Stability of separation bubbles in a channel induced by a suction slot

Jitesh S.B. Gajjar* and Hanadi H. Zahed

School of Mathematics, University of Manchester, Manchester, M13 9PL, UK

The stability of laminar separation bubbles induced by a local adverse pressure gradient in a channel with a suction slot on the upper-wall is investigated. The Navier-Stokes equations are solved numerically and steady solutions are tracked using continuation methods. We are able to compute two solutions with long and short bubbles. The stability of these solutions is investigated using methods based on global stability analysis and temporal simulations and it found that there is good agreement between the theoretical predictions and numerical computations.

Nomenclature

ψ Streamfunction
ω Vorticity
x Non-dimensional coordinate parallel to channel walls.
y Non-dimensional coordinate in wall normal direction.
ξ Mapped coordinate in x— direction
z Mapped coordinate in y— direction
zj collocation point
Re Reynolds number
β Non-dimensional suction parameter
x_{max} Length of computational domain in x— direction
y_{max} Non-dimensional width of channel
b_s Constant (=0.062)
λ Eigenvalue
N+1 Number of collocation points in z direction
M Number of points in x.
ψ_{j,k} Numerical approximation to ψ(x_j, y_k)

I. Introduction

Laminar separation bubbles are formed in many flows of aerodynamic importance such as on the upper-surface of aerofoils. Experiments, see for example the review by Tani,14 show that the presence of boundary layer separation near the leading edge of an aerofoil can lead to stall. The manner in which a laminar separated bubble becomes unstable and breaks up has continued to attract much attention over the years. Gaster (1966)6 was one of the first to conduct a detailed experimental study of the influence of an adverse pressure gradient produced by an artificially generated suction slot on a flat plate boundary layer. He was able to demonstrate the presence of long and short separation bubbles and tried to identify the critical parameters for the onset of bubble bursting.

Pauley et al. (1990)11 solved the Navier-Stokes equations numerically and studied the breakup of separation bubbles under the influence of a suction induced adverse pressure gradient. Computations were performed for different suction strengths and different Reynolds numbers and showed that for weak adverse

*Professor of Applied Mathematics, School of Mathematics, University of Manchester, Manchester M13 9PL, UK. AIAA Senior Member.
pressure gradients, closed steady separation bubbles developed. After a critical suction strength, the separation region lengthened and unsteady oscillations developed leading to vortex shedding. They suggested that the bursting observed by Gaster was related to a time-averaged periodic vortex shedding. Similar conclusions were made by Ripley & Pauley (1992) who also found suggested that the low frequency oscillations in Gaster’s experiments were due to large scale vortex motion with turbulence playing only a secondary role. This work was later extended by Hsiao & Pauley (1994) who tried to compare Navier-Stokes computations with triple-deck theory and interactive boundary layer theory. The monograph by Sychev et al. gives an excellent account of triple-deck theory and marginal separation theory which provides a rational theoretical description of laminar separation in boundary layer flows. Hsiao & Pauley (1994) were not able to compute stable laminar solutions at large Reynolds numbers to be able to make proper comparisons with the theory.

The problem has been looked at recently by many researchers including Alam & Sandham (2000) who carried out numerical simulations with short laminar separation bubbles in two and three-dimensions. The stability characteristics of the mean flow profiles were examined and they have suggested that the bursting and vortex shedding may be connected with absolute instabilities. Cassel et al. (2007) have investigated the flow geometry similar to Hsiao & Pauley (1994) and Alam & Sandham (2000) and studied aspects of optimal control and unsteady separation within the context of unsteady boundary layer theory. Hagmark et al. (2000) have studied separation bubbles in the boundary layer flow over a flat plate with an adverse pressure gradient generated by a curved upper surface. They highlight the difficulties in conducting experiments for this problem and the care which needs to be exercised in order to reproduce consistent results.

Despite all the above cited work, it is clear that the manner in which a separation becomes unstable is still not properly understood. Gaster (2005) (unpublished) in a talk at a conference on global instabilities in Crete, said that after 50 years of research we are still far short of an agreed criterion for the bursting of a separation bubble.

