive models are very similar to the massless ones [27, 26, 63, 64, 136].

<table>
<thead>
<tr>
<th>N</th>
<th>initial</th>
<th>final</th>
<th>M_P</th>
<th>M_P/N</th>
<th>M_P</th>
<th>M_P/N</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>A_{1,1}</td>
<td>A_{1,1}</td>
<td>1.438</td>
<td>1.438</td>
<td>1.373*</td>
<td>1.373</td>
</tr>
<tr>
<td>2</td>
<td>A_{2,1}</td>
<td>A_{2,1}</td>
<td>2.287*</td>
<td>1.359</td>
<td>2.188*</td>
<td>1.300</td>
</tr>
<tr>
<td>3</td>
<td>A_{3,1}</td>
<td>A_{3,1}</td>
<td>3.173*</td>
<td>1.391</td>
<td>3.041*</td>
<td>1.334</td>
</tr>
<tr>
<td>4</td>
<td>A_{4,1}</td>
<td>A_{4,1}</td>
<td>4.034*</td>
<td>1.426</td>
<td>3.862</td>
<td>1.365</td>
</tr>
<tr>
<td>5</td>
<td>A_{5,1}</td>
<td>A_{5,1}</td>
<td>4.871*</td>
<td>1.456</td>
<td>4.549*</td>
<td>1.360</td>
</tr>
<tr>
<td>6</td>
<td>A_{6,1}</td>
<td>A_{6,1}</td>
<td>5.402*</td>
<td>1.409</td>
<td>5.134*</td>
<td>1.339</td>
</tr>
<tr>
<td>7</td>
<td>A_{7,1}</td>
<td>A_{7,1}</td>
<td>6.138*</td>
<td>1.426</td>
<td>5.822*</td>
<td>1.352</td>
</tr>
<tr>
<td>8</td>
<td>A_{8,1}</td>
<td>A_{8,1}</td>
<td>6.974*</td>
<td>1.418</td>
<td>6.414*</td>
<td>1.348</td>
</tr>
</tbody>
</table>

Table 4.1: All initial conditions and final μ = 1 Hopf configurations with their respective energies. M_P and M_P denote the soliton energy with potential V_0 and V_II included, respectively. The superscript “pert.” indicates that we applied non-axial perturbations to the initial configuration. (⋆) These configurations correspond to global energy minima for given Hopf charge N. Recall that energies are given in units of 1/32π^2/√2.

Relaxing (4.3.4) with n = m = 1 reproduces the A_{1,1} static Hopf configuration, which has for V_0 an energy M_1 = 1.438 and a moment of inertia U_33 = 0.500, this agrees well with M_1 = 1.421 stated in [56]. For comparison, substituting a spherically-symmetric hedgehog form U(x) = exp(ρf(ρ) ⃗r · ⃗τ) in (4.2.6) and minimizing the energy with respect to the profile function f gives for the 1-Hopf soliton solution an energy M_1 = 1.452 and a moment of inertia U_33 = 0.502. The minimal energy N = 2 Hopf solitons are of the type A_{2,1} – axially-symmetric configurations with the linking curve twisted two times around the position curve. Applying non-axial perturbations to an A_{3,1} or a K_{2,1} initial configuration, we find the A_{3,1} 3-Hopf soliton solution to be of lowest energy for both potential choices. Here, the tilde indicates that the position curve is not lying completely in the plane but it
is bent. For completeness, we also include in Tab. 4.1 and in Fig. 4.1 minimal-energy configurations of solution type $N\mathcal{A}_{N,1}$. These axial solutions are known to be unstable for $N \geq 3$ [27, 26]. Taking perturbed axially-symmetric $\mathcal{A}_{4,1}$ and knotted $\mathcal{K}_{2,1}, \mathcal{K}_{4,1}$ configurations as our initial conditions, we identify the bent axial solution $\tilde{\mathcal{A}}_{4,1}$ as the global energy minimum for $N = 4$ and potential $V_f$ [56]. The charge-4 configuration $\mathcal{A}_{2,2}$ (created from axial and linked initial conditions) and $\mathcal{A}_{4,1}$ are local energy minima. However, for potential $V_{II}$ the minima swap with $\mathcal{A}_{2,2}$ becoming the lowest minimal-energy charge-4 soliton solution. For $N = 5$ the minimal configuration in both massive models is a link of type $L_{1,2}^{1,1}$, which we obtained by relaxing a perturbed trefoil knot $\mathcal{K}_{3,2}$. The charge-5 bent solution $\tilde{\mathcal{A}}_{5,1}$ and the toroidal $\mathcal{A}_{5,1}$ seem to be metastable local minima. For $N = 6$ we find using a variety of initial conditions that the $\mathcal{A}_{3,2}$ configuration has minimal energy, whereas the links $L_{2,2}^{1,1}$, $L_{3,1}^{1,1}$, the bent unknot $\tilde{\mathcal{A}}_{6,1}$ and the rotationally-symmetric unknot $\mathcal{A}_{6,1}$ are only local minima [56]. This differs from the massless Skyrme-Faddeev model where the link $L_{2,2}^{1,1}$ is the minimal-energy charge-6 soliton. Similar to the massless case, the trefoil knot $\mathcal{K}_{3,2}$ turns out to be the global minimum for $N = 7$ in the massive models. Charge-7 Hopf solutions like the $\mathcal{K}_{2,3}$ knot and the bent unknot $\tilde{\mathcal{A}}_{7,1}$ represent local minima. Finally, for $N = 8$ we identify $\tilde{\mathcal{A}}_{4,2}$ as the minimal-energy solution. For potential $V_f$ the trefoil knot $8\mathcal{K}_{3,2}$ can be seen within the numerical accuracy as an almost energy-degenerate state. The link $L_{3,3}^{1,1}$ which is the minimal-energy solution type in the massless model relaxes to $\tilde{\mathcal{A}}_{4,2}$.

In Fig 4.2 we show in analogy to [64] the normalized minimum energies $M_N^* = M_N / \langle M, N^{3/4} \rangle$ for both potential choices. In both cases the energies of the ground-state Hopf configurations (filled circles) follow $M_N \propto N^{3/4}$. As already pointed out in [64] for the massless case, the energies for the $2\mathcal{A}_{2,1}$ configurations in the massive models are par-
4.5. NUMERICAL RESULTS ON CLASSICALLY ISOSPINNING HOPF SOLITONS

particularly low compared to the standard level. We verify in Fig 4.2 that the normalized energies of the bent configurations $N\overline{A}_{N,1}$ with $N = 3-5$ are well described by the linear [103] fits $M_N/M_1 = 0.39 + 0.6N$ and $M_N/M_1 = 0.40 + 0.6N$ for $V_I$ and $V_{II}$, respectively. A very similar fit ($M_N/M_1 = 0.36 + 0.65N$) is given in [64] for the bent unknots in the massless Skyrme-Faddeev model. For the planar configurations $N\overline{A}_{N,1}$ with $N = 1-6$ we obtain $M_N/M_1 = 0.415 + 0.5719N + 0.009N^2$ for $V_I$ and $M_N/M_1 = 0.4348 + 0.5501N + 0.01475N^2$ for $V_{II}$ and $N = 1-4$. The corresponding quadratic fit for massless rotationally symmetric unknots is given as $M_N/M_1 = 0.39 + 0.59N + 0.015N^2$ in [64].

Figure 4.2: The normalized energies $M_N^* = M_N/\left(M_1N^{3/4}\right)$ for different minimal energy, massive Hopf soliton solutions as a function of the Hopf charge $N$. The mass parameter $\mu$ is chosen to be 1. Here, our global minima for $1 \leq N \leq 8$ are represented by filled circles (●), bent unknots $N\overline{A}_{N,1}$ by triangles (▲), rotationally symmetric unknots $N\overline{A}_{N,1}$ by diamonds (⋄) and the remaining local energy minima are displayed as open circles (○). The dashed line shows our linear fit to the bent unknots $N = 1-5$ (\(\mathcal{A}_{1,1}, \mathcal{A}_{2,1}\) included), whereas the dash dotted line represents a quadratic fit to the rotationally symmetric unknots $N = 1-6$. The expected $N^{3/4}$ power growth is represented by the horizontal line. The corresponding plots for the massless Faddeev-Skyrme model can be found in [64] and [136].

4.5 Numerical Results on Classically Isospinning Hopf Solitons

In this section, we present the results of our energy minimization simulations of isospinning Hopf solitons with charges $N$ up to 8. The variational equations derived from (4.2.9) are implemented in analogy to (4.4.1), where we include in $\alpha$ the isorotational extra terms. We use the configurations obtained in the previous sections as our start configurations for vanishing angular momentum ($K = 0$) and increase $K$ in a stepwise manner. All simulation parameters are chosen as stated in Section 4.4. In particular, we
use the mass parameter $\mu = 1$ and work on grids containing $(201)^3$ lattice points with a lattice spacing $\Delta x = 0.1$. If not stated otherwise, we use $V = V_I$ as our potential term in (4.2.1).

Note that for $\mu \leq 1$ there exists a maximal frequency $\omega_{\text{max}} = \mu$ beyond which no stable isospinning Hopf soliton solution exists. This upper limit follows from the stability analysis of the linearized Euler-Lagrange equations derived from (4.2.9).

### 4.5.1 Low Charge Hopf Solitons: $1 \leq N \leq 3$

We show in Fig. 4.3 the total energy $E_{\text{tot}}$ as a function of the rotation frequency $\omega$ and the angular momentum $K$ for isospinning Hopf solitons (of type $1A_{1,1}, 2A_{2,1}, 3\tilde{A}_{3,1}$) with charges up to 3. The corresponding plots for the moment of inertia $U_{33}$ as function of $\omega$ are also presented. For all these configurations the solution type of the isospinning soliton is the same as the one in the static case, only the soliton’s size grows with $\omega$ and $K$. As expected, the energies and moment of inertia diverge for $\omega = \mu$.

![Figure 4.3](image_url)

Figure 4.3: Energy and moment of inertia of isospinning Hopf solitons with $N = 1 \ldots 3$ are plotted as functions of the angular frequency $\omega$ and isospin $K$. For $N = 1$ the results for both potential choices are shown.

### 4.5.2 Higher Charge Hopf Solitons: $4 \leq N \leq 8$

- $N = 4$: The energy and moment of inertia plots for isospinning 4-Hopf solitons ($4\tilde{A}_{4,1}, 4A_{4,2}$) are shown in Fig. 4.5. The $4\tilde{A}_{4,1}$ configuration is found to be the solution type of lowest energy for all $\omega$ and $K$. The $4A_{4,2}$ soliton deforms for $\omega \geq 0.60$ ($K \geq 23$) into a $4L_{1,1}^4$ link, that means into a solution type which does not represent a local minimum in the static case ($\omega = 0$). The isosurface plots in Fig. 4.4 illustrate the formation of the linked configuration as $K$ increases.

- $N = 5$: We show in Fig. 4.6 the total energy $E_{\text{tot}}$ of isospinning charge-5 Hopf solitons ($5L_{1,1}^{1,2}, 5\tilde{A}_{5,1}, 5A_{5,1}$) as function of the rotation frequency $\omega$ and the angular momentum $K$. We observe that the energy curve $E_{\text{tot}}(\omega)$ of the linked unknot $5L_{1,1}^{1,2}$ crosses the one of the bent ring $\tilde{A}_{5,1}$ at $\omega \approx 0.33$. For $\omega > 0.33$ the bent ring becomes the new ground state for Hopf charge $N = 5$. However, for fixed $K$ the linked configuration continues to be the lowest energy state, see Fig. 4.6.
4.5. NUMERICAL RESULTS ON CLASSICALLY ISOSPINNING HOPF SOLITONS

Figure 4.4: Deformation of the isospinning $\mathcal{A}_{2,2}$ Hopf configuration into $\mathcal{L}_{1,1,1}^{1,1}$. Results are plotted for potential $V_I$, results for potential $V_{II}$ show the same qualitative behaviour. We visualize the linking structure by plotting tube-like isosurfaces $\phi_1 = -0.9$ (red tube) and $\phi_3 = -0.9$ (blue tube). Recall that the angular momentum $K$ is given in units of $4\pi$.

Figure 4.5: Total energy $E_{tot}$ and moment of inertia $U_{33}$ of isospinning 4-Hopf solitons calculated with potential $V_I$. The isospinning $\mathcal{A}_{2,2}$ soliton (blue) deforms into $\mathcal{L}_{1,1,1}^{1,1}$ (red). The transition occurs at $\omega \approx 0.606$, $K \approx 23$. The bent configuration $\mathcal{A}_{4,1}$ (green) exists for all $\omega \in [0, 1)$ and its size is growing with $\omega$.

- **$N = 6$:** Our simulations of isospinning 6-Hopf solitons ($6\mathcal{A}_{3,2}$, $6\mathcal{L}_{3,1,1}^{1,1}$, $6\mathcal{L}_{2,2,2}^{1,1}$, $6\mathcal{A}_{6,1}$) are summarized by the energy curves in Fig. 4.9. Here, we see an example of transmutation: the $6\mathcal{A}_{3,2}$ configuration which is the ground state at $\omega = 0$ transforms into a $6\mathcal{L}_{2,2}^{1,1}$ link when $K$ increases. For $K \geq 35$ ($\omega \geq 0.56$) the $6\mathcal{A}_{3,2}$ soliton has completely deformed into the link configuration that forms the new lowest energy state. The deformation process is visualized by the isosurface plots in Fig. 4.7. Bent Hopf configurations of solution type $6\mathcal{A}_{6,1}$ and links of type $6\mathcal{L}_{3,1,1}^{1,1}$ have higher energies for all $\omega$ and $K$. The linking curves in Fig. 4.8 show that the $6\mathcal{L}_{3,1,1}^{1,1}$ configuration is of the same qualitative shape for all $\omega$ and $K$.

- **$N = 7$:** We do not observe any crossing of the energy curves of isospinning $7\mathcal{K}_{3,2}$ and $7\mathcal{K}_{2,3}$ knot solutions. We find the $7\mathcal{K}_{3,2}$ knot as the state of lowest energy for
CHAPTER 4. HOPF SOLITONS

(a) $E_{\text{tot}}$ as function of $\omega$

(b) $E_{\text{tot}}$ as function of $K$

Figure 4.6: Total energy $E_{\text{tot}}$ of isospinning charge 5-Hopf Solitons as function of $\omega$ and $K$ ($V = V_I$). The energy curve $E_{\text{tot}}(\omega)$ of the link $5\mathcal{L}_{1,1}^{1,2}$ (green) crosses the one of $5\mathcal{A}_{5,1}$ (purple) at $\omega \approx 0.33$. The lowest-energy, isospinning soliton is of type $5\mathcal{L}_{1,1}^{1,2}$ for $\omega \in [0, 0.33)$ and $5\mathcal{A}_{5,1}$ for $\omega \in [0.33, 1)$. For comparison, we also show the axial, unstable $5\mathcal{A}_{5,1}$ solution (blue).

Figure 4.7: Deformation of the isospinning $6\mathcal{A}_{3,2}$ Hopf soliton solution into $6\mathcal{L}_{2,2}^{1,1}$. First row: We display isosurfaces $\phi_1 = -0.98$ (red tube) and $\phi_3 = -0.65$ (blue tube) to illustrate the change of the solution types. Second row: We show the linking curves for $\phi_1 = -0.98$, separately.

Figure 4.8: Linking curves of the isospinning $6\mathcal{L}_{3,1}^{1,1}$ Hopf soliton solution. The linking structure is visualized by the isosurfaces $\phi_1 = -0.95$ (red) and $\phi_3 = -0.65$ (blue).
4.6 Critical Angular Frequencies

Very recently it was found [60] that isospinning soliton solutions in models of the Skyrme family exhibit two different types of instabilities (compare Sections 4.2 and 4.3).

1By this we mean that they have the same energy within the numerical errors and hence we are unable to distinguish them.
3.3.1). We verify in Fig. 4.12 and Fig. 4.13 that the energy and isospin curves of isospinning Hopf soliton solutions for the potential choices $V_I$ and $V_{II}$ are in good agreement with the pattern of critical behaviour discussed in Section 4.2. For mass parameters $\mu \leq \omega_1 = 1$ we observe that energy and isospin diverge in both models at $\omega_2 = \mu$, that is the “meson” mass of the two models. When the mass value $\mu$ is chosen to be larger than $\omega_1$, isospinning solutions cease to exist close to $\omega_1 = 1$. In this case – completely analogous to the observations in the baby Skyrme models (see Chapter 3) – the energy and the isospin take finite values at the first critical frequency $\omega_1$.

Figure 4.12: Isospinning $N = 1$ Hopf soliton solutions of the model with the conventional potential term $V_I$ included for a range of mass values $\mu$. We plot the total energy $E_{\text{tot}}$ and isospin $K$ as function of angular frequency $\omega$ and the total energy as a function of isospin $K$ at $\mu = 0.5, 1, 1.5, 2$. The $\phi_3$-axis is chosen to be the axis of rotation. Our 3D-relaxation calculations are performed on a $(100)^3$ grid with grid spacing $\Delta x = 0.2$.

