ON THE CHOICE OF THE UNCERTAINTY STRUCTURE IN ROBUST CONTROL PROBLEMS—A DISTANCE MEASURE APPROACH

A THESIS SUBMITTED TO THE UNIVERSITY OF MANCHESTER FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN THE FACULTY OF ENGINEERING AND PHYSICAL SCIENCES

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Abstract

This thesis is concerned with the choice of the uncertainty structure in robust control problems. This choice affects the optimization carried out to obtain a robust feedback controller, and determines how robust a feedback loop will be to discrepancies in the parameters or dynamics of the plant model. Firstly, it presents readily applicable distance measures, robust stability margins and associated robust stability and robust performance theorems for several commonly used uncertainty structures for linear time-invariant systems (additive, multiplicative, inverse multiplicative, inverse additive, right coprime factor uncertainty).

Secondly, the thesis discusses the robust stabilization problem for linear plants with a coprime factor uncertainty structure where the coprime factors of the plant are not necessarily normalized. The problem considered here is a generalization of the normalized coprime factor robust stabilization problem. It is shown that the minimum of the ratio of (non-normalized) coprime factor distance over (non-normalized) coprime factor robust stability margin, termed the robustness ratio, is an important bound in robust stability and performance results. A synthesis method is proposed which maintains a lower bound on the normalized coprime factor robust stability margin (as a proxy for nominal performance) while also robustly stabilizing a particular perturbed plant, potentially far outside a normalized coprime factor neighbourhood of the nominal plant.

The coprime factor synthesis problem is also considered in a state-space framework. It is shown that it admits a simple and intuitive controller implementation in observer form. Via the solution of one Riccati equation, an optimally robust observer gain $L$ can be obtained for any state-feedback matrix $F$. One particular method for obtaining a suitable $F$ is also proposed, ensuring that the feedback loop is particularly robust to uncertain lightly damped poles and zeros.
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University’s policy on Presentation of Theses.
In thankfulness
to the risen, living Christ,
my hope and strength.
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Publications From This Thesis

Journal Papers:


2. S. Engelken, A. Lanzon: “Revisiting robust stabilization of coprime factors: The general case.” Draft manuscript, to be submitted shortly. This paper will be based on material in Chapters 4 and 5 of this thesis.

Refereed Conference Papers:

1. S. Engelken, A. Lanzon, S. Patra, G. Papageorgiou: “Distance Measures for Linear Systems with Multiplicative and Inverse Multiplicative Uncertainty Characterisation.” In Proceedings of the 49th IEEE Conference on Decision and Control, Atlanta, GA, USA, December 2010, pp. 2336-2341. This paper is based on material in Chapter 3 of this thesis.

2. S. Engelken, A. Lanzon, S. Patra: “Robustness Analysis and Controller Synthesis with Non-Normalized Coprime Factor Uncertainty Characterisation.” Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference, Orlando, FL, USA, December 2011, pp. 4201-4206. This paper is based on material in Chapter 4 in this thesis.

ence on Decision and Control, December 2012. It is based on material in Chapter 5 of this thesis.
Notation and Acronyms

Fields of Numbers

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\mathbb{R}$</td>
<td>real numbers</td>
</tr>
<tr>
<td>$\mathbb{R}^n$</td>
<td>real-valued $n$-dimensional vectors</td>
</tr>
<tr>
<td>$\mathbb{R}^{m \times n}$</td>
<td>real-valued matrices with $m$ rows and $n$ columns</td>
</tr>
<tr>
<td>$j$</td>
<td>the imaginary unit, i.e. $j = \sqrt{-1}$</td>
</tr>
<tr>
<td>$\mathbb{C}$</td>
<td>complex numbers</td>
</tr>
<tr>
<td>$\mathbb{C}_+$</td>
<td>complex numbers with strictly positive real part</td>
</tr>
<tr>
<td>$\mathbb{C}_+$</td>
<td>complex numbers with real part $\geq 0$</td>
</tr>
<tr>
<td>$\mathbb{C}_-$</td>
<td>complex numbers with strictly negative real part</td>
</tr>
<tr>
<td>$\mathbb{C}_+$</td>
<td>complex numbers with real part $\leq 0$</td>
</tr>
<tr>
<td>$\mathbb{C}^n$</td>
<td>complex-valued $n$-dimensional vectors</td>
</tr>
<tr>
<td>$\mathbb{C}^{m \times n}$</td>
<td>complex-valued matrices with $m$ rows and $n$ columns</td>
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Relational Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>$\in$</td>
<td>belongs to</td>
</tr>
<tr>
<td>$\subset$</td>
<td>subset of</td>
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<tr>
<td>$\cup$</td>
<td>union</td>
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<tr>
<td>$:=\ $</td>
<td>defined by</td>
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<tr>
<td>$&lt;$</td>
<td>less than</td>
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<tr>
<td>$\leq$</td>
<td>less than or equal to</td>
</tr>
<tr>
<td>$&gt;$</td>
<td>greater than</td>
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<tr>
<td>$\geq$</td>
<td>greater than or equal to</td>
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<tr>
<td>$\neq$</td>
<td>not equal to</td>
</tr>
<tr>
<td>$\mapsto$</td>
<td>maps to</td>
</tr>
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\( \rightarrow \) tends to
\( \Rightarrow \) implies
\( \Leftarrow \) is implied by
\( \Leftrightarrow \) is equivalent to

**Miscellaneous**

\( \exists \) there exists
\( \forall \) for all
\( \square \) end of proof
\( \Re(s) \) real part of a complex number \( s \in \mathbb{C} \)
\( \Im(s) \) imaginary part of a complex number \( s \in \mathbb{C} \)
\( \{X,Y\} \) an ordered pair of two objects \( X \) and \( Y \)
\( [P,C] \) standard positive feedback interconnection of \( P \) and \( C \)
\( \text{Ric}(J) \) stabilizing solution of an Algebraic Riccati Equation

\( x \in (a,b) \) \( a < x < b \), where \( a, x, b \in \mathbb{R} \)
\( x \in [a,b] \) \( a < x \leq b \), where \( a, x, b \in \mathbb{R} \)
\( x \in [a,b] \) \( a \leq x < b \), where \( a, x, b \in \mathbb{R} \)
\( x \in [a,b] \) \( a \leq x \leq b \), where \( a, x, b \in \mathbb{R} \)

\( \lim_{x \rightarrow a} f(x) \) limit of \( f(x) \) as \( x \) tends to \( a \)
\( \min_{x \in \mathcal{X}} f(x) \) minimum of \( f(x) \) over \( x \in \mathcal{X} \)
\( \max_{x \in \mathcal{X}} f(x) \) maximum of \( f(x) \) over \( x \in \mathcal{X} \)
\( \inf_{x \in \mathcal{X}} f(x) \) infimum of \( f(x) \) over \( x \in \mathcal{X} \)
\( \sup_{x \in \mathcal{X}} f(x) \) supremum of \( f(x) \) over \( x \in \mathcal{X} \)
\( \text{ess sup}_{x \in \mathcal{X}} f(x) \) supremum of \( f(x) \) over \( x \in \mathcal{X} \) omitting isolated points

**Matrix Operators**

\( 0 \) zero matrix of compatible dimensions
\( I \) identity matrix of compatible dimensions
\( I_n \) \( n \)-dimensional identity matrix
\( A^T \) transpose of matrix \( A \)
\( A^* \) complex conjugate transpose of matrix \( A \)
\( A^{-1} \) inverse of matrix \( A \)
\( A^\dagger \) pseudo-inverse of matrix \( A \)
\( A^{-T} \) \((A^{-1})^T = (A^T)^{-1}\)
\( A^{-*} \) \((A^{-1})^* = (A^*)^{-T}\)
\( A > 0 \) positive definite matrix, \( x^*Ax > 0 \) \( \forall x \neq 0 \)
\( A \geq 0 \) positive semi-definite matrix, \( x^*Ax \geq 0 \) \( \forall x \neq 0 \)
\( A < 0 \) negative definite matrix, \( x^*Ax < 0 \) \( \forall x \neq 0 \)
\( A \leq 0 \) negative semi-definite matrix, \( x^*Ax \leq 0 \) \( \forall x \neq 0 \)
\( \det(A) \) determinant of matrix \( A \)
\( \mathcal{F}_l(H, \Delta) \) lower linear fractional transformation of matrices \( H \) and \( \Delta \)
\( \mathcal{F}_u(H, \Delta) \) upper linear fractional transformation of matrices \( H \) and \( \Delta \)

**Function Spaces**

\( \mathcal{R} \) space of real-rational functions
\( \mathcal{L}_2 \) space of functions that are square-integrable on \( j\mathbb{R} \cup \infty \)
\( \mathcal{H}_2 \) subspace of functions in \( \mathcal{L}_2 \) that are analytic and bounded in \( \mathbb{C}_+ \)
\( \mathcal{L}_\infty \) space of functions that are bounded on \( j\mathbb{R} \cup \infty \)
\( \mathcal{H}_\infty \) subspace of functions in \( \mathcal{L}_\infty \) that are analytic and bounded in \( \mathbb{C}_+ \)
\( \mathcal{R}X \) subspace of real-rational functions in the space \( X \)
\( \mathcal{U}H_\infty \) space of functions that are units in \( \mathcal{R}H_\infty \)
\( X^{n \times m} \) matrix-valued functions in \( X \), with \( n \) rows and \( m \) columns

**Systems Operations**

\( P^T \) transpose or dual of real-rational system \( P \), i.e. \( P^T(s) = P(s)^T \)
\( P^* \) adjoint of real-rational system \( P \), i.e. \( P^*(s) = P^T(-s) \)
\( P^{-1} \) inverse of real-rational system \( P \), i.e. \( P^{-1}(s) = P(s)^{-1} \)
\( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) shorthand for state-space realization \( C(sI - A)^{-1}B + D \)
\( \text{wno}(p) \) winding number of \( p(s) \in \mathcal{R} \) evaluated on a contour indented into the right half-plane around any imaginary poles of \( P \)
\( \eta(P) \) number of open right-half plane poles of \( P \in \mathcal{R} \)
Measures of Size

\( \lambda_i(A) \) \( i \)-th largest eigenvalue of a matrix \( A \) with only real eigenvalues

\( \sigma_i(A) \) \( i \)-th largest singular value of a matrix \( A \)

\( \sigma(A) \) largest singular value of a matrix \( A \)

\( \sigma(A) \) smallest singular value of a matrix \( A \)

\( \| x \| \) Euclidean norm of a vector \( x \in \mathbb{C}^n \)

\( \| P \|_2 \) \( L_2/\mathcal{H}_2 \)-norm of system \( P \in \mathbb{R}\mathcal{L}_2/\mathbb{R}\mathcal{H}_2 \)

\( \| P \|_\infty \) \( L_\infty/\mathcal{H}_\infty \)-norm of system \( P \in \mathbb{R}\mathcal{L}_\infty/\mathbb{R}\mathcal{H}_\infty \)

Coprime Factors

\( \{ N_0, M_0 \} \) right coprime factors of a real-rational system

\( \{ N, M \} \) normalized right coprime factors of a real-rational system

\( \{ \tilde{N}_0, \tilde{M}_0 \} \) left coprime factors of a real-rational system

\( \{ \tilde{N}, \tilde{M} \} \) normalized left coprime factors of a real-rational system

\( G_0 \) right graph symbol of a real-rational system

\( G \) normalized right graph symbol of a real-rational system

\( \tilde{G}_0 \) left graph symbol of a real-rational system

\( \tilde{G} \) normalized left graph symbol of a real-rational system

\( K \) normalized right inverse graph symbol

\( \bar{K} \) normalized left inverse graph symbol

Acronyms

SISO single-input, single-output

MIMO multiple-input, multiple-output

LFT linear fractional transformation

ARE Algebraic Riccati Equation

LMI linear matrix inequality

rcf right coprime factorisation

lcf left coprime factorisation
Chapter 1

Introduction

1.1 Background and Motivation

Uncertainty is a theme that lies at the heart of control engineering and control theory. Most engineering systems are subject to unmeasured and hence uncertain disturbances; at the same time, the mathematical system models used for analysis and control design are always imprecise. Several approaches exist for dealing with this problem of uncertainty. This thesis takes the approach of robust control: a nominal system model is chosen based on identification via experiments or based on equations describing the (physical) laws responsible for the behaviour of the plant, and the inherent uncertainty is represented by a norm-bounded operator interconnected with the nominal plant. The feedback control strategy is then optimized with two main aims:

1. to ensure a large robust stability margin, i.e. to guarantee stability of the uncertain feedback loop for a large set of uncertainty operators; and

2. to bound robust performance, i.e. to ensure that the performance of the uncertain feedback loop is as similar to nominal performance as possible.

Often there is a close link between the two objectives and the optimization of a certain robust stability margin entails robust performance guarantees.

A key aspect in this framework is the way in which the nominal plant and the uncertainty block are interconnected. This is referred to as the uncertainty
structure. The choice of this structure will be examined in detail in this thesis, since it plays a crucial role for achieving the two aims set out above. Plants and uncertainty blocks are assumed to be linear, time-invariant systems. The remainder of this section contains a brief, selective overview of the development of the field of robust control, followed by the motivation of the problems considered in subsequent chapters of this thesis.

### 1.1.1 Historical Development

The problem of robustness of feedback controllers began to receive significant attention in the control and systems research community in the mid-1970s. It had become apparent that the assumptions made in optimal linear-quadratic-Gaussian (LQG) control [Ath71, KS72, AM90], the leading multivariable control paradigm at the time, specifically the absence of modelling errors in the plant description and the description of disturbances by the Gaussian distribution, were too strong for many practical applications. Design studies on applications of LQG control to aerospace and submarine systems found significant problems when the optimal controllers were tested on real-world systems or complex numerical models [SF97, ACD et al. 1977]. Initially this refocused attention on concepts such as the gain and phase margin, which had been important tools in classical control theory for single-input, single-output (SISO) systems in the 1940s and 1950s. Attempts were made to extend these margins to multivariable plant settings [Ros72, DS81], but were eventually found inadequate to guarantee stability in the multi-input, multi-output (MIMO) case [Doy78b] and references therein; see also [ZDG96, Section 9.6].

The emerging field of robust control soon focused on worst-case bounds, making use of the small-gain theorem, singular values and $H_\infty$-norms. The small-gain theorem, due to Zames [Zam66b, Zam66a] formulates a sufficient condition for the stability of a feedback loop with two stable systems: the product of the worst-case gains of the systems must be strictly less than one. This well-known theorem provided the basis for the development of $H_\infty$ methods, introduced by Zames in [Zam81]. Additionally, singular values were identified as adequate generalizations of system gain and proximity to singularity for multivariable sys-
tems [Saf80, Doy78b]. Based on these tools, stability results were formulated for systems with $\mathcal{H}_\infty$ norm-bounded (stable) uncertainty blocks. For several types of uncertainty (additive, input/output multiplicative and their inverse counterparts), the small-gain theorem was quickly specialised in [DS81] and [DWS82]. For the additive uncertainty structure only, [Glo86] extended the space of allowable systems from $\mathbb{R}\mathcal{H}_\infty$ to $\mathbb{R}\mathcal{L}_\infty$.

Naturally, the analysis results led to a concerted research effort aimed at finding $\mathcal{H}_\infty$-optimal controllers. A first approach was described by Francis and Doyle [FD87, Fra87], but it suffered from being computationally expensive. Starting out with a state-space representation of the system, Francis and Doyle reduced the $\mathcal{H}_\infty$ problem to a problem of Nehari/Hankel norm approximation that had already been solved in [Glo84]. More tractable solutions to the $\mathcal{H}_\infty$ synthesis problem were obtained using algebraic Riccati equations [GD88, DGKF89] and several years later via linear matrix inequalities [GA94]. The former method was integrated into a comprehensive controller design methodology by McFarlane and Glover [GM89, MG90, MG92], combining a designer-specified weight selection with the synthesis of an $\mathcal{H}_\infty$-optimal controller to attain both robustness and performance properties.

Both in the technical developments of the controller design methods [FD87, Fra87] and separately as a tool to describe a large set of potential uncertainties [GM89], stable (normalized) coprime factors of systems emerged as a significant mathematical tool. [NJB84, MF87, Vid85] Several fundamental results, e.g. on simultaneous stabilization of systems, can be formulated in terms of coprime factors of systems.

From a control theoretic point of view, robust stability and robust performance problems can also be interpreted in the framework of distance measures. While robust stabilization focusses on one particular nominal plant and a set of norm-bounded uncertainties, distance measures take as their starting point sets or spaces of plant models. The distance between two plants is measured by the size ($\mathcal{H}_\infty$-norm) of a perturbation that would transform one plant into the other under an assumption on the perturbation structure. If a controller is designed for a nominal plant model, then robustness margins for a perturbed closed-loop system can be obtained which are related to the distance between the nominal and any per-
turbed plant. The gap metric [ES85] and the graph metric [Vid84] were proposed as such distance measures. Optimisation of the robust stability margin in the gap metric is equivalent to optimising robust stability with respect to a normalized coprime factor uncertainty structure [GS90]. A point-wise gap-metric was also described [QD92]. The $\nu$-gap [Vin93, Vin01] is a less conservative distance measure, also in the normalized coprime factor uncertainty structure, but considering a larger class of uncertainty blocks for measuring the discrepancy between systems.

The combination of $\mathcal{H}_\infty$ optimal control for normalized coprime factors using the loopshaping method and the $\nu$-gap has proved to be a versatile method for analysis and controller synthesis for uncertain MIMO systems. Applications can be found in the literature on control for aerospace systems [HG93, Hyd95, PGSP97, GBHP02, CH06], nuclear power feedwater systems [SS10], fuel cells [WCYY08], high speed flywheels [LT05], combustion processes [CGD03], power systems [PSR07] and other areas.

Beyond the robust stabilization of linear, time-invariant systems, recent years have seen research on extending the $\nu$-gap and related theoretical concepts in the areas of distributed/irrational systems [CJK12, RCS02], infinite dimensional systems [CV02, BS12], adaptive control [ABLM01, Fre08], time-varying systems [GLA09, JC11], system identification [HG03, DV04, DV05], model validation [DC05a, DC05b], iterative identification and control [DL04] and controller discretization [CV04].

### 1.1.2 Problem Motivation

This thesis is concerned with providing mathematical tools for the choice of the uncertainty structure in robust control problems. From the above sketch of the historical developments in robust control, however, it appears as if this question has been settled once and for all, with normalized coprime factor uncertainty emerging as the clear winner. To understand why different uncertainty structures should be considered by control engineers, several remarks are made here.

In the early years of robust control theory, engineers considered a variety of uncertainty structures, including additive and various types of multiplicative uncertainty etc. [DS81, DWS82], see also [DFT92]. An overview can be found...
e.g. in [ZDG96, Table 9.1]. These structures were often motivated by specific application-oriented, uncertainty inducing phenomena. For example, multiplicative uncertainty is particularly well-suited for modelling neglected high-frequency dynamics, while inverse multiplicative uncertainty is advantageous for low-frequency modelling errors. Each structure also imposes certain restrictions on whether a perturbation may change the number of right or left half plane poles and zeros of a plant for $H_\infty$-type robust stability conditions to hold. In this context, coprime factor uncertainty appears as the most general and powerful uncertainty structure, as it can be used to model both low and high frequency discrepancies and allows changing numbers of both right and left half plane poles and zeros. This generality explains in part its widespread application in linear robust control and adjacent areas.

While coprime factor uncertainty is indeed a very powerful tool, its generality may lead to relatively conservative robustness results when the uncertainty in a system is more naturally described by one of the other structures. The lack of any distance measures for these other structures, however, meant that the choice of which structure to use was based entirely on intuition and practical experience, as there was no clear link between the physical space of system models and the sizes of the uncertainties (in terms of their $H_\infty$ norm) transforming one system into another. The emphasis of Chapter 3 lies on providing distance measures for some of the uncertainty structures of [ZDG96, Table 9.1], based on a generalized framework described by Lanzon and Papageorgiou in [LP09], together with associated robust stability and robust performance results. The framework presented in [LP09] is very general, and encompasses normalized coprime factor uncertainty as one of its special cases. For that case, the distance measures and robust stability margins reduce to the well known concepts of the $\nu$-gap and the robust stability margin $b(P,C)$ as was shown in [PL06].

The approach of [LP09] is inspired by robust model validation techniques (see e.g. [CG00, NS98, LLDA06, DLVLA09, SD92, DVdH05]). The literature on model validation also provides important motivations for seeking distance measures for uncertainty structures other than normalized coprime factors. With a view to applying model validation techniques to robust identification problems, [SD92] provides a computational algorithm for obtaining a distance-like uncer-
tainty block in a generic structured singular value framework (with the associated numerical difficulties). Explicit results for any particular uncertainty structure are not provided. In [DVdH05], robust stability and performance results for single-input, single-output systems with additive, normalized coprime factor and Youla-type uncertainty are derived, with a focus on how these uncertainty structures can be made equivalent via the choice of different nominal plants. For the Youla controller structure only, these results have been extended to the multi-variable case [OvHB09]. Model embedding in different uncertainty structures is also important in iterative identification and control redesign methods (see e.g. [ASP02, DL04, DLVL09], and the references therein), where the controller is synthesized using a nominal plant model and an uncertainty set around this nominal plant model obtained during the identification stage. These two stages of identification and control synthesis are iterated until no further improvement in performance can be achieved. Quantitative measures that enable model embedding in small uncertainty sets can hence assist in obtaining less conservative results.

Outside of robust control, different uncertainty structures are also used in model order reduction [OA01], in model approximation and in system identification [Lju99]. Most authors use an additive discrepancy criterion, due to its simplicity and ease of intuition. However, recent results on model approximation techniques invoke different discrepancy criteria between the original model and the approximated model (see [SLA06, SLA07] for model order reduction and system identification using multiplicative discrepancy criteria and [Bus07] for model order reduction using a normalized coprime factor discrepancy criterion). In order to differentiate these methods quantitatively rather than only qualitatively, and to link the resulting discrepancy between nominal and perturbed model to a degradation in closed-loop stability margins and robust performance, the distance measure approach is indispensable.

The other main focus of this thesis lies on providing both analysis and controller synthesis results for coprime factor uncertainty that is not necessarily normalized. This structure is another special case of the generalized distance measure framework of [LP09], for which results were provided in the same article. The motivation for considering this type of uncertainty comes from the application
area of lightly damped systems. It has long been known [HS93, Vin01] that normalized coprime factor uncertainty and controller optimisation with respect to this structure are problematic when the nominal plant has poles or zeros with uncertain location on or near the imaginary axis in certain regions of high or low gain. Typically this problem cannot simply be solved via the loopshaping weights of the $\mathcal{H}_\infty$ loopshaping method, since this would in effect require weights that cancel out the problematic lightly damped poles or zeros. It is argued in this thesis that choosing different coprime factors, i.e. removing the requirement for normalization, may lead to significantly less conservative robust stability and robust performance results.

1.2 Organisation of the Thesis

The thesis consists of six chapters, which following the Introduction are organised as follows.

Chapter 2: Preliminaries This chapter collects the mathematical and system theoretical concepts that are used for the derivations and results in the subsequent chapters of the thesis. It begins with several definitions on linear algebra and function spaces, before providing formal definitions for several concepts from multivariable systems theory, both in a state space and frequency domain formulation. Standard feedback loops are then introduced and several of their properties defined. Brief expositions of two solutions to the $\mathcal{H}_\infty$ control problem are then provided and the chapter concludes with an overview of some results on coprime factors.

Chapter 3: Robust stability and performance analysis for uncertain linear systems—The distance measure approach This chapter presents readily applicable distance measures, robust stability margins and associated robust stability and robust performance theorems for several commonly used uncertainty structures (additive, input/output multiplicative, output/input inverse multiplicative, inverse additive and right coprime factor uncertainty). Besides providing robust stability results for a larger uncertainty class than previously reported ($RL_\infty$ instead of $RH_\infty$), this chapter also states robust performance theorems for the above uncertainty structures. In contrast to previous methods for robust perfor-
mance analysis, they only require the computation of two infinity norms for every uncertain plant considered. The theorems in this chapter enable control engineers to choose the most suitable uncertainty structure for a family of uncertain plants, as illustrated through numerical examples. The examples also illustrate how in some cases the normalized coprime factor uncertainty structure yields results that are more conservative than those obtained for the same plant with a more specialised uncertainty structure.

Chapter 4: Analysis and controller synthesis for the non-normalized coprime factor uncertainty structure

This chapter discusses the robust stabilization problem for linear plants with a coprime factor uncertainty structure where the coprime factors of the plant are not necessarily normalized. The problem considered here is a generalization of the normalized coprime factor robust stabilization problem. Coprime factorizations minimizing the ratio of coprime factor distance between nominal and perturbed plant to coprime factor robust stability margin are described. It is shown that this minimized ratio, termed the robustness ratio, provides less conservative robust stability and robust performance conditions than can be obtained using normalized coprime factor concepts. An objective function for controller synthesis is then proposed which combines the robustness ratio and the weighted normalized coprime factor robust stability margin. This enables robust stabilization of sets with plants far away from the nominal plant in the normalized coprime factor sense, while maintaining robustness in that sense and also ensuring robust performance. A numerical example demonstrates the synthesis method on an uncertain lightly damped benchmark plant.

Chapter 5: Non-normalized coprime factor uncertainty—a state-space solution

Like the previous chapter, Chapter 5 considers the robust stabilization problem of uncertain linear-time invariant plants with coprime factor uncertainty bounded in $\mathcal{RH}_\infty$. It is shown that the problem admits a simple and intuitive controller implementation parameterized in terms of an observer-form controller with a state-feedback matrix $F$ and an observer gain $L$. The choice of a state-feedback matrix $F$ induces a topology in which distance between plants is measured. Subsequently, an observer gain $L$ can be obtained to maximize robustness of the controller in this topology via the solution of a Riccati equation. This synthesis method results in a controller of the same order as the nominal plant. It is
also shown that non-normalized coprime factorizations are a more suitable tool for obtaining robustly stabilizing controllers for uncertain lightly damped plants than normalized coprime factorizations, which only provide very limited robustness guarantees.

**Chapter 6: Conclusions** This chapter provides a summary of the contributions of this thesis and outlines some directions for future research.
Chapter 2

Mathematical Preliminaries

In this chapter, some of the basic mathematical and system theoretical tools used in the subsequent developments in this thesis are described. This includes concepts from linear algebra and functional analysis that are standard in robust control theory. Some basic results on stability theory and optimization in $\mathcal{H}_\infty$ are also described. Throughout this chapter, references to more detailed treatments of these topics are given.

2.1 Linear Algebra

2.1.1 Singular Values

In linear systems theory, the gain of an operator is of great importance. For complex-valued matrices, the singular values describe the gain in the different output directions.

**Definition 2.1.1.** Given $A \in \mathbb{C}^{m \times n}$ with $m \geq n$, define its $i$-th singular value $\sigma_i(A)$ as

$$\sigma_i(A) := \sqrt{\lambda_i(A^*A)}.$$ 

For the case, $m \leq n$, replace $A^*A$ by $AA^*$. Further define $\overline{\sigma}(A) := \sigma_1(A)$ and $\underline{\sigma}(A) := \sigma_p(A)$, where $p = \min\{m, n\}$.

From this definition, it is clear that for (possibly complex) vectors $x, y$ of ap-
Figure 2.1: An upper LFT (left) and a lower LFT (right).

appropriate dimensions related by \( y = Ax \),

\[
\sigma(A) \leq \frac{\|y\|}{\|x\|} \leq \bar{\sigma}(A).
\]

Furthermore, one can also interpret the singular values of \( A \) geometrically as the semi-axes of a hyperellipsoid \( E \) defined by

\[
E := \{ y \in \mathbb{C}^m : y = Ax, \ x \in \mathbb{C}^n, \ ||x|| = 1 \}.
\]

Each singular value is associated with a direction defined by the eigenvector \( v_i \) of \( A^*A \) (\( u_i \) of \( AA^* \)) associated with the eigenvalue \( \lambda_i(A^*A) \) (respectively \( \lambda_i(AA^*) \)).

For further reading, see [HJ90, HJ91]. Singular values played a crucial role in the development of robust control theory, as they generalize the concept of “system gain” for multivariable systems [Saf80, Doy78b].

### 2.1.2 Linear Fractional Transformation

For the analysis of feedback loops, the linear fractional transformation (LFT) has been adopted as a standard tool in the robust control literature. It describes the system obtained by closing an upper (respectively lower) loop around a block \( 2 \times 2 \) coefficient matrix \( H \) with a block \( \Delta_u \) (resp. \( \Delta_l \)) in the loop. An algebraic defition follows, with its interpretation in terms of block diagrams shown in Fig. 2.1.

**Definition 2.1.2.** Let \( H \in \mathbb{C}^{(p_1+p_2) \times (q_1+q_2)} \) be partitioned as

\[
H = \begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix},
\]
with $H_{11} \in \mathbb{C}^{p_1 \times q_1}$ and $H_{22} \in \mathbb{C}^{p_2 \times q_2}$. Given $\Delta_1 \in \mathbb{C}^{q_2 \times p_2}$ and $\Delta_u \in \mathbb{C}^{q_1 \times p_1}$, then define

- the lower LFT $\mathcal{F}_l(H, \Delta_1) := H_{11} + H_{12} \Delta_1 (I - H_{22} \Delta_1)^{-1} H_{21}$, provided that $(I - H_{22} \Delta_1)^{-1}$ exists;
- the upper LFT $\mathcal{F}_u(H, \Delta_u) := H_{22} + H_{21} \Delta_u (I - H_{11} \Delta_u)^{-1} H_{12}$, provided that $(I - H_{11} \Delta_u)^{-1}$ exists.

Further details can be found in [ZDG96, Chapter 10] and [DPZ91].

### 2.2 Function Spaces and Norms

Several spaces consisting of complex valued functions with their domain on (the imaginary axis of) the complex plane are defined in the following. For normed spaces, the associated norm is also provided.

**Definition 2.2.1.** $\mathcal{R}$ is the space consisting of all matrix valued functions $F : \mathbb{C} \rightarrow \mathbb{C}^{n \times m}$ which are proper rational functions with real coefficients.

**Definition 2.2.2.** $\mathcal{L}_2$ is the space consisting of all matrix valued functions $F : j\mathbb{R} \rightarrow \mathbb{C}^{n \times m}$ for which

$$
\int_{-\infty}^{\infty} \text{tr} [F^*(j\omega)F(j\omega)] d\omega < \infty.
$$

The $\mathcal{L}_2$-norm for a function $F \in \mathcal{L}_2$ is defined by

$$
\|F\|_2 := \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} [F^*(j\omega)F(j\omega)] d\omega}.
$$

**Definition 2.2.3.** $\mathcal{H}_2 \subset \mathcal{L}_2$ is the space of all matrix valued functions $F : \mathbb{C} \rightarrow \mathbb{C}^{n \times m} \in \mathcal{L}_2$ that are analytic in $\mathbb{C}_+$. Its norm is defined as

$$
\|F\|_2^2 := \sup_{\sigma > 0} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} [F^*(\sigma + j\omega)F(\sigma + j\omega)] d\omega \right\}
$$

$$
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} [F^*(j\omega)F(j\omega)] d\omega.
$$
It should be noted that $L_2$ contains all transfer function matrices which are square-integrable along the $j\omega$ axis\(^1\), and $H_2$ contains all stable transfer function matrices for which this property holds. The norms of both spaces are denoted hereinafter by $\|F\|_2$, since they are computed in the same way [Fra87, Chapter 2]

**Definition 2.2.4.** $L_\infty$ is the space of all matrix valued functions $F : j\mathbb{R} \to \mathbb{C}^{n \times m}$ which are essentially bounded on the imaginary axis of the complex plane, with a norm defined by

$$\|F\|_\infty := \text{ess sup}_{\omega \in \mathbb{R}} \sigma[F(j\omega)].$$

**Definition 2.2.5.** $H_\infty \subset L_\infty$ is the space of all matrix valued functions $F : \mathbb{C} \to \mathbb{C}^{n \times m} \in L_\infty$ which are analytic and bounded in $\mathbb{C}_+$. The associated norm is given by

$$\|F\|_\infty := \sup_{\Re(s) > 0} \sigma[F(s)] = \sup_{\omega \in \mathbb{R}} \sigma[F(j\omega)].$$

Note that the norm of the $L_\infty$ and $H_\infty$ spaces can again be computed in the same way [BD85] and are henceforth both denoted by $\|F\|_\infty$.

**Definition 2.2.6.** Denote by the prefix $\mathcal{R}$ those subspaces of $L_2$, $H_2$, $L_\infty$ and $H_\infty$ which contain only real-rational transfer function matrices.

**Definition 2.2.7.** Denote by $\mathcal{G}_\infty$ the subspace of $\mathcal{R}H_\infty$ whose elements are units in $\mathcal{R}H_\infty$, i.e. $R \in \mathcal{G}_\infty$ if and only if $R \in \mathcal{R}H_\infty$ and $R^{-1} \in \mathcal{R}H_\infty$.

Additional details and results can be found in [ZDG96, Fra87, DV75, GG81, You81].

### 2.3 Multivariable Systems: Operations and Properties

In this section, several aspects of multivariable systems are discussed. The definitions and results are all standard in robust control theory and can be found e.g.\(^1\)

\(^1\)There exists an isometric isomorphism between the $L_2$ space defined here and the space of square-time integrable functions. This signal space is often denoted by $L_2(-\infty, \infty)$ or just $L_2$.  

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in [ZDG96, GL00, Kai80]. The entire thesis will make frequent use of state space realizations, which are formally defined below.

**Definition 2.3.1.** Given $P \in \mathbb{R}^{p \times q}$, then the ordered quadruplet $(A, B, C, D)$ with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times q}$, $C \in \mathbb{R}^{p \times n}$ and $D \in \mathbb{R}^{p \times q}$ is called a state space realization of $P$, denoted by

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

if $P(s) = C(sI - A)^{-1}B + D$.

### 2.3.1 State Space Properties

Several important properties of state space realizations are defined in this subsection.

**Definition 2.3.2.** A matrix $A \in \mathbb{R}^{n \times n}$ is said to be Hurwitz if all eigenvalues of $A$ are in $\mathbb{C}^-$.  

**Definition 2.3.3.** The pair $(A, B)$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times q}$, is said to be controllable if the eigenvalues of $A + BF$ can be freely assigned by a suitable choice of $F \in \mathbb{R}^{q \times n}$ (under the restriction that complex eigenvalues occur in complex conjugate pairs).

Note that this definition of controllability is equivalent to the existence of a piecewise continuous time varying vector $u(t) \in \mathbb{R}^{q}$ driving the dynamical system $\dot{x}(t) = Ax(t) + Bu(t)$ from any initial condition $x(0) \in \mathbb{R}^{n}$ to any final state $x(t_1) \in \mathbb{R}^{n}$, where $t_1 > 0$.

**Definition 2.3.4.** The pair $(A, B)$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times q}$, is said to be stabilizable if there exists a matrix $F \in \mathbb{R}^{q \times n}$ such that $A + BF$ is Hurwitz.

**Definition 2.3.5.** The pair $(C, A)$, $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{p \times n}$, is said to be observable if the eigenvalues of $A + LC$ can be freely assigned by a suitable choice of $L \in \mathbb{R}^{n \times p}$ (under the restriction that complex eigenvalues occur in complex conjugate pairs).

Note that this definition of observability is equivalent to the ability to observe the initial state $x(0)$ of a dynamical system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$ at any time $t_1 > 0$ from knowledge of the history of $u(t)$ and $y(t)$ in the time interval $[0, t_1]$. 

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**Definition 2.3.6.** The pair \((C,A)\), \(A \in \mathbb{R}^{n \times n}\), \(C \in \mathbb{R}^{p \times n}\), is said to be detectable if there exists a matrix \(L \in \mathbb{R}^{q \times n}\) such that \(A + LC\) is Hurwitz.

The following standard result provides conditions for each of the above-defined system properties.

**Theorem 2.3.1** ([ZDG96]). Given \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times q}\), \(C \in \mathbb{R}^{p \times n}\), then
\[ (A,B) \text{ is controllable } \iff [A - \lambda I \ B] \text{ has full row rank } \forall \lambda \in \mathbb{C}; \]
\[ (A,B) \text{ is stabilizable } \iff [A - \lambda I \ B] \text{ has full row rank } \forall \lambda \in \mathbb{C}^+; \]
\[ (C,A) \text{ is observable } \iff [A - \lambda I \ C] \text{ has full column rank } \forall \lambda \in \mathbb{C}; \]
\[ (C,A) \text{ is detectable } \iff [A - \lambda I \ C] \text{ has full column rank } \forall \lambda \in \mathbb{C}^+. \]

**Definition 2.3.7.** A state space realization \((A,B,C,D)\) is said to be minimal if \((A,B)\) is controllable and \((C,A)\) is observable.

### 2.3.2 Operations on Systems

In the following, various basic operations on systems are described in terms of their state space realizations.

**Definition 2.3.8.** Given \(P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{p \times q}\), then the transpose or dual of \(P\) is defined as
\[ P^T := \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}. \]

**Definition 2.3.9.** Given \(P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{p \times q}\), then the \(L_2\)-adjoint of \(P\) is defined as
\[ P^* := \begin{bmatrix} -A^T & -C^T \\ B^T & D^T \end{bmatrix}. \]

Therefore, \(P^*(s) = P^T(-s)\).
**Definition 2.3.10.** Suppose \( P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{p \times p} \). If \( D \) has full rank, then the inverse of \( P \) is defined as

\[
P^{-1} := \begin{bmatrix} A - BD^{-1}C & -BD^{-1} \\ D^{-1}C & D^{-1} \end{bmatrix}.
\]

### 2.3.3 Poles and Zeros of Multivariable Systems

In this subsection, poles and zeros of multivariable systems are discussed based on state space realizations.

