COMPUTATION OF THE VIBRATION OF A WHOLE AERO-ENGINE MODEL WITH NONLINEAR BEARINGS

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# GLOSSARY OF TERMS

## DEFINITION OF COMMON USED TERMS

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<td>Journal</td>
<td>Ring fixed to the outer race of a rolling-element bearing and mechanically prevented from rotating relative to the shaft axis. Forms inner surface of SFD.</td>
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<td>Linear part</td>
<td>The system minus the squeeze film dampers.</td>
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<td>Non-linear degree of freedom</td>
<td>Those degrees of freedom of the linear part that are associated with the nonlinear forces.</td>
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<td>Receptance</td>
<td>Frequency response function that, for a given frequency, relates the force/moment applied in the direction of one degree of freedom with the consequent displacement response on another degree of freedom.</td>
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<td>Sprung SFD</td>
<td>SFD with a parallel retainer spring.</td>
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<td>Squeeze film damper (SFD)</td>
<td>Annulus of oil filling the clearance between the journal and the inner surface of the bearing housing.</td>
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<td>State space</td>
<td>A space that is used to specify the instantaneous values of the dynamical variables (displacements and velocities) and (for a forced system) the associated value of the independent variable (i.e. time).</td>
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<td>Unsupported SFD</td>
<td>An unsprung SFD in which the journal is fully eccentric within the radial clearance under the static load in the static condition.</td>
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<td>Whole-engine model</td>
<td>A mathematical model of the structure of an aero-engine (multi-shaft), whose principal constitutions include rotors supported on one casing via SFD (s).</td>
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COMMONLY USED ABBREVIATIONS

FE       Finite element
iLP, iIP, iHP The frequency components whose frequencies are equal to the i-th multiples of the rotational speeds of LP-, IP- and HP-rotors respectively
IRM       Impulsive Receptance Method (Chapter 3)
RHBM      Receptance Harmonic Balance Method (Chapter 5)
SFD       Squeeze film damper
NH        The nominal rotational speed of the HP rotor (Chapter 7)

LIST OF SYMBOLS FOR PRINCIPAL PARAMETERS

The following list is not exhaustive. However, all parameters are defined in the main text.

Greek letter symbols are listed towards the end. Vector and matrices are in bold typeface (vector in lower case and matrices in upper case).

\[ \mathbf{a}, \mathbf{b} = [a_1 \cdots a_n]^T, [b_1 \cdots b_n]^T \]
\[ \mathbf{a} \cdot \mathbf{b} = [a_1b_1 \cdots a_nb_n]^T \]
\[ \mathbf{a} / \mathbf{b} = [a_1/b_1 \cdots a_n/b_n]^T \]
\[ \mathbf{a}^\circ = [a_1^\circ \cdots a_n^\circ]^T \]
\[ \sin(\mathbf{a}), \cos(\mathbf{a}), \exp(\mathbf{a}) \]
\[ \exp(\mathbf{a}) = [\exp(a_1) \cdots \exp(a_n)]^T \]
\[ \mathbf{A}_{J_{\infty}}(\omega) \]
\[ \mathbf{B}(k\omega) \]

accelerance matrices, eqs (5.27a, b)

matrix, eq. (5.24b)
\( C_{JJ}(\omega), C_{vv}(\omega), \ldots \) receptance matrices, eqs. (5.17a-g)

\( \tilde{C}_{vv}(\omega), \tilde{C}_{vw}(\omega) \) “incomplete” receptance matrices, eq. (5.34a, b)

\( D(k\omega), E(k\omega) \) matrices, eqs. (5.24a, 5.26a, b)

\( F(k\omega) \)

\( f \) vector of squeeze-film forces, unbalance forces, static loading (eq. (3.3))

\( f_k = f(x_k, x_k, t_k) \)

\( g = \begin{bmatrix} g_{(i)}^T & \cdots & g_{(j)}^T \end{bmatrix}^T \)

\( g_{(j)} = \begin{bmatrix} M_{(j)ii} & N_{(j)ii} & \cdots & M_{(j)ij} & N_{(j)ij} \end{bmatrix}^T \)

\( G_j \) total number of gyroscopic locations on rotor no. \( j \)

\( H_f, H_x, H_g \) modal matrices defined in eqs. (3.2, 3.5, 3.18, 3.19, 3.28)

\( \tilde{H}_v, \tilde{H}_v, \tilde{H}_w \) “incomplete” modal matrices defined by eqs. (5.33, 5.41b, c)

\( h \) time step size and film thickness for Chapter 2 and Appendix A1

\( i \) counter for non-linear bearings

\( I_{(j)p} \) polar moment of inertia at gyroscopic location no. \( p \) of rotor no. \( j \)

\( I \) identity matrix

\( j \) counter for rotors

\( J \) total number of rotors

\( k \) time-step counter in Chapter 3 and Chapter 4, counter for harmonics in Chapter 5

\( K \) maximum of number of harmonics

\( L \) matrix defined by eq. (3.29a)

\( L_{(j)} \) matrix defined by eq. (5.25c)
\( \mathbf{n} \) \n \quad \text{vector of ones}

\( n_{\text{extra}} \) \n \quad \text{number of extra unknowns (\( \mathbf{\bar{q}} \))}

\( n_p \) \n \quad \text{number of points of SFD force time history (Figure 5.1)}

\( N \) \n \quad \text{number of SFDs}

\( M_{(j)p}, N_{(j)p} \) \n \quad \text{gyroscopic moments about } x, y \text{ axes respectively at gyroscopic location no. } p \text{ of rotor no. } j

\( p \) \n \quad \text{counter for concentrated gyroscopic effect location}

\( P \) \n \quad \text{total number of rigid body modes}

\( \mathbf{P} \) \n \quad \text{block diagonal matrix of diagonal sub-matrices } \mathbf{P}_{(j)}, \ j = 1...J

\( \mathbf{P}_{(j)} = \Omega_{(j)} \text{diag}(-I_{(j)g}, I_{(j)g}, ..., -I_{(j)g}, I_{(j)g}) \) 

\( \mathbf{q} \) \n \quad \text{vector of modal co-ordinates (eq. (3.1))}

\( \mathbf{q}_k, \dot{\mathbf{q}}_k \) \n \quad = \mathbf{q}(t_k), \dot{\mathbf{q}}_k(t_k)

\( \mathbf{\hat{q}}_k, \dot{\mathbf{\hat{q}}}_k \) \n \quad \text{defined by eqs. (3.9 or 3.21)}

\( \mathbf{\bar{q}}, \mathbf{\bar{q}} \) \n \quad \text{rigid and flexible mode subvectors of } \mathbf{\bar{q}} \text{ (eq. (5.31))}

\( Q \) \n \quad \text{positive integer (eq. (5.1))}

\( \mathbf{Q}_{(j)} \) \n \quad \text{matrix defined by eq. (5.26d)}

\( r \) \n \quad \text{counter for modes}

\( R \) \n \quad \text{total number of modes considered}

\( \mathbf{R}_{xf}, \mathbf{S}_{xf} \) \n \quad \text{matrices defined by eqs. (3.13, 3.22)}

\( \mathbf{R}_{xg}, \mathbf{S}_{xg}, \mathbf{S}_{0f}, \mathbf{S}_{0g} \) \n \quad \text{matrices obtained analogously to } \mathbf{R}_{xf}, \mathbf{S}_{xf}

\( s \) \n \quad \text{counter for unbalance location on rotor no. } j \text{ used in Chapter 5}

\( S_j \) \n \quad \text{maximum of } s \text{ used in Chapter 5}

\( \mathbf{s} \) \n \quad \text{vector of state variables } \mathbf{q}, \dot{\mathbf{q}} \text{ used in Chapter 3}

\( t \) \n \quad \text{time (s)}
\[ t_k = t_{k-1} + h \]

**T**

matrix defined by eq. (3.30b)

\[ \mathbf{u}_{(j)} \]

unbalance excitation vector eq. (5.10)

\[ \mathbf{u}_{(j)\cos}, \mathbf{u}_{(j)\sin} \]

component amplitudes of \( \mathbf{u}_{(j)} \) (eq. 5.11a,b)

\[ U_{(j)s} \]

unbalance at location \( s \) on rotor no. \( j \) (kg.m)

\[ \mathbf{v} \]

\[ = \mathbf{x}_J - \mathbf{x}_B = \begin{bmatrix} \mathbf{v}_1^T & \cdots & \mathbf{v}_N^T \end{bmatrix}^T \]

\[ \mathbf{v}_i \]

vector of relative Cartesian displacements at terminals of squeeze-film no. \( i \) (eq. (3.32))

\[ \mathbf{v}_{s,i} \]

vector of static Cartesian offsets at squeeze-film no. \( i \)

\[ \mathbf{w} \]

static loading distribution vector (all rotors)

\[ x, \ y, \ z \]

Cartesian frame, Figure 1.1

\[ x_{B_i}, \ y_{B_i}, \ x_{J_i}, \ y_{J_i} \]

instantaneous Cartesian displacements of the journal and housing centres \( J_i, B_i \) at SFD no. \( i \)

\[ \mathbf{x}_J, \ \mathbf{x}_B \]

\[ = \begin{bmatrix} x_{i1} & y_{i1} & \cdots & x_{iN} & y_{iN} \\ x_{B1} & y_{B1} & \cdots & x_{BN} & y_{BN} \end{bmatrix}^T \]

\[ \mathbf{x} \]

vector of dynamic Cartesian displacements at the squeeze-films, eq. (3.33)

\[ \mathbf{x}_k, \ \mathbf{\dot{x}}_k \]

\[ = \mathbf{x}(t_k), \ \mathbf{\dot{x}}(t_k) \]

\[ \mathbf{\hat{x}}_k, \ \mathbf{\hat{\dot{x}}}_k \]

defined by eqs. (3.11)

\[ x_{d,i}, \ y_{d,i} \]

dynamic part of \( \mathbf{v}_i \), eq. (3.32)

\[ \mathbf{z} \]

vector of unknowns (eq. (5.36))

\[ \mathbf{z}_0 \]

initial approximation for \( \mathbf{z} \)

\[ \alpha \]

speed ratios, Chapter 6 only; constants of proportional damping characteristics, Chapter 7 only

\[ \beta \]

coefficients of Newmark-beta Method, Chapter 4 only; speed
ratios, Chapter 6 only; constants of proportional damping characteristics, Chapter 7 only

\[ \alpha_{(j)p}, \beta_{(j)p} \]

rotational deformation about \( x, y \) axes respectively at gyroscopic location no. \( p \) of rotor no. \( j \)

\[ \delta_{k\sigma, \Omega_{(j)}} = \begin{cases} 
1 & , \quad k\sigma = \Omega_{(j)} \\
0 & , \quad k\sigma \neq \Omega_{(j)}
\end{cases} \]

\( \zeta \)
Tolerance for approximation of speed ratio, Chapter 6 only.

\( \lambda \)
end-leakage factor of a squeeze-film damper (reference [37]), Chapter 3

\( \Lambda \), \( \tilde{\Lambda} \)
diagonal matrices, eqs. (5.39, 5.41a)

\( \psi_{(r)}^{(r)}, \psi_{B}^{(r)} \)
mass-normalised eigenvectors defining the \( x, y \) displacements of the squeeze-film terminals \( J_{i}, B_{i} \) in mode no. \( r \).

\[ \psi_{v}^{(r)} = \psi_{j}^{(r)} - \psi_{B}^{(r)} \]

\( \psi_{(r)}^{(r)}_{sel} \)
sub-vector of \( \psi_{j}^{(r)} \) for selected journals (section 5.2.3)

\( \psi_{u(j)}^{(r)}, \psi_{g}^{(r)}, \psi_{w}^{(r)} \)
mass-normalised eigenvectors evaluated at degrees of freedom corresponding to directions and locations of elements in \( u_{(j)} \), \( g \), and \( w \)

\( \psi_{x}^{(r)} \)
value of \( x \) in \( r^{th} \) mass-normalised mode

\( \psi_{f}^{(r)}, \psi_{g}^{(r)} \)
mass-normalised eigenvectors evaluated at degrees of freedom corresponding to directions and locations of elements in \( f \), \( g \)

\( \psi_{0}^{(r)} \)
mass-normalised eigenvectors evaluated at degrees of freedom in \( \theta \)

\[ \theta = \begin{bmatrix} \theta_{(i)}^{T} & \cdots & \theta_{(j)}^{T} \end{bmatrix}^{T} \]

\[ \theta_{(j)} = \begin{bmatrix} \beta_{(j)1} & \alpha_{(j)1} & \cdots & \beta_{(j)g_{j}} & \alpha_{(j)g_{j}} \end{bmatrix}^{T} \]
\( \dot{\theta}_k \) \( = \theta(t_k) \)

\( \dot{\hat{\theta}}_k \) defined by eq. (3.27b)

\( \sigma \) fundamental circular frequency of the RHBM (eq. (5.1))

\( \omega \) generic circular frequency

\( \omega_r \) natural circular frequency of mode no. \( r \)

\( \omega \) vector of natural circular frequencies in eq. (3.1)

\( \Omega_{(j)} \) rotational speed of rotor no. \( j \) (rad/s)

\( \Omega_{(j)ref} \) reference unbalanced shaft speed (eq. (5.1))

\( \gamma \) coefficients of Newmark-beta Method, Chapter 4 only;

\( \gamma(j) \) angle defined in text (below eqs. (5.11a,b))

\( \phi(j) \) angle defined in text (below eqs. (5.11a,b))

\( \tau \) local time over interval \([t_{k-1}, t_k]\)

\( \chi \) vector function of \( s \) and \( t \) (section 3.3.2.1, Chapter 3) and vector function of \( z, \Omega_{(j)ref} \) and \( Q \) (eq. (5.37), Chapter 5)

\( \overline{\rho} \), \( \rho^{(k)}_{\cos}, \rho^{(k)}_{\sin} \) vectors of Fourier coefficients of all SFD forces (eq. 5.8a-c)

\( \rho \) \( = \{ \rho \} = [\rho_1^T \cdots \rho_5^T]^T \)

\( \rho_i \) \( = [Q_{x_i} \ Q_{y_i}]^T \) (Cartesian forces on journal at squeeze-film no. \( i \))

\( \cdot^T \) matrix/vector transpose

\( \sim \) modal parameters pertaining to flexible modes

\( \tilde{\cdot} \) modal parameters pertaining to rigid body modes

\( \cdot \) \( d(\cdot)/dt \)

\( \tilde{\cdot}(k), \tilde{\cdot}(k)_{\cos}, \tilde{\cdot}(k)_{\sin} \) Fourier coefficients of vector ( ), e.g. eq. (5.3)
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Aero-engine assemblies are complex structures typically involving two or three nested rotors mounted within a flexible casing via squeeze-film damper (SFD) bearings. The deployment of SFDs into such structures is highly cost-effective but requires careful calculation since they can be highly nonlinear in their performance, particularly if they are unsupported (i.e. without a retainer spring). The direct study of whole-engine models with nonlinear bearings has been severely limited by the fact that current nonlinear computational techniques are not well-suited for complex large-order systems. The main contributions of this thesis are:

- A procedure for unbalance response computation, suitable for generic whole-engine models with nonlinear bearings, which significantly extends the capability of current finite element packages. This comprises two novel nonlinear computational techniques: an implicit time domain integrator referred to as the Impulsive Receptance Method (IRM) that enables rapid computation in the time domain; a whole-engine Receptance Harmonic Balance Method (RHBM) for rapid calculation of the periodic response in the frequency domain. Both methods use modal data calculated from a one-off analysis of the linear part of the engine at zero speed.

- First-ever analyses on real twin-spool and three-spool engines. These studies illustrate the practical use of these solvers, provide an insight into the nonlinear dynamics of whole-engines and correlate with a limited amount of industrial experimental data.

Both IRM and RHBM are directly formulated in terms of the relative response at the terminals of the nonlinear bearings. This makes them practically immune to the number of modes that need to be included, which runs into several hundreds for a typical engine. The two solvers are extensively tested on two/three-shaft engine models (with 5-6 SFDs) provided by a leading engine manufacturer using an SFD model that is used in industry.

The tests show the IRM to be many times faster than an established robust conventional implicit integrator while achieving a similar level of accuracy. It is also shown to be more reliable than another popular implicit algorithm. The RHBM enables, for the first time, the frequency domain computation of the nonlinear response of whole-engine models. Its use is illustrated for both Single-Frequency Unbalance (SFU) excitation (unbalance confined to only one shaft) and Multi-Frequency Unbalance (MFU) excitation (unbalance located on two or more shafts, rotating at different speeds). Excellent correlation is demonstrated between RHBM and IRM.

The parametric studies compare and contrast the frequency spectra for SFU and MFU cases. They also reveal the varying degree of lift at the unsupported SFDs. The sensitivity of the response to end-sealing and bearing housing alignment is also illustrated. It is demonstrated that the use of suitably preloaded vertically oriented “bump-springs” at the SFDs of heavy rotors produces a significant improvement in journal lift. It is also shown that the consideration of a slight amount of distributed damping in the structure significantly affects the predicted casing vibration levels, bringing them closer to measured levels, while having little effect on the SFD orbits.
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1 INTRODUCTION

Aero-engine assemblies typically have at least two nested rotors mounted within a flexible casing via squeeze-film damper (SFD) bearings (see Figure 1.1). The role of the SFDs is to attenuate vibrations and transmitted forces. A SFD is a film of oil surrounding the outer race of a rolling element bearing. The inner surface of the damper is formed by the “journal” which is a ring fixed to the outer race of the rolling element bearing. The journal is prevented from rotating but is free to orbit within the oil-filled radial clearance in the bearing housing. Two SFD arrangements are shown in Figures 1.2 and 1.3. The journal rotation can be blocked by a retainer spring, if one is fitted, as in the arrangement of Figure 1.3. Such a spring is typically in the form of a squirrel cage and is in parallel with the SFD (i.e. supports the journal on the bearing housing). For the unsprung SFD in Figure 1.2, rotation of the journal is prevented through the use of dogs projecting from the journal. These engage with similar ones on one of the end-plates with sufficient clearance so as not to obstruct orbital motion of the journal within the annular clearance.

In a typical engine of the leading European aero-engine manufacturer, most of the SFDs are not supported by a parallel retainer spring. In the typical twin-spool engine design of Figure 1.1 it is seen that a parallel retainer spring is only used with one SFD at the end of each rotor for axial location. The rest of the SFDs are said to be “unsupported”. The omission of the retainer spring eliminates the problem of fatigue in the spring and consequent maintenance costs. However, an unsupported SFD bearing cannot support a
static load in the absence of any vibration of the journal relative to the housing. This is clearly seen in Figure 1.1, where the journals of bearings nos. 4 and 5 will be fully eccentric in their housings under the weight of their respective rotors in the absence of any relative vibration. Such bearings therefore rely on the vibrations produced by the ever-present residual unbalance in the rotors to generate a pressure that supports the static load, in addition to absorbing vibration energy.

The deployment of SFDs in aero-engines is a highly cost-effective means of introducing damping into an otherwise lightly damped structure. However, an SFD is a nonlinear element, i.e. its forces are nonlinear functions of the relative displacements and velocities across it. This is especially so for unsupported SFDs. Hence, their deployment in an aero-engine requires careful unbalance response calculations that take account of the SFDs’ nonlinearity, to ascertain smooth running, particularly at the unsupported SFDs. These calculations need to take into account the dynamics of the complete assembly, comprising unbalanced rotors rotating at different speeds interacting through the SFDs via a flexible casing. Calculations that treat each rotor in isolation, as typically done in industry, do not give a realistic picture of the performance of the SFD bearings, let alone the dynamics of the engine itself.

Calculations for the response of a nonlinear system to a periodic excitation can be done in either the time or frequency domains. The former approach uses a numerical technique to march the equations of motion forward in time past the transient stage to predict the steady-state response. Depending on the governing dynamics, this may not necessarily be periodic. The frequency domain approach, based on harmonic balance or trigonometric collocation, is much faster but is restricted to a pre-assumed periodic steady-state response. An efficient computational facility takes advantage of the relative merits of both approaches through an integrated strategy that makes effective use of both.
Most proposed computational techniques, although ostensibly generic, have almost invariably been illustrated on simple rotor-bearing systems. In fact, they are not tractable to real whole engine models due to the complexity of these structures. Moreover, as will be seen throughout this thesis, even a transformation of the problem to modal space would require the retention of a large number of modes. As a result of the inadequacy of current computational techniques, the direct study of real aero-engines has been severely hindered.

The main contribution of this thesis is to resolve this problem through the development of highly efficient time/frequency domain solvers for whole-engine models. This enables the computational analyses of the unbalance response of real twin/three spool engines, which is done for the first time in this thesis.

The proposed analysis of the nonlinear rotating assembly uses modal data calculated from a one-off analysis of the linear part of the engine at zero speed (see Figure 1.4). Any convenient finite element (FE) package (e.g. Nastran®) can be used for the linear computation. Specially-written Matlab® routines are used for the subsequent nonlinear computation. Due to the complementary nature and relative merits of time and frequency domain solution techniques, both options are available. The two novel solvers created in this project are:

- An implicit time domain solver termed the Impulsive Receptance Method (IRM). The IRM equations relate the relative displacements and velocities at the SFDs with the motion-dependent forces and other excitations acting on the linear part. Hence, the IRM’s computational efficiency is largely immune to the number of modes since the number of equations to be solved at each time step is dependent only on the number of nonlinear elements.
- A whole-engine Receptance Harmonic Balance Method (RHBM). This is a significant development of the earlier elementary RHBM (used on simple single-
shaft test rigs) that is capable of dealing with the complexities of a real engine structure.

As their names suggest, these two solvers can be regarded as time/frequency domain analogues of each other. These methods are computationally validated against conventional solvers for a realistically-sized representative twin-spool engine model provided by a leading aero-engine manufacturer. The IRM is shown to be many times faster than a conventional robust implicit solver while retaining a similar level of accuracy. It is also shown to be more reliable than another popular implicit algorithm. The RHBM is successfully tested for both single-frequency unbalance excitation (SFU- unbalance distribution confined to one rotor) and multi-frequency unbalance excitation (MFU- unbalance on both rotors, rotating at different speeds), showing excellent correlation with IRM.

The methods are then applied to a three-spool engine model provided by a leading manufacturer. The aims of the study are two fold: i) to present some preliminary results of a parametric study into a three-spool aero-engine assembly that provides insight into the dynamics of such a structure; ii) to propose a technique that makes use of both IRM and RHBM in producing the speed responses under MFU excitation (from all three rotors), with a realistic speed relation between the rotors.

The final part of the research makes a first-ever attempt to relate a limited amount of measured vibration data from a manufacturer’s test engine with the predictions from a whole engine-model of the same engine using the IRM. Motivated by this study, the IRM is developed to include damping distributed within the linear part of the structure.

The methods developed can accommodate a number of generic concentrated nonlinearities. In this thesis the nonlinearities are the SFDs and their forces are calculated from a standard model that is widely used in industry.
1.1 SUMMARY OF THESIS OBJECTIVES AND CONTRIBUTIONS

The thesis objectives are:

- To create and computationally validate an efficient procedure for the computation of the vibration of a whole aero-engine model with nonlinear bearings.

- To perform first-ever parametric analyses of real twin-spool and three-spool engines under realistic operating conditions, with the following aims:
  
  - To illustrate the most effective practical use of the algorithms developed;
  
  - To achieve a better understanding of the nonlinear dynamics of these complex structures;
  
  - To correlate the predictions with a limited amount of experimental data and draw conclusions from this.

The thesis contributions to both academia and industry are:

- A procedure for unbalance response computation, suitable for generic whole-engine models with nonlinear bearings, which significantly extends the capability of current FE packages (Figure 1.4). This comprises two novel solvers: the IRM for rapid computation in the time domain; a whole-engine RHBM for rapid calculation in the frequency domain.

- First-ever analyses on real twin-spool and three-spool engines. These studies illustrate the practical use of these solvers, provide an insight into the nonlinear dynamics of whole-engines and correlate with a limited amount of industrial experimental data.

The salient contributions to knowledge arising from the above can be summarized as follows:
• The problem of a large number of modes in aero-engines with concentrated nonlinearities, that severely reduces the computation speed of conventional implicit solvers, can be overcome through a convolution integral approach that uses “impulsive receptances”. These are the time-domain analogues of receptance functions and relate the response at the nonlinear elements with the motion-dependent forces and other excitations acting on the linear part of the structure. The IRM’s computational efficiency is therefore virtually immune to the number of modes since the number of equations to be solved at each time-step is simply four times the number of nonlinear elements. The IRM’s accuracy is on a par or superior to that of conventional implicit solvers.

• The elementary RHBM can be developed to effectively handle the various complexities of aero-engine dynamics such as: (a) statically indeterminate problems in the “zeroth” harmonic of the vibration; (b) gyroscopic effects; (c) MFU excitation.

• An efficient analysis of twin-spool and three-spool engines operating under a realistic speed relation between the rotors can be performed through an approach that exploits the relative merits of IRM and RHBM.

• The use of suitably preloaded vertically oriented “bump-springs” at the SFDs of the heavy low-pressure rotor of a typical three-spool engine produces a significant improvement in journal lift, allowing for smoother running of the shaft.

• The consideration of a slight amount of distributed damping in the linear part of the structure significantly affects the predicted casing vibration levels, bringing them closer to measured levels, while having little effect on the SFD orbits.

The work in Chapters 3-6 respectively forms the basis of the following journal publications:


Additionally, the work has been presented at two refereed conferences: IMechE VIRM9 (Exeter, UK, 2008) and ASME Turbo Expo 2009 (Florida, USA).

1.2 SUMMARY OF THESIS STRUCTURE

This chapter is followed by a critical review of previous research (Chapter 2). Chapter 3 gives the detailed documentation of the IRM and its validation against a robust conventional technique when applied to a representative two-spool engine model with around 900 modes of the engine structure taken into account. Chapter 4 provides further proof of the validity of the IRM, showing superior reliability relative to a Newmark-beta based method. An aero-engine oriented RHBM is developed in Chapter 5 and tested on the model of a twin-spool engine for both SFU and MFU excitation with a constant speed.
ratio. In Chapter 6, both IRM and RHBM are used to tackle an MFU excitation problem on a three-spool engine model. The theoretical analysis is done with more than 2000 modes included in the nonlinear computation. The variation of the ratio between the rotational speeds of the rotors is considered in a preliminary parametric analysis into the effect of SFD end-sealing, rotor unbalance distribution and the use of bump-springs. The original work of the thesis is completed in Chapter 7 showing the outcomes of an effort to relate the predicted results on a twin-shaft engine with a limited amount of experimental data provided by an aero-engine developer. The analysis in that chapter motivates the development and testing of an enhanced IRM that accommodates proportional damping. The general conclusions are drawn in Chapter 8, along with recommendations for future research.
Figure 1.1: Schematic of a representative twin-spool engine [1] (abbreviations “LP”, “HP” stand for “low pressure”, “high pressure” respectively; Bi and Ji (i=1…5) respectively denote the centres of the housings and the journals of the SFDs)
Figure 1.2 Unsprung SFD bearing assembly

Figure 1.3 Sprung SFD bearing assembly [2]
Figure 1.4: Overall computational procedure
2 REVIEW OF PREVIOUS RESEARCH

2.1 INTRODUCTION

This chapter presents a critical review of research into the modelling and analysis of squeeze-film damped rotordynamic systems. The main part of the chapter is devoted to a review of solution techniques for the unbalance response (Section 2.2). This is then followed by a review of research into nonlinear phenomena (Section 2.3). An overview of methods for modelling the SFD element is then given in Section 2.4. Finally, a brief overview of research into parametric analysis of SFD-rotor systems is given in Section 2.5.

2.2 SOLUTION TECHNIQUES FOR THE UNBALANCE RESPONSE

As stated in the Introduction, the deployment of SFDs into aero-engines requires careful unbalance response calculations that take account of the SFDs’ nonlinearity, to ascertain smooth running particularly at the unsupported SFDs. Various techniques have been proposed by academia for the computation of the unbalance vibration of simple rotor/nonlinear bearing systems. However, such tools are not well-suited for complex systems that have many degrees of freedom, thereby severely limiting the direct study of real engine structures. This is evidenced by the fact that there exists little, if any, published research on the computational analysis of such structures. Indeed, proposed computational techniques, although ostensibly generic, have almost invariably been illustrated on simple rotor-bearing systems e.g. [3, 9, 10, 19].

The solution techniques for the response of a nonlinear system subjected to periodic external excitation (e.g. rotor unbalance) can be broadly categorised into either time
domain or frequency domain techniques. Time domain (“time-marching”) techniques
march forward in time past the initial transient stage to yield the steady-state response,
which may not necessarily be periodic. Frequency domain (“periodic solution”) techniques,
which employ analytical methods like harmonic balance [3-8] or trigonometric
collocation [9], are inherently very much faster since they directly yield steady-state
solutions that are assumed to be periodic at an assumed fundamental frequency. However,
such periodic vibration may not always be physically possible, in which case the computed
periodic solutions represent a dynamic state that is unstable to the slightest physical
perturbation. In such a case, a time-marching technique is used, which yields the actual
steady-state response after the initial transients have died out. Moreover, in order to start
off a HB solution procedure over a range of speeds, a good initial approximation at just
one speed is required. This can only be reliably provided by a time domain solution,
especially for a large order system [10].

An efficient computational facility takes advantage of the relative merits of both
categories through an integrated approach that makes effective use of both [10].

### 2.2.1 Time Domain Techniques

The common strategy of all time domain techniques for the analysis of rotating systems
with concentrated nonlinearities like fluid bearings is to consider the forces from these
elements as external to the remaining linear part (i.e. “right-hand forces”). First to be
discussed will be the main types of numerical integration algorithm used for the solution of
the resulting equations of motion. The discussion then focuses on current methods to
reduce the size of the problem (i.e. number of equations to be solved).