Note that our simulations show for both models (4.2.2) slightly higher values for the first critical frequency $\omega_1$ (compare energy curves for mass values $\mu > 1$ in Fig. 4.12 and Fig. 4.13). This is consistent with the findings of Ref. [60]. Finite energy solutions with $\omega > \omega_1 = 1$ can continue to exist as saddle points of the pseudo-energy $F_\omega$ (4.2.11).
4.7 Conclusions

We have performed full 3-dimensional numerical relaxations of isospinning soliton solutions in the Skyrme-Faddeev model with mass terms included. Our computations of charge-4, -6 and -8 solitons show that the qualitative shapes of internally rotating Hopf solitons can differ from the static (ω = 0) solitons. However, in most cases (for Hopf charges N = 1, 2, 3, 5, 7) the solution types present at ω = 0 also exist for non-zero ω. The qualitative shape of the lowest energy configuration can be frequency dependent. The energy curves E_{tot}(ω) for a given N can cross and minima can swap (e.g. N = 5). In summary, we distinguish 3 different types of behaviour:

- **Crossings of E_{tot}(ω):** The energy curves E_{tot}(ω) of Hopf solitons for different solution types of the same charge N can cross which results in a rearrangement of the spectrum of minimal-energy configurations. Our simulations on isospinning charge-5 solitons illustrate this: at ω = 0 the link 5L_{1,2} \text{ is the lowest energy solution, but for } ω \approx 0.33 \text{ its energy curve crosses that of the bent unknot } 5\overline{A}_{5,1}. \text{ For } ω \geq 0.33 \text{ the lowest energy soliton is given by } 5\overline{A}_{5,1}.

- **Transmutation:** Isospinning Hopf solitons can deform into minimal-energy solutions of a type which also exists at ω = 0 (e.g. 6A_{3,2} \rightarrow 6L_{2,2}^{1,1}, 8\overline{A}_{4,2} \rightarrow 8K_{3,2}).

- **Formation of new solution types:** New solution types can emerge which are unstable for vanishing ω. For example, for N = 4 the 4A_{2,2} deforms into 4L_{1,1}^{1,1} with the later only being stable for ω ≥ 0.60

Naturally one expects these effects to be present and increasingly relevant for higher Hopf charges (N > 8) since the number of (local) energy minima grows with the Hopf charge N [136].

In this chapter we have focussed on purely classically isospinning soliton solutions in the Skyrme-Faddeev model. The relevance of classically (iso)spinning soliton solutions was discussed in [98] in the context of the Skyrme model. Here it was argued that classically spinning Skyrmions could be used to model classically the quantized Skyrmion states. For example, a spin-1/2 proton in its spin up state can be interpreted within this approximate classical description as a hedgehog Skyrmion of topological charge B = 1.
spinning anticlockwise relative to the positive $\phi_3$-axis and with its normalized pion fields $\hat{\pi}$ orientated in such a way that $\pi_3 = \pm 1$ for $z \to \pm \infty$, respectively. Analogously, classically spinning Hopf soliton solutions can classically model the quantized spectra of glueballs. To do this, it is necessary to determine the (iso)space orientations that describe the excited states of glueballs. To approximate states of non-vanishing spin, rotations in physical space have to be implemented in our computations, which significantly complicates our numerics.

Our numerical results are of relevance for the quantization of the classical soliton solutions. There are two main methods used in the literature to obtain quantized Hopf solitons: the bosonic, semiclassical collective coordinate quantization [134, 73] and the fermionic quantization [76] which is based on the Finkelstein-Rubinstein (FR) approach [52]. Both approaches assume that the symmetries of the classical Hopf configurations are not broken by centrifugal effects. In the semiclassical bosonic collective coordinate quantization procedure glueballs can be modelled by quantum mechanical states on the moduli space – the finite-dimensional space of static minimal energy Hopf solutions in a given topological sector which is generated from a single Hopf configuration by rotations and isorotations. The effective Hamiltonian on this restricted configuration space is canonically quantized. The numerical calculations presented in this chapter could be seen as a classical approximation to the collective coordinate dynamics on the moduli space. The allowed quantum states have to satisfy the FR constraints [52] which follow from the continuous and discrete symmetries of the classical Hopf configurations: For a bosonic quantum theory the FR constraints result in constraints for the wave functions defined on the configuration space, whereas fermionic quantization [76] constrains the wave functions on the covering space of configuration space.

Ground states and first excited states of Hopf solitons for charges up to 7 have been calculated in [76] using the symmetries of the classical Hopf solutions given in [64]. Here, it would be instructive to work out the spectra that emerge from the classical solutions calculated in this chapter. The isospinning, minimal energy Hopf solitons of charge $N = 5, 6, 8$ are particularly interesting since their symmetries are different from those of the static configurations which are commonly used to calculate the solitons’ possible ground states. The presentation of a self-consistent, non-rigid quantization procedure goes far beyond the scope of this chapter and is the subject of future research.

Note added

Similar results were also reported in a very recent paper [60] which appeared when our paper [19] was in preparation. The authors in [60] carried out most of their calculations with $\mu = 2$ and the potential choice $V_{II}$. Differences to our results are that they neither identify a $6A_3^2\tilde{A}_2$ nor a $8\tilde{A}_{4,2}$ configuration. Unfortunately they did not visualize the linking structure of their 6- and 8-Hopf soliton solutions, so that we could not compare them.
Differences to our results could be due to the different potential choice or to the different choice of the mass parameter $\mu$. 
5.1 Introduction

Skyrmions [129] arise as classical, topologically stable, finite-energy solutions in the Skyrme model, which is nowadays commonly interpreted as a simplified candidate for a low energy effective theory of QCD in the large $N_C$ limit [138, 153]. A very appealing idea [129] is to identify a nucleus with baryon number $B$ with a quantized Skyrmion of conserved topological charge $B$. One parametrizes the zero-mode space – the space of energy-degenerate Skyrme configurations generated from a single Skyrmion by spatial translations and rotations in space and isospace – using collective coordinates which are then taken to be time-dependent. This gives rise to additional dynamical terms in the classical Hamiltonian which can then be quantized following semiclassical quantization rules. In this way classical Skyrmion solutions acquire spin and isospin quantum numbers that can be identified with those of nuclei of given $B$.

Adkins, Nappi and Witten [8, 9] applied a simple adiabatic approximation to the rotational zero modes of the $B=1$ Skyrmion by assuming that the shape of the soliton is rotational frequency independent. Here, the nucleon isospin doublet emerges as the lowest quantum state of the rigidly spinning $B=1$ Skyrmion and the spin $3/2$ $\Delta$ multiplet forms the next lowest energy state. Traditionally, the length and energy scales of the Skyrme model are calibrated so that the physical masses of the nucleon and the delta resonance are reproduced when setting the pion mass parameter to its experimental value [9]. Generally speaking, the physical quantities calculated in the rigid-body approximation are within about 30% of the experimental values [8, 9].

The rigid-body approach when applied to the low charge Skyrmion solution with baryon number $B=2,3,4$ yields ground states of the correct spin, parity and isospin quantum numbers to represent the deuteron [35], the isospin doublet $^3\text{H}/^3\text{He}$ [37, 38] and the $\alpha$-particle [145], respectively. Here, the physically allowed quantum states are restricted by the symmetries of the classical Skyrme configuration and by the Pauli exclusion principle which is incorporated in the Finkelstein-Rubinstein (FR) constraints [52]. The FR-allowed quantum states and their excitation energies have been calculated for
Skyrmions with $B$ up to 22 [94, 23, 75, 68].

However, modeling the quantum states as rigid body states in space and isospace turns out to be too simple an approximation and a more sophisticated quantization procedure might further improve the agreement with empirical nuclei states and properties. The rigid-body quantization of the $B = 1$ Skyrmion results in an infinite tower of quantum states with half-integer spin. But allowing the Skyrme configuration to deform within an axially symmetric ansatz [21, 66, 54] sets a maximal rotation frequency $\omega_{\text{crit}}$ beyond which the Skyrmion becomes unstable due to the emission of pions of mass $\mu$. Higher spin-states with $\omega > \mu$ allowed by the rigid-body approach are now ruled out. One consequence of this is that the standard choice of Skyrme parameters can be seen as an artifact [21] of the rigid-body approximation.

A rigid collective coordinate quantization appears to be inadequate for the toroidal $B = 2$ soliton solution: In particular, the predicted deuteron binding energy is far too large and the calculated charge radius [35] is half the experimental value. However allowing the two constituent $B = 1$ Skyrmions to separate within the most attractive channel configuration [86] brings the theoretical predictions much closer to the empirical deuteron values. Furthermore, numerical calculations on spinning and axially symmetric deforming $B = 2$ Skyrmions [54] reveal that the rotational energy can be significantly lowered (factor of $\approx 4$) by rotational deformations.

The cubic symmetry of a rigidly rotating $B = 4$ Skyrmion predicts an experimentally unobserved spin 4, isospin 0 excited state of the $\alpha$-particle [94] which might be disallowed (as suggested in [23]) if the cubic symmetry is broken due to centrifugal effects.

Breaking or enlarging the symmetry groups of the static classical minimal-energy solution proves to be promising for higher charge Skyrmions as well. The $B = 5$ Skyrmion when quantized as a rigid body gives a ground state with spin 1/2 and isospin 1/2 [68] which is in contradiction with the observed spin 3/2 and isospin 1/2 lowest energy state of the $^5\text{He}/^5\text{Li}$ isodoublet. A ground state with the correct spin can be achieved by enlarging the $D_{3d}$ symmetry of the static minimal energy solution to a $D_{4h}$ symmetry [94]. In contrast, the icosahedrally symmetric $B = 7$ Skyrmion solution [93] is too symmetric to allow for the low spin states of the $^7\text{Li}/^7\text{Be}$ doublet. It turns out that the physical observed ground state of the $^7\text{Li}/^7\text{Be}$ isospin doublet with spin 3/2 and isospin 1/2 can only be modelled by significantly deforming the classical $B = 7$ solution. The rigid-body approach leads to a lowest allowed state with spin 7/2 and isospin 1/2 [68] which is experimentally observed to be the second excited state of $^7\text{Li}/^7\text{Be}$.

Consequently, in this chapter we consider isospinning soliton solutions in the Skyrme model beyond the rigid body approximation. In particular, we allow Skyrmions to deform arbitrarily as they isospin. For future work, it might be instructive to work out what the modified symmetries of the isospinning Skyrmion solutions imply concretely for the ground states and excited states of these solitons.

The outline of this chapter is as follows. Section 5.2 recalls briefly the $SU(2)$ Skyrme
model and describes how we can construct isospinning Skyrmion solutions by solving energy minimization problems numerically. In Section 5.3, we review how suitable initial conditions for Skyrme configurations of a given non-trivial topological charge $B$ and of a specific symmetry $G$ can be created for our numerical relaxation simulations. Relaxing the initial Skyrme fields generated with the methods described in Section 5.3 we find in Section 5.4 massive Skyrme soliton solutions with topological charges $B$ up to 16. Then, in Section 5.5 we investigate how the solitons’ geometric shape and energy are classically affected by the addition of isospin. A brief summary and conclusion of our results is given in Section 5.6.

For more details on the numerical methods and routines used in this chapter we refer the interested reader to the literature [29, 19] and our explanations given in Chapter 4.

Note that the work presented in this chapter has been done in collaboration with Richard Battye and very preliminary results have been reported in [17, 18]. Further note that the research described in this chapter is still very much ongoing and that some of the results reported here will form part of [16]. Finally note that we do not include in this chapter our numerical computations on spinning Skyrmion solutions (see [17, 18] for some preliminary work), the reason being that our numerics in the spinning case needs further improvement and refinement.

5.2 Spinning and Isospinning Skyrmions

The Lagrangian density of the $(3+1)$-dimensional, massive Skyrme model [129] is defined in SU(2) notation by

$$\mathcal{L} = -\frac{1}{2} \text{Tr}(R_\mu R^\mu) + \frac{1}{16} \text{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) + \mu^2 \text{Tr}(U - \frac{1}{2}) \ , \quad (5.2.1)$$

where the Skyrme field $U(t, x)$ is an SU(2)-valued scalar, $R_\mu = \left( \partial_\mu U \right) U^\dagger$ its associated right-handed chiral current and $\mu$ is the rescaled pion mass parameter. The Lagrangian (5.2.1) is expressed in terms of standard “Skyrme units” [9, 8], in which the energy and length units are given by 5.58 MeV and 0.755 fm, respectively. Throughout this chapter we often set $\mu = 1$. This parameter choice is motivated by [21, 30, 23] where it has been argued that a rescaled mass parameter $\mu$ larger than the physical pion mass in dimensionless Skyrme units (in particular, $\mu > 0.526$) yields improved results when applying the Skyrme model to nuclear physics.

Skyrmions arise as static solutions of minimal potential energy in the Skyrme model (5.2.1). They can be characterized by their conserved, integer-valued topological charge $B$ which is given by the degree of the mapping $U : \mathbb{R}^3 \rightarrow SU(2)$. To ensure fields of finite potential energy and a well-defined integer degree $B$ the Skyrme field $U(t, x)$ has to approach the vacuum configuration $U(x) = 1_2$ at spatial infinity for all $t$. Therefore, the domain can be formally compactified to a 3-sphere $S^3_{\text{space}}$ and the Skyrme field $U$ is
then given by a mapping $S^3_{\text{space}} \rightarrow SU(2) \sim S^3_{\text{iso}}$ labelled by the topological invariant $B = \pi_3(S^3) \in \mathbb{Z}$. The topological degree $B$ of a Skyrme soliton solution is explicitly given by

$$B = -\frac{1}{24\pi^2} \int \epsilon_{ijk} \text{Tr} \left(R_i R_j R_k\right) d^3x,$$  

and when modelling atomic nuclei by spinning and isospinning Skyrmion solutions, the topological charge (5.2.2) can be interpreted as the mass number or baryon number of the configuration. Throughout this chapter, the energies $M_B$ of minimal-energy solutions in the Skyrme model will be given in units of $12\pi^2$, so that the Faddeev-Bogomolny lower energy bound for a charge $B$ Skyrmion takes the form $M_B \geq |B|$.

Note that the $SU(2)$ field $U$ can be associated to the scalar meson field $\sigma$ and the pion isotriplet $\pi = (\pi_1, \pi_2, \pi_3)$ of the $O(4)$ $\sigma$-model representation $\phi = (\sigma, \pi)$ via

$$U(x) = \sigma(x)1_2 + i\pi(x) \cdot \tau,$$  

where $\tau$ denotes the triplet of standard Pauli matrices and the unit vector constraint $\phi \cdot \phi = 1$ has to be satisfied.

The Skyrme Lagrangian (5.2.1) is manifestly invariant under translations in $\mathbb{R}^3$ and rotations in space and isospace. Classically spinning and isospinning Skyrmion solutions are obtained within the collective coordinate approach [9, 8]: The 6-dimensional\(^1\) space of zero modes – the space of energy-degenerate Skyrmion solutions which only differ in their orientations in space and isospace – is parametrised by collective coordinates which are then taken to be time-dependent. The dynamical ansatz is given by

$$\hat{U}(t, x) = A(t)U_0(D(A'(t))x)A^+(t),$$  

where the matrices $A, A' \in SU(2)$ are the collective coordinates describing the isorotational and rotational fluctuations around a static minimal energy solution $U_0(x)$. Substituting (5.2.4) in (5.2.1) yields the effective Lagrangian

$$L = \frac{1}{2} \omega_i U_{ij} \omega_j + \frac{1}{2} \Omega_i V_{ij} \Omega_j - \omega_i W_{ij} \Omega_j - M_B,$$  

where $M_B$ is the classical Skyrmion mass given by

$$M_B = \int \left\{ (\partial_i \phi \cdot \partial_i \phi) + \frac{1}{2} (\partial_i \phi \cdot \partial_i \phi)^2 - (\partial_i \phi \cdot \partial_j \phi)^2 \right\} d^3x,$$  

and $\Omega_k = -i \text{Tr} \left( \tau_k \dot{A} A'^\tau \right)$ and $\omega_k = -i \text{Tr} \left( \tau_k A \dot{A}^\dagger \right)$ are the rotational and isorotational angular velocities, respectively. The inertia tensors $U_{ij}, V_{ij}, W_{ij}$ are given explicitly by the

---

\(^1\)Translational degrees of freedom are ignored.
integrals

\[ U_{ij} = 2 \int \left\{ (\pi_d \pi^d \delta_{ij} - \pi_i \pi_j) (1 + \partial_k \phi \cdot \partial_k \phi) - \epsilon_{ide} \epsilon_{jfg} \left( \nabla^d \nabla^e \right) \left( \pi^l \partial_k \pi^e \right) \right\} \, d^3 x, \quad (5.2.7a) \]

\[ V_{ij} = 2 \int \epsilon_{ilm} \epsilon_{jnp} \left( \partial_m \phi \cdot \partial_p \phi - (\partial_k \phi \cdot \partial_m \phi) \left( \partial_k \phi \cdot \partial_p \phi \right) + (\partial_k \phi \cdot \partial_k \phi) \left( \partial_m \phi \cdot \partial_p \phi \right) \right) \, d^3 x, \quad (5.2.7b) \]

\[ W_{ij} = 2 \int \epsilon_{ilm} \left( \epsilon_{ide} \pi^d \partial_m \pi^e (1 + \partial_k \phi \cdot \partial_k \phi) - (\partial_k \phi \cdot \partial_m \phi) \left( \epsilon_{jfg} \pi^l \partial_k \pi^g \right) \right) \, d^3 x. \quad (5.2.7c) \]

In this chapter, we focus on the construction of isospinning Skyrmion solutions and consequently (5.2.5) simplifies to

\[ L = \frac{1}{2} \omega_i U_{ij} \omega_j - M_B. \quad (5.2.8) \]

Uniformly isospinning soliton solutions in Skyrme models are obtained by solving one of the following, precisely equivalent variational problems \[60\] for \( \phi \):

1. Extremize the pseudoenergy functional \( F_\omega (\phi) = -L \) for fixed \( |\omega| \),

2. Extremize the Hamiltonian \( H = M_B + \frac{1}{2} K_i U_{ij}^{-1} K_j \) for fixed isospin \( K_i = U_{ij} \omega_j \).