**Definition 2.3.11.** Given a system \( P \in \mathbb{R}^{p \times q} \) with a minimal state space realization \((A, B, C, D)\), then the poles of \( P \) are the eigenvalues of \( A \).

Now consider the following definition of normal rank, which is needed to define system zeros.

**Definition 2.3.12.** Given \( P \in \mathbb{R}^{p \times q} \), define the normal rank of \( P \) as the maximally possible rank of \( P(s) \) for at least one \( s \in \mathbb{C} \).

The following definition of transmission zeros is made independent of any state space realization. The link to state space is provided subsequently.

**Definition 2.3.13.** Given \( P \in \mathbb{R}^{p \times q} \). Then \( z_0 \in \mathbb{C} \) is a transmission zero of \( P \) if either

- \( P \) has full column normal rank and \( \exists \neq u_0 \in \mathbb{C}^q \) s.t. \( P(z_0)u_0 = 0 \); or
- \( P \) has full row normal rank and \( \exists \neq \eta_0 \in \mathbb{C}^p \) s.t. \( \eta_0^* P(z_0) = 0 \).

The lemma below provides conditions for the existence of transmission zeros based on a minimal state space realization of a system. It follows immediately from the combination of [ZDG96, Lemma 3.31-3.33] and [ZDG96, Theorem 3.34].

**Lemma 2.3.2.** Given \( P \in \mathbb{R}^{p \times q} \) with a minimal state space realization \((A, B, C, D)\), where \( A \in \mathbb{R}^{n \times n} \).
Figure 2.2: The feedback interconnection \([P, C]\).

- if \(P\) has full column normal rank, \(z_0 \in \mathbb{C}\) is a transmission zero of \(P\) if and only if \(\exists 0 \neq x \in \mathbb{C}^n, u \in \mathbb{C}^q\) such that
  \[
  \begin{bmatrix}
  A - z_0 I & B \\
  C & D
  \end{bmatrix}
  \begin{bmatrix}
  x \\
  u
  \end{bmatrix} = 0;
  \]

- if \(P\) has full row normal rank, \(z_0 \in \mathbb{C}\) is a transmission zero of \(P\) if and only if \(\exists 0 \neq y \in \mathbb{C}^n, v \in \mathbb{C}^p\) such that
  \[
  \begin{bmatrix}
  y^* & v^*
  \end{bmatrix}
  \begin{bmatrix}
  A - z_0 I & B \\
  C & D
  \end{bmatrix} = 0.
  \]

### 2.4 Feedback Loops and Robustness

In control systems, feedback is one of the fundamental principles employed to reduce the impact of uncertainty on the performance of the system. Uncertainty always exists with respect to the physical model of a system, with respect to the parameters used in that model, and with respect to exogeneous inputs.

In the remainder of this thesis, the standard feedback loop displayed in Fig. 2.2 will be considered. Here, \(P \in \mathbb{R}^{p \times q}\) represents the multidimensional plant to be controlled, and \(C \in \mathbb{R}^{q \times p}\) is the controller in the feedback path. This interconnection will be denoted by \([P, C]\).

Two important properties of feedback loops are now introduced and algebraic conditions given under which these properties hold.

**Definition 2.4.1.** A feedback interconnection of systems in \(\mathcal{R}\) is said to be well-posed if all closed-loop transfer function matrices are well-defined and proper.
Lemma 2.4.1 ([ZDG96, Lemma 5.1]). The feedback interconnection $[P,C]$ is well-posed if and only if

$$I - C(\infty)P(\infty)$$

is invertible.

Definition 2.4.2. A feedback interconnection of systems in $\mathcal{R}$ is said to be internally stable if it is well-posed and all closed-loop transfer function matrices belong to $\mathcal{RH}_\infty$.

Lemma 2.4.2 ([ZDG96, Lemma 5.3]). The feedback interconnection $[P,C]$ is internally stable if and only if

$$\begin{bmatrix} I & -C \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} C \\ I \end{bmatrix} (I - PC)^{-1} \begin{bmatrix} P & I \\ 0 & 0 \end{bmatrix} \in \mathcal{RH}_\infty.$$

As alluded to in the initial paragraph of this section, focusing on the feedback loop with the nominal plant alone however is not sufficient in most control applications. Instead, the behaviour of the feedback loop with perturbed versions of the plant, denoted by $P_\Delta \in \mathcal{RH}_p^{p \times q}$, is of great interest. The Small Gain Theorem, given by Zames in 1966 [Zam66b, Zam66a], describes stability conditions for a feedback interconnection of norm-bounded systems. This theorem forms the basis for much of robust control theory. Its setting can be seen in Fig. 2.3. A version of the theorem based on stable real-rational systems is given here.

Theorem 2.4.3 (Small Gain Theorem [ZDG96, Theorem 9.1]). Given $M \in \mathcal{RH}_\infty^{p \times q}$ and a scalar $\gamma > 0$. Then $[M,\Delta]$ is well-posed and internally stable for all $\Delta \in \mathcal{RH}_\infty^{q \times p}$ such that $\|\Delta\|_\infty \leq \gamma^{-1}$ (respectively $\|\Delta\|_\infty < \gamma^{-1}$) if and only if $\|M\|_\infty < \gamma$ (respectively $\|M\|_\infty \leq \gamma$).
Chapter 3 will return to this theorem and its interpretation in the context of distance measures. An extension to systems in $\mathcal{H}_\infty$ will also be stated there, which was first developed in [Glo84] for systems with additive uncertainty.

In the formulation of the Small Gain Theorem above, the system $M$ is deliberately not denoted simply as the plant $P$, because this block typically involves all known parts of the system, most commonly a LFT $\mathcal{P}_1(H,C)$ of a generalized plant $H$ (involving the nominal plant model $P$ as one of its sub-blocks) and the controller $C$. Such a setup is depicted in Fig. 2.4, and will be denoted by $\langle H, C \rangle$.

**Definition 2.4.3.** Given $H \in \mathbb{R}^{(p_1+p_2) \times (q_1+q_2)}$ and $C \in \mathbb{R}^{q_2 \times p_2}$, denote by $\langle H, C \rangle$ the feedback interconnection depicted in Fig. 2.4.

Based on this setup, the following definition of an admissible controller is given.

**Definition 2.4.4.** A controller $C \in \mathbb{R}^{q_2 \times p_2}$ is said to be an admissible controller for a generalized plant $H \in \mathbb{R}^{(p_1+p_2) \times (q_1+q_2)}$ if it internally stabilizes $\langle H, C \rangle$.

### 2.5 $\mathcal{H}_\infty$ Control

The Small Gain Theorem implies that minimizing $\|\mathcal{P}_1(H,C)\|_\infty$, where $H$ is a generalized plant and $C$ is a controller, both of appropriate dimensions, improves the robustness of a feedback interconnection of $\mathcal{P}_1(H,C)$ with an uncertainty block $\Delta$, i.e. guarantees internal stability for a larger set of uncertainties. The fundamental problem of finding a controller $C$ such that $\mathcal{P}_1(H,C)$ fulfills an $\mathcal{H}_\infty$-norm bound received a large amount of attention in the 1980s. In this section, the seminal result of Doyle, Glover, Khargonekar and Francis on suboptimal $\mathcal{H}_\infty$
control [DGKF89] is stated. Additionally, a separate $\mathcal{H}_\infty$ control result due to Gahinet and Apkarian [GA94] is also given. This latter result is based on a different mathematical solution technique (Linear Matrix Inequalities instead of Algebraic Riccati Equations) and has somewhat less restrictive assumptions. However, the controller reconstruction does not allow for a straightforward interpretation as is the case for the results of [DGKF89]. Both results are used in this thesis to solve two separate $\mathcal{H}_\infty$ control problems.

**Remark 2.5.1.** Subsequently, this thesis will occasionally use the notation of $C_\infty$ for the controller to avoid ambiguity with the state-space matrix $C$.

### 2.5.1 The Algebraic Riccati Equation-based Solution to the $\mathcal{H}_\infty$ Control Problem

Prior to the main result, this subsection contains the definition of several concepts related to Algebraic Riccati Equations. These are treated extremely briefly here; more details and additional results may be found e.g. in [ZDG96, LR95].

**Definition 2.5.1.** Given $A \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{n \times n}$, the Algebraic Riccati Equation (ARE) in the variable $X \in \mathbb{C}^{n \times n}$ associated with the Hamiltonian matrix $J := \begin{bmatrix} A & R \\ -Q & A^* \end{bmatrix}$ is given by

$$A^*X + XA + XRX + Q = 0.$$  

**Definition 2.5.2.** Denote by Ric the function

$$\text{Ric} : J \in \text{dom}(\text{Ric}) \subset \mathbb{R}^{2n \times 2n} \longmapsto X \in \mathbb{R}^{n \times n}$$

mapping a Hamiltonian matrix $J \in \text{dom}(\text{Ric})$ to the solution $X \in \mathbb{R}^{n \times n}$ of the ARE associated with $J$, where $\text{dom}(\text{Ric})$ is the set of Hamiltonian matrices with the following properties

- $J$ has no imaginary eigenvalues;
the n-dimensional invariant spectral subspace $\mathcal{X}(J)$ corresponding to eigenvalues in $\mathbb{C}$ has a basis $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, where $X_1, X_2 \in \mathbb{R}^{n \times n}$ and $X_1$ is invertible.

Note that the Ric operator always yield a real solution matrix $X$. In fact, $X$ is also symmetric and stabilizing, in the sense that $A + RX$ is Hurwitz [ZDG96, Theorem 13.5]. Analytic conditions for the existence of some types of AREs which are of great interest from a control perspective exist (see e.g. [ZDG96, Theorem 13.7]); however, these results are beyond the scope of this chapter. Algebraic Riccati Equations can be solved efficiently using standard numerical software.

With these preliminaries defined, the main result on an ARE-based solution to the $H_\infty$ control problem can now be stated.

**Theorem 2.5.1** ([ZDG96, Theorem 17.1]). Given

$$H = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A \in \mathbb{R}^{n \times n}, B_1 \in \mathbb{R}^{n \times q_1}, B_2 \in \mathbb{R}^{n \times q_2}, C_1 \in \mathbb{R}^{p_1 \times n}, C_2 \in \mathbb{R}^{p_2 \times n}, D_{11} \in \mathbb{R}^{p_1 \times q_1}, D_{12} \in \mathbb{R}^{p_1 \times q_2}$ and $D_{21} \in \mathbb{R}^{p_2 \times q_1},$ and fulfilling the following assumptions:

1. **(A1)** $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable;

2. **(A2)** $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix};$

3. **(A3)** $\begin{bmatrix} A - j \omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega$;

4. **(A4)** $\begin{bmatrix} A - j \omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\omega$;

and given $\gamma > 0$, define the following:

$$D_{1\bullet} := \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}; \quad D_{\bullet 1} := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix};$$
\( R := D_1^* D_1 \bullet - \begin{bmatrix} \gamma^2 I_{q_1} & 0 \\ 0 & 0 \end{bmatrix}; \)

\( \tilde{R} := D_1^* D_1 \star - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix}; \)

\( H_\infty := \begin{bmatrix} A & 0 \\ -C_1^* C_1 & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C_1^* D_1 \bullet \end{bmatrix} R^{-1} \begin{bmatrix} D_1^* C_1 & B^* \end{bmatrix}; \)

\( J_\infty := \begin{bmatrix} A^* & 0 \\ -B_1 B_1^* & -A \end{bmatrix} - \begin{bmatrix} C^* \\ -B_1 D_1^* \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_1 B_1^* & C \end{bmatrix}; \)

\( X_\infty := \text{Ric}(H_\infty); \quad Y_\infty := \text{Ric}(J_\infty); \)

\( F := \begin{bmatrix} F_{1\infty} \\ F_{2\infty} \end{bmatrix} := -R^{-1} \begin{bmatrix} D_1^* C_1 + B^* X_\infty \end{bmatrix}; \)

\( L := \begin{bmatrix} L_{1\infty} & L_{2\infty} \end{bmatrix} := - \begin{bmatrix} B_1 D_1^* + Y_\infty C^* \end{bmatrix} \tilde{R}^{-1}. \)

Also introduce the following partitioning for \( D, F_\infty \) and \( L_\infty \):

\[
\begin{bmatrix}
\overset{F^*}{ullet} \\
\overset{L^*}{\bullet}
\end{bmatrix} =
\begin{bmatrix}
\overset{F_{1\infty}^*}{L^*_{1\infty}} & \overset{F_{12\infty}^*}{D_{1111}} & \overset{F_{2\infty}^*}{D_{1112}} \\
\overset{L^*_{1\infty}}{D_{1111}} & \overset{D_{1112}}{D_{1121}} & 0 \\
\overset{L^*_{12\infty}}{D_{1121}} & \overset{D_{1122}}{D_{1221}} & 1 \\
\overset{L^*_{2\infty}}{0} & \overset{1}{0} & 0
\end{bmatrix}.
\]

There exists an admissible controller \( C_\infty \) such that \( \| \mathcal{F}_1 (H, C_\infty) \|_\infty < \gamma \) if and only if

1. \( \gamma > \max \left( \sigma \left( \begin{bmatrix} D_{1111} & D_{1112} \end{bmatrix} \right), \sigma \left( \begin{bmatrix} D_1^* & D_{1121}^* \end{bmatrix} \right) \right); \)
2. \( H_\infty \in \text{dom}(\text{Ric}) \text{ with } X_\infty = \text{Ric}(H_\infty) \geq 0; \)
3. \( J_\infty \in \text{dom}(\text{Ric}) \text{ with } Y_\infty = \text{Ric}(J_\infty) \geq 0; \)
4. \( \rho (X_\infty Y_\infty) < \gamma^2. \)

If conditions (i)-(iv) are satisfied, then all rational internally stabilizing controllers \( C_\infty \) satisfying \( \| \mathcal{F}_1 (H, C_\infty) \|_\infty < \gamma \) are given by \( C_\infty = \mathcal{F}_1 (M_\infty, Q) \), where
\( Q \in \mathcal{RH}_\infty \) satisfies \( \|Q\|_\infty < \gamma \) and
\[
M_\infty := \begin{bmatrix}
\hat{A} & \hat{B}_1 & \hat{B}_2 \\
\hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\
\hat{C}_2 & \hat{D}_{21} & 0
\end{bmatrix},
\]

where
\[
\hat{D}_{11} := -D_{1121}D_{1111}^* (\gamma^2 I - D_{1111}D_{1111}^*)^{-1} D_{1112} - D_{1122},
\]
\( \hat{D}_{12} \in \mathbb{R}^{q_2 \times q_2} \) and \( \hat{D}_{21} \in \mathbb{R}^{p_2 \times p_2} \) satisfy
\[
\hat{D}_{12}^* \hat{D}_{12} = I - D_{1121} (\gamma^2 I - D_{1111}D_{1111}^*)^{-1} D_{1112},
\]
\[
\hat{D}_{21}^* \hat{D}_{12} = I - D_{1112} (\gamma^2 I - D_{1111}D_{1111}^*)^{-1} D_{1112},
\]
and
\[
\hat{B}_2 := Z_\infty (B_2 + L_{12\infty}) \hat{D}_{12},
\hat{C}_2 := -\hat{D}_{21} (C_2 + F_{12\infty}),
\hat{B}_1 := -Z_\infty L_{2\infty} + \hat{B}_2 \hat{D}_{12}^{-1} \hat{D}_{11},
\hat{C}_1 := F_{2\infty} + \hat{D}_{11} \hat{D}_{21}^{-1} \hat{C}_2,
\hat{A} := A + BF + \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2
\]

where \( Z_\infty := (I - \gamma^{-2} Y_{\infty} X_{\infty})^{-1} \).

**Remark 2.5.2.** Some of the assumptions in the above theorem can be relaxed or lifted. Various loop shifting operations can be performed on the generalized plant \( H \) if assumption (A2) does not hold, but \( D_{12} \) and \( D_{21} \) have full column and row rank, respectively, or if \( D_{22} \neq 0 \) [SLC89]. If \( D_{12} \) and \( D_{21} \) do not have full rank, the problem becomes singular. A solution to that case can be found in [Sto92].
2.5.2 The Linear Matrix Inequality-based Solution to the $\mathcal{H}_\infty$ Control Problem

The suboptimal $\mathcal{H}_\infty$ control problem can also be formulated using Linear Matrix Inequalities (LMI). These are convex constraints formulated as a linear function dependent on a decision variable vector or matrix. Details are beyond the scope of this chapter and can be found e.g. in [BEGFB94]. Numerical software solvers for LMIs are available.

**Theorem 2.5.2 ([GA94]).** Given

$$H = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A \in \mathbb{R}^{n \times n}$, $B_1 \in \mathbb{R}^{n \times q_1}$, $B_2 \in \mathbb{R}^{n \times q_2}$, $C_1 \in \mathbb{R}^{p_1 \times n}$, $C_2 \in \mathbb{R}^{p_2 \times n}$, $D_{11} \in \mathbb{R}^{p_1 \times q_1}$, $D_{12} \in \mathbb{R}^{p_1 \times q_2}$ and $D_{21} \in \mathbb{R}^{p_2 \times q_1}$, fulfilling the assumption

(A1) $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable,

and given $\gamma > 0$, there exists an admissible controller $C_\infty$ such that $\| \mathcal{F}_1(H, C_\infty) \|_\infty < \gamma$ if and only if there exist symmetric matrices $R, S$ satisfying the following system of LMIs:

$$\begin{bmatrix} \mathcal{N}_R ; 0 \\ 0 ; I \end{bmatrix}^* \begin{bmatrix} AR + RA^* & RC_1^* \\ C_1R & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{N}_R ; 0 \\ 0 ; I \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathcal{N}_S ; 0 \\ 0 ; I \end{bmatrix}^* \begin{bmatrix} A^* + SA & SB_1 \\ B_1^* & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{N}_S ; 0 \\ 0 ; I \end{bmatrix} < 0,$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0,$$

where $\mathcal{N}_R$ and $\mathcal{N}_S$ denote bases of the null spaces of $(B_2^*, D_{12}^*)$ and $(C_2, D_{21})$, respectively.
The advantage of this solution is that Theorem 2.5.2 makes less assumptions on the generalized plant $H$. Note that from the four assumptions in Theorem 2.5.1, only assumption (A1) remains in Theorem 2.5.2. On the other hand, the reconstruction of the resulting controller is not as straightforward from the LMI solutions $R$ and $S$ as it is from the solutions to the AREs in Theorem 2.5.1. An algorithm for controller reconstruction can be found in [Gah96]. This algorithm is used in Chapter 4, and more details are given there.

## 2.6 Coprime Factors

The concept of coprime factorizations of systems is described in this section. A coprime factorization can be said to represent generalized numerators and denominators of rational transfer function matrices. Formal definitions follow below. Coprime factorizations have been used as tools to solve a number of fundamental problems in robust control. For extensive treatments of the topic, see among others [Vid85, Fra87] and additional references provided therein.

The results in this section are standard, with the exception of Theorem 2.6.4. Consider the following basic definition.

**Definition 2.6.1.** Two matrices $M \in RH_\infty$ and $N \in RH_\infty$ are right coprime over $RH_\infty$ if they have the same number of columns and there exist matrices $X_r, Y_r \in RH_\infty$ of compatible dimensions such that

$$X_r M + Y_r N = I.$$

Similarly, two matrices $\tilde{M} \in RH_\infty$ and $\tilde{N} \in RH_\infty$ are left coprime over $RH_\infty$ if they have the same number of rows and there exist matrices $X_l, Y_l \in RH_\infty$ of compatible dimensions such that

$$\tilde{M} X_l + \tilde{N} Y_l = I.$$

With the general concept of coprimeness defined, the left and coprime factorizations of a system $P$ are now introduced.
Definition 2.6.2. The ordered pair \( \{N, M\} \) with \( M \in \mathcal{RH}_{q \times q} \) and \( N \in \mathcal{RH}_{p \times q} \) is a right coprime factorization (rcf) of \( P \in \mathcal{R}^{p \times q} \) if

(i) \( M \) is invertible in \( \mathcal{R} \),

(ii) \( P = NM^{-1} \), and

(iii) \( N \) and \( M \) are right coprime.

Similarly, the ordered pair \( \{\tilde{N}, \tilde{M}\} \) with \( \tilde{M} \in \mathcal{RH}_{p \times p} \) and \( \tilde{N} \in \mathcal{RH}_{p \times q} \) is a left coprime factorization (lcf) of \( P \in \mathcal{R}^{p \times q} \) if

(i) \( \tilde{M} \) is invertible in \( \mathcal{R} \),

(ii) \( P = \tilde{M}^{-1}\tilde{N} \), and

(iii) \( \tilde{N} \) and \( \tilde{M} \) are left coprime.

Lemma 2.6.1 ([Vin01, Propositions 1.4 & 1.6]). Any \( P \in \mathcal{R}^{p \times q} \) has both an rcf \( \{N, M\} \) and an lcf \( \{\tilde{N}, \tilde{M}\} \). Furthermore, any such representations are unique to within right multiplication (in the case of an rcf) or left multiplication (in the case of an lcf) by a matrix \( R \in \mathcal{Gq} \).

As a consequence, given any rcf \( \{N, M\} \) (lcf \( \{\tilde{N}, \tilde{M}\} \)) of \( P \), all possible rcfs (lcfs) are generated by \( \{NR, MR\} \) (\( \{R\tilde{N}, R\tilde{M}\} \)), where \( R \in \mathcal{Gq} \) is of appropriate dimension.

Under a specific condition, a coprime factorization is referred to as normalized.

Definition 2.6.3. The rcf \( \{N, M\} \) of \( P \in \mathcal{R}^{p \times q} \) is a normalized rcf of \( P \) if

\[ M^*M + N^*N = I. \]

Similarly, the lcf \( \{\tilde{N}, \tilde{M}\} \) of \( P \in \mathcal{R}^{p \times q} \) is a normalized lcf of \( P \) if

\[ \tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I. \]
Frequently in this thesis, when a coprime factorization is not normalized, the subscript \( \{N_0, M_0\} \) is added for clarity.

To simplify notation and because of their fundamental importance in several theorems, the following so called graph symbols are introduced.

**Definition 2.6.4.** Given an rcf \( \{N_0, M_0\} \) and an lcf \( \{\tilde{N}_0, \tilde{M}_0\} \) of \( P \in \mathbb{R}^{p \times q} \). Define, respectively, the right and left graph symbols of \( P \) as

\[
G_0 := \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} \quad \text{and} \quad \tilde{G}_0 := \begin{bmatrix} -\tilde{M}_0 & \tilde{N}_0 \end{bmatrix}.
\]

Denote by \( G \) and \( \tilde{G} \) the right and left graph symbols obtained from a normalized coprime factorization.

For some of the robust stability results of the thesis, the inverse graph symbols of a controller \( C \) will be required. These are only defined for the normalized case.

**Definition 2.6.5.** Given a normalized rcf \( \{U, V\} \) and a normalized lcf \( \{\tilde{U}, \tilde{V}\} \) of \( C \in \mathbb{R}^{q \times p} \), define, respectively, the right and left inverse normalized graph symbols of \( C \) as

\[
K := \begin{bmatrix} V \\ U \end{bmatrix} \quad \text{and} \quad \tilde{K} := \begin{bmatrix} -\tilde{U} & \tilde{V} \end{bmatrix}.
\]

In light of this definition, the condition for a coprime factorization to be normalized can be reformulated as \( G \) being inner or \( \tilde{G} \) being co-inner in the cases of right and left coprime factorizations, respectively.

The above definitions of right and left, normalized and non-normalized coprime factorizations will now be given state space interpretations.

**Theorem 2.6.2** ([Vid84, ZDG96]). Given \( P \in \mathbb{R}^{p \times q} \) with a state space realization \((A, B, C, D)\), \( A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times q}, \ C \in \mathbb{R}^{p \times n}, \ D \in \mathbb{R}^{p \times q} \) such that \((A, B)\) is stabilizable and \((C, A)\) is detectable, and given \( F \in \mathbb{R}^{q \times n} \) and \( L \in \mathbb{R}^{n \times p} \) such that
$A + BF$ and $A + LC$ are Hurwitz, define

$$
\begin{bmatrix}
M & -Y_l \\
N & X_l
\end{bmatrix} :=
\begin{bmatrix}
A + BF & B & -L \\
F & I & 0 \\
C + DF & D & I
\end{bmatrix};
$$

$$
\begin{bmatrix}
X_r & Y_r \\
-\tilde{N} & \tilde{M}
\end{bmatrix} :=
\begin{bmatrix}
A + LC & -(B + LD) & L \\
F & I & 0 \\
C & -D & I
\end{bmatrix}.
$$

Then $\{N,M\}$ and $\{\tilde{N},\tilde{M}\}$ are an rcf and an lcf of $P$, respectively, and, furthermore,

$$
\begin{bmatrix}
X_r & Y_r \\
-\tilde{N} & \tilde{M}
\end{bmatrix}
\begin{bmatrix}
M & -Y_l \\
N & X_l
\end{bmatrix} = I.
$$

The following theorem describes choices of $F$ and $L$ that result in the right and left coprime factorization of $P$, respectively, being normalized.

**Theorem 2.6.3 ([ZDG96, Theorem 13.37]).** Given $P \in \mathbb{R}^{p \times q}$ with a state space realization $(A,B,C,D)$, define $R := I + D^*D$ and $\tilde{R} := I + DD^*$.

1. Assume that $(A,B)$ is stabilizable and $(C,A)$ has no unobservable modes on the imaginary axis. Then there is a normalized right coprime factorization $\{N,M\}$ over $\mathbb{RH}_\infty$ of $P$ given by

$$
\begin{bmatrix}
M \\
N
\end{bmatrix} :=
\begin{bmatrix}
A + BF & BR^{-\frac{1}{2}} \\
F & R^{-\frac{1}{2}} \\
C + DF & DR^{-\frac{1}{2}}
\end{bmatrix},
$$

where $F := -R^{-\frac{1}{2}}(B^*X + D^*C)$ and

$$
X := \text{Ric} \begin{bmatrix}
A - BR^{-1}D^*C & -BR^{-1}B^* \\
-C^*R^{-1}C & -(A - BR^{-1}D^*C)^*
\end{bmatrix} \succeq 0. \tag{2.1}
$$

2. Assume that $(C,A)$ is detectable and $(A,B)$ has no uncontrollable modes on the imaginary axis. Then there is a normalized left coprime factorization
\{\tilde{N}, \tilde{M}\} over \mathcal{RH}_\infty of P given by

\[
\begin{bmatrix}
\tilde{M} & \tilde{N}
\end{bmatrix} := \begin{bmatrix}
A + LC & L & B + LD \\
\tilde{R}^{-\frac{1}{2}}C & \tilde{R}^{-\frac{1}{2}} & \tilde{R}^{-\frac{1}{2}}D
\end{bmatrix},
\]

where \(L := -(BD^* + YC^*)\tilde{R}^{-\frac{1}{2}}\) and

\[
Y := \text{Ric} \begin{bmatrix}
(A - BD^*\tilde{R}^{-1}C)^* & -C^*\tilde{R}^{-1}C \\
-BR^{-1}B^* & -(A - BD^*\tilde{R}^{-1}C)
\end{bmatrix} \geq 0. \tag{2.2}
\]

Finally, the following theorem provides a state space realization of a right multiplier \(R\) that links a normalized coprime factorization \(\{N, M\}\) of \(P\) to a right coprime factorization \(\{NR, MR\}\) of the same order that is not necessarily normalized.

**Theorem 2.6.4.** Given \(P \in \mathcal{RH}_{\infty}^{p \times d}\) with a state space realization \((A, B, C, 0)\) such that \((A, B)\) is stabilizable and \((C, A)\) has no unobservable modes on the imaginary axis, and given a normalized rcf \(\{N, M\}\) of \(P\) and a non-normalized rcf \(\{N_0, M_0\}\) of \(P\), with

\[
\begin{bmatrix}
M_0 \\
N_0
\end{bmatrix} := \begin{bmatrix}
A + BF & B \\
F & I \\
C & 0
\end{bmatrix}
\]

and \(F\) such that \(A + BF\) is Hurwitz, then a matrix \(R \in \mathcal{RH}_\infty\) such that \(\{N_0, M_0\} = \{NR, MR\}\) is given by

\[
R := \begin{bmatrix}
A + BF & B \\
B^*X + F & I
\end{bmatrix},
\]

where \(X\) is given by (2.1).

**Proof.** Since \(M\) is invertible,

\[R = M^{-1}M_0.\]
Now, $M^{-1} = \begin{bmatrix} A & B \\ B^*X & I \end{bmatrix}$, where $X$ is given in (2.1). It then follows that

\[
R = \begin{bmatrix} B^*X (sI - A)^{-1} B + I \\ F (sI - A - BF)^{-1} B + I \end{bmatrix},
\]

\[
= B^*X (sI - A)^{-1} B \left[ F (sI - A - BF)^{-1} B + I \right] + \left[ F (sI - A - BF)^{-1} B + I \right],
\]

\[
= B^*X (sI - A)^{-1} \left[ BF + sI - A - BF \right] (sI - A - BF)^{-1} B 
+ \left[ F (sI - A - BF)^{-1} B + I \right],
\]

\[
= B^*X (sI - A - BF)^{-1} B + \left[ F (sI - A - BF)^{-1} B + I \right],
\]

\[
= [B^*X + F] (sI - A - BF)^{-1} B + I.
\]

A similar result can be obtained for left coprime factorizations in an analogous fashion.
Chapter 3

Robust Stability and Performance Analysis for Uncertain Linear Systems — The Distance Measure Approach

3.1 Introduction

This chapter is concerned with the robust stability and robust performance analysis for multiple-input, multiple-output, linear time-invariant systems with additive, multiplicative, inverse multiplicative, inverse additive and right coprime factor uncertainty structures via the generic distance measures of [LP09]. The results for the additive case mainly serve a didactic purpose by illustrating in an intuitive manner how distance is measured, and how it affects the stability and performance of the perturbed system. The analysis for the multiplicative and inverse additive cases is much more intricate. For multiplicative, inverse multiplicative and inverse additive structures, the first difficulty lies in finding a minimal-size uncertainty that captures the difference between a nominal and a perturbed system; e.g. in the input multiplicative case a $\Delta \in RL_\infty$ that fulfills

$$P_\Delta = P(I - \Delta)$$
for given $P, P_\Delta \in \mathbb{R}L_\infty$. First, a parameterization of all solutions $\Delta$ must be found, and then the smallest of these solutions must be chosen. The parameterization depends on the dimension of the plant, i.e. it is different for square, tall and fat plants. For inverse multiplicative and inverse additive uncertainty, a second difficulty arises in ensuring that the perturbed plant description is well-posed. From this requirement, additional constraints on the set containing all possible solutions $\Delta$ are derived. Via analytic expressions of all minimal-size solutions $\Delta$ for multiplicative, inverse multiplicative and inverse additive uncertainty structures, the distance measures and of robust stability and robust performance theorems for these structures are formulated in concise fashion. The results for the right coprime factor uncertainty structure follow partially from duality with the left coprime factor uncertainty structure, for which results were derived in [LP09]. However, some conditions differ non-trivially in the right-coprime factor case from their duals in the left-coprime factor case. In view of the importance of this structure for derivations in Chapter 5, the results are included here for completeness.

Taken together, the readily-applicable theorems of this chapter on robust stability and performance for additive, various types of multiplicative, inverse additive and right coprime factor structures form a valuable tool for the practising engineer to determine the most suitable uncertainty structure for a given family of uncertain plant models. In fact, engineers have been using similar approaches without rigorous theoretical underpinning for many years (see e.g. how a minimal-size multiplicative uncertainty is constructed in [Enn91]). In contrast to the structured singular value approach [DPZ91, PD93, ZDG96], the robust performance theorems in this chapter require only two infinity norms to be computed for each uncertain plant for which performance guarantees are sought.

The choice of uncertainty structure greatly influences the resulting stability and performance guarantees. This fact is illustrated through several physically-motivated examples. In one of these examples, uncertainty arising from neglected actuator dynamics is analyzed using multiplicative distance measures. The results are compared to $\nu$-gap theory analysis for the same plant, and it is shown that for this case, multiplicative distance measures enable less conservative robust stability and performance guarantees.

The remainder of this chapter is structured as follows: Section 3.2 summa-
rizes the theory of generic distance measures. In Section 3.3, the uncertainty structures studied in this chapter are introduced, for which the robust stability and performance theorems are derived in Sections 3.4-3.7, respectively. The example follows in Section 3.8, while Section 3.9 concludes the chapter.

3.2 Robust Stability and Performance via Generic Distance Measures

Different measurement or modelling methods may result in situations where the same physical system is described by various models, leading to uncertainty in the model description. Let us consider a nominal plant \( P \) and one of its perturbed versions \( P_\Delta \). Often a controller is designed based on the nominal plant but is supposed to work well with a family of perturbed plants. In such a situation, the control engineer will have to answer two questions: keeping the controller unchanged, will the system remain stable when \( P \) is replaced by \( P_\Delta \)? How will the performance of the system be affected by the model change? To answer these, the engineer needs to quantify how different \( P_\Delta \) is from \( P \) in a closed-loop sense. This difference is called the distance between \( P \) and \( P_\Delta \). Of course, there are numerous ways to measure distance in a function space [Vid84, ES85, GS90, QD92, Vin01], which depend on the underlying metric topology (or allowable uncertainty structure). The distance notion used in this chapter is generic in the sense that it can capture many different uncertainty structures, and therefore allows an easy comparison of distances under various different uncertainty structures. This section contains an introduction to and engineering motivation of, the distance measure and related concepts and theorems first defined in [LP09].

In robust control, families of uncertain systems are often described via a LFT [ZDG96, Ch. 10] as shown in Fig. 3.1, where

\[
H = \begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix} \in \mathcal{R}
\]

is the generalised plant and \( \Delta \in \mathcal{R} \) is an uncertainty. If the element \( H_{22} \) describes the nominal plant \( P \in \mathcal{R}^{p \times q} \), the size of \( \Delta \in \mathcal{R} \) corresponds to the “radius” of
Figure 3.1: An uncertain plant family parameterised by the perturbation $\Delta$.

an uncertain region around $P$. The shape of this region depends on the other elements of $H$, which describe the uncertainty structure, i.e. the allowable way in which the uncertainty can alter the plant model. For any given $\Delta$, it is easy to compute the resulting perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$ (under the assumption that $(I - H_{11}\Delta)^{-1} \in \mathcal{R}$):

$$P_\Delta = \mathcal{F}_u(H, \Delta) = H_{22} + H_{21}(I - H_{11}\Delta)^{-1}H_{12}. \quad (3.1)$$

When $\Delta = 0$, it is easy to see that $\mathcal{F}_u(H, 0) = P$, corresponding to the centre of this uncertain family. This leads to the question: which $\Delta$ yields a given perturbed plant $P_\Delta$, if connected with $H$ as shown in Fig. 3.1? Two conditions on the solutions $\Delta$ follow from (3.1): they must satisfy a well-posedness condition $((I - H_{11}\Delta)^{-1} \in \mathcal{R})$, and also the consistency equation ($P_\Delta = \mathcal{F}_u(H, \Delta)$). In the general case, there is a set of solutions $\Delta \subset \mathcal{R}$ for (3.1) which may have multiple elements, or none at all, or exactly one. By allowing perturbations $\Delta \in \mathcal{R}L_\infty$, it is assured that the size of $\Delta$ can still be measured with the infinity norm. Hence, $\|\Delta\|_\infty$ can be used to measure the distance between $P$ and $P_\Delta$. This distance measurement is unambiguous if there is just a single solution $\Delta \in \mathcal{R}L_\infty$ satisfying (3.1), or no solution at all (in which case the distance is infinite). If the set $\Delta$ contains several elements $\Delta$ that satisfy the consistency equation relating $P$ and $P_\Delta$ and also the well-posedness condition, the smallest $\|\Delta\|_\infty$ is chosen from among the possible solutions that explain the change from $P$ to $P_\Delta$. The distance measure can now be formally defined.

**Definition 3.2.1.** Given a plant $P \in \mathcal{R}^{p \times q}$, a generalized plant $H \in \mathcal{R}$ with $H_{22} = P$, and a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$, let the set of all admissible perturbations be
given by \( \Delta = \{ \Delta \in \mathcal{RL}_\infty : (I - H_{11}\Delta)^{-1} \in \mathcal{R}, P_\Delta = \mathcal{F}_u(H, \Delta) \} \). Define the distance measure \( d^H(P, P_\Delta) \) between plants \( P \) and \( P_\Delta \) for the uncertainty structure implied by \( H \) as:

\[
d^H(P, P_\Delta) := \begin{cases} 
\inf_{\Delta \in \Delta} \| \Delta \|_\infty & \text{if } \Delta \neq \emptyset, \\
\infty & \text{otherwise.}
\end{cases}
\]

Note that if \( \Delta \neq \emptyset \) then the infimum always exists in \( \mathbb{R} \) as \( \Delta \in \mathcal{RL}_\infty \). If multiple solutions \( \Delta \) exist, it may be that several of them have the same, smallest size.