Algorithms for the solution of initial value problems involving a system of ordinary
differential equations (ODE) can be either explicit or implicit [11]. Hulmes and Holmes
[12], when investigating the simplest model of a system incorporating squeeze-film dampers, used the Runge-Kutta-Merson procedure to solve the equation of motion. This kind of algorithm uses explicit formulae to find the solution at a certain time step from the solution at the previous step. Explicit methods are not suitable for systems that are numerically “stiff” [13]. With such systems, in order to satisfy a prescribed tolerance, an explicit solver requires increasingly small step sizes, making the solution highly time consuming and causing problems with data storage. Craven and Holmes [14] observed that even a simple rotor-bearing system is prone to this problem. Hence, implicit time-marching schemes are preferred [13].

In an early research on the effect of clearance in bearings, Craven and Holmes [14] proposed a fast numerical integration method for solving the full dynamical equations governing the motion of a crankshaft mounted on journal bearings. Later on, this method was introduced for squeeze-film damped systems by Chu and Holmes [15]. This method implemented an implicit one-step algorithm, developed from the trapezium rule of integration, to transform the ODE into a system of nonlinear algebraic equations which was to be solved at each time step to get the instantaneous response. This method was shown to be much faster than the (explicit) Runge-Kutta technique when applied to a simple flexible rotor system with one SFD. However, the constant step-size approach used in [15] encountered divergence in some cases that were not clearly identified.

At around the same time as Chu and Holmes [15], Mathworks introduced into their Matlab code a whole suite of solvers suitable for stiff systems, based on more sophisticated rules such as the Modified Rosenbrock method used in solver *ode23s* [13, 16]. This solver is equipped with an adaptive step control algorithm which helps overcome problems with high numerical stiffness generally.

Another implicit algorithm is the Newmark-Beta method [17]. An implementation of this is solver *SOL 129* provided by the MD Nastran® software [18]. This has no adaptive
step control and the convergence of solution when using this solver is quoted to be uncertain [18]. It is to be mentioned that this time domain solver facility is considered by the author to be very restrictive for whole-engine calculations in other ways: (i) limitations on the choice of the model for the nonlinear bearing element (limited to either “long” or “short” bearings, see Section 2.4); (ii) it restricts its application to the direct integration of the finite-element (FE) equations (i.e. does lend itself to the modal transformation method discussed below); (iii) unsuitability of this solver and the aforementioned implicit solvers to whole engine modelling due to the issues described in the following part of this section.

When dealing with large-order systems, the direct integration of the finite-element (FE) equations is highly time-consuming and so, is generally avoided [17]. Hence, prior to implementing the above-mentioned algorithms to such applications, a model reduction is performed. Two alternative model-reduction approaches have been proposed in the literature.

One model-reduction approach is to model the system using a time-domain transfer matrix method e.g. [19, 20]. This technique is an alternative to the FE approach of modelling the structure. The transient response is evolved step by step using a time marching algorithm (e.g. the Newmark Method). However, instead of solving the assembled FE equations of the whole rotor, this technique uses the equilibrium equation of each element divided along the rotor axis and the continuity in the connection between successive elements to deliver a recursion scheme. The instantaneous displacements of one node can be formulated from displacements of the successive node along the shaft. For instance, with a rotor divided into a number of elements (say, \( n \)), at each time-step of the solution, all nodal displacements are found after inverting consecutively \( n \) matrices which are half the size of the stiffness matrix of the corresponding elements. Such a method helps save computation time and instantaneous memory resource in comparison with solving the global FE equation. However, there are some restrictions to this method.
For example, all nodal displacements in time-history have to be calculated, which appears to be unnecessary in practice. Furthermore, the iteration formulae are not applicable to rotors with complex structure. Moreover, unlike FE, the transfer matrix approach is not suited for modelling complex sub-systems such as the engine casing. For this reason, the transfer matrix approach has not crossed over to industrial whole-engine modelling.

The other model-reduction approach, which is preferred since it exploits the modelling capability of FE, is to transform the FE problem into modal space and retain only a limited number of modal equations of motion [17]. The resulting equations are then solved by an implicit time-marching scheme. For example, the solver \textit{ode23s}© was used extensively by Bonello et al. [10, 21-23] to integrate the modal equations of motion of various test rigs. In [24], the modal equations of motion of a rotor-bearing system were integrated using an iterative predictor-corrector scheme combining the Central Difference and Newmark-Beta methods. The system considered there was a gas turbine rotor mounted on two bearing pedestals simplified as mass-spring systems.

Conventional implicit integration schemes transform the differential equations of motion (whether FE or modal) into an equal number of algebraic equations which then have to be solved at each time step to obtain the current state variables. Moreover, if the system contains nonlinear motion-dependent forces, then this solution has to be obtained by iteration. If the number of modes retained is small then a conventional implicit solver works efficiently. However, as shall be shown in Chapter 3, with a real engine a modal transformation would still necessitate the retention of a very large number of modes. Hence, due to the large number of modal equations, the time-marching process slows down to impractical levels. It is this fact that motivated the development of a novel implicit “Impulsive Receptance Method” (IRM) to analyse an aero-engine (Chapter 3).
2.2.2 Frequency Domain Techniques

Various frequency-domain techniques have been proposed for the computation of the unbalance vibration and illustrated on simple rotor-bearing systems. Some of the earlier proposed methods were limited to synchronous motion e.g. [4, 25]. Later methods e.g. [6] admitted harmonics in the response. However, the main flaw with all such methods is that they are even less tractable to complex systems like aero-engines than their time domain counterparts. Indeed, commercial Finite-Element (FE) codes do not possess any frequency domain solver for nonlinear whole-engine modelling.

The common strategy of all proposed frequency domain techniques for the analysis of rotating systems with concentrated nonlinearities like fluid bearings is to consider the forces from these elements as external to the remaining linear part. Assuming that the steady-state response is periodic at an assumed fundamental frequency with an adequate number of harmonics, the principle of harmonic balance (HB) [3-8] or trigonometric collocation (TC) [9] can be applied to the system. Regardless of whether HB or TC is used, what is ultimately solved is a set of nonlinear algebraic equations whose unknowns are the Fourier coefficients of the degrees of freedom at the nonlinearities only. The difference between the various frequency domain techniques lies mainly in the way these equations are generated (i.e. how HB or TC was applied to the system). Such techniques can then be classified as follows: (a) Transfer Matrix approach e.g. [5]; (b) Direct Finite Element (FE) approach e.g. [6-8]; (c) Direct modal approach [9]; (d) Receptance Harmonic Balance approach [10, 21-23]. The transfer matrix approach is not suited for modelling complex sub-systems such as the engine casing. The direct FE approach applies the HB or TC method to the full FE equations of the system. A condensation technique, involving the inversion of large matrices, can then be applied to reduce the number of unknowns to those associated only with the nonlinear degrees of freedom. Such a process is not feasible for a whole-engine model. As stated in [9], approach (c) is more tractable to large systems
since it uses component mode synthesis. However, this requires the prior solution of the eigenproblem of the rotating linear part and modal truncation. This latter will be shown in the following chapters to be a problem with a real engine due to a high modal density at relatively low frequencies. Moreover, gyroscopic effects would necessitate the solution of the eigenproblem for each speed.

The Receptance Harmonic Balance method (RHBM), as presented in [10, 21-23], shows great potential for the analysis of complex structures. The nonlinear algebraic equations are yielded without the need of any costly condensation, by applying the force-displacement relationships for each harmonic of the vibration using the frequency response functions (receptances) of the linear part. There is no problem with modal truncation since any desired number of modes can be included in the receptance expressions. This concept was first introduced by Ren and Beards [26] for a simple non-rotating structure. The RHBM was adapted by Bonello et al. [10, 21-23] for a rotating system comprising a single shaft with negligible gyroscopic effect running on nonlinear bearings housed within a flexible support structure. Among the issues considered in [10, 21-23] was the question of how to derive the equations for the zero-frequency harmonic (i.e. the mean component of the vibration) when the presence of rigid body modes of the linear part cause the relevant receptance functions to be undefined at zero-frequency. Two common examples of this case are: a rotor pivoted at one end by a self-aligning rolling-element bearing and running on an unsupported SFD at the other end [10, 21]; or a rotor running only on two unsupported SFDs [23]. The linear part of the system then comprises a pinned-free rotor in the former case, and a free-free rotor in the latter case. The solution provided in [10, 21, 23] was to consider the fact that the mean components of both velocity and acceleration of the vibration of the nonlinear system are zero. Hence, at zero-frequency such a rotor is in a state of “pseudo-static” equilibrium under the action of the static external forces and the mean SFD forces. The term “pseudo” is used here since the latter forces are themselves
generated by the vibration. The works in [10, 21, 23] then proceeded to solve for the vibration response of a \textit{statically-determinate} rotor with unsupported SFDs by including the pseudo-static equilibrium equations with the rest of the dynamic frequency response equations.

If the work in [10, 21-23] is to be used as the basis for solving the unbalance response of a whole aero-engine model the following outstanding issues need to be resolved:

- The presence of rotors that introduce rigid body modes in the linear part but are \textit{statically indeterminate} when the nonlinear bearings are in place: this is clearly the case for the low pressure spool in Figure 1.1, where the rigid body modes define pivoting motion about J$_1$.

- Gyroscopic effects: Receptances are computed from a modal series summation [17]. Hence, gyroscopy is an issue since, for RHBM to be feasible, the receptances should pertain to the linear part under non-rotational conditions, making them independent of the rotors’ rotational speeds, thus allowing a one-off eigenproblem. Hence, a means has to be found of including gyroscopic effects into the nonlinear problem without adding to the number of unknowns.

- An efficient post-solution recovery of the full set of degrees of freedom.

- Unbalance excitation from more than one rotor: for single shaft systems the external excitation (unbalance) is at a single frequency – Single Frequency Unbalance (SFU). This also applies for multi-spool systems where the unbalance distribution is confined to only one of the shafts. Frequency domain techniques have so far been used only on SFU problems. In practice, the unbalance may be located on more than one rotor and the unbalanced rotors will turn at different speeds, resulting in Multi-Frequency Unbalance (MFU). It is clearly desirable for the method to be able to handle MFU conditions.
The above issues are resolved in Chapter 5, where a whole-engine RHBM is developed.

2.3 NONLINEAR PHENOMENA IN SFD ROTOR-BEARING SYSTEMS

A number of research works have investigated nonlinear vibration phenomena in rotating systems incorporating SFD bearings. As mentioned previously, despite the good damping capacity of SFDs in attenuating the level of vibrations and transmitted forces, they introduce undesirable nonlinear phenomena. Examples of nonlinear phenomena that are typical of squeeze-film damped rotating systems are amplitude jumps, non-synchronous vibrations and bilinear oscillator effects.

The “jump phenomenon” refers to the abrupt increase or decrease in amplitude of vibration during run-up or run-down of the rotational speed. An illustration of such phenomena can be found in [27]. In this research work, Bonello applied the arc-length continuation technique to advance the solution process of the RHBM over a range of speeds and applied this approach to various SFD-rotor systems. In a system comprising a rigid rotor running in one or two SFD bearings supported by flexible pedestal foundations, the speed response curves revealed amplitude jumps occurring when the rotational speed goes into the bistable regions. Another illustration for this phenomenon can be found in a research by McLean and Hahn [28].

Non-synchronous vibrations, which contain frequency components differing from the excitation frequencies, have been observed in many researches. Such phenomena may be in form of sub-harmonic, super-harmonic vibrations, or a combination of two or more frequencies that are not in a simple integer relation (which could result in quasi-periodicity [27]). Experimental observations and a theoretical explanation for their occurrence were presented by Holmes and Dede [29]. Chaotic motion, wherein the frequency spectrum is continuous, is also possible [27].
Ehrich [30, 31] observed bilinear-oscillator phenomena in the vibration of a rotor operating within the radial clearance between the journal and a flexibly supported bearing housing. A bilinear oscillator phenomenon occurs when the journal of the rotor vibrates in the local contact with the stator, normally, when the unbalance excitation frequency is lower than the bounce natural frequency of the system. Such bilinear oscillator phenomena were also observed by Bonello [27, 21, 23] in various configurations where unsupported SFDs were used.

It was stated in Section 2.2 that periodic vibration may not always be physically possible in a periodically excited nonlinear system. In this case the periodic solutions calculated by a frequency domain solver like HB represent a dynamic state that is unstable to the slightest physical perturbation. For example, as the rotational speed is increased, periodic solutions of period $T$ (where $T$ is the period of excitation) may become unstable [32, 27, 23]. In this case a “bifurcation” [32] occurs and the stable motion is then either: (i) a different periodic motion of the same period (tangent bifurcation, associated with amplitude jumps); (ii) a periodic motion of twice the period (“period doubling bifurcation”) or a quasi-periodic motion containing an additional frequency that is irrationally related to the original one (“secondary Hopf bifurcation”). Bonello [27] observed all these in both theory and experiment. In addition, the vibration can descend into a chaotic state through successive period-doubling bifurcations, as observed in [33, 34]. Bonello [27] also mentions quasi-periodicity as a potential route to chaos.

Periodic solutions can be tested for stability by two methods [27]: (a) cross-checking with time-marching, which always gives the stable solution; (b) use of a stability-check algorithm. In approach (a) the time domain solver is started off with initial conditions on the period orbit. If the time domain trajectory remains approximately on the periodic orbit, then the orbit is stable. If it diverges then the orbit is unstable. Approach (b) involves the application of either Floquet Theory [10, 27, 32] or Hill’s Method [8] to test the stability of
the periodic solution to an infinitesimal perturbation. The Floquet theory approach involves the calculation of the “monodromy” matrix that describes the evolution of the perturbation. The stability of the periodic solution is governed by the eigenvalues of this matrix [32]. Calculation of the monodromy matrix by the standard method described in [32] is time-consuming but the fast approximate method of Hsu [35] can be used to accelerate the process. The applications of this method can be found in [6, 33, 34, 36, 27, 21-23]. In Hill’s Method, used in [8], the harmonic balance equation is used to establish the perturbation equation and this leads to a conventional eigenvalue problem.

The main problem with the methods of both Hsu and Hill is that they are not feasible for systems with a very large number of modes like aero-engines. Finding a solution to this problem is outside the scope of this thesis. Hence, stability-check routines will not be developed or used in this thesis. The combined use of time/frequency domain techniques will be shown to be more than adequate for whole-engine modelling.

2.4 THE SFD MODEL

The SFD model governs the relationship between the SFD force and the motion of the journal relative to the bearing housing. In operation, the condition of the oil film depends on many parameters, such as geometric dimensions, pressure, temperature and physical features of the oil film. The reliability of the results of the solution techniques described in Section 2.2 clearly depends on the choice of the SFD model used.

The calculation of the SFD forces is typically based on the incompressible Reynolds Equation, which governs the spatial distribution of the instantaneous pressure over one land of the SFD. With reference to Figure 2.1, this is expressed as [37]:

\[
\frac{1}{R^2} \frac{\partial}{\partial \theta} \left( h^3 \frac{\partial \hat{P}}{\partial \theta} \right) + \frac{\partial}{\partial \zeta} \left( h^3 \frac{\partial \hat{P}}{\partial \zeta} \right) = 12\eta c (\dot{\epsilon} \cos \theta + \epsilon \psi \sin \theta) \tag{2.1}
\]
where \( h = e(1 + e \cos \theta) \) is the oil film thickness, \( e = \epsilon/c \) is the non-dimensional eccentricity of the journal centre J from the bearing housing centre B and \( \psi \) is the attitude angle. \( R \) is the bearing housing bore radius. Among other things [38], the Reynolds equation (2.1) assumes no fluid inertia, an incompressible fluid (i.e. density independent of pressure) and constant viscosity \( \eta \). Eq. (2.1) can be solved for given boundary conditions to obtain the pressure distribution \( p(\theta, z) \) at any instant in time. The instantaneous radial and tangential squeeze film forces \( Q_R, Q_T \) respectively, acting on the journal, are then obtained by integrating the pressure distribution \( p(\theta, z) \) after truncating it below a minimum pressure at which the oil film is assumed to cavitate (i.e. rupture due to the formation of bubbles) [10]:

\[
\begin{bmatrix}
Q_R \\
Q_T
\end{bmatrix}
= -n_L R L \int_{-L/2}^{L/2} \int_0^{2\pi} \left[ \frac{\cos \theta}{\sin \theta} \right] p_{t}(\theta, Z) \, d\theta \, dZ
\]

where \( n_L \) is the number of SFD lands, \( R \) the bearing radius, \( L \) the land length, \( p_{t}(\theta, Z) \) is the truncated pressure distribution in the oil film [10]:

\[
p_{t}(\theta, Z) = \begin{cases} 
p(\theta, Z) & \text{if } p(\theta, Z) > p_{cav} \\
p_{cav} & \text{if } p(\theta, Z) \leq p_{cav}
\end{cases}
\]

and \( p_{cav} \) is the cavitation pressure. Feng and Hahn [38] state that the film cavitation can be either vaporous or gaseous. The former involves oil vaporisation together with the release of dissolved gases. The latter involves ambient air entering the oil film.

Since fluid inertia is neglected, the Reynolds equation (2.1) still applies when B is moving (Figure 4.2). The SFD forces are thus (nonlinear) functions of the relative displacements and velocities across the damper.

There are two important issues that need consideration: (i) the choice of the expression for \( p(\theta, z) \); (ii) the choice of cavitation model.
Analytical expressions for \( p(\theta, z) \) are only possible for two conditions. If the damper is unsealed and axially short \((L/(2R) \leq 0.25\) [39]), the *short bearing approximation* of the Reynolds equation applies. In this approximation, it is assumed that the pressure gradient in the circumferential \((\theta)\) direction is negligible relative to that in the axial \((z)\) direction (i.e. \(\partial p/\partial z >> \partial p/\partial \theta\)) so that the first term on the left hand side of eq. (2.1) can be neglected. Integration of the resulting equation with respect to \(z\) and application of the pressure boundary conditions yields the short bearing pressure expression \( p_{\text{short}}(\theta, z) \). If the damper is axially very long or completely sealed, then the long bearing approximation applies. In this case, it is assumed there is no flow in the axial direction and so the second term of eq. (2.1) can be dropped. Integration of the resulting equation yields the *long bearing pressure approximation* \( p_{\text{long}}(\theta) \) [37].

SFDs may be partially sealed at their outlets with devices like end-plates, piston rings or O-rings [37, 40], as shown in Figures 1.2, 1.3 and 2.1. For these configurations, neither the short nor the long bearing approximations are satisfactory. In this case two alternative approaches are normally used: (a) numerical solution of the full two-dimensional Reynolds Equation; (b) use of a combination of the short and long bearing expressions.

The numerical solution of the Reynolds Equation is typically obtained through the method of finite-difference (FD). This method discretises pressure distribution into a grid of pressure points covering the squeeze film. The partial derivatives of the Reynolds governing equation are approximated by the three-point central difference formulae. The resulting finite difference equations can then be solved by recursion using the Column Method proposed by Castelli and Shapiro [41]. The finite-difference technique helps retain the high level of generality of an oil film model, capable of handling various configurations of inlet and outlet (feed ports, grooves, seals). Besides, with this technique, fine details like deformation of the housing can be accommodated. However, especially
for the two-dimensional problem, FD is highly time-consuming because of the burden of matrix-inversion and the sequential nature of Castelli’s Column Method, which processes need to be performed at each considered instant in time. The commercial FE package MD Nastran® [42] employs one-dimensional adaptations of Castelli’s Column Method developed by Adams et al [43]. These adaptations make the recursion fast but restrict it to either a short bearing or a long bearing. The main difference between the short/long bearing approximations in MD Nastran® and their analytical counterparts (discussed previously) is that the former accommodate oil holes for supply and drain in the mid-section of the housing. However, they cannot accommodate oil grooves, making them unsuitable for most aero-engine applications.

Dede et al [37] established an analytical/empirical model to represent an end-sealed squeeze film damper bearing supplied by oil at a prescribed pressure through a deep groove. The pressure distribution was formed by combining the analytical short bearing approximation $p_{\text{short}}(\theta, z)$ with the analytical long bearing approximation $p_{\text{long}}(\theta)$ through an empirical “end-leakage” factor $\lambda$ which accounts for the degree of end-sealing. This factor ranges from 0 (no sealing, full leakage) to 1 (full sealing, no leakage). Holmes and Dogan [44] also used this model to investigate the behaviour of a sealed squeeze-film damper in a flexible support structure. As can be seen in that work, the introduction of the end-leakage factor resulted in successful simulation of the vibration. However, the outcomes showed that such an end-leakage factor depended not only on the end-clearance but also on the viscosity of the lubricating oil. These findings suggested that, for successful prediction of vibration, the value of the end-leakage factor may need to be adjusted to specific operating conditions and the system configuration. Since the $\lambda$-factor SFD model is based on the analytical expressions $p_{\text{short}}(\theta, z)$ and $p_{\text{long}}(\theta)$ it introduces little computational burden, making it feasible for aero-engine calculations. Moreover, it accommodates an external pressure supply through an oil groove, as used in industry. It is
indeed the method preferred by industry. For all these reasons it will be used throughout this thesis.

Having established the instantaneous pressure distribution \( p(\theta, z) \), it needs to be integrated to obtain the SFD forces, after due account of cavitation, as per formulae in eqs. (2.2, 2.3).

Analytical expressions for the forces \( Q_r, Q_T \) are only possible for two idealised conditions [12]: (i) a short unsealed bearing that does not cavitate (the “full film” or “\( 2\pi \)-film” model) i.e. \( p_{\text{cav}} = \infty \) in eq. (2.3); (ii) a short unsealed bearing without external pressurisation and in which it is assumed that the oil film ruptures in the region where \( p(\theta, z) \leq 0 \) i.e. \( p_{\text{cav}} = 0 \) (atmospheric pressure, i.e. gaseous cavitation) in eq. (2.3). The latter model is called the “half film” or “\( \pi \)-film” model since the cavitated region forms half of the circumferential extent of the oil annulus. A \( \pi \)-film model was employed by Zhao et al [33, 34] as the basis for the illustration of sub-harmonic, quasi-periodic and chaotic phenomena occurring in a rigid rotor-sprung SFD system. Although these models describe the nature of high nonlinearity of squeeze film damper better than linearization, and take advantage of analytical formulae for the SFD forces, they hardly returned good agreement between predicted and experimental orbits [12]. By combining the full and half film models in various proportions by trial and error and comparing predicted journal orbits with experimental orbits, Humes and Holmes [12] concluded that the oil film can support a degree of subatmospheric pressure without rupturing. This means that \( p_{\text{cav}} < 0 \) i.e. the cavitation is typically vaporous rather than gaseous. Moreover, the extent of the cavitated region influences the performance of the squeeze film bearing.

In view of the above, for both unsealed and sealed dampers, it is better to use the general cavitation model of eq. (2.3) since this allows for the possibility of different values of the cavitation pressure \( p_{\text{cav}} \), depending on the operating conditions (e.g. unbalance
force magnitude, supply pressure, etc.). The forces $Q_R$, $Q_T$ are then determined through numerical integration of the truncated pressure distribution. The value of $p_{cav}$ under particular operating conditions can be estimated experimentally from pressure probe recordings, as in [37, 44, 45]. In the absence of experimental values, a fixed value for $p_{cav}$ equal to $-101.325 \times 10^3$ Pa (absolute zero pressure) is taken. This is reasonable since, in most cases, this value of pressure is very close to the vapour pressure of the oil (i.e. the pressure at which it will spontaneously change phase from liquid to gas) [27]. Feng and Hahn [46], working on a simple rigid rotor rig with a centralised SFD, found that better agreement between measurements and theoretical predictions was achieved when $p_{cav}$ was taken as $-101.325 \times 10^3$ Pa rather than 0 (i.e. atmospheric pressure). Bonello [27] cites cases where an oil film can momentarily support tension (i.e. $p_{cav} < -101.325 \times 10^3$ Pa). Experimental recordings from a pressure probe located at the bottom of an unsupported SFD in [45] revealed a tension spike followed by recovery to absolute zero pressure. In that work, predictions were obtained by using a value for $p_{cav}$ below $-101.325 \times 10^3$ Pa that was an average taking account of the tension spike. However, it stated in [45] that the area of the spike is so small that the value of $p_{cav}$ might as well be taken as $-101.325 \times 10^3$ Pa. Bonello [27] used an unsealed SFD model with $p_{cav}$ equal to $-101.325 \times 10^3$ Pa. The validity of this model was confirmed in a series of works by Bonello et al [27, 10, 21-23] that accurately predicted the measured nonlinear performance of various systems, from rigid to flexible rotors, with sprung or unsprung SFD configurations.

The remainder of this section is devoted to two issues that have not been considered so far: (a) compressibility of the fluid; (b) fluid inertia.
The cavitation model of eq. (2.3) assumes that the oil film fully reconstitutes itself at a given location where it is ruptured when the instantaneous untruncated pressure \( p(\theta, z) \) at that location is restored to a value above \( p_{\text{cav}} \). Such an SFD model is referred to as an “incompressible model” by Feng and Hahn [39, 46]. In that work, they stated that experimental observations on unpressurised squeeze film dampers indicate that cavitation bubbles, once formed, do not completely redissolve upon restoration of the super-cavitation pressure. Instead, one is left with a spongy compressible fluid. In [38], they assumed this fluid to be a homogeneous gas-liquid mixture and proceeded to solve the compressible form of the Reynolds equation in which density and viscosity were a function of pressure. This solution procedure is too involved to incorporate in practical rotor-dynamic solution techniques. However, in [39], theoretical results for a simple rigid rotor with a centralised SFD showed that the compressible model results gave very good agreement with the incompressible model results when \( p_{\text{cav}} \) was taken as \(-101.325 \times 10^3\) Pa in the latter model. Moreover, these findings were confirmed by experiments in [46, 27, 10, 21-23], as previously mentioned.

The consideration of fluid inertia necessitates the replacement of the Reynolds Equation by the Navier-Stokes equation, as in the SFD model proposed by San Andres and Vance [47]. The numerical solution of this equation at each instant in time is highly time-consuming. Hence, in order to enable efficient solution for just one SFD, the system dynamics had to be restricted in [47] to circular centred journal orbits or small-amplitude orbits about the bearing centre. In [48] San Andres and Vance provide a formula for a mass addition to the SFD journal to account for the fluid inertia effect. This addition depends on the “Gap Reynolds Number” which indicates the relative magnitudes of viscous and inertia forces. However, this mass addition was derived on the basis of a short unsealed SFD with uncavitated film and a journal executing small amplitude vibrations about the centre. Moreover, the bearing housing was fixed in [47, 48] and so it is not clear
how fluid inertia can be accounted for in the situation where the bearing housing also vibrates. All these restrictions render such an approach unsuitable for the dynamic analysis of real engines like those considered in this thesis. For this reason, fluid inertia is not considered in this thesis, as most references cited.

2.5 PARAMETRIC STUDIES ON SFD-ROTOR SYSTEMS

A number of research works have investigated the influences of various SFD parameters on the overall operation of SFD-rotor systems. The findings of a selection of these are briefly summarised here.

In a research work on the performance of sealed SFD bearing, Holmes and Dogan [44] showed the effectiveness of the supply pressure in suppressing cavitation in the oil film. Another valuable finding that came out from this work is the relationship between the sealing gap and the end-leakage factor of SFD bearings. That work indicated that an SFD can be regarded as practically unsealed if the sealing gap dimension is comparable to the radial clearance. Feng and Hahn [49], through experimental work, proved that when the supply pressure for an SFD bearing was increased, the possibility for an amplitude jump to take place was reduced. Meanwhile, Bonello et al [22], working on a flexible rotor test rig with one spring-supported SFD in rigid housing, concluded that cavitation is promoted by increasing static eccentricity and/or unbalance level.

Domes and Levesley [40] carried out various experiments on a one-shaft test rig incorporating a SFD bearing that is used on a real engine BR710 (Rolls-Royce). The results from this work indicated that the oil supply configuration, such as the oil holes’ diameters and their arrangement, had a certain effect on the performance of the SFD-rotor system.
Sykes and Holmes [50] experimented on a test-rig that had one rotor supported by three bearings: one location bearing and the other two being SFD bearings. Misalignment between bearing housings was applied by raising the housing of one SFD bearing while the others were fixed. The results showed that such misalignment had significant effect on jumps and subharmonic resonances. However, the rotor in the test-rig was rigid and there was little compensation from the pedestal structure. It remains to be seen to what extent housing misalignment can affect the response of real aero-engine model where flexibility of the casing is considerable.

2.6 CONCLUSIONS

This chapter presented a critical review of research into the modelling and analysis of squeeze-film damped rotordynamic systems. A review of existing solution techniques gave reasons for their inadequacy for large-order systems like aero-engines. This inadequacy has severely limited the direct study of the vibration of real engine structures. This is evidenced by the fact that research into nonlinear phenomena in SFD systems, as well as parametric studies, has been restricted to simple rotor-bearing systems. This review has therefore made a strong case for the development of the novel time/frequency domain solvers in Chapters 3 and 5.

A review of modelling techniques for the SFD element revealed that the most appropriate model to use within the solvers to be developed is the “λ -theory” model for a partly-sealed SFD with oil feeding via a groove at its central section or at one end. This SFD model is indeed the one preferred by industry.
Figure 2.1: Transverse and axial cross-sections of SFD
3 THE IMPULSIVE RECEPTANCE METHOD

3.1 INTRODUCTION

This chapter presents the first stage of a project aimed at the development and validation of a suite of computational techniques for unbalance response computation, suitable for generic whole-engine models, which will significantly extend the capability of current finite element packages. The proposed analysis of the nonlinear rotating assembly uses the modal parameters of the undamped linear part of the assembly under non-rotating conditions. Implementation is possible through the integration of Nastran®, used for the linear computation, and specially-written Matlab® routines for the subsequent nonlinear computation (Figure 1.4).