Most of our numerical results on isospinning Skyrme solitons are based on formulation 2. In order to avoid precession effects in our numerical simulations the Skyrmion solutions have to be oriented in isospace so that their principal axes are approximately aligned with the chosen rotation axes.

### 5.3 Initial Conditions

We use the product ansatz [129], the rational map ansatz [67] and the multi-layer rational map ansatz [99, 51] to create approximate Skyrme fields of non-trivial topological charge \( B \) and of given symmetry type \( G \). Relaxing these initial field configurations with a full 3-dimensional numerical relaxation algorithm we obtain static, minimum energy solutions of the Skyrme model with pion mass \( \mu = 1 \) and baryon number up to 16.

- **Product Ansatz**

Approximate Skyrme fields of the product ansatz form \( U(x) = \prod_{n=1}^{N} U^n(x) \) with baryon number \( B = N \) can be generated by superposing \( N \) well-separated \( B = 1 \) Skyrmions. The individual Skyrmions are initially arranged so that the amount of attraction is maximal between each pair of Skyrmions. The Skyrmion-Skyrmion interaction [69] is maximal when one Skyrmion is rotated in isospace relative to the other through angle \( \pi \) about an axis perpendicular to the line of separation of
the two Skyrmions. Initial Skyrme field configuration obtained with the product
ansatz approximation have for example been used successfully in [24, 25, 29, 22]
to compute minimal-energy Skyrmion solutions.

- **Single Rational Map Ansatz**

  The main idea is to approximate charge $B$ Skyrme configurations $\phi$ which can be
  seen as maps from a 3-sphere in space to a 3-sphere in the target $SU(2)$ by ra-
tional maps $R : S^2 \mapsto S^2$ of degree $B$. Within the rational map ansatz the angular
dependence of the Skyrme field $\phi$ is described by a rational function

\[
R(z) = \frac{p(z)}{q(z)},
\]

where $p$ and $q$ are polynomials in the complex Riemann sphere coordinate $z$. The $z$-
coordinate can be expressed via standard stereographic projection, $z = \tan(\theta/2) e^{i\phi}$,
in terms of the conventional spherical polar coordinates $\theta$ and $\phi$. The radial de-
pendence is encoded in the radial profile function $f(r)$ which has to satisfy $f(0) = \pi$
and $f(\infty) = 0$ to ensure a well-defined behaviour at the origin and a configuration
of finite energy.

The rational function $R(z)$ takes values on the target $S^2$ and its value is associated
via stereographic projection with the cartesian unit vector

\[
\hat{n}_R = \frac{1}{1 + |R|^2} \left( R + \overline{R}, i(\overline{R} - R), 1 - |R|^2 \right).
\]

The rational map approximation for the Skyrme field is given in terms of the isoscalar
$\sigma$ and the pion isotriplet $\pi$ of the non-linear sigma model notation by

\[
\sigma = \cos f(r), \quad \pi = \sin f(r) \hat{n}_R(z).
\]

Substituting (5.3.3) in the Skyrme energy functional (5.2.6) results in an angular
integral $I$ which depends on the rational map $R(z)$ and a radial part only de-
pendent on the monotonic function $f(r)$. To find low energy Skyrmion solutions of
a given topological charge $B$ and pion mass $\mu$ one minimizes $I$ over all maps of
algebraic degree $B$ and then solves the Euler-Lagrange equation for $f(r)$ with the
minimized $I$, $\mu$ and $B$ occuring as parameters. As starting point for our numerical
relaxations we choose initial Skyrme fields generated with the rational maps $R(z)$
given in [67, 29, 94, 99, 39]. Note that the rational maps given in these references
are the optimal maps for Skyrmions with massless pions. However, since the angu-
lar integral $I$ is independent of the mass parameter $\mu$, the same rational maps are
also the minimizing maps for non-zero $\mu$ and the main effect of the pion mass is to
change the shape function $f(r)$.

- **Multi-Layer Rational Map Ansatz**

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For Skyrmion solutions of higher baryon number $B$ and massive pions a generalized version of the single rational map ansatz – the multi-layer rational map ansatz – has proven to be helpful [99, 22, 51]. The $K$-layer ansatz uses $K$ rational maps $R^k(z)$ with $k = 1, \ldots, K$ and a single profile function $f(r)$ satisfying $f(0) = K\pi$ and $f(\infty) = 0$. The Skyrme field can be constructed via (5.3.3), where $R^k(z)$ is used in the region $r_{k-1} < r < r_k$ and the monotonically decreasing profile function has to take the values $f(r_k) = (K - k)\pi$ at a radius $r_k$, with $r_0 = 0$ and $r_K = \infty$. The total baryon number $B$ of the resulting Skyrme configuration is given by summing up all the degrees of the $K$ rational maps.

In this chapter, we use the double rational map ansatz in particular to approximate Skyrmion solutions for $12 \leq B \leq 16$. For example, a $B = 12$ Skyrme configuration with approximate $D_{3h}$ symmetry can be realized by placing a spherically-symmetric $B = 1$ Skyrmion inside a $D_{3h}$-symmetric $B = 11$ Skyrme soliton. Suitable rational maps [22] for the spherically-symmetric degree 1 inner map $R^1(z)$ and the degree 11 outer map $R^2(z)$ are given explicitly by

$$R^1(z) = \frac{-1}{z},$$

$$R^2(z) = \frac{z^2(1 + az^3 + bz^6 + cz^9)}{c + bz^3 + az^6 + z^9},$$

where the real coefficients are $a = -2.47$, $b = -0.84$ and $c = -0.13$. Setting $R(z) = R^1(z)$ for $0 \leq r < r_1$ and $R(z) = R^2(z)$ for $r_1 \leq r < \infty$ in (5.3.3) together with a monotonic radial profile function $f(r)$ chosen so that $f(0) = 2\pi$, $f(\infty) = 0$ and $f(r_1) = \pi$ gives after numerical relaxation a soliton solution of exact $D_{3h}$ symmetry.

### 5.4 Static Skyrmion Solutions

We compute low energy static Skyrmion solutions with baryon numbers up to 16 and with the rescaled pion mass set to $\mu = 1$ by solving the full Skyrme field equations with a numerical three-dimensional relaxation algorithm [29]. Most of our simulations are performed on regular, cubic grids of typically $(201)^3$ grid points and a lattice spacing $\Delta x = 0.1$. Only the numerical results for $B = 1$ Skyrmions were obtained using grids with larger grid spacings ($\Delta x = 0.2$) and $(100)^3$ points. The finite difference scheme used is fourth order accurate in the spatial derivatives. Suitable initial Skyrme field configurations of given topological charge $B$ were created with the methods described in the previous section. For more detailed information on our relaxation procedure we refer the interested reader to the literature [29, 19].
5.4.1 Lower Charge Skyrmions: $1 \leq B \leq 8$

We list in Table 5.1 the energies and the diagonal elements of the isospin inertia tensor $U_{ij}$ for Skyrmions with baryon numbers up to 8 and of symmetry group $G$. The diagonal elements of the spin and mixed inertia tensor and all the off-diagonal elements are given in tabular form in Appendix B. The obtained baryon density isosurfaces can be found in Fig. 5.1.

**B=1**

The minimal-energy $B = 1$ Skyrmion solution is spherically-symmetric and substituting the rational map $R(z) = z$ in (5.3.3) reproduces the standard hedgehog form

$$
\sigma = \cos f(r), \quad \pi = \sin f(r) \hat{r},
$$

(5.4.1)

where $r = |x|$, $f(0) = \pi$ and $f(\infty) = 0$. Solving\(^1\) the Skyrme field equation for a $B = 1$ hedgehog Skyrmion (5.4.1) gives an energy

$$
M_1 = \frac{1}{3\pi} \int_0^\infty \left\{ r^2 f'^2 + 2 \sin^2 f \left( 1 + f'^2 \right) + \frac{\sin^4 f}{r^2} + 2\mu^2 (1 - \cos f) r^2 \right\} dr = 1.416.
$$

(5.4.2)

Note that the inertia tensors (5.2.7) take for an $O(3)$-symmetric Skyrme configuration (5.4.1) the simple form $U_{ij} = V_{ij} = W_{ij} = \Lambda \delta_{ij}$ with

$$
\Lambda = \frac{16\pi}{3} \int_0^\infty r^2 \sin^2 f \left( 1 + f'^2 + \frac{\sin^2 f}{r^2} \right) dr = 47.625.
$$

(5.4.3)

Our full 3D simulations give an approximately 0.9% lower energy value $M_1 = 1.403$ and within our numerical accuracy the inertia tensors (5.2.7) are proportional to the unit matrix with $\Lambda \approx 46.5$.

Note that we cannot confirm the energy value $M_1 = 1.465$ calculated in the recent article [51] for a $B = 1$ hedgehog Skyrmion of mass $\mu = 1$. We double-checked our results using two very different numerical approaches (a collocation method [11, 12] and a one-dimensional gradient flow method). In both cases we obtain an energy value $M_1 = 1.416$. We conclude that the energy value stated in Ref. [51] is incorrect.

**B=2**

The $B = 2$ Skyrmion has toroidal symmetry $D_{\infty h}$ and it can be approximated by choosing the rational map $R(z) = z^2$ in (5.3.3). Relaxing this rational map generated initial Skyrme field configuration with a 3D relaxation code, we verify that all inertia tensors

\(^1\)We used the collocation method [11, 12] to determine the profile function $f(r)$ which minimizes $M_1$ (5.2.6). The rational map approximations for the higher charge Skyrmion solutions were generated with the profile function calculated in the charge-1 sector.
are diagonal, with $U_{11} = U_{22} = 97.039$, $V_{11} = V_{22} = 153.995$ and $W_{11} = W_{22} = 0$. Our numerically calculated charge-2 configuration fulfills the relation $U_{33} = \frac{1}{2} W_{33} = \frac{1}{4} V_{33}$ [35, 66] – a manifestation of the axial symmetry. The soliton’s energy is $M_2 = 2.720$, which is reasonable close to the energy value $M_2 = 2.77$ given in [51]. Finally, we can check the results of our fully three-dimensional numerical relaxation with those obtained by minimizing the two-dimensional, total energy functional of an axially symmetric Skyrme configuration. An axially-symmetric ansatz [77] is given by

$$\sigma = \psi_3, \quad \pi_1 = \psi_1 \cos n\theta, \quad \pi_2 = \psi_1 \sin n\theta, \quad \pi_3 = \psi_2,$$ (5.4.4)

where $\psi(\rho, z) = (\psi_1, \psi_2, \psi_3)$ is a unit vector that is dependent on the cylindrical coordinates $\rho$ and $z$. Here, the non-zero winding number $n \in \mathbb{Z}$ counts the windings of the Skyrme fields in the $(x_1, x_2)$-plane. Substituting (5.4.4) in (5.2.6) results in the classical soliton mass

$$M_B = 2\pi \int_0^\infty d\rho \int_{-\infty}^{+\infty} dz \rho \left\{ \left( \partial_\rho \psi \cdot \partial_\rho \psi + \partial_z \psi \cdot \partial_z \psi \right) \left( 1 + \frac{n^2}{\rho^2} \psi_1^2 \right) + \left| \partial_\rho \psi \times \partial_z \psi \right|^2 + \frac{n^2}{\rho^2} \psi_1^2 + 2\mu^2 (1 - \psi_3^2) \right\},$$ (5.4.5)

and the baryon number $B$ (5.2.2) of an axially-symmetric configuration $\psi$ is given by

$$B = \frac{n}{\pi} \int_0^\infty d\rho \int_{-\infty}^{+\infty} dz \left| \psi_1 \psi_2 \left| \partial_\rho \psi \times \partial_z \psi \right| \right|.$$ (5.4.6)

To ensure a configuration of finite energy $M_B$ the unit vector $\psi$ has to satisfy the boundary condition $\psi \to (0, 0, 1)$ as $\rho^2 + z^2 \to \infty$ together with $\psi_1 = 0$ and $\partial_\rho \psi_2 = \partial_\rho \psi_3 = 0$ at $\rho = 0$. A suitable start configuration with baryon number $B = n$ is given in [77] by

$$\psi_1 = \frac{\rho}{r} \sin f(r), \quad \psi_2 = \frac{z}{r} \sin f(r), \quad \psi_3 = \cos f(r),$$ (5.4.7)

where $r = \sqrt{\rho^2 + z^2}$ and $f(r)$ denotes, as usual, a monotonically decreasing profile function satisfying the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$. For $B = 2$ minimization of the static energy functional (5.4.5) with a simple gradient flow algorithm on a rectangular grid in the $(\rho, z)$-plane containing $(401)^2$ grid points with a lattice spacing $\Delta x = 0.05$, so that $(\rho, z) \in [0, 20] \times [-10, 10]$, gives $M_2 = 2.720$ which agrees with our 3D results. The nonvanishing components of the isospin inertia tensor $U_{ij}$ (5.2.7a) are given for the
axially-symmetric ansatz (5.4.4) by [54]

\begin{align}
U_{33} &= 4\pi \int_0^\infty d\rho \int_{-\infty}^{+\infty} dz \rho \left\{ \psi_1^2 \left( \partial_\rho \psi \cdot \partial_\rho \psi + \partial_z \psi \cdot \partial_z \psi + 1 \right) \right\}, \\
U_{11} &= U_{22} = 2\pi \int_0^\infty d\rho \int_{-\infty}^{+\infty} dz \rho \left\{ \psi_1^2 + 2\psi_2^2 + \left( \partial_\rho \psi_3 \right)^2 + \left( \partial_z \psi_3 \right)^2 \right\} \\
&\quad + \left( \partial_\rho \psi \cdot \partial_\rho \psi + \partial_z \psi \cdot \partial_z \psi + n^2 \frac{\psi_1^2}{\rho^2} \right) \psi_2^2 + n^2 \frac{\psi_1^4}{\rho^2} \right\}.
\end{align}

\textbf{B=3}

The minimal-energy \(B=3\) Skyrmion has \(T_d\) symmetry and can be created with the rational map [94]

\[ R(z) = \frac{\sqrt{3}iz^2 - 1}{z^3 - \sqrt{3}iz}. \]  

Relaxing the tetrahedrally-symmetric \(B=3\) Skyrme configuration (5.4.9), we verify that the inertia tensors (5.2.7) are all diagonal: \(U_{ij} = u \delta_{ij}, V_{ij} = v \delta_{ij}\) and \(W_{ij} = w \delta_{ij}\) with \(u = 124.05, v = 402.82\) and \(w = 85.23\). We find for the total energy \(M_3 = 3.969\) which is slightly lower than the numerical value \(M_3 = 4.02\) given in [51].

\textbf{B=4}

The minimal-energy Skyrmion solution with \(B=4\) has octahedral symmetry \(O_h\) and can be approximated by the rational map [94]

\[ R(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}. \]

The inertia tensors \(U_{ij}, V_{ij}\) and \(W_{ij}\) for the cubically-symmetric, numerically relaxed charge-4 configuration (5.4.10) are determined to be diagonal, fulfilling \(U_{11} = U_{22} = 148.04\) and \(V_{ij} = v \delta_{ij}\) with \(v = 666.76\) and with the cross-term \(W_{ij}\) vanishing within the limits of our numerical accuracy. Our calculated energy value \(M_4 = 5.177\) agrees very well with the one \((M_4 = 5.18\) stated in [51].