Hence, the following definition is introduced for ease of notation:

**Definition 3.2.2.** Given a plant \( P \in \mathbb{R}^{p \times q} \), a generalized plant \( H \in \mathcal{R} \) with \( H_{22} = P \), and a perturbed plant \( P_\Delta \in \mathbb{R}^{p \times q} \), define

\[
\Delta^{\text{min}} := \{ \Delta \in \Delta : \| \Delta \|_\infty = d^H(P, P_\Delta) \}.
\]

These developments lead to the question of robust stability. The nominal closed-loop system is shown in Fig. 3.2. If an uncertainty is present in the system, the plant \( P \) is replaced by \( P_\Delta \), which corresponds to an LFT structure as depicted in Fig. 3.3. Classical robust stability analysis for uncertainties in \( \mathcal{RH}_\infty \) relies on the standard small-gain theorem \([ZDG96, p. 218]\) as follows: the loop is broken into two parts, \( \Delta \) and \( \mathcal{F}_1(H, C) \), and the loop gain is computed by multiplying the infinity norms of both of these parts. A loop gain strictly smaller than unity implies internal stability. Now define a generic stability margin notion indicating the size of the smallest uncertainty (measured by the infinity norm) that violates the classical small-gain stability condition:

**Definition 3.2.3.** Given a plant \( P \in \mathbb{R}^{p \times q} \), a generalized plant \( H \in \mathcal{R} \) with \( H_{22} = P \), and a controller \( C \in \mathbb{R}^{q \times p} \), define the stability margin \( b^H(P, C) \) of the feedback interconnection \( \langle H, C \rangle \) as:

\[
b^H(P, C) := \begin{cases} 
\| \mathcal{F}_1(H, C) \|_\infty^{-1} & \text{if } 0 \neq \mathcal{F}_1(H, C) \in \mathcal{RL}_\infty \\
0 & \text{and } [P, C] \text{ is internally stable},
\end{cases}
\]

otherwise.
Figure 3.2: The nominal closed-loop system.

Figure 3.3: The closed-loop system when a perturbation \( \Delta \) is present.

Notice how this stability margin is related to the distance measure of Definition 3.2.1: the distance measure allows statements on the minimal size of a perturbation that yields a plant \( P_{\Delta} \) when connected with a nominal plant \( P \). Given \( P \) and \( P_{\Delta} \), the smallest size of a \( \Delta \) which explains the difference is defined. In conjunction with the stability margin of Definition 3.2.3 the loop gain of the closed-loop system of Fig. 3.3 can then be computed when \( d^H(P, P_{\Delta}) \) is finite and \( b^H(P, C) \neq 0 \).

A small gain-type condition using \( \Delta \in \Delta_{\text{min}} \) can be stated as follows:

\[
\|\Delta\|_{\infty} \|\mathcal{F}_1(H, C)\|_{\infty} < 1 \iff d^H(P, P_{\Delta}) < b^H(P, C).
\]

The above classical small-gain condition is valid only for stable perturbations. As mentioned earlier, the perturbations \( \Delta \) are not restricted to \( \mathcal{RH}_{\infty} \), and hence need a more powerful stability theorem. The small-gain theorem can be extended to systems in \( \mathcal{RL}_{\infty} \) [Glo84, Vin01], and this comes at the cost of introducing an additional condition on the winding numbers similar to generalised forms of the Nyquist stability theorem. Simply ensuring that the loop gain of the system is
The following theorem uses the generic distance measure and stability margin defined above to give necessary and sufficient stability conditions for systems in $\mathcal{RL}_\infty$.

**Theorem 3.2.1 (Robust Stability).** [LP09, Section III] Given a plant $P \in \mathbb{R}^{p \times q}$, a stabilizable generalized plant $H \in \mathbb{R}$ with $H_{22} = P$, a perturbed plant $P_\Delta \in \mathbb{R}^{p \times q}$ and a controller $C \in \mathbb{R}^{q \times p}$ such that $d^H(P, P_\Delta) < b^H(P, C)$ and $\Delta^{\min} \neq 0$, then the following statements are equivalent:

$$(a) \ [P_\Delta, C] \text{ is internally stable};$$

$$(b) \ \forall \Delta \in \Delta^{\min}, \ \eta(P_\Delta) = \eta(P) + \text{wnodet}(I - H_{11}\Delta);$$

$$(c) \ \exists \Delta \in \Delta^{\min}: \eta(P_\Delta) = \eta(P) + \text{wnodet}(I - H_{11}\Delta). \quad (3.2)$$

A very powerful implication of $(c) \Rightarrow (b)$, or equivalently its contrapositive

$$\exists \Delta \in \Delta^{\min}: \eta(P_\Delta) \neq \eta(P) + \text{wnodet}(I - H_{11}\Delta)$$

$$\Rightarrow \eta(P_\Delta) \neq \eta(P) + \text{wnodet}(I - H_{11}\Delta) \ \forall \Delta \in \Delta^{\min},$$

is that winding number condition (3.2) is true or false for all $\Delta \in \Delta^{\min}$. Therefore, to check whether or not $[P_\Delta, C]$ is internally stable it suffices to compute the winding number in (3.2) for a single arbitrary $\Delta \in \Delta^{\min}$. From this theorem, stability for any perturbed plant $P_\Delta$ which can be obtained from a nominal plant $P$ through a perturbation $\Delta \in \mathcal{RL}_\infty$ (in an LFT fashion) can be checked. However, the robust performance of an uncertain family of plants is often of similar or even greater importance than robust stability itself. In the control of manned aircraft for example, the delay due to pilot reaction places strict robust performance limitations on the non-human parts of the control loop in order to achieve overall robust stability. Hence, it is desirable to characterize the worst-case performance degradation when the nominal plant $P$ is replaced by $P_\Delta$. To this end, the structure of the generalized plant $H$ is defined more specifically. Fig. 3.4 shows a left four-block structure [ES91], which will be used to express different generalized plants throughout this chapter. It can be specialised to many common uncertainty structures through the suitable choice of two selection/filtering matrices $S_w, \ S_z \in \mathbb{R}$ (see Section 3.3.
Figure 3.4: The left 4-block interconnection.

For specific values of $S_w$ and $S_z$ and the resulting uncertainty structures), where the connection between the signals $w$ and $z$ of Fig. 3.3 and $w_1$, $w_2$, $z_1$ and $z_2$ of Fig. 3.4 is given by $z = S_z \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = S_w w$. In this setting, $H$ takes the following form:

$$H = \begin{bmatrix} S_z & I \\ -P & P \end{bmatrix} \begin{bmatrix} I & -P & P \\ 0 & 0 & I \\ I & -P & P \end{bmatrix} \begin{bmatrix} S_w \\ I \\ I \end{bmatrix}. \quad (3.3)$$

Having established the structure of the generalised plant $H$, a unified framework for the analysis of robust performance and possible degradation of the robust stability margin of uncertain systems can be defined. The following theorem answers the question: how will the worst case closed-loop performance of the system be affected if $P$ is replaced by $P_\Delta$ while the controller $C$ remains unchanged? This question has been asked by [Vid84, QD92, GS90, Vin01] before, but these authors answered it only in a normalised coprime factor setting. The following theorem is more general because it captures normalised coprime factor uncertainty (or four-block uncertainty) when $S_w$ and $S_z$ are simply the identity matrices [PL06], but it can also be reduced to additive, input/output multiplicative, input/output inverse multiplicative, feedback uncertainty (also known as inverse additive uncertainty), non-normalised coprime factors [LP09] etc., which are all cases of interest to the practising engineer. In order for the performance measures to be well-defined, stability of the perturbed closed-loop system is required. Hence, the conditions of
Theorem 3.2.1 are assumed. Using the distance measure and the robust stability margin defined above, the change in robust stability margin, and the worst-case robust performance can be quantified.

**Theorem 3.2.2 (Robust Performance).** \([LP09, \text{Section V}]\) Given a nominal plant \(P \in \mathbb{R}^{p \times q}\), a stabilizable generalized plant

\[
H = \begin{bmatrix}
S_z & I \\
- & - \\
I & -P
\end{bmatrix}
\begin{bmatrix}
I & -P \cdot P \\
0 & I \\
I & -P \cdot P
\end{bmatrix}
\begin{bmatrix}
S_w & I \\
0 & - \\
- & I
\end{bmatrix}
\]

where \(S_w, S_z \in \mathbb{R}\), a perturbed plant \(P_\Delta \in \mathbb{R}^{p \times q}\) and a controller \(C \in \mathbb{R}^{q \times p}\) such that \(d^H(P, P_\Delta) < b^H(P, C)\) and \(\Delta_{\text{min}} \neq \emptyset\), assume that there exists a \(\Delta \in \Delta_{\text{min}}\) that satisfies \(\eta(P_\Delta) = \eta(P) + \text{winding}\ det(I - H_1\Delta)\), where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of \(P\) and \(P_\Delta\). Suppose furthermore that \(S_{\Delta} = S_w(I - k\Delta S_w)^{-1} \in \mathbb{R}\) for a given \(k \in \{0, 1\}\), \(S = (1 - k)S_z S_w\) and

\[
H_\Delta = \begin{bmatrix}
S_z & I \\
- & - \\
I & -P_\Delta
\end{bmatrix}
\begin{bmatrix}
I & -P_\Delta \\
0 & I \\
I & -P_\Delta
\end{bmatrix}
\begin{bmatrix}
S_{\Delta} & I \\
0 & - \\
- & I
\end{bmatrix}
\]

Then the following results hold when \(S \in \mathbb{R}L_\infty\) and \((I - \Delta S)^{-1} \in \mathbb{R}\):

(a) \(0 \neq \mathcal{F}_I(H_\Delta, C) \in \mathbb{R}L_\infty\) and \([P_\Delta, C]\) is internally stable;

(b) \(|b^{H}(P_\Delta, C) - b^{H}(P, C)| \leq \|\mathcal{F}_I(H_\Delta, C) - S\|_\infty \cdot b^{H}(P_\Delta, C) \cdot d^H(P, P_\Delta);\) and

(c) \(\|\mathcal{F}_I(H_\Delta, C) - \mathcal{F}_I(H, C)\|_\infty \leq \|\mathcal{F}_I(H_\Delta, C) - S\|_\infty \frac{d^H(P, P_\Delta)}{b^{H}(P, C)}\).

Result (b) gives an upper bound on the variation between \(b^{H}(P, C)\) and \(b^{H}(P_\Delta, C)\). Also result (c) gives an upper bound on the worst-case discrepancy between the transfer functions of interest when \(P\) changes to \(P_\Delta\). Whenever \(\|\mathcal{F}_I(H_\Delta, C) - S\|_\infty \cdot b^{H}(P_\Delta, C) \leq 1\) (as is the case for additive uncertainty, normalised coprime factor uncertainty and four-block uncertainty [LP09]), result (b) reduces to the familiar \(|b^{H}(P_\Delta, C) - b^{H}(P, C)| \leq d^H(P, P_\Delta)\), and result (c) reduces
to $\| F_l(H, C) - F_l(H, C) \|_{\infty} \leq \frac{d_H(P, P_\Delta)}{d_H(P, P_\Delta) + d_H(C, C_\Delta)}$, which have been derived for only the normalised coprime-factor case in the gap and $\nu$-gap theory. From results (b) and (c), intuitively, a small distance between plants will generally result in tight bounds on the robust performance variation.

### 3.3 Specific Uncertainty Structures

In the following sections, Theorems 3.2.1 and 3.2.2 will be specialized for the cases of additive, input/output multiplicative, input/output inverse multiplicative, inverse additive and right coprime factor uncertainty. The generalised plant for the left four-block structure given in (3.3) can capture all of these uncertainty structures except the right coprime factor structure through the choice of the selection matrices $S_w$ and $S_z$. Table 3.1 lists the left four-block based structures with their consistency equations, their corresponding $S_w$ and $S_z$ matrices, the resulting generalized plant $H$ as given in (3.3), and the input-output transfer function $F_l(H, C)$ as required for computation of the stability margin given in Definition 3.2.3. Results for the left four-block/normalized left coprime factor case [PL06] and the left coprime factor case (not necessarily normalized) [LP09] can be found elsewhere.

The right coprime factor structure, i.e. $P_\Delta = (N_0 + \Delta_N)(M_0 + \Delta_M)^{-1}$, where $\{N_0, M_0\}$ is a right coprime factorization of the nominal plant $P$, can be obtained from a right four-block generalized plant. For the right coprime factor structure, the generalised plant $H_{\text{RCF}}$ (corresponding to column 5 of Table 3.1) is given by

$$H_{\text{RCF}} := \begin{bmatrix} S_z \cdot 0 & 0 & P \cdot P \\ -I & I & -I \\ 0 & 0 & -I \end{bmatrix} \begin{bmatrix} 0 & P \cdot P \\ 0 & I \cdot I \\ -I & -I \end{bmatrix} \begin{bmatrix} S_w \cdot 0 & 0 \\ 0 & -I \end{bmatrix},$$

(3.4)

with $S_w = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, $S_z = \begin{bmatrix} 0 & M_0^{-1} \\ 0 & -I \end{bmatrix}$.

Using these basic results, much of the work will be to express the distance measure and the winding number condition described in Theorem 3.2.1 independent of $\Delta$ for each specific uncertainty characterisation, thereby giving readily computable formulae for the distance measure and readily computable robust sta-
Table 3.1: The basic uncertainty structures considered in this chapter.

<table>
<thead>
<tr>
<th>Uncertainty structure</th>
<th>( P_\Delta = )</th>
<th>( S_w = )</th>
<th>( S_z = )</th>
<th>( H = )</th>
<th>( \mathcal{F}_I(H,C) = )</th>
</tr>
</thead>
</table>
| Additive              | \( P + \Delta \) | \[
\begin{bmatrix}
I \\
0
\end{bmatrix}
\] | \[
\begin{bmatrix}
0 & I \\
\end{bmatrix}
\] | \[
\begin{bmatrix}
0 & I \\
\end{bmatrix}
\] | \( C(I - PC)^{-1} \) |
| Input multiplicative  | \( P(I - \Delta) \) | \[
\begin{bmatrix}
0 & I \\
I
\end{bmatrix}
\] | \[
\begin{bmatrix}
0 & I \\
-I & P
\end{bmatrix}
\] | - | \( -C(I - PC)^{-1} P \) |
| Output inverse multiplicative | \( (I - \Delta)^{-1} P \) | \[
\begin{bmatrix}
I \\
0
\end{bmatrix}
\] | \[
\begin{bmatrix}
I & P \\
\end{bmatrix}
\] | \( \begin{bmatrix}
0 & I \\
\end{bmatrix}
\] | \( (I - PC)^{-1} \) |
| Inverse additive      | \( (I + P\Delta)^{-1} P \) | \[
\begin{bmatrix}
0 & I \\
I
\end{bmatrix}
\] | \[
\begin{bmatrix}
-P & P \\
\end{bmatrix}
\] | \( \begin{bmatrix}
0 & I \\
\end{bmatrix}
\] | \( (I - PC)^{-1} P \) |

3.4 Additive Uncertainty Characterization with an Additive Performance Measure

In the case of additive uncertainty, the analysis is uniquely straightforward. Plugging \( \mathcal{F}_I(H,C) \) from Table 3.1 into Definition 3.2.3, the stability margin \( b_a(P,C) \) for an additive uncertainty characterisation is given by:

\[
b_a(P,C) := \begin{cases} \|C(I - PC)^{-1}\|_\infty^{-1} & \text{if } [P,C] \text{ is internally stable,} \\ 0 & \text{otherwise.} \end{cases}
\]
The uncertainty $\Delta$ that yields $P_\Delta$ when connected to $P$ is also easily characterised as $\Delta = P_\Delta - P$. Thus, for this specific case, a necessary and sufficient condition for the existence of a $\Delta \in \mathcal{RL}_\infty$ that satisfies the consistency equation $\Delta = P_\Delta - P$ is $P_\Delta - P \in \mathcal{RL}_\infty$, with an obvious simple sufficient condition being $P, P_\Delta \in \mathcal{RL}_\infty$. Also, given any $P, P_\Delta$ pair that satisfies $P_\Delta - P \in \mathcal{RL}_\infty$, there exists a unique solution for $\Delta \in \mathcal{RL}_\infty$ given by $\Delta = P_\Delta - P$. Note that the LFT $\mathcal{F}_u(H, \Delta)$ is always well-posed (i.e. det$(I - H_{11} \Delta) \neq 0$) for any $\Delta$ as $H_{11} = 0$ in this simple case as seen from Table 3.1. Consequently, as shown above,

$$P_\Delta - P = \Delta \iff P_\Delta = \mathcal{F}_u(H, \Delta).$$

Hence, given a nominal plant $P$ and a perturbed plant $P_\Delta$, it has been shown above that $P_\Delta - P \in \mathcal{RL}_\infty$ is a necessary and sufficient condition for there to exist a $\Delta \in \mathcal{RL}_\infty$ satisfying $P_\Delta = \mathcal{F}_u(H, \Delta)$. Also, when this condition is satisfied, there only exists a unique solution $\Delta = P_\Delta - P \in \mathcal{RL}_\infty$ for $P_\Delta = \mathcal{F}_u(H, \Delta)$. Straight from Definition 3.2.1, the solution set $\Delta$ reduces to

$$\Delta = \begin{cases} \{\Delta = P_\Delta - P\} & \text{when } (P_\Delta - P) \in \mathcal{RL}_\infty, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that here $\Delta$ is either empty or contains only one element. It easily follows straight from Definition 3.2.1 that the distance measure $d_a(P, P_\Delta)$ for additive uncertainty characterisations is given by:

$$d_a(P, P_\Delta) := \begin{cases} \|P_\Delta - P\|_\infty & \text{when } (P_\Delta - P) \in \mathcal{RL}_\infty, \\ \infty & \text{otherwise.} \end{cases} \quad (3.6)$$

It is clear that $d_a(P, P_\Delta)$ is a metric on $\mathcal{RL}_\infty$ in this specific case.\(^1\) Now, the winding number condition (3.2) must be rewritten independently of $\Delta$. For additive uncertainty characterisations, this process is trivial since $H_{11} = 0$. Consequently,

\(^1\)This allows the use of mathematical results on metric spaces for further analysis. Note that the multiplicative distances in the following section generally violate the symmetry property of metrics, but can be turned into metrics by taking the maximum over both directed distances, as is done for the gap metric [Vin01, Ch. 7.1].
winding number condition (3.2) reduces to \( \eta(P_\Delta) = \eta(P) \), which is easily com-
putable as it is independent of \( \Delta \). All these computations now enable us to give specific versions of the generic Robust Stability Theorem 3.2.1 and the generic
Robust Performance Theorem 3.2.2 for additive uncertainty.

**Theorem 3.4.1** (Robust Stability — Additive). Given a plant \( P \in \mathcal{R}^{p \times q} \), a per-
turbed plant \( P_\Delta \in \mathcal{R}^{p \times q} \) and a controller \( C \in \mathcal{R}^{q \times p} \). Define a stability margin \( b_a(P, C) \) as in (3.5) and a distance measure \( d_a(P, P_\Delta) \) as in (3.6). Furthermore, suppose \( d_a(P, P_\Delta) < b_a(P, C) \). Then

\[ [P_\Delta, C] \text{ is internally stable } \Leftrightarrow \eta(P_\Delta) = \eta(P). \]

**Proof.** This theorem specialises Theorem 3.2.1 using formulae derived in the above section. Note that the supposition that “\( H \) is stabilizable” is automatically fulfilled in this specific design case, and hence does not need to be independently enforced. Also, the supposition \( d_a(P, P_\Delta) < b_a(P, C) \) guarantees that \( d_a(P, P_\Delta) < \infty \)
which then implies via definition (3.6) that \( (P_\Delta - P) \in \mathcal{RL}_\infty \).

A similar extension of the small-gain theorem for systems in \( \mathcal{RL}_\infty \) was first described by [Glo84, Glo86] for additive uncertainty only. Robust performance guarantees, however, were not given at the time, and are here expressed for additive uncertainty for the first time in the following theorem:

**Theorem 3.4.2** (Robust Performance — Additive). Given the suppositions of The-
orem 3.4.1 and furthermore assuming \( \eta(P_\Delta) = \eta(P) \). Then

\[ |b_a(P_\Delta, C) - b_a(P, C)| \leq d_a(P, P_\Delta) \tag{3.7} \]

and

\[ \| \mathcal{F}_1(H_\Delta, C) - \mathcal{F}_1(H, C) \|_\infty \leq \frac{d_a(P, P_\Delta)}{b_a(P, C)b_a(P_\Delta, C)}, \tag{3.8} \]

where \( H = \begin{bmatrix} 0 & I \\ -I & P \end{bmatrix} \) and \( H_\Delta = \begin{bmatrix} 0 & I \\ -I & P_\Delta \end{bmatrix} \).

**Proof.** This theorem specialises Theorem 3.2.2 using formulae derived in the above section. Specifically, choosing \( k = 0 \) and noting that \( S = S_zS_w = 0 \in \mathcal{RL}_\infty \).
gives $\|\mathcal{F}_l(H_\Delta, C) - S\|_\infty = \frac{1}{b_a(P_\Delta, C)}$. This equality follows from (3.5) for $b_a(P_\Delta, C)$ on noting that $[P_\Delta, C]$ is internally stable via Theorem 3.4.1.

3.5 Multiplicative Uncertainty Characterization with a Multiplicative Performance Measure

In this section, robust stability and robust performance results for the input multiplicative and output inverse multiplicative uncertainty characterisation are derived, and it is shown how these results can be modified to get dual results for the output multiplicative and input inverse multiplicative cases.

3.5.1 Input Multiplicative

Define the stability margin $b_{im}(P, C)$

Using Table 3.1 and Definition 3.2.3, the stability margin $b_{im}(P, C)$ for an input multiplicative uncertainty characterisation is given by:

$$b_{im}(P, C) := \begin{cases} \|C(I - PC)^{-1}P\|_\infty^{-1} & \text{if } [P, C] \text{ is internally stable,} \\ 0 & \text{otherwise.} \end{cases} \quad (3.9)$$

Solve consistency equation for all $\Delta \in \mathcal{P}_L$.

Using the structure of $H$ defined in (3.3) the consistency equation $P_\Delta = \mathcal{F}_u(H, \Delta)$ is reformulated as follows:

$$P_\Delta = \mathcal{F}_u(H, \Delta) \iff P_\Delta - P = \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta S_z \begin{bmatrix} P_\Delta \\ I \end{bmatrix} \quad (3.10)$$

whenever well-posedness of $\mathcal{F}_u(H, \Delta)$ can be assumed (i.e. $\det(I - H_{11}\Delta)(\infty) \neq 0$). (3.10) further reduces to

$$P_\Delta - P = -P\Delta \quad (3.11)$$
on substituting $S_w = \begin{bmatrix} 0 & I \end{bmatrix}^T$ and $S_z = \begin{bmatrix} 0 & I \end{bmatrix}$ from Table 3.1. It is assumed in this section that $P(\infty)$ has full rank, which is imposed for mathematical convenience.\(^2\)

To proceed, the analysis is split into three cases — i.e. for square plants, fat plants and tall plants — since one of these cases has a unique solution, one case can possibly have no solution and one has multiple solutions.

**Square Plants:**

Assume (in this square plants case) that $P, P_\Delta \in \mathbb{R}^{p \times q}$ is such that $p = q$ and $P(\infty)$ has full rank. Then it is easy to see that equation (3.11) can be equivalently rearranged into

$$\Delta = P^{-1}(P - P_\Delta). \tag{3.12}$$

Consequently, for this case, a necessary and sufficient condition for there to exist a $\Delta \in \mathcal{RL}_\infty$ that satisfies consistency of equations is $P^{-1}P_\Delta \in \mathcal{RL}_\infty$, with a sufficient condition being $P^{-1}, P_\Delta \in \mathcal{RL}_\infty$. Also, given any $P, P_\Delta$ pair that satisfies $P^{-1}P_\Delta \in \mathcal{RL}_\infty$, there exists a unique solution for $\Delta \in \mathcal{RL}_\infty$ given by equation (3.12).

**Fat Plants:**

Assume (in this fat plants case) that $P, P_\Delta \in \mathbb{R}^{p \times q}$ is such that $p < q$ and $P(\infty)$ has full rank. Let $P$ have the state-space realisation $P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $D$ having full row rank, and define

$$\bar{P} := \begin{bmatrix} A - BD^\dagger C & -BD^* \\ D^\dagger C & D^\perp \end{bmatrix} \in \mathbb{R}^{q \times (q-p)}$$

where $D^\dagger$ is the Moore-Penrose inverse of $D$ and $D^\perp$ satisfies

$$\begin{bmatrix} D \\ D^\perp \end{bmatrix} \begin{bmatrix} D^\dagger & D^\perp \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Find an $X \in \mathbb{R}^{p \times p}$ satisfying

$$XX^* = PP^* \tag{3.13}$$

and a $Y \in \mathbb{R}^{(q-p) \times (q-p)}$ satisfying $Y^*Y = \bar{P}^*\bar{P}$. Note that since $X$ (resp. $Y$) is

\(^2\)If $P$ does not satisfy this assumption, one can always negligibly perturb $P$ at infinite frequency to satisfy this assumption.
square and $DD^*$ (resp. $D \perp D^*_\perp$) is nonsingular, it follows that $XX^* = PP^*$ (resp. $Y^*Y = \bar{P}^*\bar{P}$) implicitly implies that $X^{-1} \in \mathcal{R}^{p \times p}$ (resp. $Y^{-1} \in \mathcal{R}^{(q-p) \times (q-p)}$). Define
\[
\Phi := \begin{bmatrix} P^*X^{-*} & \bar{P}Y^{-1} \end{bmatrix} \in \mathcal{R}^{q \times q}
\] (3.14)
and note that $\Phi^*\Phi = I$ since $P\bar{P} = 0$. Since $\Phi$ is also square, $\Phi^{-1} = \Phi^*$. Now it is easy to see that equation (3.11) can be equivalently rearranged into
\[
P - P_\Delta = P\Delta
\]
\[
\Leftrightarrow (P - P_\Delta) = \begin{bmatrix} X & 0 \end{bmatrix} \Phi^{-1}\Delta
\]
\[
\Leftrightarrow \Delta = \Phi \begin{bmatrix} X^{-1}(P - P_\Delta) \\ Q \end{bmatrix}
\]
for any $Q \in \mathcal{R}^{(q-p) \times q}$.

Consequently, for this specific case, since $\Phi$ is a unit in $\mathcal{RL}_\infty$, a necessary and sufficient condition for there to exist a $\Delta \in \mathcal{RL}_\infty$ that satisfies consistency of equations is $X^{-1}P_\Delta \in \mathcal{RL}_\infty$, with a simple sufficient condition being $P$ having no transmission zeros on $j(\mathbb{R} \cup \{\infty\})$ and $P_\Delta \in \mathcal{RL}_\infty$. Then, given any $P, P_\Delta$ pair that satisfies $X^{-1}P_\Delta \in \mathcal{RL}_\infty$, there always exist multiple solutions for $\Delta \in \mathcal{RL}_\infty$ given by
\[
\Delta = \Phi \begin{bmatrix} X^{-1}(P - P_\Delta) \\ Q \end{bmatrix}
\]
for any $Q \in \mathcal{RL}_\infty^{(q-p) \times q}$. (3.15)

Tall Plants:

Assume (in this tall plants case) that $P, P_\Delta \in \mathcal{R}^{p \times q}$ is such that $p > q$ and $P(\infty)$ has full rank. Let $P$ have the state-space realisation $P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $D$ having full column rank, and define
\[
\bar{P} := \begin{bmatrix} A - BD^*C & -BD^* \\ D^*C & D^*_\perp \end{bmatrix} \in \mathcal{R}^{(p-q) \times p}
\] (3.16)
where $D^*$ is the Moore-Penrose inverse of $D$ and $D^*_\perp$ satisfies
\[
\begin{bmatrix} D^* \\ D^*_\perp \end{bmatrix} \begin{bmatrix} D & D^* \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.
\]
Find a \( U \in \mathbb{R}^{q \times q} \) satisfying \( U^*U = P^*P \) and a \( V \in \mathbb{R}^{(p-q) \times (p-q)} \) satisfying \( VV^* = \tilde{P}\tilde{P}^* \). Note that since \( U \) (resp. \( V \)) is square and \( D^*D \) (resp. \( D^*_\perp D^*_\perp \)) is nonsingular, it follows that \( U^*U = P^*P \) (resp. \( VV^* = \tilde{P}\tilde{P}^* \)) implicitly implies that \( U^{-1} \in \mathbb{R}^{q \times q} \) (resp. \( V^{-1} \in \mathbb{R}^{(p-q) \times (p-q)} \)). Define

\[
\Psi := \begin{bmatrix} U^{-*}P^* \\ V^{-1}\tilde{P} \end{bmatrix} \in \mathbb{R}^{p \times p}
\]

and note that \( \Psi^*\Psi = I \) since \( \tilde{P}P = 0 \). Since \( \Psi \) is also square, \( \Psi^{-1} = \Psi^* \). Now it is easy to see that equation (3.11) can be equivalently rearranged into

\[
P - P_\Delta = P\Delta \quad \Leftrightarrow \quad \Psi(P - P_\Delta) = \Psi P\Delta
\]

\[
\Leftrightarrow \begin{bmatrix} U^{-1}U^{-*}P^*(P - P_\Delta) \\ \tilde{P}P_\Delta \end{bmatrix} = \begin{bmatrix} \Delta \\ 0 \end{bmatrix}.
\]

Consequently, for this specific case, necessary and sufficient conditions for there to exist a \( \Delta \in \mathcal{RL}_\infty \) that satisfies consistency of equations are

\[
(P^*P)^{-1}P^*P_\Delta \in \mathcal{RL}_\infty \quad \text{and} \quad \tilde{P}P_\Delta = 0, \tag{3.17}
\]

with a sufficient condition being \( P \) having no transmission zeros on \( j(\mathbb{R} \cup \{\infty\}) \), \( P_\Delta \in \mathcal{RL}_\infty \) and \( \tilde{P}P_\Delta = 0 \). Then, given any \( P, P_\Delta \) pair that satisfies condition (3.17), there exists only a unique solution for \( \Delta \in \mathcal{RL}_\infty \) given by

\[
\Delta = (P^*P)^{-1}P^*(P - P_\Delta). \tag{3.18}
\]

It is worth pointing out at this stage that input multiplicative uncertainty characterisations for strictly tall plants have limited authority to alter the nominal plant dynamics.\(^3\) This engineering insight is captured above by the fact that the condition \( \tilde{P}P_\Delta = 0 \) is not easy to satisfy in practice, hence making this form of uncertainty modelling of limited engineering relevance.

\(^3\)The dimension of \( \Delta \) in this case is \( q \times q \), where \( P \in \mathbb{R}^{p \times q} \) and \( p > q \). The image space created by right multiplication of \( P \) by a vector \( u \in \mathbb{C}^q \) (e.g. one generated by \( u = (I - \Delta)\hat{u} \), with \( \hat{u} \in \mathbb{C}^q \)) will not cover the entire space \( \mathbb{C}^p \), even if \( P \) has full normal column rank.
**Derive conditions to guarantee well-posedness of** $F_u(H, \Delta)$

Like in the additive case, since $H_{11} = 0$, no extra conditions need to be imposed on the data $P, P_\Delta$ for well-posedness of the linear fractional transformation $F_u(H, \Delta)$. Consequently, given a nominal plant $P$ and a perturbed plant $P_\Delta$, it was shown above that one of the following three conditions is a necessary and sufficient condition for there to exist a $\Delta \in \mathcal{RL}_\infty$ satisfying $P_\Delta = F_u(H, \Delta)$:

- **Condition I** means $P, P_\Delta \in \mathbb{R}^{p \times q}$ with $p = q$ satisfying $P(\infty)$ having full rank and $P^{-1}P_\Delta \in \mathcal{RL}_\infty$;

- **Condition II** means $P, P_\Delta \in \mathbb{R}^{p \times q}$ with $p < q$ satisfying $P(\infty)$ having full rank and $X^{-1}P_\Delta \in \mathcal{RL}_\infty$ (where $X \in \mathbb{R}^{p \times p}$ satisfies $XX^* = PP^*$);

- **Condition III** means $P, P_\Delta \in \mathbb{R}^{p \times q}$ with $p > q$ satisfying $P(\infty)$ having full rank, $(P^*P)^{-1}P^*P_\Delta \in \mathcal{RL}_\infty$ and $\tilde{P}P_\Delta = 0$ (where $\tilde{P}$ is defined in (3.16)).

Also, for the equation $P_\Delta = F_u(H, \Delta)$, when:

- **Condition I** is satisfied (square plant case), there exists a unique solution $\Delta \in \mathcal{RL}_\infty$ given by equation (3.12);

- **Condition II** is satisfied (fat plant case), there always exist multiple solutions $\Delta \in \mathcal{RL}_\infty$ given by equation (3.15);

- **Condition III** is satisfied (tall plant case), there exists a unique solution $\Delta \in \mathcal{RL}_\infty$ given by equation (3.18).

**Define the solution set $\Delta$ and distance measure $d_{lm}(P, P_\Delta)$**

Straight from Definition 3.2.1, the solution set $\Delta$ reduces to

$$
\Delta = \begin{cases} 
\{P^{-1}(P - P_\Delta)\} & \text{when Condition I holds,} \\
\{\Phi \left[ X^{-1}(P - P_\Delta) \right] : Q \in \mathcal{RL}_\infty(q-p) \times q \} & \text{when Condition II holds,} \\
\{((P^*P)^{-1}P^*(P - P_\Delta))\} & \text{when Condition III holds,} \\
\emptyset & \text{otherwise}
\end{cases}
$$
in this specific case. The objects $X, \Phi$ in the definition of set $\Delta$ above are given by equations (3.13) and (3.14). The definition of the distance measure $d_{\text{im}}(P, P_{\Delta})$ when Condition I or Condition III holds is trivial since there is only one element in the set $\Delta$. When Condition II holds, and since $\Phi$ is allpass, $\inf_{\Delta \in \Delta} \| \Delta \|_{\infty} = \| X^{-1}(P - P_{\Delta}) \|_{\infty}$. Then, it easily follows straight from Definition 3.2.1 that—for an $X \in \mathbb{R}^{p \times p}$ that satisfies $XX^* = PP^*$—the distance measure $d_{\text{im}}(P, P_{\Delta})$ for input multiplicative uncertainty characterisations is given by:

$$d_{\text{im}}(P, P_{\Delta}) := \begin{cases} \| X^{-1}(P - P_{\Delta}) \|_{\infty} & \text{when Condition I or II holds,} \\ \| (P^*P)^{-1}P^*(P - P_{\Delta}) \|_{\infty} & \text{when Condition III holds,} \\ \infty & \text{otherwise.} \end{cases}$$

(3.19)

**Write the winding number condition independent of $\Delta$**

For input multiplicative uncertainty characterisations as considered here, this process is trivial since $H_{11} = 0$. Winding number condition (3.2) reduces to $\eta(P_{\Delta}) = \eta(P)$, which is easily computable as it is independent of $\Delta$.

**State robust stability and robust performance theorems**

**Theorem 3.5.1** (Robust Stability — Input Multiplicative). Given a plant $P \in \mathbb{R}^{p \times q}$, a perturbed plant $P_{\Delta} \in \mathbb{R}^{p \times q}$ and a controller $C \in \mathbb{R}^{q \times p}$, define a stability margin $b_{\text{im}}(P, C)$ as in (3.9) and a distance measure $d_{\text{im}}(P, P_{\Delta})$ as in (3.19). Furthermore, suppose $d_{\text{im}}(P, P_{\Delta}) < b_{\text{im}}(P, C)$. Then

$[P_{\Delta}, C]$ is internally stable $\iff \eta(P_{\Delta}) = \eta(P)$.

**Proof.** This theorem specialises Theorem 3.2.1 using formulae derived in the above subsection. Like in the additive case, the supposition that “$H$ is stabilizable” is automatically fulfilled because of the special form of $H$. \hfill \Box

---

4Note that in the case of square plants (Condition I), $X^{-1} = P^{-1}$. 

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Theorem 3.5.2 (Robust Performance — Input Multiplicative). Given the suppositions of Theorem 3.5.1 and furthermore assuming $\eta(P_\Delta) = \eta(P)$. Then

$$\left| 1 - \frac{b_{im}(P, C)}{b_{im}(P_\Delta, C)} \right| \leq \|(I - CP_\Delta)^{-1}\|_\infty d_{im}(P, P_\Delta)$$ (3.20)

and

$$\frac{\| \mathcal{F}_1(H_\Delta, C) - \mathcal{F}_1(H, C) \|_\infty}{\| \mathcal{F}_1(H, C) \|_\infty} \leq \|(I - CP_\Delta)^{-1}\|_\infty d_{im}(P, P_\Delta),$$ (3.21)

where $H = \begin{bmatrix} 0 & I \\ -P & I \\ -P_\Delta & P_\Delta \end{bmatrix}$ and $H_\Delta = \begin{bmatrix} 0 & I \\ -P & I \end{bmatrix}$.

Proof. This theorem specialises Theorem 3.2.2 using formulae derived in the above subsection. The result follows on choosing $k = 0$ and noting that $S = S_z S_w = I \in \mathcal{L}_\infty$ thereby giving $\| \mathcal{F}_1(H_\Delta, C) - S \|_\infty = \|(I - CP_\Delta)^{-1}\|_\infty$. The result given in inequality (3.20) follows from that of inequality (b) in Theorem 3.2.2 on dividing both sides with $b_{im}(P_\Delta, C)$, since this is guaranteed to be strictly greater than zero via Theorem 3.5.1. The result given in inequality (3.21) follows from that of inequality (c) in Theorem 3.2.2 on multiplying both sides with $b_{im}(P, C) = \frac{1}{\| \mathcal{F}_1(H, C) \|_\infty}$, since this too is guaranteed to be strictly greater than zero via the suppositions of this theorem.