Due to the complementary nature and relative merits of time and frequency domain solution techniques [10] both approaches were developed for the nonlinear computation part. As discussed in the literature review of Chapter 2, frequency domain techniques, most notably Harmonic Balance (HB), are inherently much faster since they directly yield steady-state solutions that are assumed to be periodic at an assumed fundamental frequency. However, such periodic vibration is not always physically possible since a nonlinear system subjected to periodic external excitation (e.g. rotor unbalance) is capable of non-periodic steady-state vibration. In such a case, a time-marching technique is used, which yields the actual steady-state response after the initial transients have died out. Moreover, in order to start off a HB solution procedure over a range of speeds, a good initial approximation at just one speed is required. This can only be reliably provided by a time domain solution, especially for a large order system [10].
This chapter will focus on the development of the time domain solver. The need for a novel implicit time domain solver for whole-engine modelling was justified in the literature review of Chapter 2. As stated there, conventional implicit integration schemes transform the differential equations of motion into an equal number of nonlinear algebraic equations which then have to be solved at each time step to obtain the current state variables. As shall be shown in this chapter, with a real engine a modal transformation would still necessitate the retention of a very large number of modes. Hence, due to the large number of modal equations, the time-marching process slows down to impractical levels. For this reason, a novel implicit “Impulsive Receptance Method” (IRM) was developed.

The choice of the name for the method derives from the fact that the structural response at a certain degree of freedom due to unit impulse in another degree of freedom is termed in [51] as the “impulsive receptance” connecting the two degrees of freedom. This quantity is also the inverse Fourier transform of the corresponding frequency response function (receptance). The IRM equations derived in this chapter are essentially the inverse Fourier transform of the frequency domain receptance/mobility equations relating the relative displacements and velocities at the SFDs with the motion-dependent forces and other excitations acting on the linear part of the structure. Hence, the IRM’s computational efficiency is largely immune to the number of modes used and dependent only on the number of nonlinear elements. This means that, apart from retaining numerical accuracy, a much more physically accurate solution is achievable within a short timeframe.

The work presented in the following sections has been published as a journal paper [1]. The theory of the IRM is presented in the following section. In section 3.3 the method is tested on a realistically sized representative twin-spool aero-engine model and validated against a conventional technique. The results of a preliminary parametric analysis are also presented.
3.2 THEORY

3.2.1 Outline

The following notation for operations on vectors (i.e. column matrices) shall be adopted: if
\[ a = [a_1 \cdots a_n]^T, \quad b = [b_1 \cdots b_n]^T, \] then
\[ a \cdot b = [a_1 b_1 \cdots a_n b_n]^T, \quad a \cdot b' = [a_1^p \cdots a_n^p]^T, \quad \sin(a) = [\sin a_1 \cdots \sin a_n]^T, \quad \text{etc.} \]

This notation is borrowed from Matlab [52] and allows not only the compact expression of the theory, but highly efficient computational implementation.

With reference to engine schematic of Figure 1.1, the modal parameters in all the theory pertain to the linear part of the assembly under non-rotating conditions. By “linear part” is meant the structure that remains in Figure 1.1 when the SFDs are replaced by gaps. The damping in the linear part of an engine is commonly regarded as negligible, but the following analysis can be modified to accommodate proportional damping in the linear part at no computational cost. The gyroscopic effect is neglected in this sub-section but is considered in sub-section 3.2.3.

The equation of motion in modal space is:

\[ \ddot{q} + (\omega^2) \cdot q = H_t^T f \quad (3.1) \]

In eq. (3.1), \( q \) is the \( R \times 1 \) vector of modal coordinates, \( \omega \) the vector of natural frequencies, and \( H_t \) is the matrix of mass-normalised eigenvectors \( \psi^{(r)}_t \ (r=1\ldots R) \) taken at the degrees of freedom corresponding to the directions and locations of the elements of \( f \):

\[ H_t = [\psi^{(1)}_t \cdots \psi^{(R)}_t] \quad (3.2) \]
f contains the forces external to the linear part: squeeze-film forces, unbalance forces, static forces. The inertia of the fluid films at the bearings is commonly regarded as negligible (e.g. e.g. [1, 3-10, 12,...etc]). Hence, one can write:

$$ f = f(x, \dot{x}, t) \quad (3.3) $$

where \(x, \dot{x}\) are the vectors of the relative \(x, y\) dynamic displacements and velocities respectively, at the squeeze-film terminals (i.e. journal centre \(J_i\) relative to housing centre \(B_i\), \(i = 1...5\), in Figure 1.1):

$$ x = H_x q, \quad \dot{x} = H_x \dot{q} \quad (3.4a, b) $$

\(H_x\) being the matrix whose columns \(\psi_x^{(r)}\) define the values of \(x\) in the respective mass-normalised modes:

$$ H_x = \begin{bmatrix} \psi_x^{(1)} & \cdots & \psi_x^{(r)} \end{bmatrix} \quad (3.5) $$

The \(\psi_x^{(r)}\) are loosely referred to here as “eigenvectors”. They are in fact the difference of the eigenvectors \(\psi_J^{(r)}, \psi_B^{(r)}\) defining the \(x, y\) displacements of \(J_i, B_i\) in mode no. \(r\).

Let \(q_k = q(t_k)\) and \(\dot{q}_k = \dot{q}(t_k)\). \(q_k, \dot{q}_k\) are unknown whereas \(q_{k-1}, \dot{q}_{k-1}\) are known. Using the Duhamel integral method [17] over the interval \([t_{k-1}, t_k]\), where \(t_k = t_{k-1} + h\):

$$ q_k = \dot{q}_{k-1} \cdot \sin(\omega h) \cdot \omega + q_{k-1} \cdot \cos(\omega h) + \int_{t_{k-1}}^{t_k} H_t^T f(\tau) \cdot \sin(\omega h - \omega \tau) \cdot \omega \ d\tau \quad (3.6a) $$

$$ \dot{q}_k = \dot{q}_{k-1} \cdot \cos(\omega h) - q_{k-1} \cdot \omega \cdot \sin(\omega h) + \int_{t_{k-1}}^{t_k} H_t^T f(\tau) \cdot \cos(\omega h - \omega \tau) \ d\tau \quad (3.6b) $$
Over the interval \( 0 \leq \tau \leq h \), \( f(\tau) \) is approximated as:

\[
f(\tau) = f_{k-1} + \left( f_k - f_{k-1} \right) \frac{\tau}{h}
\]

(3.7)

where \( f_k = f(x_k, \dot{x}_k, t_k) \) and \( x_k = x(t_k), \dot{x}_k = \dot{x}(t_k) \). Hence, after substituting eq. (3.7) into (3.6a, b), expanding the trigonometric functions under the integral signs, evaluating the resulting integrals and simplifying, one obtains:

\[
q_k = \dot{q}_k + \left( H^T f_k - H^T f_{k-1} \right) \cdot \left\{ \omega - \frac{1}{h} \sin(\omega h) \right\}, \left\lfloor \omega^2 \right\rfloor
\]

(3.8a)

\[
\dot{q}_k = \dot{\dot{q}}_k + \left( H^T f_k - H^T f_{k-1} \right) \cdot \left\{ n - \cos(\omega h) \right\}, \left\lfloor h\omega^2 \right\rfloor
\]

(3.8b)

where:

\[
\dot{q}_k = q_{k-1} \cdot \sin(\omega h), \frac{\omega}{h} + q_{k-1} \cdot \cos(\omega h) + H^T f_{k-1} \cdot \left\{ n - \cos(\omega h) \right\}, \left\lfloor \omega^2 \right\rfloor
\]

(3.9a)

\[
\dot{\dot{q}}_k = q_{k-1} \cdot \cos(\omega h) - q_{k-1} \cdot \omega \cdot \sin(\omega h) + H^T f_{k-1} \cdot \left\{ \sin(\omega h) \right\}, \frac{1}{h}\omega^2
\]

(3.9b)

In eqs. (3.8b, 3.9a), \( n \) is a vector of ones (“1’s”) of the appropriate size. Multiplying both sides of (3.8a) and (3.8b) by \( H_x \) and noting eqs. (3.4a, b) yields:

\[
x_k = \dot{x}_k + H_x \left[ \left\{ H^T f_k - H^T f_{k-1} \right\}, \left\{ \omega - \frac{1}{h} \sin(\omega h) \right\}, \left\lfloor \omega^2 \right\rfloor \right]
\]

(3.10a)

\[
\dot{x}_k = \dot{\dot{x}}_k + H_x \left[ \left\{ H^T f_k - H^T f_{k-1} \right\}, \left\{ n - \cos(\omega h) \right\}, \left\lfloor h\omega^2 \right\rfloor \right]
\]

(3.10b)

where:
\[ \dot{x}_k = H_x \dot{q}_k, \quad \ddot{x}_k = H_x \ddot{q}_k \]  \hspace{1cm} (3.11a, b)

Eqs (3.10a, b) can be simplified to

\[ x_k = \dot{x}_k + \mathbf{R}_{xt}(h)(\mathbf{f}_k(x_k, \ddot{x}_k, t_k) - \mathbf{f}_{k-1}) \]  \hspace{1cm} (3.12a)

\[ \ddot{x}_k = \dddot{x}_k + \mathbf{S}_{xt}(h)(\mathbf{f}_k(x_k, \ddot{x}_k, t_k) - \mathbf{f}_{k-1}) \]  \hspace{1cm} (3.12b)

where the discrete time domain analogues of the receptance and mobility matrices are given by:

\[ \mathbf{R}_{xt}(h) = \sum_{r=1}^{R} \left\{ \psi_x^{(r)}(r^{(r)}_T \left( \omega_r - \frac{1}{h} \sin \omega_r h \right) / \omega_r^3 \right\} \]  \hspace{1cm} (3.13a)

\[ \mathbf{S}_{xt}(h) = \sum_{r=1}^{R} \left\{ \psi_x^{(r)}(r^{(r)}_T \left( 1 - \cos \omega_r h \right) / \omega_r^2 \right\} \]  \hspace{1cm} (3.13b)

Eqs. (3.12a, b) are a set of nonlinear algebraic equations with \( x_k, \ddot{x}_k \) as unknowns. Hence, one can use an iterative method to solve for \( x_k, \dddot{x}_k \). *The number of equations to be solved is hence only four times the number of SFDs.* From eqs. (3.12a, b) the initial approximation can be taken as \( x_k \approx \dot{x}_k, \dddot{x}_k \approx \dddot{x}_k \). Once \( x_k, \dddot{x}_k \) are found, \( f_k \) is determined, hence from eqs. (3.8a, b) one can determine \( q_k, \dddot{q}_k \) and progress the solution.

### 3.2.2 Modification for Rigid Body Modes

Some of the above expressions need to be modified to account for the presence of rigid body modes since, in the linear part of Figure 1.1, each rotor is supported at only one point (\( J_1 \) or \( J_3 \)) about which it is free to pivot. The rigid body mode expressions are obtained by taking the limit \( \omega \to 0 \). From (3.9a, b) one can show that,
\[
\lim_{\omega \to 0} \ddot{q}_k = h\dot{q}_{k-1} + q_{k-1} + \frac{h^2}{2} H_r^T f_{k-1}, \quad \lim_{\omega \to 0} \dddot{q}_k = \dddot{q}_{k-1} + h H_r^T f_{k-1} \tag{3.14a, b}
\]

Also, from (3.13a, b):

\[
\lim_{\omega \to 0} R_{xt}(h) = \frac{h^2}{6} H_x H_r^T \quad \text{and} \quad \lim_{\omega \to 0} S_{xt}(h) = \frac{h}{2} H_x H_r^T \tag{3.15a, b}
\]

From (3.8a, b):

\[
\lim_{\omega \to 0} q_k = \lim_{\omega \to 0} \ddot{q}_k + \frac{h^2}{6} (H_r^T f_k - H_r^T f_{k-1}) \tag{3.16a}
\]

\[
\lim_{\omega \to 0} \dddot{q}_k = \lim_{\omega \to 0} \dddot{q}_k + \frac{h}{2} (H_r^T f_k - H_r^T f_{k-1}) \tag{3.16b}
\]

If the system has \( P \) rigid body modes, the modal parameters and coordinates are given by:

\[
\omega = \begin{bmatrix} 0 \\ \tilde{\omega} \end{bmatrix}, \quad H_r = \begin{bmatrix} \tilde{H}_f & \tilde{H}_r \end{bmatrix}, \quad H_x = \begin{bmatrix} \tilde{H}_x & \tilde{H}_x \end{bmatrix}, \quad q = \begin{bmatrix} \tilde{q} \\ \tilde{q} \end{bmatrix} \tag{3.17-20}
\]

…where \( \omega \) has a \( P \)-length sub-vector of zeros, \( \tilde{H}_{fx} \) contain the eigenvectors of the rigid body modes, \( \tilde{H}_{fx}, \tilde{\omega} \) contain the flexible mode parameters, \( \tilde{q} \) and \( \tilde{q} \) respectively denote the sub-vectors of rigid and flexible modal coordinates. The equations (3.12a, b), (3.13a, b), (3.8a, b) and (3.9a, b) are then modified as follows, by combining the above rigid body expressions with the flexible mode expressions of the previous section:

\[
\ddot{q}_k = \begin{bmatrix} h\tilde{q}_{k-1} + q_{k-1} + \frac{h^2}{2} \tilde{H}_r^T f_{k-1} \\
\tilde{q}_{k-1} \cdot \sin(\tilde{\omega}h) \cdot \tilde{\omega} + \tilde{q}_{k-1} \cdot \cos(\tilde{\omega}h) + \tilde{H}_r^T f_{k-1} \cdot \{ n - \cos(\tilde{\omega}h) \} \end{bmatrix} \frac{h^2}{2} \tilde{H}_r^T f_{k-1} \tag{3.21a}
\]
\[
\ddot{q}_k = \begin{bmatrix}
\ddot{q}_{k-1} \cdot \ast \cos(\bar{\omega}h) - \ddot{\bar{q}}_{k-1} \cdot \ast \bar{\omega} \cdot \ast \sin(\bar{\omega}h) + \bar{\bar{H}}^T_f f_{k-1} \cdot \ast \sin(\bar{\omega}h) / \bar{\omega}
\end{bmatrix}
\] (3.21b)

\[
R_{xf}(h) = \frac{h^2}{6} \bar{H} \bar{H}^T_t + \sum_{r=p+1}^{k} \left\{ \psi_x^{(r)} \psi_t^{(r)^T} \left[ \omega_r - \frac{1}{h} \sin(\omega_r h) / \omega_r^3 \right] \right\}
\] (3.22a)

\[
S_{xf}(h) = \frac{h}{2} \bar{H} \bar{H}^T_t + \sum_{r=p+1}^{k} \left\{ \psi_x^{(r)} \psi_t^{(r)^T} \left( 1 - \cos(\omega_r h) / (h\omega_r^2) \right) \right\}
\] (3.22b)

\[
q_k = \dot{q}_k + \frac{h^2}{6} \left( \bar{H}^T_f f_k - \bar{H}^T_t f_{k-1} \right)
\]

\[
\dot{q}_k = \ddot{q}_k + \frac{h}{2} \left( \bar{H}^T_f f_k - \bar{H}^T_t f_{k-1} \right)
\] (3.23a)

### 3.2.3 Inclusion of the Gyroscopic Effect

In order to include the gyroscopic effects, a vector \( g \) is introduced containing the gyroscopic moments, which, like the forces in \( f \), are considered as external to the non-rotating linear part. For the \( j^{th} \) rotor ( \( j = 1 \ldots J \) ) with speed \( \Omega_{(j)} \), the gyroscopic effects are concentrated at \( G_j \) points. Hence:

\[
g = P\dot{\theta}
\] (3.24)

where \( P \) is a block-diagonal matrix of diagonal sub-matrices
\[ p_{(j)} = \Omega_{(j)} \text{diag} \left\{ -I_{(j)_1}, I_{(j)_1}, \ldots, -I_{(j)_G}, I_{(j)_G} \right\}, \quad \theta = [\theta_{(j)}], \quad g = [g_{(j)}], \]

\[ g_{(j)} = \begin{bmatrix} M_{(j)_h} & N_{(j)_h} & \ldots & M_{(j)_G} & N_{(j)_G} \end{bmatrix}^T, \]

\[ \theta_{(j)} = \begin{bmatrix} \beta_{(j)_h} & \alpha_{(j)_h} & \ldots & \beta_{(j)_G} & \alpha_{(j)_G} \end{bmatrix}^T. \]

For the \( j^{th} \) rotor: \( I_{(j)_p} \) is the polar moment of inertia at location \( p \) (\( p = 1 \ldots G)\); \( M_{(j)_p}, \alpha_{(j)_p} \) are the gyroscopic moment and rotation respectively at location \( p \) about the \( x \) axis and \( N_{(j)_p}, \beta_{(j)_p} \) are the gyroscopic moment and rotation respectively at location \( p \) about the \( y \) axis (see the coordinate system of Figure 1.1).

The gyroscopic effect is added to the right side of eq. (3.1), which now becomes:

\[ \ddot{q} + \left( \omega \cdot \omega \right) \cdot q = H_f^T f + H_g^T g \tag{3.25} \]

Extending eqs (3.12a, b):

\[ x_k = \dot{x}_k + R_{xf} \{f_k - f_{k-1}\} + R_{xg} \{g_k - g_{k-1}\} \tag{3.26a} \]

\[ \dot{x}_k = \dot{x}_k + S_{xf} \{f_k - f_{k-1}\} + S_{xg} \{g_k - g_{k-1}\} \tag{3.26b} \]

\[ \dot{\theta}_k = \dot{\theta}_k + S_{\theta f} \{f_k - f_{k-1}\} + S_{\theta g} \{g_k - g_{k-1}\} \tag{3.26c} \]

where the matrices \( R_{xg}, S_{xg}, S_{\theta f}, S_{\theta g} \) are obtained in a similar way to eqs (3.22a, b), using modal vectors \( \psi^{(r)}_\theta, \psi^{(r)}_g \) taken at the degrees of freedom corresponding to the elements in \( \theta \) and \( g \) respectively and
\[
\hat{\theta}_k = H_0 \hat{q}_k, \quad \hat{\theta}_k = H_0 \hat{q}_k
\]  
(3.27a, b)

\[
H_0 = \begin{bmatrix} \psi_0^{(1)} & \cdots & \psi_0^{(k)} \end{bmatrix}^T = [\tilde{H}_0 \quad \tilde{H}_0], \quad H_g = \begin{bmatrix} \psi_g^{(1)} & \cdots & \psi_g^{(k)} \end{bmatrix}^T = [\tilde{H}_g \quad \tilde{H}_g]  
\]  
(3.28a, b)

Using eq. (3.24) in eq. (3.26c) and rearranging:

\[
\hat{\theta}_k = L \hat{\theta}_k + LS_{of} (f_k - f_{k-1}) - LT \hat{\theta}_{k-1}
\]  
(3.29)

where

\[
L = L \left( \Omega_{(1...j)} \right) = \left( I - T \right)^{-1}, \quad T = T \left( \Omega_{(1...j)} \right) = S_{ug} (h) P  
\]  
(3.30a, b)

where \( I \) being the identity matrix. Hence, using eq. (3.24) in eqs. (3.26a, b) and substituting eq. (3.29) yields:

\[
x_k = \hat{x}_k + R_{sg} PL \hat{\theta}_k - \left( R_{sg} PLT + R_{sg} P \right) \hat{\theta}_{k-1} + \left( R_{sf} + R_{sg} PLS_{of} \right) (f_k - f_{k-1})  
\]  
(3.31a)

\[
\dot{x}_k = \hat{x}_k + S_{sg} PL \hat{\theta}_k - \left( S_{sg} PLT + S_{sg} P \right) \hat{\theta}_{k-1} + \left( S_{sf} + S_{sg} PLS_{of} \right) (f_k - f_{k-1})  
\]  
(3.31b)

The above eqs (3.31a, b) now replace eqs (3.12a, b) and are solved for \( x_k, \dot{x}_k \) in the same way. One can use the sum of the first three terms in each equation as a first approximation. \( \hat{\theta}_k \) is then updated according to eq. (3.29). Eqs. (3.23a, b) are then used to determine \( q_k, \dot{q}_k \) and progress the solution.

The IRM computational sequence is illustrated in Figure 3.1. For \( k = 1 \), the data \( f_{k-1}, g_{k-1}, \hat{\theta}_{k-1} \) in the first block of the loop in Figure 3.1a are generated by the starting sequence depicted in Figure 3.1b. It is noted that equations (3.21a, b) and (3.23a, b) are to
be used with the terms $\tilde{H}_f^T f_k$, $\tilde{H}_f^T f_{k-1}$, $\tilde{H}_g^T g_k$, $\tilde{H}_g^T g_{k-1}$ replaced by $\tilde{H}_f^T f_k + \tilde{H}_g^T g_k$.

$\tilde{H}_f^T f_k + \tilde{H}_g^T g_k$, $\tilde{H}_f^T f_{k-1} + \tilde{H}_g^T g_{k-1}$, $\tilde{H}_f^T f_k + \tilde{H}_g^T g_k$, $\tilde{H}_f^T f_{k-1} + \tilde{H}_g^T g_{k-1}$ respectively.

Finally, it is to be noted that any concentrated viscous damping elements can be dealt with in a similar manner to the gyroscopic moments, without affecting the size of the problem.

3.3 SIMULATIONS

The computational method was applied to a representative twin-spool aero-engine having the schematic layout in Figure 1.1, using a realistically sized whole-engine finite-element (FE) model provided by an engine manufacturer [53]. Figure 3.1 illustrates the overall computational procedure. All simulations were performed in Matlab on a standard 2006-issue desktop pc with Intel® Pentium® D CPU 3GHz processor.

3.3.1 Linear computation

In order to assess the influence of the computed modes, the frequency response functions (receptances) of the undamped linear part for zero rotational speed were computed from the modal parameters. Figure 3.2 shows the $y$ displacement at $B_1$ (Figure 1.1) per unit $y$ force at the same point over the range 0 to 1kHz, which contains a total of 934 modes. It is clear that there is a high modal density at relatively low frequencies, considering that the high pressure shaft runs at speeds up to 16000rpm and that the nonlinear response will contain engine orders. One should note that the sudden reduction in modal density beyond 500Hz is merely an artefact of the degree-of-freedom reduction technique used by the FE modellers.

This exercise clearly shows that, for the nonlinear processing, an attempt at time-saving by integrating a restricted number of modal equations is not advisable. The use of a
residual flexibility technique [17] is not helpful since this only corrects the static influence of truncated high frequency modes. Moreover, any attempt at sifting out some of those low frequency modes deemed not to significantly affect the vibration is bound to be both time-consuming and prone to significant errors. It is for this reason that certain whole-engine modelling specialists actually opt for the highly time-consuming direct integration of the FE equations (i.e. in physical coordinates).

3.3.2 Nonlinear Computation for the Unbalance Response

Figure 3.3 gives information relating to the constituents of the force vector \( \mathbf{f} \) in eq. (3.3). This vector contains a static sub-vector defining the distributed weights of the two rotors.

The motion-dependence of \( \mathbf{f} \) is due to the sub-vector of SFD forces \( \mathbf{p} = \begin{bmatrix} p_1^T & \cdots & p_n^T \end{bmatrix}^T \) where \( p_i = \begin{bmatrix} Q_{xi} & Q_{yi} \end{bmatrix}^T \) defines the Cartesian forces at SFD no. \( i \). Now \( p_i = \rho_i (v_i, \dot{v}_i) \) [10] where the vector \( \mathbf{v}_i \) defines the instantaneous Cartesian displacements of \( J_i \) relative to \( B_i \) (see Figure 1.1):

\[
\mathbf{v}_i = \mathbf{v}_{s,i} + \begin{bmatrix} x_{d,i} & y_{d,i} \end{bmatrix}^T
\]

(3.32)

\( \mathbf{v}_{s,i} \) defines the static offset of \( J_i \) relative to \( B_i \) in the \( x, y \) directions (under no rotor loading). The vector \( \mathbf{x} \) in section 3.2 comprises the dynamic components of the relative displacements in eq. (3.32), i.e.:

\[
\mathbf{x} = \begin{bmatrix} x_{d,1} & y_{d,1} & \cdots & x_{d,5} & y_{d,5} \end{bmatrix}^T
\]

(3.33)

The SFDs considered for this illustrative study were single-land and end-fed with oil of viscosity 0.0049Ns\(^{-2}\) at a pressure of 3bar (gauge). The bearing diameters and radial
clearances were typically 200mm, 0.1mm respectively and the land lengths ranged from 16 to 34mm. As in current industrial practice, the instantaneous pressure distribution in each SFD was approximated by a combining the short and long bearing expressions through an “end-leakage factor” \( \lambda \) representing the degree of end-sealing [37]. Each distribution was truncated below a cavitation pressure of absolute zero (\(-101.325\) kPa) and numerically integrated across the oil film to yield the associated SFD forces. The SFD force computation procedure was described in Section 3.2.4 and is presented in detail in Appendix A1. For these preliminary calculations, the bearing housings were assumed to be perfectly aligned with each other prior to rotor assembly (i.e. \( v_{v,i} = 0 \), \( i = 1...5 \)).

For all simulations presented in this paper, the unbalance was restricted to two locations U1 and U2 on the low-pressure (LP) rotor as shown in Figure 3.3. The unbalance mass-radius products at these locations were set to be 6.3gm in-phase. The rotational speeds of both rotors were 10000 rpm in the same sense (i.e. co-rotation).

The gyroscopic effect was discretised at 7 points on the LP rotor and 12 points on the high-pressure (HP) rotor.

### 3.3.2.1 Testing and Validation of IRM

An unbalance response problem was solved in two ways: (i) IRM; (ii) “CIM”: the Conventional Implicit integration of the Modal equations (3.1) using the solver \( \text{ode23s} \) available in the Matlab package. In either case all 934 modes were considered. In the case of method (ii) the equations (3.1) were cast in the standard form \( \dot{s} = \chi(t,s) \) where \( s \) is the vector of modal state variables \( q, \dot{q}, \) and \( \chi \) is a vector function of \( s \) and \( t \). Both methods required a Jacobian matrix at each time step. Method (i) (IRM) required the Jacobian of eqs. (3.31a,b) for solution by the Newton-Raphson method. During the iterative process at a given time step, the inverse of the Jacobian was updated using Broyden’s method [54]. In the case of method (ii) (CIM), the solver was supplied with a
user-derived expression for the Jacobian $\partial \mathbf{s} / \partial \mathbf{x}$, as recommended in [13], since this considerably accelerated the solution compared to letting the solver compute the Jacobian from first principles. Although the required Jacobians were different for the two methods (i, ii), in either case the effort in the Jacobian computation lay in the calculation of the partial derivatives of the SFD forces with respect to the associated relative displacements and velocities (as can be seen in [55] for the case of CIM). The SFD force partial derivatives calculation was performed in both methods (i, ii) by the same routine. Hence, the ensuing differences in computational speed were attributable purely to the different time-marching algorithms.

Both solvers used an adaptive control of the time step-size $h$ in order to efficiently maintain the numerical error within a prescribed tolerance. In the case of the IRM, the iterative solution of eqs. (3.31a, b) at a given time step was deemed successful if, among other conditions, the magnitude of the difference between consecutive iterates fell below the prescribed tolerance and the solution was completed within a set number of iterations. An estimate for a new step-size $h_{\text{new, est}}$ was computed on the basis of the degree of convergence obtained with the previous step-size. In order to save time, the program had the facility to pick a step-size closely matching $h_{\text{new, est}}$ from a range of discrete step-sizes for which those matrices in eqs. (3.31a, b) that depend on $h$ (like $\mathbf{R}_{\text{sf}}, \mathbf{S}_{\text{sf}}, \ldots$ etc.) had been pre-computed at the start of the time-marching procedure.

Figures 3.4, 3.5 show that the results obtained by the two methods are in excellent agreement, and that the IRM was found to be at least 40 times faster than the CIM. The number of equations solved by CIM at each time step was $934 \times 2$, whereas the number of equations to be solved by the IRM was fixed at a mere 20, regardless of the number of modes included. For the tolerance setting used to obtain the IRM results in Figure 3.4b, the difference graphs in Figure 3.4c were invariant for tolerance settings of the CIM solver below 1% of the average radial clearance (the CIM computational time quoted in Figure
3.4a is for this latter tolerance). For coarser tolerance settings of the CIM, the difference was greater than that shown in Figure 3.4c. This suggests that the IRM has also the potential for greater numerical accuracy. The reason for this is that the unknowns of algebraic equations solved by the CIM are the modal state variables \( q, \dot{q} \). Hence, in order to obtain the relative displacements and velocities necessary to evaluate the SFD forces, the CIM had to perform a modal transformation each time. The CIM solver applies its prescribed tolerance to the modal state variables, rather than the relative displacements and velocities at the SFDs. On the other hand the IRM equations (3.31a, b) are directly formulated in terms of the relative displacements and velocities at the SFDs.

Figure 3.5 shows that the close agreement between the two methods was maintained as the solution was progressed. More simulations performed at other LP rotors speeds also showed that the IRM was at least 40 times faster than the CIM.