The background described above provides the motivation for the current work. Our primary objectives are to revisit the model problem of a separation bubble induced by a locally adverse pressure gradient due to the presence of a suction slot, and conduct a systematic investigation of the flow properties. In particular we are interested in understanding when and how the separation bubble becomes unstable. In our work, we use modern numerical techniques to solve the governing equations. We adopt boundary conditions similar (but not identical) to those in Hsiao & Pauley (1994) and Alam & Sandham (2000) that correspond to having slip on the upper-wall. We have used the same geometry as in Ref. 8, see Figure 1 which gives a sketch of the geometry with the suction slot. The numerical techniques we have used allow us to compute and track unstable basic states using continuation methods and are quite different to those employed by others for this problem. Briefly the Navier-Stokes equations are solved using an accurate hybrid method with finite-differences and spectral collocation. To investigate the instability, we have used a combination of global stability analysis in which we look for perturbed solutions of the form $e^{-\lambda t} f(x, y)$ which leads to the solution of a generalised partial differential eigenvalue problem. The stability is also investigated by doing simulations of the unsteady equations. The techniques we have adopted are robust and accurate and have been used successfully in other related problems, see Boppana & Gajjar (2010,2011). Below we have given a brief description of the problem and numerical method followed by a discussion of the results.

II. Problem Formulation

Consider the flow of an incompressible fluid through a channel as shown in the Figure 1. The upper wall contains a suction port between $x = 1$ and $x = 1 + b_s$, and by adjusting the suction strength one can create an adverse pressure gradient which leads to marginal separation conditions on the lower wall.

The governing equations are the two-dimensional Navier-Stokes equations for an incompressible fluid which in terms of streamfunction ($\psi$) and vorticity ($\omega$) can be written as,

\begin{align}
\omega_t + \psi_y \omega_x - \psi_x \omega_y &= \frac{1}{Re} (\omega_{xx} + \omega_{yy}), \\
\omega &= (\psi_{xx} + \psi_{yy}).
\end{align}

The Reynolds number here is given by $Re = \frac{UL}{\nu}$, while $L$ is the characteristic entrance length, and the inlet velocity is $U$, with $\nu$ being the kinematic viscosity of the fluid.

The computational flow domain is $0 \leq x \leq x_{max}$ and $0 \leq y \leq y_{max}$.
II.A. Steady analysis

The steady version of the equations (1) (with the $\partial/\partial t$ terms set to zero), are solved first to obtain the steady flow solutions. These are then used in the unsteady analysis of the flow to determine stability.

The corresponding boundary conditions for the flow (see Figure 1) are given by

$$
\begin{align*}
\psi &= 0, \quad \psi_y = 0 \quad \text{for} \quad y = 0, \quad 0 \leq x \leq x_{max}, \\
\psi_y &= 1 - \beta \hat{u}_s, \quad \omega = 0 \quad \text{for} \quad y = y_{max}, \quad 0 \leq x \leq x_{max}, \\
\psi &= y, \quad \omega = 0 \quad \text{for} \quad x = 0, \quad 0 \leq y \leq y_{max}, \\
\psi_{xx} &= 0, \quad \omega - \psi_{yy} = 0 \quad \text{for} \quad x = x_{max}, \quad 0 \leq y \leq y_{max},
\end{align*}
$$

where the function $\hat{u}_s$ is defined by

$$
\hat{u}_s = \left[ \frac{1}{2} - \frac{y_{max}}{2b_s \pi} \log \left( \frac{\sinh \left( \frac{\pi(x-1-b_s)}{2y_{max}} \right)}{\sinh \left( \frac{\pi(1-b_s)}{2y_{max}} \right)} \right) \right],
$$

and we have taken $b_s = 0.062$.

The function $\hat{u}_s$ is the same as that used in Ref. 4 and corresponds to taking a distribution of source singularities between $x = 1$ and $x = 1 + b_s$ for the outer inviscid flow. A plot of $\hat{u}_s(x)$ is shown in figure 2.

The equations with the boundary condition have to be solved to obtain $\psi$ and $\omega$. In the results reported below we have taken $y_{max} = 0.3$ which is different from the value of $y_{max} = 0.2096$ used in Ref. 8.