It is worth pointing out that the discrepancy between the nominal and perturbed stability margin and also the discrepancy between the nominal and perturbed closed-loop in this theorem are naturally written in multiplicative form since this section is concerned with multiplicative uncertainty characterisations and performance measures. Also, note that the object $\| (I - CP_\Delta)^{-1} \|_\infty$ corrupts the right side of inequalities (3.20) and (3.21). Considered on a frequency-by-frequency basis, the corresponding largest singular value is typically very small in the pass-band, close to unity in the stop-band and not too big around crossover for a good design. Hence, this factor assists in tightening the inequalities in the passband.
3.5.2 Output Multiplicative

It should be noted that output multiplicative uncertainty, i.e. \( P_\Delta = (I - \Delta)P \), is simply a dual to input multiplicative uncertainty. The results for this dual case can easily be obtained by the substitutions \( P \rightarrow P^T \), \( P_\Delta \rightarrow P_\Delta^T \) and \( \Delta \rightarrow \Delta^T \) in the relevant definitions, equations and theorems in Subsection 3.5.1.

3.5.3 Output Inverse Multiplicative

For inverse multiplicative uncertainty, it is suitable to begin with the output inverse case, as this exhibits some degree of similarity with the input multiplicative case. However, the problem of output inverse multiplicative uncertainty characterisation with output inverse multiplicative performance measure is not a simple dual problem to that derived above in Subsection 3.5.1.

**Define the stability margin** \( b_{\text{oim}}(P, C) \)

Using Table 3.1 and Definition 3.2.3, the stability margin \( b_{\text{oim}}(P, C) \) for an output inverse multiplicative uncertainty characterisation is given by:

\[
b_{\text{oim}}(P, C) := \begin{cases} 
\| (I - PC)^{-1} \|^{-1}_\infty & \text{if } [P, C] \text{ is internally stable}, \\
0 & \text{otherwise}. 
\end{cases}
\]  

(3.22)

**Solve consistency equation for all** \( \Delta \in \mathcal{RH}_\infty \)

In this specific case, (3.10) reduces to

\[
P_\Delta - P = \Delta P_\Delta 
\]  

(3.23)

on substituting \( S_w = \begin{bmatrix} I & 0 \end{bmatrix}^T \) and \( S_c = \begin{bmatrix} I & 0 \end{bmatrix} \). It is assumed in this section that \( P_\Delta(\infty) \) has full rank, which is imposed for mathematical convenience. If the perturbed plant \( P_\Delta \) does not satisfy this assumption, one can always negligibly perturb \( P_\Delta \) at infinite frequency so as to satisfy this assumption. Luckily, to answer the question of solving consistency equation (3.23) for all \( \Delta \in \mathcal{RH}_\infty \), the derivation process of Subsection 3.5.1 does not have to be repeated in its entirety, as a
quick comparison of (3.23) with (3.11) reveals that the required results follow by performing the substitutions $P \rightarrow P\Delta^T$, $P\Delta \rightarrow P\Delta^T$ and $\Delta \rightarrow \Delta^T$ in the derivation of Subsection 3.5.1.

**Derive conditions to guarantee well-posedness of $\mathcal{F}_u(H,\Delta)$**

The derivations here depart from the similarity observed before. It is shown how to make a connection between consistency equation (3.23) and the uncertainty characterisation $P\Delta = F_u(H,\Delta)$. Since $P\Delta = F_u(H,\Delta)$ is defined to be well-posed when $\det(I - H_1\Delta)(\infty) \neq 0$, firstly $\det(I - H_1\Delta)(\infty)$ has to be expressed independently of $\Delta$. Since $H_1 = I$ (a very important difference from the additive and input multiplicative cases), it follows (after some simple algebra) that:

$$
\det(I - H_1\Delta)(\infty) \neq 0 \Leftrightarrow \begin{cases}
\det(P(\infty)) \neq 0 & \text{if } p = q, \\
\det \left[ P \left( \tilde{P}_\Delta^* V^{-*} - Q \right) \right](\infty) \neq 0 & \text{if } p > q,
\end{cases}
$$

(3.24)

from the results derived from Subsection 3.5.1 via the substitutions given in the preceding step of the procedure. In equivalence (3.24), $Q \in \mathcal{RL}_{p \times (p-q)}$ is arbitrary, $V \in \mathcal{R}^{(p-q) \times (p-q)}$ satisfies $VV^* = \tilde{P}_\Delta^* \tilde{P}_\Delta$ and

$$
\tilde{P}_\Delta := \begin{bmatrix} A - BD^\dagger C & -BD^\dagger \\
D^\dagger C & D^\perp \end{bmatrix} \in \mathcal{R}^{(p-q) \times p},
$$

in which $P_\Delta = \begin{bmatrix} A & B \\
C & D \end{bmatrix}$, $D^\dagger$ is the Moore-Penrose inverse of $D$ and $D^\perp$ satisfies

$$
\begin{bmatrix} D^\dagger \\
D^\perp \end{bmatrix} \begin{bmatrix} D & D^\perp \end{bmatrix} = \begin{bmatrix} I & 0 \\
0 & I \end{bmatrix}.
$$

The inequalities in (3.24) restrict the allowable $P(\infty)$, $P_\Delta(\infty)$ data and $Q(\infty)$ for well-posedness of the linear fractional transformation $\mathcal{F}_u(H,\Delta)$. The following technical lemma is needed to simplify condition (3.24).
Lemma 3.5.3. 1. When \( p = q \), condition (3.24) is equivalent to \( P(\infty) \) having full rank;

2. when \( p > q \), \( \exists Q \in \mathcal{RL}_\infty \) so that condition (3.24) is fulfilled if and only if \( P(\infty) \) has full rank;

3. when \( p < q \), condition (3.24) is equivalent to \( P(\infty) \) having full rank under the supposition \( P\bar{\Delta}P = 0 \).

Proof. 1. Trivial.

2. Since \( \bar{\Delta}P^*V^* \in \mathcal{RL}_\infty \), \( \exists Q \in \mathcal{RL}_\infty \) so that condition (3.24) is fulfilled if and only if \( \exists \hat{Q} \in \mathcal{RL}_\infty \) so that \( \det \begin{bmatrix} P & \hat{Q} \end{bmatrix}(\infty) \neq 0 \) if and only if \( P(\infty) \) has full rank.

3. Since

\[
\text{rank}(P(\infty)) = \text{rank} \left( P(\infty) \begin{bmatrix} P_\Delta(\infty)^* & \bar{P}_\Delta(\infty) \end{bmatrix} \right) = \text{rank} \left( \begin{bmatrix} (PP_\Delta^*)(\infty) & 0 \end{bmatrix} \right),
\]

it easily follows that \( P(\infty) \) has full rank if and only if \( \text{det}(PP_\Delta^*)(\infty) \neq 0 \).

Under the restrictions imposed by (3.24), it follows that

\[
P_\Delta - P = \Delta P_\Delta \iff P_\Delta = \mathcal{F}_u(H,\Delta),
\]

as shown above in (3.10). Consequently, given a nominal plant \( P \) and a perturbed plant \( P_\Delta \), it was shown above that one of the following three conditions is a necessary and sufficient condition for there to exist a \( \Delta \in \mathcal{RL}_\infty \) satisfying \( P_\Delta = \mathcal{F}_u(H,\Delta) \):

- **Condition I** means \( P,P_\Delta \in \mathcal{R}^{p \times q} \) with \( p = q \) satisfying \( P(\infty), P_\Delta(\infty) \) having full rank and \( PP_\Delta^{-1} \in \mathcal{RL}_\infty \);

- **Condition II** means \( P,P_\Delta \in \mathcal{R}^{p \times q} \) with \( p > q \) satisfying \( P(\infty), P_\Delta(\infty) \) having full rank and \( PU^{-1} \in \mathcal{RL}_\infty \) (where \( U \in \mathcal{R}^{q \times q} \) satisfies \( U^*U = P_\Delta^*P_\Delta \));

- **Condition III** means \( P,P_\Delta \in \mathcal{R}^{p \times q} \) with \( p < q \) satisfying \( P(\infty), P_\Delta(\infty) \) having full rank, \( PP_\Delta(P_\Delta^*P_\Delta)^{-1} \in \mathcal{RL}_\infty \) and \( P\bar{\Delta} = 0 \).
Also, for the equation \( P_{\Delta} = \mathcal{F}_u (H, \Delta) \), when:

- **Condition I** is satisfied (square plant case), there exists a unique solution \( \Delta \in \mathcal{RL}_\infty \) given by
  \[
  \Delta = (P_{\Delta} - P)P_{\Delta}^{-1};
  \]
  \hspace{1cm} (3.25)

- **Condition II** is satisfied (tall plant case), there always exist multiple solutions \( \Delta \in \mathcal{RL}_\infty \) given by
  \[
  \Delta = [(P_{\Delta} - P)U^{-1} \ Q] \Psi
  \]
  \hspace{1cm} (3.26)
  for any \( Q \in \mathcal{RL}_\infty^{p \times (p-q)} \) that satisfies (3.24), where \( \Psi = \begin{bmatrix} U^{-*}P^* \\ V^{-1}\hat{P} \end{bmatrix} \in \mathcal{R}^{p \times p} \);

- **Condition III** is satisfied (fat plant case), there exists a unique solution \( \Delta \in \mathcal{RL}_\infty \) given by
  \[
  \Delta = (P_{\Delta} - P)P_{\Delta}^*(P_{\Delta}P_{\Delta}^*)^{-1}.
  \]
  \hspace{1cm} (3.27)

**Define the solution set \( \Delta \) and distance measure \( d_{oim}(P,P_{\Delta}) \)**

From Definition 3.2.1, the solution set \( \Delta \) is characterised as follows:

\[
\Delta = \begin{cases} 
(P_{\Delta} - P)P_{\Delta}^{-1} & \text{when Condition I holds,} \\
(P_{\Delta} - P)U^{-1}Q \Psi & Q \in \mathcal{RL}_\infty^{p \times (p-q)}, \\
\det [P (\hat{P}^*V^{-*} - Q)] (\infty) \neq 0 & \text{when Condition II holds,} \\
(P_{\Delta} - P)P_{\Delta}^*(P_{\Delta}P_{\Delta}^*)^{-1} & \text{when Condition III holds,} \\
\emptyset & \text{otherwise}
\end{cases}
\]

in this specific case. The definition of the distance measure \( d_{oim}(P,P_{\Delta}) \) when Condition I or Condition III holds is trivial since there is only one element in the set \( \Delta \). When Condition II holds, and since \( \Psi \) is allpass, \( \inf_{\Delta \in \Delta} \| \Delta \|_\infty = \| (P_{\Delta} - P)U^{-1} \|_\infty \).

The following technical lemma states under what conditions can \( Q(\infty) = 0 \) be chosen.

**Lemma 3.5.4.** The choice \( Q(\infty) = 0 \) in the solution \( \Delta = [(P_{\Delta} - P)U^{-1} \ Q] \Psi \) for the \( p > q \) case gives a \( \Delta(\infty) \) that satisfies \( \det(I - H_{11}\Delta)(\infty) \neq 0 \) if and only if \( \det(P_{\Delta}^*P)(\infty) \neq 0 \).
Proof. Choose \( Q(\infty) = 0 \) in (3.24) for \( p > q \). Then

\[
\det(I - H_{11}\Delta)(\infty) \neq 0 \iff \det \begin{bmatrix} P & \tilde{P}_\Delta^* \end{bmatrix}(\infty) \neq 0
\]

\[
\iff \det \begin{bmatrix} P_\Delta^* \\ \tilde{P}_\Delta \\ P \\ \tilde{P}_\Delta \end{bmatrix}(\infty) \neq 0
\]

\[
\iff \det(P_\Delta^*P)(\infty) \neq 0
\]

since \( \tilde{P}_\Delta P_\Delta = 0 \).

Then, it follows from Definition 3.2.1 that—for a \( U \in \mathbb{R}^{q \times q} \) that satisfies \( U^*U = P_\Delta^*P_\Delta \) and under the simplifying assumption that \( \det(P_\Delta^*P)(\infty) \neq 0 \)—the distance measure \( d_{\text{oim}}(P,P_\Delta) \) for output inverse multiplicative uncertainty characterisations is given by:

\[
d_{\text{oim}}(P,P_\Delta) := \begin{cases} 
\| (P_\Delta - P)U^{-1} \|_\infty & \text{when Condition I or II holds,} \\
\| (P_\Delta - P)P_\Delta^*P_\Delta^*(-1) \|_\infty & \text{when Condition III holds,} \\
\infty & \text{otherwise.}
\end{cases}
\]

(3.28)

Write the winding number condition independent of \( \Delta \)

The problem needs to be split again in three cases: square, tall and fat plants.

Square Plants:

When Condition I is satisfied, using \( \Delta \in \Delta^{\text{min}} \) given by equation (3.25) in winding number condition (3.2) gives

\[
\eta(P_\Delta) - \eta(P) = \text{wnodet}(PP_\Delta^{-1}) = \text{wnodet}(PP_\Delta^*).
\]

(3.29)

Note that equation (3.29) can be simplified to \( z(P_\Delta) = z(P) \), where \( z(\cdot) \) are the number of open right half plane zeros of (\( \cdot \)), but for consistency with the tall/fat plant cases, this simpler formulation is not preferred.

Tall Plants:

Before tackling this case, note that an immediate corollary to Lemma 3.5.4 is as follows:
**Corollary 3.5.5.** Choosing $Q = 0$ in equation (3.26) for the $p > q$ case gives a $\Delta \in \Delta_{\text{min}}$ if and only if $\det(P_\Delta^* P) (\infty) \neq 0$.

Consequently, when Condition II and $\det(P_\Delta^* P) (\infty) \neq 0$ are satisfied, using $\Delta \in \Delta_{\text{min}}$ given by (3.26) with $Q = 0$ in winding number condition (3.2) gives

$$
\eta(P_\Delta) - \eta(P) = \text{wno det}(I - (P_\Delta - P)(P_\Delta^* P)^{-1}P_\Delta^*)
= \text{wno det}(P_\Delta^* P).
$$

(3.30)

**Fat Plants:**

When Condition III is satisfied, using $\Delta \in \Delta_{\text{min}}$ given by (3.27) in winding number condition (3.2) gives

$$
\eta(P_\Delta) - \eta(P) = \text{wno det}(PP_\Delta^*(P_\Delta^* P)^{-1})
= \text{wno det}(P_\Delta^* P).
$$

(3.31)

**State robust stability and robust performance theorems**

**Theorem 3.5.6** (Robust Stability — Output Inverse Multiplicative). Given a plant $P \in \mathcal{R}^{p \times q}$, a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$ and a controller $C \in \mathcal{R}^{q \times p}$, define a stability margin $b_{\text{oim}}(P, C)$ as in (3.22), a distance measure $d_{\text{oim}}(P, P_\Delta)$ as in (3.28), and an object

$$
\Xi := \begin{cases} 
PP_\Delta^* & \text{when } p \leq q \\
P_\Delta^* P & \text{otherwise}
\end{cases}.
$$

Furthermore, suppose $d_{\text{oim}}(P, P_\Delta) < b_{\text{oim}}(P, C)$ and when $p > q$, suppose also $\det(P_\Delta^* P) (\infty) \neq 0$. Then

$$
[P_\Delta, C] \text{ is internally stable } \iff \text{wno det}(\Xi) = \eta(P_\Delta) - \eta(P),
$$

where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of $P$ and $P_\Delta$.

**Proof.** This theorem specialises Theorem 3.2.1 using formulae derived in the above subsection. The supposition $d_{\text{oim}}(P, P_\Delta) < b_{\text{oim}}(P, C)$ implies that either Condition I or II or III must hold since $d_{\text{oim}}(P, P_\Delta) < b_{\text{oim}}(P, C) \leq \infty$. Note also
that the supposition that “H is stabilizable” is automatically fulfilled in this specific design case.

\[ \text{Theorem 3.5.7 (Robust Performance — Output Inverse Multiplicative). Given the suppositions of Theorem 3.5.6 and furthermore assuming } \text{wn} \\det(\Xi) = \eta(P_{\Delta}) - \eta(P), \text{ where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of } P \text{ and } P_{\Delta}, \text{ then} \]

\[ \left| 1 - \frac{b_{\text{oim}}(P, C)}{b_{\text{oim}}(P_{\Delta}, C)} \right| \leq \|P_{\Delta}(I - CP_{\Delta})^{-1}C\|_{\infty} d_{\text{oim}}(P, P_{\Delta}) \quad (3.32) \]

and

\[ \frac{\|F_1(H_{\Delta}, C) - F_1(H, C)\|_{\infty}}{\|F_1(H, C)\|_{\infty}} \leq \|P_{\Delta}(I - CP_{\Delta})^{-1}C\|_{\infty} d_{\text{oim}}(P, P_{\Delta}), \quad (3.33) \]

where \( H = \begin{bmatrix} I & P \\ I & P \end{bmatrix} \) and \( H_{\Delta} = \begin{bmatrix} I & P_{\Delta} \\ I & P_{\Delta} \end{bmatrix} \).

\[ \text{Proof.} \text{ This theorem specialises Theorem 3.2.2 using formulae derived in the above subsection. The result follows on choosing } k = 0 \text{ and noting that } S = S_{z}S_{w} = I \in R.L_{\infty} \text{ thereby giving } \|F_1(H_{\Delta}, C) - S\|_{\infty} = \|P_{\Delta}(I - CP_{\Delta})^{-1}C\|_{\infty}. \]

Note that the object \( \|P_{\Delta}(I - CP_{\Delta})^{-1}C\|_{\infty} \) corrupts the right side of inequalities (3.32) and (3.33). Considered on a frequency-by-frequency basis, the corresponding largest singular value is typically very close to unity in the pass-band, very small in the stop-band and not too big around crossover. Hence, it is a factor that assists in tightening the inequalities in the stopband. Again, the discrepancy between nominal and perturbed stability margin and closed-loop transfer function given in inequalities (3.32) and (3.33) appear naturally in multiplicative form.

3.5.4 Input Inverse Multiplicative

The problem of input inverse multiplicative uncertainty characterisation, i.e. \( P_{\Delta} = P(I - \Delta)^{-1} \), with an input inverse multiplicative performance measure is simply a dual problem to that discussed above for output inverse multiplicative uncertainty characterisation, i.e. \( P_{\Delta} = (I - \Delta)^{-1}P \), with an output inverse multiplicative
performance measure. Consequently, the results in this case follow trivially by performing the simple substitutions $P \rightarrow P^T$, $P_\Delta \rightarrow P_\Delta^T$ and $\Delta \rightarrow \Delta^T$ in the definitions, theorems and results of Subsection 3.5.3.

3.6 Inverse Additive Uncertainty Characterization with an Inverse Additive Performance Measure

In this section, robust stability and robust performance results for the inverse additive uncertainty structure are derived.

**Define the Stability Margin $b_{ia}(P, C)$**

Using Table 3.1 and Definition 3.2.3, the stability margin for inverse additive uncertainty characterization is given by

$$b_{ia}(P, C) := \begin{cases} \| P(I - CP)^{-1} \|_\infty & \text{if } [P, C] \text{ is internally stable,} \\ 0 & \text{otherwise.} \end{cases} \quad (3.34)$$

**Solve the Consistency Equation for all $\Delta \in \mathcal{RL}_\infty$**

For inverse additive uncertainty, the consistency equation (3.1) reduces to

$$P_\Delta - P = -P \Delta P_\Delta. \quad (3.35)$$

The analysis must be split into separate cases for square, fat and tall plants.

**Square plants**

Assume that $P(\infty)$ and $P_\Delta(\infty)$ have full rank. Then, (3.35) can be equivalently rearranged as

$$\Delta = P_\Delta^{-1} - P^{-1}. \quad (3.36)$$

Consequently, for this case, a necessary and sufficient condition for there to exist a $\Delta \in \mathcal{RL}_\infty$ that satisfies consistency of equations is $P_\Delta^{-1} - P^{-1} \in \mathcal{RL}_\infty$, with an
obvious sufficient condition being that $P$ and $P_\Delta$ have no zeros on the imaginary axis. Given any $P, P_\Delta$ pair that satisfy $P_\Delta^{-1} - P^{-1} \in \mathcal{RL}_\infty$, there exists a unique solution for $\Delta \in \mathcal{RL}_\infty$ given by (3.36).

**Fat plants**

Assume in this case that $P, P_\Delta \in \mathcal{R}^{p \times q}$ are such that $p < q$ and $P(\infty), P_\Delta(\infty)$ have full rank.

Let $P$ have a state space realization $P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $D$ having full row rank. Define

$$\bar{P} := \begin{bmatrix} A - BD_\Delta^\dagger C & -BD_\Delta^* \\ D_\Delta^\dagger C & D_\Delta^* \end{bmatrix} \in \mathcal{R}^{q \times (q-p)}$$

where $D_\Delta^\dagger$ is the Moore-Penrose inverse of $D_\Delta$ and $D_\Delta^\perp$ satisfies

$$\begin{bmatrix} D \\ D_\Delta^\perp \end{bmatrix} \begin{bmatrix} D_\Delta^\dagger & D_\Delta^\perp \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Find an $X_1 \in \mathcal{R}^{p \times p}$ satisfying

$$X_1X_1^* = PP^*$$

and a $Y_1 \in \mathcal{R}^{(q-p) \times (q-p)}$ satisfying $Y_1^*Y_1 = \bar{P}^* \bar{P}$. Define

$$\Phi_1 := \begin{bmatrix} P^*X_1^{-*} & \bar{P}Y_1^{-1} \end{bmatrix} \in \mathcal{R}^{q \times q}.$$

Also, let $P_\Delta$ have a state space realization $P_\Delta = \begin{bmatrix} A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix}$, with $D_\Delta$ having full row rank. Define

$$\bar{P}_\Delta := \begin{bmatrix} A_\Delta - B_\Delta D_\Delta^\dagger C_\Delta & -B_\Delta D_\Delta^* \\ D_\Delta^\dagger C_\Delta & D_\Delta^* \end{bmatrix} \in \mathcal{R}^{q \times (q-p)}$$

where $D_\Delta^\dagger$ is the Moore-Penrose pseudoinverse of $D_\Delta$ and $D_\Delta^\perp$ satisfies

$$\begin{bmatrix} D_\Delta \\ D_\Delta^\perp \end{bmatrix} \begin{bmatrix} D_\Delta^\dagger & D_\Delta^\perp \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$
Using techniques similar to those used in the derivations for multiplicative uncertainty characterizations in Section 3.5, it can then be shown that (3.35) can be equivalently rearranged as

\[
\begin{cases}
\Delta = -\Phi_1 \left[ X^{-1}(P_\Delta - P)P_\Delta^*(P_\Delta P_\Delta^*)^{-1}\right] Q \\
P\bar{P}_\Delta = 0.
\end{cases}
\]

for any \( Q \in \mathbb{R}^{(q-p)\times p} \), \( (3.40) \)

Consequently, for this specific case, since \( \Phi_1 \) is a unit in \( \mathcal{RL}_\infty \), necessary and sufficient conditions for there to exist a \( \Delta \in \mathcal{RL}_\infty \) that satisfies consistency of equations are

\[
X^{-1}(P_\Delta - P)P_\Delta^*(P_\Delta P_\Delta^*)^{-1} \in \mathcal{RL}_\infty \quad \text{and} \quad P\bar{P}_\Delta = 0.
\]

Then, given any \( P, P_\Delta \) pair that satisfy these conditions, there always exist multiple solutions given by (3.40), with \( Q \) restricted to \( Q \in \mathcal{RL}_\infty^{(q-p)\times p} \).

**Tall plants**

Assume in this case that \( P, P_\Delta \in \mathbb{R}^{p\times q} \) are such that \( p > q \) and \( P(\infty), P_\Delta(\infty) \) have full rank.

Let \( P \) have the state space realization \( P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) with \( D \) having full column rank. Define

\[
\bar{P} := \begin{bmatrix} A - BD^\dagger C & -BD^\dagger \\ D^\dagger C & D^\dagger \end{bmatrix} \in \mathbb{R}^{(p-q)\times p},
\]

where \( D^\dagger \) is the Moore-Penrose inverse of \( D \) and \( D_{\perp} \) satisfies

\[
\begin{bmatrix} D^\dagger \\ D_{\perp} \end{bmatrix} \begin{bmatrix} D & D_{\perp} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.
\]

Find a \( U \in \mathbb{R}^{q\times q} \) and a \( V \in \mathbb{R}^{(p-q)\times (p-q)} \) satisfying

\[
U^*U = P^*P,
\]

\[
VV^* = \bar{P}\bar{P}^*.
\]
Define
\[ \Psi := \begin{bmatrix} U^{-*} \tilde{P}^* \\ V^{-1} \tilde{p} \end{bmatrix} \in R^{p \times p}, \tag{3.42} \]
and note that \( \Psi \Psi^* = I \) since \( \tilde{P} P = 0 \). Since \( \Psi \) is square, \( \Psi^{-1} = \Psi^* \).

Also, let \( P_\Delta \) have the state space realization \( P_\Delta = \begin{bmatrix} A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} \) with \( D_\Delta \) having full column rank. Define
\[ \tilde{P}_\Delta := \begin{bmatrix} A_\Delta - B_\Delta D_\Delta^\dagger C_\Delta \\ D_\Delta^\dagger C_\Delta \end{bmatrix} \in R^{(p-q) \times p}, \]
where \( D_\Delta^\dagger \) is the Moore-Penrose inverse of \( D_\Delta \) and \( D_\Delta^\perp \) satisfies
\[ \begin{bmatrix} D_\Delta^\dagger \\ D_\Delta^\perp \end{bmatrix} \begin{bmatrix} A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \]

Find an \( X_2 \in R^{q \times q} \) satisfying
\[ X_2^* X_2 = P_\Delta^* P_\Delta \tag{3.43} \]
and a \( Y_2 \in R^{(p-q) \times (p-q)} \) satisfying \( Y_2 Y_2^* = \tilde{P}_\Delta \tilde{P}_\Delta^* \). Define
\[ \Phi_2 := \begin{bmatrix} X_2^{-*} P_\Delta^* \\ Y_2^{-1} \tilde{P}_\Delta \end{bmatrix} \in R^{p \times p}. \tag{3.44} \]

Using techniques similar to those used in the derivations for multiplicative uncertainty characterizations in Section 3.5, it can then be shown that (3.35) can be equivalently rearranged as
\[ \begin{align*}
\Delta &= \left((P^* P)^{-1} P^* (P - P_\Delta) X_2^{-1} \right) Q \Phi_2 \\
\tilde{P} P_\Delta &= 0.
\end{align*} \tag{3.45} \]

Consequently, for this specific case, since \( \Phi_2 \) is a unit in \( RL_\infty \), necessary and sufficient conditions for there to exist a \( \Delta \in RL_\infty \) that satisfies consistency of
equations are

\[(P^*P)^{-1}P^*(P-P_\Delta)X_2^{-1} \in \mathcal{RL}_\infty \quad \text{and} \quad \tilde{P}P_\Delta = 0.\]

Then, given any \(P, P_\Delta\) pair that satisfy these conditions, there always exist multiple solutions \(\Delta \in \mathbb{R}L^{q \times p}_\infty\) given by (3.45), with \(Q\) restricted to \(Q \in \mathbb{R}L^{q \times (p-q)}_\infty\).

**Derive Conditions to Guarantee Well-posedness of \(\mathcal{F}_u(H, \Delta)\)**

For the case of inverse additive uncertainty, the well-posedness condition is formulated as

\[
\det(I - H_{11}\Delta)(\infty) \neq 0 \iff \det(I + P\Delta)(\infty) \neq 0.
\]

(3.46)

Again, the derivations need to be carried out separately for the case of square, fat and tall plants.

**Square plants**

In the square plants case it is obvious that

\[
\det(I + P\Delta)(\infty) \neq 0 \iff \det(P)(\infty) \neq 0.
\]

**Fat plants**

In the fat plants case, substitution of the solution \(\Delta\) in (3.40) into (3.46) yields

\[
\det(I + P\Delta)(\infty) \neq 0 \iff \det \left( I - P\Phi_1 \left[ \begin{array}{c}
X_1^{-1}(P_\Delta - P)P_\Delta^*(P_\Delta P_\Delta^*)^{-1}
\end{array} \right] \right)(\infty) \neq 0
\]

\[
\iff \det \left( I - (P_\Delta - P)P_\Delta^*(P_\Delta P_\Delta^*)^{-1} \right)(\infty) \neq 0
\]

\[
\iff \det \left( P\Phi_1 Q \right)(\infty) \neq 0.
\]
Tall plants

In the tall plants case, substitution of the solution \( \Delta \) in (3.45) into (3.46) yields

\[
\det(I + P\Delta)(\infty) \neq 0
\]

\[
\iff \det \left( I + P \left[ (P^* P)^{-1} P^* (P - P_\Delta) X_2^{-1} Q \right] \Phi_2 \right)(\infty) \neq 0
\]

\[
\iff \det \left( \Phi_2^* + P \left[ (P^* P)^{-1} P^* (P - P_\Delta) X_2^{-1} Q \right] \right)(\infty) \neq 0
\]

\[
\iff \det \left( [PX_2^{-1} + (I - P(P^* P)^{-1} P^*) P_\Delta X_2^{-1} \tilde{P}_\Delta^* Y^{-*} + PQ] \right)(\infty) \neq 0.
\]

The results for all three cases are summarised and simplified in the following lemma.

**Lemma 3.6.1.** Given \( P, P_\Delta \in \mathbb{R}^{p \times q} \), assume that \( P(\infty) \) has full rank. Furthermore when \( p < q \) assume that \( P\tilde{P}_\Delta = 0 \), where \( \tilde{P}_\Delta \) is given by (3.39), and when \( p > q \) that \( \tilde{P}P_\Delta = 0 \), where \( \tilde{P} \) is given by (3.41). Then, for \( p \leq q \),

\[
\det(I - H_{11}\Delta)(\infty) \neq 0,
\]

(3.47)

and \( \exists Q \in \mathbb{R}^{L \times (p-q)} \) such that the above holds when \( p > q \).

**Proof.** The lemma is obvious for \( p = q \). In the fat plants case, i.e. \( p < q \), the proof follows from Lemma 3.5.3. For the tall plants case, note that for \( \Psi \) as defined in (3.42),

\[
\Psi^* \Psi = I \iff P(P^* P)^{-1} P^* = P(U^* U)^{-1} P^* = I - \tilde{P}^* V^{-*} V^{-1} \tilde{P}.
\]

Now, under the assumption that \( \tilde{P}P_\Delta = 0 \),

\[
\det(I - H_{11}\Delta)(\infty) \neq 0 \iff \det \left( \left[ P \tilde{P}_\Delta^* Y^{-*} + PQ \right] \right)(\infty) \neq 0.
\]

This can be satisfied if and only if \( P(\infty) \) has full rank, since for any \( \hat{Q} \in \mathbb{R}L_\infty \) such that

\[
\det \left( \left[ P \hat{Q} \right] \right)(\infty) \neq 0
\]

there exists a \( Q \in \mathbb{R}L_\infty \) such that \( \hat{Q} = \tilde{P}_\Delta^* Y^{-*} + PQ \) if and only if \( P \) has full rank (i.e. a left inverse of \( P \) exists).

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Under the restrictions imposed by the above derivations, it follows that

\[ P_\Delta - P = -P \Delta P_\Delta \iff P_\Delta = \mathcal{F}_u (H, \Delta). \]

Consequently, given a nominal plant \( P \) and a perturbed plant \( P_\Delta \), it was shown above that one of the following conditions is a necessary and sufficient condition for there to exist a \( \Delta \in \mathbb{R} \mathcal{L}_{\infty}^{q \times p} \) satisfying \( P_\Delta = \mathcal{F}_u (H, \Delta) \):

- **Condition I** means \( P, P_\Delta \in \mathbb{R}^{p \times q} \) with \( p = q \), satisfying \( P(\infty), P_\Delta(\infty) \) having full rank and \( P^{-1} - P_\Delta^{-1} \in \mathbb{R} \mathcal{L}_{\infty} \).

- **Condition II** means \( P, P_\Delta \in \mathbb{R}^{p \times q} \) with \( p < q \), satisfying \( P(\infty), P_\Delta(\infty) \) having full rank, \( X_1^{-1}(P_\Delta - P)P_\Delta^*(P_\Delta P_\Delta^*)^{-1} \in \mathbb{R} \mathcal{L}_{\infty} \), where \( X_1 \) satisfies \( X_1 X_1^* = PP^* \), and \( P \tilde{P}_\Delta = 0 \), where \( \tilde{P}_\Delta \) is given in (3.39).

- **Condition III** means \( P, P_\Delta \in \mathbb{R}^{p \times q} \) with \( p > q \), satisfying \( P(\infty), P_\Delta(\infty) \) having full rank, \( (P^*P)^{-1}P^*(P - P_\Delta)X_2^{-1} \in \mathbb{R} \mathcal{L}_{\infty} \), where \( X_2 \) satisfies \( X_2^*X_2 = P_\Delta^*P_\Delta \), and \( P \tilde{P}_\Delta = 0 \), where \( \tilde{P} \) is given in (3.41).

Also, for the equation \( P_\Delta = \mathcal{F}_u (H, \Delta) \), when

- **Condition I** is satisfied (square plants), there exists a unique solution \( \Delta \in \mathbb{R} \mathcal{L}_{\infty}^{q \times p} \) given by (3.36);

- **Condition II (respectively III)** is satisfied (fat, respectively tall plants), there always exist multiple solutions \( \Delta \in \mathbb{R} \mathcal{L}_{\infty}^{q \times p} \) given by (3.40) (respectively (3.45)) with \( Q \) restricted to \( Q \in \mathbb{R} \mathcal{L}_{\infty} \) of appropriate dimension.
**Define the Solution Set \( \Delta \) and Distance Measure \( d_{ia}(P,P_\Delta) \)**

Straight from Definition 3.2.1, the solution set \( \Delta \) for inverse additive uncertainty reduces to

\[
\Delta = \begin{cases}
P_\Delta^{-1} - P^{-1} & \text{when Condition I holds,} \\
-\Phi_1 \left[ X_1^{-1}(P_\Delta - P)P_\Delta X_2^{-1} \right] & \text{when Condition II holds,} \\
\left[ (P^*P)^{-1}P^*(P - P_\Delta)X_2^{-1} \right] \Phi_2 & \text{when Condition III holds,} \\
\emptyset & \text{otherwise.}
\end{cases}
\]

The definition of the distance measure \( d_{ia}(P,P_\Delta) \) is trivial when Condition I or II hold, since there is only one element (square plants) or the choice \( Q = 0 \) results in the \( \Delta \) of smallest size (fat plants). When Condition III holds, \( Q(\infty) = 0 \) may be chosen whenever \( \det(P_\Delta^*P)(\infty) \neq 0 \) via Lemma 3.5.4, which will be assumed from here on for notational simplicity. Therefore, with \( X_1 \in \mathbb{R}^{p \times p} \) satisfying \( X_1 X_1^* = PP^* \) and \( X_2 \in \mathbb{R}^{q \times q} \) satisfying \( X_2^* X_2 = P_\Delta^* P_\Delta \) and under the assumption \( \det(P_\Delta^*P)(\infty) \neq 0 \), immediately from Definition 3.2.1, the distance measure for inverse additive uncertainty \( d_{ia}(P,P_\Delta) \) is given by

\[
d_{ia}(P,P_\Delta) := \begin{cases}
\| P_\Delta^{-1} - P^{-1} \|_\infty & \text{when Condition I holds,} \\
\| X_1^{-1}(P_\Delta - P)P_\Delta X_2^{-1} \|_\infty & \text{when Condition II holds,} \\
\| (P^*P)^{-1}P^*(P - P_\Delta)X_2^{-1} \|_\infty & \text{when Condition III holds,} \\
\infty & \text{otherwise.}
\end{cases}
\]

\( (3.48) \)

**Write the Winding Number Condition Independent of \( \Delta \)**

As in the previous subsections, the analysis is carried out separately for square, fat and tall plants.

**Square plants** When Condition I holds, substituting \( \Delta \) from (3.36) into the winding
number condition (3.2) results in
\[
\eta(P_\Delta) - \eta(P) = \text{wnodet}(PP_\Delta^{-1}) = \text{wnodet}(PP^*_\Delta).
\]

This can be reformulated to the condition that the number of open right half plane zeros of \(P\) and \(P_\Delta\) be the same (see Section 3.5).