3.3.2.2 Some Preliminary Results of a Parametric Analysis

Figure 3.6 shows the final steady-state orbits for the above discussed simulations, for which the end-leakage factor \( \lambda = 0.03 \) (typical value used for engine simulations in industry). SFDs 1, 2 and 5 were on the LP shaft, which carried the unbalance. Vibration transmission between the two rotors is through the bearing housings, whose vibration is shown in Figure 3.6b. It is to be noted that SFDs 1 and 3 were spring-supported and the orbit offsets within the clearances of these SFDs were mainly due to the respective rotor weights. The size, shape and position of the orbits at the other (unsupported) SFDs were influenced by their static and dynamic loading. Figure 3.7 shows that the steady-state vibration was also periodic for a lower degree of sealing of \( \lambda = 0.015 \) (all other parameters kept the same). The orbits in Figures 3.6 and 3.7 described periodic vibration with a fundamental frequency equal to the rotational speed of the unbalanced rotor. As the end-leakage factor was further reduced to \( \lambda = 0.010 \) no such periodic orbits were obtained
after several thousand revolutions, as can be seen in Figures 3.8a, b. Hence, it appears that, for the unbalance distribution and rotational speed considered, the nature of the predicted engine vibration changes from periodic to quasi-periodic as the degree of sealing is reduced to a very low level. Such aperiodic vibration is similar to that predicted and measured on other, much simpler, systems with unsealed (i.e. $\lambda \to 0$) squeeze-film damper bearings e.g. [10, 23].

3.4 CONCLUSIONS

A novel implicit “Impulsive Receptance Method” has been developed for the time domain computation of the vibration of a whole-engine model with nonlinear bearings. The IRM’s computational efficiency is largely immune to the number of modes used and dependent only on the number of nonlinear elements. This means that, apart from retaining numerical accuracy, a much more physically accurate solution is achievable within a short timeframe. Simulation tests on a realistically sized representative twin-spool engine showed that the new method was around 40 times faster than a conventional implicit integration scheme. Preliminary results illustrated the varying degree of lift and orbit size at the squeeze-film damper bearings for given unbalance distribution, and the effect of sealing. The new method is of course equally applicable with other types of concentrated nonlinearities and should greatly facilitate the hitherto highly restricted nonlinear dynamic analysis of realistic engine structures.
Figure 3.1: Flow chart of IRM computational sequence

* See note on penultimate paragraph of section 3.2.3
**Figure 3.2:** Point receptance frequency response at B in y direction

**Figure 3.3:** Rotor weight distribution, unbalance locations and bearing locations
a. Relative orbits of journals within the respective clearance circles by CIM (CPU time: 86mins.)

b. Relative orbits of journals within the respective clearance circles by IRM (CPU time: 2mins.)

c. Difference between CIM and IRM orbits (normalised with respect to corresponding radial clearance)

Figure 3.4: Validation of IRM against CIM: first 10 LP-shaft revs.
Figure 3.5: Validation of IRM against CIM: last 10 out of first 100 LP-shaft revs

Figure 3.6: Steady-state orbits for $\lambda = 0.03$: last 5 out of first 4000 LP-shaft revs (orbits normalised with respect to corresponding radial clearances)
a. Relative orbits of journals within the respective clearance circles

b. Absolute orbits of journal (—) and housing (−−−) centres

**Figure 3.7:** Steady state orbits for \( \lambda = 0.015 \): last 5 out of first 4000 LP-shaft revs (orbits normalised with respect to corresponding radial clearances)

a. Last 5 out of first 4000 LP-shaft revs

b. Last 50 out of first 4000 LP-shaft revs

**Figure 3.8:** Steady state orbits for \( \lambda = 0.01 \): last 50 out of first 4000 LP-shaft revs (orbits normalised with respect to corresponding radial clearances)
4 AN INVESTIGATION INTO THE USE OF THE NEWMARK-BETA METHOD FOR WHOLE-ENGINE ANALYSIS

4.1 INTRODUCTION

In the previous chapter the Impulsive Receptance Method (IRM) was validated against the highly robust Modified Rosenbrock algorithm used by Matlab solver \textit{ode23s}® [13] and found to be about 40 times faster for a realistic engine problem. The Modified Rosenbrock algorithm is an example of what can be described as “standard generic form implicit” (SGFI) solvers. By the term “SGFI” is meant an implicit integration routine that requires expression of the equations of motion (eq. (3.1)) in standard first order (state-space) form, or second order form, prior to integration. The Newmark-Beta method (NBM) [17] is an implicit alternative to SGFI solvers. In its conventional formulation the NBM suffers from the same inefficiency problem when subjected to a large order system. However, this chapter shows for the first time that the NBM can be reformulated to avoid this problem without introducing any new assumptions or approximations. The resulting Fast Newmark-Beta Method (FNBM) would then be as fast as the IRM and use the same input (modal data of linear part at zero rotational speed). Nonetheless, it will be shown that the NBM can be unreliable for a real engine-type problem. This will be illustrated by revisiting the whole-engine example of the previous chapter and solving it using the FNBM.

The work presented in the following sections has been published in a journal paper [56]. The theory of the FNBM is presented in the following section. In section 4.3 the results of the computational investigation are presented and discussed.
4.2 FAST NEWMARK-BETA METHOD (FNBM)

The governing equation of motion, given by eq. (3.1), is rewritten here as:

\[
\ddot{\mathbf{q}} + \Lambda \mathbf{q} = \mathbf{H}_t^T \mathbf{f}(\mathbf{x}, \dot{x}, t)
\]  

(4.1)

where \( \Lambda \) is a diagonal matrix of the squares of the natural frequencies \( \omega_1, \ldots, \omega_r \). The gyroscopic effect is excluded merely for ease of presentation.

Applying the Newmark-Beta method to the modal coordinate space, it is assumed that [17]:

\[
\mathbf{q}_k = \mathbf{q}_{k-1} + h \dot{\mathbf{q}}_{k-1} + \frac{1}{2} h^2 \left[ (0.5 - \beta) \ddot{\mathbf{q}}_{k-1} + \beta \dddot{\mathbf{q}}_k \right]
\]  

(4.2a)

\[
\dot{\mathbf{q}}_k = \dot{\mathbf{q}}_{k-1} + h \left[ (1 - \gamma) \dddot{\mathbf{q}}_{k-1} + \gamma \dddot{\mathbf{q}}_k \right]
\]  

(4.2b)

where \( \beta \) and \( \gamma \) are constants that are arbitrarily chosen with a view to accuracy and stability of the solution. For \( \beta = 0.25 \) and \( \gamma = 0.50 \) the method becomes the Constant Average Acceleration method [17]. Rearranging eq. (4.2a):

\[
\ddot{\mathbf{q}}_k = h^{-2} \beta^{-1} (\mathbf{q}_k - \mathbf{q}_{k-1}) - (h \beta)^{-1} \ddot{\mathbf{q}}_{k-1} - (0.5 \beta^{-1} - 1) \dddot{\mathbf{q}}_{k-1}
\]  

(4.3)

Substituting eq. (4.3) into eq. (4.2b) gives:

\[
\dot{\mathbf{q}}_k = \gamma (h \beta)^{-1} (\mathbf{q}_k - \mathbf{q}_{k-1}) + (1 - \gamma \beta^{-1}) \dddot{\mathbf{q}}_{k-1} + h \left( 1 - 0.5 \gamma \beta^{-1} \right) \dddot{\mathbf{q}}_{k-1}
\]  

(4.4)
Substituting eq. (4.3) into eq. (4.1) and solving for \( q_k \) gives:

\[
q_k = \tilde{q}_k + DH^T \{ f(x_k, \dot{x}_k, t) - f_{k-1} \}
\]  

(4.5)

where:

\[
D(h) = \left( h^{-2} \beta^{-1} I + \Lambda \right)^{-1}
\]  

(4.6)

\[
\tilde{q}_k = D \left[ h^{-2} \beta^{-1} q_{k-1} + (h\beta)^{-1} \dot{q}_{k-1} + \left( 0.5 \beta^{-1} - 1 \right) \ddot{q}_{k-1} + H^T f_{k-1} \right]
\]  

(4.7)

Substituting for \( q_k \) from eq. (4.5) into eq. (4.4) gives:

\[
\dot{q}_k = \tilde{q}_k + \gamma(h\beta)^{-1} DH^T \{ f(x_k, \dot{x}_k, t) - f_{k-1} \}
\]  

(4.8)

where:

\[
\tilde{q}_k = \gamma(h\beta)^{-1} (\tilde{q}_k - q_{k-1}) + \left( 1 - \gamma \beta^{-1} \right)^{-1} \dot{q}_{k-1} + h(1 - 0.5 \beta^{-1}) \ddot{q}_{k-1}
\]  

(4.9)

It is noted that eqs. (4.2-4.9) are basically the same as those in [17] except that they have been adapted to the modal coordinate space.

In eqs. (4.5, 4.8) it is noted that \( x_k = H_q q_k, \dot{x}_k = H_q \dot{q}_k \). Hence, eqs. (4.5, 4.8) comprise a set of \( 2R \) nonlinear algebraic equations in an equal number of unknowns contained in state variables \( q_k, \dot{q}_k \). In the conventional implementation of the Newmark-
Beta method to nonlinear systems (e.g. [24]) it is state-variable equations such as these that are solved at each time-step, making the process unwieldy for a very large number of modes. This problem is overcome in this chapter through the following slight modification to the formulation.

Multiplying both sides of eqs. (4.5, 4.8) by $H_x$ yields:

\begin{align*}
x_k &= \tilde{x}_k + R'_{xf} \left[ f_k \left( x_k, \dot{x}_k, t_k \right) - f_{k-1} \right] \quad \text{(4.10a)} \\
\dot{x}_k &= \tilde{\dot{x}}_k + S'_{xf} \left[ f_k \left( x_k, \dot{x}_k, t_k \right) - f_{k-1} \right] \quad \text{(4.10b)}
\end{align*}

where

\begin{align*}
\tilde{x}_k &= H_x \tilde{q}_k, \quad \tilde{x}_k = H_x \tilde{q}_k \\
R'_{xf} (h) &= H_x D H_x^T, \quad S'_{xf} (h) = \gamma (h \beta)^{-1} R'_{xf} (h) 
\end{align*}

The analogy with the IRM equations (4.10a, b) is evident. Eqs. (4.10a, b) comprise only $4N$ equations in an equal number of unknowns in $x_k, \dot{x}_k$, where $N$ is the number of SFDs. It is these equations that are solved at each time step. The definitive value of $f_k$ is then determined, enabling the updating of $q_k, \dot{q}_k$ through eqs. (4.5, 4.8).

The gyroscopic effect can be added to the above FNBM analysis in a similar manner to the IRM (see Section 3.2.3). It is noted that all simulations presented in Section 4.3 include the gyroscopic effect.

### 4.3 DISCUSSION

The problem solved in section 3.3 of the previous chapter was reworked with FNBM. Figures 4.1(a-d) compare the results obtained by the three methods for the steady-state
SFD orbits for LP, HP rotor speeds of 10000, 12000 rpm respectively. The time domain results required up to 4000 LP shaft revolutions to reach steady-state from default initial conditions (corresponding to zero relative displacements and velocities at each SFD). Figures 4.1(a) and (b) show that there was significant disagreement between the IRM and FNBM orbits, particularly at the unsupported SFD no. 5. The FNBM result was also found to be sensitive to slight variations in $\beta$ and $\gamma$, without improving the correlation with the IRM (Figure 4.1(c)). It is noted that, for both IRM and FNBM, a variable step size $h$ was used to efficiently maintain the accuracy within a preset tolerance and that $h$ was capped at a maximum value of $1/250$ of the period of rotation of the unbalanced shaft. In addition, the problem was also solved using the Receptance Harmonic Balance Method (RHBM), which will be described in the following chapter. The Fourier coefficients of the FNBM result shown in Figure 4.1(b) were used to provide the initial approximation for the RHBM iterative process, which subsequently converged to a solution that was in virtual perfect agreement with the IRM in Figure 4.1(d). The RHBM convergence to the IRM result proved conclusively that the FNBM results were erroneous, and not some alternative solution. Similarly erroneous FNBM results were observed at other speeds.

Prior to analyzing the whole-engine model, the three methods were tested on the simple system shown in Figure 4.2 which was a simplification of the real twin-spool engine. The details of this test rig are given in Appendix A3. A total of 10 modes were used to analyse this simple system and excellent agreement was generally achieved between the IRM and FNBM, as can be seen in Figure 4.3. The FNBM was also observed to be robust with respect to slight changes in the values of $\beta$ and $\gamma$ about their typical values of 0.25 and 0.50 respectively. The discrepancy between IRM and FNBM in the subsequent real engine analysis lead to the conclusion that the underlying assumptions of the Newmark-Beta method, expressed by eqs. (4.2a, b), may become problematic in the presence of a large number of modes and several nonlinear elements. The relationships in eqs. (4.2a, b) effectively pre-lock the state-variables into a relationship and this restriction
is carried through the computation. In fact, comparing the FNBM eqs. (4.10, 4.12) with the IRM eqs. (3.12, 3.13), it is evident that the matrices $\mathbf{R}'_{xf}, \mathbf{S}'_{xf}$ differ by just a scalar factor, in contrast to $\mathbf{R}_{xf}, \mathbf{S}_{xf}$. The use of the FNBM was therefore discontinued for the remaining simulations presented in this thesis.

4.4 CONCLUSIONS

This chapter investigated the potential use of the popular Newmark-Beta method for the time domain computation of the unbalance response of a whole aero-engine model with nonlinear bearings. It was shown that the Newmark-Beta method could be reformulated in order to make it virtually immune to the number of modes of the system, just like the IRM, thereby allowing whole-engine analysis to be performed with comparable computation speeds. However, simulations on a whole-engine model using this scheme gave erroneous results. Since the reformulation introduced no new assumptions or approximations to the conventional Newmark-Beta method, this study has provided important evidence that the Newmark-Beta method can be unreliable for large order nonlinear systems.
Figure 4.1: Computed periodic steady-state orbits of bearing journals relative to their housings for SFU (orbits normalized with respect to the respective radial clearances)
Figure 4.2: Schematic of simple twin-shaft system

Figure 4.3: Orbit of bearing journal within the radial clearance circle computed by three methods

(a) IRM
(b) FNBM (β = 0.25, γ = 0.55)
(c) FNBM (β = 0.30, γ = 0.60)
(d) RHBM (Ω_{ref} = Ω_0, Q = 1, K = 20 defined in Chapter 5)

(for test rig in Figure 4.2; orbit normalised to the respective radial clearance)
5 A WHOLE-ENGINE RHBM

5.1 INTRODUCTION

This chapter describes the second stage of the development and validation of the suite of computational techniques for unbalance response computation of generic whole-engine models illustrated in Figure 1.4. As mentioned previously, the proposed analysis of the nonlinear rotating assembly uses the modal parameters of the undamped linear part of the assembly under non-rotating conditions. Nastran® is used for the linear computation and specially-written Matlab® routines are used for the subsequent nonlinear computation; these routines give both time domain and frequency domain options. The complementary nature and relative merits of time and frequency domain approaches have already been discussed. Suffice to repeat here that the attraction of the frequency domain approach is that it is inherently very much faster since it directly yields steady-state solutions that are assumed to be periodic at an assumed fundamental frequency. The first stage of development of the suite in Figure 1.4, described in Chapter 3 and in reference [1], produced a novel time domain solver called the Impulsive Receptance Method (IRM). This was shown to be much faster than a typical conventional time domain solver for whole-engine analysis. The present chapter deals with the development of the whole-engine frequency domain solver. For reasons given in Chapter 2, it is based on the Receptance Harmonic Balance Method (RHBM).

As mentioned in Chapter 2, the RHBM was developed in [10] for the frequency domain analysis of a rotating system comprising a single shaft with negligible gyroscopic effect running on flexibly-housed SFDs. This approach is ideal for a large-order system since it
simplifies the problem by using the receptances of the linear part to extract equations
governing the harmonics of the response at the SFDs only. The present chapter resolves
outstanding issues with the RHBM which are essential for its application to whole-engine
models. These issues were bulleted in the final part of Section 2.2.2. Foremost among
these are: (a) the consideration of statically indeterminate problems in the “zero
harmonic of the vibration [10] and (b) gyroscopic effects. In addition, the method
proposed here reduces the number of unknowns by about half that in previous research by
formulating the equations in terms of the relative rather than the absolute degrees of
freedom at the nonlinearities. This reduction is the result of taking advantage of the SFD
model used in this thesis (Section 2.4). In this SFD model, the effect of fluid inertia is
neglected and the oil-film pressure distribution is established based on the relative motion
between the journal and the housing, regardless of the absolute movements. Therefore,
only the relative motion at the nonlinearities appears in the equation of motion, as can be
noticed with the Eqs. (3.1) and (3.3). It is noted that fluid inertia is commonly neglected
eq. []. Moreover, as discussed at the end of Section 2.4, approximations for the fluid
inertia effect are based on the assumption of a fixed bearing housing, among other
simplifications. Hence, the neglect of fluid inertia is necessary to make the computation
feasible for the present case where the bearing housing also vibrates.

The work presented in the following sections has been published as a journal paper
[57]. The theory of whole-engine RHBM is presented in the following section. In section
5.3 the method is tested on a realistically sized representative twin-spool aero-engine
model and validated against the IRM.

5.2 THEORY

5.2.1 System Description
With reference to the engine schematic of Figure 1.1, the modal parameters in all the theory pertain to the linear part of the assembly under non-rotating conditions. As stated before, by “linear part” is meant the structure that remains in Figure 1.1 when the SFDs are replaced by gaps. The damping in the linear part of an engine is commonly regarded as negligible [10]. The linear part is acted on by the SFD forces, unbalance forces, static loading on the rotors, and the gyroscopic moments on the rotors. The engine is assumed to have \( J \) rotors each with speeds \( \Omega_{(j)} \), \( j = 1 \ldots J \).

In the RHBM, the vibration is assumed to be periodic with a fundamental frequency of \( \alpha \) where:

\[
\alpha = \frac{\Omega_{(j_{\text{ref}})}}{Q}
\]

(5.1)

\( Q \) is a positive integer and \( \Omega_{(j_{\text{ref}})} \) is termed the “reference unbalanced shaft speed”. In the case of SFU, \( \Omega_{(j_{\text{ref}})} \) is the speed of the (only) unbalanced shaft and \( Q \) is commonly taken as 1. In the case of MFU, \( \Omega_{(j_{\text{ref}})} \) is the speed of the slowest unbalanced shaft and the value of \( Q \) will be chosen according to the ratio of the speeds of the unbalanced shafts (section 5.3.2.2).

Let \( x_{J_i}, y_{J_i} \) and \( x_{B_i}, y_{B_i} \) be the instantaneous Cartesian displacements of the journal and housing centres \( J_i, B_i \) at SFD no. \( i \), both measured from the static equilibrium position of \( B_i \) under no rotor loading. Hence, if \( x_J = [x_{J_1} \ y_{J_1} \ \cdots \ x_{J_N} \ y_{J_N}]^T \), \( x_B = [x_{B_1} \ y_{B_1} \ \cdots \ x_{B_N} \ y_{B_N}]^T \), then:

\[
x_J = \bar{x}_J + \sum_{k=1}^{K} \{x_{J_k}^\cos \cos k \alpha t + x_{J_k}^\sin \sin k \alpha t \}
\]

(5.2a)

\[
x_B = \bar{x}_B + \sum_{k=1}^{K} \{x_{B_k}^\cos \cos k \alpha t + x_{B_k}^\sin \sin k \alpha t \}
\]

(5.2b)
Let \( \mathbf{p}_i = \begin{bmatrix} Q_{x_i} & Q_{y_i} \end{bmatrix}^T \) define the Cartesian forces on the journal \( J_i \) of SFD no. \( i \), \( i = 1 \ldots N \). The forces on the corresponding bearing housing \( B_i \) are hence given by \(-\mathbf{p}_i\).

Now \( \mathbf{p}_i = \mathbf{p}_i(v_i, \dot{v}_i) \) where \( v_i = [x_{J_i} - x_{B_i}, y_{J_i} - y_{B_i}]^T \) and \( \mathbf{p}_i(v_i, \dot{v}_i) \) is calculated from the physical model of the SFD, as illustrated in Section 2.4. Hence,

\[
\mathbf{v}_i = \mathbf{\bar{v}}_i + \sum_{k=1}^{K} \left\{ v^{(k)}_{i\cos} \cos k \omega t + v^{(k)}_{i\sin} \sin k \omega t \right\} \tag{5.3}
\]

\[
\mathbf{p}_i = \mathbf{\bar{p}}_i + \sum_{k=1}^{K} \left\{ \mathbf{p}^{(k)}_{i\cos} \cos k \omega t + \mathbf{p}^{(k)}_{i\sin} \sin k \omega t \right\} \tag{5.4}
\]

where:

\[
\mathbf{p}^{(k)}_{i\cos} = \mathbf{p}^{(k)}_{i\cos}(v_i, v^{(1)}_i, v^{(2)}_i, \ldots, v^{(K)}_i) = (\alpha/(2\pi)) \int_0^{2\pi/\alpha} \mathbf{p}_i(v_i, \dot{v}_i) \cos k \omega t \, dt \tag{5.5a}
\]

\[
\mathbf{p}^{(k)}_{i\sin} = \mathbf{p}^{(k)}_{i\sin}(v_i, v^{(1)}_i, v^{(2)}_i, \ldots, v^{(K)}_i) = (\alpha/(\pi)) \int_0^{2\pi/\alpha} \mathbf{p}_i(v_i, \dot{v}_i) \sin k \omega t \, dt \tag{5.5b}
\]

Hence, if \( \mathbf{v} = \mathbf{x}_J - \mathbf{x}_B = \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_N \end{bmatrix} \)

\[
\mathbf{\bar{v}} = \begin{bmatrix} \mathbf{\bar{v}}_1 \\ \vdots \\ \mathbf{\bar{v}}_N \end{bmatrix}, \quad \mathbf{v}^{(k)}_{i\cos} = \begin{bmatrix} v^{(k)}_{i\cos} \\ \vdots \\ v^{(k)}_{N\cos} \end{bmatrix}, \quad \mathbf{v}^{(k)}_{i\sin} = \begin{bmatrix} v^{(k)}_{i\sin} \\ \vdots \\ v^{(k)}_{N\sin} \end{bmatrix} \tag{5.7a, b, c}
\]

then:

\[
\mathbf{\bar{p}} = \begin{bmatrix} \mathbf{\bar{p}}_1 \\ \vdots \\ \mathbf{\bar{p}}_N \end{bmatrix} = \mathbf{\bar{p}}(\mathbf{\bar{v}}^{(1)}_{i\cos}, \mathbf{\bar{v}}^{(1)}_{i\sin}, \ldots, \mathbf{\bar{v}}^{(K)}_{i\cos}, \mathbf{\bar{v}}^{(K)}_{i\sin}) \tag{5.8a}
\]
\[ \rho_{\text{cos}}^{(k)} = \begin{bmatrix} \rho_{1\text{cos}}^{(k)} \\ \vdots \\ \rho_{N\text{cos}}^{(k)} \end{bmatrix} = \rho_{\text{cos}}^{(k)} \begin{pmatrix} v, & v(1), & \ldots, & v^{(K)}, & v_{\text{sin}}^{(K)} \end{pmatrix} \]  
\text{(5.8b)}

\[ \rho_{\text{sin}}^{(k)} = \begin{bmatrix} \rho_{1\text{sin}}^{(k)} \\ \vdots \\ \rho_{N\text{sin}}^{(k)} \end{bmatrix} = \rho_{\text{sin}}^{(k)} \begin{pmatrix} \bar{v}, & v_{\text{cos}}^{(1)}, & \ldots, & v_{\text{cos}}^{(K)}, & v_{\text{sin}}^{(K)} \end{pmatrix} \]  
\text{(5.8c)}

It is recalled from Chapter 3 (eq. (3.32)), that \( v \) can be expressed as:

\[ v = v_s + x \]  
\text{(5.9)}

where \( v_s \) is the vector defining the static offsets of the SFD journals relative to their housings, in the \( x, y \) directions, under no rotor loading. \( x \) was defined in Section 3.2 as the “dynamic part” i.e. that component of \( v \) that is induced by the excitations \( f, g \) on the linear part.

For the \( j \)th rotor the unbalances \( U_{(j)1}, \ldots, U_{(j)s_j} \) are concentrated at \( S_j \) points. The vector of unbalance excitations on rotor no. \( j \) can be written as:

\[ u_{(j)} = u_{(j)\text{cos}} \Omega_{(j)t} + u_{(j)\text{sin}} \sin \Omega_{(j)t} \]  
\text{(5.10)}

where:

\[ u_{(j)\text{cos}} = \Omega_{(j)}^2 \begin{bmatrix} U_{(j)1} \sin(\phi_{(j)} + \gamma_{(j)1}) \\ -U_{(j)1} \cos(\phi_{(j)} + \gamma_{(j)1}) \\ \vdots \\ U_{(j)s_j} \sin(\phi_{(j)} + \gamma_{(j)s_j}) \\ -U_{(j)s_j} \cos(\phi_{(j)} + \gamma_{(j)s_j}) \end{bmatrix}, \quad u_{(j)\text{sin}} = \Omega_{(j)}^2 \begin{bmatrix} U_{(j)1} \cos(\phi_{(j)} + \gamma_{(j)1}) \\ U_{(j)1} \sin(\phi_{(j)} + \gamma_{(j)1}) \\ \vdots \\ U_{(j)s_j} \cos(\phi_{(j)} + \gamma_{(j)s_j}) \\ U_{(j)s_j} \sin(\phi_{(j)} + \gamma_{(j)s_j}) \end{bmatrix} \]  
\text{(5.11a, b)}
\( \gamma(j)s \), \( s = 1 \ldots S_j \), is the angular position of the unbalance at location no. \( s \) of rotor no. \( j \) relative to the angular position of unbalance no. 1 on the same rotor (\( \gamma(j)1 = 0 \) for all \( j \)). \( \phi(j) \) is the angular position of unbalance no. 1 on rotor no. \( j \) relative to the angular position of unbalance no. 1 on rotor no. 1 at the instant \( t = 0 \) (\( \phi(1) = 0 \)). It is to be noted that, for a given unbalance distribution on the engine, the values of \( \gamma(j,s) \) are fixed but the values of \( \phi(j) \) (\( j = 2, 3 \ldots \)) are arbitrary since the angular positions of the rotors relative to each other at the reference time \( t = 0 \) are arbitrary.