\textbf{B=5}

For \(B=5\) we construct numerically four different types of Skyrmion solutions: the dihedral-symmetric Skyrme solitons \(5D_{2d}\) and \(5D_{4h}\), the octahedrally-symmetric configuration \(5O_h\) and the toroidal Skyrme field \(5D_{oh}\). The minimal energy Skyrmion solution in the charge-5 sector has \(D_{2d}\) symmetry and can be obtained by relaxing a Skyrme configuration generated by the rational map [94]

\[ R(z) = \frac{z \left( z^4 + ibz^2 + a \right)}{az^4 + ibz^2 + 1}, \]  

(5.4.11)
where the two real parameters $a, b$ are taken to be $a = -3.07$ and $b = 3.94$. The corresponding inertia tensors are diagonal with $U_{11} = U_{22} = 200.84$, $V_{11} = V_{22} = 1214.82$ and $W_{11} = W_{22} = -70.07$. Numerically we calculate for the static classical Skyrmion mass $M_5 = 6.472$. Following [94] we can construct a $D_{4h}$-symmetric initial charge-5 Skyrme configuration by setting $b = 0$ in (5.4.11) and choosing the real parameter $a \neq -5$. For the minimal energy $5D_{4h}$ Skyrmion solution the inertia tensors are diagonal and given by $U_{11} = U_{22} = 198.52$, $V_{11} = V_{22} = 1109.89$ and $W_{11} = W_{22} = -186.84$. The relaxed $D_{4h}$ solution has an energy $M_5 = 6.621$. Octahedral symmetry $O_h$ can be achieved when setting $a = -5$ and $b = 0$ in the rational map (5.4.11). Within the limits of our numerical accuracy the inertia tensors of a relaxed $5O_h$ configuration are diagonal and proportional to the unit matrix: $U_{ij} = u\delta_{ij}$, $V_{ij} = v\delta_{ij}$ and $W_{ij} = w\delta_{ij}$ with $u \approx 197.46$, $v \approx 1090.75$ and $w \approx -184.66$. The energy of the relaxed octahedrally-symmetric charge-5 Skyrme soliton does not differ within our numerical accuracy from that of the relaxed $D_{4h}$ solution. Finally, there exists a toroidal saddle point solution $D_{ooh}$ with $U_{11} = U_{22}$, $V_{11} = V_{22}$ $W_{11} = W_{22}$ and $U_{33} = \frac{1}{3}W_{33} = \frac{1}{3}V_{33}$ of higher energy ($M_5 = 6.845$) which can be obtained by relaxing a rational map with $a = b = 0$ in (5.4.11).

**B=6**

To approximate a minimal-energy charge-6 Skyrmion solution we use the $D_{4d}$-symmetric rational map given in [94]

\[
R(z) = \frac{z^4 + ia}{z^2(1 + iaz)} ,
\]

(5.4.12)

where the real parameter $a = 0.16$. The inertia tensors of the relaxed $D_{4d}$-symmetric configuration are determined to be diagonal, fulfilling $U_{11} = U_{22} \approx 228.59$, $V_{11} = V_{22} \approx 1720.93$ and $W_{11} = W_{22} \approx 0$.

**B=7**

In the charge-7 sector we compute minimal energy Skyrme configurations for four different symmetry groups: the icosahedrally-symmetric configurations $7Y_h$, the $C_3$ symmetry breaking $7D_{3d}$ Skyrme soliton, the tetrahedrally-symmetric configuration $7T_d$ and the octahedrally-symmetric solution $7O_h$. The $B = 7$ Skyrme solution of lowest energy has icosahedral symmetry and is obtained by minimizing an initial Skyrme configuration created with the rational map [94]

\[
R(z) = \frac{az^5 + 1}{z^2(z^5 - a)} ,
\]

(5.4.13)

where the parameter $a$ is chosen to be 7. Our energy mimimization code gives $M_7 = 9.12$ for the classical Skyrmion mass (5.2.6) of the relaxed, icosahedrally-symmetric field configuration (5.4.13). The dihedrally-symmetric $D_{5d}$ soliton solution which can be con-
structured by setting $a \neq 7$ in (5.4.13) is found to be of comparable energy. Further, we create initial conditions with the single rational map approximation of a tetrahedrally-symmetric charge-7 Skyrmion [99]

$$R(z) = \frac{bz^6 - 7z^4 - bz^2 - 1}{z(z^8 + bz^4 + 7z^2 - b)},$$  \hspace{1cm} (5.4.14)

where $b$ is chosen to be imaginary: $b = 0.5i$. Using (5.4.14) as start configuration in our energy minimization algorithm, we calculate $M_7 = 9.489$ and $U_{ij} = u\delta_{ij}$ with $u \approx 277.62$. Finally, we construct an octahedrally-symmetric Skyrme field [99] using the rational map (5.4.14) with $b = 0$. The inertia tensors of the numerically relaxed $O_h$ field configuration (5.4.14) satisfy $U_{11} = U_{22} = U_{33} \approx 276.74$, $V_{11} = V_{22} = V_{33} \approx 2311.81$ and $W_{11} = W_{22} = W_{33} \approx -305.80$. The associated energy of the cube-shaped Skyrme soliton is calculated to be $M_7 = 9.54$.

**B=8**

For baryon number $B = 8$, we use the single rational map ansatz [67] as well as the product ansatz [129] to generate suitable initial conditions for our numerical relaxation simulations. We approximate an initial field configuration with $D_{6d}$ symmetry by the rational map [94]

$$R(z) = \frac{z^6 - ia}{z^2(iaz^6 - 1)},$$  \hspace{1cm} (5.4.15)

where the free parameter is set to $a = 0.14$. The obtained relaxed Skyrme field resembles a hollow polyhedron: a ring of twelve pentagons with a hexagon at the top and at the bottom (see baryon density isosurface plot in Fig. 5.1). Furthermore, there is a $D_{4h}$-symmetric Skyrme configuration described by the rational map [94]

$$R(z) = \frac{z^8 + bz^6 - az^4 + bz^2 + 1}{z^8 - bz^6 - az^4 - bz^2 + 1},$$  \hspace{1cm} (5.4.16)

where we choose $a = 5$ and $b = 2$. We verify that the inertia tensors of the relaxed $D_{4h}$-symmetric $B = 8$ Skyrme field (5.4.16) take the form $U_{11} \neq U_{33}$ and $V_{33} \neq V_{11} = V_{22}$ and $W_{ij} = 0$. Note that our calculated energy value $M_8 = 10.28$ for the dihedral $B = 8$ solution agrees well with the values ($M_8' = 10.28$ and $M_8'' = 10.25$) given in [51]. An octahedrally-symmetric configuration can be obtained [94] by setting $a = 10$ and $b = 4\sqrt{3}$ in the map (5.4.16). For the numerically relaxed solution with $O_h$ symmetry the inertia tensors (5.2.7) are computed to be diagonal and to satisfy $U_{11} = U_{33} \approx 305.789$, $V_{11} = V_{22} = V_{33} \approx 2481.23$ and $W_{ij} = 0$.

\footnote{Note that these energy values have been calculated by relaxing start configurations created with the product ansatz of two $B = 4$ cubes.}
5.4. STATIC SKYRMION SOLUTIONS

1O(3) 2D_{coh} 3T_d 4O_h 5D_{2d}

5D_{4h} 5O_h 5D_{coh} 6D_{td} 7Y_h

7D_{5d} 7O_h 8D_{td} 8D_{4h} 8O_h

Figure 5.1: Lower charge Skyrmions: Baryon density isosurfaces of minimal-energy Skyrmion solutions with baryon number $B = 1 – 8$ and with pion mass parameter $\mu = 1$.

5.4.2 Higher Charge Skyrmions: $9 \leq B \leq 16$

We display in Fig. 5.2 the baryon density isosurfaces of Skyrmion solutions for topological charge $9 \leq B \leq 16$ and with the rescaled mass parameter $\mu$ set to 1. The corresponding numerical values for the classical Skyrmion mass $M_B$ (5.2.6) and the diagonal elements of the isorotational inertia tensor $U_{ij}$ (5.2.7) can be found in Table 5.2. Additionally, we include in Appendix B all the diagonal elements of the rotational inertia tensor $V_{ij}$ and of the cross term $W_{ij}$ together with all the off-diagonal elements. For further details on the numerical methods used the reader is referred to to the literature [29, 19].

B=9

For baryon number $B = 9$ we construct numerically three different types of Skyrmion solutions: dihedrally-symmetric $9D_{4d}$ solitons, tetrahedrally-symmetric configurations $9T_d$ and icosahedrally-symmetric Skyrme fields $9Y_h$. The solution of lowest energy in the charge-9 sector is believed to possess $D_{4d}$ symmetry [30], that is the minimal-energy Skyrmion shares the same symmetries as the one in the massless case [29]. The associated baryon density isosurface (see Fig. 5.2) resembles a flat polyhedron whose top and bottom face are made up of four pentagons and both faces are linked by a ring of eight pentagons which point alternately up- and downwards. Due to the $D_{4d}$ symmetry the top and bottom face are rotated $45^\circ$ relative to each other. To find the $B = 9$ minimum energy soliton solution, we relax a configuration created by the rational map [29]

$$R(z) = \frac{z(a + ibz^4 + z^8)}{1 + ibz^4 + az^8}, \quad (5.4.17)$$

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Table 5.1: Skyrmions of baryon number 1 ≤ B ≤ 8. We list the energies $M_B$, the energy per baryon $M_B/B$, the diagonal elements of the inertia tensor $U_{ij}$ and the symmetries $G$ of the Skyrme solitons. Note that energies $M_B$ are given in units of 12π² and that the mass parameter is chosen to be $\mu = 1$. (⋆) These configurations correspond to global energy minima for given baryon number $B$.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$G$</th>
<th>$M_B$</th>
<th>$M_B/B$</th>
<th>$U_{11}$</th>
<th>$U_{22}$</th>
<th>$U_{33}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$O(3)$</td>
<td>1.403*</td>
<td>1.403</td>
<td>46.592</td>
<td>46.591</td>
<td>46.591</td>
</tr>
<tr>
<td>2</td>
<td>$D_{soh}$</td>
<td>2.720*</td>
<td>1.360</td>
<td>97.039</td>
<td>97.058</td>
<td>68.848</td>
</tr>
<tr>
<td>3</td>
<td>$T_d$</td>
<td>3.969*</td>
<td>1.323</td>
<td>124.059</td>
<td>124.059</td>
<td>124.059</td>
</tr>
<tr>
<td>4</td>
<td>$O_h$</td>
<td>5.177*</td>
<td>1.294</td>
<td>148.044</td>
<td>148.034</td>
<td>177.221</td>
</tr>
<tr>
<td>5</td>
<td>$D_{2d}$</td>
<td>6.472*</td>
<td>1.294</td>
<td>200.845</td>
<td>200.847</td>
<td>180.145</td>
</tr>
<tr>
<td></td>
<td>$D_{4h}$</td>
<td>6.621</td>
<td>1.324</td>
<td>198.520</td>
<td>198.525</td>
<td>196.778</td>
</tr>
<tr>
<td></td>
<td>$O_h$</td>
<td>6.621</td>
<td>1.286</td>
<td>196.481</td>
<td>196.482</td>
<td>197.231</td>
</tr>
<tr>
<td></td>
<td>$D_{soh}$</td>
<td>7.292</td>
<td>1.459</td>
<td>262.295</td>
<td>262.291</td>
<td>119.959</td>
</tr>
<tr>
<td>6</td>
<td>$D_{4d}$</td>
<td>7.712*</td>
<td>1.285</td>
<td>228.594</td>
<td>228.485</td>
<td>226.800</td>
</tr>
<tr>
<td>7</td>
<td>$Y_h$</td>
<td>9.111</td>
<td>1.302</td>
<td>280.622</td>
<td>290.767</td>
<td>297.466</td>
</tr>
<tr>
<td></td>
<td>$D_{5d}$</td>
<td>9.110</td>
<td>1.301</td>
<td>280.467</td>
<td>290.688</td>
<td>297.497</td>
</tr>
<tr>
<td></td>
<td>$T_d$</td>
<td>9.364</td>
<td>1.338</td>
<td>281.798</td>
<td>281.795</td>
<td>281.814</td>
</tr>
<tr>
<td></td>
<td>$O_h$</td>
<td>9.478</td>
<td>1.354</td>
<td>279.943</td>
<td>279.939</td>
<td>279.933</td>
</tr>
<tr>
<td>8</td>
<td>$D_{5d}$</td>
<td>10.237*</td>
<td>1.279</td>
<td>295.882</td>
<td>295.900</td>
<td>282.661</td>
</tr>
<tr>
<td></td>
<td>$D_{4h}$</td>
<td>10.288</td>
<td>1.286</td>
<td>299.609</td>
<td>260.983</td>
<td>302.328</td>
</tr>
<tr>
<td></td>
<td>$O_h$</td>
<td>10.286</td>
<td>1.286</td>
<td>305.789</td>
<td>265.244</td>
<td>305.788</td>
</tr>
</tbody>
</table>

where $a = -3.38$ and $b = -11.19$. The inertia tensors of the relaxed dihedrally-symmetric Skyrme configuration (5.4.17) are found to have the symmetries $U_{11} = U_{22} \approx 314.55$, $V_{11} = V_{22} \approx 2837.60$ and $W_{11} = W_{22} \approx 315.44$. The static Skyrmion mass (5.2.6) is calculated to be $M_9 = 11.521$. Note that there also exists a saddle point solution with $T_d$ symmetry and of slightly larger energy $M_9 = 11.55$. The associated Skyrme field can be thought of being made up of four hexagons located at the vertices of a tetrahedron and linked by four triplets of pentagons [67, 29] (compare Fig. 5.2). As initial Skyrme configuration for our energy minimization calculations we use the tetrahedral rational map approximation given in [67]

$$R(z) = \frac{5i \sqrt{3} z^6 - 9z^4 + 3i \sqrt{3} z^2 + 1 + az^2 (z^6 - i \sqrt{3} z^4 - z^2 + i \sqrt{3})}{z^3 (-z^6 - 3i \sqrt{3} z^4 + 9z^2 - 5i \sqrt{3}) + az (-i \sqrt{3} z^6 + z^4 + i \sqrt{3} z^2 - 1)}, \quad (5.4.18)$$

where $a = -1.98$. The associated inertia tensors are found to satisfy within the limits of our numerical accuracy $U_{ij} = u \delta_{ij}$ with $u \approx 326.72$, $V_{11} = V_{22} = V_{33} \approx 3155.44$ and $W_{ij} = w \delta_{ij}$ with $w \approx -118.69$.

**B=10**

For $B = 10$ we compute five different, dihedrally-symmetric Skyrmion solutions ($D_{2h}$, $D_{5d}$, $D_{4d}$, $D_{3h}$) whose energies (5.2.6) are very close together. The minimal energy $B = 10$ Skyrmion for $\mu = 1$ has $D_{2h}$ symmetry [29]. As an suitable initial condition with
5.4. STATIC SKYRMION SOLUTIONS

Figure 5.2: Higher charge Skyrmions: Baryon density isosurfaces of minimal-energy Skyrmion solutions with baryon number $B = 9 - 16$ and with pion mass parameter $\mu = 1$.

this symmetry we take the optimal rational map [39] for $\mu = 1$:

$$R(z) = \frac{\alpha + \beta z^2 + \gamma z^4 + \delta z^6 + \epsilon z^8 + z^{10}}{1 + \epsilon z^2 + \delta z^4 + \gamma z^6 + \beta z^8 + \alpha z^{10}},$$

(5.4.19)

where $\alpha = 0.2772$, $\beta = -9.3594$, $\gamma = 14.81$, $\delta = 4.997$ and $\epsilon = 3.015$. The structure of the solution presented in Fig. 5.2 can be interpreted in terms of a pair of deformed $B = 4$ cubes separated by two single $B = 1$ Skyrmions. A $D_3$ symmetric configuration can be obtained from a rational map of the form [29]

$$R(z) = \frac{z(1 + az^3 + bz^6 + cz^9)}{c + bz^3 + az^6 + z^9},$$

(5.4.20)

where $a$, $b$, $c$ denote complex parameters. Choosing the values $a = 4.40 - 1.72i$, $b = -2.38 + 3.10i$ and $c = -0.12 + 0.19i$ in (5.4.20) gives the energy minimizing map $R(z)$ for the massless $D_3$ Skyrmion. With this choice of initial conditions our energy relaxation with $\mu = 1$ results in a field configuration with $U_{11} = U_{22} \approx 354.60$ and static energy $M_{10} \approx 12.81$. The baryon density of the $D_3$ solution can be thought [29] of being composed of two pentagon triples with three hexagons and three pentagons placed alternately around each pentagon triple. A Skyrme configuration of very similar structure – only differing in how the hexagons and pentagons are distributed around the three pentagon triples – can be obtained when setting $a = 20.40i$, $b = -30.22$ and $c = -4.69i$ in (5.4.20). The
attained Skyrme field has $D_{3d}$ symmetry and the corresponding inertia tensors are found to be diagonal satisfying $U_{11} = U_{22} \approx 360.33$, $V_{11} = V_{22} \approx 4252.80$ and $W_{11} = W_{22} \approx 0$. A $D_{3h}$ Skyrmion solution is attained when the coefficients $a$, $b$, $c$ are chosen to be real. In particular, we take as initial condition for our numerical relaxation simulations the minimal energy $D_{3h}$ symmetric Skyrmion for massless pions which can be constructed with $a = -5.14$, $b = -2.20$ and $c = -0.36$ in (5.4.20). Finally, we relax a charge-10 configuration with the four-fold symmetry $D_{4d}$ generated with the rational map [29]

$$R(z) = \frac{z^2(a + ibz^4 + z^8)}{1 + ibz^4 + az^8},$$  \hspace{1cm} (5.4.21)

where the optimal values in the massless case have been determined to be $a = -8.67$ and $b = 14.75$.

Table 5.2: Same as Table 5.1 but for Skyrme solutions of baryon number $9 \leq B \leq 16$.