III. Numerical Method

Equations (1) are discretized using a hybrid method based on spectral Chebychev collocation in the $y-$ direction and second-order finite differences in $x$. In the $y-$ direction the domain is transformed into $z \in [-1,1]$ and the collocation points used are $z = z_j = -\cos \left( \frac{j\pi}{N} \right), \quad j = 0, 1, \ldots, N$ with the physical points given by $y = y_j = \frac{y_{max}}{2} (1 + z_j)$.

In the $x-$ direction we first use a mapping $x = f(\xi)$ and map the domain $x \in [0,x_{max}]$ to $\xi \in [\xi_{min},\xi_{max}]$, and use a uniform grid in $\xi$ with $M$ points. Also the mapping function used is

$$
x = 1 + \pi \tan \left( \frac{\xi}{2} \right).
$$

Figure 1. Sketch of channel with suction with boundary conditions.
In our work we have taken $\xi_{\text{max}} = 0.93$ and $\xi_{\text{min}}$ defines the point $x = 0$.

If we define $\psi_{k,j} = \psi(x_k, y_j)$, $\omega_{k,j} = \omega(x_k, y_j)$ then the discretization leads to a set of nonlinear algebraic equations for the unknowns $\psi_{k,j}, \omega_{k,j}$, $1 \leq k \leq M$, $0 \leq j \leq N$. These equations are solved by first using Newton linearization and writing the linear system in the form of a block-tridiagonal matrix system which is solved directly. For further details of the numerical methods used see Zahed (2010), and also Boppana & Gajjar (2010, 2011), where similar techniques are adopted to solve for the flow in a lid-driven cavity and flow past a row of circular cylinders.

In our work we also used pseudo arc-length continuation to follow solution branches for different values of the two parameters $Re$ and $\beta$.

III.A. Unsteady Analysis

We used two approaches to determine the stability of the steady states computed in the previous section. We suppose that small perturbations $\tilde{\psi}(x, y, t), \tilde{\omega}(x, y, t)$ are superimposed on the basic steady flow with the streamfunction and vorticity given by

$$\psi(x, y, t) = \bar{\psi}(x, y) + \tilde{\psi}(x, y, t), \quad \omega(x, y, t) = \bar{\omega}(x, y) + \tilde{\omega}(x, y, t),$$

where the barred quantities denote the steady flow solutions. The Navier-Stokes equations (1) are linearized for small perturbations and this leads to the unsteady perturbation equations for the disturbances $\tilde{\psi}, \tilde{\omega}$:

$$\tilde{\omega}_t + \bar{\psi}_y \tilde{\omega}_x + \bar{\psi}_x \tilde{\omega}_y - \bar{\psi}_x \omega_y - \bar{\psi}_x \omega_y = \frac{1}{Re} (\bar{\omega}_{xx} + \bar{\omega}_{yy}), \quad (2a)$$

$$\tilde{\omega} = \left( \bar{\psi}_{xx} + \bar{\psi}_{yy} \right), \quad (2b)$$

and with homogenous boundary conditions,

III.A.1. Global stability analysis

In the global stability analysis we look for normal mode solutions to (2) of the form

$$\tilde{\psi} = e^{-\lambda t} \bar{\Psi}(x, y), \quad \tilde{\omega} = e^{-\lambda t} \bar{\Omega}(x, y).$$

Substituting this into the equations (2) leads to a partial differential eigenvalue problem for determining the eigenvalues $\lambda$ and eigensolutions $\bar{\Psi}(x, y), \bar{\Omega}(x, y)$. This is handled by using the same discretization as for the basic flow and solving the resultant generalised eigenvalue problem with the software library package ARPACK. See also Boppana & Gajjar (2010).

III.A.2. Temporal simulations

In additional to the global stability analysis we have performed direct simulations of the unsteady linearized equations for the perturbations. The linearized equations (2) were rewritten in terms of the total flow
ψ = ψ + ψ̃,  ω = ω + ω̃, and solved for ψ, ω. The boundary conditions were perturbed by modifying the no
slip condition on lower wall to

ψ = h(x, t),  ψ_y = 0,  for  y = 0.

Here the forcing function h was taken as

\[ h(x, t) = e^{-20(x-1.5)^2} e^{-50(t-1)^2/(1000\sqrt{\pi})} \]

which represents a short pulse centered on x = 1.5 near the suction slot and which decays rapidly in time. In
the simulations, a Crank-Nicholson scheme was used for the first time step after which we switched to a
second order backward difference scheme in time. The spatial discretizations were identical to those used
for computing the basic flow. The discrete equations can again be represented as a block tridiagonal system
which was solved directly.