For the \( j^{th} \) rotor the gyroscopic effects are concentrated at \( G_j \) points. Let \( g \) be the vector containing the gyroscopic moments on all the rotors. Hence:

\[
g = P \theta
\]

(5.12)

where \( g \), \( P \), and the vector \( \theta \) of flexural rotations at the gyroscopic locations are defined in the Nomenclature. Due to the periodicity of the vibration and as a result of eq. (5.12), one can write:

\[
\theta = \bar{\theta} + \sum_{k=1}^{K} \left\{ \theta_{\cos}^{(k)} \cos k \omega t + \theta_{\sin}^{(k)} \sin k \omega t \right\}
\]

(5.13)

\[
g = \sum_{k=1}^{K} \left\{ g_{\cos}^{(k)} \cos k \omega t + g_{\sin}^{(k)} \sin k \omega t \right\}
\]

(5.14)

where:

\[
g_{\cos}^{(k)} = k \omega \theta_{\cos}^{(k)}, \quad g_{\sin}^{(k)} = -k \omega \theta_{\cos}^{(k)}
\]

(5.15a, b)

### 5.2.2 Derivation of the Block of Dynamic Equations

For each harmonic \( k \omega \), \( k = 1 \ldots K \), of the response at the SFDs (eqs. (5.2a, b)) one can write the following force-displacement relationships:
\[
\begin{bmatrix}
\mathbf{x}_{J,\cos}^{(k)} \\
\mathbf{x}_{J,\sin}^{(k)}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{C}_{JJ}(k) & 0 \\
0 & \mathbf{C}_{JJ}(k)
\end{bmatrix}
\begin{bmatrix}
\mathbf{p}_{\cos}^{(k)} \\
\mathbf{p}_{\sin}^{(k)}
\end{bmatrix} +
\begin{bmatrix}
\mathbf{C}_{JB}(k) & 0 \\
0 & \mathbf{C}_{JB}(k)
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_{(j)\cos}^{(k)} \\
\mathbf{u}_{(j)\sin}^{(k)}
\end{bmatrix},
\]

\[
\begin{bmatrix}
\mathbf{x}_{B,\cos}^{(k)} \\
\mathbf{x}_{B,\sin}^{(k)}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{C}_{BJ}(k) & 0 \\
0 & \mathbf{C}_{BJ}(k)
\end{bmatrix}
\begin{bmatrix}
\mathbf{p}_{\cos}^{(k)} \\
\mathbf{p}_{\sin}^{(k)}
\end{bmatrix} +
\begin{bmatrix}
\mathbf{C}_{BB}(k) & 0 \\
0 & \mathbf{C}_{BB}(k)
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_{(j)\cos}^{(k)} \\
\mathbf{u}_{(j)\sin}^{(k)}
\end{bmatrix},
\]

\[k = 1 \ldots K \quad (5.16a)\]

\[k = 1 \ldots K \quad (5.16b)\]

In above equations, the receptance (or “compliance”) matrices are defined from modal theory [58] as:

\[
\mathbf{C}_{JJ}(\omega) = \sum_{r=1}^{R} \frac{\mathbf{\Psi}_J^{(r)\,\mathbf{\Psi}_J^{(r)\,T}}}{\omega_r^2 - \omega^2}, \quad \mathbf{C}_{JB}(\omega) = \mathbf{C}_{BJ}(\omega) = \sum_{r=1}^{R} \frac{\mathbf{\Psi}_B^{(r)\,\mathbf{\Psi}_B^{(r)\,T}}}{\omega_r^2 - \omega^2}, \quad \mathbf{C}_{Jg}(\omega) = \sum_{r=1}^{R} \frac{\mathbf{\Psi}_J^{(r)\,\mathbf{\Psi}_g^{(r)\,T}}}{\omega_r^2 - \omega^2},
\]

\[
\mathbf{C}_{Ju(j)}(\omega) = \sum_{r=1}^{R} \frac{\mathbf{\Psi}_J^{(r)\,\mathbf{\Psi}_{u(j)}^{(r)\,T}}}{\omega_r^2 - \omega^2}, \quad \mathbf{C}_{BB}(\omega) = \sum_{r=1}^{R} \frac{\mathbf{\Psi}_B^{(r)\,\mathbf{\Psi}_B^{(r)\,T}}}{\omega_r^2 - \omega^2}, \quad \mathbf{C}_{Bg}(\omega) = \sum_{r=1}^{R} \frac{\mathbf{\Psi}_B^{(r)\,\mathbf{\Psi}_g^{(r)\,T}}}{\omega_r^2 - \omega^2},
\]

\[
\mathbf{C}_{Bu(j)}(\omega) = \sum_{r=1}^{R} \frac{\mathbf{\Psi}_B^{(r)\,\mathbf{\Psi}_{u(j)}^{(r)\,T}}}{\omega_r^2 - \omega^2}.
\]

(5.17a-g)

In the above expressions, \(\omega_r\ (r = 1 \ldots R)\) are the undamped natural frequencies of the linear part. \(\mathbf{\Psi}_J^{(r)}, \mathbf{\Psi}_B^{(r)}\) are the mass-normalised eigenvectors defining the \(x, y\) displacements of the squeeze-film terminals \(J, B\) in mode no. \(r\). Similarly, \(\mathbf{\Psi}_{u(j)}^{(r)}, \mathbf{\Psi}_g^{(r)}\)
are mass-normalised eigenvectors taken at the degrees of freedom corresponding to the
directions and locations of the elements of \( u_{(j)} \) and \( g \) respectively. In Chapter 3, \( \psi_x^{(r)} \) was
defined as:

\[
\psi_x^{(r)} = \psi_j^{(r)} - \psi_B^{(r)}
\]  \( (5.18) \)

Hence, subtracting eq. (5.16b) from eq. (5.16a), and using relations (5.17a-g) one obtains:

\[
\begin{bmatrix}
  v_{\cos}^{(k)} \\
v_{\sin}^{(k)}
\end{bmatrix}
= \begin{bmatrix}
  C_{xx}(k\omega) & 0 \\
  0 & C_{xx}(k\omega)
\end{bmatrix}
\begin{bmatrix}
  \rho_{\cos}^{(k)} \\
  \rho_{\sin}^{(k)}
\end{bmatrix}
+ \begin{bmatrix}
  C_{xg}(k\omega) & 0 \\
  0 & C_{xg}(k\omega)
\end{bmatrix}
\begin{bmatrix}
  \nu_{\cos}^{(k)} \\
  \nu_{\sin}^{(k)}
\end{bmatrix}
+ \sum_{j=1}^J \delta_{k\omega,\Omega_{(j)}} \begin{bmatrix}
  C_{xu_{(j)}}(k\omega) & 0 \\
  0 & C_{xu_{(j)}}(k\omega)
\end{bmatrix}
\begin{bmatrix}
  u_{(j)\cos} \\
  u_{(j)\sin}
\end{bmatrix}, \ k = 1\ldots K
\]  \( (5.19) \)

where:

\[
C_{xx}(\omega) = \sum_{r=1}^R \frac{\psi_x^{(r)(r)^T}}{\omega_r^2 - \omega^2}, \ C_{xg}(\omega) = \sum_{r=1}^R \frac{\psi_x^{(r)(r)^T}}{\omega_r^2 - \omega^2}, \ C_{xu_{(j)}}(\omega) = \sum_{r=1}^R \frac{\psi_x^{(r)(r)^T}}{\omega_r^2 - \omega^2}
\]  \( (5.20a, b, c) \)

The above matrices can be regarded as compliance matrices relating the relative responses
at the SFD terminals to the various excitations.

The gyroscopic terms \( \nu_{\cos}^{(k)} \), \( \nu_{\sin}^{(k)} \) can be eliminated from the right hand side of eq.
(5.19) as follows. By analogy with eq. (5.19), one can write a force-displacement
relationship for each harmonic \( k\omega \) of the flexural angular displacement \( \theta \) at the
gyroscopic locations:

\[
\begin{bmatrix}
  \theta_{\cos}^{(k)} \\
  \theta_{\sin}^{(k)}
\end{bmatrix}
= \begin{bmatrix}
  C_{\theta x}(k\omega) & 0 \\
  0 & C_{\theta x}(k\omega)
\end{bmatrix}
\begin{bmatrix}
  \rho_{\cos}^{(k)} \\
  \rho_{\sin}^{(k)}
\end{bmatrix}
+ \begin{bmatrix}
  C_{\theta g}(k\omega) & 0 \\
  0 & C_{\theta g}(k\omega)
\end{bmatrix}
\begin{bmatrix}
  \nu_{\cos}^{(k)} \\
  \nu_{\sin}^{(k)}
\end{bmatrix}
+ \sum_{j=1}^J \delta_{k\omega,\Omega_{(j)}} \begin{bmatrix}
  C_{\theta u_{(j)}}(k\omega) & 0 \\
  0 & C_{\theta u_{(j)}}(k\omega)
\end{bmatrix}
\begin{bmatrix}
  u_{(j)\cos} \\
  u_{(j)\sin}
\end{bmatrix}, \ k = 1\ldots K
\]
\[ + \sum_{j=1}^{J} \delta_{k\omega,\omega(j)} \begin{bmatrix} C_{0u(j)}(k\omega) & 0 \\ 0 & C_{0u(j)}(k\omega) \end{bmatrix} \begin{bmatrix} u_{(j)\cos} \\ u_{(j)\sin} \end{bmatrix} \]  

(5.21)

where:

\[ C_{0x}(k\omega) = \sum_{r=1}^{R} \frac{\Psi^{(r)}_0 \Psi^{(r)\dagger}_x}{\omega_r^2 - \omega^2}, \quad C_{0g}(k\omega) = \sum_{r=1}^{R} \frac{\Psi^{(r)}_0 \Psi^{(r)\dagger}_g}{\omega_r^2 - \omega^2}, \quad C_{0u(j)}(k\omega) = \sum_{r=1}^{R} \frac{\Psi^{(r)}_0 \Psi^{(r)\dagger}_{u(j)}}{\omega_r^2 - \omega^2} \]

(5.22a, b, c)

with \( \Psi^{(r)}_0 \) being mass-normalised eigenvectors taken at the degrees of freedom corresponding to the elements in \( \omega \). Substituting for \( g_{\cos}^{(k)}, g_{\sin}^{(k)} \) from eqs. (5.15a, b) into eq. (5.21) and solving the resulting equations for \( \omega_{\cos}^{(k)}, \omega_{\sin}^{(k)} \) one obtains:

\[
\begin{bmatrix} \omega_{\sin}^{(k)} \\ \omega_{\cos}^{(k)} \end{bmatrix} = \begin{bmatrix} -D(k\omega)B(k\omega)C_{0x}(k\omega) & D(k\omega)C_{0x}(k\omega) \\ D(k\omega)C_{0x}(k\omega) & D(k\omega)B(k\omega)C_{0x}(k\omega) \end{bmatrix} \begin{bmatrix} \omega_{\cos}^{(k)} \\ \omega_{\sin}^{(k)} \end{bmatrix} + \sum_{j=1}^{J} \delta_{k\omega,\omega(j)} \begin{bmatrix} -D(k\omega)B(k\omega)C_{0u(j)}(k\omega) & D(k\omega)C_{0u(j)}(k\omega) \\ D(k\omega)C_{0u(j)}(k\omega) & D(k\omega)B(k\omega)C_{0u(j)}(k\omega) \end{bmatrix} \begin{bmatrix} u_{(j)\cos} \\ u_{(j)\sin} \end{bmatrix} \]

(5.23)

where:

\[ D(k\omega) = \left( I + B(k\omega)^2 \right)^{-1}, \quad B(k\omega) = k\omega \quad C_{0g}(k\omega) \quad P \]

(5.24a, b)

Hence, from eqs. (5.15a, b) and eq. (5.23), one can eliminate \( g_{\cos}^{(k)}, g_{\sin}^{(k)} \) from eq. (5.19):

\[
\begin{bmatrix} v_{\cos}^{(k)} \\ v_{\sin}^{(k)} \end{bmatrix} = \begin{bmatrix} C_{xx}(k\omega) - E(k\omega) & F(k\omega) \\ -F(k\omega) & C_{xx}(k\omega) - E(k\omega) \end{bmatrix} \begin{bmatrix} \omega_{\cos}^{(k)} \\ \omega_{\sin}^{(k)} \end{bmatrix} \begin{bmatrix} \omega_{\cos}^{(k)} \\ \omega_{\sin}^{(k)} \end{bmatrix} \]

(5.25)
\[
\sum_{j=1}^{J} \delta_{k\sigma,\omega(j)} \left[ C_{xu(j)}(k\sigma) - L_{(j)}(k\sigma) \right] - Q_{(j)}(k\sigma) \left[ C_{xu(j)}(k\sigma) - L_{(j)}(k\sigma) \right] \left[ \begin{array}{c} u_{(j)\cos} \\ u_{(j)\sin} \end{array} \right], \quad k = 1\ldots K
\]

(5.25)

where:

\[
E(k\sigma) = k\sigma C_{xg}(k\sigma)PD(k\sigma)B(k\sigma)C_{ox}(k\sigma)
\]

(5.26a)

\[
F(k\sigma) = k\sigma C_{xg}(k\sigma)PD(k\sigma)C_{ox}(k\sigma)
\]

(5.26b)

\[
L_{(j)}(k\sigma) = k\sigma C_{xg}(k\sigma)PD(k\sigma)B(k\sigma)C_{0u(j)}(k\sigma)
\]

(5.26c)

\[
Q_{(j)}(k\sigma) = k\sigma C_{xg}(k\sigma)PD(k\sigma)C_{0u(j)}(k\sigma)
\]

(5.26d)

Equations (5.25) constitute the dynamic block of the RHBM equations. By consideration of the relationships in eqs. (5.8a-c), it is clear that eqs. (5.25) constitute a set of \(2N \times 2K\) equations in \(2N \times (2K + 1)\) unknowns (which are the Fourier coefficients of the relative displacements at the SFDs, eqs. (5.7a-c)). A block of “pseudo-static” equations (defining force-response relations at zero-frequency) will complete the equation set.

### 5.2.3 Derivation of the Block of Pseudo-Static Equations

As can be seen from eq. (5.20a), the presence of rigid body modes (for which \(\omega_r = 0\)) in the linear part, would result in the receptance matrix \(C_{xx}(0) \rightarrow \pm \infty\). Note that the rigid body modes define rigid body motion of one or more rotors in the \(xz\) or \(yz\) planes (the superfluous modes defining rigid body spin of each rotor about its axis are removed). As discussed in the Introduction, \(\bar{\rho}\) and the static loading \(w\) are in equilibrium. This principle can be applied in a systematic fashion by writing zero-frequency force-acceleration response equations at selected journals \(J_i\) as follows:

\[
0 = A_{J_{ax}}(0)\bar{\rho} + A_{J_{aw}}(0)w
\]

(5.27)
where the accelerance matrices are defined as:

\[
A_{J_{\text{sel}}}^{(x)}(\omega) = \sum_{r=1}^{R} -\omega^2 \frac{\psi_{J_{\text{sel}}}^{(r)} \psi_{x}^{(r)T}}{\omega_r^2 - \omega^2}, \quad A_{J_{\text{sel}}}^{(w)}(\omega) = \sum_{r=1}^{R} -\omega^2 \frac{\psi_{J_{\text{sel}}}^{(r)} \psi_{w}^{(r)T}}{\omega_r^2 - \omega^2}
\]  

(5.28a, b)

\[\psi_{J_{\text{sel}}}^{(r)}\] are the mass-normalised eigenvectors defining the \(x, y\) displacements at the selected journals \(J_i\). If the linear part has a total of \(P\) rigid body modes i.e. \(\omega_1 \ldots \omega_p = 0\), then:

\[
A_{J_{\text{sel}}}^{(x)}(0) = \sum_{r=1}^{P} \psi_{J_{\text{sel}}}^{(r)} \psi_{x}^{(r)T}, \quad A_{J_{\text{sel}}}^{(w)}(0) = \sum_{r=1}^{P} \psi_{J_{\text{sel}}}^{(r)} \psi_{w}^{(r)T}
\]  

(5.29a, b)

In the case of the engine shown in Figure 1.1, \(P = 4\), since each rotor has two rigid body modes: one per plane \(xz, yz\), defining pivoting motion about \(J_1\) or \(J_3\). Eqs. (5.27) effectively define moment equilibrium in each plane about each pivot. In general, the method of eq. (5.27) yields a maximum of \(P\) independent equations. Hence, it is for this reason that eq. (5.27) is only applied at selected journals \(J_i\). The selection is arbitrary provided that none of the chosen journals is a node (pivot) in the corresponding rotor’s rigid body modes since it would then introduce trivial “\(0 = 0\)” equations in the corresponding rows of eq. (5.27).

Since the total number of unknown Fourier coefficients of the relative displacement at the SFDs (eqs. (5.7a-c)) is \(2N \times (2K + 1)\), then from the above it is apparent that if \(P < 2N\) additional zero-frequency equations need to be found to supplement eqs. (5.26).

Let \(q\) be the vector of instantaneous modal coordinates, which represent the vibration of the structure in modal space. Due to the periodicity of the vibration:

\[
q = \bar{q} + \sum_{k=1}^{K} \left\{ \begin{array}{c} q_{\cos}^{(k)} \cos k \omega t + q_{\sin}^{(k)} \sin k \omega t \end{array} \right\}
\]  

(5.30)
Let the mean component vector $\bar{q}$ be partitioned thus:

$$\bar{q} = \begin{bmatrix} q^T & \bar{q}^T \end{bmatrix}$$  \hspace{1cm} (5.31)$$

where $\bar{q}$ is the $P \times 1$ vector of mean modal coordinates associated with the rigid body modes. Hence, recalling eq. (5.9) and noting that $x = \bar{H}_x q$, the additional block of zero-frequency equations is given as follows, by splitting the response at zero frequency into rigid and flexible modal contributions:

$$\bar{v} - v_s = \bar{H}_x \bar{q} + \tilde{C}_{xx}(0)\bar{p} + \tilde{C}_{xw}(0)w$$  \hspace{1cm} (5.32)$$

$\bar{H}_x$ is the rigid body modal matrix defined by:

$$\bar{H}_x = \begin{bmatrix} \psi_x^{(1)} & \cdots & \psi_x^{(P)} \end{bmatrix}$$  \hspace{1cm} (5.33)$$

$\tilde{C}_{xx}(0)$, $\tilde{C}_{xw}(0)$ are zero-frequency “incomplete” compliance matrices with the rigid body mode contribution excluded i.e.:

$$\tilde{C}_{xx}(\omega) = \sum_{r=1}^{R} \frac{\psi_x^{(r)} \psi_x^{(r)T}}{\omega_r^2 - \omega^2}, \quad \tilde{C}_{xw}(\omega) = \sum_{r=1}^{R} \frac{\psi_x^{(r)} \psi_w^{(r)T}}{\omega_r^2 - \omega^2}$$  \hspace{1cm} (5.34a, b)$$

The block of zero-frequency equations can thus be expressed as follows, for the most general case:

$$\begin{bmatrix} 0 \\ \bar{v} - v_s \end{bmatrix} = \begin{bmatrix} 0_{P \times P} \\ \bar{H}_x \end{bmatrix} \bar{q} + \begin{bmatrix} A_{J_{xx}(0)} \\ \tilde{C}_{xx}(0) \end{bmatrix} \bar{p} + \begin{bmatrix} A_{J_{xw}(0)}w \\ \tilde{C}_{xw}(0)w \end{bmatrix}$$  \hspace{1cm} (5.35)$$
It is observed that an extra $P$ zero-frequency unknowns, contained in $\bar{q}$, have been introduced into the system, in addition to the $2N$ zero-frequency unknowns in $\bar{v}$. However, this is not a problem since there are $P+2N$ equations in eq. (5.35). The extra unknowns term $\bar{q}$ in eq. (5.35) either vanishes or is not required for the following two special cases:

a) $P = 2N$ - zero-frequency equations “fully implicit”: in this case eq. (5.32) (i.e. the lower row set of eq. (5.35), and consequently $\bar{q}$) is not required. One example of this case would be if each rotor in Figure 1.1 had only two bearings, all of which were unsupported SFDs;

b) $P = 0$ - zero-frequency equations “fully explicit”: in this case the upper row set of eq. (5.35) and $\bar{H}_x$ are null and only the lower row set is used. This would occur if each rotor in Figure 1.1 had retainer springs at two or more SFDs.

The introduction of the extra unknowns term $\bar{q}$ allows the resolution of the first of the outstanding issues bulleted in the latter part of the Introduction i.e. the solution of systems involving at least one rotor that is statically indeterminate with the nonlinear elements in place and either does not have any linear support or has just one linear point support (as clearly evident for the low pressure rotor in Figure 1.1).

### 5.2.4 Solution of the Equations

Equations (5.25) and (5.35) together constitute the full set of $2N(2K+1)+n_{\text{extra}}$ nonlinear algebraic equations in an equal number of unknowns (where $n_{\text{extra}} = P$, except for the special cases (a), (b) in previous section where $n_{\text{extra}} = 0$). Collecting the unknowns into one vector
\[ \mathbf{z} = \begin{bmatrix} -T & \mathbf{V}^T & \mathbf{v}^{(1)}_\cos & \mathbf{v}^{(1)}_\sin & \cdots & \mathbf{v}^{(K)}_\cos & \mathbf{v}^{(K)}_\sin \end{bmatrix}^T \] (5.36)

and moving all terms of eqs. (5.25) and (5.35) to the left hand side, the system of equations can be expressed as:

\[ \chi(z, \Omega_{\text{ref}}, Q) = 0 \] (5.37)

where \( \chi \) is a nonlinear vector function of \( z \), \( \Omega_{\text{ref}} \) and \( Q \). This is due to the fact that, for given \( \Omega_{\text{ref}}, Q \) (eq. (1)) and an assumed \( z \), all terms in eqs. (5.25) and (5.35) are fully determined. In particular, the terms \( \overline{p}, \mathbf{p}^{(k)}_\cos, \mathbf{p}^{(k)}_\sin \) are calculated as indicated in the flow chart of Figure 5.1.

Hence, for given \( \Omega_{\text{ref}} \) and \( Q \) the equation (5.37) can be solved for \( z \) by iteration. Eq. (5.37) can be solved over a range of \( \Omega_{\text{ref}} \) for given \( Q \) to yield a set of solutions defining a “speed response curve”. In the case of MFU, a fixed \( Q \) implies that the ratio of the speeds of the rotors is kept fixed as the speed \( \Omega_{\text{ref}} \) is varied. The continuation technique used to advance the solution procedure along the curve uses a predictor-corrector approach [10] where the initial approximation (or “predictor”) \( z_0 \) for the solution \( z \) at a point on the curve is obtained from the previous points. Equation (5.37) is then solved by the damped Newton-Raphson method (the “corrector”) [54]. The initial approximation for the first point on the curve is furnished by the Fourier coefficients of a time-marching solution.

At each point on the speed response, the Newton-Raphson method requires the calculation of the Jacobian matrix \( \frac{\partial \chi}{\partial z}|_{z=z_0} \), which is then inverted. As \( z \) iterates towards its correct value the inverse of the Jacobian is efficiently updated using Broyden’s
Method [54]. In view of the large number of unknowns and the process of Figure 5.1 (which is applied as many times as necessary during the iterative process), an efficient means of computing the Jacobian is imperative. From eqs. (5.25) and (5.35) it is clear that the computational burden lies in the evaluation of the matrices $\partial \mathbf{p}/\partial z$, $\partial \mathbf{p}^{(k)}_{\cos}/\partial z$, $\partial \mathbf{p}^{(k)}_{\sin}/\partial z$. Appendix A2 gives a method to assemble these matrices efficiently from the partial derivatives of the Fourier coefficients of the forces at each SFD no. $i$ with respect to the Fourier coefficients of its own relative displacements.

5.2.5 Recovery of the Full Set of Degrees of Freedom

Having solved for $z$, the definitive values of the Fourier coefficients of the SFD forces and the gyroscopic moments can be determined (the latter found from eqs. (5.15a, b) and (5.23)). These enable the computation of the Fourier coefficients of the vector of modal coordinates $\mathbf{q}$ (eq. (5.29)). This in turn allows the computation of the time history of the response at a set of arbitrary degrees of freedom $x_P$ since $x_P = H_P \mathbf{q}$ where $H_P$ is a matrix of mass-normalised eigenvectors evaluated at the degrees of freedom $x_P$. The Fourier coefficients of $\mathbf{q}$ are found as follows, by considering the modal-space equivalent of eq. (5.25):

$$
\begin{bmatrix}
\mathbf{q}^{(k)}_{\cos} \\
\mathbf{q}^{(k)}_{\sin}
\end{bmatrix} =
\begin{bmatrix}
\Lambda - k^2 \omega^2 I & 0 \\
0 & \Lambda - k^2 \omega^2 I
\end{bmatrix}^{-1}
\left[
\begin{bmatrix}
\mathbf{H}_x \mathbf{p}^{(k)}_{\cos} \\
\mathbf{H}_x \mathbf{p}^{(k)}_{\sin}
\end{bmatrix}
+ \sum_{j=1}^{J} \delta_{k \sigma, \Omega(j)}
\begin{bmatrix}
\mathbf{H}_u^{T} \mathbf{u}^{(j)}_{\cos} \\
\mathbf{H}_u^{T} \mathbf{u}^{(j)}_{\sin}
\end{bmatrix}
\right]
$$

where:

$$
\Lambda = \text{diag}\{\omega_1^2, \ldots, \omega_R^2\}
$$

With reference to eq. (5.31), the components $\mathbf{q}$ were found as part of the solution process of the previous section. As for the non-rigid components $\mathbf{q}$ in eq. (5.31), these are given by the equation:
\[
\tilde{q} = \tilde{\Lambda}^{-1}\{\tilde{\mathbf{H}}_x^T \tilde{p} + \tilde{\mathbf{H}}_w^T \mathbf{w}\}
\]  

(5.40)

where:

\[
\tilde{\Lambda} = \text{diag}\{\omega_{p+1}^2, \ldots, \omega_R^2\}, \quad \tilde{\mathbf{H}}_x = [\psi_x^{(p+1)} \ldots \psi_x^{(R)}], \quad \tilde{\mathbf{H}}_w = [\psi_w^{(p+1)} \ldots \psi_w^{(R)}]
\]

(5.41a, b, c)

### 5.2.6 Some Observations

It is to be noted that concentrated viscous damping forces can be dealt with in a similar manner to the gyroscopic effect in Section 5.2.2. Distributed damping of the proportional type [58] in the linear part can also be accommodated into the above analysis. In this case, the receptance matrix expressions in eqs. (5.20) are simply modified by changing their denominator to \( \omega_r^2 - \omega^2 + j2\zeta_r \omega \), where \( j = \sqrt{-1} \) and \( \zeta_r \) is the modal damping ratio [58]. Hence, in the first term of the right hand side of eq. (5.19), the matrices \( \mathbf{C}_{xx} \) are replaced by \( \text{Re}\{\mathbf{C}_{xx}\} \) and the off-diagonal zero matrices are replaced by \( \text{Im}\{\mathbf{C}_{xx}\} \), \( - \text{Im}\{\mathbf{C}_{xx}\} \) - similarly for the other terms.

Stability analysis is not performed here since available techniques are not useful for systems with a large number of modes, as discussed in Section 2.3, Chapter 2. However, a spot check can be performed at an arbitrary solution point on a speed response by running a time-marching analysis from initial conditions on the particular RHBM solution that is being tested: if the trajectory remains on the RHBM orbit then it is stable [10]. The initial state variables \( \mathbf{q}(t = 0) \), \( \dot{\mathbf{q}}(t = 0) \) are obtained through eq. (5.30) and eqs. (5.38, 5.40).

### 5.3 SIMULATIONS
The RHBM was applied to a representative twin-spool aero-engine having the schematic layout in Figure 1.1, using a realistically sized whole-engine finite-element (FE) model provided by an engine manufacturer [53]. This engine-model was used in Chapter 3 to test the IRM. In this section, the RHBM results for both SFU and MFU response are validated against the corresponding IRM results. All simulations were performed in Matlab on a standard 2006-issue desktop pc with Intel® Pentium® D CPU 3GHz processor.

5.3.1 Linear Computation

As indicated in Figure 1.4, a one-off eigenvalue analysis was performed on the linear part under non-rotational conditions. The results of the eigenvalue analysis were used during the course of the subsequent RHBM process to compute the required receptance matrices (eqs. (5.20, 5.22)). By way of illustration of a typical receptance function, one may look at Figure 3.2 in Chapter 3. All 934 modes over the frequency range 0 to 1kHz were included in the subsequent RHBM analysis due to the high shaft speeds and harmonics in the response. As observed in Chapter 3, the sudden reduction in modal density beyond 500Hz is merely an artefact of the degree-of-freedom reduction technique used by the FE modellers.

5.3.2 Nonlinear Computation for the Unbalance Response

The RHBM was tested for two cases A, B involving SFU and MFU respectively, as illustrated in Table 5.1. In Case A the unbalance was restricted to the low pressure (LP) rotor only. For each rotor the unbalance was concentrated at two locations. Figure 5.2 indicates the positions of these locations, as well as the locations of the SFDs and the distribution of the weights of the two rotors. The gyroscopic effect was discretised at 7 points on the LP rotor and 12 points on the high-pressure (HP) rotor. For each Case A, B,
RHBM speed response curves were computed for a fixed ratio $\Omega(2)/\Omega(1) = 1.2$ where $\Omega(1), \Omega(2)$ are the LP, HP speeds respectively. It should be mentioned that MFU results are affected by the angular position $\phi(2)$ of the HP rotor relative to the LP rotor at the reference time $t = 0$. For the MFU results presented here $\phi(2) = 0$.

<table>
<thead>
<tr>
<th>Case</th>
<th>Low Pressure (LP) rotor</th>
<th>High Pressure (HP) rotor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$U_{(1)1}$</td>
<td>$U_{(1)2}$</td>
</tr>
<tr>
<td>A: SFU</td>
<td>6.3</td>
<td>6.3</td>
</tr>
<tr>
<td>B: MFU</td>
<td>6.3</td>
<td>6.3</td>
</tr>
</tbody>
</table>

Table 5.1: Test cases for RHBM

For these preliminary calculations, the bearing housings were assumed to be perfectly aligned with each other prior to rotor assembly (i.e. $v_s = 0$ in eq. (5.35)). As in Chapter 3, the SFDs considered for this study were single-land and end-fed with oil of viscosity $0.0049\text{Nsm}^{-2}$ at a pressure of 3bar (gauge). The bearing diameters and radial clearances were typically 200mm, 0.1mm respectively and the land lengths ranged from 16 to 34mm. Each iteration of the RHBM solution process required the evaluation of the Fourier coefficients of the SFD forces. As indicated in Figure 5.1 this necessitated the generation of an $n_F$-point time history of the forces $p_i(v_i, v_j)$ at each SFD (where $n_F$ is a suitable number of points [10]). For each of these points, the forces at SFD no. $i$ were evaluated by numerical integration of the pressure distribution across the oil film at the instantaneous dynamic condition $v_i, \dot{v}_i$, as described in Section 5.2.4. An “end-leakage factor” $\lambda = 0.03$ was used to account for the sealing at the ends of the dampers, as in Chapter 3. The SFD force calculation procedure is detailed in Appendix A1.

5.3.2.1 Case A: SFU

In this case, $\Omega_{(\text{ref})} = \Omega(1)$ in eq. (5.1) since the LP rotor carried the unbalance. A speed response curve was constructed for $Q = 1$ in eq. (5.1) and $\Omega(2)/\Omega(1) = 1.2$. Eight
harmonics of the fundamental frequency $\Omega_1$ were used (i.e. $K = 8$). The number of unknowns for each solution point was therefore 174 (including the $P = 4$ extra unknowns). With reference to the iterative process described in Figure 5.1, $n_F = 19$ was adequate for the SFU computations described.

The first solution point on the speed response corresponded to LP, HP rotor speeds of 10000, 12000 rpm respectively. For this first point (only), the initial approximation was provided by the Fourier coefficients of the transient time domain solution obtained by the IRM. Figure 5.3 shows the IRM solution over 3000 LP shaft revolutions starting from initial conditions corresponding to zero relative displacements and velocities at each SFD. It is evident that the system dynamics dictated that a time-march through a very large number of revolutions (up to 4000) was necessary to reach complete steady-state (regardless of the type of time domain solver used). However, for the purpose of generating the RHBM approximation, the solution over the first 100 LP shaft revolutions (i.e. Figure 5.3a) was sufficient: this took about 20 mins to generate. Using this transient approximation, the RHBM took 5 seconds to converge to the steady-state periodic solution shown in Figure 5.4. It is also noted that the RHBM also converged successfully in around the same time using a transient approximation generated in considerably less than 20 mins by using cruder numerical accuracy (tolerance) settings for the IRM solver. Figure 5.4 shows the excellent agreement between the RHBM and the steady-state IRM solution. This figure also shows the absolute vibration of the bearing housings which, in the case of RHBM, was recovered by the method described in section 5.3.2.5.