<table>
<thead>
<tr>
<th>B</th>
<th>G</th>
<th>$M_B$</th>
<th>$M_B/B$</th>
<th>$U_{11}$</th>
<th>$U_{22}$</th>
<th>$U_{33}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>$D_{4d}$</td>
<td>11.521*</td>
<td>1.280</td>
<td>314.556</td>
<td>314.553</td>
<td>339.622</td>
</tr>
<tr>
<td></td>
<td>$T_d$</td>
<td>11.556</td>
<td>1.284</td>
<td>326.725</td>
<td>326.727</td>
<td>326.726</td>
</tr>
<tr>
<td>10</td>
<td>$D_{2h}$</td>
<td>12.752*</td>
<td>1.275</td>
<td>366.133</td>
<td>351.990</td>
<td>359.097</td>
</tr>
<tr>
<td></td>
<td>$D_{3d}$</td>
<td>12.752</td>
<td>1.275</td>
<td>360.337</td>
<td>360.761</td>
<td>360.623</td>
</tr>
<tr>
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<td>$D_{4d}$</td>
<td>12.802</td>
<td>1.280</td>
<td>359.288</td>
<td>359.283</td>
<td>348.443</td>
</tr>
<tr>
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<td>$D_3$</td>
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<td>1.282</td>
<td>354.609</td>
<td>354.635</td>
<td>360.045</td>
</tr>
<tr>
<td></td>
<td>$D_{3h}$</td>
<td>12.827</td>
<td>1.283</td>
<td>368.828</td>
<td>368.753</td>
<td>355.904</td>
</tr>
<tr>
<td>11</td>
<td>$D_{3h}$</td>
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<td>1.281</td>
<td>407.003</td>
<td>406.785</td>
<td>392.382</td>
</tr>
<tr>
<td>12</td>
<td>$T_d$</td>
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<td>1.280</td>
<td>429.610</td>
<td>429.586</td>
<td>413.360</td>
</tr>
<tr>
<td></td>
<td>$D_{3h}$</td>
<td>15.428</td>
<td>1.286</td>
<td>455.821</td>
<td>455.817</td>
<td>459.930</td>
</tr>
<tr>
<td></td>
<td>$O_h$</td>
<td>15.564</td>
<td>1.297</td>
<td>500.375</td>
<td>500.357</td>
<td>444.477</td>
</tr>
<tr>
<td>13</td>
<td>$O$</td>
<td>16.653*</td>
<td>1.281</td>
<td>448.191</td>
<td>448.332</td>
<td>448.126</td>
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<tr>
<td></td>
<td>$D_{4d}$</td>
<td>16.653</td>
<td>1.281</td>
<td>451.031</td>
<td>450.825</td>
<td>446.853</td>
</tr>
<tr>
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<td>$O_h$</td>
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<td>1.296</td>
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<td>466.842</td>
<td>466.971</td>
</tr>
<tr>
<td>14</td>
<td>$D_2$</td>
<td>17.964</td>
<td>1.283</td>
<td>488.331</td>
<td>488.229</td>
<td>480.712</td>
</tr>
<tr>
<td></td>
<td>$O_h$</td>
<td>18.788</td>
<td>1.342</td>
<td>617.533</td>
<td>617.753</td>
<td>617.702</td>
</tr>
<tr>
<td>15</td>
<td>$T$</td>
<td>19.289*</td>
<td>1.285</td>
<td>510.964</td>
<td>509.279</td>
<td>511.750</td>
</tr>
<tr>
<td></td>
<td>$T_d$</td>
<td>19.334</td>
<td>1.288</td>
<td>520.986</td>
<td>520.995</td>
<td>521.107</td>
</tr>
<tr>
<td>16</td>
<td>$T_d'$</td>
<td>20.515</td>
<td>1.282</td>
<td>638.278</td>
<td>638.265</td>
<td>624.806</td>
</tr>
<tr>
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<td>$D_2$</td>
<td>20.558</td>
<td>1.285</td>
<td>540.346</td>
<td>537.700</td>
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</tr>
<tr>
<td></td>
<td>$D_3$</td>
<td>20.566</td>
<td>1.285</td>
<td>540.749</td>
<td>540.805</td>
<td>537.684</td>
</tr>
<tr>
<td></td>
<td>$T_d''$</td>
<td>20.841</td>
<td>1.303</td>
<td>628.354</td>
<td>628.296</td>
<td>624.189</td>
</tr>
</tbody>
</table>

B=11

The minimal rational map of degree 11 has been found [29] to possess $D_{3h}$ symmetry in the massless pion limit. Explicitly, a suitable $D_{3h}$ symmetric map can be written in the
form [29]

\[ R(z) = \frac{z^2 \left(1 + az^3 + bz^6 + cz^9\right)}{c + bz^3 + az^6 + z^9}, \quad (5.4.22) \]

where the real parameters are chosen to take the values \( a = -2.47, b = -0.84 \) and \( c = -0.13 \). Relaxing the Skyrme field configuration (5.4.22) with the rescaled mass parameter \( \mu = 1 \) results in a local minimal energy solution with \( M_{11} \approx 14.10 \) and the isorotational inertia tensor is diagonal with \( U_{11} = U_{22} \approx 407.003 \).

**B=12**

For baryon number \( B = 12 \), we relax a field configuration with exact \( T_d \) symmetry [29]

\[ R(z) = \frac{ap_+^3 + bp_-^3}{p_+^2 p_-}, \quad (5.4.23) \]

where \( p_\pm = z^4 \pm 2\sqrt{3}iz^2 + 1 \) are the Klein polynomials and the real parameters are chosen analogous to the massless Skyrme model \( a = -0.53, b = 0.78 \). Numerically, we obtain for the static energy of the relaxed field configuration (5.4.23) \( M_{12} \approx 15.363 \) and the inertia tensor satisfy \( U_{11} = U_{22} \approx 429.61, V_{11} = V_{22} \approx 5365.81 \) and \( W_{11} \approx 0 \). Further there exists a triangular \( B = 12 \) soliton solution [22] in the massive Skyrme model which can be thought of being made up of three \( B = 4 \) cubes placed so that their centres lie on the vertices of an equilateral triangle. Such a Skyrme field configuration with \( D_{3h} \) symmetry can be constructed via the double rational map ansatz (5.3.4): The outer map is given by the \( D_{3h} \)-symmetric, degree 11 map (5.4.22) and the inner map is chosen to be a spherically-symmetric degree 1 mapping (5.3.4a). Relaxing this field configuration with exact \( D_{3h} \) symmetry produces a saddle point solution with static energy \( M_{12} = 15.428 \) and inertia tensors satisfying the relations \( U_{11} = U_{22} \approx 455.821 \) and \( V_{11} = V_{22} \approx 4711.04 \). To construct a \( B = 12 \) solution of cubic symmetry \( O_h \) [99], we use a double rational map ansatz with the inner map chosen to be the tetrahedrally-symmetric charge-5 map (5.4.11) with \( a = -5 \) and \( b = 0 \) and with the outer rational map given by the tetrahedrally-symmetric charge-7 map (5.4.14). After relaxation, we obtain a cubically-symmetric solution (most likely a saddle point solution) of energy \( M_{12} = 15.564 \) and with the isospin inertia tensor fulfilling \( U_{11} = U_{22} \approx 500.37 \).

**B=13**

A \( B = 13 \) Skyrme solution with cubic symmetry group \( O \) can be constructed from the single rational map [29]

\[ R(z) = \frac{z \left(a + (6a - 39)z^4 - (7a + 26)z^8 + z^{12}\right)}{1 - (7a + 26)z^4 + (6a - 39)z^8 + az^{12}}, \quad (5.4.24) \]
where \( a = 0.4 + 5.18i \). Relaxing the Skyrme configuration (5.4.24) gives \( M_{13} = 16.653 \) and the isospin inertia tensor is diagonal and proportional to the unit matrix: \( U_{ij} = u \delta_{ij} \) with \( u = 448.12 \). \( O_h \) symmetry is attained for real \( a \) in (5.4.24). Taking for \( a \) the optimal value for the massless Skyrme model \( a = 7.2 \) and relaxing the start configuration (5.4.24) results for \( \mu = 1 \) in an \( O_h \)-symmetric solution of slightly higher energy \( M_{13} = 16.85 \) which is believed to be a saddle point. The isorotational moments of inertia are diagonal and satisfy \( U_{ij} = u \delta_{ij} \) with \( u \approx 466.82 \). Further we minimize a \( D_{4d} \)-symmetric initial field configuration created with the rational map [22]

\[
R(z) = \frac{z\left(ia + bz^4 + icz^8 + z^{12}\right)}{1 + iaz^4 + bz^8 + iacz^{12}},
\]

(5.4.25)

where the real parameters are chosen to be the ones computed in the massless case [29]: \( a = -5.15, b = -50.46 \) and \( c = 46.31 \). The energy value \( M_{12} = 16.653 \) calculated for the relaxed \( B = 13 \) Skyrme configuration (5.4.25) is very close to the one obtained from the relaxation of the \( O \)-symmetric map (5.4.24). Both configuration – the \( O \)-symmetric and the \( D_{4d} \)-symmetric – can be seen as energy-degenerate within the limits of our numerical accuracy.

**B=14**

We construct three different types of \( B = 14 \) Skyrmion solutions: a \( D_2 \)-symmetric soliton solution [29], a cubically-symmetric solution \( O_h \) [99] and a \( B = 14 \) crystal chunk [98]. A Skyrmion with \( D_2 \)-symmetry can be approximated by the degree 14 rational map [29]

\[
R(z) = \left(\sum_{j=0}^{7} a_j z^{2j}\right) / \left(\sum_{j=0}^{7} a_{7-j} z^{2j}\right),
\]

(5.4.26)

where the complex coefficients \( a_j \) (optimized for massless pions) can be found in Table 3 of [29]. Relaxing (5.4.26) for the mass parameter \( \mu = 1 \) we obtain a minimum energy, \( D_2 \)-symmetric Skyrmion with energy \( M_{14} \approx 17.964 \). The isospin inertia tensor is approximately diagonal within our numerical accuracy and shows the symmetry \( U_{11} = U_{22} \approx 488.32 \). A cubically-symmetric \( B = 14 \) solution can be achieved by using a double rational map approximation [99]: One can combine two dodecahedral degree 7 rational maps so that the inner map is given by (5.4.14) with \( b = 7/\sqrt{5} \) and the outer map is described by (5.4.14) with \( b = -7/\sqrt{5} \). The resulting, relaxed cubically-symmetric solution with \( T_h \) symmetry has an energy \( M_{14} \approx 18.78 \). The associated isorotational moments of inertia take the form \( U_{ij} = u \delta_{ij} \) with \( u = 617.53 \).

Finally, we construct a cubic crystal chunk with baryon number \( B = 14 \) by relaxing
the double rational map ansatz given in [98]:

\[ R^1(z) = z, \]  
\[ R^2(z) = z \left( \frac{z_{12}^{12} - (7a + 26)z^8 + (6a - 39)z^4 + a}{az_{12}^{12} + (6a - 39)z^8 - (7a + 26)z^4 + 1} \right), \]

where we choose the real parameter \( a = 0.5 \). The two-shell ansatz (5.4.27) involves a spherically symmetric degree 1 inner map \( R^1(z) \) together with an \( O_h \)-symmetric outer map \( R^2(z) \) of degree 13 (5.4.24). The crystal chunk (5.4.27) can be viewed as being made up of an \( O_h \)-symmetric \( B = 13 \) Skyrmion with a \( B = 1 \) Skyrmion placed at its centre. When relaxed, we find the static energy to be \( M_{14} = 18.42 \) and the isospin inertia tensor is diagonal and proportional to the unit matrix: \( U_{ij} = u \delta_{ij} \) with \( u \approx 580.38 \).

Note that the two-shell rational map ansatz (5.4.27), although proposed by Manton in [98], has never actually been relaxed before to construct a cubic \( B = 14 \) crystal chunk (with massive pions). Remarkably, relaxing it we do not reproduce the cubically symmetric solution which we constructed above by using two degree 7 maps with icosahedral symmetry [99]. However, we find that the double layer ansatz (5.4.27), with \( B = 1 \) inside and \( B = 13 \) outside, relaxes to a new \( B = 14 \) Skyrmion solution with cubic symmetry and of lower energy.

**B=15**

For baryon number \( B = 15 \) we construct numerically two different tetrahedrally-symmetric Skyrmion solutions: one with \( T \) symmetry and the other with \( T_d \) symmetry. As an initial condition we take the rational map approximation for the minimal energy charge-15 Skyrmion for massless pions which takes the form \( R(z) = p(z)/q(z) \) [29] where

\[
p(z) = i \sqrt{3} (1 + a - b) z^{15} + (77 - 99a - 5b) z^{13} \\
+ i \sqrt{3} (637 + 21a + 35b) z^{11} + (1001 + 561a - 65b) z^9 \\
+ i \sqrt{3} (-429 + 99a + 45b) z^7 + (-1001 - 297a - 127b) z^5 \\
- i \sqrt{3} (273 + 185a + 15b) z^3 + (115 + 27a + 5b) z,
\]

and \( q(z) = z^{15} p(1/z) \). A minimum energy, \( T \)-symmetric rational map for \( B = 15 \) soliton solutions in the massless Skyrme model is obtained for the complex parameter choice: \( a = 0.16 + 2.06i \) and \( b = -4.47 - 8.57i \). After relaxation, we find for the energy of the solution \( M_{15} = 19.28 \). When \( a \) and \( b \) are chosen to be real, a \( T_d \)-symmetric Skyrme configuration of slightly higher energy can be achieved. Minimization of the Skyrme field configuration (5.4.28) with the optimal values \( a = 4.64 \) and \( b = -20.45 \) for massless pions gives for the classical Skyrmion mass (5.2.6) \( M_{15} = 19.33 \). \( T_d \) symmetry implies that the isospin inertia tensor is diagonal with \( U_{ij} = u \delta_{ij} \), where \( u \approx 520.99 \).
For $B = 16$ we consider four different Skyrmion solutions: two dihedrally-symmetric ($D_2$, $D_3$) configurations and two tetrahedrally-symmetric ($T_d$) solutions. A solution (most likely a saddle point) with $D_3$ symmetry in the massless Skyme model is known to be well approximated by a $D_3$-symmetric rational map [29]

$$R(z) = \left( \sum_{j=0}^{\delta} a_j z^{3j+1} \right) / \left( \sum_{j=0}^{\delta} a_{5-j} z^{3j} \right), \quad (5.4.29)$$

where the optimal values for the complex parameters $a_j$ can be found in Table 4 of [29]. Relaxation of the initial configuration (5.4.29) produces for the rescaled mass parameter $\mu = 1$ a $D_3$-symmetric solution with energy $M_{16} = 20.567$ and diagonal isospin inertia tensor fulfilling $U_{11} = U_{22} \approx 540.75$. A $D_2$ configuration can be reproduced by the single rational map [29]

$$R(z) = \left( \sum_{j=0}^{\delta} a_j z^{2j} \right) / \left( \sum_{j=0}^{\delta} a_{8-j} z^{2j} \right), \quad (5.4.30)$$

where the optimal values of the complex coefficients $a_j$ are listed in Table 5 of [29].

Taking (5.4.30) as start configuration of our energy minimization algorithm, we find $M_{16} = 20.559$ for the static energy of a $\mu = 1$ Skyrmion. The $D_2$ and $D_3$ configurations can be regarded as energy-degenerate within the limits of our numerical accuracy.

Further we construct two tetrahedral configurations which both can be thought of being composed of $B = 4$ cubes. One of them is the tetrahedral configuration labelled as $T_d^{II}$ created with the single rational ansatz given in [22]

$$\tilde{R}(z) = cR(z)^4, \quad (5.4.31)$$

where $c = 0.5$ and $R(z)$ is given by (5.4.32a). Relaxation of (5.4.31) gives a static energy $M_{16} = 20.841$. The diagonal isorotational inertia tensor has the symmetry $U_{33} \neq U_{11} = U_{22} \approx 628.35$. Another tetrahedral $T_{d,\prime}$-symmetric solution type – here denoted as $T_d^{I}$ – can be obtained via the double rational map approximation [22]

$$R^1(z) = \frac{p_+}{p_-}, \quad (5.4.32a)$$

$$R^2(z) = \frac{a p_+^3 + b p_-^3}{p_+^2 p_-}, \quad (5.4.32b)$$

where $p_\pm(z) = z^4 \pm 2\sqrt{3}iz^2 + 1$ are the Klein polynomials and the real constants $a$, $b$ are given by $a = -0.53$ and $b = 0.78$. When relaxed, we find numerically for the energy of the solution (5.4.32) $M_{16} = 20.515$. The associated isospin inertia tensor is diagonal with $U_{11} = U_{22} \approx 638.27$ and $U_{33} \approx 624.81$. 

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5.5 Numerical Results on Isospinning Skyrme Solitons

In this section, we investigate how the inclusion of isospin affects the geometrical shape and the total energy of the classical Skyrmion solutions with baryon numbers \( B = 1 - 5, 8 - 10, 12 \) computed in the previous section. Recall that in our numerical simulations we do not impose any spatial symmetries on the isospinning Skyrme soliton solutions and we do not assume that the solitons’ shape is independent of the angular frequency \( \omega \). Analogous calculations of isospinning solitons in the Skyrme-Faddeev model [46, 47] that go beyond the rigid body type approximation have been performed in [19, 60].