IV. Results

Extensive grid size checks have been performed to validate the results as documented in Ref. 15. In the
results below, for clarity, only a partial domain in x is shown.

IV.A. Basic flow results

By keeping the Reynolds number fixed and varying the suction parameter it is possible to generate the
conditions for marginal separation. Beyond a critical suction parameter the flow separates and a separation
bubble is formed at the lower wall as shown in the streamfunction contours in Figure 3. By following the
solutions for different β it is found that there is a critical value of β at which there is a turning point and
steady solutions no longer exist beyond this critical value. For values of β less than this critical value, it
is found that two solutions can arise, one with a short separation bubble and one with a long separation
double. Two such solutions are shown in Figure 3 for a Reynolds number \( Re = 50000 \). In the Reynolds
number range studied \( 5000 \leq Re \leq 80000 \) it is found that the critical value of β for the onset of separation
decreases with increasing Reynolds number, see Figure 4.

The contours are equispaced and dashed contours indicate negative values.

In Figure 5 the bubble separation length versus suction parameter shows clearly the two solution branches.
The figure also shows results for varying M and N and a trend towards convergence. The calculations are
more sensitive to changes in M than N where for fixed M and varying N it is possible to obtain converged
solutions for \( N = 80 \) points.

In Figure 6 we have shown typical scaled skin friction \( \hat{\tau}(x) = \psi_{yy}(y = 0)/\sqrt{Re} \) and wall pressure for the
lower-branch (short bubble) and upper-branch (long bubble) solutions. The wall pressure shows a plateau
effect in the bubble region for the upper-branch solutions and rises sharply after re-attachment before levelling
off. This is similar to that found by Gaster in his experiments. The skin friction plots show a similar plateau
in the separation region and a sharp drop and rise close to re-attachment.
By taking the values for the scaled skin friction at a particular \( x \) value and varying the suction parameter \( \beta \) it is possible to generate the so called fundamental curve for separation. This is shown in Figure 7 where we have plotted \( \tau_0 = \psi_{yy}(x = 1.5, y = 0)/\sqrt{Re} \) versus the suction parameter \( \beta \). The typical fundamental curve is predicted in theoretical studies of marginal separation, see Braun,\(^4\) although the precise shape found is not the same. This could be because of various factors such as the location chosen for the measurements, the values of the Reynolds numbers used. In the parameter regime studied here, only two solutions were found whereas marginal separation theory Braun,\(^4\) predicts that there are some parameter values for which many more solutions exist.

**IV.B. Global stability results**

The basic flows calculated were examined for stability using the global stability analysis. Since disturbances are proportional to \( e^{-\lambda t} \), if eigenvalues can be found for which \( \Re(\lambda) \) is negative, the flow is linearly globally unstable. For the basic flows calculated, we found that the solutions with short separation bubble were stable and the long bubble solutions unstable. Typical eigenvalue spectrums for the stable and unstable solutions are shown in figure 8 where it can be observed that the unstable solution has one real and negative eigenvalue, whereas for the stable solution the least stable eigenvalue is real and positive. As the suction parameter approaches the critical value for the turning point, the real eigenvalue moves towards the origin, which is the typical behaviour for a turning point bifurcation, and provides further support for the presence of a turning point in the earlier steady state computations.

In figures 9 we show typical eigenfunctions for the disturbance streamfunction for various grid sizes, on the unstable branch with \( Re = 50000, N = 100 \), and \( \beta = 0.0997 \). Note that since the eigenvalue is real, the imaginary part of the eigenfunction can be taken to be zero. The eigenfunction is normalised so that the
Figure 6. A plot of (a) the wall pressure and (b) scaled skin friction $\tau(x) = \psi_{yy}(y = 0)/\sqrt{Re}$ with $Re = 50000, M = 1501, N = 100$ and $\beta = 0.0997$.

Figure 7. Fundamental curve showing a plot of $\beta$ versus $\hat{\tau}_0 = \psi_{yy}(1.5, 0)/\sqrt{Re}$.

maximum value is unity in the region $0 < x < 20$. The contour plots in figure 9 show that the eigenvector is converged for different grid sizes in the region shown. Note that because of numerical errors the calculated imaginary part of the eigenfunction was not zero but exhibited an oscillation, but with zero mean, for any fixed $y$ value.