Having generated this first RHBM solution point, a speed response curve was constructed as described in section 5.3.2.4. This is expressed in Figure 5.5 as a set of graphs showing the variation with LP shaft speed of the half-peak-to-peak amplitude of the relative $y$ displacement at each SFD. The steady-state IRM results at discrete speeds are overlaid on the same axes and show excellent correlation with the RHBM.
5.3.2.2 Case B: MFU

In this case, $\Omega_{j,ret} = \Omega_{(1)}$ in eq. (5.1) since it is the slower shaft. Since $\Omega_{(2)}/\Omega_{(1)} = 1.2 = 6/5$, the value of $Q$ in eq. (5.1) was taken to be 5. Thirty-three harmonics of the fundamental frequency $\Omega_{(1)}/5$ were used (i.e. $K = 33$), resulting in 674 unknowns for each solution point. With reference to the iterative process described in Figure 5.1, the value of $n_F$ had to be increased to 69.

As for the SFU case, the first solution point on the speed response corresponded to LP, HP rotor speeds of 10000, 12000 rpm respectively. The initial approximation to this point was again provided by the Fourier coefficients of the transient IRM solution over the first 100 LP shaft revolutions. This approximation took roughly 20mins to produce and, using this, the RHBM took 50 seconds to converge to the steady-state periodic solution shown in Figure 5.6. This figure shows the excellent agreement between the RHBM solution and the steady-state IRM solution (achieved after a total of 4000 LP shaft revolutions).

From this first RHBM solution point, the speed response curve of RHBM solutions depicted in Figure 5.7 was generated. Excellent correlation is again seen between the RHBM and the discrete IRM results plotted in Figure 5.7.

5.3.2.3 Discussion

It is to be noted that, as indicated in Figure 1.1, SFDs 1 and 3 were spring-supported and the orbit offsets within the clearances of these SFDs were mainly due to the respective rotor weights (see Figure 5.4a). The rest of the SFDs, particularly SFDs 4 and 5, relied on the relative vibration (between their journal and housing) to generate a lifted mean position within their clearances. Hence, it is the prediction of the response at these unsupported
SFDs that posed the major computational challenge for the iterative process, particularly in the presence of a large number of harmonics.

Figure 5.8 shows the development of the harmonic content of the RHBM SFU speed response at one of the unsupported SFDs. This figure refers to the relative y displacement (mean removed) and velocity, the latter being included since the SFD force is a function of both relative displacement and velocity. This spectrum simply features the LP-synchronous frequency (“1LP”) and its harmonics: the HP-synchronous frequency is absent since there is no HP unbalance. It is also evident that an adequate number of harmonics has to be retained due to their prominence in the velocity spectrum.

Figure 5.9 shows the development of the harmonic content of the RHBM MFU speed response at one of the unsupported SFDs. This spectrum features both LP and HP-synchronous components, along with non-synchronous frequencies which are multiples of \( \Omega_{\{1\}}/5 \). It is evident that the sub-synchronous harmonics are prominent in the relative displacement signals whereas the super-synchronous signals are prominent in the relative velocity signals.

Figure 5.10 shows the variation of the computation time per solution point for both SFU and MFU speed response curves. The typical solution times per point for the two cases were 5s and 50s respectively. The total (cumulative) times to generate the speed response curve for the two cases were 85mins and 550mins respectively. However, these total times can be significantly cut through improvement of the algorithms used for the continuation procedure. Such numerical issues are outside the scope of this thesis. Suffice to say that the continuation procedure used here was a rudimentary one in which the LP-speed was the control parameter and the solution at a prescribed speed-step was solved using the solution at the previous step as the predictor. When the iteration showed signs of divergence, it automatically restarted with a reduced step-size as many times as necessary until the solution converged. The spikes in Figure 5.10 indicate that a large portion of the
total time was taken up by a few very narrow regions where the process struggled to advance the solution. It is likely that a switch to arc-length continuation [10] in these regions would significantly reduce the number of failed attempts, thereby cutting the total computation time.

Finally, it should be emphasised that the RHBM is highly useful even in the absence of a continuation procedure. As illustrated in section 5.3.2.1, “unfinished” time-domain solutions at a number of discrete speeds, each generated with crude tolerance settings over a small number of revolutions can each be “finished off” and refined to steady-state in a matter of seconds using the RHBM. This of course is conditional on the time-domain solution being indeed periodic in the steady-state, as in all the results presented here. As observed in Chapter 3, reducing the degree of sealing from $\lambda = 0.03$ (as used in the this thesis) to a very low level ($\lambda = 0.01$), all other parameters being kept the same, resulted in a steady-state quasi-periodic response which could only be computed by a time-domain technique, as discussed in the Introduction.

5.4 CONCLUSIONS

In this chapter, a whole-engine Receptance Harmonic Balance Method has been devised that, for the first time, has allowed the frequency domain computation of the steady-state periodic unbalance vibration of a whole aero-engine model with nonlinear bearings. The method uses the receptance functions of the linear part of the structure under non-rotational conditions, obtained from a one-off eigenvalue analysis, to set up the equations for the rotating nonlinear assembly. The unknowns solved for are the Fourier coefficients of the relative displacements at the nonlinear bearings plus a few extra unknowns. These latter unknowns enable solution of the problem in the presence of statically indeterminate rotors that have just one linear point support or none at all. Simulation tests were performed on a
realistically sized representative twin-spool engine with rotors running at different speeds for both single-frequency unbalance (SFU- unbalance distribution confined to one rotor) and multi-frequency unbalance (MFU- unbalance on both rotors). In either case, excellent correlation with time-marching results was obtained. For the cases studied, the computation time for MFU was about 10 times greater than SFU due to the larger number of harmonics necessary to describe the solution. However, for either case, it has been demonstrated that, when used in conjunction with a time-marching solver like the recently developed Impulsive Receptance Method, the RHBM is a very powerful tool that should greatly facilitate the hitherto highly restricted nonlinear dynamic analysis of realistic engine structures.
From eqs. (5.36), (5.7), extract \( \bar{v}_i, v^{(k)}_{\cos}, v^{(k)}_{\sin}, k = 1...K, i = 1...N \).

From eq. (5.3) and its time derivative construct an \( n_F \)-point time history of each \( \bar{v}_i, \dot{v}_i, i = 1...N \).

From the physical model of the SFD, evaluate the \( n_F \)-point time history of each \( p_i(v, \dot{v}_i), i = 1...N \).

From eqs. (5.5) evaluate the Fourier coefficients of the SFD forces and assemble these as in eqs. (5.8).

**Figure 5.1:** Calculation of SFD force Fourier coefficients \( \bar{\rho}, \rho_{\cos}^{(k)}, \rho_{\sin}^{(k)} \) for assumed \( z \).

**Figure 5.2:** Unbalance locations, bearing locations and rotor weight distribution
Figure 5.3: Periodic steady-state solution obtained using a time-domain approach (IRM) (orbits normalised with respect to the corresponding radial clearances)
b. Absolute displacement of the bearing-housing centres

**Figure 5.4:** Validation of RHBM (—) against steady-state IRM (---) for the starting solution of the SFU response curve (orbits normalised with respect to the respective radial clearances)

**Figure 5.5:** SFU speed response curves of y relative displacements at SFDs (vertical axes give half peak-peak amplitude normalised by radial clearance)
a. Orbits of the bearing journals within the radial clearance circles

b. Absolute displacement of the bearing-housing centres

Figure 5.6: Validation of RHBM (---) against steady-state IRM (-----) for the starting solution of the MFU response curve (orbits normalised with respect to the respective radial clearances)
Figure 5.7: MFU speed response curves of $y$ relative displacements at SFDs (vertical axes give half peak-peak amplitude normalised by radial clearance)
Figure 5.8: Development of harmonics in RHBM SFU response at an unsupported SFD (harmonic amplitudes normalised with respect to greatest amplitude in the respective diagram)
Figure 5.9: Development of harmonics in RHBM MFU response at an unsupported SFD (harmonic amplitudes normalised with respect to greatest amplitude in the respective diagram)
(a) Case A: SFU

(b) Case B: MFU

**Figure 5.10**: Point-to-point computation time in generating the RHBM speed response curves using a rudimentary speed-control continuation process
6 A COMPUTATIONAL PARAMETRIC ANALYSIS OF THE VIBRATION OF A THREE-SPOOL AERO-ENGINE UNDER MULTI-FREQUENCY UNBALANCE EXCITATION

6.1 INTRODUCTION

Of all rotating machines that use SFD bearings, three-spool aero-engines can be considered as the most complex structures. The integration of SFDs into bearing assemblies has proved to be a very cost-effective technical solution to the problem of attenuating vibration caused by rotor unbalance. However, as mentioned before, in order to achieve the best deployment of SFDs in an engine it is necessary to carry out thorough calculations on the whole-engine structure taking due account of the SFDs’ nonlinearity. This is especially so considering that, most of the bearings of typical two/three-spool engines of European design are unsupported by a centralising spring.

The nonlinear solvers used to calculate the unbalance response work either in the time domain or in the frequency domain. Time domain solvers progress forward in time until a steady-state response is obtained that may not necessarily be periodic. Frequency domain solvers extract steady-state solutions that are assumed to be periodic of given fundamental frequency. The previous chapter has illustrated the benefit of a computational facility that takes advantage of the relative merits of both time and frequency domain methods through an integrated approach that makes effective use of both. Prior to the research of Chapters 3 and 5, published in [1, 57], such time/frequency domain calculations on realistic aero-engine models are prohibitive due to the large number of assembly modes that need to be considered. This problem has been overcome using the novel IRM and RHBM, which efficiently solve the nonlinear problem in the time and frequency domains respectively. In
these works, these two methods have been illustrated on a realistic twin-spool engine model and have been shown to be effective for both SFU excitation (unbalance on a single rotor) and MFU excitation (unbalance on both rotors).

In this chapter the IRM and RHBM are applied to the analysis of the vibration of a realistically-sized representative three-shaft whole aero-engine model subject to MFU excitation. The two main goals are:

- To carry out a first-ever parametric analysis of such a system, examining the effect of SFD end-sealing and the use of simple uni-directional “bump-spring” supports at certain SFDs on the overall dynamic performance.
- To propose a technique that makes use of both IRM and RHBM to produce the speed response under MFU excitation (from all three rotors), with a realistic speed relation between the rotors.

The latter technique is necessary since the RHBM solution-set over the speed range (the “speed response”) generated for the MFU problem in the previous study [58] assumed a constant ratio between the rotor speeds. The fundamental frequency of the RHBM solution is \( \Omega_{\text{ref}} / Q \) rad/s where \( Q \) is a positive integer and \( \Omega_{\text{ref}} \) is speed of the slowest unbalanced shaft. Hence, a constant rotor speed ratio means that \( Q \) stays constant along the speed response curve. In practice, the ratio of the speeds of the rotors will vary as the speed of one of the rotors changes. This variation is governed by the engine speed characteristic. Under these conditions \( Q \) will vary as the speed changes. The approach used here is to approximate the engine speed characteristic by one in which the speed ratios are ratios of low integers, thereby minimising \( Q \). This enables the use of RHBM to finish off (to steady-state) time-transient solutions obtained through IRM.

The work in this chapter has been accepted for publication in [59]. Section 6.2 describes the whole-engine model. The results of the simulations are presented and discussed in section 6.3.
6.2 DESCRIPTION OF WHOLE-ENGINE MODEL

Figure 6.1 portrays the schematic layout of the representative three-spool aero-engine considered in this research. Its finite element (FE) model was devised by a leading engine manufacturer [53]. The whole structure is an assembly which incorporates three rotors supported by a casing via six SFD bearings and a location bearing. The SFD bearings each consist of a rolling-element bearing whose outer race is the non-rotating journal of an oil annulus. The journal of the ‘HP front’ SFD bearing is supported on the casing via a squirrel cage spring, which is therefore in parallel with the oil annulus. The retainer spring is used merely to provide axial location to the HP rotor. The rest of the SFD bearings are unsupported. The location bearing locates the IP rotor in the casing. There is also an inter-shaft bearing, which has no SFD, that fixes the LP rotor to the IP rotor axially.

A full description of the workings of the IRM and RHBM solvers was given in Chapters 3 and 5 respectively. The complete nonlinear rotordynamic assembly is regarded as a non-rotating linear part acted on by “external” forces. By “linear part” is meant the structure left after all SFDs in the schematic of Figure 6.1 are replaced by ‘gaps’.

The IRM and RHBM require a preliminary one-off eigenvalue analysis of the undamped linear part at zero rotational speed (Figure 1.4). The modal displacements (eigenvectors) were then extracted at the degrees of freedom of interest (e.g. those at the SFD positions) and used as input data for the IRM and RHBM solvers to calculate the unbalance response of the complete nonlinear rotating system.

In linear part, the inter-shaft bearing, location bearing and HP front bearing allow full pivoting motion (in the xy and yz planes) of the LP, IP and HP shafts respectively. Hence, apart from the flexible modes, there were six rigid-body modes found. These rigid-body modes represented the pivoting motions of the three rotors in the xz and yz planes, about the inter-shaft, location and HP front bearings. The rigid-body modes that describe the spinning motions of the rotors do not enter in the nonlinear calculations and so were
removed. This is necessary to avoid confusion in the RHBM solver which separates the rigid body modes from the flexible body modes as part of its calculation of the zero\textsuperscript{th} harmonic of the vibration (Section 5.2.3). Apart from being a necessary part of the calculation, the zero\textsuperscript{th} harmonic defines the mean operating position of the SFD journals in their housings (i.e. the degree of lift) and is therefore a performance indicator for unsupported SFDs. For each harmonic of the response, the RHBM generates a sub-set of equations that use receptances – frequency response functions that relate displacement with applied force at a given frequency. However, in the presence of rigid body modes, the receptances are undefined at the zero\textsuperscript{th} harmonic (Section 5.2.3). The RHBM solver developed in Chapter 5 overcomes this problem by considering that the zero\textsuperscript{th} harmonics of the SFD forces are in equilibrium with the distributed weight of the rotors. For the present case of the three-spool engine (as for the twin-spool engine considered previously) this equilibrium problem is statically indeterminate, as is evident from the interconnected LP and IP shafts in Figure 6.1.

6.2.1 Linear Computation

The modal displacements of the linear part at the relevant degrees of freedom were supplied by the manufacturer for all the modes in the range 0-500 Hz. That frequency range was considered sufficient to cover the engine speed range considered in the nonlinear analysis. Figure 6.2 shows a receptance at a point on the linear part, calculated from the results of the eigenvalue analysis as per the receptance expansion formula in Section 5.2.2.

As in the case of the twin-spool engine of the previous chapters, it is evident that there is a high modal density at relatively low frequencies. With conventional nonlinear solvers, the analyst would find it very difficult to decide which modes to omit to avoid prohibitively long computation times. On the other hand, IRM and RHBM allow the
inclusion of all the 2014 modes covering the frequency range 0-500 Hz while retaining manageable computation times.

6.2.2 Non-linear computation for unbalance response

The forces external to the linear part comprise the rotor unbalance excitations, rotor weights, gyroscopic moments and SFD forces. These are indicated in Figure 6.3. The distributed weight of each rotor was taken into account by discretising it at a selected number of stations along its axis. The same approach was used to discretise the distributed polar moment of inertia of the rotors for the calculation of gyroscopic moments in nonlinear analyses. Figure 6.3 show the weight and polar moment of inertia distributions of the three rotors.

Incorporated in this engine design are SFDs with radial clearances \( c \) typically of 0.1mm, bore diameters \( D \) ranging from 170mm to 340mm and land lengths \( L \) from 16mm to 52mm. The SFDs were either end-fed or centre-fed with oil viscosity \( \eta \) of 0.0049Nsm\(^{-2}\) at a pressure of 3 bar (gauge). The calculation of SFD forces in this study was based on the “\( \lambda \)-theory” model in which the “end-sealing factor” \( \lambda \) was chosen to represent the assumed degree of sealing at the ends of the SFD. A cavitation assumption was applied to the calculation of the oil film pressure distribution, with the oil cavitation pressure \( p_{cav} \) set at \(-101.325 \) kPa. Appendix A1 gives a full description of the calculation of the SFD forces.

With such an engine it is common practice to use a unidirectional bump-spring at one or both of the SFD bearings that support the heavy LP rotor. This is a cost-effective approach of enhancing the bearing performance by increasing the lift without introducing the installation and maintenance complications of a retainer spring (which is only used on the HP front bearing as discussed previously). Figure 6.4 illustrates the author’s idea of
how a bump spring working in tandem with a SFD bearing to support the rotor in the $y$ direction can be implemented.

Figure 6.4a defines the status of “zero preloading”. In words, “zero preloading” means that, when the journal is centred in its housing, the bump-spring just about makes contact with it, i.e. it is at its natural (undeformed) length. With the journal held centred, adjusting the position of the ‘fastening screw’ results in different levels of preloading, which is defined as the upward or downward displacement $\delta$ of the screw relative to the position of zero preloading. The bump-spring, unlike the retainer spring, is not part of the linear part. Hence, its presence in the complete nonlinear assembly is accounted for by modifying the vertical component of the respective SFD force as follows:

$$Q'_y = \begin{cases} Q_y + k_{bump} \delta - y_{rel} & \text{if } y_{rel} < \delta \\ Q_y & \text{if } y_{rel} \geq \delta \end{cases}$$  \hspace{1cm} (6.1a) \hspace{1cm} (6.1b)

where $Q'_y$ is total force in the $y$ direction, $Q_y$ is the SFD $y$ force given by eq. (A1), $k_{bump}$ the stiffness of the bump spring and $y_{rel}$ is the $y$ displacement of the journal relative to the housing.

The vibration of the system is ultimately excited by the rotor unbalances. For the study of the MFU problem, the unbalance was assumed to be present on all three rotors at chosen positions diagrammed in Figure 6.3. The unbalance mass-radius products are given in the Table 6.1.

<table>
<thead>
<tr>
<th>Rotor</th>
<th>$U_{j,1}$ (gmm)</th>
<th>$U_{j,2}$ (gmm)</th>
<th>Phase shift (°)</th>
<th>'Co-phased'</th>
<th>'Anti-Phased'</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP ($j=1$)</td>
<td>20000</td>
<td>20000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IP ($j=2$)</td>
<td>5000</td>
<td>5000</td>
<td>0</td>
<td>180</td>
<td></td>
</tr>
<tr>
<td>HP ($j=3$)</td>
<td>5000</td>
<td>5000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 6.1**: Unbalances (see Figure 6.3)
In practical operation, the speeds of the rotors are controlled to conforming predetermined relations, when sped up or slowed down. In this research work the speed responses were calculated taking realistic rotor-speed relations into consideration. Figure 6.5 graphs the speed relation used. As can be seen from this plot, the ratio of speeds of any two of the three rotors varies over the speed range. For convenience of expression, in the following sections the 'speed point number' will be used to refer to the combination of the rotational speeds of the three rotors at each point of time in the run-up simulation, as shown in Figure 6.5.

6.3 SIMULATION RESULTS

With the scope of this work focused on the response of the SFD bearings to MFU excitation, the results presented show the orbits of the SFD journals relative to the respective housings. The bearing housings were assumed to be perfectly aligned with each other. IRM and/or RHBM were used to calculate the response. The speed range considered was roughly covered by the 17 points indicated in Figure 6.5. At each of these points, the steady-state solution in time-domain was calculated using the time-domain solver IRM. All the resultant frequency spectra were calculated using a Matlab® function $\text{fft}$ for Fast Fourier Transform with frequency resolution of 0.5Hz, sampling rate of 2048Hz, and Hanning window [60] applied to snapshots of data.

6.3.1 Speed Response through Time-domain Solution

The vibration of SFD-rotor systems depends significantly on the regime in which each SFD in the system operates. Most practical designs of SFD assemblies use sealing devices like end-plates or piston-rings in order to apply a certain level of end-sealing to the oil film. The purpose of using a sealed oil-film is to generate higher and desirable amounts of damping for the whole structure. The end-sealing level depends not only on the end
clearance but also on the viscosity of the lubricating oil [44]. Therefore, the end-sealing factor at each SFD is one of the parameters that need to be investigated. In practice, slight sealing or unsealed models appear to be preferred and attract greater interest from manufacturers since these types of sealing are easier to achieve for various technical reasons, such as simple format of seals and less tight machining tolerance. In this particular engine, the SFD 'HP front' assembly is set up with a tight sealing device, namely piston rings. The remaining bearings are either unsealed or slightly sealed. For that reason, the following analyses will consider two cases: (i) “unsealed”; (ii) “slightly sealed” (Table 6.2). In both cases the HP front bearing is tightly sealed. The factor $\hat{B}$ in Table 6.2 quantifies the strength of the dynamic pressure term of the short bearing pressure expression (eq. (A1.5)), which is the dominant component of the combined pressure distribution (eq. (A1.4)).

<table>
<thead>
<tr>
<th>SFD</th>
<th>End-Sealing Factor $\lambda$</th>
<th>$\hat{B} = \eta \left( \frac{L}{c} \right)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>'Unsealed'</td>
<td>'Slightly Sealed'</td>
</tr>
<tr>
<td>LP front</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
<td>IP front</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
<td>HP front</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>HP rear</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
<td>IP rear</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
<td>LP rear</td>
<td>0.00</td>
<td>0.01</td>
</tr>
</tbody>
</table>

**Table 6.2:** SFD parameters

6.3.1.1 ‘Unsealed’ Case, no bump-springs, ‘Co-phased’ unbalances

Figures 6.6-6.8 show the response of the system to the excitation of 'co-phased' unbalances (Table 6.1) for the unsealed case (Table 6.2), with no bump-springs used. As can be seen in Figure 6.6, the response at all SFDs contains three frequency components that are synchronous with the rotational speeds of the rotors, namely 1LP, 1IP and 1HP. Frequency components which are the combinations between the synchronous frequencies
are also prominent in the graph for SFDs 'HP rear' and 'IP rear'. This phenomenon is indicative of the nonlinear characteristics of the SFD models used in the analyses.

Figure 6.7 shows the variation of the mean position of the journals within the corresponding radial clearances over the speed range. The level of lift at SFD ‘HP front’ is high and independent of the rotor speeds, since it is supported by a parallel retainer spring. For an unsupported SFD, a “healthy” (moderate) level of vibration of its journal relative to the housing is essential to generate a reasonable journal lift to counter the static loading. It also reduces transmission of dynamic forces to other parts of the engine via the housing. Negligible relative vibration results in a bottomed-out journal that vibrates as one with the housing (SFD “locked”) [21]. Figure 6.7 indicates that the LP’s SFDs are mainly bottomed-out.

Figure 6.8 shows the variation of the amplitudes of the synchronous components over the speed range. At SFDs that support LP rotor, the 1LP component is dominant. The 1IP and 1HP components dominate the IP and HP SFDs respectively in the low-speed regime. However, in the high-speed regime, the 1LP component surpasses the others at the 'IP rear' and 'HP rear'. This behaviour was also observed in the other cases to be presented later. The cause of such behaviour is explained by considering the two points below.

- Firstly, for the case studied, the unbalance level attached to the LP rotor was 4 times greater than those attached to the HP and IP rotors (Table 6.1), whereas the speed ratios change from about 13 to 3 between HP and LP, and from almost 9 to 2 between IP and LP, as shown in Figure 6.5. As a result, there is a drastic change in the relative magnitudes of the unbalance forces from the low-speed regime to the high-speed regime. For instance, at speed point number 1, the LP-unbalance excitation is 45 and 20 times weaker than the IP- and HP-unbalance excitations, respectively. However, at speed point number 17, the respective differences are reduced to only 2.2 and 1.1 times. Thus, this partly accounts for the strength of the 1LP component, compared to other components, at the high-speed end of the response.
• As previously mentioned, the SFD coefficient $\hat{B}$, given in Table 6.2, indicates how largely the dynamical term of the pressure distribution changes with respect to the movement of the journal relative to the housing. For an unsupported SFD under high static and/or dynamic loading, a large $\hat{B}$ could result in a “locked” damper that generates large forces akin to a rigid connection. The behaviour of the LP rotor and its supporting SFDs appears to be similar to what observed in [21], a research work on a system with a rotor running in an unsupported SFD. The rotor was highly flexible relative to the SFD’s pedestal and the journal was mostly bottomed-out under the static loading over the regime of operation, behaving like a pinned connection. In the present case the LP rotor has a lower dynamic stiffness than the other rotors, due to the concentration the mass at two ends and the long, slim shaft stem. Moreover, the high weight of the LP rotor leads to a high static loading on its SFDs, evident from the low levels of lift at these SFDs. The vibration caused by the LP-unbalance excitation tends to be transmitted to other SFDs, through the casing and intershaft connection, rather than to reside as a relative vibration in the annular clearances of the ‘LP front’ and ‘LP rear’ bearings.

6.3.1.2 ‘Slightly sealed’ (no bump-springs) versus ‘Unsealed’ (no bump-springs)

Figures 6.7 and 6.9 illustrate the response of the system to the case of ‘slightly sealed’ SFDs (Table 6.2) and ‘co-phased’ unbalances (Table 6.1). Although the overall features of the response look similar to the ‘unsealed’ case, it can be easily noticed that the lifts are lower (Figure 6.7) and so are the amplitudes (comparing Figure 6.9 with Figure 6.8).

6.3.1.3 ‘Co-phased’ versus ‘Anti-phased’ unbalances (no bump springs)

There are clear differences in the response of the system in the case of ‘anti-phased’ unbalances (Table 6.1) in comparison with that in the ‘co-phased’ unbalances case.
can be seen in Figure 6.10, the vibration amplitude at SFD ‘HP front’ is increased considerably, while it is largely reduced at SFD ‘HP rear’, resulting in poor lift at this bearing. It should be noted that, although ‘HP front’ is tightly-sealed and has a higher value of the coefficient $\hat{B}$, it has a retainer spring that provides a good static lift, thereby preventing locking of the SFD. Furthermore, in the anti-phased positions, the unbalances, which have equal magnitudes, seem to act like a moment. Such effect becomes particularly sharp with the HP rotor being a rather rigid structure. The axial positions of the HP unbalances probably would then have minor effect on the response at ‘HP front’ and ‘HP rear’. In the previous ‘co-phased’ case (Figures 6.8, 6.9), the fact that the HP-rotor unbalances were mounted closer to ‘HP rear’ can be seen as the main reason for the more equal levels of the 1HP responses at the two HP-rotor SFDs.

6.3.1.4 Use of bump-springs on LP rotor’s SFD bearings

From all the cases presented in the previous sections, it can be noticed from Figure 6.7 that there is poor lift at the SFDs supporting the LP rotor. The reasons for this were mentioned above, namely the heavy weights concentrated at the two ends of a long flexible rotor and the features of the connected SFDs. This section is devoted to an investigation on the effect that bump-spring supports can bring to the response of the system. For that purpose, two linear bump-springs having stiffness coefficients $k_{LP\text{front}}$ and $k_{LP\text{rear}}$ of 3.5MNm$^{-1}$ and 1.75MNm$^{-1}$ were added to the model, supporting SFDs ‘LP front’ and ‘LP rear’ respectively in $y$ direction. Two cases of preloading, as described in section 6.2.2, were considered:

i) ‘Preloading 1’: $\delta_{LP\text{front}} = 0$ and $\delta_{LP\text{rear}} = 20c_{LP\text{rear}}$

ii) ‘Preloading 2’: $\delta_{LP\text{front}} = 10c_{LP\text{front}}$ and $\delta_{LP\text{rear}} = 20c_{LP\text{rear}}$
where $c_{\text{LPfront}}$, $c_{\text{LPrear}}$ are radial clearances of SFDs ‘LP front’ and ‘LP rear’ respectively and $\delta_{\text{LPfront}}$, $\delta_{\text{LPrear}}$ the preloading of their bump-springs.

Figures 6.11-13 show the response of the system at ‘preloading 1’ and ‘preloading 2’, both cases for the ‘slightly sealed’ SFD case (Table 6.2) and ‘co-phased’ unbalances (Table 6.1). With ‘preloading 1’, both the 1LP amplitude and the lift at ‘LP rear’ were clearly improved even in the low-speed regime, while the response showed little change at ‘LP front’ (comparing Fig. 6.11 with Fig. 6.7 and Fig. 6.12 with Fig. 6.9). It seems the preloading level of zero was not strong enough to provide essential static lift to the journal of the SFD ‘LP Front’. When the preloading of the spring at ‘LP front’ was increased to $\delta_{\text{LPfront}} = 10c_{\text{LPfront}}$, the lift and the displacement amplitudes were reasonably improved (comparing Fig. 6.11 with Fig. 6.7 and Fig. 6.13 with Fig. 6.9).

Comparing the bump-spring-supported cases (Figures 6.11-13) with the case that includes no bump-spring support (Figure 6.7-10), a trend that can be seen is that when the lifts and amplitudes at ‘LP front’ and ‘LP rear’ improved, these bearings became more receptive to the 1IP and 1HP. This observation agrees with the expectation that a proper lift level can “loosen” a locked SFD. This means that, by promoting the relative journal motion within the oil annuli, the bump-springs improved the dissipation of vibration energy of the whole system.

6.3.1.5 Note on computation time

It should be noted that all simulations were performed in Matlab® on a standard 2006-issue desktop pc with Intel® Pentium® D CPU 3GHz processor. With all 2014 modes included in the analysis, the computation time consumed to complete 1s length of data by IRM was from about 5 to 30 minutes, generally depending on the lift levels of the orbits at SFDs in the system. The solution required a few thousand shaft revolutions to reach
steady-state from initial conditions corresponding to zero relative displacements and velocities at the SFD journals.

6.3.2 Application of RHBM – non-constant speed ratio

This section is focused on the application of RHBM to accelerate the calculation of the system response over the speed range.