We construct stationary isospinning soliton solutions by numerically solving the energy minimization problem formulated in Section 5.2: For fixed isospin \( K_i = U_i \omega_j \) we calculate the Skyrme configuration of given baryon number \( B \) that minimizes the total energy \( H = M_B + \frac{1}{2} K_i U^{-1} K_j \). As initial field configurations, we take the static solutions at \( \omega = 0 \) (see previous section) and increase the angular momentum \( |K| \) stepwise. Relaxed solutions at lower \( |K| \) serve as initial conditions for higher \( |K| \). The chosen rotation axes coincide with the principal axes. The energy minimization calculations are performed with the relaxation algorithm used in the previous section, and for further details the interested reader is referred to literature [29, 19] and our explanations given in Chapter 4. The discretization scheme and grid sizes are chosen as stated in Section 5.4. Most of our numerical simulations are performed for a range of mass values \( \mu \).

Recall that the different orientations of Skyrmion solitons in isospace can be visualized using the field colouring scheme described in detail in [98]. We illustrate the colouring for a \( B = 1 \) Skyrmion solution in Fig. 5.3: The points where the normalised pion isotriplet \( \hat{\pi} \) takes the values \( \hat{\pi}_1 = \hat{\pi}_2 = 0 \) and \( \hat{\pi}_3 = +1 \) are shown in white and those where \( \hat{\pi}_1 = \hat{\pi}_2 = 0 \) and \( \hat{\pi}_3 = -1 \) are coloured black. The red, blue and green regions indicate where \( \hat{\pi}_1 + i \hat{\pi}_2 \) takes the values 1, \( e^{2\pi i/3} \), \( e^{4\pi i/3} \), respectively and the associated complementary colors in the RGB colour scheme (cyan, yellow and magenta) show the segments where \( \hat{\pi}_1 + i \hat{\pi}_2 = -1, e^{3\pi i/3}, e^{\pi i/3} \).

Note that all our calculations are carried out in dimensionless Skyrme units which are related to natural units \( (\hbar = c = 1) \) by rescaling the units of energy by \( F_\pi/4e \) and those of length by \( 2/e F_\pi \) where the parameters \( F_\pi \) and \( e \) are the pion decay constant and the dimensionless Skyrme constant. The moment of inertia is given in units of \( 1/e^3 F_\pi \) and
consequently the isorotational energy contributions are expressed in units of $e^3 F_\pi$. Note that adopting the standard choice [8] of Skyrme parameters ($e = 4.84, F_\pi = 108$ MeV) gives¹ in our geometrical units $\hbar = 46.8$ [86]. Different parameter sets like the ones suggested by [21] ($e = 4.90, F_\pi = 90.5$ MeV) and by [94] ($e = 3.26, F_\pi = 75.2$ MeV) yield $\hbar = 48.1$ and $\hbar = 21.3$, respectively.

### 5.5.1 Lower Charge Isospinning Skyrmions: $B = 1 − 5, 8$

In this section, we present some of our numerical results on isospinning soliton solutions of topological charges $B = 1 − 5$ and $B = 8$. Most of our numerical simulations are performed for a range of mass values $\mu$.

- **$B = 1$:**

  For the $O(3)$ symmetric charge-1 Skyrmion solution we choose the $z$-axis as the axis of rotation. This particular choice is motivated by [98], where it was argued that spin-polarised protons and neutrons are best modeled by hedgehog $B = 1$ Skyrmions classically spinning relative to the $z$-axis. We briefly illustrate in Fig. 5.4 Manton’s idea how spin and isospin excitations of nucleons can be approximated by classically spinning Skyrmion solutions with different internal orientations [98].

Note that in our numerical simulations the $B = 1$ Skyrmion is chosen to be in its standard position and orientation, that is the white-black axis (see Fig. 5.4) coincides with the rotation axis, with white up and black down. The results of our fully three-dimensional numerical relaxation calculations of isospinning $B = 1$ Skyrmion solutions with mass parameter $\mu = 1$ are shown in Fig. 5.5. Note that for a spherically-symmetric, hedgehog Skyrme configuration rotation in physical space and isospace are equivalent. Thus, we expect the same energy curves for a $B = 1$ Skyrmion classically rotating about the $z$-axis. As expected from the discussion on critical frequencies, the soliton’s energy $E_{\text{tot}}(\omega) = M_B + \omega^2 U_{33}/2$ and the moment of inertia $U_{33}$ diverge at $\omega_{\text{crit}} = 1$ (see Fig.5.5 (a),(c)). Stable, internally spinning solutions cease to exist beyond this critical value. We verify that the $E_{\text{tot}}(\omega)$ graph shows the same behaviour as predicted by an axially-symmetric spinning, charge-1 Skyrme configuration (5.4.4). Axially-symmetric deforming, (iso)spinning configurations are constructed by minimizing the total energy $E_{\text{tot}} = M_1 + K^2/(2U_{33})$ for fixed isospin $K$ with a 2D gradient flow algorithm, where the classical soliton mass $M_1$ is given by (5.4.5) and the relevant moment of inertia $U_{33}$ can be found in

¹To estimate the value of $\hbar$ [86] for different Skyrme parameter sets we used $\hbar = 197.3$ MeV fm. Thus, for the standard choice of Skyrme parameters we can relate Skyrme units to conventional units via $F_\pi/4e = 5.58$ MeV and $2/e F_\pi = 0.755$ fm. It follows that

$$\hbar = 46.8 \left( \frac{F_\pi}{4e} \right) \left( \frac{2}{e F_\pi} \right).$$

Hence in standard Skyrme units we have $F_\pi/4e = 2/e F_\pi = 1$ and $\hbar = 46.8$. Analogous calculations for different choices of Skyrme parameters $e$ and $F_\pi$ result in the $\hbar$ values stated above.
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Figure 5.4: Classical approximation of spin and isospin states of nucleons. Left: Proton and neutron in spin up and spin down states modelled as hedgehog $B = 1$ Skyrmions classically spinning about their white-black axis. The colouring indicates the different orientations in isospace: The points where the normalised isotriplet $\vec{\pi}$ takes the values $\vec{\pi}_1 = \vec{\pi}_2 = 0$ and $\vec{\pi}_3 = +1$ are shown in white and those where $\vec{\pi}_1 = \vec{\pi}_2 = 0$ and $\vec{\pi}_3 = -1$ are coloured black. For the neutron the spin is always anticlockwise relative to the white-black axis, whereas protons are always spinning clockwise relative to the white-black axis. (Here, we illustrate some of the ideas given in [98]). Right: As described in [98], the $B = 1$ Skyrmion when rotating around the white-black axis can be used to model classically the nucleon doublet and its spin excitations.

Both energy curves agree within the limits of our numerical accuracy (see Fig. 5.5 (a)). In Fig. 5.5 (b), we display the mass-isospin relationship $E_{\text{tot}}(K)$ for an isospinning $B = 1$ Skyrmion solution calculated without imposing any symmetry constraints on the configuration. Imposing axial symmetry, we reproduce the same mass-isospin relation. As shown in Fig. 5.5 (b), the rigid body formula proves to be a good approximation for small isospins ($K \leq 3.52 \times 4\pi$), whereas for higher isospin values $E_{\text{tot}}(K)$ deviates from the quadratic behaviour. Close to the critical angular frequency $\omega_{\text{crit}} = 1 (K \approx 6.20 \times 4\pi)$ the rigid body approximation gives an approximate 30% larger energy value for the spinning soliton solution. The associated isospin inertia tensor $U_{ij}$ (5.2.7a) is diagonal and its diagonal elements as function of isospin $K$ are shown in Fig. 5.5 (d). For small isospin values the Skyrme configuration possesses $O(3)$ symmetry ($U_{ij} = V_{ij} = W_{ij} = \Lambda \delta_{ij}$ with $\Lambda \approx 46.5$) and as $K$ (or $L$) increases the soliton solution deforms breaking the spherical symmetry to an axial symmetry (the tensors of inertia (5.2.7) are all diagonal and satisfy $U_{11} = U_{22} = u$, $V_{11} = V_{22} = v$, $W_{11} = W_{22} = w$ and $U_{33} = V_{33} = W_{33}$). Close to the maximal angular frequency $\omega_{\text{crit}} = 1 (K \approx 6.20 \cdot 4\pi)$ we find numerically $u \approx 64.81$, $v \approx 65.80$, $w \approx 58.52$ and $U_{33} \approx 77.76$.

We compare in Fig. 5.6 the frequency-isospin relation $\omega(K)$ for (iso)spinning $B = 1$ Skyrmions (with $\mu = 1$) obtained when we do not impose any constraints on the spatial symmetries of the (iso)spinning soliton solution with those calculated when only considering deformations within a spherically-symmetric (5.4.1) [120] and an axially symmetric ansatz (5.4.4) [21, 66, 54], respectively. For a spherically symmetric charge-1 Skyrme configuration ($\mu = 1$) we solve the variational equation for the radial profile function $f(r)$ [120] derived from the minimization of the pseu-
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Figure 5.5: (Iso)spinning $B = 1$ Skyrmion ($\mu = 1$). A suitable start configuration of topological charge 1 is numerically minimized using 3D modified Newtonian flow on a $(100)^3$ grid with a lattice spacing of $\Delta x = 0.2$ and a time step size $\Delta t = 0.02$. We choose the $z$-axis as our rotation axis. Our results are compared with those obtained assuming an axially-symmetric deforming $B = 1$ Skyrme configuration (5.4.4). Furthermore, we include the energy curve for a rigidly rotating Skyrme configuration.

do energy $F_\omega(f) = M_1(f) - \frac{1}{2}\Lambda(f)\omega^2$, where the classical soliton mass $M_1$ is given by (5.4.2) and the associated moment of inertia $\Lambda$ can be found in (5.4.3). The underlying two point boundary value problem $-f(0) = \pi$ and $f(\infty) = 0$ is solved with the collocation method [11] for fixed angular frequency $\omega$. Taking the asymptotic limit ($r \to \infty, f \to 0$) of the nonlinear equation for $f(r)$ reveals that the existence of a stable, (iso)spinning soliton solution requires that the rotation frequency satisfies $\omega \leq \sqrt{3/2}\mu$. The classically spinning Skyrmion will lower its rotational energy by pion emission [21, 120] when spinning faster than the critical angular frequency $\omega_{\text{crit}} = \sqrt{3/2}\mu$. Similarly, the linearization of the partial differential equations for an axially-symmetrical deforming Skyrme soliton (5.4.4) yields a critical frequency $\omega_{\text{crit}} = \mu$, beyond which the (iso)spinning soliton solutions becomes unstable against pion emission. Clearly, the frequency-isospin relation $\omega(K)$ of the (iso)spinning $B = 1$ solution calculated without imposing any symmetry constraints agrees with the one of the axially-symmetric deforming solution. The energy density contour plots of the $B = 1$ Skyrmion solution ($\mu = 1$) as function of
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Figure 5.6: Angular frequencies $\omega$ as function of isospin $K$ for $B = 1$ Skyrmions ($\mu = 1$). We compare our results on arbitrarily deforming charge-1 Skyrme configurations with those obtained when allowing for spherically symmetric and axially symmetric deformations, respectively.

Angular momentum $K$ are presented in Fig. 5.7

Figure 5.7: Energy density contour plots of the isospinning $B = 1$ Skyrmion solution as function of isospin $K$. To visualize the axially symmetric deformation, we show a slice through the centre of the energy density distribution in the $xz$-plane. The rotation axis is chosen to be the $z$-axis. Note that the angular momentum $K$ is given in units of $4\pi$. The numerical calculations were performed with a full 3D relaxation algorithm. The mass parameter $\mu$ is chosen to be 1.

- $B = 2$

For the toroidal $B = 2$ Skyrmion solution we choose two different isospin axes [98]. One is the axis of symmetry – the $z$-axis – with the torus spinning in the $xy$-plane and the other is an axis orthogonal [35, 86, 53] to it – the $y$-axis – so that the symmetry axis rotates in the $xz$-plane. It was argued in [98] that these two spatial orientations are the relevant ones for describing the rotational states of the deuteron.

With the $z$-axis chosen as our rotation axis and the mass parameter $\mu$ set to 1, we obtain the energy and moment of inertia curves presented in Fig. 5.9 and the corresponding energy density contour plots are displayed in Fig. 5.10. We verify in Fig. 5.9 (a),(b) that the total energy $E_{\text{tot}}$ as function of $\omega$ and $K$ of the isospinning Skyrme configuration follows for angular frequencies $\omega \leq 0.6$ ($K \leq 4 \times 4\pi$).
Figure 5.8: The deviation \( \Delta E_{\text{tot}} = \left( E_{\text{Rigid}} - E_{\text{tot}} \right) / E_{\text{tot}} \) from the rigid body approximation for charge-1 and charge-2 Skyrmions as a function of isospin \( K \) for various rescaled mass values \( \mu \). The rotation axis is chosen to be the \( z \)-axis.

in good approximation the behaviour expected from a rotationally-symmetric deforming charge-2 Skyrme soliton. An isospinning, axially-symmetric Skyrme configuration can be computed by minimizing the two-dimensional energy functional \( E_{\text{tot}} = M^2 + K^2 / (2U_{33}) \) for fixed isospin \( K \), where the classical soliton mass \( M^2 \) is given by (5.4.5) with winding number \( n = 2 \) and the relevant moment of inertia \( U_{33} \) can be found in (5.4.8a). Again, we observe that both the energy and the isorotational moment of inertia \( U_{33} \) diverge at \( \omega_{\text{crit}} = \mu \). The rigid-body approximation is shown in Fig. 5.9 (b) to be a valid simplification for small isospin values \( K \leq 4 \times 4\pi \) (\( \omega \approx 0.673 \)). For higher angular frequencies the energy values for spinning soliton solutions predicted by the rigid-body formula are too large at a given angular momentum \( K \). For example, at the critical frequency \( \omega_{\text{crit}} \approx 1.09 \) (\( K \approx 9.5 \times 4\pi \)) we observe that the energy of the isospinning charge-2 solution is about 4.7\% smaller than the one calculated with the rigid-body formula. Finally we show in Fig. 5.9 (d) the diagonal elements \( U_{ii} \) of the isospin inertia tensor as function of isospin \( K \).

We verify numerically that the tensors of inertia are all diagonal and satisfy for all \( K \) the relations \( U_{11} = U_{22} \neq U_{33} \), \( V_{11} = V_{22} \neq V_{33} \), \( W_{11} \approx W_{22} \neq W_{33} \) and \( U_{33} \approx \frac{1}{2} W_{33} \approx \frac{3}{4} V_{33} \) which are consequences of the cylindrical symmetry [66, 35]. The energy density contour plots in Fig. 5.10 indicate that close to the critical angular frequency \( \omega_{\text{crit}} \) the isospinning \( B = 2 \) Skyrmion solution slowly starts to deform into its \( B = 1 \) components. As \( \omega \) approaches the critical value \( \omega_{\text{crit}} \) the rotational invariance of the Skyrme configuration becomes slightly broken and then the total energy and the angular momentum diverge. Recall that we observed similar critical behaviour for mass values \( \mu \leq 1 \) in our simulations on isospinning Skyrmion solutions in the two-dimensional version of (5.2.1) (see Fig. 3.15). When repeating our relaxation calculations for higher mass values \( \mu \), we find (analogous to our observations in the conventional baby Skyrme model presented in Section 3.6.1) that the isospinning charge-2 Skyrmion solution breaks up at some critical value \( \omega \) into
5.5. NUMERICAL RESULTS ON ISOSPINNING SKYRME SOLITONS

its charge-1 constituents that move apart from each other. We display in Fig. 5.10 baryon density isosurface and contour plots for a range of mass values $\mu$. Note that the break-up of isospinning $B = 2$ Skyrmion solutions into their charge-1 constituents differs from the one observed for isospinning solutions in the conventional baby Skyrme model in that it does not appear as spontaneously.

Figure 5.9: Isospinning $B = 2$ Skyrmion ($\mu = 1$). A start configuration of topological charge 2 is numerically minimized using 3D modified Newtonian flow on a $(200)^3$ grid with a lattice spacing of $\Delta x = 0.1$ and a time step size $\Delta t = 0.01$. We choose the $z$-axis as our rotation axis. Our full 3-dimensional relaxation calculations are compared with the energy values for an axially-symmetric deforming charge-2 Skyrme configuration (5.4.4). Additionally, we include the energy values obtained when assuming a rigid rotor.

The deviations from the rigid-body approximation plotted as function of the angular momentum $K$ are shown for various mass values $\mu$ in Fig. 5.8 for charge-1 and charge-2 solutions isospinning around their $z$-axis. As the mass value $\mu$ (or the topological charge $B$) increases, the rigid body approximation provides more accurate results for the isospinning solutions of the model.