**Fat plants** When Condition II holds, substituting \(\Delta \in \Delta^{\text{min}}\) (i.e. with \(Q = 0\)) from (3.40) into winding number condition (3.2) results in
\[
\eta(P_\Delta) - \eta(P) = \text{wnodet}(PP^*_\Delta(P_\Delta P^*_\Delta)^{-1}) = \text{wnodet}(PP^*_\Delta).
\]

**Tall plants** When Condition III holds and furthermore \(\det(P^*_\Delta P)(\infty) \neq 0\), substitution of a \(\Delta \in \Delta^{\text{min}}\) from (3.45) (i.e. with \(Q = 0\) since \(\det(P^*_\Delta P)(\infty) \neq 0\) is assumed) into the winding number condition (3.2) results in
\[
\eta(P_\Delta) - \eta(P) = \text{wnodet}\left(I + P\begin{bmatrix}(P^*_\Delta)^{-1}P^*(P - P_\Delta)X_2^{-1} & 0 \\ X_2^{-*}P^*_\Delta \\ Y_2^{-1}P_\Delta \end{bmatrix}\right)
\]
\[
= \text{wnodet}(I + P(P^*_\Delta P_\Delta)^{-1}P^*_\Delta)
\]
\[
= \text{wnodet}(P^*_\Delta(P_\Delta + P)).
\]

**State Robust Stability and Robust Performance Theorems**

**Theorem 3.6.2.** Given a plant \(P \in \mathbb{R}^{p \times q}\), a perturbed plant \(P_\Delta \in \mathbb{R}^{p \times q}\) and a controller \(C \in \mathbb{R}^{q \times p}\), define a stability margin \(b_{ia}(P, C)\) as in (3.34), a distance measure \(d_{ia}(P, P_\Delta)\) as in (3.48) and an object
\[
\Xi := \begin{cases} PP^*_\Delta & \text{when } p \leq q, \\ P^*_\Delta(P_\Delta + P) & \text{otherwise}. \end{cases}
\]

Furthermore, suppose \(d_{ia}(P, P_\Delta) < b_{ia}(P, C)\) and if \(p > q\) that \(\det(P^*_\Delta P)(\infty) \neq 0\). Then
\[
[P_\Delta, C] \text{ is internally stable } \iff \text{wnodet}(\Xi) = \eta(P_\Delta) - \eta(P),
\]
where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of \(P\) and \(P_\Delta\).
Proof. This theorem specializes Theorem 3.2.1 using formulae derived in this section. The supposition $d_{ia}(P, P_\Delta) < b_{ia}(P, C)$ implies that either Condition I, II or III holds, since $b_{ia}(P, C) \leq \infty$. The stabilizability of $H$ is easily fulfilled, e.g. when a state space realization is constructed using $P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $(A, B)$ stabilizable and $(C, A)$ detectable.

\textbf{Theorem 3.6.3.} Given the suppositions of Theorem 3.6.2 and furthermore assuming $\text{wndet}(\Xi) = \eta(P_\Delta) - \eta(P)$, where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of $P$ and $P_\Delta$, then

$$|b_{ia}(P_\Delta, C) - b_{ia}(P, C)| \leq d_{ia}(P, P_\Delta),$$

$$\|\mathcal{F}_1(H_\Delta, C) - \mathcal{F}_1(H, C)\|_\infty \leq \frac{d_{ia}(P, P_\Delta)}{b_{ia}(P, C)b_{ia}(P_\Delta, C)},$$

where

$$H = \begin{bmatrix} -P & P \\ -P & P \end{bmatrix}, \quad H_\Delta = \begin{bmatrix} -P_\Delta & P_\Delta \\ -P_\Delta & P_\Delta \end{bmatrix}.$$ 

\textbf{Proof.} This theorem specializes Theorem 3.2.2 using formulae derived in this section. The results follow upon noting that $S_{w\Delta} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $S = 0$. 

\subsection*{3.7 Right Coprime Factor Uncertainty Characterization with a Right Coprime Factor Performance Measure}

In this section, robust stability and robust performance theorems for the right coprime factor uncertainty structure are derived. The equivalent results for the left coprime factor uncertainty structure were reported in [LP09, Section VII]. The right coprime factor structure is dual to the left coprime factor structure, which means that most—though not all—results follow from simple substitutions. However, the substitutions cannot be applied at any point in the derivations. As a result, a winding number condition and a condition on the gain of the product
of two graph symbols at infinite frequency are non-trivially different in the right coprime factor case and are therefore separately derived in this section.

Given a nominal plant $P \in \mathbb{R}^{p \times q}$ with a right coprime factorization (not necessarily normalized) $\{N_0, M_0\}$, then a perturbed plant with right coprime factor uncertainty is given by

$$P_\Delta = (N_0 + \Delta N) (M_0 + \Delta M)^{-1}. \quad (3.49)$$

It can be shown that

$$P_\Delta = \mathcal{F}_u (H_{RCF}, \Delta) = \mathcal{F}_u \left( \begin{bmatrix} 0 & -M_0^{-1} & M_0^{-1} \\ I & -P & I \\ 0 & 0 & P \end{bmatrix}, \Delta \right), \quad (3.50)$$

with $\Delta = \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}$. This is a dual formulation to the left coprime factor problem, i.e.

$$P_\Delta = \mathcal{F}_u (H_{LCF}, \Delta) = \mathcal{F}_u \left( \begin{bmatrix} \tilde{M}_0^{-1} & P \\ 0 & I \\ M_0^{-1} & P \end{bmatrix}, \Delta \right).$$

It is therefore easy to verify that the following substitutions transform some of the objects relating to the left coprime factor problem into the equivalent objects in the right coprime factor problem.

<table>
<thead>
<tr>
<th>Left coprime factor</th>
<th>$\rightarrow$</th>
<th>Right coprime factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>$\rightarrow$</td>
<td>$P^T$</td>
</tr>
<tr>
<td>${\tilde{N}_0, \tilde{M}_0}$</td>
<td>$\rightarrow$</td>
<td>${N_0^T, M_0^T}$</td>
</tr>
<tr>
<td>$P_\Delta$</td>
<td>$\rightarrow$</td>
<td>$P_\Delta^T$</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>$\rightarrow$</td>
<td>$\Delta^T \begin{bmatrix} 0 &amp; I \ -I &amp; 0 \end{bmatrix}$</td>
</tr>
<tr>
<td>$C$</td>
<td>$\rightarrow$</td>
<td>$C^T$</td>
</tr>
<tr>
<td>$\mathcal{F}_I (H, C)$</td>
<td>$\rightarrow$</td>
<td>$0 \begin{bmatrix} -I \ I \end{bmatrix}$ $\mathcal{F}_I (H, C)^T$</td>
</tr>
<tr>
<td>$\mathcal{F}<em>I (H</em>\Delta, C)$</td>
<td>$\rightarrow$</td>
<td>$0 \begin{bmatrix} -I \ I \end{bmatrix}$ $\mathcal{F}<em>I (H</em>\Delta, C)^T$</td>
</tr>
</tbody>
</table>
Define the Stability Margin $b_{rcf}(P,C)$

Immediately from Definition 3.2.3 and simple manipulation, the right coprime factor robust stability margin for a given $P \in \mathbb{R}^{p \times q}$ with a right coprime factorization (not necessarily normalized) $\{N_0, M_0\}$ and $C \in \mathbb{R}^{q \times p}$ is obtained as:

$$b_{rcf}(P,C) := \begin{cases} \left\| M_0^{-1} (I - CP)^{-1} \begin{bmatrix} C & I \end{bmatrix} \right\|^{-1}_\infty & \text{if } [P,C] \text{ is internally stable}, \\ 0 & \text{otherwise.} \end{cases}$$

(3.51)

In this case, as in the case of left coprime factors, $0 \neq \mathcal{F}_1(H,C) \in \mathcal{RL}_\infty$ is dropped, since internal stability of $[P,C]$ and the structure of the generalized plant automatically guarantee this.

Solve the Consistency Equation

The consistency equation for the right coprime factor case is obtained from [LP09, (15)] by substitutions as outlined at the beginning of this section, resulting in

$$P_\Delta - P = \begin{bmatrix} I & -P_\Delta \end{bmatrix} \Delta M_0^{-1}.$$  

(3.52)

Before obtaining an admissible $\Delta$ by manipulation of this equation, consider the following objects. Let $\{N,M\}$ be normalized right coprime factors of $P$ over $\mathcal{RH}_\infty$ and $\{N_0 = NR, M_0 = MR\}$ be not necessarily normalized right coprime factors of $P$ over $\mathcal{RH}_\infty$, related to the normalized coprime factors by $R \in \mathcal{GH}_\infty$. Also, let $\{\tilde{N}_\Delta, \tilde{M}_\Delta\}$ be normalized left coprime factors of $P_\Delta$ over $\mathcal{RH}_\infty$. Now, (3.52)
can be rearranged as follows:

\[
P_\Delta - P = \begin{bmatrix} I & -P_\Delta \end{bmatrix} \Delta M_0^{-1} \]
\[
\Leftrightarrow (P - P_\Delta) MR = \begin{bmatrix} I & -P_\Delta \end{bmatrix} \Delta \]
\[
\Leftrightarrow \tilde{M}_\Delta (P - P_\Delta) MR = \begin{bmatrix} \tilde{M}_\Delta & -\tilde{N}_\Delta \end{bmatrix} \Delta \]
\[
\Leftrightarrow (\tilde{M}_\Delta N - \tilde{N}_\Delta M) R = -\tilde{G}_\Delta \Delta \]
\[
\Leftrightarrow -\tilde{G}_\Delta GR = -\tilde{G}_\Delta \Delta \]
\[
\Leftrightarrow -\tilde{G}_\Delta GR = -\tilde{G}_\Delta \begin{bmatrix} G_\Delta & \tilde{G}_\Delta^* \end{bmatrix} \begin{bmatrix} G_\Delta^* \\ \tilde{G}_\Delta \end{bmatrix} \Delta,
\]

where the last equivalence is due to [Vin01, (2.14)]. Then,

\[
-\tilde{G}_\Delta GR = -\begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} G_\Delta^* \\ \tilde{G}_\Delta \end{bmatrix} \Delta
\]
\[
\Leftrightarrow \Delta = \begin{bmatrix} G_\Delta & \tilde{G}_\Delta^* \end{bmatrix} \begin{bmatrix} Q \\ \tilde{G}_\Delta GR \end{bmatrix}, \tag{3.53}
\]

where \(Q\) is an arbitrary object of appropriate dimension. It will be restricted to \(Q \in RL_\infty\), since \(\Delta \in RL_\infty \Leftrightarrow Q \in RL_\infty\).

**Derive Conditions to Guarantee Well-posedness of**

\(P_u (H, \Delta)\)

Having characterized all \(\Delta \in RL_\infty\) for which (3.52) holds, the next step is to ensure that \((I - H_{11} \Delta)^{-1} \in \mathcal{B}\). From (3.50),

\[
\det (I - H_{11} \Delta) = \det \left( I - \begin{bmatrix} 0 & -M_0^{-1} \end{bmatrix} \Delta \right),
\]
or equivalently via Sylvester’s determinant identity,

\[
\det(I - H_{11}\Delta) = \det\left(I - \Delta \begin{bmatrix} 0 & -M_0^{-1} \end{bmatrix}\right)
= \det\left(I - G_\Delta \tilde{G}_\Delta^* \begin{bmatrix} Q \\ \tilde{G}_\Delta \tilde{G}_\Delta^* \end{bmatrix} R^{-1} \begin{bmatrix} 0 & -M^{-1} \end{bmatrix}\right)
= \det\left(I - \begin{bmatrix} G_\Delta^* \\ \tilde{G}_\Delta \end{bmatrix} \begin{bmatrix} G_\Delta & \tilde{G}_\Delta^* \end{bmatrix} - \begin{bmatrix} Q \\ \tilde{G}_\Delta \tilde{G}_\Delta^* \end{bmatrix} R^{-1} \begin{bmatrix} 0 & -M^{-1} \end{bmatrix} \begin{bmatrix} G_\Delta & \tilde{G}_\Delta^* \end{bmatrix}\right)
= \det\left(I - \begin{bmatrix} 0 & -QR^{-1}M^{-1} \\ 0 & -\tilde{G}_\Delta GM^{-1} \end{bmatrix} \begin{bmatrix} G_\Delta & \tilde{G}_\Delta^* \end{bmatrix}\right)
= \det\left(I - \begin{bmatrix} 0 & -QR^{-1}M^{-1} \\ 0 & -\tilde{G}_\Delta \end{bmatrix} \begin{bmatrix} P \\ I \end{bmatrix} \begin{bmatrix} G_\Delta & \tilde{G}_\Delta^* \end{bmatrix}\right)
= \det\left(I - \begin{bmatrix} 0 & -QR^{-1}M^{-1} \\ 0 & -\tilde{G}_\Delta \end{bmatrix} \begin{bmatrix} G_\Delta & \tilde{G}_\Delta^* \end{bmatrix}\right)
= \det\left(I - \begin{bmatrix} G_\Delta \tilde{G}_\Delta^* \end{bmatrix} \begin{bmatrix} 0 & -QR^{-1}M^{-1} \\ 0 & -\tilde{G}_\Delta \end{bmatrix} \begin{bmatrix} G_\Delta & \tilde{G}_\Delta^* \end{bmatrix}\right)
= \det\left(I - \begin{bmatrix} G_\Delta \tilde{G}_\Delta^* \end{bmatrix} \begin{bmatrix} \tilde{G}_\Delta \tilde{G}_\Delta^* \end{bmatrix} \begin{bmatrix} 0 & -QR^{-1}M^{-1} \\ 0 & -\tilde{G}_\Delta \end{bmatrix} \begin{bmatrix} G_\Delta & \tilde{G}_\Delta^* \end{bmatrix}\right)
= \det\left(G_\Delta + \begin{bmatrix} 0 & QR^{-1}M^{-1} \\ \tilde{G}_\Delta \end{bmatrix} \begin{bmatrix} G_\Delta & \tilde{G}_\Delta^* \end{bmatrix}\right).
Consequently, since $\det \left( \begin{bmatrix} G_\Delta & \tilde{G}_\Delta^* \end{bmatrix} \right) (\infty) \neq 0$ and $\det \left( \begin{bmatrix} I & 0 \\ 0 & \tilde{M} \tilde{M}_\Delta^{-1} \end{bmatrix} \right) (\infty) \neq 0$,

$$\det (I - H_{11} \Delta) (\infty) \neq 0 \iff \det \left( \begin{bmatrix} G_\Delta^* + 0 & 0 \\ \tilde{G} & 0 \end{bmatrix} \right) (\infty) \neq 0$$

$$\iff \det \left( \begin{bmatrix} N_\Delta^* & M_\Delta^* + QR^{-1}M_\Delta^{-1} \\ -\tilde{M} & \tilde{N} \end{bmatrix} \right) (\infty) \neq 0$$

$$\iff \det \left( M_\Delta^* + QR^{-1}M_\Delta^{-1} + N_\Delta^* \tilde{M}^{-1} \tilde{N} \right) (\infty) \neq 0$$

$$\iff \det \left( QR^{-1} + M_\Delta^* M + N_\Delta^* \tilde{N} \right) (\infty) = \det \left( QR^{-1} + G_\Delta^* G \right) (\infty) \neq 0. \quad (3.54)$$

Under a specific condition, a particularly simple $Q$ may be chosen, as made clear by the following lemma.

**Lemma 3.7.1.** The choice $Q(\infty) = 0$ in (3.53) gives a $\Delta(\infty)$ that satisfies $\det (I - H_{11} \Delta) (\infty) \neq 0$ if and only if $\sigma \left( G_\Delta^* \tilde{G}_\Delta^* \right) (\infty) < 1$.

**Proof.** Choose $Q(\infty) = 0$ in (3.54). Then

$$\det (I - H_{11} \Delta) (\infty) \neq 0$$

$$\iff \sigma \left( G_\Delta^* G \right) (\infty) > 0$$

$$\iff \sigma \left( G_\Delta^* \tilde{G}_\Delta^* \right) (\infty) < 1.$$

\[ \square \]

**Define the Solution Set $\Delta$ and the Distance Measure**

$\text{}$

$d_{\text{ref}}(P, P_\Delta)$

All feasible solutions $\Delta \in \mathbb{R} \mathcal{L}_\infty$ satisfying $P_\Delta = \mathcal{F}_u (H_{\text{RCF}}, C)$ can now be parameterized as follows:

$$\Delta = \left\{ \Delta = \begin{bmatrix} G_\Delta & \tilde{G}_\Delta^* \end{bmatrix} \begin{bmatrix} Q \\ \tilde{G}_\Delta G \end{bmatrix} : Q \in \mathbb{R} \mathcal{L}_\infty, \ det \left( QR^{-1} + G_\Delta^* G \right) (\infty) \neq 0 \right\}.$$
The set $\Delta$ is never empty. Immediately from Definition 3.2.1 and under the simplifying assumption that $\sigma \left( G_\Delta^* \tilde{G}^* \right) (\infty) < 1$, the distance measure $d_{\text{rcf}}(P, P_\Delta)$ for a right coprime factor uncertainty characterization is given by

\[
d_{\text{rcf}}(P, P_\Delta) = \inf_{\Delta \in \Delta} \| \Delta \|_{\infty}
\]

\[
= \inf_{Q \in \mathcal{RL}_\infty} \left\| \begin{bmatrix} G_\Delta & \tilde{G}_\Delta \\ \tilde{G}_\Delta & \tilde{G}_\Delta GR \end{bmatrix} \right\|_{\infty}
\]

\[
= \inf_{Q \in \mathcal{RL}_\infty} \left\| \begin{bmatrix} Q \\ \tilde{G}_\Delta GR \end{bmatrix} \right\|_{\infty}
\]

\[
= \| \tilde{G}_\Delta GR \|_{\infty}.
\] (3.55)

Write the Winding Number Condition Independent of $\Delta$

From Lemma 3.7.1, the following corollary is obtained.

**Corollary 3.7.2.** The choice $Q = 0$ in (3.53) gives a $\Delta \in \Delta^{\min}$ if and only if $\sigma \left( G_\Delta^* \tilde{G}^* \right) (\infty) < 1$.

Consequently, for right coprime factor uncertainty and whenever $\sigma \left( G_\Delta^* \tilde{G}^* \right) (\infty) < 1$ is assumed, winding number condition (3.2) reduces to

\[
\eta \left( P_\Delta \right) - \eta \left( P \right) = \text{wno det} \left( I - H_{11} \Delta \right)
\]

\[
\iff \eta \left( P_\Delta \right) - \eta \left( P \right) = \text{wno det} \left( \begin{bmatrix} I & 0 \\ \tilde{M}_\Delta \tilde{M}^{-1} \tilde{G} \tilde{G}_\Delta & \tilde{M}_\Delta \tilde{M}^{-1} \tilde{G} \tilde{G}_\Delta^* \end{bmatrix} \right)
\]

\[
\iff \eta \left( P_\Delta \right) - \eta \left( P \right) = \text{wno det} \left( \tilde{M}_\Delta \tilde{M}^{-1} \tilde{G} \tilde{G}_\Delta^* \right)
\]

\[
\iff \eta \left( P_\Delta \right) - \eta \left( P \right) = \text{wno det} \left( \tilde{M} \right) - \text{wno det} \left( \tilde{M} \Delta \right) + \text{wno det} \left( \tilde{G} \tilde{G}_\Delta \right)
\]

\[
\iff \text{wno det} \left( \tilde{G} \tilde{G}_\Delta^* \right) = 0.
\]

**State Robust Stability and Robust Performance Theorems**

All derivations are now in place to give succinctly the specific robust stability and robust performance theorems for right coprime factor uncertainty.
**Theorem 3.7.3** (Robust Stability–Right Coprime Factor). Given a plant $P \in \mathbb{R}^{p \times q}$, a perturbed plant $P_{\Delta} \in \mathbb{R}^{p \times q}$, a controller $C \in \mathbb{R}^{q \times p}$ and not necessarily normalized right coprime factors $\{N_0, M_0\}$ of $P$ over $\mathcal{RH}_\infty$, define normalized graph symbols $G, \tilde{G}, G_\Delta$ and $\tilde{G}_\Delta$ as in Chapter 2 and let $R \in \mathcal{RH}_\infty$ satisfy $\begin{bmatrix} N_0 & M_0 \end{bmatrix} = GR$. Define a stability margin $b_{rcf}(P, C)$ as in (3.51) and a distance measure $d_{rcf}(P, P_{\Delta})$ as in (3.55). Furthermore, suppose $d_{rcf}(P, P_{\Delta}) < b_{rcf}(P, C)$ and $\sigma \left( G_\Delta^{*} \tilde{G}^{*} \right) (\infty) < 1$.

Then,$\left[ P_{\Delta}, C \right]$ is internally stable $\iff$ $\text{wnodet} \left( \tilde{G}_\Delta^{*} \right) = 0$,

where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of $P$ and $P_{\Delta}$.

**Proof.** This theorem follows from 3.2.1, upon noticing that the assumption that

$$H_{RCF} = \begin{bmatrix} 0 & -M_0^{-1} & M_0^{-1} \\ I & -P & P \end{bmatrix}$$

be stabilizable is automatically fulfilled, since $[P, C]$ is internally stable via the assumptions of the theorem, which implies that $\mathcal{F}_l(H_{RCF}, C) \in \mathcal{RH}_\infty$ via an argument similar to that in the proof of [LP09, Theorem 7]. This in turn implies internal stability of $\langle H_{RCF}, C \rangle$ which proves the stabilizability of $H_{RCF}$. The supposition $\sigma \left( G_\Delta^{*} \tilde{G}^{*} \right) (\infty) < 1$ ensures that Corollary 3.7.2 is applicable.

The following lemma shows that for $d_{rcf}(P, P_{\Delta}) < b_{rcf}(P, C)$,

$$\left\{ N_{\Delta,0} = N_\Delta + \Delta N, M_{\Delta,0} = M_\Delta + \Delta M \right\}$$

is a right coprime factorization of $P_{\Delta}$ for all $\Delta \in \Delta^{\text{min}}$.

**Lemma 3.7.4.** Given the suppositions of Theorem 3.7.3. Then

$$\left\{ N_{\Delta,0} = N_\Delta + \Delta N, M_{\Delta,0} = M_\Delta + \Delta M \right\}$$

is a right coprime factorization of $P_{\Delta}$ over $\mathcal{RL}_\infty$ for all $\Delta = \begin{bmatrix} \Delta N \\ \Delta M \end{bmatrix} \in \Delta^{\text{min}}$.

**Proof.** This lemma is dual to [LP09, Lemma 8].
Theorem 3.7.5 (Robust Performance–Right Coprime Factors). Given the suppositions of Theorem 3.7.3 and furthermore assuming $\det (\tilde{G}\tilde{G}_\Delta^*) = 0$, where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of $P$ and $P_\Delta$. Then,

$$|b_{rcf}(P_\Delta, C) - b_{rcf}(P, C)| \leq d_{rcf}(P,P_\Delta),$$  \hspace{1cm} (3.56)

and

$$\|\mathcal{F}_1(H_{\Delta, rcf}, C) - \mathcal{F}_1(H_{rcf}, C)\|_\infty \leq \frac{d_{rcf}(P,P_\Delta)}{b_{rcf}(P,C)b_{rcf}(P_\Delta,C)},$$  \hspace{1cm} (3.57)

where

$$H_{rcf} = \begin{bmatrix} 0 & -M_0^{-1} & M_0^{-1} \\ I & -P & P \end{bmatrix}$$

and

$$H_{\Delta, rcf} = \begin{bmatrix} 0 & -M_{\Delta,0}^{-1} & M_{\Delta,0}^{-1} \\ I & -P_\Delta & P_\Delta \end{bmatrix}.$$  

Proof. This theorem is dual to [LP09, Theorem 9] and follows upon trivial substitutions according to the table at the beginning of this section, with the winding number condition and well-posedness condition obtained separately above. \Box

Consequently, this section has shown that the robust stability and robust performance theorems for right coprime factor uncertainty that is not necessarily normalized are similar to those obtained in [LP09] for left coprime factor uncertainty, but that important details differ non-trivially, i.e. cannot be obtained simply by substitutions in the theorem statements. The right coprime factor uncertainty structure is used in Chapter 5, where controller synthesis in this structure is explored.

3.8 Numerical Examples

A physically motivated numerical example is studied in this section to illustrate the effectiveness of the robust stability and robust performance results derived in
the preceding sections for different uncertainty structures. The considered example is a spring-mass-damper system which is a small variation of a benchmark problem that has been studied in robust control theory ([WB92] and references therein; see also [LP08]). The schematic diagram of the system is depicted in Fig. 3.5. The plant is a single-input single-output (SISO) system, where the input is a force applied to one mass, and the output is the measured acceleration of the same mass amplified with a sensor gain \( k = 10 \). This resembles a practical situation in the control of mechanical structures, where an accelerometer is used to capture information about the state of the system. Elementary mechanical modelling yields the following nominal SISO plant transfer function:

\[
P(s) = 10 \frac{s^2 (s^2 + 2s + 2)}{s^4 + 4s^3 + 7s^2 + 6s + 3}.
\]

Based on the nominal plant, an output feedback controller \( C \) is synthesized using the Matlab routine “ncfsyn”. It achieves an optimal robust stability margin in the \( \nu \)-gap metric, i.e. with respect to normalized coprime factor uncertainty [GM89]. The synthesized controller is of third order, and its transfer function is given by

\[
C(s) = -\frac{0.4587s^3 + 1.165s^2 + 1.456s + 0.5978}{s^3 + 2.173s^2 + 2.349s + 0.35}.
\]

This robust controller achieves a normalized coprime factor robust stability margin of \( b(P, C) = 0.5053 \).
3.8.1 Example 1: Specific Perturbed Plants

The Bode plot of $P(s)$ is displayed in Fig. 3.6, along with the Bode plots of three perturbed versions of the plant, denoted by $P_{\Delta 1}$, $P_{\Delta 2}$ and $P_{\Delta 3}$, whose transfer functions are given below.

$$P_{\Delta 1}(s) = 10 \frac{s^2 (0.1s^3 + 9.2s^2 + 18.2s + 18)}{(s^5 + 14s^4 + 47s^3 + 76s^2 + 63s + 30)};$$
$$P_{\Delta 2}(s) = \frac{10s^4 + 20s^3 + 20s^2 + 0.0324s + 1.62}{s^4 + 4s^3 + 7s^2 + 6s + 3};$$
$$P_{\Delta 3}(s) = 10 \frac{s^2 (s^3 + 11s^2 + 20s + 18)}{0.7s^5 + 5.3s^4 + 14.9s^3 + 21.7s^2 + 17.1s + 7.5}. $$

The robust stability of the positive feedback loop with controller $C$ when the nominal plant is replaced by one of three perturbed plants $P_{\Delta i}(s)$ is now analysed. The perturbed plants are chosen to represent different types of perturbations. $P_{\Delta 1}(s)$ differs from $P(s)$ mostly at high frequencies, as is common when high-frequency dynamics of the plant (e.g. actuator dynamics) are neglected. In the case of $P_{\Delta 2}(s)$, only the zeros of the system are perturbed. This may be caused
e.g. by uncertainties in the measurement setup for lumped mass systems. Finally, $P_{Δ3}(s)$ differs from $P(s)$ only at low-to-medium frequencies. Using these exemplary systems, the application of the robust stability and robust performance theorems of this chapter as a tool for choosing the most suitable uncertainty structure to represent perturbations in a system is now demonstrated. For comparison, the normalized coprime factor uncertainty structure is also included, the formulae and theorems for which can be found e.g. in [Vin01].

The robust stability margins of $[P, C]$ with respect to several uncertainty structures can be found below.

<table>
<thead>
<tr>
<th>$b_a(P, C)$</th>
<th>$b_{im}(P, C)$</th>
<th>$b_{oim}(P, C)$</th>
<th>$b(P, C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5856</td>
<td>1.0142</td>
<td>0.7920</td>
<td>0.5053</td>
</tr>
</tbody>
</table>

The robust stability margins must be seen in conjunction with the distances between the nominal plant and the perturbed plants under different uncertainty structures, which are as follows $^5$.

<table>
<thead>
<tr>
<th>$P_{Δi}$</th>
<th>$d_a(P, P_{Δi})$</th>
<th>$d_{im}(P, P_{Δi})$</th>
<th>$d_{oim}(P, P_{Δi})$</th>
<th>$δ_ν(P, P_{Δi})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{Δ1}$</td>
<td>9.0000</td>
<td>0.9000</td>
<td>9.0000</td>
<td>0.6332</td>
</tr>
<tr>
<td>$P_{Δ2}$</td>
<td>0.5780</td>
<td>-</td>
<td>3.6914</td>
<td>0.5077</td>
</tr>
<tr>
<td>$P_{Δ3}$</td>
<td>18.4961</td>
<td>2.7016</td>
<td>0.7222</td>
<td>0.5646</td>
</tr>
</tbody>
</table>

The quantities for additive, input multiplicative and output inverse multiplicative uncertainty were calculated using the formulae derived in Sections 3.4 and 3.5. The robust stability margin for left four-block/normalized coprime factor uncertainty is given by [Vin01, (2.1)]

$$b(P, C) := \left\| \begin{bmatrix} I \\ C \end{bmatrix} (I - PC)^{-1} \begin{bmatrix} I & -P \end{bmatrix} \right\|^{-1}_∞ = 0.5053.$$ 

The distance measure for the four-block uncertainty structure can also be obtained from the generalized distance measures described in Section 3.2, see [PL06, Section III]. It is equivalent to the $ν$-gap metric [Vin01, Definition 3.1]. $^6$ Its formula

$^5$In the input multiplicative case, the consistency equation for $P_{Δ2}$ and $P$ can not be satisfied.

$^6$Vinnicombe’s definition of the $ν$-gap includes a winding number condition which in the distance measure framework used in this chapter corresponds to an assumption of the stability and performance theorems.
is given by
\[ \delta_{\nu}(P, P_{\Delta i}) = \| \tilde{G} G_{\Delta i} \|_{\infty}. \]

In addition to robust stability tests via the distance measures and robust stability margins, robust performance results for all three perturbed systems can also be obtained. In each of the cases displayed below, the performance bounds are derived using an uncertainty structure which guarantees robust stability of the perturbed plant in question.

<table>
<thead>
<tr>
<th>$P_{\Delta i}$</th>
<th>Robust perf. thm.</th>
<th>$b^{H_\Delta}(P_{\Delta}, C) \geq x$</th>
<th>$| \mathcal{F}<em>1(H</em>{\Delta}, C) - \mathcal{F}<em>1(H, C) |</em>{\infty} \leq y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{\Delta 1}$</td>
<td>Thm. 3.5.2</td>
<td>$x = 0.4813$</td>
<td>$y = 1.0916$</td>
</tr>
<tr>
<td>$P_{\Delta 2}$</td>
<td>Thm. 3.4.2</td>
<td>$x = 0.0076$</td>
<td>$y = 1.1466$</td>
</tr>
<tr>
<td>$P_{\Delta 3}$</td>
<td>Thm. 3.5.7</td>
<td>$x = 0.4029$</td>
<td>$y = 1.2193$</td>
</tr>
</tbody>
</table>

Note that upper bounds on the residual stability margin could also be computed, but are typically not relevant.

It will be explained how the tables above can be interpreted in terms of robust stability guarantees, by looking in detail at the results for $P_{\Delta 1}$. The nominal plant $P$ and $P_{\Delta 1}$ differ at high frequencies. Such discrepancies are common when high-frequency dynamics of the plant (e.g. actuator dynamics) are neglected. A multiplicative uncertainty structure is particularly suitable for the robustness analysis in this case. The solution set $\Delta$ contains only one element in this case:

\[ \Delta_{\text{im}}(s) = \left( 1 - P(s)^{-1} P_{\Delta 1}(s) \right) = \frac{0.9s + 1}{s + 10}. \]

Using an input multiplicative uncertainty characterisation, Theorem 3.5.1 is applied to test whether the closed-loop system is robustly stable with respect to this perturbation. Since $[P, C]$ is internally stable, (3.9) implies that $b_{\text{im}}(P, C) = \left\| C \left( 1 - PC \right)^{-1} P \right\|_{\infty}^{-1} = 1.0142$. Additionally, since Condition I of section 3.5.1 is fulfilled ($P$ is by-proper and therefore $P(\infty) \neq 0$), (3.19) implies that $d_{\text{im}}(P, P_{\Delta 1}) = \left\| P^{-1} (P - P_{\Delta 1}) \right\|_{\infty} = 0.9$. Having found that the assumptions of Theorem 3.5.1—most significantly $d_{\text{im}}(P, P_{\Delta 1}) < b_{\text{im}}(P, C)$—are fulfilled, it is concluded that $[P_{\Delta 1}, C]$ is internally stable because $\eta(P_{\Delta 1}) = \eta(P) = 0$. Theorem 3.5.2 can therefore be applied to obtain bounds on the robust performance, as seen above.
The same perturbed system \([P_{\Delta 1}, C]\) can also be analyzed using four-block/normalized coprime factor analysis results. Recall that \(C\) was chosen to be optimal for the four-block framework. As can be seen from the tables above, \(\delta_{\nu}(P, P_{\Delta 1}) = 0.6332 > b_{lfb}(P, C)\). Hence the assumptions of the robust stability theorem for the \(\nu\)-gap metric (see e.g. [Vin01, PL06, Theorem 3]) are not fulfilled (with the distance being larger than the stability margin), and the internal stability of \([P_{\Delta 1}, C]\) cannot be guaranteed using \(\nu\)-gap theory.

Hence, for this perturbed plant, an input multiplicative uncertainty characterisation yields a less conservative robust stability test, and therefore also robust performance bounds that cannot be obtained using existing \(\nu\)-gap machinery (even though the controller design favoured normalized coprime factor stability). It should be noted that additive and output inverse multiplicative uncertainty characterisations will in this case also not be able to guarantee stability of the perturbed feedback system, because the distance measures are larger than the corresponding robust stability margins in both cases. This shows the importance of using an appropriate uncertainty structure (rather than a very general one like normalized coprime factor in each case) for the envisaged perturbations in the particular problem at hand (e.g. making use of the theory in this chapter).

In a similar fashion, robust stability and performance results can be obtained for \(P_{\Delta 2}\) using an additive uncertainty structure (and Theorem 3.4.1), and for \(P_{\Delta 3}\) using an output inverse multiplicative uncertainty structure (and Theorem 3.5.6). For both of these perturbed plants, all other uncertainty structures do not guarantee robust stability, because the corresponding distances are larger than the respective stability margins (input multiplicative uncertainty cannot be applied to \(P_{\Delta 2}\) because there is no \(\Delta \in \mathbb{R}_L\) which fulfills the consistency equation for that case).

When robust stability holds for a particular uncertainty structure, one of the five robust performance theorems of this chapter can be applied. Bounds on the residual stability margin of \([P_{\Delta}, C]\), as well as bounds on the robust performance were listed above. For each perturbed plant, the applicable robust performance theorem is stated.

It must be stressed again that these results were obtained with a controller optimized with respect to the normalized coprime factor robust stability margin. Of course, using the insights gained from the robust stability analysis in this section,
a designer could now change the controller and optimize with respect to the robust stability margin in the uncertainty structure yielding the least conservative stability results, and hence obtain even better robust stability margins and robust performance.

Hence, the tools developed in this chapter can serve as the basis for a more systematic control law design methodology. Given a group of uncertain plant models, the current practice is to pick a nominal plant based on experience, to design a suitable control law, and then to evaluate robust stability and performance on all members of the uncertain plant family, iterating over these steps until a satisfactory performance is achieved. Based on the tools of this chapter, a future design methodology would involve choosing a suitable uncertainty structure together with an associated optimal controller, which provides optimal guarantees for the whole family of uncertain plants (through the theorems of this chapter), thereby eliminating the need for iterative stability and performance checks and controller redesigns.

### 3.8.2 Example 2: Neglected Actuator Dynamics

Given the combination of nominal plant \( P \) and controller \( C \) as defined above, the impact of neglected actuator dynamics on the stability and performance of the feedback system is studied in this subsection. Let

\[
U(s) = \frac{as + 9.9}{s + 10}U'(s),
\]

where \( u' \) is a calibrated input signal and \( a \in \mathbb{R} \) is an uncertain parameter. This represents a low-pass filter with uncertain high frequency gain. As a consequence, the uncertain plant becomes

\[
P_{\Delta}(s, a) = P(s)\frac{as + 9.9}{s + 10}
\]

and study the effect that varying values of \( a \) have on stability and performance. Fig. 3.7 shows the Bode plots of several members of the uncertain plant family \( P_{\Delta}(s, a) \) for values \( a \in [-0.1, 0.99] \). The nominal plant \( P \) is almost recovered for \( a = 0.99 \), except for a 1% change in gain.
This example focuses on multiplicative uncertainty, for which the robust stability margin is given in (3.9), and is calculated as \( b_{im}(P,C) = 1.0142 \). For a set of values of \( a \in \left[-0.21, 0.99\right] \), the distance between the nominal and the perturbed plant is calculated using both the multiplicative distance measure given in (3.19), and the \( \nu \)-gap distance for normalized coprime factor uncertainty. From (3.19), and since \( P(\infty) \) has full rank and \( P^{-1}P_\Delta \in \mathcal{RL}_\infty \), it follows that

\[
d_{im}(P,P_\Delta) = \left\|P^{-1}(P - P_\Delta)\right\|_\infty.
\]

Similarly, the \( \nu \)-gap distance measure for normalized coprime factor uncertainty has been given above (c.f. [Vin01]) as \( \delta_\nu(P,P_\Delta) = \left\|\tilde{G}\tilde{G}_\Delta\right\|_\infty \) (once the winding number condition is assumed). Both these distance measures are plotted in Fig. 3.8 for the interval \( a \in \left[-0.21, 0.99\right] \) together with the corresponding robust stability margins. Using Theorem 3.5.1, robust stability is guaranteed for all plants for which \( d_{im}(P,P_\Delta) < b_{im}(P,C) \) and \( \eta(P) = \eta(P_\Delta) \). Hence, all \( P_\Delta(a,s) \) with \( a \in \left[-0.014, 0.99\right] \) are guaranteed to be stable using the methods proposed in this chapter for input multiplicative uncertainty. Similarly, using \( \nu \)-gap theory, robust stability can be guaranteed for all plants for which \( \delta_\nu(P,P_\Delta) < b(P,C) \) (as-
assuming that certain winding number conditions hold, which are satisfied in this case; cf. [Vin01]). Therefore, using normalized coprime factor uncertainty, robust stability is guaranteed for the set of uncertain plants with $a \in [0.138, 0.99]$.