In the method proposed in this section, for each point of the speed response, IRM is used only to generate an unfinished (i.e. transient) solution. This solution is then approximated by a Fourier series containing $K$ harmonics with a fundamental frequency $\sigma$ to feed RHBM scheme that delivers the steady-state periodic solution. With the speed ratios changing over the speed range as shown in Figure 6.5, it is inevitable that both the number of harmonics and the ratio of the fundamental frequency to the rotor speeds will vary along the speed range. As can be seen in the analysis presented in the previous sections, various combinations of the excitation frequencies were found in the response. Hence, the fundamental frequency $\sigma$ is the lowest one among all the possible combinations [29], and normally is the common-integer dividend of the excitation frequencies or even the integer-fractions of those frequencies. For example, let the ratio of two excitation frequencies $\Omega_{(1)}$ and $\Omega_{(2)}$ be represented by the ratio of two whole numbers $p$ and $q$ ($p > q$), then the fundamental frequency $\sigma$ of the response of a nonlinear system excited by $\Omega_{(1)}$ and $\Omega_{(2)}$ is normally equal to $\Omega_{(2)}/q$. It is clear that if $q$ is a large number, then the fundamental frequency will be small and a large number of harmonics needs to be included in the RHBM solution. The consequence of this is that the size of the system of harmonic balance equations to be solved can be so large that it degrades the relative advantage of harmonic balance method. For that reason, in the proposed approach, the speed relation was replaced by an approximated one in which the fundamental frequency at each point in the speed range was as high as possible.
For the system under consideration, the approximation of the speed relation was carried out using the following procedure:

- Let

\[
\alpha = \frac{\Omega_{HP}}{\Omega_{LP}}, \quad \beta = \frac{\Omega_{HP}}{\Omega_{LP}}
\]  

(6.2)

- Find whole numbers \( p, q, r \) and \( s \) with \( q \) and \( s \) smallest possible so that

\[
|\alpha - p/q| \leq \zeta \alpha, \quad |\beta - r/s| \leq \zeta
\]

(6.3)

in which \( \zeta \) was a prescribed tolerance.

- Find the least common multiple number \( Q \) of \( q \) and \( s \)

- Establish the fundamental frequency using the formula

\[
\sigma = \frac{\Omega_{LP}}{Q}
\]

(6.4)

Figure 6.14 show the approximated rotor speeds obtained using the procedure described above with tolerance \( \zeta = 0.05 \). The effect of using these new approximated speeds instead of the previous exact ones was investigated by redoing the IRM analysis for the ‘co-phased’ unbalances, ‘slightly-sealed’ SFDs with bump-springs at ‘preloading 2’ at these speeds. By comparing the results in Figure 6.15 (for the approximated speeds) with those in Figure 6.13 (for the exact speeds), it can be seen that the change resulting from the approximation of the rotor speeds at such a high level of tolerance was minor.

Figure 6.16 compares the RHBM solutions obtained at the approximated speeds with the IRM solutions at the same speeds. In each figure, a steady-state SFD orbit covering 1s length of data generated using IRM (gray – solid line) is superimposed upon by the corresponding RHBM solution (red – dashed line) obtained using \( Q \) and \( K \) given in Table 6.3. The number of unknowns solved for by the RHBM at each speed point was
\(2N(2K+1)+P\) (Chapter 5 and [59]) where \(N\) is the number of SFDs, \(K\) is the order of the highest harmonic and \(P\) the total number of rigid body modes of the assembly (section 6.2).

The excellent correlation between these two vastly different methods lends a definitive validation to the computational soundness of both methods when applied to a three-spool engine. Moreover, the computation time for RHBM to deliver a steady-state solution from an unfinished (transient) IRM solution as its approximation was merely a matter of a few minutes.

<table>
<thead>
<tr>
<th>Figure</th>
<th>Speed point number</th>
<th>(Q)</th>
<th>(K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15a</td>
<td>2</td>
<td>1</td>
<td>38</td>
</tr>
<tr>
<td>15b</td>
<td>4</td>
<td>1</td>
<td>32</td>
</tr>
<tr>
<td>15c</td>
<td>9</td>
<td>1</td>
<td>23</td>
</tr>
<tr>
<td>15d</td>
<td>10</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>15e</td>
<td>11</td>
<td>3</td>
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</tr>
<tr>
<td>15f</td>
<td>13</td>
<td>2</td>
<td>25</td>
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<td>15g</td>
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</tr>
<tr>
<td>15h</td>
<td>17</td>
<td>2</td>
<td>22</td>
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</tbody>
</table>

Table 6.3: RHBM solution parameters

6.4 CONCLUSIONS

In this chapter the IRM and RHBM were applied to the analysis of the vibration of a realistically-sized representative three-shaft whole aero-engine model subject to unbalance excitation from all three rotors (MFU excitation).

A first-ever parametric analysis of such a system was carried out that studied the effect of SFD end-sealing and the use of simple unidirectional “bump-springs” at certain SFDs on the overall dynamic performance. A mathematical model describing how such
springs can be implemented was presented. The results show that the use of suitably preloaded vertically oriented bump-springs at the SFDs of the heavy LP rotor produced a significant improvement in journal lift, allowing for smoother running of the shaft. The results obtained also give a better understanding of the interaction between the SFDs and rotors in an aero-engine assembly.

A technique was proposed that makes use of both IRM and RHBM to efficiently produce the steady-state system response under MFU excitation over a range of speeds accounting for a realistic speed relation between the rotors. The proposed approach approximated the engine speed characteristic by one in which the speed ratios were ratios of low integers. This enabled the use of RHBM to finish off (to steady-state) transient solutions obtained by IRM in a matter of minutes. As for the twin-spool engine in [57, 59], this first-ever application of a frequency-domain solver to a three-spool engine was computationally validated by excellent correlation with the time-domain solver. The proposed combined use of these two complementary solvers represents a step-change in the analysis of realistic whole-engine models.
Figure 6.1: Schematic of a representative three-spool engine

Figure 6.2: Point receptance frequency response at centre of the housing of the SFD ‘LP front’ in y direction
Figure 6.3: Axial distributions of weight (—) and polar moment of inertia (——); locations of SFD bearings and unbalances $U_{j,1}, U_{j,2}$ on rotor no. $j$
Figure 6.4: Bump-spring support with adjustable preloading

Figure 6.5: Variation of rotor speeds and speed ratios
Figure 6.6: Frequency spectra of relative velocity in y direction with magnitudes max-normalised.
'Co-phased', 'Unsealed', No bump-springs
Figure 6.7: Lifts of journal centres, with magnitudes normalised to the corresponding radial clearances. ‘No bump-springs’

- Red: ‘Co-Phased’, ‘Unsealed’
- Blue: ‘Co-Phased’, ‘Slightly sealed’
- Black: ‘Anti-Phased’, ‘Slightly sealed’
Figure 6.8: Half peak-to-peak displacement amplitudes taken at synchronous frequencies in y direction, with magnitudes normalised to the corresponding radial clearances.

'Co-phased', 'Unsealed', No bump-springs

- LPfront
- LPrear
- IPfront
- IPrear
- HPfront
- HPrear
Figure 6.9: Half peak-to-peak displacement amplitudes taken at synchronous frequencies in y direction, with magnitudes normalised to the corresponding radial clearances.

'Co-phased', 'Slightly sealed', No bump-springs
Figure 6.10: Half peak-to-peak displacement amplitudes taken at synchronous frequencies in \( y \) direction, with magnitudes normalised to the corresponding radial clearances.

'Anti-phased', 'Slightly sealed', No bump-springs

<table>
<thead>
<tr>
<th>1LP</th>
<th>1IP</th>
<th>1HP</th>
</tr>
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<tbody>
<tr>
<td>1LP</td>
<td>1IP</td>
<td>1HP</td>
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</table>
Figure 6.11: Lifts of journal centres, with magnitudes normalised to the corresponding radial clearances. ‘Co-Phased’, ‘Slightly sealed’

- Red square: ‘Preloading 1’
- Blue triangle: ‘Preloading 2’
Figure 6.12: Half peak-to-peak displacement amplitudes taken at synchronous frequencies in $y$ direction, with magnitudes normalised to the corresponding radial clearances.

‘Co-phased’, ‘Slightly sealed’, ‘Preloading 1’
Figure 6.13: Half peak-to-peak displacement amplitudes taken at synchronous frequencies in y direction, with magnitudes normalised to the corresponding radial clearances.

‘Co-phased’, ‘Slightly sealed’, ‘Preloading 2’
Figure 6.14: Approximated speeds with 5% rounded-off
**Figure 6.15**: Half peak-to-peak displacement amplitudes taken at synchronous frequencies in y direction, with magnitudes normalised to the corresponding radial clearances. ‘Co-phased’, ‘Slightly sealed’, ‘Preloading 2’, ‘Approximated’ speeds.
Figure 6.16: Comparison between RHBM (-----) and IRM (——) orbits of SFD ‘LP rear’, with the approximated rotor speeds
7 PARAMETRIC ANALYSIS OF THE UNBALANCE RESPONSE OF A REAL TWIN-SPOOL AERO-ENGINE

7.1 INTRODUCTION

Attempts to directly relate measured vibration data from squeeze-film damped systems with predictions from computational analysis have so far been restricted to simple test rigs. The reasons for this are two-fold:

a) Experimental data from real engines under operational conditions are very hard to obtain due to the financial cost of the tests and manufacturers’ confidentiality concerns;

b) The size of the computational (Finite Element/modal) model of a real engine renders most conventional nonlinear time/frequency domain solvers unsuitable for the determination of the unbalance response.

The main contribution of this thesis has been to overcome the limitation in (b) through the development of the novel Impulsive Receptance Method (IRM) and the Receptance Harmonic Balance Method (RHBM). In Chapters 3-6 these methods have been successfully tested on realistically-sized representative whole-engine models of two/three-shaft engines under conditions of single/multi-frequency excitation. They have been shown to be computationally sound and far more efficient than conventional methods.

Chapters 3-5 presented preliminary computational studies on a twin-spool engine and the previous chapter presented a preliminary computational parametric analysis of a three-shaft engine. The present chapter presents the final part of the research, in which a first-ever attempt is made to relate measured vibration data from a manufacturer’s test engine with the predictions from a whole engine-model of the same engine using the IRM. It
should be understood that the data provided is limited, restricted to the engine casing, and that the author was not involved in performing the tests. Moreover, the accuracy of the predictions is bound to be limited by the potential influence of unaccounted factors such as residual unbalance and aeroelastic phenomena. Hence, the aim of the comparison is merely restricted to examining the order of magnitude of the vibration levels and general trends in the data.

The first part of the theoretical analysis uses the IRM of the previous chapters, which is based on the assumption of negligible damping in the linear part of the structure (i.e. SFDs constitute the only damping forces). The effect of bearing housing misalignment on the predicted vibration levels is investigated. Unsatisfactory correlation of the predicted casing vibration levels with the measured ones motivated the second part of the theoretical analysis, in which the IRM is developed to include damping distributed within the linear part of the structure. This chapter therefore significantly enhances the applicability of this effective solver. Moreover, previous research into squeeze-film damped turbomachinery has tended to neglect damping in the structure e.g. [10]. Hence, this investigation into the influence of distributed damping should be of prime interest to SFD analysts.

Section 7.2 describes the test engine. Section 7.3 presents and discusses a parametric analysis that neglects distributed damping. Section 7.4 develops an IRM that accounts for proportional distributed damping in the structure and presents its results.

7.2 DESCRIPTION OF TEST ENGINE

This section describes the test engine used in this parametric study. The engine is a twin-spool engine of the same series of engine-models as that considered in Chapters 3-5. Its structural data and measured vibration data were provided by a leading aero-engine manufacturer [53].
7.2.1 Overview

The schematic of the test engine is depicted in Figure 7.1. The structure is an assembly which incorporates two rotors supported by a casing via five SFD bearings. The casing itself is mounted on the engine test bed. The SFD bearings each consist of a rolling-element bearing whose outer race is the non-rotating journal of an oil annulus. The journals of the ‘LP rear’ and ‘HP front’ SFD bearings are supported on the casing via squirrel-cage springs, which are therefore in parallel with the respective oil annuli. The retainer springs are used merely to provide axial location to the rotors. The rest of the SFD bearings are unsupported.

The SFDs have bearing bore diameters $D$ ranging from 161mm to 250mm, radial clearances $c$ from 0.05mm to 0.13mm, land lengths $L$ from 16mm to 33mm. All SFDs are end-fed with lubricating oil. The viscosity of the oil is estimated at $7 \times 10^{-3}$Nsm$^{-2}$ for all SFDs located in the front bearing chamber, i.e. SFDs ‘LP front’, ‘LP mid’ and ‘HP front’, and of $2.5 \times 10^{-3}$Nsm$^{-2}$ for SFDs ‘HP rear’ and ‘LP rear’ which reside in the rear bearing chamber. The pressure of the oil at the inlets of all SFD bearings was sustained at 3atm (303.975kPa, gauge), with zero pressure at the outlet.

The unbalance-response test investigated in this chapter is a case of single-frequency excitation, in which one unbalance with the mass-radius product of 4100gmm was mounted onto the HP rotor. The response of the engine assembly was captured by accelerometers at three different locations on the casing, as illustrated in Figure 7.2. These sensors $A_i$ ($i=1, 2, 3$) were arranged at different radial positions around the casing centreline. The test was carried out for a typical engine run-up condition. Figure 7.3 shows the variation of rotational speed of the two rotors in the test. The whole set of readings covered an area of the speed range in which the speed of the HP rotor was increased from about 60% to 100% of the nominal speed NH of 16000rpm, and was accomplished in a period of 200s approximately.
7.2.2 Structural Model

As discussed in the previous chapters, the IRM and RHBM solvers require the modal data of the undamped non-rotating linear part of the structure. To retain confidentiality, this eigenvalue analysis was performed by the manufacturer [53] and its results provided to the author. The frequency range considered was 0-300 Hz and 921 modes were found. A higher frequency range, as used in Chapters 3-5, would have been preferable, but this is what was provided by the manufacturer. Figure 7.4 shows a point receptance frequency response function (FRF) at a particular location of the casing, calculated from the modal data using the FRF modal expansion formulae of eq. (3.13). The characteristic high modal density of the assembly at relatively low frequencies observed in the previous chapters is once again evident. As discussed, this feature makes conventional nonlinear time/frequency domain solvers unsuitable since it is difficult to judge which modes to retain. As shown in Chapters 3 and 5, the IRM and RHBM circumvent this problem since they allow the inclusion of the full number of modes without compromising the computational efficiency. Since, in the “linear part”, each of the two rotors is in pinned-free connection with the casing, the non-linear computation of unbalance response of the whole system has to take into account the rotor weights. As in Chapters 3-6, the weight of each rotor was discretised at a selected number of points along the rotor axis. Figure 7.5a gives an indicative picture of the axial weight distributions of the two rotors. Figure 7.5a also shows the location of the single applied unbalance on the HP rotor. For the inclusion of the gyroscopic effect, the polar moments of inertia of the rotors were discretised at the same set of rotor stations, as indicated in Figure 7.4b.

7.3 PARAMETRIC ANALYSIS – UNDAMPED LINEAR PART

The parametric analysis was performed for the above-described case of a single unbalance applied to the HP rotor. The analysis assumed that there was no residual unbalance on the
HP rotor and that the LP rotor was perfectly balanced. The solver used was the IRM developed in Chapter 3, which neglects distributed damping in the linear part of the structure.

The non-linear computation of the unbalance response in this chapter applied the "\( \lambda \) - theory" model to the calculation of the SFD forces. Without knowledge of the actual end-sealing settings at the each SFD of the test engine, the effect of the end-sealing on the performance of the whole system was investigated by considering two cases of extreme conditions: i) 'Tightly sealed': \( \lambda = 0.03 \), for all SFDs; ii) 'Unsealed': \( \lambda = 0 \), for all SFDs. In both cases, the pressure distribution in the oil film was truncated where the pressure was lower than a cavitation threshold set at the absolute zero pressure (-101.325kPa), as discussed in Section 2.4 and detailed in Appendix A1.

The speed responses of the system for both sealing conditions were calculated at 50 speed points over the rotor-speed range considered (Figure 7.3). For each point, the IRM was used to generate an unfinished transient solution which then was used to feed RHBM solver to rapidly reach the steady state, as described in Chapter 5. Once the steady state solution was obtained, IRM was used to generate a data length of 2s for the further analysis, where the modal coordinates were the by-product generated using eqs. (3.23a, b).

The physical velocities in both \( x \) (horizontal) and \( y \) (vertical) directions at the locations of the three accelerometers \( A_i \) (\( i = 1, 2, 3 \)) were then extracted according to eq. (3.4b). The radial motion corresponding to the direction in which each accelerometer was installed (see Fig. 7.4) was then calculated combining the \( x \) and \( y \) components.

7.3.1 Aligned Bearing Housings

Figures 7.6-7.11 shows the predicted and measured variations of the 1HP component of the velocity at the three accelerometers. The predictions considered in this sub-section are for
the case of all bearing housings perfectly aligned (i.e. \( v_{s,i} = 0, i = 1...5 \) in eq. (3.32)).

Figures 7.6, 7.8, 7.10 also include predictions for misaligned housings, but these will be discussed in Section 7.3.2. It can be noticed that there are rough correlations between the predicted and measured curves. For example in Figure 7.6, a peak is predicted at 65%NH for both cases of end-sealing and a peak is also evident in the measurement in the same speed range (Fig. 7.7). In the same sense, the predicted responses at A2 (Fig. 7.8) exhibit fairly good agreement with the measured ones (Fig. 7.9) in terms of the locations of peaks, i.e. at 70%NH, at 80-90%NH, and at slightly lower than 100%NH, where NH is the nominal rotational speed of the HP rotor. As for the terminal A3, (Figures 7.10, 7.11) apart from the peak at 70%NH, the spiky area from 75 to 90%NH of the calculated results can be roughly matched by those of the corresponding measured data.

Regarding the levels of vibration, the predicted results over-estimate the response of the system, especially around the peak locations. Of the two cases of end-sealing, the ‘Tightly Sealed’ case resulted in far higher amplitude of vibration than what was obtained from the measured data, while with the ‘Unsealed’ settings to the SFDs, the responses are clearly milder. These features of the responses can be explained by the fact that, with increased end-sealing, SFDs become more rigid resulting in larger dynamic forces being transmitted to the casing. The issue of unsatisfactory correlation at the response peaks is attributed to the absence of damping in the linear part of the structure and this will be addressed in Section 7.4.

Figs. 7.12-7.14 confirm the presence of the super-synchronous components in the response of the system under the HP-unbalance excitation. These figures compare the predicted and measured frequency spectra of the response at the monitored terminals A\(_i\) (\( i = 1,2,3 \)). The spectra are presented in “z-modulation” form, in which the relative strength of the frequency components is indicated by the intensity of the shading (i.e. the darker the shading the stronger the frequency component). It can also be noticed from
these figures that the frequency compositions of the predicted responses are in good overall agreement with the measured ones. However, the measurements show unpredicted frequency components that are non-integer multiples of the HP rotor speed (e.g. the component indicated by an arrow in the z-mod for A₁ (Fig. 7.12), that also appears in the z-mod for A₃ (Fig. 7.14) but not in that for A₂ (Fig. 7.13)). One should bear in mind factors that are unaccounted for in the model (like aeroelastic phenomena). Also, the whole speed range was covered in just 200s in the test, whereas the calculations assume full steady-state conditions at each speed point. More importantly, one has to note that the model is not capable of predicting strong high frequency response like that indicated by the arrows in Figures 7.12, 7.14 since the manufacturer only provided modal parameters up to 300Hz.

A closer look at the contribution of all frequency components is given in Figs. 7.15-7.17. Figure 7.15a picks out the frequency spectrum of the response at A₁, for one single speed point where the HP rotor speed reached 80%NH. Figure 7.15b shows the computed waterfall diagram of the frequency spectra for the full speed range. In both figures, the multi-HP components appear to be dominated by the 1HP. The dominance of the 1HP component is also clearly indicated in Figs. 7.16a and 7.16b, with the best agreement between the predicted response and the measured counterpart shown in Figures. 7.17a and 7.17b, where the higher order components are negligible. It is observed that the measured super-synchronous frequency components in Figures 7.15, 7.16 are considerably larger than the predicted ones. The main reason for this is the limitation of the modal data to 300Hz, as previously mentioned. In fact, from Figures 7.18a-e it is clear that the super-synchronous components of the predicted response at the SFDs are considerable, implying that the predicted super-synchronous frequency components of the SFD forces are significant. However, the limitation of the modal data to 300 Hz results in the computational model of the casing behaving like an isolator that attenuates the transmission of the high-frequency components of the predicted SFD forces to the outer
parts of the casing. This limitation underlines the observation made earlier that a higher frequency range for the preliminary eigenvalue analysis would have been preferable.

### 7.3.2 Effect of SFD-housing misalignment

The LP rotor is supported on the casing via three SFD bearings. There is a possibility of the three SFD housings not being in line ‘before’ the rotor is put in place. The misalignment can be the consequence of the machining/assembling errors or the static deflection of the casing under its own weight. When one considers that the SFD radial clearance is of the order of a tenth of a millimetre it is pertinent to investigate the influence of misalignment on the predicted response. Such an investigation was first carried out in [50] with respect to a simple single-rotor test rig. A preliminary investigation into the effect of bearing housing misalignment on the unbalance response calculation of a whole-engine model was considered in this thesis. It was assumed that the offset of the housing of the ‘LP mid’ from the line going through the centres of the housings of the ‘LP front’ and ‘LP rear’ is in the $y$ direction and equal to $-0.3c_{LPmid}$, where $c_{LPmid}$ is the radial clearance of the SFD ‘LP mid’. This means that in eq. (3.32) $v_{s,2} = [0 \ 0.3c_{LPmid}]^T$ and $v_{s,i} = 0, \ i = 1, 3, 4, 5$. The responses were shown in Figs. 7.6, 7.8 and 7.10. One can notice that, although the amplitudes at some peaks are significantly affected, the effect of the misalignment appears negligible in most areas in this particular case.

### 7.4 Parametric Analysis – Proportionally Damped Linear Part

As already observed, the analysis of the previous section over-estimated the response of the casing vibration, especially around the peaks of the unbalance response (Figs. 7.6-7.11). It was surmised that the omission of damping distributed in the linear part of the structure was the likely cause. While it is certainly correct to say that the damping forces in an engine are mainly concentrated at the SFD bearings [10], the effect of even a tiny
amount of distributed damping is bound to have some (beneficial) effect. This hypothesis was tested by developing an IRM that accommodates distributed damping in the linear part and repeating the unbalance response calculations. The theory is developed in Section 7.4.1, tested against a slow conventional integrator in Section 7.4.2. Its results for the unbalance response are presented and discussed in Section 7.4.3.

Distributed damping in the linear part of the structure arises both from damping within the material of the individual components and, to a greater extent, from sliding contacts between the various assembled components. Distributed damping is very hard to quantify and is usually approximated from experiments [58]. Viscous damping is the preferred type of linear damping since it is not restricted to harmonic excitation. Proportional (or “Rayleigh type” [17]) damping is the most computationally convenient form of viscous damping. The developed IRM assumes this form of damping.

7.4.1 Theory

The derivation follows a similar procedure and notation to that of Chapter 3. The equation of motion in physical space is given by

\[
M\ddot{x} + C\dot{x} + Kx = f
\]  

(7.1)

where, for the proportionally damped linear part, the damping matrix \( C \) can be expressed as a linear combination of the mass matrix \( M \) and the stiffness matrix \( K \) as follows:

\[
C = \alpha M + \beta K
\]  

(7.2)
in which $\alpha$ and $\beta$ are constants arbitrarily chosen to represent damping distributed in the structure. For such a system, the equation of motion, can be transformed into modal space using the eigenvalues and eigenvectors of the undamped non-rotating linear part:

$$ \ddot{\mathbf{q}} + 2\zeta \omega \dot{\mathbf{q}} + \left( \omega^2 \right) \mathbf{q} = \mathbf{H}_f^T \mathbf{f} \quad (7.3) $$

i.e. $\omega$ remains the vector of undamped natural frequencies, and $\mathbf{H}_f$ the matrix of the corresponding mass-normalised eigenvectors. $\zeta$ is the vector of damping ratios given by the following formula [17]:

$$ \zeta_r = \begin{cases} (\alpha \omega_r + \beta \omega_r) / 2 & \text{for flexible modes } (\omega_r \neq 0) \\ 0 & \text{for rigid-body modes } (\omega_r = 0) \end{cases} \quad (7.4) $$

for $r = 1, 2, \ldots, R$. For the damped structure, it is reasonably assumed that all modes are under-damped, i.e. $\zeta < 1$ for $r = 1, 2, \ldots, R$. Let:

$$ \sin(\mu_r) = \zeta_r \quad (7.5) $$

Hence, $\cos(\mu_r) = \sqrt{1 - \zeta_r^2}$ and the damped natural frequencies are given by:

$$ \omega_{d,r} = \omega_r \cos(\mu_r) \quad (7.6) $$

As in Chapter 3, let $\mathbf{q}_k = \mathbf{q}(t_k)$ and $\dot{\mathbf{q}}_k = \dot{\mathbf{q}}(t_k)$. $\mathbf{q}_k$, $\dot{\mathbf{q}}_k$ are unknown whereas $\mathbf{q}_{k-1}$, $\dot{\mathbf{q}}_{k-1}$ are known. Using the Duhamel integral method [1, 17] over the interval $[t_{k-1}, t_k]$, where $t_k = t_{k-1} + h$: 

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\[ q_k = \dot{q}_{k-1} \cdot \exp(-\xi \cdot \omega h) \cdot \sin(\omega_d h) / \omega_d + \]
\[ q_{k-1} \cdot \exp(-\xi \cdot \omega h) \cdot \left[ \cos(\omega_d h) + \xi \cdot \omega \cdot \sin(\omega_d h) / \omega_d \right] + \]
\[ \int_0^h \mathbf{H}_i^T \mathbf{f}(\tau) \cdot \exp(\xi \cdot (\omega h - \omega_d \tau)) \cdot \sin(\omega_d h - \omega_d \tau) / \omega_d \ d\tau \]

(7.7a)

\[ \dot{q}_k = \dot{q}_{k-1} \cdot \exp(-\xi \cdot \omega h) \cdot \left[ \cos(\omega_d h) - \xi \cdot \omega \cdot \sin(\omega_d h) / \omega_d \right] - \]
\[ q_{k-1} \cdot \exp(-\xi \cdot \omega h) \cdot \xi \cdot \omega \cdot \sin(\omega_d h) / \omega_d + \]
\[ \int_0^h \mathbf{H}_i^T \mathbf{f}(\tau) \cdot \exp(\xi \cdot (\omega h - \omega_d \tau)) \cdot \left\{ \cos(\omega_d h - \omega_d \tau) - \xi \cdot \omega \cdot \sin(\omega_d h - \omega_d \tau) / \omega_d \right\} \ d\tau \]

(7.7b)

Over the interval \( 0 \leq \tau \leq h \), \( f(\tau) \) is approximated as:

\[ f(\tau) = f_{k-1} + (f_k - f_{k-1}) \frac{\tau}{h} \]

(7.8)

where \( f_k = f(x_k, \dot{x}_k, \tau_k) \) and \( x_k = x(t_k), \dot{x}_k = \dot{x}(t_k) \). Hence, after substituting eq. (7.8) into (7.7a, b), expanding the trigonometric functions under the integral signs, evaluating the resulting integrals and simplifying, one obtains:

\[ q_k = \dot{q}_k + \]
\[ \left( \mathbf{H}_i^T f_k - \mathbf{H}_i^T f_{k-1} \right) \cdot \left\{ \omega h - 2 \sin(\mu) - \exp(-\xi \cdot \omega h) \cdot \sin(\omega_d h - 2 \mu) / \cos(\mu) \right\} / \left\{ h \omega \right\}^3 \]

(7.9a)

\[ \dot{q}_k = \dot{q}_k + \left( \mathbf{H}_i^T f_k - \mathbf{H}_i^T f_{k-1} \right) \cdot \left\{ \mu - \exp(-\xi \cdot \omega h) \cdot \cos(\omega_d h - \mu) / \cos(\mu) \right\} / \left\{ h \omega \right\}^2 \]

(7.9b)

where \( \mu = [\mu_1 \ldots \mu_r]^T \) and
\[ \dot{q}_k = \dot{q}_{k-1} \cdot \exp(-\xi \omega h) \cdot \sin(\omega_d h) / \omega_d + \]
\[ q_{k-1} \cdot \exp(-\xi \omega h) \cdot \cos(\omega_d h - \mu) / \cos(\mu) + \]
\[ H_T f_{k-1} \cdot \left[ \mathbf{n} - \exp(-\xi \omega h) \cdot \cos(\omega_d h - \mu) / \cos(\mu) \right] / (\omega^2) \] (7.10a)

\[ \ddot{q}_k = \dot{q}_{k-1} \cdot \exp(-\xi \omega h) \cdot \cos(\omega_d h + \mu) / \cos(\mu) - \]
\[ q_{k-1} \cdot \exp(-\xi \omega h) \cdot \omega \cdot \sin(\omega_d h) / \cos(\mu) + \]
\[ H_T f_{k-1} \cdot \exp(-\xi \omega h) \cdot \sin(\omega_d h) / \omega_d \] (7.10b)

Pre-multiplying eqs. (7.9 a, b) by \( H_x \) results in the IRM equations:

\[ x_k = \dot{x}_k + R_{xf}(h) f_k(x_k, \dot{x}_k, t_k) - f_{k-1} \] (7.11a)
\[ \ddot{x}_k = \dot{x}_k + S_{xf}(h) f_k(x_k, \dot{x}_k, t_k) - f_{k-1} \] (7.11b)

where the discrete time domain analogues of the receptance and mobility matrices are given by:

\[ R_{xf}(h) = \sum_{r=1}^{R} \left\{ \psi_x^{(r)} \psi_T^{(r)T} \left[ \omega h - 2 \sin(\mu_r) - \exp(-\xi \omega_r h) \sin(\omega_d h - 2 \mu_r) / \cos(\mu_r) \right] / (\omega h^2) \right\} \] (7.12a)
\[ S_{xf}(h) = \sum_{r=1}^{R} \left\{ \psi_x^{(r)} \psi_T^{(r)T} \left[ 1 - \exp(-\xi \omega_r h) \cos(\omega_d h - \mu_r) / \cos(\mu_r) \right] / (\omega h^2) \right\} \] (7.12b)

The formulae for the rigid-body modes and for the inclusion of gyroscopic effect are exactly the same as those presented in Chapter 3.
7.4.2 Computational Validation of Proportionally-damped IRM

This section presents the validation of the IRM against a conventional integration technique (CIM), e.g. solver ode23s© available in Matlab®. For illustrative purposes, the validation was carried out using the transient solutions for two conditions of rotational speed: (A) $\Omega_{LP} = 1440$ rpm, $\Omega_{HP} = 10080$ rpm, (B) $\Omega_{LP} = 2602$ rpm, $\Omega_{HP} = 12562$ rpm. In both cases, the time-marching solutions were started from conditions corresponding to zero relative displacements and velocities of the SFD journals relative to their housings.