When choosing the $y$-axis as our rotation axis, we obtain for $\mu = 1$ the energy and moment of inertia curves shown in Fig. 5.11. The corresponding baryon density isosurface plots can be found in Fig. 5.12. As expected, with the mass parameter $\mu$ chosen to be 1 the total energy and the relevant isospin inertia tensor entry $U_{22}$ diverge at $\omega_{\text{crit}} \approx 1.05$ (see Fig. 5.11 (a),(c)). The rigid-body approximation proves
Figure 5.10: Baryon density isosurfaces and energy density contour plots of isospinning $B = 2$ Skyrmion solutions for a range of mass parameters $\mu$. The axis of rotation is chosen to be the $z$-axis. We illustrate the orientations in isospace by using Manton & Sutcliffe’s colour scheme [98]. Note that the energy minimization calculations for $\mu = 1, 2$ have been performed on a grid with $(200)^3$ grid points and a spacing $\Delta x = 0.1$, whereas we used for $\mu = 1.5$ a smaller grid with $(100)^3$ points and $\Delta x = 0.1$. 

$\mu = 1$

$\mu = 1.5$

$\mu = 2$
to be a valid simplification for isospin values $K \leq 8 \times 4\pi$ ($\omega \approx 0.84$). However, close to the critical frequency $\omega_{\text{crit}} \approx 1.05$ ($K \approx 15 \times 4\pi$) the energy values turn out to be $\approx 9\%$ lower than the ones predicted by the rigid-body approximation.

![Graphs](image1.png)

(a) Total energy vs angular frequency  
(b) Mass-Spin relationship  
(c) Inertia vs angular frequency  
(d) Inertia-Spin relationship

Figure 5.11: Same as Fig. 5.9 but with the $y$-axis chosen as our rotation axis.

The baryon density isosurfaces of isospinning charge-2 Skyrmions for a range of mass values are also displayed in Fig. 5.12. Note that we do not observe for this mass range a break up into charge-1 constituents.

- $B = 3$

For $B = 3$, we isorotate the minimal-energy tetrahedron about its 2-fold rotation axis (see Fig. 5.13). Performing a damped field evolution (as used above) we obtain, for the rescaled mass value $\mu = 1$, the energy and inertia dependencies on $\omega$ and $K$ displayed in Fig. 5.13. The corresponding baryon density isosurfaces for a range of mass values $\mu$ can be found in Fig. 5.14 where we used Manton’s & Sutcliffe’s field colouring scheme [98, 51, 22] to visualize the tetrahedron’s orientations in isospace.

As $\mu$ increases, the soliton’s deformations due to centrifugal effects become more apparent. For $\mu = 2$ the isospinning charge-3 Skyrmion solution breaks into a toroidal $B = 2$ Skyrmion solution and a $B = 1$ Skyrmion before reaching its upper frequency limit $\omega_{\text{crit}} = \mu$. Note that (with increasing angular velocity $\omega$) the
isospinning $B = 3$ Skyrmion seems to pass through a distorted “pretzel” configuration – a state that has previously been found to be meta-stable [143] for vanishing isospin $K$. For mass value $\mu = 1.5$ the tetrahedral charge-3 Skyrmion does not break into lower-charge Skyrmions when increasing the angular frequency $\omega$. As $\omega$ increases, the charge-3 tetrahedron slowly deforms into the “pretzel”-like configuration which appears to be of lower energy than an isospinning solution with tetrahedral symmetry. Even for $\mu = 1$ the tetrahedral symmetry is broken as $\omega$ increases (see inertia-spin relationship shown in Fig. 5.13 (d) and baryon density isosurfaces in Fig. 5.14). For angular frequencies $\omega$ approximately larger than 0.06 ($K \approx 1.12 \times 4\pi$) the isospinning $B = 3$ Skyrme soliton (with $\mu = 1$) starts to violate tetrahedral symmetry. Note that with increasing mass parameter $\mu$ the breaking of tetrahedral symmetry occurs at increasingly higher rotation frequencies. For example, we find the critical angular velocities $\omega \approx 0.1$ ($K \approx 0.79 \times 4\pi$) and $\omega \approx 0.12$ ($K \approx 0.51 \times 4\pi$) for $B = 3$ Skyrmions with mass $\mu = 1.5$ and $\mu = 2$, respectively.

Fig. 5.17 shows the deviation of the deformable isospinning Skyrmion solution from its rigid rotor approximation as a function of $K$ for various mass values. Again, we note that the soliton’s energy $E_{\text{tot}}(K)$ can be significantly lower than the one of the rigidly isospinning Skyrmion solution. For example for $\mu = 1$ the rigid body
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Figure 5.13: Isospinning $B = 3$ Skyrmion. ($\mu = 1$). A start configuration is numerically minimized using 3D modified Newtonian flow on a $(200)^3$ grid with a lattice spacing of $\Delta x = 0.1$ and a time step size $\Delta t = 0.01$. We choose the 2-fold axis of the tetrahedron shown in (f) as our rotation axis.

approach predicts an approximately 10% larger energy value close to the cutoff frequency $\omega_{\text{crit}} = \mu = 1$ ($K \approx 20 \times 4\pi$). As already observed for $B = 1$ and $B = 2$ Skyrmion solutions, the accuracy of the rigid body approximation improves for a given isospin value $K$ with increasing mass parameter $\mu$ (see Fig. 5.17).

- $B = 4$

For the minimal energy $B = 4$ Skyrmion solution we do not observe any breaking of the octahedral symmetry when isospinning the configuration around the $z$-axis.
Figure 5.14: Baryon density isosurfaces of isospinning, tetrahedrally-symmetric charge-3 Skyrmion solutions for a range of mass values $\mu$. The rotation axis is chosen as illustrated in Fig. 5.13 (f).

(see the baryon density isosurfaces shown in Fig. 5.16 as function of $K$ for various mass values $\mu$ and the chosen rotation axis is illustrated in Fig. 5.15 (f)). Plotting the elements of the isospin, spin and mixed inertia tensor as a function of $K$ confirms that the isospinning charge-4 Skyrmion preserves $O_h$ symmetry up to the maximal angular frequency $\omega = \mu$. In particular, the isospin inertia tensor $U_{ij}$ is diagonal and satisfies $U_{11} = U_{22} ≠ U_{33}$ for all allowed values of $K$ (compare Fig. 5.15 (d) for mass value $\mu = 1$). Although, the Skyrmion’s shape turns out to be largely independent of the angular frequency $\omega$, the Skyrmion’s size increases monotonically with $\omega$. We compare in Fig. 5.17 our numerical energy values for arbitrarily deforming Skyrmion solutions with those obtained when assuming a rigidly rotating, cubically-symmetric $B = 4$ Skyrmion solution. Evidently, the accuracy of the rigid-body approximation improves with increasing soliton mass and baryon number $B$. For example, for the mass value $\mu = 2$ we find that the rigid body formula predicts for the $B = 4$ Skyrmion energy values which are for rotation frequencies $\omega ≈ \omega_{\text{crit}} = \mu$ roughly 7% larger than those obtained for the deformed, isospinning solutions. For comparison, the rigid-body approximation resulted for the $B = 1$ Skyrmion (with mass value $\mu = 2$ and $\omega ≈ \omega_{\text{crit}}$) in energy values which were 25% larger than the ones for the deforming soliton solutions.

- $B = 5$

We display in Fig. 5.18 the resulting baryon density isosurfaces when isospinning
5.5. NUMERICAL RESULTS ON ISOSPINNING SKYRME SOLITONS

Figure 5.15: Isospinning $B = 4$ Skyrmion. ($\mu = 1$). A start configuration is numerically minimized using 3D modified Newtonian flow on a $(200)^3$ grid with a lattice spacing of $\Delta x = 0.1$ and a time step size $\Delta t = 0.01$. We choose the $(0, 0, 1)$ axis as our isorotation axis.

$D_{2d}$-symmetric charge-5 Skyrmions with different mass values $\mu$ and without making any assumptions about the spatial symmetries on the Skyrme fields. The rotation axis is chosen so that the $5D_{2d}$ Skyrmion is isospinning relative to a white-black axis when adopting Manton’s & Sutcliffe’s field colouring scheme [98, 51, 22].

We show in Fig. 5.20 the diagonal elements of the isospin inertia tensor $U_{ij}$ as a function of isospin $K$ for mass values $\mu = 1, 1.5, 2$. From the intercept points in Fig. 5.20, we deduce that for $\mu = 1$ the isospinning soliton preserves its $D_{2d}$ sym-
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Figure 5.16: We display the baryon density isosurfaces (not to scale) of isospinning, octahedrally-symmetric charge-4 Skyrmion solutions for a range of mass values. The rotation axis is chosen as illustrated in Fig. 5.15 (f).

Figure 5.17: The deviation $\Delta E_{\text{tot}} = (E_{\text{Rigid}} - E_{\text{tot}})/E_{\text{tot}}$ from the rigid body approximation for charge-3 and charge-4 Skyrmions as a function of isospin $K$ for various rescaled mass values $\mu$. The isorotation axes are chosen to be the (0, 0, 1) axes.

Figure 5.18: The deviation $\Delta E_{\text{tot}} = (E_{\text{Rigid}} - E_{\text{tot}})/E_{\text{tot}}$ from the rigid body approximation for charge-3 and charge-4 Skyrmions as a function of isospin $K$ for various rescaled mass values $\mu$. The isorotation axes are chosen to be the (0, 0, 1) axes.
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Figure 5.18: Baryon density isosurfaces of isospinning $5D_{2d}$-symmetric Skyrmion solutions for a range of mass values. The isospinning $5D_{2d}$-symmetric Skyrmion configuration is found to break up into a $B = 2$ torus and a $B = 3$ “pretzel”-like configuration (see Fig. 5.19 for a zoomed in version of these plots). The rotation axis is chosen to be the $(0, 0, 1)$ axis. Note that isospinning $5D_{4h}$-symmetric Skyrmion solutions are found to be of higher energy and unstable to decay into $5D_{2d}$-symmetric configurations.

lower charge Skyrmions happens at higher angular frequencies as $\mu$ increases.

When relaxing $D_{4h}$-symmetric $B = 5$ Skyrmion solutions we find that the isospinning configurations are of higher energy and unstable to decay into the $5D_{2d}$-symmetric Skyrmions shown in Fig. 5.18.

Figure 5.19: Break-up up of the isospinning $5D_{2d}$-symmetric Skyrmie soliton into a $B = 2$ torus and a $B = 3$ “pretzel”-like Skyrmie configuration. We display the baryon density isosurfaces for the mass parameter $\mu = 2$. We zoom into the Skyrmie configurations. The isorotation axis is chosen to be the $(0, 0, 1)$ axis.

- $B = 8$
CHAPTER 5. SKYRMIONS

5.5.2 Higher Charge Isospinning Skyrmions: $B = 9, 10, 12$

In this section, we present some of our numerical results on isospinning soliton solutions of topological charges $B = 9, 10, 12$. All the numerical simulations in this Section

Figure 5.20: Diagonal elements of the isospin inertia tensor $U_{ij}$ for isospinning $5D_{2d}$ Skyrmion solutions as function of $K$ for $\mu = 1, 1.5, 2$.

Figure 5.21: Baryon density isosurfaces of isospinning $8D_{4h}$-symmetric Skyrmion solutions for mass parameter $\mu = 1$. Each baryon density isosurface corresponds to a value $B = 0.15$. The isorotation axis is chosen to be the $(0, 0, 1)$ axis.

For baryon number $B = 8$ we investigate the effect of isospin on $8D_{4h}$ and $8D_{6d}$ symmetric Skyrme configurations (with rescaled mass value $\mu$ set to 1). For $D_{6d}$ symmetric Skyrme solitons we find that there exists a break-up frequency at which the isospinning solution splits into four $B = 2$ tori (compare baryon density isosurfaces and rotation axis shown in Fig. 5.21). Similar, we observe that the $8D_{6d}$ configuration when isospinning about its $(0, 1, 0)$ axis breaks apart into four $B = 2$ tori (see baryon density isosurfaces given in Fig. 5.23). Choosing the $(0, 0, 1)$ axis as rotation axis (see Fig. 5.22) preserves $D_{6d}$ symmetry.

Fig. 5.24 shows in particular the total energy for $D_{4h}$ and $D_{6d}$ Skyrmions as a function of isospin $K$ when isospinning around their $(0, 0, 1)$ axis (see Fig. 5.21 and Fig. 5.22 for the Skyrmion’s orientation in isospace). For isospin values larger than $K \approx 23.3 \times 4\pi$ the isospinning $D_{4h}$ solution is found to be of lower energy.
5.5. NUMERICAL RESULTS ON ISOSPINNING SKYRME SOLITONS

Figure 5.22: Baryon density isosurfaces of isospinning $8D_{6d}$-symmetric Skyrmion solutions for mass parameter $\mu = 1$. Each baryon density isosurface corresponds to a value $B = 0.12$. The isorotation axis is chosen to be the $(0, 0, 1)$ axis.

Figure 5.23: Baryon density isosurfaces of isospinning $8D_{6d}$-symmetric Skyrmion solutions for mass parameter $\mu = 1$. Each baryon density isosurface corresponds to a value $B = 0.22$. The isorotation axis is chosen to be the $(0, 1, 0)$ axis.

Figure 5.24: Total energy $E_{\text{tot}}$ for isospinning charge-8 Skyrmion solutions ($\mu = 1$) as function of $\omega$ and $K$. Further, we display the inertia tensor diagonal element $U_{33}$ as a function of $\omega$. The isorotation axes are chosen to be the $(0, 0, 1)$ axes.

are performed with the mass parameter $\mu$ set to 1.

- $B = 9$

Baryon density isosurface plots of isospinning $9T_d$ and $9D_{4d}$ symmetric Skyrmion solutions (with $\mu = 1$) can be found in Fig. 5.25 and Fig. 5.26, respectively. The rotation axes are as indicated. We observe that the tetrahedrally-symmetric $B = 9$ con-
Figure 5.25: Baryon density isosurfaces of isospinning $9T_d$-symmetric Skyrmion solutions for mass parameter $\mu = 1$. Each baryon density isosurface corresponds to a value $B = 0.08$. The isorotation axis is chosen to be the $(0, 0, 1)$ axis.

Figure 5.26: Baryon density isosurfaces of isospinning $9D_{4d}$-symmetric Skyrmion solutions for mass parameter $\mu = 1$. Each baryon density isosurface corresponds to a value $B = 0.12$. The isorotation axis is chosen to be the $(0, 0, 1)$ axis.

Figure 5.27 cross at $K \approx 21.04 \times 4\pi$ ($\omega \approx 0.64$). For larger isospin values we find the layer formed by nine single $B = 1$ Skyrmion to be of lower energy.

- **$B = 10$**

For baryon number $B = 10$ we isorotate five different dihedrally-symmetric Skyrmion solutions generated with $D_{2h}, D_{4d}, D_{3h}, D_{3d}$ and $D_3$ symmetric rational maps (see Section 5.4.2). We observe that the geometrical shape of the charge-10 soliton solutions can be dramatically altered when isospin is included. For example, the $10D_{2h}$ symmetric solution (see baryon density isosurfaces shown as a function of isospin $K$ and with mass value $\mu = 1$ in Fig. 5.28) breaks apart at some critical frequency value into two $B = 4$ cubes and two single $B = 1$ hedgehog Skyrmions when isospinning around its $(0, 0, 1)$ axis. We show in detail in Fig. 5.29 the constituents in which the $10D_{2h}$ configuration breaks apart and their respective orientations in...
5.5. NUMERICAL RESULTS ON ISOSPINNING SKYRME SOLITONS

(a) Total energy vs angular frequency
(b) Mass-Spin relationship

Figure 5.27: Total energy $E_{\text{tot}}$ for isospinning charge-9 Skyrmion solutions ($\mu = 1$) as function of $\omega$ and $K$. The isorotation axes are chosen to be the (0, 0, 1) axes.

Figure 5.28: Baryon density isosurfaces of isospinning $10D_{2h}$-symmetric Skyrmion solutions for mass parameter $\mu = 1$. Each baryon density isosurface corresponds to a value $B = 0.08$. The isorotation axis is chosen to be the (0, 0, 1) axis. See Fig. 5.29 for a zoomed in version of the baryon density isosurface plot for $K = 40 \times 4\pi$.

Figure 5.29: Break-up up of the isospinning $10D_{2h}$-symmetric Skyrme soliton into two $B = 4$ cubes and two single $B = 1$ hedgehog Skyrmions. We display three different views of the baryon density isosurfaces (mass parameter $\mu = 1$) for isospin value $K = 40$ ($\omega = 0.964$). Each baryon density isosurface corresponds to a value $B = 0.08$. We zoom into the Skyrme configuration. The isorotation axis is chosen to be the (0, 0, 1) axis.
Another example of a charge-10 Skyrmion configuration breaking apart due to centrifugal effects can be observed for an isospinning $D_{3h}$ configuration. As $\omega$ increases, the $D_{3h}$ polyhedron splits into its constituents: three “pretzel”-like $B = 3$ Skyrmions forming a belt of nine Skyrmions and a single $B = 1$ hedgehog Skyrmion placed at the centre (see baryon density isosurface plots shown in Fig. 5.30).

When isospinning a $D_{4d}$ Skyrme configuration (with $\mu = 1$) we do not observe a break-up into lower charge solitons but the formation of a completely different charge-10 solution type (compare baryon density isosurface plots given in Fig. 5.31 as functions of $K$ and $\omega$). At vanishing $K$ the minimal energy $D_{4d}$ solution is known to be very spherical [29] and well-described by the rational map (5.4.21). However, as the angular frequency $\omega$ increases, the $D_{4d}$ configuration becomes more and more elongated.