Consequently, for this example, the multiplicative distance measure and robust stability margin of this chapter enable less conservative robust stability guarantees for the plant with uncertain actuator dynamics. One of the strengths of the distance measure concept is that it also allows the design engineer to bound robust performance degradation. For comparison, lower bounds on the remaining robust stability margin in both the multiplicative and normalised coprime factor measures are computed. In the multiplicative case, Theorem 3.5.2 is used and—when all assumptions hold—reformulate inequality (3.20) as follows:

$$b_{\text{im}}(P_\Delta, C) \geq \frac{b_{\text{im}}(P, C)}{1 + \| (I - CP_\Delta)^{-1} \|_\infty} d_{\text{im}}(P, P_\Delta).$$

Similarly, from [Vin01, Theorem 3.8], it follows that

$$b(P_\Delta, C) \geq \sin(\arcsin(b(P, C)) - \arcsin(\delta_v(P, P_\Delta))).$$
Figure 3.9: Lower bounds on the robust stability margins provided by Theorem 3.5.2 and [Vin01, Theorem 3.8] together with the actual robust stability margins. Note that the performance theorems only apply to those plants where the distance is less than the corresponding robust stability margin.

Fig. 3.9 plots the lower bounds on the robust stability margin for the perturbed plant that can be obtained for the multiplicative case from Theorem 3.5.2 and the corresponding $\nu$-gap theorem [Vin01, Theorem 3.8], together with the actual values of the respective robust stability margins as computed for each value of $a$. It can be seen that both multiplicative and normalized coprime factor theorems provide conservative guarantees, with the multiplicative lower bound declining relatively less strongly as it approaches the point where $d_{im}(P, P_{\Delta}) = b_{im}(P, C)$, at which point Theorem 3.5.1 is no longer valid.

As a final remark, it must be stressed that the insight that multiplicative uncertainty enables less conservative robust stability and performance guarantees should lead the designer to reconsider the choice of the controller, which was chosen to maximize the normalized coprime factor robust stability margin in this example. The input multiplicative robust stability margin would be an obvious choice for an objective function in this specific case, but synthesis methods are beyond the scope of this chapter.
3.9 Conclusions

This chapter has derived specific robust stability tests as well as bounds on robust performance for several uncertainty structures (additive, input/output multiplicative, output/input inverse multiplicative, inverse additive and right coprime factor uncertainty) commonly used in control engineering practice. The robust stability theorems of this chapter go beyond previously known results for additive and multiplicative cases by extending the space of allowable perturbations from $\mathcal{RH}_\infty$ to $\mathcal{RL}_\infty$. Furthermore, the degradation of performance of a given feedback loop upon replacing the nominal plant by the perturbed plant is quantified in various robust performance theorems relying only on the computation of two infinity norms for each uncertain plant being considered. Physically motivated examples show that a suitably chosen uncertainty characterisation is essential in ensuring the least conservative stability and performance guarantees for a given perturbation class. Four-block uncertainty—though able to capture a large set of possible uncertainties—cannot always guarantee less conservative robust stability results for specific classes of perturbed plants for which other less general uncertainty structures are better suited (as shown in the example).

\footnote{Though not formally defined, [ZDG96, Table 9.1] contains robust stability margins for these cases. Distance measures were absent from the literature.}
Chapter 4

Analysis and Controller Synthesis for the Non-normalized Coprime Factor Uncertainty Structure

4.1 Introduction

In this chapter, the robust stabilization problem for plants with a left coprime factor uncertainty structure is considered. The robust stabilization problem for linear plants with a normalized left coprime factor uncertainty characterization was solved in [GM89], based on a solution to the $\mathcal{H}_\infty$ control problem for generalized plants [DGKF89]. Coprime factors are a powerful tool for characterizing plant uncertainty since they capture both stable and unstable pole and zero uncertainty [Vid85, Fra87]. It forms the basis for the $\mathcal{H}_\infty$ loop-shaping controller design method [MG90, MG92], in which a stabilizing controller is synthesized for a plant shaped using input/output weights chosen to achieve performance objectives.

For every closed-loop system, the robust stability margin $b(P, C)$ [GS90] quantifies the normalized coprime factor\(^1\) neighbourhood around the nominal plant (in the $\mathcal{H}_\infty$ sense) for which robust stability is guaranteed. An optimal value $b_{\text{opt}}(P)$ exists for every plant [GM89]. This largest possible neighbourhood for which ro-

\(^1\)Throughout the chapter, normalized coprime factor uncertainty is also referred to as four-block uncertainty.
bust stability can be guaranteed in the normalized coprime factor topology may however exclude certain plants which would intuitively be considered “close” to the nominal plant [HS93, Vin01, LT05]. This is the case e.g. for lightly damped plants, where small deviations in pole or zero frequencies may result in distances easily exceeding $b_{opt}(P)$, and often leads to actual loss of stability of the closed-loop system when a controller is used that is optimal in the normalized coprime factor sense. Distance between the nominal plant $P$ and any perturbed plant $P_\Delta$ is here measured by the $\nu$-gap [Vin93, Vin01], which computes the smallest infinity norm of a perturbation block $\Delta$ yielding $P_\Delta$ when connected to $P$ in a normalized left coprime factor setting.

A given plant may be factorized in any of an infinite number of coprime factorizations, i.e. normalization is only one option [Vid85, Fra87]. In [LP09], a robust stability margin $b_{lcf}(P, C)$ and distance measure $d_{lcf}(P, P_\Delta)$ for general left coprime factor uncertainty were proposed, enabling a systematic and quantitative comparison of different factorizations. These concepts apply to plants with performance weights, and it is assumed in the following that all plants are already “shaped”.

The contributions of this chapter are twofold. Firstly, it is shown that particular coprime factorizations minimize the ratio of coprime factor distance to robust stability margin for given $P$, $P_\Delta$, $C$. In a series of corollaries and theorems it is shown that with these factorizations—i.e. without changing the controller, plant or introducing weights—a larger set of plants can be guaranteed to be robustly stable than with a normalized coprime factorization. Also, tighter robust performance guarantees (in both normalized and non-normalized coprime factor sense) can be obtained than for normalized coprime factors. Secondly, a combined normalized/non-normalized coprime factor cost function is proposed for controller synthesis in an $\mathcal{H}_\infty$ framework. The proposed synthesis method achieves a strategic shaping of the guaranteed-stability neighbourhood around the nominal plant to include a particular perturbed plant $P_\Delta$, e.g. a plant with shifted lightly-damped pole frequency, by optimizing the controller with respect to the ratio of coprime factor distance to robust stability margin for a given $P_\Delta$. This objective is traded off with the normalized coprime factor robust stability margin of

$^2$Several $P_\Delta$’s can also be considered, at the cost of increased controller order.
the nominal closed-loop system. Via this second objective, nominal performance can also be maintained, since \( b(P, C) \) can be related to classical measures like gain and phase margin and hence performance [GVP00, Lan05]. There are some similarities of this method to the simultaneous \( \mathcal{H}_\infty \) stabilization problem solved in [LS04]. However, the approach of this chapter is to assume a particular nominal plant \( P \), and then to optimize robustness with respect to \( P_\Delta \) while maintaining a certain lower bound on \( b(P, C) \). This is in contrast to [LS04], where performance bounds are necessarily the same for all plants considered.

### 4.2 Coprime Factor Uncertainty and Robust Stability

Let a left coprime factorization of \( P \) (not necessarily normalized) be given by \( \{ \tilde{M}_0 = RM, \tilde{N}_0 = RN \} \), where \( \{ \tilde{M}, \tilde{N} \} \) is normalized and \( R \in \mathcal{G}\mathcal{H}_\infty \) is a denormalization factor. The perturbed plant is given by

\[
P_\Delta = \tilde{M}_{\Delta,0}^{-1}\tilde{N}_{\Delta,0} = (\tilde{M}_0 - \Delta_M)^{-1}(\tilde{N}_0 + \Delta_N).
\]

For the purpose of stating robust stability and robust performance theorems for left coprime factor uncertainty (that is not necessarily normalized), recall the following definitions from [LP09].

**Definition 4.2.1.** [LP09] Given two plants \( P, P_\Delta \in \mathbb{R}^{p \times q} \) with normalized graph symbols \( \tilde{G} \) and \( G_\Delta \) as defined in Definition 2.6.4, and a denormalization factor \( R \in \mathcal{G}\mathcal{H}_\infty \), define the distance measure for left coprime factor uncertainty structure as implied by \( R \) as

\[
d^R_{\text{lef}}(P, P_\Delta) := \|RGG_\Delta\|_\infty. \tag{4.1}
\]

**Definition 4.2.2.** [LP09] Given a positive feedback interconnection \([P, C]\) of a plant \( P \in \mathbb{R}^{p \times q} \) with left coprime factorization \( \{ \tilde{M}_0, \tilde{N}_0 \} = \{ RM, RN \} \), where \( R \in \mathcal{G}\mathcal{H}_\infty \) is a denormalization factor, and a controller \( C \in \mathbb{R}^{q \times p} \), define the
robust stability margin in the left coprime factor uncertainty structure as

\[
\begin{align*}
\text{b}_{\text{lcf}}^R(P,C) := \begin{cases} 
\left\| \begin{bmatrix} I & (I-PC)^{-1}\tilde{M}_0^{-1} 
\end{bmatrix} \right\|^{-1}_\infty & \text{if } [P,C] \text{ is internally stable;} \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

(4.2)

It is easy to show that when \([P,C]\) is internally stable, \(b_{\text{lcf}}^R(P,C) = \left\| (R\tilde{G}K)^{-1} \right\|^{-1}_\infty\), where \(\tilde{G}\) is a normalized left graph symbol of \(P\) and \(K\) is a normalized inverse right graph symbol of \(C\), as defined in Definition 2.6.4 and Definition 2.6.5, respectively. The superscript \(R\) denotes the dependence on the denormalization factor \(R\) of the coprime factorization \(\{\tilde{N}_0 = R\tilde{N}, M_0 = R\tilde{M}\}\). The significant difference between coprime factor uncertainty and four-block uncertainty lies in the additional degree of freedom offered by the denormalization factor \(R\). Note that when \(R\) is unitary, the two uncertainty structures are identical. In that case, \(b_{\text{lcf}}^R(P,C) = b(P,C)\) as defined in [GS90] and \(d_{\text{lcf}}^R(P,P_\Delta) = \delta_\nu(P,P_\Delta)\) as defined in [Vin93].

The ratio \(d_{\text{lcf}}^R(P,P_\Delta)/b_{\text{lcf}}^R(P,C)\) is crucial for quantifying robust stability and robust performance, as will be shown in the following. Its infimum over \(R \in \mathcal{KH}_\infty\) is given by \(\left\| (\tilde{G}K)^{-1} \tilde{G}G_\Delta \right\|_\infty\) when \([P,C]\) is internally stable.\(^3\) The robustness ratio is precisely this infimum and is given below.

**Definition 4.2.3.** Given plants \(P, P_\Delta \in \mathcal{H}^{p\times q}\) and a controller \(C \in \mathcal{H}^{q\times p}\), with graph symbols \(\tilde{G}, G_\Delta\) and \(K\) as defined in Definitions 2.6.4 and 2.6.5, respectively, define the robustness ratio

\[
r(P,P_\Delta,C) := \begin{cases} 
\left\| (\tilde{G}K)^{-1} \tilde{G}G_\Delta \right\|_\infty & \text{if } [P,C] \text{ is internally stable;} \\
\infty & \text{otherwise.}
\end{cases}
\]

\(^3\)A non-unique choice of \(R\) achieving this infimum is \(R = (\tilde{G}K)^{-1}\):

\[
\inf_{R \in \mathcal{KH}_\infty} \frac{d_{\text{lcf}}^R(P,P_\Delta)}{b_{\text{lcf}}^R(P,C)} = \inf_{R \in \mathcal{KH}_\infty} \left\| R\tilde{G}G_\Delta \right\|_\infty \left\| (R\tilde{G}K)^{-1} \right\|_\infty \geq \inf_{R \in \mathcal{KH}_\infty} \left\| (GK)^{-1} R^{-1} R\tilde{G}G_\Delta \right\|_\infty
\]

\[
= \left\| (\tilde{G}K)^{-1} \tilde{G}G_\Delta \right\|_\infty.
\]

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Using the definition of the robustness ratio \( r(P, P_\Delta, C) \), the following corollary to \([LP09, \text{Theorem 7}]\) describes the set of plants guaranteed to be robustly stable given a nominal closed-loop system.

**Corollary 4.2.1.** Given a plant \( P \in \mathbb{R}^{p \times q} \), a controller \( C \in \mathbb{R}^{q \times p} \), define normalized graph symbols \( G, \tilde{G}, G_\Delta \) as in Definition 2.6.4 and \( r(P, P_\Delta, C) \) as in Definition 4.2.3. Then the following set of feedback systems is internally stable, where \( P_\Delta \in \mathbb{R}^{p \times q} \):

\[
P_{cf} := \{ P_\Delta \in \mathbb{R}^{p \times q} : r(P, P_\Delta, C) < 1, \sigma(\tilde{G}G_\Delta)(\infty) < 1, \text{wnodet}(G_\Delta^*G) = 0 \}.
\]

Furthermore \( \mathcal{P}_v \subseteq \mathcal{P}_{cf} \), where

\[
\mathcal{P}_v := \{ P_\Delta \in \mathbb{R}^{p \times q} : b(P, C) < \delta_v(P, P_\Delta), \sigma(\tilde{G}G_\Delta)(\infty) < 1, \text{wnodet}(G_\Delta^*G) = 0 \}.
\]

**Remark 4.2.1.** Independently of \( C \), \( \mathcal{P}_v \subseteq \mathcal{P}_{cf} \), and the difference between the two sets will be especially marked when the infimum of the frequency-wise four-block robust stability margin \( \sigma \left[ (\tilde{G}K)^{-1} \right]^{-1}(j\omega) \) occurs in a different channel and/or frequency region than the maximum frequency-wise four-block distance measure \( \sigma(\tilde{G}G_\Delta)(j\omega) \).

### 4.3 Coprime Factor Uncertainty and Coprime Factor Performance

In typical control problems, robust performance is at least as important as robust stability. This section describes bounds on the robust performance degradation under perturbation (in a fractional form). A corollary to \([LP09, \text{Theorem 9}]\) shows that the performance of the perturbed system is bounded by the robustness ratio \( r(P, P_\Delta, C) \). The generalized plant \( H \) for a fractional interconnection with left
coprime factor uncertainty as in Fig. 3.3 is given by

\[
H := \begin{bmatrix}
\tilde{M}_0^{-1} & P \\
0 & I \\
\tilde{M}_0^{-1} & P
\end{bmatrix},
\]

(4.4)

and \( H_\Delta \) is obtained by replacing \( \tilde{M}_0 \) by \( \tilde{M}_\Delta \).

**Corollary 4.3.1.** Given the suppositions of Corollary 4.2.1 and given \( P_\Delta \in \mathcal{P}_{cf} \) as defined in (4.3), then

\[
1 - r(P, P_\Delta, C) \leq b_{cf}(P_\Delta, C) \leq 1 + r(P, P_\Delta, C),
\]

(4.5)

\[
\frac{\| \mathcal{F}_I (H_\Delta, C) - \mathcal{F}_I (H, C) \|_\infty}{\| \mathcal{F}_I (H, C) \|_\infty} \leq \frac{\| (\tilde{G}_\Delta K)^{-1} \tilde{G} K - I \|_\infty}{1 - r(P, P_\Delta, C)}.
\]

(4.6)

Referring back to Corollaries 4.2.1 and 4.3.1, it is clear that \( r(P, P_\Delta, C) < 1 \)—together with the winding number condition—is a sufficient condition for robust stability of \([P_\Delta, C]\), and that a small value of \( r(P, P_\Delta, C) \) indicates that performance for \( P_\Delta \) will be very similar to nominal performance.

### 4.4 Coprime Factor Uncertainty and Four-block Performance

The same uncertainty structure with a left coprime factor plant description can also be analyzed using a normalized coprime factor performance measure. This corresponds to the setting shown in Fig. 4.1. Consider the following lemma, which will be useful for a subsequent proof.

**Lemma 4.4.1.** Given \( x, y \in [0, 1] \),

\[
\arcsin(xy) \leq \arcsin(x) \arcsin(y).
\]

Proof. Clearly \( \arcsin(x) \arcsin(y) - \arcsin(xy) = 0 \) for \( (x = 0, y \in [0, 1]) \) and
Figure 4.1: Coprime factor uncertainty with four-block performance measure.

\[(x \in [0, 1), y = 0).\] It then suffices to show that for \(x, y \in (0, 1),\)

\[\frac{\partial}{\partial x} \left[ \arcsin x \arcsin y - \arcsin xy \right] = \frac{1}{\sqrt{1-x^2}} \arcsin y - \frac{1}{\sqrt{1-x^2}y^2} y > 0.\]

This follows from the observation that \(\frac{1}{\sqrt{1-x^2}} > \frac{1}{\sqrt{1-x^2}y^2}\) and \(\arcsin y > y\) for \(x, y \in (0, 1).\) Similarly it can be shown that \(\frac{\partial}{\partial y} \left[ \arcsin x \arcsin y - \arcsin xy \right] > 0\) for \(x, y \in (0, 1).\)

Using the structure of Figure 4.1 as a basis, the following theorem bounds the ratio \(\frac{b(P, C)}{b(P, C)}\), i.e. the relative change in four-block robust stability margin when \(P\) is replaced by \(P_\Delta\) (under the assumption that robust stability of \([P_\Delta, C]\) is guaranteed). These new bounds are tighter than previous bounds obtained e.g. from [Vin01, Theorem 3.8].

**Theorem 4.4.2** (Four-block performance measurement under non-normalized coprime factor uncertainty). *Given the suppositions of Corollary 4.3.1, additionally define a robust stability margin for four-block uncertainty \(b(P, C)\) as in [GS90]. Then,

\[1 - \arcsin r(P, P_\Delta, C) \leq \frac{\arcsin b(P_\Delta, C)}{\arcsin b(P, C)} \leq 1 + \arcsin r(P, P_\Delta, C). \quad (4.7)\]

*Proof.* Letting \(z = \begin{bmatrix} z_1^* & z_2^* \end{bmatrix}^*, e = [e_1^* \ e_2^*]^* \) and \(d = [d_1^* \ d_2^*]^*,\) analysis of the
feedback loops of Fig. 4.1 reveals the following input-output relationships:

\[
\begin{bmatrix}
  e \\
  w
\end{bmatrix} =
\begin{bmatrix}
  Z_0 & -Y \\
  Z_0 & X
\end{bmatrix}
\begin{bmatrix}
  w \\
  d
\end{bmatrix},
\quad w = \Delta z,
\tag{4.8}
\]

where

\[
Z_0 := \begin{bmatrix}
  I \\
  C
\end{bmatrix} (I - PC)^{-1} M_0^{-1} = -K(\tilde{G}_0 K)^{-1},
\]

\[
X := \begin{bmatrix}
  I \\
  C
\end{bmatrix} (I - PC)^{-1} \begin{bmatrix}
  I & -P
\end{bmatrix} = K(\tilde{G} K)^{-1} \tilde{G},
\]

\[
Y := \begin{bmatrix}
  P \\
  I
\end{bmatrix} (I - CP)^{-1} \begin{bmatrix}
  -C & I
\end{bmatrix} = G(\tilde{K} G)^{-1} \tilde{K},
\]

and \(G, \tilde{G}, \tilde{G}_0, K\) and \(\tilde{K}\) are graph symbols of the plant and inverse graph symbols of the controller as defined in Definitions 2.6.4 and 2.6.5, respectively. From (4.8), note that

\[
e = \left[ X - Z_0 \Delta (I - Z_0 \Delta)^{-1} Y \right] d.
\]

Let \(X_\Delta := \left[ X - Z_0 \Delta (I - Z_0 \Delta)^{-1} Y \right] = K(\tilde{G}_\Delta K)^{-1} \tilde{G}_\Delta\). Then,

\[
X_\Delta = X - (I - Z_0 \Delta)^{-1} Z_0 \Delta Y \iff X_\Delta = X - Z_0 \Delta (X + Y - X_\Delta)
\]

\[
\iff X_\Delta = X - Z_0 \Delta Y_\Delta,
\tag{4.9}
\]

upon noting that \(X + Y = I\) and \(I - X_\Delta = Y_\Delta\). From (4.9), we have (via substitution of graph symbols)

\[
K(\tilde{G}_\Delta K)^{-1} \tilde{G}_\Delta = K(\tilde{G} K)^{-1} \tilde{G} + K(\tilde{G}_0 K)^{-1} \Delta G_\Delta (\tilde{K} G_\Delta)^{-1} \tilde{K},
\]

which implies

\[
(\tilde{G}_\Delta K)^{-1} \tilde{G}_\Delta \tilde{K}^* = (\tilde{G} K)^{-1} \tilde{G} \tilde{K}^* - (\tilde{G}_0 K)^{-1} \tilde{G}_0 G_\Delta (\tilde{K} G_\Delta)^{-1},
\tag{4.10}
\]

by pre-multiplying by \(K^*\), post-multiplying by \(\tilde{K}^*\) and substituting
\[ \Delta G_{\Delta} = -\tilde{G}_0 G_{\Delta} \] [LP09, (16)]. Taking singular values, (4.10) implies that

\[ \bar{\sigma}((G_{\Delta} K)^{-1}\tilde{G}_{\Delta} \tilde{K}^*) \leq \sigma((\tilde{G} K)^{-1}\tilde{G} \tilde{K}^*) + \frac{\bar{\sigma}(\tilde{G}_0 G_{\Delta})}{\sigma(\tilde{G}_0 K) \sigma(\tilde{K} G_{\Delta})}, \]

which is equivalent to

\[ \frac{\sqrt{1 - \sigma(\tilde{G}_0 K)^2}}{\sigma(\tilde{G}_0 K)} \leq \frac{\sqrt{1 - \sigma(\tilde{G}_0 K)^2}}{\sigma(\tilde{G}_0 K)} + \frac{\bar{\sigma}(\tilde{G}_0 G_{\Delta})}{\sigma(\tilde{G}_0 K) \sigma(\tilde{G}_0 K)}, \] (4.11)

via [Vin01, Lemma 2.2, ii]). Using the notation \( \alpha = \sigma(\tilde{G}_0 K), \beta = \sigma(\tilde{G} K), \gamma = \frac{\sigma(\tilde{G}_0 G_{\Delta})}{\sigma(\tilde{G}_0 K)} \sigma(\tilde{G} K), \) \( \hat{\alpha} = \arcsin \alpha, \hat{\beta} = \arcsin \beta, \) and \( \hat{\gamma} = \arcsin \gamma, \) (4.11) can be equivalently reformulated as

\[ \beta \sqrt{1 - \alpha^2} - \alpha \sqrt{1 - \beta^2} \leq \gamma \]

\[ \Leftrightarrow \begin{cases} \sin (\hat{\beta} - \hat{\alpha}) \leq \sin (\hat{\gamma}), \\ \sin (\hat{\alpha} - \hat{\beta}) \leq \sin (\hat{\gamma}), \end{cases} \]

\[ \Leftrightarrow \begin{cases} \arcsin \beta - \arcsin \alpha \leq \arcsin \gamma, \\ \arcsin \alpha - \arcsin \beta \leq \arcsin \gamma. \end{cases} \] (4.12)

Reverting to graph symbol notation, (4.12) is equivalently expressed as

\[ \begin{cases} \arcsin \sigma(\tilde{G} K) - \arcsin \sigma(\tilde{G}_0 K) \leq \arcsin \left( \frac{\sigma(\tilde{G}_0 G_{\Delta})}{\sigma(\tilde{G}_0 K)} \sigma(\tilde{G} K) \right), \\ \arcsin \sigma(\tilde{G}_0 K) - \arcsin \sigma(\tilde{G} K) \leq \arcsin \left( \frac{\sigma(\tilde{G}_0 G_{\Delta})}{\sigma(\tilde{G}_0 K)} \sigma(\tilde{G} K) \right). \end{cases} \] (4.13)

By assumption, \( d_{\text{lcl}}^R(P, P_{\Delta}) < b_{\text{lcl}}^R(P, C) \Rightarrow \bar{\sigma}(\tilde{G}_0 G_{\Delta}) \cdot (\sigma(\tilde{G}_0 K))^{-1} < 1. \) Also,
\( \sigma(\tilde{G}K) < 1 \) since \( b(P, C) < 1 \). Hence, via Lemma 4.4.1, (4.13) implies that

\[
\begin{align*}
\arcsin \sigma(\tilde{G}_\Lambda K) & \geq \arcsin \sigma(\tilde{G}K) \left( 1 - \arcsin \frac{\hat{\sigma}(\tilde{G}_0 G_\Lambda)}{\hat{\sigma}(\tilde{G}_0 K)} \right), \\
\arcsin \sigma(\tilde{G}_\Lambda K) & \leq \arcsin \sigma(\tilde{G}K) \left( 1 + \arcsin \frac{\hat{\sigma}(\tilde{G}_0 G_\Lambda)}{\hat{\sigma}(\tilde{G}_0 K)} \right),
\end{align*}
\]

\[\Rightarrow \]

\[
\begin{align*}
\arcsin \inf_\omega \sigma(\tilde{G}_\Lambda K) & \geq \arcsin \inf_\omega \sigma(\tilde{G}K) \left( 1 - \arcsin \frac{\sup_\omega \hat{\sigma}(\tilde{G}_0 G_\Lambda)}{\inf_\omega \hat{\sigma}(\tilde{G}_0 K)} \right), \\
\arcsin \inf_\omega \sigma(\tilde{G}_\Lambda K) & \leq \arcsin \inf_\omega \sigma(\tilde{G}K) \left( 1 + \arcsin \frac{\sup_\omega \hat{\sigma}(\tilde{G}_0 G_\Lambda)}{\inf_\omega \hat{\sigma}(\tilde{G}_0 K)} \right),
\end{align*}
\]

from which (4.7) follows from the definitions of \( b(P, C) \), \( b(P_\Delta, C) \), \( b^R_{\text{lcf}}(P, C) \), \( d^R_{\text{lc}}(P, P_\Delta) \) and \( r(P, P_\Delta, C) \).

A slightly more conservative formulation without trigonometric functions can also be stated.

**Theorem 4.4.3.** Given the suppositions of Theorem 4.4.2. Then,

\[
1 - r(P, P_\Delta, C) \leq \frac{b(P_\Delta, C)}{b(P, C)} \leq 1 + r(P, P_\Delta, C).
\] (4.14)

**Proof.** The proof follows Theorem 4.4.2, making use of [Vin01, Corollary 3.2].

**Remark 4.4.1.** The bounds provided by Theorem 4.4.3 are tighter bounds on \( b(P_\Delta, C) \) than those that can be obtained via standard \( \nu \)-gap theory, e.g. [Vin01, Theorem 3.8], since

\[
|b(P_\Delta, C) - b(P, C)| \leq \delta_\nu(P, P_\Delta)
\]

\[\Leftrightarrow 1 - \frac{\delta_\nu(P, P_\Delta)}{b(P, C)} \leq \frac{b(P_\Delta, C)}{b(P, C)} \leq 1 + \frac{\delta_\nu(P, P_\Delta)}{b(P, C)}
\]

and \( r(P, P_\Delta, C) \leq \frac{\delta_\nu(P, P_\Delta)}{b(P, C)} \). The arcsin result of Theorem 4.4.2 is also tighter than the corresponding arcsin result in \( \nu \)-gap theory [Vin01, Theorem 3.8].
4.5 Controller Synthesis Using a Coprime Factor Uncertainty Structure

In the previous sections, the importance of the robustness ratio $r(P, P_\Delta, C)$ for bounding deviations in the robust stability margin and the performance given a nominal plant $P$, a perturbed plant $P_\Delta$ and a controller $C$ was demonstrated. These insights can be exploited for controller synthesis, resulting in a controller which is particularly robust with respect to the perturbations embodied by a specific plant $P_\Delta$. Consider the following synthesis problem.

Optimal controller synthesis for the robustness ratio. Given plants $P, P_\Delta \in \mathbb{R}^{p \times q}$ with graph symbols $\tilde{G}$ and $G_\Delta$ as defined in Definition 2.6.4, define $r(P, P_\Delta, C)$ as in Definition 4.2.3. Let $\gamma_0 = \inf_{C \text{ stab}} r(P, P_\Delta, C)$ and $\gamma$ be any number $\gamma > \gamma_0$. Find a controller $C \in \mathbb{R}^{q \times p}$ such that

$$ [P, C] \text{ is internally stable and } r(P, P_\Delta, C) < \gamma. \quad (4.15) $$

This controller stabilizes $[P, C]$ and minimizes the robustness ratio $r(P, P_\Delta, C)$, with tight bounds on degradation in performance and the two robust stability margins $b(P, C)$ and $b_{cf}(P, C)|\left((\tilde{G}K)^{-1}\right)$ when $P$ is replaced by $P_\Delta$, as outlined in the previous two sections. However, an optimization of $r(P, P_\Delta, C)$ alone does not ensure good performance for the nominal feedback loop, or robust stability with respect to perturbations in frequency regions and/or channels in which $P_\Delta$ is very similar to $P$. By introducing $b(P, C) = \left\| (\tilde{G}K)^{-1} \right\|_\infty$, weighted by a positive scalar $\varepsilon$, as an additional optimization criterion, however, nominal performance and robustness in a normalized coprime factor neighbourhood around $P$ can be guaranteed. This enables a trade-off between nominal performance via $b(P, C)$ and similarly good performance for $P_\Delta$ as for $P$ via $r(P, P_\Delta, C)$. The combined optimization criterion can also be interpreted as a strategic deformation of the ball of four-block uncertainty around $P$ for which robust stability is guaranteed to include a specific $P_\Delta$.

The following theorem states the robust stability and performance guarantees

\footnote{It is well known that $b(P, C)$ can be related to classical concepts like gain and phase margin, and that $b(P, C) \approx 0.3$ often results in acceptable performance [GVP00, Lan05].}
obtained from the proposed optimization.

**Theorem 4.5.1.** Given $P, P_\Delta \in \mathbb{R}^{p \times q}$ along with graph symbols $G, \tilde{G}$ and $G_\Delta$ as defined in Definition 2.6.4, and given a generalized plant $H$ for $P$ and $H_\Delta$ for $P_\Delta$ as in (4.4), assume that $\sigma(\tilde{G}G_\Delta)(\infty) < 1$ and $\text{wdet}(G_\Delta^*G) = 0$. Choose $\varepsilon \in (0, b_{\text{opt}}(P))$. If $\exists C \in \mathbb{R}^{q \times p}$ such that $[P, C]$ is internally stable and

$$\|\begin{bmatrix} (\tilde{G}K)^{-1} \tilde{G} G_\Delta & \varepsilon (\tilde{G}K)^{-1} \end{bmatrix}\|_\infty < \gamma \leq 1.$$  \hspace{1cm} (4.16)

Then

(a) $b(P, C) > \frac{\varepsilon}{\gamma}$;

(b) $b_{\text{ref}}^R(P, C)|_{R=(\tilde{G}K)^{-1}} = 1$;

c) $[P_\Delta, C]$ is internally stable;

d) $b(P_\Delta, C) > b(P, C)(1 - \gamma)$;

e) $b_{\text{ref}}^R(P_\Delta, C)|_{R=(\tilde{G}K)^{-1}} > 1 - \gamma$.

Furthermore, if $\gamma < 1$, then

(f) $$\frac{\|\mathcal{F}_l(H_\Delta, C) - \mathcal{F}_l(H, C)\|_\infty}{\|\mathcal{F}_l(H, C)\|_\infty} \bigg|_{R=(\tilde{G}K)^{-1}} < \frac{\gamma}{1 - \gamma}. \hspace{1cm} (4.17)$$

**Proof.**

(a) This follows from noting that $b(P, C) = \| (\tilde{G}K)^{-1} \|_\infty^{-1}$, since $[P, C]$ is internally stable. From (4.16) it follows that $\| (\tilde{G}K)^{-1} \|_\infty < \frac{\varepsilon}{\gamma}$, from which a) is easily derived.

(b) This follows from simple substitution of $R = (\tilde{G}K)^{-1}$ into Definition 4.2.2.

c) Under the assumptions of this theorem, robust stability of $[P_\Delta, C]$ follows from $r(P, P_\Delta, C) < 1$ via Corollary 4.2.1.

d) This is a consequence of Theorem 4.4.3 and $r(P, P_\Delta, C) < \gamma$ via (4.16).

e) Follows from Corollary 4.3.1 upon noting that $r(P, P_\Delta, C) < \gamma$ via (4.16).
f) Follows from Corollary 4.3.1 upon noting that \( r(P, P_\Delta, C) < \gamma \) via (4.16).

**Remark 4.5.1.** While the robust stability guarantee in Theorem 4.5.1 is formulated for \([P_\Delta, C]\) only, the controller is guaranteed to stabilize the entire set of plants

\[
\{ \hat{P} \in \mathcal{R}_0^{p	imes q} : \text{wnodet}(\hat{G}^* G) = 0, \ (\hat{G}) (\infty) < 1, \ \| (\hat{G})^{-1} \hat{G}\|_\infty < 1 \},
\]

via an argument as in part a) of the proof for each \( \hat{P} \). The set (4.18) includes as a subset \( \{ P' \in \mathcal{R}_0^{p	imes q} : \delta_v(P, P') \leq \frac{\varepsilon}{\gamma} \} \), i.e. the set of plants that are within a distance of \( \frac{\varepsilon}{\gamma} \) as measured by the \( \nu \)-gap, since from statement a), \( b(P, C) > \frac{\varepsilon}{\gamma} \).

But, as pointed out in the previous sections, the stability guarantees obtained by using non-normalized coprime factors are less conservative than those obtained using \( \nu \)-gap theory. In particular, the set (4.18) may contain plants for which \( \delta_v(P, P_\Delta) > b_{\text{opt}}(P) \).

Based on the above theorem, the following optimization problem is proposed.