<table>
<thead>
<tr>
<th>Damping distribution</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\xi_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>‘Damping 1’</td>
<td>0</td>
<td>2.6525×10^{-6}</td>
<td>2.5×10^{-3}</td>
</tr>
<tr>
<td>‘Damping 2’</td>
<td>0.1346</td>
<td>2.6161×10^{-6}</td>
<td>2.5×10^{-3}</td>
</tr>
<tr>
<td>‘Damping 3’</td>
<td>0</td>
<td>1.0610×10^{-5}</td>
<td>1.0×10^{-2}</td>
</tr>
<tr>
<td>‘Damping 4’</td>
<td>0.5463</td>
<td>0</td>
<td>1.0×10^{-2}</td>
</tr>
</tbody>
</table>

Table 7.1: Damping distributions

Four cases of damping are considered. Table 7.1 give the values of the coefficients $\alpha$ and $\beta$ chosen for different damping distributions. The resultant damping ratios corresponding to the flexible modes are shown in Fig. 7.19. As can be seen, the damping ratio does not exceed 1%.

Figures 7.20a-c show the first HP-revolution of the transient solution for speed condition (A) obtained using CIM and IRM for the case ‘Damping 1’ (Table 7.1), with ‘Unsealed’ SFDs (as defined in the beginning of Section 7.3). In row (a) are the orbits of the journals of the SFDs within the corresponding clearance circles. Meanwhile, the variation of the velocities of the journals relative to the respective housings are shown in...
row (b), in which each plot was scaled by the factor attached to the top side of the plot axes. Row (c) presents the SFD forces with magnitudes normalised in the same way as for the velocity plots. In each of the plots, the IRM solution (red-dotted line) was superimposed on the CIM one (blue-solid line). The agreement between the two sets of solution is seen to be excellent. Such a good agreement was maintained when the transient solution was carried on, as shown in Figs. 7.21a-c over the 80th HP-revolution and in Figs. 7.22a-c for the 5600th HP-revolution, where the steady-state was reached. Figures 7.23a-c give another example of the agreement between the two computational techniques, using the solution obtained for speed condition (B), with the ‘Damping 1’ and ‘Unsealed’ conditions applied.

While ‘Damping 1’ is very slight, the damping ratio associated with each mode in the case of ‘Damping 3’ is 4 times greater. The IRM was tested for this case of damping, “Tightly sealed” SFDs (see beginning of Section 7.3.0) and speed case (A). Figures 7.24a-c indicate that, for such tight end-sealing level and high damping, the agreement between the IRM and CIM are also very good.

Both ‘Damping 1’ and ‘Damping 3’ presume that the damping matrix of the structure is independent of the mass matrix \((\alpha = 0)\). ‘Damping 2’ pattern was used as an example of damping matrix being linearly dependent on both the mass and the stiffness matrices. Constants \(\alpha\) and \(\beta\) were chosen simply to introduce greater damping ratios for the low-frequency modes as shown in Fig. 7.19. Like the previous cases of damping, the CIM solutions were matched perfectly by the IRM one, as can be seen in Figs. 7.25 and 7.26.

The new improved IRM has been ascertained to be at least as accurate as the CIM. Moreover, it retains all the advantages highlighted in Chapter 3. In fact, it was observed through the examples presented that the inclusion of proportional damping to the IRM had little effect on the time-efficiency of the IRM e.g. for the calculation of Figure 7.20, the
IRM took 15 mins to integrate over 80 HP shaft revolutions whereas the CIM took 450 mins.

7.4.3 Unbalance response of proportionally-damped engine structure

The improved IRM was used to investigate the effect that a slight amount of distributed damping potentially has on the predicted engine vibration. Figures 7.27-7.29 show the full speed-responses calculated for the ‘Tightly sealed’ case, where ‘Damping 1’ and ‘Damping 3’ were used (Table 7.1), and for the ‘Unsealed’ case with all four distributions of damping (Table 7.1). Comparing these figures with those for the undamped linear part (Figures 7.6, 7.8 and 7.10) it is clear that the distributed damping has a profound effect on the predicted vibration levels at the casing, bringing these significantly closer to the measured levels in Figures 7.7, 7.9 and 7.11. Of all the cases considered in Figures 7.27-7.29, the combination of ‘Damping 1’ and ‘Unsealed’ appears to return the closest agreement with the measured responses. Figure 7.30 shows that such a slight level of distributed damping has a very minor effect on the relative motions at the SFD bearings, despite making a significant contribution to the dissipation of the energy of vibration transmitted through the casing.

7.5 CONCLUSIONS

This chapter presented the final part of the research, in which a first-ever attempt was made to relate a limited amount of measured vibration data from a manufacturer’s test engine with the predictions from a whole engine-model of the same engine. The aim of the comparison was merely restricted to examining the order of magnitude of the vibration levels and general trends in the data. The test analysed involved the monitoring of the vibration response at three points on the casing due to the application of unbalance to a single point on the HP rotor with the rotor speeds following a typical run-up procedure.
Residual unbalance and aeroelastic phenomena were not accounted for in the modelling. The first part of the theoretical analysis used the IRM based on the assumption of negligible damping in the linear part of the structure. Reasonable agreement between predicted and measured frequency spectra was obtained, although high frequency components in the casing vibration were under-predicted due to limitations in the modal model provided. The predictions over-estimated the casing vibration levels, especially around the peaks of the unbalance response. It was ascertained that this discrepancy could not be attributed to bearing housing misalignment, which for this particular case only marginally affected the results. The disagreement in vibration levels was attributed to the omission of structural damping in the linear part of the assembly. This hypothesis motivated the development of the existing IRM to accommodate proportional damping. The enhanced IRM was checked against a conventional integrator and shown to retain all its advantages with regard to accuracy and computation speed. The parametric analysis using the damped IRM showed that a slight amount of distributed damping in the linear part of the structure significantly affected the vibration levels, bringing them closer to the measured levels, while having little effect on the SFD orbits. Hence, the important conclusion drawn from this part of the research is that a slight amount of distributed damping in the structure should be included if the analyst is interested in predicting casing vibration levels. If, on the other hand, the analyst is only interested in bearing performance i.e. smooth running of the rotors (determined by the SFD orbits – their size and position in the clearance), then distributed damping can be omitted, as is typically done in SFD research. However, since the inclusion of distributed damping in the IRM hardly introduces any additional cost, one may just as well include it.
Figure 7.1: Schematic of the test engine

Figure 7.2: Locations of sensors
Figure 7.3: Variation of rotor speeds (NH = 16000rpm)

Figure 7.4: Point-receptance function of undamped non-rotating linear part evaluated at the centre $B_4$ of the housing of the SFD ‘HP rear’ in $y$ direction
Figure 7.5: Axial distributions of weight (a) and polar moment of inertia (b) of the rotors
Figure 7.6: 1HP predicted at 'A_1'; Undamped structure

Figure 7.7: 1HP measured at 'A_1'

Figure 7.8: 1HP predicted at 'A_2'; Undamped structure

Figure 7.9: 1HP measured at 'A_2'
Figure 7.10: 1HP predicted at 'A3'; Undamped structure

Figure 7.11: 1HP measured at 'A3'
**Figure 7.12:** Z-modulation plot of frequency spectra of velocity at point 'A1'

(a) – Measured;
(b) – Predicted, with undamped structure and 'Unsealed' SFDs
Figure 7.13: z-modulation plot of frequency spectra of velocity at point 'A2'
(a) – Measured;
(b) – Predicted, with undamped structure and 'Unsealed' SFDs
Figure 7.14: \( z \)-modulation plot of frequency spectra of velocity at point 'A3'
(a) – Measured;
(b) – Predicted, with undamped structure and 'Unsealed' SFDs
Figure 7.15a: Measured frequency spectrum of the velocity at $A_1$ with HP speed at 80% NH

Figure 7.15b: Predicted frequency spectra of the velocity at $A_1$
Figure 7.16a: Measured frequency spectrum of the velocity at A_2 with HP speed at 96% NH

Figure 7.16b: Predicted frequency spectra of the velocity at A_2
Figure 7.17a: Measured frequency spectrum of the velocity at A₃ with HP speed at 94% NH

Figure 7.17b: Predicted frequency spectra of the velocity at A₃
Figure 7.18a: Predicted frequency spectra of the relative velocity at ‘LP front’, with magnitude max-normalised

Figure 7.18b: Predicted frequency spectra of the relative velocity at ‘LP mid’, with magnitude max-normalised

Figure 7.18c: Predicted frequency spectra of the relative velocity at ‘HP front’, with magnitude max-normalised
Figure 7.18d: Predicted frequency spectra of the relative velocity at ‘HP rear’, with magnitude max-normalised.

Figure 7.18e: Predicted frequency spectra of the relative velocity at ‘LP rear’, with magnitude max-normalised.

Figure 7.19: Assumed proportional damping.
Figure 7.20: Validation of IRM-damped (red-dotted line) against CIM (blue-solid line); Damping 1 - Unsealed SFDs; Speed case (A), the 1st HP revolution;
(a) – SFD orbits
(b) – SFD velocities (mm/s)
(c) – SFD forces (N)
Figure 7.21: Validation of IRM-damped (red-dotted line) against CIM (blue-solid line);
Damping 1 - Unsealed SFDs;
Speed case (A), the 80th HP revolution;
(a) – SFD orbits
(b) – SFD velocities (mm/s)
(c) – SFD forces (N)
Figure 7.22: Validation of IRM-damped (red-dotted line) against CIM (blue-solid line); Damping 1 - Unsealed SFDs; Speed case (A), the 5600th HP revolution; (a) – SFD orbits (b) – SFD velocities (mm/s) (c) – SFD forces (N)
Figure 7.23: Validation of IRM-damped (red-dotted line) against CIM (blue-solid line); Damping 1 - Unsealed SFDs; Speed case (B), the 1600th HP revolution; 
(a) – SFD orbits 
(b) – SFD velocities (mm/s) 
(c) – SFD forces (N)
Figure 7.24: Validation of IRM-damped (red-dotted line) against CIM (blue-solid line);  
Damping 3 - Tightly sealed SFDs;  
Speed case (A), the 1360th HP revolution;  
(a) – SFD orbits  
(b) – SFD velocities (mm/s)  
(c) – SFD forces (N)
**Figure 7.25:** Validation of IRM-damped (red-dotted line) against CIM (blue-solid line); Damping 2 - Unsealed SFDs; Speed case (A), the 1st HP revolution;  
(a) – SFD orbits  
(b) – SFD velocities (mm/s)  
(c) – SFD forces (N)
Figure 7.26: Validation of IRM-damped (red-dotted line) against CIM (blue-solid line); Damping 2 - Unsealed SFDs; Speed case (A), the 800th HP revolution; (a) – SFD orbits (b) – SFD velocities (mm/s) (c) – SFD forces (N)
Figure 7.27: 1HP predicted at 'A1'; Proportionally-damped structure
(a) – 'Tightly sealed'; (b) - 'Unsealed'
Figure 7.28: 1HP predicted at point 'A2'; Proportionally-damped structure (a) – 'Tightly sealed'; (b) - 'Unsealed'
Figure 7.29: 1HP predicted at 'A3'; Proportionally-damped structure (a) – 'Tightly sealed'; (b) - 'Unsealed'
Figure 7.30: 1HP of relative velocity at SFDs

- **LPfront**
- **LPmid**
- **HPfront**
- **HPrear**
- **LPrear**

**Legend:**
- Undamped
- Damping 1
8 CONCLUSIONS AND PROPOSALS FOR FUTURE RESEARCH

8.1 CONCLUSIONS

Prior to the work of this thesis, the nonlinear dynamic analysis of real aero-engines was severely restricted. The reason for this was the inadequacy of the various available computational techniques when dealing with a very large number of structural modes. The main contribution of this thesis has been to resolve this problem through the development of highly efficient time/frequency domain solvers for whole-engine models. This enabled the computational analyses of the unbalance response of real twin/three spool engines, which was done for the first time in this thesis.

The proposed analysis of the nonlinear rotating assembly used modal data calculated from a one-off analysis of the linear part of the engine at zero speed. Any convenient finite element (FE) package (e.g. Nastran®) can be used for the linear computation. Specially-written Matlab® routines were used for the subsequent nonlinear computation. Due to the complementary nature and relative merits of time and frequency domain solution techniques, both options were developed. The two novel solvers created in this project were:

- An implicit time domain solver termed the Impulsive Receptance Method (IRM).

- A whole-engine Receptance Harmonic Balance Method (RHBM) for the frequency domain calculation of periodic solutions.

As shown in Chapter 3, the IRM equations relate the relative displacements and velocities at the SFDs with the motion-dependent forces and other excitations acting on the linear part. Hence, the IRM’s computational efficiency is largely immune to the number of
modes since the number of equations to be solved at each time step is dependent only on the number of nonlinear elements. This solver has been tested extensively on two/three spool engine models and shown to be remarkably faster than a highly robust conventional implicit solver while achieving similar levels of accuracy. It has also been shown in Chapter 4 to be far more reliable than another popular implicit solver (Newmark-beta). In Chapter 7 the IRM was enhanced to accommodate distributed damping in proportional form.

The whole-engine RHBM developed in Chapter 5 is a significant development of the earlier elementary RHBM (used on simple single-shaft test rigs) that is capable of dealing with the complexities of a real engine structure. Among other complexities, the whole-engine RHBM can account for statically indeterminate rotors on unsupported SFDs, such as the LP rotor of the twin-spool engine in Chapters 3, 4, 5 and 7 or the assembly of the LP and IP rotors in the three-shaft engine in Chapter 6. The size of the RHBM system of equations depends on the number of nonlinear elements and the number of harmonics of the Fourier series chosen to describe the steady-state solution. RHBM was tested for both single-frequency unbalance excitation (SFU- unbalance distribution confined to one rotor) and multi-frequency unbalance excitation (MFU- unbalance on more than one rotor, rotating at different speeds). The arc-length continuation technique used in [10] was successfully implemented to advance the RHBM solution process over a range of speeds for SFU. Arc-length continuation also worked for MFU with constant speed ratio between the rotors. MFU computation times were longer than for SFU due to the larger number of harmonics required in the former case. However, for MFU as well as SFU, RHBM resulted in vast time-savings relative to the time domain approach and returned excellent correlation with IRM. The excellent correlation between these two vastly different methods for both two-shaft and three-shaft engines lent definitive validation to the computational soundness of both methods.
Chapter 6 considered MFU excitation of a three-spool engine with a realistic speed relation between the rotors. This presented two challenges when implementing RHBM: (a) the variation of the rotor speed ratios with rotational speed, making the available continuation technique unsuitable; (b) the necessity to include a very large number of harmonics in order to accommodate the exact speed ratios. Chapter 6 proposed a solution in which the rotational speeds of the unbalanced rotors were slightly altered to yield a manageable number of required harmonics. The minor changes to the response in terms of the magnitudes of its main frequency components suggest that such strategy can be used in the practical analysis of real whole-engine models. Moreover, instead of implementing RHBM with arc-length continuation, the RHBM was used to “finish off” to steady-state unfinished transient IRM solutions obtained by integrating over a small number of shaft revolutions with crude tolerance settings. The efficiency of this combined use of RHBM and IRM was demonstrated for the three-shaft engine.

The theoretical parametric study of the three-spool engine (Chapter 6) clearly showed the dynamic interaction between the rotors and the influence of each SFD on the overall performance of the whole structure. The outcomes indicated that, with tighter sealed SFDs, the sizes of SFD orbits tend to reduce and the forces transmitted to the casing increase. This characteristic was also observed with the two-shaft models in Chapters 3 and 7. The contrasting responses of the system to different unbalance distributions and the influence of the two main geometric parameters of the SFD (land length and radial clearance) have been presented. Moreover, the use of suitably preloaded vertically oriented “bump-springs” at the SFDs of the heavy low-pressure rotor was shown to produce a significant improvement in journal lift, allowing for smoother running of the shaft. The results of this parametric study suggest that an optimal configuration of SFDs can be achieved. They also clearly show that findings from studies on a single SFD or one rotor considered in isolation are limited in meaning.
The final part of the research (Chapter 7) was a first-ever attempt to relate measured vibration data from a manufacturer’s test engine with the predictions from a whole engine-model of the same engine using the IRM. The most important finding from this chapter was that the consideration of a slight amount of distributed damping in the linear part of the structure significantly affected the predicted casing vibration levels, bringing them closer to measured levels. This damping however had little effect on the SFD orbits.

The reader is referred to the specific conclusions of Chapters 3-7 for further details.

8.2 PROPOSALS FOR FUTURE RESEARCH

The research of this thesis can be developed in the following ways:

- **An experimentally validated parametric and optimisation analysis.** This would involve an application of the computation techniques developed here to a thorough parametric analysis of a real aero-engine to obtain optimal parameters of the SFDs used. Such a study would be validated by comprehensive experimental data, which would also include MFU excitation.

- **Development of the RHBM to better deal with MFU excitation, particularly for the case of realistic speed relations between the rotors.** There are two areas that can be considered:
  
  o **Development of the arc-length continuation scheme.** The current continuation scheme, developed in [10], works for MFU with constant speed ratio between the unbalanced rotors. The reason is that it was designed for the case where the fundamental frequency of the solution does not change its ratios relative to the unbalance excitation frequencies as the solution progresses along the speed response. For each point on the speed response, the proposed continuation scheme will approximate the exact speed ratios with ratios of low integers as
proposed in Chapter 6 (Section 6.3.2). The initial approximation for the current point on the speed response is then obtained from the previously computed solution point after having accounted for the fact that the fundamental frequency of the solution has changed its ratio relative to the unbalance excitation frequencies. Moreover, the size of the problem can be cut down by omitting harmonics that do not contribute to the response. Information on which harmonics are not likely to contribute to the response could be obtained from the previously computed solution point.

- **Development of a quasi-periodic solution technique.** This would use more than one fundamental frequency to represent the steady-state solution, as proposed in [61].

- **Development of a stability-check routine for the whole-engine RHBM.** Although the absence of a stability-check routine was not problematic in this thesis, it is clearly desirable to have one since it minimises the use of time marching. However, as discussed in Section 2.3, current techniques cannot handle the huge number of modes considered in the applications of this thesis. Hence, the development of such a routine is a very challenging task. The Floquet perturbation problem has to be formulated in terms of all the perturbed modal state variables [10]. However, it may be possible to condense it using the impulsive receptance approach introduced in this thesis.

- **The inclusion of more advanced SFD models in the whole-engine solvers.** The end-seal factor model used in this thesis (Appendix A1) requires an empirical factor. Attempts to theoretically relate this factor to the SFD parameters have proved elusive [37]. Moreover, the SFD model cannot account for details such as shallow grooves and feed-ports. The use of the full finite difference approach (Section 2.4) to compute the SFD forces for each of the 5-6 SFDs in a typical engine would cripple even the fast
solvers developed in this thesis. However, it is possible to use identification techniques to accurately identify an advanced numerical model of an SFD [62]. This would enable the identified model to be used in the dynamic analysis of any system in which that particular bearing is installed, thereby enabling the deployment of advanced SFD models in whole-engine analysis.
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APPENDICES

A1 “λ-theory” MODEL FOR SFD

With reference to Figure A1.1, where B and J are the centres of the bearing housing and the journal respectively, the Cartesian SFD forces \( Q_{x,y} \), are given by:

\[
\begin{bmatrix}
Q_x \\
Q_y
\end{bmatrix} =
\begin{bmatrix}
-\sin \psi & -\cos \psi \\
\cos \psi & -\sin \psi
\end{bmatrix}
\begin{bmatrix}
Q_R \\
Q_T
\end{bmatrix}
\]

(A1.1)

where

\[
\begin{bmatrix}
Q_R \\
Q_T
\end{bmatrix} = -n_L RL \int_{-L/2}^{L/2} \int_{0}^{2\pi} p_L(\theta,Z) \begin{bmatrix}
\cos \theta \\
\sin \theta
\end{bmatrix} d\theta dZ
\]

(A1.2)

where \( n_L \) is the number of SFD lands, \( R \) the bearing radius, \( L \) the land length and \( p_L(\theta,Z) \) is the truncated pressure distribution in the oil film:

\[
p_L(\theta,Z) = \begin{cases} 
p(\theta,Z) & \text{if } p(\theta,Z) > p_{cav} \\
p_{cav} & \text{if } p(\theta,Z) \leq p_{cav}
\end{cases}
\]

(A1.3)

…where \( p_{cav} \) is the cavitation pressure. The untruncated pressure distribution \( p(\theta,Z) \) is the combination of the short and long bearing solutions of the incompressible Reynolds Equation:

\[
p(\theta,Z) = p_{\text{short}}(\theta,Z) + \lambda p_{\text{long}}(\theta)(0.5 - Z/L)
\]

(A1.4)

…where \( \lambda \) is the end-sealing factor and:
\[ p_{\text{short}}(\theta,Z) = 6\eta \left( \frac{L}{c} \right)^2 \frac{(e \psi \sin \theta + \dot{\epsilon} \cos \theta)(Z^2 - 0.25)}{(1 + \epsilon \cos \theta)^3} + p_S \left( \frac{Z}{L} + 0.5 \right) \] (A1.5)

\[ p_{\text{long}}(\theta) = \eta \left( \frac{R}{c} \right)^2 \left\{ -12\psi \left( \frac{e}{2 + \epsilon^2} \right) \frac{(2 + \epsilon \cos \theta)\sin \theta}{(1 + \epsilon \cos \theta)^2} \right\} - \eta \left( \frac{L}{c} \right)^2 \frac{\dot{\epsilon}}{(1 + \epsilon)} + p_S \] (A1.6)

…where \( \epsilon = e/c \), \( c \) is the radial clearance, \( \eta \) the oil viscosity and \( p_S \) the supply pressure.

Figure A1.1: Transverse and axial cross-sections of SFD
For each SFD no. \( i \), \( i = 1 \ldots N \), calculate the partial derivatives:

\[
\frac{\partial}{\partial \vec{v}_i} \begin{bmatrix}
\vec{p}_i \\
\rho_{i,\cos}
\end{bmatrix}, \quad \frac{\partial}{\partial \vec{v}_i} \begin{bmatrix}
\vec{p}_i \\
\rho_{i,\sin}
\end{bmatrix}, \quad \frac{\partial}{\partial \vec{v}_i} \begin{bmatrix}
\vec{p}_i \\
\rho_{i,\cos}
\end{bmatrix}, \quad \frac{\partial}{\partial \vec{v}_i} \begin{bmatrix}
\vec{p}_i \\
\rho_{i,\sin}
\end{bmatrix}, \quad k = 1 \ldots K
\]  

(A2.1)

Now:

\[
\frac{\partial \vec{p}}{\partial \vec{z}} = \begin{bmatrix}
0_{2N \times n_{\text{extra}}}
\frac{\partial \vec{p}}{\partial \vec{v}} & \frac{\partial \vec{p}}{\partial \vec{v}_{(1)\cos}} & \frac{\partial \vec{p}}{\partial \vec{v}_{(1)\sin}} & \cdots & \frac{\partial \vec{p}}{\partial \vec{v}_{(K)\cos}} & \frac{\partial \vec{p}}{\partial \vec{v}_{(K)\sin}}
\end{bmatrix}
\]  

(A2.2)

where:

\[
\frac{\partial \vec{p}}{\partial \vec{v}} = \text{blkdiag} \left\{ \frac{\partial \vec{p}_1}{\partial \vec{v}_1}, \ldots, \frac{\partial \vec{p}_N}{\partial \vec{v}_N} \right\},
\]

(A2.3a)

\[
\frac{\partial \vec{p}}{\partial \vec{v}_{(m)\cos}} = \text{blkdiag} \left\{ \frac{\partial \vec{p}_1}{\partial \vec{v}_{1\cos}}, \ldots, \frac{\partial \vec{p}_N}{\partial \vec{v}_{N\cos}} \right\}, \quad \frac{\partial \vec{p}}{\partial \vec{v}_{(m)\sin}} = \text{blkdiag} \left\{ \frac{\partial \vec{p}_1}{\partial \vec{v}_{1\sin}}, \ldots, \frac{\partial \vec{p}_N}{\partial \vec{v}_{N\sin}} \right\}, \quad m = 1 \ldots K
\]

(A2.3b, c)

For \( k = 1 \ldots K \):

\[
\frac{\partial \rho_{(k)\cos}}{\partial \vec{z}} = \begin{bmatrix}
0_{2N \times n_{\text{extra}}}
\frac{\partial \rho_{(k)\cos}}{\partial \vec{v}} & \frac{\partial \rho_{(k)\cos}}{\partial \vec{v}_{(1)\cos}} & \frac{\partial \rho_{(k)\cos}}{\partial \vec{v}_{(1)\sin}} & \cdots & \frac{\partial \rho_{(k)\cos}}{\partial \vec{v}_{(K)\cos}} & \frac{\partial \rho_{(k)\cos}}{\partial \vec{v}_{(K)\sin}}
\end{bmatrix}
\]

(A2.4)

where, for each \( k \):

\[
\frac{\partial \rho_{(k)\cos}}{\partial \vec{v}} = \text{blkdiag} \left\{ \frac{\partial \rho_{(k)\cos}}{\partial \vec{v}_1}, \ldots, \frac{\partial \rho_{(k)\cos}}{\partial \vec{v}_N} \right\}
\]

(A2.5a)
For $\partial p^{(k)}_{\text{sin}} / \partial z$, replace the subscript “cos” by “sin” in the numerators of the partial derivative expressions of eqs. (A2.4), (A2.5).

A3 VALIDATION OF THE FAST NEWMARK-BETA METHOD

In this section presents a validation of the suggested Fast Newmark-beta Method against the IRM and RHBM. Figure A3.1 shows the schematic of a simple twin-shaft model.

The system is an assembly of two subsystems: a) rotor 1; b) rotor 2 and the support structure. The modal parameters of the model are given in Tables A3.1-4. The rotational speed of rotor 1 is $\Omega_1 = 3000 \text{ rpm}$ and rotor 2 $\Omega_2 = 6000 \text{ rpm}$. The equivalent weight of rotor at J is equal to 267.64 N in y direction.
<table>
<thead>
<tr>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\omega_r}{2\pi}$ (Hz)</td>
<td>0</td>
<td>0</td>
<td>286.45</td>
<td>286.45</td>
<td>912.14</td>
<td>912.14</td>
<td>94.61</td>
<td>95.07</td>
<td>552.76</td>
<td>552.77</td>
</tr>
<tr>
<td>$\Psi_J^{(r)}$ (kg$^{-0.5}$)</td>
<td>$\begin{bmatrix} -0.229 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ 0.229 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.228 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ -0.228 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.195 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ 0.195 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\Psi_B^{(r)}$ (kg$^{-0.5}$)</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.240 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -0.158 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ 0.160 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\Psi_{u(1)}^{(r)}$ (kg$^{-0.5}$)</td>
<td>$\begin{bmatrix} -0.204 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ 0.204 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.119 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ -0.119 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.008 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ 0.008 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\Psi_{u(3)}^{(r)}$ (kg$^{-0.5}$)</td>
<td>$\begin{bmatrix} 0 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.235 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.085 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ -0.085 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\Psi_B^{(r)}$ (kg$^{-0.5} m^{-1}$)</td>
<td>$\begin{bmatrix} 0 \ -0.241 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -0.241 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ 0.307 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.307 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1.238 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

Table A3.1 Modal parameter of the system in Figure 4.2
\( I_{p1} \) (rotor 1) \hspace{1cm} 0.055216 \text{ (kgm}^2\text{)}

\( I_{p2} \) (rotor 2) \hspace{1cm} 0.010215 \text{ (kgm}^2\text{)}

**Table A3.2** Polar moments of inertia

\( U_1 \) (on rotor 1) \hspace{1cm} 0.545\times10^{-3} \text{ (kgm)}

\( U_2 \) (on rotor 2) \hspace{1cm} 0.2225\times10^{-3} \text{ (kgm)}

**Table A3.3** Mass-Radius Imbalance Product

<table>
<thead>
<tr>
<th>Radial Clearance (mm)</th>
<th>Land Length (mm)</th>
<th>Number of Land</th>
<th>Radius (mm)</th>
<th>Oil Viscosity (Nsm}^{-2}\text{)}</th>
<th>Inlet Oil Pressure (psi)</th>
<th>Ambient Pressure (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.11</td>
<td>6.26</td>
<td>2</td>
<td>69.89</td>
<td>10</td>
<td>30</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table A3.4** SFD parameters