Unlike the $D_{4d}$ solution, the $D_{3d}$ Skyrme soliton is already much more elongated [29] for vanishing isospin $K$ (see Fig. 5.32). With increasing angular velocity $\omega$ the $D_{3d}$ solution seems to form $B = 4$ cubes.

Finally, we display in Fig. 5.33 the total energy $E_{\text{tot}}$ for isospinning charge-10 Skyrmion solutions as a function of isospin $K$ and angular frequency $\omega$. Note that for vanishing isospin $K$ the $D_{2h}$ and $D_{3d}$ solution can be seen as energy-degenerate within the limits of our numerical accuracy. Same is true for the $10D_{4d}$, $10D_{3}$ and $10D_{3h}$ configurations. As $K$ increases only $10D_{3h}$ and $10D_{4d}$ remain energy-degenerate. We find that for our rotation axes choices and the considered symmetry
5.5. NUMERICAL RESULTS ON ISOSPinning SKYRME SOLITONS

![Figure 5.31: Baryon density isosurfaces of isospinning 10D4d-symmetric Skyrmion solutions (two different views) for mass parameter μ = 1. Each baryon density isosurface corresponds to a value B = 0.14. The isorotation axis is chosen to be the (0, 0, 1) axis.](image)

5.5.3 Critical Angular Frequencies

It has been observed [60, 19, 20, 59] that isospinning soliton solutions in models of the Skyrme family (see baby Skyrmions in Chapter 3 and Hopf solitons in Chapter 4) suffer from two different types of instabilities: One is related to the deformed metric in the pseudo-energy functional $F_\omega (\phi) \ (3.3.5)$ becoming singular at $\omega_1 = 1$ and the other to the Hamiltonian not longer being bounded from below at $\omega_2$. The first critical frequency
Figure 5.32: Baryon density isosurfaces of isospinning $10D_{3d}$-symmetric Skyrmion solutions (two different views) for mass parameter $\mu = 1$. Each baryon density isosurface corresponds to a value $B = 0.11$. The isorotation axis is chosen to be the $(0, 0, 1)$ axis.

Figure 5.33: Total energy $E_{\text{tot}}$ for isospinning charge-10 Skyrmion solutions ($\mu = 1$) as function of $\omega$ and $K$. The isorotation axes are chosen to be the $(0, 0, 1)$ axes.

is independent of the concrete potential choice, whereas the second critical frequency $\omega_2$ crucially depends on the particular choice of the potential term. See Section 3.3.1 for a more detailed discussion of the critical frequencies in Skyrme-like models.

Note that we do not observe the same pattern of critical frequencies in the full Skyrme model (5.2.1). In Fig. 5.35 we plot energy $E_{\text{tot}}$ and isospin $K$ of the $B = 1$ Skyrmion
NUMERICAL RESULTS ON ISOSPINNING SKYRME SOLITONS

Figure 5.34: Baryon density isosurfaces of isospinning 12O$_{16}$-symmetric Skyrmion solutions for mass parameter $\mu = 1$. Each baryon density isosurface corresponds to a value $B = 0.07$. The isorotation axis is chosen to be the (0,0,1) axis.

with the standard potential term $V = 2\mu^2 (1 - \sigma)$ as functions of the angular frequency $\omega$ for a range of values of the mass parameter $\mu$. In addition, the energy is shown as a function of angular momentum $K$. These plots should be compared with the curves in the two-dimensional version of the Skyrme model – the old baby Skyrme model – shown in Fig. 3.12. Different to our observations in the baby Skyrme model (see Fig. 3.12) and in the Skyrme-Faddeev model (see Fig. 4.12) both the energy $E_{\text{tot}}$ and the isospin $K$ diverge at $\omega_{\text{crit}} = \mu$ for all mass values $\mu$. In particular, there does not exist a critical behaviour at $\omega = \omega_1$.

Figure 5.35: Isospinning $B = 1$ Skyrmion solutions for a range of mass values $\mu$. We plot the total energy $E_{\text{tot}}$ and isospin $K$ as function of angular frequency $\omega$ and the total energy as a function of isospin $K$ at $\mu = 0.5, 1, 1.5, 2$. The $z$-axis is chosen to be the axis of rotation. Our 3D-relaxation calculations are performed on a (100) grid with grid spacing $\Delta x = 0.2$. Note that the corresponding plots in the two-dimensional version of the Skyrme model can be found in Fig. 3.12.

When choosing the double vacuum potential $V = 2\mu^2 (1 - \sigma^2)$ in (5.2.6), we obtain for the isospinning $B = 1$ Skyrmion solution the energy and isospin curves shown in Fig. 5.36. Again, we observe that Skyrmions cannot spin with angular frequencies $\omega > \sqrt{2}\mu$ – the meson mass of the model – since they become unstable to the emission of radiation.

5.5.4 Mean Charge Radii

We display in Fig. 5.37 as a function of isospin $K$ the mean charge radii of $B = 1 - 3$ soliton solutions in the Skyrme model (5.2.1) with the rescaled mass parameter $\mu$ chosen
to be 1. In addition, we show for $B = 1$ the mean charge radii for a range of mass values. We define the mean charge radius of a Skyrmion solution as the square root of the second moment of the topological charge density $B(x)$ (5.2.2)

$$< r^2 > = \frac{\int r^2 B(x) \, d^2 x}{\int B(x) \, d^2 x}. \quad (5.5.1)$$

Similar to our observations of isospinning soliton solutions in the standard baby Skyrme model in Section 3.6.1, we note that the changes in the Skyrmion’s shape are reflected by the changes in slopes of the mean charge radius curves in Fig. 5.37. For example for $B = 1$ and mass value $\mu = 1$ we find that the radius $< r^2 >^{1/2}$ grows approximately linear with the isospin $K$ for $K > 3 \times 4\pi$. For $B = 2$ and $B = 3$ we observe that the linear growth starts at higher isospin values $K \approx 4.3 \times 4\pi$ and $K \approx 8.4 \times 4\pi$, respectively. These changes in slopes agree with the maximal isospin values stated in Section 5.5.1 up to which the rigid-body approximation is a valid simplification. For comparison, we found in Section

Figure 5.37: Mean charge radii $< r^2 >^{1/2}$ (5.5.1) for Skyrmion solutions of topological charge $1 \leq B \leq 3$ as function of isospin $K$. These calculations have been performed with the conventional potential term $V = 2\mu^2 (1 - \sigma^2)$ with mass value $\mu = 0.5, 1, 1.5, 2$ (left) and $\mu = 1$ (right).
5.5.1 $K \approx 3.52 \times 4\pi$ and $K \approx 4 \times 4\pi$ for $B = 1$ and $B = 2$, respectively. Furthermore, as $\mu$ increases the mean charge radius $<r^2>^{1/2}$ starts to grow linearly at increasingly higher isospin values. This confirms that the rigid body approximation becomes more accurate as $B$ or $\mu$ increases.

5.6 Conclusions

In this chapter we have performed full three-dimensional numerical relaxations of isospinning soliton solutions with topological charges $B = 1-5, 8-10, 12$ in the Skyrme model with the conventional mass term included. Our calculations show that the qualitative shape of isospinning Skyrmion solutions can differ drastically from the ones of the static ($\omega = 0$) solitons. The deformations become increasingly pronounced as the mass value $\mu$ increases. Briefly summarized, we distinguish the following types of behaviour:

- **Break-up into lower charge Skyrmions**: Isospinning Skyrmion solutions can split into lower charge Skyrmions at some critical break-up frequency value. Examples are the break-up (for $\mu$ sufficiently large) of the $B = 2$ solution into two $B = 1$ Skyrmions when isospinning about the $z$-axis; the break-up of the charge-3 Skyrmion into a $B = 1$ hedgehog and a $B = 2$ torus; the break-up of the $5D_{2d}$ symmetric solution into a $B = 2$ torus and a $B = 3$ “pretzel” configuration; the break-up of the $8D_{4h}$ and $8D_{6d}$ Skyrme configurations into four $B = 2$ tori etc.

Recall that these break-up processes do not occur as spontaneously as observed for isospinning soliton solutions in the (2+1)-dimensional version of the Skyrme model (compare Chapter 3.6.1).

- **Formation of new solution types**: Isospinning Skyrmion solutions can deform into configurations that do not exist at vanishing $\omega$ or are only metastable at $\omega = 0$. An example is the tetrahedral $B = 3$ Skyrmion (with $\mu = 1.5$) which evolves with increasing $\omega$ into a “pretzel”-like configuration – a state that is only metastable at $\omega = 0$.

- **Crossings of $E_{\text{tot}}(K)$**: The energy curves $E_{\text{tot}}(K)$ of Skyrmion solutions of different symmetry groups (but with the same topological charge $B$) can cross resulting in a rearrangement of the spectrum of minimal-energy solutions. For example, the energy curves of the isospinning $8D_{4h}$ and $8D_{6d}$ soliton configurations cross. For $\omega = 0$ the state of lowest energy is given by the $D_{6d}$-symmetric $B = 8$ solution. However, for larger isospin values $(23.3 \times 4\pi < K < 35 \times 4\pi)$ the $D_{4h}$ symmetric solution is found to be of lower energy. For isospin values $K > 35 \times 4\pi$, $D_{4h}$ symmetry is broken and the $8D_{4h}$ configuration breaks apart into four $B = 2$ tori.

- **Transmutation**: Isospinning Skyrmion solutions can evolve into a solution type that also exists at $\omega = 0$. For example the $9T_d$ Skyrme configuration deforms into a
layer formed by nine $B = 1$ hedgehog Skyrmions.
6.1 Summary

In this thesis we performed numerical simulations of classically isospinning topological soliton solutions in nonlinear $O(N)$ sigma models in $(2+1)$- and $(3+1)$- dimensional spacetime. Here, the crucial difference to previous work was that we considered isospinning soliton solutions beyond the rigid-body approximation, that is we explicitly allowed the soliton solutions to deform and to break the symmetries of the static configurations. Furthermore, we did not impose any spatial symmetry constraints on the geometrical shape of the isospinning soliton solutions.

Naturally, such an approach is a huge computational challenge since constructing even minimal-energy solutions for vanishing angular frequencies typically requires substantial computing resources. However, our numerical investigations revealed interesting properties of isospinning soliton solutions. A lot of these properties are inaccessible by purely analytical methods. Briefly summarized, we found the following properties of internally rotating soliton solutions in the

- **Baby Skyrme Model (see Chapter 3):**
  Stable isospinning baby Skyrme solutions can be constructed for all angular frequencies $\omega \leq \min(\omega_2, 1)$, where $\omega_2$ depends on the specific choice of the $SO(2)$ invariant potential term. The geometric shape of isospinning baby Skyrmion solutions is found to depend strongly on the choice of the mass term.

- **Skyrme-Faddeev Model (see Chapter 4):**
  The Skyrme-Faddeev model with its rich spectrum of soliton solutions, often of similar energy, allows for transmutations, formation of new solution types and the rearrangement of the spectrum of minimal-energy solitons in a given topological sector when isospin is added. We observe that the shape of isospinning Hopf solitons can differ qualitatively from that of the static solution. In particular the solution type of the lowest energy soliton can change.
• *Skyrme Model (see Chapter 5):*

Isospinning Skyrmion solutions can break-up into lower charge Skyrmions, can deform into new solution types that do not exist at vanishing angular frequency $\omega$ or can evolve into solution types that also exist at $\omega = 0$. Furthermore, the energy curves $E_{tot}(K)$ of Skyrmion solutions of different symmetry groups (but with the same topological charge) can cross resulting in a rearrangement of the spectrum of minimal-energy solutions in a given topological sector.
Previous numerical and analytical results [109, 57] on isospinning charge-1 and -2 soliton solutions in the conventional baby Skyrme model (the “old” baby Skyrme model) are largely based on the assumption that deformations are only happening within a rotationally-symmetric hedgehog ansatz (3.4.1). Consequently, previous work has been mainly concerned with the solution of Eq. (3.6.1). In this Appendix, we briefly demonstrate that the pattern of critical behaviour observed for rotationally-symmetric deforming Skyrmion solutions differs significantly from the one we found when allowing for arbitrary deformations (see Chapter 3 and Section 3.6.1).

A.1 Isospinning charge-1 and -2 baby hedgehog solitons

For mass values $\mu \leq 1$ the asymptotic behavior is governed by the $O(3)$ sigma model term and the potential term in the Skyrme Lagrangian (3.2.1), whereas the Skyrme term is effectively removed. The linearized field equations give a critical angular frequency $\omega_{\text{crit}} = \omega_2 = \mu$ and the spinning solitons are exponentially localized for $\omega < \omega_{\text{crit}}$ [109]. However, for larger $\mu$ ($\mu > 1$) the Skyrme and potential term become increasingly dominant. For $\mu \to \infty$ the model is effectively described by the quartic (Skyrme) term and the potential. In this limit the model is often referred to as Bogomolny-Prasad-Sommerfield (BPS) baby Skyrme model [57, 6, 133] because its infinitely many exact static (multi)soliton solutions saturate the corresponding Bogomolny lower energy bound [41, 6, 133]. Using various numerical methods (collocation [11], simple gradient flow evolution and Newton iteration [15]) we solve (3.6.1) for isospinning charge-1 and -2 baby hedgehog solitons within the mass range $0 < \mu \leq 16$. The obtained critical angular frequencies $\omega_{\text{crit}}$ are shown in Fig. A.1 as a function of the mass parameter $\mu$. For comparison, we also display in Fig. A.1 the analytically calculated [57] critical frequencies for charge-1 and -2 rotationally-invariant Skyrme solitons in the $\mu \to \infty$ limit of the conventional baby Skyrme model. In this BPS limit the maximal rotation frequency has been calculated analytically [57] to be given by $\omega_{\text{crit}} = \sqrt{B\mu/2 \sqrt{2}}$.

We observe that the $\omega_{\text{crit}}(\mu)$ curves for isospinning hedgehog solutions in the full baby
Figure A.1: Critical frequency $\omega_{\text{crit}}$ as a function of the mass parameter $\mu$ for isospinning $B = 1$ and $B = 2$ hedgehog soliton solutions in the full baby Skyrme model compared with the ones in the BPS baby Skyrme model [57]. Note that solid lines represent the critical frequencies obtained by solving (3.6.1) numerically, whereas dashed lines show the frequencies calculated analytically [57] in the infinite mass limit of the standard baby Skyrme model.

Skyrme model and in the BPS model are in qualitative agreement: In particular, the graphs show approximately the same asymptotic behaviour and a crucially different behaviour for low and higher mass values.

However, our full two-dimensional relaxation calculations in the standard baby Skyrme model reveal that isospinning soliton solutions are only stable up to angular frequencies $\omega \leq \min(\mu, 1)$ and that the higher frequency value shown for $\mu > 1$ in Fig. A.1 are purely an artefact of the hedgehog approximation.
SKYRMION INERTIA TENSORS

For completeness, we explicitly list in this Appendix all diagonal elements of the spin and mixed inertia tensors together with the off-diagonal inertia tensor elements for static Skyrmion solutions with $1 \leq B \leq 16$ and rescaled mass parameter $\mu = 1$. The corresponding isospin inertia tensor diagonal elements have already been given in Table 5.1 and 5.2 in Chapter 5.

### B.0.1 Elements of the inertia tensors $U_{ij}, V_{ij}, W_{ij}$

Table B.1: Skyrmions of baryon number $1 \leq B \leq 8$. We list the diagonal elements of the spin ($V_{ij}$) and mixed ($W_{ij}$) inertia tensors and the symmetries $G$ of the Skyrme solitons.

<table>
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<th>$V_{22}$</th>
<th>$V_{33}$</th>
<th>$W_{11}$</th>
<th>$W_{22}$</th>
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### APPENDIX B. SKYRMION INERTIA TENSORS

Table B.2: Skyrmions of baryon number $9 \leq B \leq 16$. We list the diagonal elements of the spin ($V_{ij}$) and mixed ($W_{ij}$) inertia tensors and the symmetries $G$ of the Skyrme solitons.

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<th>$G$</th>
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<th>$V_{22}$</th>
<th>$V_{33}$</th>
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Table B.3: Skyrmions of baryon number $1 \leq B \leq 8$. We list the off-diagonal elements of the isospin ($U_{ij}$) and spin ($V_{ij}$) inertia tensors and the symmetries $G$ of the Skyrme solitons.

<table>
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<th>$U_{23}$</th>
<th>$V_{12}$</th>
<th>$V_{13}$</th>
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Table B.4: Skyrmions of baryon number $1 \leq B \leq 8$. We list the off-diagonal elements of the mixed inertia tensor $W_{ij}$ and the symmetries $G$ of the Skyrme solitons.

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Table B.5: Skyrmions of baryon number $9 \leq B \leq 16$. We list the off-diagonal elements of the isospin ($U_{ij}$) and spin ($V_{ij}$) inertia tensors and the symmetries $G$ of the Skyrme solitons.

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Table B.6: Skyrmions of baryon number $9 \leq B \leq 16$. We list the off-diagonal elements of the mixed inertia tensor $W_{ij}$ and the symmetries $G$ of the Skyrme solitons.

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