Figure 8. Plot showing the eigenvalue spectrum for 100 eigenvalues for stable (shown by an asterix) and unstable solutions (shown by circles) at $R = 50000, N = 100, M = 1501$, (a) $\beta = 0.0997$, (b) $\beta$ values just either side of the turning point at $\beta = 0.1054$. 

7 of 10

American Institute of Aeronautics and Astronautics
Further validation of the global stability results are provided by the temporal simulation results discussed below.

**IV.C. Temporal simulation results**

The temporal simulations confirm the findings above that perturbations imposed on the short bubble solutions generally decay whereas on the long bubble solutions the disturbance grow. Suppose that in the simulation the velocity disturbance is of the form \( u(x, y, t) = e^{kt} \hat{u}(x, y) \). Then linear growth rates can be estimated by computing the growth rate \( k \) via

\[
k = \frac{G_t(t)}{G(t)}
\]

using a finite difference formula for \( G_t \) and with various measures \( G(t) \). In the Figure 10 we have shown results using

\[
G(t) = \left( \int \int u^2 \, dx \, dy \right)^{1/2},
\]

where the integral is taken over the computational domain and approximated using a trapezoidal rule. From Figure 10 it is evident that the numerically computed growth rate is in excellent agreement to that predicted from the global stability analysis. The contour plot of the perturbation streamfunction from the simulations at a time \( t = 199.9 \) is shown in Figure 11 and compares well with 9 from the global stability analysis.

![Figure 9](image1.png)

**Figure 9.** Contours of the real part of the eigenfunction \( \psi \) for varying \( M \) and \( \text{Re} = 50000, N = 100 \) and \( \beta = 0.997 \).

![Figure 10](image2.png)

**Figure 10.** A comparison of the numerically calculated growth rate (solid line) with that from the global stability analysis (\( \text{Re}(k) = 0.021961 \)).
V. Discussion and Conclusions

Using continuation methods together with a robust numerical method it has been possible for the first time to compute steady short and long bubble solutions to the Navier-Stokes equations for marginal separation. The stability of these solutions has also been examined and it is found that for the parameter regimes studied here, the short bubble solutions are stable and the long bubble solutions globally unstable. The instability takes the form of a turning point bifurcation. For suction parameter values larger than the critical value at the turning point, there is no steady flow solution.

Direct comparisons with previous work is difficult because in many cases very few details of the actual computations performed are presented and the parameter regimes studied are also different. In the work of Hsiao & Pauley, Pauley et. al. (1990) it is found that for increased suction the steady flow regime was replaced by one with vortex shedding beyond a critical suction parameter value of 0.12. In Hsiao & Pauley the $y_{\text{max}}$ value used is 0.2096 and the lowest Reynolds number studied was $Re = 59629$. In our computations with $Re = 50000$ and with a suction value of $\beta = 0.12$ and $y_{\text{max}} = 0.2096$ we also do not find any stable solutions beyond $\beta = 0.094$.

Very few two-dimensional results are available in Alam & Sandham (2000) to compare with. They like other researchers report self-sustained vortex shedding after some critical suction parameter values, but Alam & Sandham (2000) also note that the self-sustained vortex shedding observed in many of these simulations may not be physical because of an artificially generated feedback loop via the boundary conditions used for the simulations.

Cassel et al. (2007) were not able to find multiple solutions, but this may be attributed to the different equations and conditions used at the upper-wall which can change the character of the problem. In Zahed (2010) it was found that using different boundary conditions at the upper-wall resulted in completely different results to those presented here and in some cases only one bubble solution could be found from the solution of the steady equations.

Our work has shown that more systematic studies are required to understand the manner in which a two-dimensional laminar separation bubble becomes unsteady. The numerical computations are sensitive to the boundary conditions used at the upper-wall as well as far downstream. With the conditions used in our work, it is possible to generate two types of bubble solutions, and identify critical parameters when the steady solutions cease to exist. However for larger Reynolds numbers the presence of Tollmien-Schlichting type globally unstable modes also complicates the analysis, see Logue et al.

Acknowledgments

References