**Controller synthesis using both coprime factor and four-block robustness measures.** Given plants \( P, P_\Delta \in \mathcal{R}_0^{p	imes q} \) with graph symbols \( \hat{G} \) and \( G_\Delta \) as defined in Definition 2.6.4, choose \( \varepsilon \in (0, b_{\text{opt}}(P)) \). Find a controller \( C \in \mathcal{R}_0^{q \times p} \) such that \([P, C]\) is internally stable which satisfies

\[
\| \left[ \begin{array}{cc} (\hat{G})^{-1} \hat{G}_\Delta & \varepsilon (\hat{G})^{-1} \end{array} \right] \|_\infty < \gamma,
\]

where \( \gamma > \gamma_0 \) and

\[
\gamma_0 = \inf_{C_{\text{stab}}} \| \left[ \begin{array}{cc} (\hat{G})^{-1} \hat{G}_\Delta & \varepsilon (\hat{G})^{-1} \end{array} \right] \|_\infty. \tag{4.19}
\]

The next theorem collects all the technical machinery that is required to solve the \( \mathcal{H}_\infty \) norm problem formulated above, using the linear matrix inequality (LMI) approach (see Section 2.5.2 and also [GA94]) via an augmented state-space realization for the problem.
Theorem 4.5.2. Given $P, P_{\Delta} \in \mathbb{R}^{p \times q}$ with graph symbols $\tilde{G}, G_{\Delta}$ and minimal state-space realizations $P = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}$ and $P_{\Delta} = \begin{bmatrix} A_{\Delta} & B_{\Delta} \\ C_{\Delta} & D_{\Delta} \end{bmatrix}$, and given $\varepsilon \in (0, b_{\text{opt}}(P))$, define

\[ R_{\Delta} := I + D_{\Delta}^* D_{\Delta}; \quad \tilde{R}_{\Delta} := I + D_{\Delta} D_{\Delta}^*; \]
\[ X_{\Delta} := \text{Ric} \begin{bmatrix} A_{\Delta} - B_{\Delta} R_{\Delta}^{-1} D_{\Delta}^* C_{\Delta} & -B_{\Delta} R_{\Delta}^{-1} B_{\Delta}^* \\ -C_{\Delta}^* \tilde{R}_{\Delta}^{-1} C_{\Delta} & -(A_{\Delta} - B_{\Delta} R_{\Delta}^{-1} D_{\Delta}^* C_{\Delta})^* \end{bmatrix}; \]
\[ F_{\Delta} := -R_{\Delta}^{-1} (B_{\Delta}^* X_{\Delta} + D_{\Delta}^* C_{\Delta}); \]

Furthermore, define an augmented generalized plant $H_e(s)$ with the following state-space realization:

\[ H_e(s) := \begin{bmatrix} A_H & B_{H1} & B_{H2} \\ C_{H1} & D_{H11} & D_{H12} \\ C_{H2} & D_{H21} & D_{H22} \end{bmatrix} \begin{bmatrix} A_0 & -B_0 F_{\Delta} & -B_0 R_{\Delta}^{-1/2} & 0 & -\varepsilon B_0 & B_0 \\ 0 & A_\Delta + B_\Delta F_{\Delta} & B_{\Delta} R_{\Delta}^{-1/2} & 0 & 0 & 0 \\ C_0 & C_\Delta + D_\Delta F_{\Delta} & D_{\Delta} R_{\Delta}^{-1/2} & \varepsilon I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ C_0 & C_\Delta + D_\Delta F_{\Delta} & D_{\Delta} R_{\Delta}^{-1/2} & \varepsilon I & 0 & 0 \end{bmatrix}. \]

Then there exists a suboptimal controller $C \in \mathbb{R}^{q \times p}$ with graph symbol $K$ such that $[P, C]$ is internally stable and

\[ \| (\tilde{G}K)^{-1} \tilde{G} G_{\Delta} \varepsilon (\tilde{G}K)^{-1} \|_\infty < \gamma \]
if and only if there exist symmetric matrices $R, S$ satisfying

\[
\begin{bmatrix}
A^HR + RA^*_H - \gamma B^*H2B^*_H & RC^*_H & B^*_H \\
C^*_H & -\gamma I & D^*_H \\
B^*_H & D^*_H & -\gamma I
\end{bmatrix} < 0, \quad (4.22)
\]

\[
\begin{bmatrix}
A^*HS + SA^*_H - \frac{\gamma}{\varepsilon} C^*_H C^*_H & S\hat{B}^*_H1 - \frac{\gamma}{\varepsilon^2} C^*_H1 D^*\Delta R^*_{\Delta} & 0 \\
S\hat{B}^*_H1 - \frac{\gamma}{\varepsilon^2} C^*_H1 D^*\Delta R^*_{\Delta} & I + \frac{1}{\varepsilon^2} D^*_{\Delta} D^* \Delta R^*_{\Delta} & 0 \\
0 & 0 & I
\end{bmatrix} < 0, \quad (4.23)
\]

\[
\begin{bmatrix}
R & I \\
I & S
\end{bmatrix} \geq 0, \quad (4.24)
\]

where

\[
\hat{B}^*_H1 := \begin{bmatrix}
-B^*_{0\Delta} & -\varepsilon B^*_0 \\
B^*_{\Delta \Delta} & 0
\end{bmatrix}.
\]

**Proof.** It can be shown by straightforward calculation that $H_e(s)$ as given in (4.20) is an augmented state-space for the optimization problem (4.21), i.e.

\[
\mathcal{F}_l(H_e(s), C) = \begin{bmatrix}(\tilde{G}K)^{-1} \tilde{G}G_\Delta & \varepsilon (\tilde{G}K)^{-1}\end{bmatrix},
\]

by assuming without loss of generality that $D_0 = 0$.\(^5\) The LMIs (4.22)-(4.24) follow from [GA94, (6.1)-(6.3)] via the choice

\[
\mathcal{N}_R := \begin{pmatrix} I & 0 \\ 0 & I_p \end{pmatrix}; \quad \mathcal{N}_S := \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\frac{1}{\varepsilon} C^*_H & -\frac{1}{\varepsilon} D^* \Delta R^*_{\Delta} & 0 \\
0 & 0 & I_q
\end{pmatrix}.
\]

Finally, stabilizability of $(A^*_H, B^*_H)$ and detectability of $(C^*_H, A^*_H)$ are given whenever the state-space realizations of $P$ and $P^\Delta$ are minimal, as assumed in this theorem.

\(^5\)See Lemma 4.5.3 for a note on why this assumption can easily be made.
The corresponding suboptimal controller is reconstructed from the solutions \( R, S \) of the LMI’s (4.22)-(4.24) via an algorithm described in [Gah96]. The formulae are collected in the lemma below.

**Lemma 4.5.3.** Given \( P, P_{\Delta} \in \mathbb{R}^{p \times q} \) with minimal state-space realizations

\[
P = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}, \quad P_{\Delta} = \begin{bmatrix} A_{\Delta} & B_{\Delta} \\ C_{\Delta} & D_{\Delta} \end{bmatrix}
\]

and graph symbols \( \tilde{G} \) and \( G_{\Delta} \), and given \( \varepsilon \in (0, b_{\text{opt}}(P)) \), define \( R_{\Delta}, \tilde{R}_{\Delta}, X_{\Delta}, F_{\Delta} \) and

\[
\begin{bmatrix} A_H & B_{H1} \\ C_{H1} & D_{H11} \\ C_{H2} & D_{H12} \end{bmatrix}
\]

as in Theorem 4.5.2. Assume that real symmetric solutions \( R, S \) to the inequalities (4.22)-(4.24) exist. Then a suboptimal controller \( C \in \mathbb{R}^{q \times p} \) stabilizing \( [P, C] \) and satisfying

\[
\left\Vert \left[ \left( \tilde{G}K \right)^{-1} \tilde{G}_{\Delta} \varepsilon \left( \tilde{G}K \right)^{-1} \right] \right\Vert_\infty < \gamma
\]

is given by

\[
C = C' \left( I + D_0 C' \right)^{-1}
\]

where

\[
C' = \begin{bmatrix} A_{C'} & B_{C'} \\ C_C' & 0 \end{bmatrix}
\]

with

\[
A_{C'} := - (I - SR)^{-1} \left[ SB_{H2} \begin{bmatrix} -\gamma B_0^* & 0 \end{bmatrix} - \Gamma C_{H2} R + SA_{H} R + A_{H} + \left[ (SB_{H1} - \Gamma^* D_{H21}) \right]^* \Delta^{-1} \begin{bmatrix} B_{H1}^* \\ C_{H1} R - D_{H12} \begin{bmatrix} \gamma B_0^* & 0 \end{bmatrix} \end{bmatrix} \right]
\]

\[
B_{C'} := - (I - SR)^{-1} \Gamma
\]

\[
C_{C'} := \begin{bmatrix} -\gamma B_0^* & 0 \end{bmatrix}
\]
and $\Delta := \begin{bmatrix} \gamma I & -D_{H1}^* \\ -D_{H1} & \gamma I \end{bmatrix}$, $\Gamma := (SB_{H1} + \gamma C_{H2}^* (D_{H21} D_{H21}^*)^{-1}$.

Proof. The state-space matrices (4.25)-(4.27) follow from [Gah96, Algorithm 3.1], upon noting that $D_1 = 0$ minimizes $\sigma (D_{H11} + D_{H12} D_1 D_{H21})$, implying that $D_{C'} = 0$. The loop shifting operation $C = C' (I + D_0 C')^{-1}$ is a consequence of the assumption in the LMI method that $D_{H22} = D_0 = 0$.

The order of the controller is the sum of the order of $P$ and $P_\Delta$. This is a consequence of optimizing with respect to both a nominal plant and a perturbation embodied in $P_\Delta$.

### 4.5.1 Synthesis Procedure

The following synthesis procedure makes use of Theorem 4.5.2 and Lemma 4.5.3 to arrive at a controller with good nominal performance and robustness properties. Note that this procedure is indispensable when $\delta_\nu (P, P_\Delta) \geq b_{\text{opt}} (P)$ and hence standard $\nu$-gap theory cannot guarantee the existence of a robustly stabilizing controller.

1. Define a nominal plant $P$ with a minimal state-space realization.

2. Obtain a “worst-case” perturbed plant $P_\Delta$ with minimal state-space realization for which the controller is meant to be robustly stabilizing. There are several possible ways to obtain such a perturbed plant:

   a A particular $P_\Delta$ might be known to the designer.

   b Based on information about the expected size of the uncertainty in every channel and frequency range, an uncertainty block $\Delta$ can be generated, from which one can obtain a perturbed plant $P_\Delta = (N + \Delta_N) (M + \Delta_M)^{-1}$, based on a right coprime factorization $[N, M]$ of the nominal plant $P$.

   c Given a family of perturbed plants $P_{\Delta, i}$, the designer can compute a worst-case point-wise distance

   $$\kappa_{wc}(P, P_{\Delta, i})(j \omega) := \max_i \{ \kappa(P, P_{\Delta, i})(j \omega) \}$$
between the nominal plant $P$ and the family of perturbed plants, where
\[ \kappa(P,P_{\Delta,i})(j\omega) = \sigma(\tilde{G}_{\Delta})(j\omega) \]
is the point-wise \( \nu \)-gap. Note that this imparts some degree of conservativeness due to the use of singular values. A more precise method would be to carry out this procedure for every channel separately.

3. Compute \( b_{\text{opt}}(P) \) and
\[
\gamma_{\text{opt}} := \inf_{C \text{ stab.}} r(P,P_{\Delta},C) = \inf_{C \text{ stab.}} \left\| (\tilde{G}K)^{-1} \tilde{G}_{\Delta} \right\|_{\infty}
\]
via an interval search using a LMI solver.\(^6\) If \( \gamma_{\text{opt}} > 1 \), a less “ambitious” \( P_{\Delta} \) must be chosen, since robust stability cannot be guaranteed via Theorem 4.5.1.

4. Choose \( \varepsilon = 0.9 \frac{b_{\text{opt}}(P)}{\gamma_{\text{opt}}} \) (or indeed a lower number if \( b_{\text{opt}}(P) \) is relatively large).

5. Compute
\[
\gamma = \inf_{C \text{ stab.}} \left\| \left[ (\tilde{G}K)^{-1} \tilde{G}_{\Delta} \quad \varepsilon (\tilde{G}K)^{-1} \right] \right\|_{\infty}
\]
using an LMI solver and Theorem 4.5.2. The controller \( C \) (reconstructed via Lemma 4.5.3) which achieves the approximate infimum guarantees nominal performance as well as robust stability and robust performances as outlined in Theorem 4.5.1. If \( \gamma > 1 \), return to the previous step and choose a lower \( \varepsilon \). Iterate until \( \gamma \leq 1 \).

### 4.6 Numerical Examples

In this section, the synthesis methodology described in the previous section will be applied to two numerical examples to demonstrate the easy applicability and the resulting balance of robustness and performance. The first example is a physically motivated single-input, single-output (SISO) robust motion control problem which has been proposed as a benchmark example in [WB92] (see also [Vin01]). The second example is an expanded multi-input, multi-output (MIMO) example.

\(^6\)Note that this only requires a simplified version of the state-space of \( H_e(s) \) of Theorem 4.5.2.
This serves to demonstrate the applicability of the synthesis method to MIMO systems with significant coupling between channels.

### 4.6.1 SISO Motion Control Problem

The synthesis methodology described in the previous section is here applied to a physically motivated robust motion control problem which has been proposed as a benchmark example in [WB92] (c.f. [HS93, Vin01]). The system consists of two masses $M_1, M_2$, coupled by a spring of stiffness $k$, with the assembly sliding on a frictionless table. The input $u$ is a force applied to $M_1$, the output $x$ is the displacement of $M_1$. The nominal transfer function is:

$$P_0(s) = \frac{x(s)}{u(s)} = \frac{M_2s^2 + k}{s^2(M_1M_2s^2 + (M_1 + M_2)K)}.$$

Slight changes in the location of the poles and zeros of this system can cause very large distances as measured by the $\nu$-gap [Vin01]. Consider the nominal system $P_0(s)$ ($M_1 = M_2 = 1$kg, $k = 1\frac{N}{m}$, sensor gain of 10) and the perturbed system $P_1(s)$, respectively defined as

$$P_0(s) = \frac{10(s^2 + 1)}{s^2(s^2 + 2)}, \quad P_1(s) = \frac{10(s^2 + 1.1)}{s^2(s^2 + 2)}.$$

The Bode plots of these transfer functions are displayed in Fig. 4.3. The $\nu$-gap between both systems is $\delta_\nu(P_0, P_1) = 0.8012$, i.e. very large compared to the optimal normalized coprime factor robust stability margin $b_{opt}(P_0) = 0.3919$. Robust stability of $[P_1, C]$ cannot be guaranteed using four-block methods since $\delta_\nu(P_0, P_1) > b_{opt}(P_0)$. And indeed, the controller achieving $b(P_0, C) = b_{opt}(P_0)$ does not stabilize $[P_1, C]$ [Vin01]. Using Matlab’s “mincx” LMI solver, The-
Theorem 4.5.2 with $\varepsilon = 0.3$ is used to obtain a feasible robustly stabilizing controller, with $\gamma \approx 0.8874$. The Bode plot of the controller is displayed in Fig. 4.3. Theorem 4.5.1 provides a number of performance guarantees for this controller. It achieves a normalized coprime factor robust stability margin of $b(P_0, C) = 0.3385 > \frac{\varepsilon}{\gamma}$ for the nominal plant, with that for the perturbed system being only marginally smaller ($b(P_1, C) = 0.3360$). While robustly stabilizing controllers for these single-input, single-output plants could also be obtained using other methods, this example demonstrates that the proposed synthesis method can strategically extend the robust stability neighbourhood around $P$ far beyond the normalized coprime factor optimum, while maintaining good $b(P_0, C)$ and near-nominal performance for $P_1$.

Its transfer function is given by

$$C(s) = \frac{0.072s^7 + 0.398s^6 + 1.356s^5 + 2.222s^4 + 3.790s^3 + 3.468s^2 + 2.674s + 1.601}{10^{-6}s^8 + 0.026s^7 + 0.311s^6 + 1.133s^5 + 2.704s^4 + 2.873s^3 + 6.686s^2 + 2.019s + 4.450}$$

Note that the order could be reduced easily by standard model order reduction.

Lead-lag controllers in particular provide good robustness in this case.

Figure 4.3: Bode plots of $P_0$, $P_1$ and of the optimal controller $C$. 

-100
-50
0
50
100
Magnitude (dB)

-180
-90
0
90
180
270
Phase (deg)

10
10
10
10
Frequency (rad/sec)

-180
0
Phase (deg)

$P_0$
$P_1$
$C$
### 4.6.2 MIMO Motion Control Problem

The mechanical systems presented above and depicted in Fig. 4.2 can also be considered as a MIMO motion control problem, with force inputs $u_1$ and $u_2$ acting on masses $M_1$ and $M_2$, respectively, and two position outputs $x_1$ and $x_2$. Setting all nominal values to unity (with appropriate units as above) and reducing the coupling between the channels (by reducing the magnitude of the off-diagonal elements of the transfer-function matrix), the $2 \times 2$ nominal transfer function matrix from $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ to $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is

$$P_0(s) = \begin{bmatrix} s^2 + 1 & 0.25 \\ 0.25 & s^2 + 1 \end{bmatrix} \frac{1}{s^2(s^2 + 2)}.$$

A small deviation in the stiffness parameter $k$ causes a large distance as measured by the $\nu$-gap. Consider the perturbed system

$$P_1(s) = \begin{bmatrix} s^2 + 0.9 & 0.225 \\ 0.225 & s^2 + 0.9 \end{bmatrix} \frac{1}{s^2(s^2 + 2)}.$$

The Bode plots for both $P_0(s)$ and $P_1(s)$ are displayed in Fig. 4.4. It can be seen that the location of both the poles and the zeros of the system changes as a consequence of the perturbation. Again, the aim is to design a controller that stabilizes $[P_0, C]$ and $[P_1, C]$, and to ensure some level of nominal performance. The $\nu$-gap is computed as $\delta_\nu(P_0, P_1) = 0.5498$, which is larger than the optimal four-block robust stability margin of $b_{opt}(P_0) = 0.3916$, and therefore makes a straightforward application of four-block methods impossible.

As in the SISO example, the synthesis method described in Section 4.5.1 is applied, starting at Step 3. The optimal robustness ratio for this case is

$$\gamma_{opt} = \inf_{C \text{ stab}} r(P, P_\Delta, C) \approx 0.5498,$$

indicating that a robustly stabilizing controller for $[P_1, C]$ exists among the controllers stabilizing $[P_0, C]$. The next step consists of choosing a suitable $\varepsilon$. The recommended start value of $\varepsilon = 0.9 \frac{b_{opt}(P_0)}{\gamma_{opt}} \approx 0.7$ results in an infeasible combined
optimisation in the next step ($\gamma > 1$). This is an indication that a small robustness ratio $r(P_0, P_1, C)$—indicative of small degradation of performance and four-block stability margin—and a large nominal four-block stability margin $b(P_0, C)$ are not achievable at the same time for this particular perturbed plant. In iterative steps, $\varepsilon$ is reduced and $\gamma$ recomputed. A value of $\varepsilon = 0.25$ results in a good combination of $r(P_0, P_1, C) = 0.6142$ and $b(P_0, C) = 0.3703$, with $\gamma = 0.6752$.

Bode plots of the controller $C$ and of the controller optimizing only the robustness ratio, $C_r$, are depicted in Fig. 4.5. The controller $C$ shows some basic lead-lag features in all channels, augmented with some stronger features around $\omega = 1$, i.e. near the frequency of the shifted poles. Additional insights into the robustness properties of the two controllers can be gained from Fig. 4.6. The frequency-wise equivalent of the four-block robust stability margin, $\rho(P_0, C)(j\omega)$, has a peak around the frequency region where the uncertainty is largest as seen by the peak in both the frequency-wise equivalent of the $\nu$-gap, $\kappa(P_0, P_1)(j\omega)$, and of the robustness ratio. Finally, $\rho(P_0, C_r)(j\omega)$ drops to much lower minima than $\rho(P_0, C)(j\omega)$.

\textsuperscript{9}$C$ is of order 15. Its state-space realization is not given here for the sake of brevity.
\[ \rho(P_0, C)(j\omega) \]. This indicates that the controller \( C_r \) optimizing only the robustness ratio does not achieve as much robustness as \( C \) with respect to four-block uncertainty that is not modelled in the particular perturbed plant, i.e. \( P_1 \) in this case.

### 4.7 Conclusions

It has been shown that both robust stability/performance analysis and robust controller synthesis benefit from using coprime factorizations of uncertain plants that are not necessarily normalized. A strategically chosen denormalization factor \( R \) causes the ratio of coprime factor distance to coprime factor robust stability margin to be minimal, resulting in less conservative robust stability guarantees and tighter performance degradation bounds than can be obtained using the normalized coprime factor/four-block structure. The synthesis method proposed in this chapter takes advantages of these tighter bounds, and optimizes an objective function involving both the robustness ratio (to guarantee good robustness via the non-

Figure 4.5: Bode plots of controllers for the MIMO problem.
Figure 4.6: Pointwise distances and robust stability margins for the MIMO problem.

(normalized coprime factor theory) and the well-known normalized coprime factor robust stability margin (to guarantee nominal robustness/performance). The particular optimization method chosen in this chapter results in controllers of a high order (equal to the sum of the orders of the nominal and perturbed plants). However, it enables significantly more robust controller synthesis for lightly damped systems, where normalized coprime factor theory reaches limitations.
Chapter 5

Non-normalized Coprime Factor Uncertainty: A State-space Solution

5.1 Introduction

Like the previous chapter, this chapter considers the robust stabilization problem of plants with coprime factor uncertainty, where the coprime factors of the plant are not necessarily normalized. In contrast to the previous chapter, the emphasis here lies on a state space approach, which generalizes an observer-form controller formulation originating from the normalized coprime factor synthesis problem. While normalized coprime factor uncertainty is extremely versatile, it has been known for some time that it can be problematic for plants with uncertain lightly damped poles and zeros [HS93, Vin01, LP09, ELP11]. Even optimal robust controllers cannot be guaranteed to stabilize plants with even small changes in the location of particular lightly damped poles or zeros. In this chapter, lower bounds for the $\nu$-gap between multiple-input, multiple-output (MIMO) systems with such features are provided, showing that this is indeed a severe problem in the normalized coprime factor framework. Upper bounds on the robust stability margin in the presence of poles/zeros on the imaginary axis are also provided, showing that the problem compounds on both sides.

It was shown in the previous chapter that coprime factor uncertainty (not normalized) provides robust stability guarantees which may be less conservative than
the normalized case. This was exploited in Chapter 4 to synthesize robustly stabilizing controllers for combinations of a nominal and one or multiple perturbed plants. In this chapter, a comprehensive and systematic interpretation of the general coprime factor robust stabilization problem based on a state-space approach is provided. It is a well known fact that the central controller of the normalized coprime factor stabilization problem can be implemented in a simple observer-controller compensator form, characterized by a state-feedback matrix $F$ and an observer gain $L$ [ZDG96, GL00]. If one does not restrict the coprime factors to be normalized, one of these two matrices may be freely chosen by the designer (subject to a stability constraint), with the other being synthesized for optimal robustness. The approach taken herein is to allow the designer to choose a state-feedback $F$ and to then synthesize an optimally robust observer gain $L$. It is shown that the choice of $F$ induces a particular topology in which distance is measured, and subsequently $L$ determines how robust the resulting controller is in the topology induced by $F$. This provides enormous freedom to tailor the robustness optimization to particular uncertainty. This freedom is exploited in the final section of the chapter to obtain a state-feedback $F$ which places the poles of a coprime factorization of the plant within a circular region in the left-half plane, which induces a topology in which uncertain lightly damped poles/zeros can be more easily stabilized.

Consequently, this work also indicates that once a state feedback design is specified, the observer gain $L$ should be designed to robustify the state feedback control law to coprime factor uncertainty, unlike the common practice of choosing a Kalman filter with Kalman gain $L$ whenever an observer is required due to missing state measurements.

All plants in this chapter are assumed to be strictly proper, linear-time invariant systems. In the context of the $\mathcal{H}_\infty$ loopshaping procedure, the plants considered here are assumed to be shaped plants, i.e. with performance weights included, and are therefore denoted $P_s$. The controller in this chapter is $C_\infty$, i.e. the controller before the wrapping around of the loopshaping weights.

The remainder of this chapter is structured as follows. Section 5.2 reviews results and definitions on state-space realizations of coprime factors, distance measures and robust stability margins. Subsequently, Section 5.3 describes the diffi-
culties related to lightly damped uncertain systems in the \( \nu \)-gap metric via several theorems. Section 5.4 is the key section of this chapter, containing the main synthesis theorem and a number of remarks concerning its interpretation. Section 5.5 describes a method for obtaining coprime factorizations inducing topologies which are more suitable for handling uncertainty in lightly damped systems, while the final section contains a more comprehensive numerical example demonstrating the application of the method to a MIMO system.

### 5.2 Coprime Factors, Distance Measures and Robust Stability Margins

This section recalls results on the state-space realizations of coprime factors of rational transfer function matrices, as well as distance measures and robust stability margins for plants with coprime factor uncertainty characterization. It contains several key observations on the connections between the state-space realizations and observers and state-feedback. In contrast to Chapter 4 (see also [LP09, ELP11]), where the distance measures and robust stability margins for general coprime factor uncertainty were previously defined in operator terms, the notation in this chapter is updated to reflect the state-space approach.

Now recall the following theorem, which describes state-space realizations of left and right coprime factors of a system. The results in this chapter assume strictly proper plants for mathematical convenience, but this assumption is not restrictive since pre- and post-compensator weights in loopshaping are typically chosen such that the gain at high frequency approaches zero [MG90, MG92].

**Theorem 5.2.1.** [Vid84, ZDG96] Given \( P_s \in \mathbb{R}^{p \times q} \) with a stabilizable and detectable state-space realization

\[
P_s = \begin{bmatrix}
A & B \\
C & 0
\end{bmatrix},
\]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times q}, C \in \mathbb{R}^{p \times n}, \) let \( F \in \mathbb{R}^{q \times n} \) and \( L \in \mathbb{R}^{n \times p} \) be such that
$A + BF$ and $A + LC$ are Hurwitz. Define

$$
\begin{bmatrix}
N_0 \\
M_0
\end{bmatrix} :=
\begin{bmatrix}
A + BF & B \\
C & 0 \\
F & I
\end{bmatrix},
$$

(5.1)

$$
\begin{bmatrix}
-\tilde{M}_0 & \tilde{N}_0
\end{bmatrix} :=
\begin{bmatrix}
A + LC & -L & B \\
C & -I & 0
\end{bmatrix}.
$$

(5.2)

Then $\{N_0, M_0\}$ is a right coprime factorization and $\{\tilde{N}_0, \tilde{M}_0\}$ is a left coprime factorization of $P_s$, respectively.

The matrices $F$ in (5.1) and $L$ in (5.2) are free parameters (subject to the stability constraint) that induce a specific right and left coprime factorization, respectively. A particular choice of $F$ or $L$ will result in the coprime factorizations being normalized, as can be seen from the following theorem.

**Theorem 5.2.2.** Given $P_s \in \mathbb{R}^{p \times q}$ with state and output equations given by $\dot{x} = Ax + Bu$, $y = Cx$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times q}$, $C \in \mathbb{R}^{p \times n}$, assume $(A, B)$ is controllable and $(C, A)$ has no unobservable modes on the imaginary axis. Let $F = -B^TX$, where $X \geq 0$ is the stabilizing solution to

$$
XA + A^TX - XBB^TX + C^TC = 0.
$$

(5.3)

Then,

1. the right coprime factorization $\{N, M\}$ of $P_s$ given in (5.1) and induced by $F$ is a normalized right coprime factorization; and

2. the unique state-feedback $u = r + Fx$ applied to $P_s$ minimizes

$$
\|T_{zr}\|^2 = \left\|[N] \right\|^2,
$$

over all $F \in \mathbb{R}^{q \times n}$ for $z = [y^\ast \ (u - r)^\ast]^\ast$, with the resulting minimum cost given by $\|T_{zr}\|^2 = \text{trace}(B^*XB)$.  

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Figure 5.1: A plant with right coprime factor uncertainty.

**Proof.** 1. corresponds to [ZDG96, Theorem 13.37, a)] and 2. corresponds to the standard $\mathcal{H}_2$ result for state-feedback, see e.g. [ZDG96, Section 14.8.1].

**Remark 5.2.1.** The connection between normalized coprime factors and $\mathcal{H}_2$ optimal control is well known [ZDG96, MF87], see also [GL00, Section 10.5]. It is here restated formally to point out that the state-feedback matrix $F$ which normalizes the rcf of $P_s$ is essentially an $\mathcal{H}_2$-optimal state feedback. Later sections of this chapter will show that this particular choice of $F$ may under certain conditions compromise the achievable robustness of controllers optimized with respect to coprime factor uncertainty.

The corresponding result for normalized left coprime factorizations is omitted, as subsequent developments focus mostly on right coprime factorizations.

Consider now a plant with right coprime factor uncertainty, as shown in Fig. 5.1. This setup corresponds to the following equation for a perturbed plant $P_\Delta$, with $\{N_0,M_0\}$ being a rcf (not necessarily normalized) of the nominal plant $P_s$:

$$P_\Delta = (N_0 + \Delta_N)(M_0 + \Delta_M)^{-1}.$$

The following definition (see Chapter 4 and also [LP09, Section VII] and [ELP11]) of a generalized right coprime factor distance measure (between a nominal plant $P_s$ and a perturbed plant $P_\Delta$) is parameterized in terms of the rcf of a nominal plant induced by the matrix $F$. The perturbed plant $P_\Delta$ enters the definition via a normalized lcf, for which there is no free parameter as is clear from Theorem 5.2.2.

**Definition 5.2.1.** Given $P_s, P_\Delta \in \mathbb{R}^{p \times q}$ and $F \in \mathbb{R}^{q \times n}$, let $\tilde{G}_\Delta$ be the normalized left graph symbol of $P_\Delta$ and $G_0$ the not necessarily normalized right graph symbol
of \( P_s \) induced by \( F \) via (5.1). Define the right coprime factor distance measure as

\[
d_{\text{rcf}}(P_s, P_\Delta; F) := \|\tilde{G}_\Delta G_0\|_\infty.
\]

This distance measure reduces to the well known \( \nu \)-gap metric [Vin93, Vin01] if \( F \) is chosen to normalize the right coprime factorization of \( P_s \) as in Theorem 5.2.2. A distance measure is typically considered in conjunction with a robust stability margin which quantifies up to which distance from the nominal plant robust stability is guaranteed in a certain topology. See [Vin01, LP09, ELPP10, LEPP12, ELP11] and the previous chapters of this thesis for further remarks and robust stability and performance theorems for various uncertainty structures. Robust stability is ensured for \( P_\Delta \)'s with a distance less than the robust stability margin, which also fulfill a winding number constraint (see Theorem 3.7.3 for the robust stability theorem of general, not necessarily normalized right coprime factors). Consider a positive feedback interconnection of \( P_s \in \mathbb{R}^{p \times q} \) with state-space realization \( P_s = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \) and a control law \( u = C_\infty \begin{bmatrix} y \\ r \end{bmatrix} \), where \( C_\infty \) is implemented in observer form as in Fig. 5.2. The following state-space realization of \( C_\infty \) is illustrated in Fig. 5.2:

\[
C_\infty := \begin{bmatrix} A + BF + LC & -L & B \\ F & 0 & I \end{bmatrix}.
\]

(5.4)

Denote by \( \hat{C}_\infty \) the column of \( C_\infty \) corresponding to the transfer function from \( y \) to \( u \). This corresponds to the controller \( \hat{C}_\infty \) in Fig. 5.1. Full equivalence between Fig. 5.1 and Fig. 5.2 will be shown later (see Theorem 5.4.1).

Clearly, with \( P_s \) given, \( C_\infty \) depends only on the choices of \( F \) and \( L \). This implementation leads to the following definition of a generalized right coprime factor robust stability margin, for which only the \( \hat{C}_\infty \) subpart of the controller is relevant.

**Definition 5.2.2.** Given a positive feedback interconnection \( [P_s, C_\infty] \) of \( P_s \in \mathbb{R}^{p \times q} \) and a controller \( C_\infty \in \mathbb{R}^{q \times p} \) in observer form induced by given matrices \( F \in \mathbb{R}^{q \times n} \) and \( L \in \mathbb{R}^{n \times p} \) via (5.4), let \( \{N_0, M_0\} \) be the not necessarily normalized rcf of \( P_s \) induced by \( F \) as in (5.1). Define the right coprime factor robust stability margin
Figure 5.2: Observer form implementation of the controller.

of \([P_s, C_\infty]\) as

\[
  b_{\text{rcf}}(P_s; F, L) := \begin{cases} 
  \left\| M_0^{-1} (I - \hat{C}_\infty P)^{-1} \left[ I \quad \hat{C}_\infty \right] \right\|_\infty^{-1} & \text{if } [P, C_\infty] \text{ is internally stable;} \\
  0 & \text{otherwise.}
  \end{cases}
\]

**Remark 5.2.2.** It should be noted that the right coprime factor robust stability margin can also be defined for a generically structured \(C_\infty\), as is done in Chapter 4 (see also [LP09, ELP11]). The above formulation is chosen to highlight the impact of the state-feedback matrix \(F\) and observer-gain matrix \(L\) of the observercontroller compensator implementation. There exists a controller in this implementation for any achievable robust stability margin, as will be shown subsequently. Taking together the definition of the distance measure (Definition 5.2.1) with Definition 5.2.2, the distinct roles of \(F\) and \(L\) become apparent: \(F\) induces a right coprime factorization of \(P_s\), and thereby the topology in which distance is measured. The choice of \(L\) will then affect the robust stability margin of the feedback interconnection in the topology defined by \(F\).

**Remark 5.2.3.** If \(F\) is chosen such that the rcf of \(P_s\) is normalized, the right coprime factor robust stability margin reduces to the well known four-block/normalized coprime factor robust stability margin \(b(P_s, C_\infty)\) [GS90, Vin93, Vin01]. Typically, \(b(P_s, C_\infty)\) is defined using a lcf of \(P_s\). As is clear from Theorem 5.2.1, the observer-gain \(L\) induces a left coprime factorization. The definitions in this section can be mirrored for left coprime factor uncertainty [LP09, ELP11], but
this chapter deliberately opts for a right coprime formulation as induced by a
state-feedback $F$, for reasons laid out in Remark 5.4.2. Normalized coprime fac-
tor uncertainty has the advantage of an analytically computable optimal robust
stability margin $b_{opt}(P_s)$ [GM89]. For generalized coprime factor uncertainty, the
optimal stability margin can be computed via a line search.

5.3 Lack of Robustness to Uncertainty in Location
of Lightly Damped Poles and Zeros

Before setting out a controller synthesis theory for general, not necessarily nor-
malized coprime factor uncertainty in the following section, this section highlights
the problems that uncertain lightly damped poles and zeros can cause for robust
stability analysis and synthesis in a normalized coprime factor setting. The first
two theorems describe bounds on the $\nu$-gap when the nominal plant has a pair
of zeros and poles, respectively, on the imaginary axis. The subsequent the-
orems show that these lightly damped features also impose constraints on the
magnitude of the controller transfer function at the zero/pole frequency. To il-
lustrate these difficulties, a simple example will be used throughout this section,
which consists of a plant with uncertainty in the location of lightly damped zeros.
This example has frequently been used as a benchmark example in the litera-
ture [WB92, HS93, Vin01, ELP11].

Example 1. Consider the nominal plant

$$P_s = \frac{10 (s^2 + 1)}{s^2 (s^2 + 2)}. \quad (5.5)$$

A normalized rcf of $P_s$ is given by

$$\begin{bmatrix} N \\ M \end{bmatrix} = \frac{1}{s^4 + 4.332s^3 + 11.38s^2 + 5.256s + 10} \begin{bmatrix} 10 (s^2 + 1) \\ s^2 (s^2 + 2) \end{bmatrix}. $$

This coprime factorization has two pairs of complex conjugate poles, with one
of these pairs being extremely lightly damped (damping ratio $\zeta = 0.05$ at $\omega_n =
±1.0025. In terms of robustness, this is an undesirable effect of the requirement that the coprime factorization be normalized, as will be highlighted by the subsequent theorems in this section.

The following two theorems provide lower bounds on the ν-gap for plants with uncertain lightly damped zeros and poles, respectively. It will be seen that the ν-gap becomes very large even for small uncertainty if the uncertain zeros/poles are in particular frequency regions.

**Theorem 5.3.1.** Given a nominal plant \( P_s \in \mathbb{R}^{p \times q} \) with a pair of transmission zeros at \( s = \pm j \omega_0 \). Then for any \( P_\Delta \in \mathbb{R}^{p \times q} \),

\[
\delta_\nu(P_s, P_\Delta) \geq \sqrt{\frac{\sigma^2(P_\Delta(j \omega_0))}{1 + \sigma^2(P_\Delta(j \omega_0))}}. \tag{5.6}
\]

**Proof.** Assume, firstly, that \( P_s \) has full column normal rank. The case of full row normal rank is treated later. Let \( \{N, M\} \) be a normalized rcf of \( P_s \). Then

\[
\exists u_0 \neq 0 \in \mathbb{C}^q \text{ s.t. } P_s(j \omega_0)u_0 = 0.
\]

Furthermore, \( s = \pm j \omega_0 \) must also be a transmission zero of \( N \) as it cannot be a pole of \( M \in \mathcal{RH}_\infty \). Therefore,

\[
\exists z_0 \neq 0 \in \mathbb{C}^q \text{ s.t. } N(j \omega_0)z_0 = 0.
\]

For a normalized rcf, the following identity holds (see Definition 2.6.3):

\[
M(j \omega_0)^*M(j \omega_0) + N(j \omega_0)^*N(j \omega_0) = I,
\]

But \( \omega_0 \) is the frequency of a transmission zero in the direction implied by vector \( z_0 \), and therefore

\[
z_0^*M(j \omega_0)^*M(j \omega_0)z_0 + z_0^*N(j \omega_0)^*N(j \omega_0)z_0 = \|M(j \omega_0)z_0\|^2 = z_0^*z_0 = \|z_0\|^2.
\]

Without loss of generality, let \( \|z_0\| = 1 \). Recall from the definition of the ν-gap
that
\[
\delta_\nu(P_s, P_\Delta) \geq \sigma \left( \tilde{N}_\Delta(j\omega_0)M(j\omega_0) - \tilde{M}_\Delta(j\omega_0)N(j\omega_0) \right),
\]
\[
\geq \|\tilde{N}_\Delta(j\omega_0)M(j\omega_0)z_0 - \tilde{M}_\Delta(j\omega_0)N(j\omega_0)z_0\|,
\]
\[
= \|\tilde{N}_\Delta(j\omega_0)M(j\omega_0)z_0\| \geq \sigma \left( \tilde{N}_\Delta(j\omega_0) \right).
\]

The theorem statement then follows upon noting that for a normalized lcf \( \{\tilde{N}_\Delta, \tilde{M}_\Delta\} \) of \( P_\Delta \),
\[
\tilde{N}_\Delta(j\omega)\tilde{N}_\Delta(j\omega)^* = I - \tilde{M}_\Delta(j\omega)\tilde{M}_\Delta(j\omega)^*,
\] (5.7)

and that, with all eigenvalues of \( \tilde{M}_\Delta(j\omega)\tilde{M}_\Delta(j\omega)^* \) being positive,
\[
\sigma^2(\tilde{N}_\Delta(j\omega)) = \lambda (I - \tilde{M}_\Delta(j\omega)\tilde{M}_\Delta(j\omega)^*) = 1 - \lambda (\tilde{M}_\Delta(j\omega)\tilde{M}_\Delta(j\omega)^*)
\]
\[
= 1 - \sigma^2(\tilde{M}_\Delta(j\omega)).
\]

Furthermore, equivalently to (5.7),
\[
I + \tilde{M}_\Delta^{-1}(j\omega)\tilde{N}_\Delta(j\omega)\tilde{N}_\Delta^*(j\omega)\tilde{M}_\Delta^{-*}(j\omega) = \tilde{M}_\Delta^{-1}(j\omega)\tilde{M}_\Delta^{-*}(j\omega)
\]
\[
\Leftrightarrow P_\Delta(j\omega)P_\Delta^*(j\omega) = \tilde{M}_\Delta^{-1}(j\omega)\tilde{M}_\Delta^{-*}(j\omega) - I,
\]

which implies
\[
\sigma^2(P_\Delta(j\omega)) = \frac{1}{\sigma^2(\tilde{M}_\Delta(j\omega))} - 1.
\]

This concludes the proof for the case of full column normal rank. Assume now that \( P(s) \) has full row normal rank. Let \( \{\tilde{N}, \tilde{M}\} \) be a normalized lcf of \( P_s \). Then
\[
\exists \eta_0 \neq 0 \in \mathbb{C}^n \text{ s.t. } \eta_0^*P_s(j\omega_0) = 0.
\]

The derivations carried out for the full column rank case can then be mirrored, using left coprime factors instead of right coprime factors and complex conjugate
transposes where appropriate, and the lower bound on the $\nu$-gap given by

$$\delta_\nu(P_s, P_\Delta) \geq \sigma \left( \tilde{N}(j\omega_0)M_\Delta(j\omega_0) - \tilde{M}(j\omega_0)N_\Delta(j\omega_0) \right).$$

\[ \Box \]

**Theorem 5.3.2.** Given a nominal plant $P_s \in \mathcal{RH}_\infty$ with a pair of poles at $s = \pm j\omega_0$, then for any $P_\Delta \in \mathcal{RH}_\infty$,

$$\delta_\nu(P_s, P_\Delta) \geq \sqrt{\frac{1}{1 + \sigma^2 (P_\Delta(j\omega_0))}}.$$  

(5.8)

**Proof.** Let $\{N, M\}$ be a normalized rcf of $P_s$. Since $N \in \mathcal{RH}_\infty$ and $M$ has full normal rank,

$$\exists 0 \neq z_0 \in \mathbb{C}^q \text{ s.t. } M(j\omega_0)z_0 = 0.$$ 

Without loss of generality, $\|z_0\| = 1$. It follows that

$$\delta_\nu(P_s, P_\Delta) \geq \sigma \left( \tilde{N}(j\omega_0)M(j\omega_0) - \tilde{M}(j\omega_0)N(j\omega_0) \right),$$

$$\geq \|\tilde{N}(j\omega_0)M(j\omega_0)z_0 - \tilde{M}(j\omega_0)N(j\omega_0)z_0\|,$$

$$= \|\tilde{M}(j\omega_0)N(j\omega_0)z_0\|,$$

from which the theorem statement can be deduced in a similar fashion as in the proof of Theorem 5.3.1. \[ \Box \]

**Remark 5.3.1.** These two theorems give lower bounds on the $\nu$-gap for systems with a pole or zero on the imaginary axis. From Theorem 5.3.1, it is clear that an uncertain undamped zero in an otherwise high-gain frequency range is problematic. If the zero occurs at a slightly different frequency in $P_\Delta$, then $\sigma (P_\Delta(j\omega_0)) >> 1$, and therefore $\delta_\nu(P_s, P_\Delta) \approx 1$. A similar problem arises for uncertain undamped poles in an otherwise low-gain frequency range, as can be seen from Theorem 5.3.2. Therefore, while the state-feedback $F$ that normalizes the rcf $\{N, M\}$ of $P_s$ may allow a high robust stability margin $b(P_s, C) \leq b_{\text{opt}}(P_s)$, it can be seen from the results of this section that this measure of robustness is deficient around lightly damped poles and zeros. The distance between $P_s$ and any $P_\Delta$ with very slightly
differing lightly damped pole/zero locations will easily exceed any robust stability margin achieved by the controller for the nominal plant.

**Remark 5.3.2.** In the SISO case, these results simplify to the bounds given in [HS93] for the gap metric, which is itself bounded from below by the $\nu$-gap metric [Vin93, Vin01].

**Example 2.** Consider again the plant $P_s$ given in (5.5), together with a perturbed plant model $P_\Delta$ given by

$$P_\Delta(s) = \frac{10 (s^2 + 1.1)}{s^2 (s^2 + 2)}.$$  

(5.9)

The location of the zeros has been shifted slightly. From Theorem 5.3.1,

$$\delta_\nu(P_s, P_\Delta) \geq \sqrt{\frac{\sigma^2(P_\Delta(j))}{1 + \sigma^2(P_\Delta(j))}} = 0.7071.$$  

This distance is large compared to the optimal normalized coprime factor robust stability margin $b_{\text{opt}}(P_s) = 0.3919$. The four-block or normalized coprime factor distance measure and robust stability margin do not provide any guarantee for the existence of a stabilizing controller for $P_s$ that also robustly stabilizes $P_\Delta$. Hence the $\nu$-gap and the standard $\mathcal{H}_\infty$ loopshaping theories abandon the designer here.

The following two results provide upper bounds on the achievable $b(P_s, C)$ for plants with undamped zeros and poles, respectively.

**Theorem 5.3.3.** Given a plant $P_s \in \mathbb{R}^{p \times q}$ with transmission zeros at $s = \pm j\omega_0$. Then for any controller $C \in \mathbb{R}^{q \times p}$,

$$b(P_s, C) \leq \min \left\{ \sqrt{\frac{1}{1 + \sigma^2(C(j\omega_0))}}, b_{\text{opt}}(P_s) \right\}. \quad (5.10)$$

**Proof.** Consider first the case that $[P_s, C]$ is not internally stable. Then by definition $b(P_s, C) = 0$ and (5.10) is automatically fulfilled. Otherwise, $b(P_s, C) = \left\| (\tilde{G}K)^{-1} \right\|^{-1}_\infty$, where $K$ is a normalized right inverse graph symbol of $C$. An obvious upper bound is given by $b_{\text{opt}}(P_s)$. For a normalized lcf $\{\tilde{\mathcal{N}}, \tilde{\mathcal{M}}\}$ of $P_s$, a

\[1\text{The actual } \nu\text{-gap is even larger than the bound: } \delta_\nu(P_s, P_\Delta) = 0.8012.\]
transmission zero at $s = j\omega_0$ implies that $\sigma(\tilde{N}(j\omega_0)) = 0$. Assume that $\tilde{N}$ has full row normal rank. Then,

$$
\exists 0 \neq \eta_0 \in \mathbb{C}^p \text{ s.t. } \eta_0^* \tilde{N}(j\omega_0) = 0,
$$

where without loss of generality, $\|\eta_0\| = 1$. Let $\{V, U\}$ be a normalized rcf of the controller $C$. Then,

$$
\sigma(\tilde{G}(j\omega_0)K(j\omega_0)) \leq \|\eta_0^* \tilde{N}(j\omega_0)V(j\omega_0) - \eta_0^* \tilde{M}(j\omega_0)U(j\omega_0)\|
$$

$$
= \|\eta_0^* \tilde{M}(j\omega_0)U(j\omega_0)\|,
$$

$$
\leq \sigma(U(j\omega_0)).
$$

Consequently,

$$
b(P_s, C) \leq \sigma(U(j\omega_0)).
$$

Furthermore, using arguments akin to those in the proof of Theorem 5.3.1, it can be shown that $\forall \omega \in \mathbb{R},$

$$
\sigma(U(j\omega)) = \sqrt{\frac{1}{1 + \sigma^2(C(j\omega))}},
$$

from which the theorem statement follows. The proof can be mirrored for tall plants using right coprime factors of the plant and left coprime factors of the controller. \qed

**Theorem 5.3.4.** Given $P_s \in \mathbb{R}^{p \times q}$ with poles at $s = \pm j\omega_0$, then for any $C \in \mathbb{R}^{q \times p},$

$$
b(P_s, C) \leq \min \left\{ \sqrt{\frac{\sigma^2(C(j\omega_0))}{1 + \sigma^2(C(j\omega_0))}}, b_{\text{opt}}(P_s) \right\}.
$$

**Proof.** The proof mirrors the proof of Theorem 5.3.3, after noting that for a normalized lcf $\{\tilde{N}, \tilde{M}\}$ of $P_s$, $\sigma(\tilde{M}(j\omega_0)) = 0$, and that, given a normalized rcf $\{V, U\}$ of $C$,

$$
\sigma(V(j\omega)) = \sqrt{\frac{\sigma^2(C(j\omega))}{1 + \sigma^2(C(j\omega))}}.
$$
Remark 5.3.3. The two results bounding the four-block or normalized coprime factor robust stability margin \( b(P_s, C) \) from above provide some insights into constraints on the controller imposed by the optimization with respect to \( b(P_s, C) \). At a lightly damped zero of the plant (Theorem 5.3.3), it is beneficial if \( \sigma(C(j\omega_0)) \) is also small, whereas at a lightly damped pole (Theorem 5.3.4), \( \sigma(C(j\omega_0)) \) should be large. Hence, for good robustness in a normalized coprime factor sense, the controller should essentially “mimic” the behavior of \( P_s \).

5.4 Right Coprime Factor Synthesis

This section contains the key result for generalized (right) coprime factor synthesis. The main theorem describes an observer form controller as in Fig. 5.2 achieving a certain level of robust stability margin. Several subsequent observations provide a comprehensive interpretation of the synthesis theorem in view of the observer form implementation of the central controller. The crucial aspect is that the state-feedback matrix \( F \) may be chosen freely by the designer, resulting in coprime factorizations that are not necessarily normalized. The system diagram of a plant with right coprime factor uncertainty is displayed in Fig. 5.1.

Theorem 5.4.1 (Non-normalized right coprime factor synthesis). Given a plant \( P_s \in \mathbb{R}^{p \times q} \) with a stabilizable and detectable state-space realization

\[
P_s = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix},
\]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times q}, C \in \mathbb{R}^{p \times n} \) and also given \( \epsilon > 0 \) and \( F \in \mathbb{R}^{q \times n} \) such that \( A + BF \) is Hurwitz, then there exists an observer gain \( L_\infty \in \mathbb{R}^{n \times p} \) such that

\[
b_{rcf}(P_s; F, L_\infty) > \epsilon
\]

if and only if \( \epsilon < 1 \) and there exists a stabilizing solution \( Y_\infty \geq 0 \) solving

\[
\hat{A}Y_\infty + Y_\infty \hat{A}^T + Y_\infty \left( \frac{\epsilon^2}{1 - \epsilon^2} F^T F - C^T C \right) Y_\infty + \frac{1}{1 - \epsilon^2} BB^T = 0,
\]

\[
(5.12)
\]
\[ \hat{A} := \left( A - \frac{\varepsilon^2}{1+\varepsilon^2} BF \right) \]. If these conditions are satisfied, one such controller is given by

\[ C_\infty(s) = \begin{bmatrix} A + BF + L_\infty C & -L_\infty B \\ F & I \end{bmatrix}, \tag{5.13} \]

with \( L_\infty = -Y_\infty C^* \).

**Proof.** This theorem is a dual to [ZDG96, Theorem 18.1]. The resulting central controller achieving \( b_{\text{rcf}}(P; F, L_\infty) > \varepsilon \) is given by

\[ \hat{C}_\infty = \begin{bmatrix} A + BF + L_\infty C & -L_\infty \\ F & 0 \end{bmatrix}. \]

It is then possible to show that equivalence between the configuration in Fig. 5.2 and Fig. 5.1 is achieved by letting \( r = \hat{V} \hat{r} \), with \( \{ \hat{U}, \hat{V} \} \) being a lcf of \( \hat{C}_\infty \) for which \( \hat{V}M - \hat{U}N = I \), where existence of such \( \{ \hat{U}, \hat{V} \} \) is guaranteed by \( \hat{C}_\infty \) being stabilizing.

**Remark 5.4.1.** The controller \( C_\infty \) given in (5.13) is in the observer form with observer gain matrix \( L_\infty = -Y_\infty C^T \) and state-feedback gain matrix \( F \). This implementation is shown in Fig. 5.2. The state-feedback matrix \( F \) may be freely chosen by the designer, under the obvious restriction that \( A + BF \) be Hurwitz. One can therefore interpret the right coprime factor \( \mathcal{H}_\infty \) optimization problem of maximizing \( \varepsilon \) in (5.11) as finding the optimally robust observer for a given state feedback gain matrix \( F \). This is in contrast to the Kalman filter, which is optimal in an \( \mathcal{H}_2 \) sense, but provides no robustness guarantees when used as an observer for state-feedback control [ZDG96, Section 14.10], [Doy78a]. In fact, it is well known that \( L_\infty = -Y_\infty C^* \) converges to the Kalman filter gain when \( \varepsilon \to 0 \), i.e. when no robustness is required [ZDG96, Section 16.2],[GL00, Section 10.5]. As can be seen from the definitions of the coprime factor distance measure \( d_{\text{rcf}}(P, P_\Delta, F) \) and the coprime factor robust stability margin \( b_{\text{rcf}}(P_\Delta; F, L) \), \( F \) and \( L \) have distinct roles in the robust control setting. Choosing the state-feedback matrix \( F \) determines the topology in which distance is measured. Once \( F \) has been chosen, \( b_{\text{rcf}}(P_\Delta; F, L) \) depends only on the observer gain \( L \). The above theorem shows that there exists a unique observer gain \( L_\infty \) which optimizes the robust stability margin for a given \( F \), or a given topology (where optimization can be achieved via \( \varepsilon \)-iterations).
Remark 5.4.2. The robust stabilization problem of a plant with left coprime factor uncertainty [ZDG96, Theorem 18.1], conversely, can be interpreted as finding an optimally robust state-feedback gain matrix $F_\infty = -B^T X_\infty$ given a particular observer gain matrix $L$. While this is possible and indeed the synthesis theorem is just a dual of the result given in this chapter, it seems fruitful, in light of this interpretation, to consider the right coprime factor synthesis problem, since a procedure that allows the designer to specify a state-feedback gain and then automatically synthesizes an optimal observer will be more intuitive to apply in practice.

Remark 5.4.3. The restriction in the above theorem that $\varepsilon < 1$ is due to an upper bound given by evaluation of $b_{\text{rcf}}(P, C_\infty)$ at $\omega = \infty$ [SLC89], which is in turn bounded from above by $\overline{\sigma}(M_0(\infty)) = 1$ (via the state-space realization for the rcf $\{N_0, M_0\}$ of Theorem 5.2.1). In Chapter 4 (see also [LP09, ELP11]) it was shown that $b_{\text{rcf}}(P, C_\infty)$ is not restricted to the interval $[0, 1]$. However, the right graph symbol $G_0$ can be scaled [Vin01] to ensure $b_{\text{rcf}}(P, C_\infty) < 1$, with a similar effect on the distance measure $d_{\text{rcf}}(P, P_\Delta)$, s.t. the ratio of the two remains unaffected (see Chapter 4).

Remark 5.4.4. The connection between coprime factorizations and state-space representations was established in [Vid84, NJB84, MF87], though its meaning in an optimization sense as described in Remark 5.4.1 seems to have remained little understood. With the increasing focus on synthesis for normalized coprime factorizations [GM89, MG90, MG92], the design freedom available via the variation of $F$ seems to have become less noticed. The problem of non-normalized coprime factor synthesis was also addressed in Chapter 4, but with a method that resulted in a controller of order larger than the order of the plant. This is avoided here.

5.4.1 Observer-based Implementation

For the robust tracking problem, an observer-based controller implementation is of crucial importance. It is a well-known fact [Vid85, Section 5.6], [Vin01, Section 4.5] that a two-degree-of-freedom implementation of the controller is of essential importance for reducing the impact of right half-plane (RHP) poles and zeros of
the controller on the tracking performance. Given a left coprime factorization 
\( \{\tilde{U}, \tilde{V}\} \) of the controller \( C_{\infty} \) which fulfills the Bezout identity

\[
\tilde{V}M - \tilde{U}N = I, \tag{5.14}
\]

where \( \{N,M\} \) is a right coprime factorization of the plant \( P_s \), the existence of \( \tilde{U}, \tilde{V} \) satisfying (5.14) is equivalent to \( C_{\infty} \) being a stabilizing controller. Then \( \tilde{U} \) contains any RHP zeros of \( C_{\infty} \), while \( \tilde{V} \) contains any RHP poles. Unless \( \tilde{U} \) is implemented in the feedback path and \( \tilde{V}^{-1} \) in the feedforward path as shown in Fig. 5.3, the transfer function from reference input \( r \) to output \( y \) may contain RHP zeros which cannot be cancelled by a stable pre-compensator at the reference input. The observer-based controller structure of Theorem 5.4.1, also seen in Fig. 5.2, can be shown to be an implementation of precisely the desired two-degree-of-freedom architecture [Vin01, Appendix A.3]. An alternative equivalent structure, which shows that the controller in fact does not need to be split, can be found in [DLVLA09, DALLV07]. There, two pre-filters (which are stable coprime factors of the plant) are used to inject a modified reference signal at two points into the loop.

The following transfer functions from reference input \( r \) to plant input \( u \) and output \( y \) result from the observer-based implementation:

\[
T_{ur}(s) = M(s) = \begin{bmatrix}
A + BF & B \\
F & I
\end{bmatrix},
\]

\[
T_{yr}(s) = N(s) = \begin{bmatrix}
A + BF & B \\
C & 0
\end{bmatrix}.
\]

From this formulation, it is apparent that the choice of state-feedback \( F \) not only affects the topology of the space of uncertain plants, but also has a direct impact in the closed loop. In the case of normalization, it has been shown in Section 5.3, that \( A + BF \) may have eigenvalues very close to the imaginary axis, which may be undesirable for tracking.

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Figure 5.3: Closed-loop controller implementation for tracking performance.

5.5 Ensuring Well-dampedness Via a Circular Pole Constraint

As has been shown in the previous section, the state-feedback matrix $F$ need not be chosen in such a way that the pcf of $P_s$ induced by $F$ is normalized. In fact, as shown in Section 5.3, normalization may have undesired effects in terms of robustness with respect to uncertain lightly damped poles and zeros. This section describes one particular method for choosing $F$ in such a way that the poles of the coprime factorization of $P_s$ induced by $F$ are sufficiently well damped, which in turn leads to good robustness with respect to uncertainty in lightly damped pole or zero locations.

Firstly, consider the following definition of a circular region in the closed left-half complex plane with center at $s = -q$ and a radius of $\rho \leq q$.

**Definition 5.5.1.** Let $q > 0$, $\rho > 0$ with $q \geq \rho$. Denote by $\mathcal{C}(q, \rho)$ the set

$$\left\{ s \in \mathbb{C}^- : \sqrt{(\Re(s) + q)^2 + \Im(s)^2} < \rho \right\}.$$

The following theorem describes a method for ensuring sufficient damping of the poles of a pcf induced by a state feedback $F$. As has been shown in the previous sections, a coprime factorization with lightly damped poles induces a topology in which uncertainty in the lightly damped poles and zeros of a plant can be difficult to manage.

**Theorem 5.5.1.** Given a system $P_s \in \mathbb{R}^{p \times q}$ with state and output equation $\dot{x} = Ax + Bu$, $y = Cx$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times q}$, $C \in \mathbb{R}^{p \times n}$. Assume $(A, B)$ is controllable and $(C, A)$ is observable. Also given $q > 0$, $\rho > 0$ s.t. $q \geq \rho$. Define
\( A_{q,\rho} := \frac{1}{\rho} (A + qI) \) and \( B_{\rho} := \frac{1}{\rho} B \). Let

\[
F := - \left( I + B_{\rho}^* X B_{\rho} \right)^{-1} \left( B_{\rho}^* X A_{q,\rho} \right),
\]

where \( X \geq 0 \) is the stabilizing solution to the discrete-time algebraic Riccati equation

\[
A_{q,\rho}^* X \left( I + B_{\rho} B_{\rho}^* X \right)^{-1} A_{q,\rho} - X + C^* C = 0. \tag{5.15}
\]

Then,

1. the right coprime factorization \( \{N_0, M_0\} \) of \( P_s \) induced by \( F \) via 5.1 has all its poles in \( \mathcal{C}(q, \rho) \); and

2. the unique state-feedback \( u = r + Fx \) minimizes

\[
v(X) = \frac{1}{\rho^2} \text{trace} (B^* XB),
\]

where \( v(X) \) fulfills the following inequality:

\[
\frac{1}{\rho} \left\| \begin{bmatrix} N_0 \\ M_0 - I \end{bmatrix} \right\|^2 \leq v(X),
\]

where \( \{N, M\} \) is the rcf of \( P \) induced by \( F \).

**Proof.** This result follows from [SKK94, Theorem 4.3]. \( \square \)

**Remark 5.5.1.** For \( q, \rho \to \infty \), the circular pole constraint region \( \mathcal{C}(q, \rho) \) approaches the open left-half complex plane, and the results of Theorem 5.5.1 converge to those of Theorem 5.2.2, i.e. \( \{N_0, M_0\} \) becomes a normalized rcf of \( P_s \).

**Remark 5.5.2.** It is possible to relax the assumptions that \( (A, B) \) is controllable. This theorem only requires \( (A, B) \) to be \( \mathcal{C}(q, \rho) \)-stabilizable, i.e. all poles outside of the region \( \mathcal{C}(q, \rho) \) can be moved by state feedback into the region. Secondly, \( ^2 \)Note that \( X \) in Theorem 5.5.1 is \( X \) in Theorem 5.2.2 scaled by \( \rho \).
the assumption that \((C,A)\) is observable can be replaced by
\[
\begin{bmatrix}
A - \lambda I & B \\
C & 0
\end{bmatrix}
\]
has full rank \(\forall \lambda \in \partial \mathcal{C}(q, \rho)\),
where \(\partial \mathcal{C}(q, \rho)\) is the boundary of \(\mathcal{C}(q, \rho)\).

**Example 3.** Consider again \(P_s\) and \(P\Delta\) given in eqs. (5.5) and (5.9), respectively. It was noted above in Example 2 that the existence of a robustly stabilizing controller for \(P\Delta\) cannot be guaranteed on the basis of using a normalized coprime factorization of \(P_s\). However, by choosing a state-feedback matrix \(F\) which ensures that the eigenvalues of \(A + BF\) are more strongly damped (via a circular pole constraint) than those of the normalized rcf given in Example 1, a robustly stabilizing controller can be obtained via Theorem 5.4.1. Let \(q = 2.9, \rho = 2.5\). Then, via Theorem 5.5.1, \(F = \begin{bmatrix} -1.5531 & -1.6026 & -3.9737 & -1.7731 \end{bmatrix} \). To obtain an optimally robust observer gain, Theorem 5.4.1 is applied. Via a line search on \(\varepsilon\), the optimal \(b_{rcf}(P_s; F, L_{\infty}) = 0.0835\) is found. The corresponding observer gain is \(L_{\infty} = \begin{bmatrix} -2.0649 & 10.4187 & -4.8039 & -7.6397 \end{bmatrix}^T\). While \(b_{rcf}(P_s; F, L_{\infty})\) is much smaller than \(b_{opt}(P_s)\), crucially, the topology induced by \(F\) is much more benign for the uncertain lightly damped zeros of \(P_s\): \(d_{rcf}(P_s, P\Delta; F) = 0.0804\). This is less than the robust stability margin, and hence (with a winding number condition also holding) robust stability of \([P\Delta, C_{\infty}]\) is guaranteed via Thm. 3.7.3.

**5.6 Numerical Example**

In this section, the synthesis of a controller based on a coprime factorization with a circular pole constraint as in Theorem 5.5.1 is described for a more complex numerical example. Consider a booster rocket ascending to orbit. Its pitch dynamics are affected by aerodynamic forces, causing it to tumble. Feedback from pitch rate to thrust vectoring is employed to control these dynamics. A tenth-order model of a booster rocket was described in [Enn91], of which the most relevant features can be captured in a fourth-order, two-by-two transfer function matrix (c.f. [LP09]). The nominal model is given by:
The interesting features are a pair of lightly damped zeros below the cross-over frequency in the 1-1 element of \( P \) \((\omega_n = \pm 0.4, \zeta = 0.005)\) and a pair of lightly damped poles above the cross-over frequency in the 2-2 element of \( P \) \((\omega_n = \pm 4, \zeta = 0.005)\). Typically, however, the location of these lightly damped poles and zeros is subject to some uncertainty [Enn91], requiring a robust control design. Such a controller can be obtained through various methods, among them the normalized coprime factor synthesis method [GM89, MG90, MG92] and the results presented in Sections 4.5 and 5.5 based on non-normalized coprime factorizations of the plant. For the purposes of this design example, it is assumed that there are no performance requirements necessitating the introduction of loopshaping weights, and therefore \( P_s = P \) as given in (5.16). For the evaluation of robustness, consider the following parameter dependent perturbed plant, where \( \delta_z \) and \( \delta_p \) denote relative changes in the natural frequency of the zeros and poles of the 1-1 and 2-2 element, respectively:

\[
P_{\Delta} (\delta_z, \delta_p) = \begin{bmatrix}
5 \ s^2 + 0.004 s + 0.36 \\
0.001 s + 1 \\
0.001 s + 1 \\
0.001 s + 1
\end{bmatrix} \begin{bmatrix}
0.1 \\
0.001 s + 1 \\
0.001 s + 1 \\
0.001 s + 1
\end{bmatrix} \begin{bmatrix}
5 \ s^2 + 0.004 s + 0.36 \delta_z^2 \\
0.001 s + 1 \\
0.001 s + 1 \\
0.001 s + 1
\end{bmatrix} \begin{bmatrix}
0.1 \\
0.001 s + 1 \\
0.001 s + 1 \\
0.001 s + 1
\end{bmatrix} \begin{bmatrix}
5 \ s^2 + 0.004 s + 0.36 \delta_p^2 \\
0.001 s + 1 \\
0.001 s + 1 \\
0.001 s + 1
\end{bmatrix}.
\]

(5.17)

Clearly, \( P_{\Delta} (1, 1) = P_s \).

The first method to be evaluated is the normalized coprime factor design. A stabilizing controller \( C_{\infty, 1} \) for \( P_s \) is obtained using the Matlab routine “ncf-syn”. It achieves \( b(P_s, C_{\infty, 1}) = b_{\text{opt}}(P_s) = 0.3797 \). This appears to be a good value, guaranteeing sufficient robustness. However, extremely small perturbations in the location of the lightly damped zeros and poles in the 1-1 and 2-2 element, respectively, result in perturbed plants with distances as measured by the \( \nu \)-gap much larger than \( b(P_s, C_{\infty, 1}) \), e.g. \( \delta_{\nu} (P_s, P_{\Delta}(0.99, 1)) = 0.6664 \) and \( \delta_{\nu} (P_s, P_{\Delta}(1, 0.99)) = 0.7081 \). Fig. 5.4 displays those \( P_{\Delta} (\delta_z, \delta_p) \) in the vicinity of the nominal plant for which \( \delta_{\nu}(P_s, P_{\Delta}(\delta_z, \delta_p)) < b(P_s, C_{\infty, 1}) \), i.e. a sufficient condition for internal stability holds. The evaluation of actual internal stability sheds further light on the problematic robust stability properties of this controller.
Figure 5.4: Scatter plot of $P_\Delta (\delta_z, \delta_p)$ for which $\delta_\nu (P_s, P_\Delta (\delta_z, \delta_p)) < b (P_s, C_{\infty, 1})$ in the range $\delta_z, \delta_p \in [0.985, 1.01]$, evaluated on a grid with spacing of 0.0005.

Neither $[P_\Delta (0.99, 1), C_{\infty, 1}]$ nor $[P_\Delta (1, 0.99), C_{\infty, 1}]$ are internally stable. Fig. 5.5 depicts as data points those $[P_\Delta (\delta_z, \delta_p), C_{\infty, 1}]$ that are internally stable for the range $\delta_z, \delta_p \in [0.5, 1.5]$. A region of instability can be seen extremely close to the nominal plant at $(\delta_z, \delta_p) = (1, 1)$ for which optimal robustness was sought (0.5% and 0.7% change in the natural frequencies of the zeros and poles, respectively, are sufficient to lose internal stability).

Secondly, a controller is designed using the non-normalized coprime factor synthesis result of Section 5.4 and the circular pole constraint described in Section 5.5. The rationale here is to increase damping on the eigenvalues of $A + BF$, some of which are extremely lightly damped when $F$ is chosen to normalize the coprime factorization. The problematic poles are those at $\omega_n = \pm 0.3999$ with $\zeta = 0.0072$ (almost cancelling the lightly damped zeros in the 1-1 element of $P_s$) and those at $\omega_n = \pm 3.9998$ with $\zeta = 0.0084$ (almost cancelling out the lightly damped poles in the 2-2 element of $P_s$). To remove these poles further from the imaginary axis, a circular pole constraint is imposed, using the parameters $q = 50, \rho = 49.96$. By solving eqs. (5.15) and (5.12) for $\epsilon = 0.65^4$, $F$ and $L$ ma-

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\textsuperscript{3}Evaluated with a spacing of 0.005 between adjacent $\delta_z, \delta_p$.

\textsuperscript{4}The Algebraic Riccati Equations of the form of (5.12) arising from plants with lightly damped
trices are obtained, from which the controller $C_{\infty,2}$ follows via (5.13). It achieves $b_{\text{ref}} (P_s; F, L) = 0.67 > \varepsilon$. This optimization with respect to a different distance measure (as induced by $F$) results in a larger region around the nominal plant in which the distance measure is less than the corresponding robust stability margin, i.e. $d_{\text{ref}} (P_s, P_{\Delta} (\delta_z, \delta_p); F) < b_{\text{ref}} (P_s; F, L)$. Fig. 5.6 shows the plants $P_\Delta (\delta_z, \delta_p)$ in the vicinity of the nominal plant for which $d_{\text{ref}} (P_s, P_{\Delta} (\delta_z, \delta_p); F) < b_{\text{ref}} (P_s; F, L)$. The region is about twice the size of the region in Fig. 5.4 for the controller optimized with respect to normalized coprime factor uncertainty, with about double the uncertainty in the zero location now covered by the distance-based sufficient stability condition. The effect on actual internal stability of $[P_{\Delta} (\delta_z, \delta_p), C_{\infty,2}]$ is even more dramatic. This is visible when comparing the plot of internally stable $[P_{\Delta} (\delta_z, \delta_p), C_{\infty,2}]$ in Fig. 5.7 with the corresponding plot for the normalized coprime factor-based controller in Fig. 5.5: The region of instability near the nominal plant has completely disappeared.

In summary, it has been shown for this 2-by-2 system with lightly damped uncertain zeros and poles that

poles or zeros can be solved numerically in a reliable manner using the iterative procedure in [LFAR08].

Figure 5.5: Scatter plot of internally stable $[P_{\Delta} (\delta_z, \delta_p), C_{\infty,1}]$ in the range $\delta_z, \delta_p \in [0.5, 1.5]$, evaluated on a grid with spacing of 0.005.
• despite a good normalized coprime factor robust stability margin, the closed loop with a controller optimized for normalized coprime factor uncertainty may be destabilized by minuscule perturbations in the parameters of the system;

• an improvement in the size of the region in which distance is less than the corresponding robust stability margin can be achieved by choosing a coprime factorization that is less lightly damped than a normalized coprime factorization of the system via the circular pole criterion described in Section 5.5;

• the controller optimized with respect to the non-normalized coprime factor uncertainty structure implied by the distance measure internally stabilizes plants in a much larger region around the nominal plant than the controller optimized with respect to normalized coprime factor uncertainty.

Time-domain performance of the synthesized controllers is deliberately not considered in this example, since—as outlined in Subsection 5.4.1—in the chosen implementation of the controller, a stable pre-filter can be implemented in the reference input to obtain any desired time-domain step response.

5.7 Conclusions

This chapter has considered the robust stabilization problem of uncertain linear-time invariant plants with coprime factor uncertainty bounded in \( \mathcal{RH}_\infty \). The problem considered here is a generalization of the normalized coprime factor robust stabilization problem. It has been shown that the problem admits a simple and intuitive controller implementation parameterized in terms of a state-feedback matrix \( F \) and observer gain \( L \). The choice of a state-feedback matrix \( F \) induces a topology in which distance between plants is measured. Subsequently, an observer gain \( L \) can be obtained to maximize robustness of the controller in this topology via the solution of a Riccati equation. This synthesis method results in a controller of the same order as the nominal plant. It has also been shown that
Figure 5.6: Scatter plot of $P_{\Delta}(\delta_z, \delta_p)$ for which $d_{\text{rcf}}(P_s, P_{\Delta}(\delta_z, \delta_p); F) < b_{\text{rcf}}(P; F, L)$ in the range $\delta_z, \delta_p \in [0.985, 1.01]$, evaluated on a grid with spacing of 0.0005.

Figure 5.7: Scatter plot of internally stable $[P_{\Delta}(\delta_z, \delta_p), C_{\infty, 2}]$ in the range $\delta_z, \delta_p \in [0.5, 1.5]$, evaluated on a grid with spacing of 0.005.
non-normalized coprime factorizations are a more suitable tool for obtaining robustly stabilizing controllers for uncertain lightly damped plants than normalized coprime factorizations, which only provide very limited robustness guarantees. This has been illustrated through a numerical example, motivated by the thrust-vectoring to pitch-rate control of a booster rocket model with uncertain lightly damped poles and zeros. In the example, it has been shown that a controller achieving a good normalized coprime factor robust stability margin for the nominal plant does not internally stabilize perturbed plants with just 1% variation in pole and zero location. Based on a non-normalized coprime factorization which fulfills a circular left-half plane pole constraint, a controller has been synthesized which internally stabilizes all perturbed plants with up to nearly 40% variation in pole and zero location. This underscores the design freedom that becomes available when the controller can be optimized with respect to coprime factor uncertainty that is not necessarily normalized.
Chapter 6

Conclusions

This chapter summarises the main contributions of this thesis and provides a brief outlook on avenues for future research. The main thrust of the thesis is to provide tools for choosing an appropriate uncertainty structure for robust control problems, and to provide synthesis tools to deal with uncertainty that is particularly challenging for the existing robust controller synthesis methods.

6.1 Contributions

The main contributions are summarized as follows, grouped into contributions to analysis and synthesis.

In the area of robust stability and robust performance analysis:

- Readily applicable distance measures, robust stability margins and associated robust stability and robust performance theorems for several commonly used uncertainty structures (additive, multiplicative, inverse multiplicative, inverse additive and right coprime factor uncertainty) are described. These lead to robust stability results for a larger uncertainty class than previously reported ($\mathcal{RL}_\infty$ instead of $\mathcal{RH}_\infty$), and also enable robust performance theorems for the above uncertainty structures. In contrast to previous methods for robust performance analysis, they only require the computation of two infinity norms for every uncertain plant considered. Physically motivated numerical examples highlight the potential for obtaining less conservative
results than may be obtained with the normalized coprime factor structure by choosing a more specialized structure.

- A comprehensive analysis is carried out for coprime factor uncertainty when the coprime factors of the nominal plant are not restricted to be normalized. It is shown how robust stability and robust performance bounds can be optimised by choosing a particular coprime factorization involving a combination of a nominal plant, a perturbed plant and a controller. The term robustness ratio is introduced for this bound. It is also shown how the robustness ratio provides bounds on the degradation of the normalized coprime factor robust stability margin that are tighter than bounds involving the $\nu$-gap.

- For plants with uncertain lightly damped poles and zeros in regions with low and high gain, respectively, it is shown that the normalized coprime factor uncertainty framework does not provide good measures of robustness and objectives for controller optimisation. Lower bounds on the $\nu$-gap (for MIMO plants) and upper bounds on the robust stability margin are obtained for such cases, showing that small deviations in pole/zero locations are sufficient to void any robust stability guarantees in the normalized coprime factor framework. This is also demonstrated through numerical examples.

In the area of robust controller synthesis:

- An $H_\infty$ optimisation problem is formulated in an LMI framework which enables the synthesis of controllers which are optimally robust in terms of the robustness ratio for the perturbations implied by a perturbed plant or a family of perturbed plants, while maintaining a certain level of normalized coprime factor robust stability margin (as a proxy of nominal performance). This comes with the penalty of increased controller order, but enables dramatic extensions in specific directions of the sets of plants which are robustly stable. This is demonstrated through numerical examples.

- It is shown that the $H_\infty$ robust stabilization problem for non-normalized coprime factor uncertainty admits a simple and intuitive solution in terms of an observer form controller. Connections to the extensive literature on co-
prime factor synthesis are made, and important re-interpretations are highlighted. In particular, it is shown that for right coprime factor uncertainty, a state-feedback matrix $F$ induces a right coprime factorization of the nominal plant, implying a topology in which distance can be measured. An optimally robust observer gain $L$ in that topology can then be obtained by solving a single Algebraic Riccati Equation.

- One particular method exploiting this newly highlighted freedom in choosing a state-feedback matrix for the robust stabilisation problem is proposed. It places the poles of the coprime factorization of the nominal plant within a circular region in the left half plane, thereby ensuring minimal damping ratios of all poles. This in turn has significant effects on the robustness of the resulting controller with respect to uncertain lightly damped poles and zeros, as highlighted through a numerical example.

### 6.2 Future Work

The developments of this thesis lead to a number of potentially interesting extensions and further research questions.

- The generalised distance measure and robust stability margin proposed in [LP09] and specialised in Chapter 3 for additive, multiplicative, inverse multiplicative and right coprime factor uncertainty appears to be extendable also for structured uncertainty.

- Connections between optimization in the robustness ratio and the simultaneous stabilization problem need to be explored more fully. The simultaneous stabilization problem [Vid85] is also formulated in terms of coprime factors (not necessarily normalized) of two plants and a controller.

- The freedom in choosing a state feedback matrix $F$ in the observer-form controller described in Chapter 5 could be exploited to ensure certain other properties hold for some parts of the feedback loop, e.g. passivity, negative imaginaryness etc.
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